UNSW Mathematics Society

Engineering Mathematics 2D/2E

Seminar I / II

Presented by: Gerald Huang

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Seminar Overview

Part I: Functions of Several Variables

Precursor: One variable derivatives

One variable derivatives

Recall that, for a single variable function,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

We can rewrite the h to denote "some change in x" as

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Partial differentiation

For two or more variables, this definition is insufficient. Instead, we use **partial derivatives**.

Definition 1.1: Partial derivatives

Let $z = f(x, y)$ be some function of two variables. Then

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x},$$

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}.$$

Computing partial derivatives

Consider the function $f(x, y)$. Since x and y are independent of each other, then x is a **constant** in terms of y and y is a **constant** in terms of x.

Computing partial derivatives

Example 1: Computing partial derivatives

Let $f(x,y) = \dfrac{y}{x+y}$. Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial x} = -\frac{y}{(x+y)^2}, \quad \frac{\partial f}{\partial y} = \frac{x}{(x+y)^2}.$$

Multivariable chain rule

If $z = f(x, y)$, $x = x(t)$, $y = y(t)$, then:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

If $z = f(x, y)$, $x = x(u, v)$, $y = y(u, v)$, then:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u},$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Multivariable Chain Rule

Example 2

Suppose that $z = x^2 + 4xy$ where $x = u^3 \ln v$ and $y = uv^2$. Find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

$$\frac{\partial z}{\partial u} = 2u^3 \ln(v)(8v^2 + 3u^2 \ln(v)),$$
$$\frac{\partial z}{\partial v} = \frac{2(2u^4 v^2 + u^6 \ln(v) + 4u^4 v^2 \ln(v))}{v}$$

Taylor series of single variable functions

The **Taylor series** of a single variable function $f(x)$ at the point $(a, f(a))$ is given by

$$f(x) \approx \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Extend this to multivariable functions...

Taylor series of multivariable functions

The **Taylor series** of a multivariable function $f(x, y)$ at the point (a, b) is given by

$$f(x, y) = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x-a) + \frac{\partial f}{\partial y}(a, b)(y-b)$$
$$+ \frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}(a, b)(x-a)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(a, b)(x-a)(y-b)\right.$$
$$\left. + \frac{\partial^2 f}{\partial y^2}(a, b)(y-b)^2\right] + \cdots$$

- Red signifies **first derivative**; Blue signifies **second derivative**.
- Very rarely does MATH2018/2019 deal with third derivative and higher.

Taylor series of multivariable functions

Example: (17S2, Q1ai)

Calculate the Taylor series expansion of the function $f(x, y) = \ln(x + y)$ about the point $(1, 0)$ up to and including quadratic terms.

$$f(x, y) \approx (x-1) + y - \frac{1}{2}\left[(x-1)^2 + 2y(x-1) + y^2\right].$$

Error approximation

Definition 1.4: Error approximation

$$|\Delta f| \leq \left|\frac{\partial f}{\partial x}\right| |\Delta x| + \left|\frac{\partial f}{\partial y}\right| |\Delta y|$$

This equation gives you the maximum error in f in terms of the errors in x and y.

Error approximation

Application of Error Approximation

The volume V of a cone with radius r and perpendicular height h is given by $V = \frac{1}{3}\pi r^2 h$. Determine the maximum absolute error and the maximum percentage error in calculating V given that $r = 5$ cm and $h = 3$ cm to the nearest millimetre.

Leibniz Rule

Definition 1.5: Leibniz Rule

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(x,t)\,dt = \int_{u(x)}^{v(x)} \frac{\partial f}{\partial x}\,dt + f(x,v(x))\frac{dv}{dx} - f(x,u(x))\frac{du}{dx}.$$

Leibniz Rule

Example: (18S2, Q1 iv)

You are given that

$$\int_0^\infty \frac{1}{\alpha^2 + x^2}\, dx = \frac{\pi}{2}\alpha^{-1}.$$

Use Leibniz' theorem to find the following integral in terms of α

$$\int_0^\infty \frac{1}{(\alpha^2 + x^2)^2}\, dx.$$

Leibniz Rule

Example: (17S2, Q1 e)

You are given the following integral

$$\int_0^a \frac{1}{(x^2+a^2)^{1/2}}\,dx = \sinh^{-1}(1).$$

Use Leibniz' rule to evaluate

$$\int_0^a \frac{1}{(x^2+a^2)^{3/2}}\,dx.$$

Part II: Extreme values

Critical points I

Back in 1131/1141... we found the **critical points** of a single variable function through differentiation. But for multivariable functions, we lose the meaning of *differentiation*.

Finding critical points

To find the critical points of multivariable functions,

1. Calculate $\dfrac{\partial f}{\partial x} = 0$.
2. Calculate $\dfrac{\partial f}{\partial y} = 0$.
3. Solve equations simultaneously.

Critical points II (classification)

Define $D = \dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial^2 f}{\partial y^2} - \dfrac{\partial^2 f}{\partial x \partial y}$ at the point (a, b).

- If $D < 0$, then (a, b) is a **saddle point**.
- If $D > 0$ and $\dfrac{\partial^2 f}{\partial x^2} < 0$, then (a, b) is a **local maximum**.
- If $D > 0$ and $\dfrac{\partial^2 f}{\partial x^2} > 0$, then (a, b) is a **local minimum**.
- If $D = 0$, then the test is inconclusive.

Critical points II

Example: (15S2, Q1d)

Find and classify the critical points of

$$h(x, y) = 2x^3 + 3x^2y + y^2 - y.$$

Also give the function value at the critical points.

Lagrange Multipliers

We may want to find critical points of a function f over a constraint g. To do this, we apply the method of **Lagrange multipliers**; we find the critical points that satisfy the equation $\nabla f = \lambda \nabla g$ and obtain the equations

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$$
$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$$
$$\vdots$$
$$g(x, y, \dots) = 0.$$

Finally, we solve for possible values of our points and determine which point(s) yield us with the maximum or minimum of f.

Lagrange Multipliers

Example: (Lagrange multipliers)

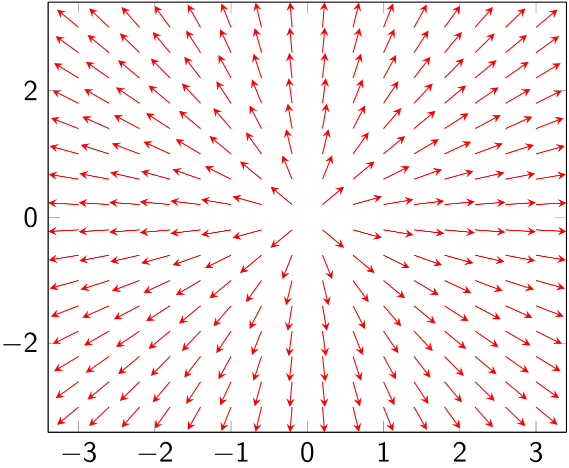
Find the extreme value(s) of $z = f(x, y) = x^4 + y^4$ subject to the condition $x + y - 1 = 0$.

Part III: Vector field theory

Introduction to vector field theory I

Vector field

A **vector field** assigns a vector to every point in some field.



Introduction to vector field theory II

Scalar field

A **scalar field** assigns a scalar value to every point in some field.

- $f(x, y) = x^2 + y^2$.
- $f(x, y, z) = x^2 + 2xyz + z^2$.

Vector fields and scalar fields

Let $\phi(x,y,z)$ be a scalar field and $\mathbf{F}(x,y,z) = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ be a vector field. Then

$$\nabla\phi = \operatorname{grad}\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} \qquad \text{scalar} \to \text{vector}$$

$$\nabla\cdot\mathbf{F} = \operatorname{div}\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \qquad \text{vector} \to \text{scalar}$$

$$\nabla\times\mathbf{F} = \operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \qquad \text{vector} \to \text{vector}$$

The vector differential operator ∇ is given by

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

Gradient, Divergence and Curl

The **divergence** of a vector field tells us how much net flow is coming **out** at a particular point. Positive divergence means that there is more outflow than inflow.
The **curl** of a vector field tells us how much a particle rotates.

Let $\phi(x, y, z)$ be a scalar field and $\mathbf{F}$ be a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$. Then

$$\operatorname{curl}(\operatorname{grad}\phi) = 0.$$

That is, the **curl** of the **gradient** of a scalar field is zero.

A vector field is **irrotational** if $\nabla \times \mathbf{F} = 0$. We shall see at a later time that irrotational vector fields (ie conservative vector fields) have some nice properties attached to it.

Line integrals I

Line integrals calculate the *work* done in moving a particle P from a point A to a point B along some path $\mathcal{C}$ through a force field $\mathbf{F}$.

$$\text{Work} = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}} F_1\, dx + F_2\, dy + F_3\, dz.$$

Line integrals II

Calculating line integrals

1. Parameterise the curve $\mathcal{C}$.
   - Circles of radius r parameterise to $x = r\cos\theta$, $y = r\sin\theta$.
   - Lines parameterise to $tA + (1-t)B$.
2. Rewrite the expressions in terms of the new variable (noting bounds and variable changes).
3. Express the line integral in terms of the new variable and integrate as normal.

Line integrals II

Example: (15S2, Q3cii)

Given a vector field

$$\mathbf{F} = 8e^{-x}\mathbf{i} + \cosh z\mathbf{j} - y^2\mathbf{k}$$

calculate the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ where $\mathcal{C}$ is the straight line path from $A(0, 1, 0)$ to $B(\ln(2), 1, 2)$.

Line integrals III

Properties of line integrals

- Let $\mathcal{C}$ be a path from A to B. If $\mathcal{C}'$ is the same path but starting at B and ending at A, then

$$\int_{\mathcal{C}'} \mathbf{F} \cdot d\mathbf{r} = -\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}.$$

- Let $\mathcal{C}$ be composed of two separate paths, $\mathcal{C}_1$ and $\mathcal{C}_2$. Then

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$$

Line integrals IV (conservative vector fields)

If $\nabla \times \mathbf{F} = 0$, then $\mathbf{F}$ is *conservative*. Additionally, there exists a scalar field $\phi(x, y, z)$ such that $\mathbf{F} = \nabla\phi(x, y, z)$; this is called the **scalar potential** of $\mathbf{F}$.

Line integrals on conservative fields

All line integrals are **path-independent** on conservative fields; that is, we **only** care about the points A and B, and not how we get from A to B.
Thus,

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \phi\,|_B - \phi\,|_A .$$

Line integrals IV (conservative vector fields)

Example: (18S1, Q1a)

Consider the scalar field

$$\phi(x, y, z) = xe^{z-1} + \cos y$$

and let $\mathbf{F} = \nabla\phi$.

- What is $\nabla \times \mathbf{F}$?
- Hence, or otherwise, calculate the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ along the straight line path $\mathcal{C}$ from $(1, 0, 1)$ to $(5, \pi, 1)$.

Line integrals IV (conservative vector fields)

Example: (18S2, Q4i)

Consider the vector field

$$\mathbf{F} = yz^2\mathbf{i} + xz^2\mathbf{j} + (2xyz + 3)\mathbf{k}.$$

- Show that $\mathbf{F}$ is conservative by evaluating curl($\mathbf{F}$).
- The path $\mathcal{C}$ in $\mathbb{R}^3$ starts at the point $(3, 4, 7)$ and subsequently travels anticlockwise four complete revolutions around the circle $x^2 + y^2 = 25$ within the plane $z = 7$, returning to the starting point $(3, 4, 7)$. Using the first part or otherwise, evaluate the work integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

Double integrals I

A **double integral** calculates the volume of a surface over a region Ω.

Double integrals on Ω

Let Ω be a region of integration and let $f(x, y)$ be the function over Ω. Then the double integral is written as

$$\text{Volume} = \iint_\Omega f(x, y)\, dA,$$

where dA is the infinitesimal area given by either $dxdy$ or $dydx$.

- Calculate the inside integral first and then the outside integral.
- The outer limits **must** be constants.
- The inner limits may be constants or functions of the other variable.

Double integrals II

Sometimes our region of integration Ω is expressed geometrically. In this case, we will need to extract the integral limits ourselves.

Calculating double integrals

1. Sketch the region Ω.
2. Determine the appropriate limits of integration and determine the order in which you would like to integrate.
3. Evaluate the inner integral and then evaluate the outer integral.

Double integrals II

Example: (double integrals)

Evaluate $\iint_\Omega x\,dA$ where Ω is the region in the first quadrant bounded by the parabola $y = 4 - x^2$ and the coordinate axes.

Double integrals III (changing the order of integration)

Double integrals can be evaluated using either ordering $dA = dxdy$ or $dA = dydx$. However, one of these orderings can make the integration process a lot less sufferable. Hence it is crucial to know how to convert between $\iint_{\Omega} f(x, y)\, dxdy$ and $\iint_{\Omega} f(x, y)\, dydx$.

Conversion between the two integrals

1. Sketch the region of integration.
2. Determine the new bounds with respect to the outer variable first and then determine the constant bounds with respect to the inner variable.
3. Swap the order of the variables and integrate.

Double integrals III (changing the order of integration)

Example: changing the order of integration

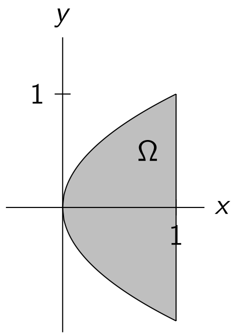
Evaluate $\int_{-1}^{1}\int_{y^2}^{1} 2\sqrt{x}e^{x^2}\, dxdy$ by first changing the order of integration.

Double integrals III (changing the order of integration)

Example: changing the order of integration

Evaluate $\int_{-1}^{1}\int_{y^2}^{1} 2\sqrt{x}e^{x^2}\,dxdy$ by first changing the order of integration.

- Sketch the region of integration.

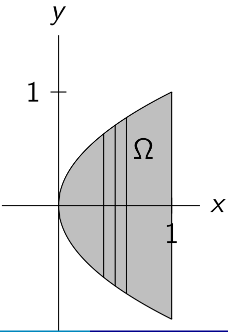


Double integrals III (changing the order of integration)

Example: changing the order of integration

Evaluate $\int_{-1}^{1} \int_{y^2}^{1} 2\sqrt{x} e^{x^2}\, dxdy$ by first changing the order of integration.

- Determine the new bounds of integration by considering strips parallel to the y axis.



Double integrals III (changing the order of integration)

Example: changing the order of integration

Evaluate $\int_{-1}^{1}\int_{y^2}^{1} 2\sqrt{x}e^{x^2}\,dxdy$ by first changing the order of integration.

- Rewrite the integral in terms of the new order of integration

$$\int_{-1}^{1}\int_{y^2}^{1} f(x,y)\,dxdy = \int_{0}^{1}\int_{-\sqrt{x}}^{\sqrt{x}} f(x,y)\,dydx$$

and evaluate the integral.

Double integrals III (changing the order of integration)

Example: (18S2, Q2ii)

Consider the double integral

$$I = \int_0^4 \int_{\sqrt{x}}^2 10x \, dydx.$$

Evaluate I with the order of integration reversed.

Double integrals IV (Polar coordinates)

Sometimes, a certain region becomes simpler to deal with if we express our integral in terms of an angle and magnitude; introducing the conversion to polar coordinates!
This is especially useful if our region is something like a circle.

Changes to our integrals

- $dA = rdrd\theta$.
- $x = r\cos\theta$, $y = r\sin\theta$.
- $\sqrt{x^2 + y^2} = r$.

Double integrals IV (Polar coordinates)

Example: (Polar coordinate conversion)

Evaluate $\iint_{\Omega} 2xy\, dydx$ where Ω is the region in the first quadrant between the circles of radius 2 and radius 5 centred at the origin.

Areas and Volumes

The volume of a surface enclosed by a region Ω is given by

$$\text{Volume} = \left| \iint_\Omega f(x, y)\, dA \right| .$$

The area of the region can be found by setting $f(x, y) = 1$

$$\text{Area} = \left| \iint_\Omega 1\, dA \right| .$$

The volume between two surfaces is given by

$$\text{Volume} = \left| \iint_\Omega \left(f_1(x, y) - f_2(x, y) \right) \, dA \right| .$$

Centre of Mass I

The **density** at a point (x, y) is represented by $\delta(x, y)$. The **mass** of a lamina Ω is

$$M = \iint_\Omega \delta(x, y)\, dA.$$

The **first moment** of Ω about the x and y axes respectively are

$$M_x = \iint_\Omega y\delta(x, y)\, dA, \quad M_y = \iint_\Omega x\delta(x, y)\, dA.$$

The centre of mass Ω is $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}.$$

Centre of Mass II

The **moments of inertia** about the x and y axes are

$$I_x = \iint_\Omega y^2 \delta(x, y)\, dA, \quad I_y = \iint_\Omega x^2 \delta(x, y)\, dA.$$

Part IV: Matrices

Revision from 1231/1241

Back in MATH1231/1241, you learned about...

- **Transposes** and **Inverses**
- **Eigenvalues** and **Eigenvectors**

Transpose and Inverse I

Tranpose of matrix

Let A be an $m \times n$ matrix. Then the **transpose** of A, $A^\intercal$, is an $n \times m$ matrix where the rows and columns of A are interchanged.

If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, then $A^\intercal = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$.

Properties of transposes

- $(AB)^\intercal = B^\intercal A^\intercal$.
- $\det(A) = \det(A^\intercal)$.
- $(A + B)^\intercal = A^\intercal + B^\intercal$.

Transpose and Inverse II

Inverse of matrix

Let A be a **square** matrix. Then, if $\det(A) \neq 0$, A will have an inverse A^{-1} such that

$$AA^{-1} = A^{-1}A = I.$$

- Review your MATH1131/1141 notes (or review the MATH1131/1141 seminar slides) to find methods of calculating matrix inverses.

Properties of inverses

- $(AB)^{-1} = B^{-1}A^{-1}$.
- $\det(A^{-1}) = \det(A)^{-1}$.

Eigenvalues and Eigenvectors I

For a square matrix A, an eigenvector $\mathbf{v}$ has a corresponding eigenvalue λ that satisfies the equation

$$A\mathbf{v} = \lambda\mathbf{v}.$$

- For an $n \times n$ matrix, A will have n linearly independent eigenvectors and n eigenvalues (not necessarily distinct).

Eigenvalues and Eigenvectors II

Calculating the eigenvalues

1. Solve the **characteristic equation** $\det(A - \lambda I) = 0$ for λ.
2. The **nonzero** solutions to the characteristic equation are the eigenvalues of A.

Calculating the eigenvectors

1. Substitute the value of λ into the expression $A - \lambda I$, and row reduce the matrix into row reduced form.
2. The vector is an **eigenvector** with corresponding eigenvalue λ.
3. Note: The number of zero rows tell you how many eigenvectors to find; if there are two zero rows, then there are two eigenvectors corresponding with the eigenvalue.

Eigenvalues and Eigenvectors III (15S2, Q1b)

The matrix B is given by

$$B = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

1. Show that the vector

$$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

   is an eigenvector of the matrix B and find the corresponding eigenvalue.
2. Given that the other two eigenvalues of B are -1 and 2, find the eigenvectors corresponding to these two eigenvalues.

Eigenvalues and Eigenvectors IV

The **trace** of a matrix A is the **sum of the diagonal entries** of A.

For example the trace of the previous matrix

$$B = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

is $0 + 0 + 2 = 2$.

Important property about traces and eigenvalues

- The trace of a matrix A is the **sum of the eigenvalues** of A.

Eigenvalues and Eigenvectors V (18S1, Q2b i)

A **real symmetric** 3×3 matrix A has eigenvalues denoted by λ_1, λ_2 and λ_3.
A student is given the following information about A:

- $\text{trace}(A) = 0$,
- $\lambda_1 = 2$ and $\lambda_3 = 4$.

What is the value of the remaining eigenvalue, namely λ_2?

$$\lambda_2 = -6.$$

Diagonalising a matrix I

An $n \times n$ matrix A is said to be diagonalisable if it has n distinct eigenvalues. Note that if A is diagonalisable, then it may or may not have n distinct eigenvalues.

- If A is diagonalisable, then there exists a matrix Q such that

$$A = QDQ^{-1} \iff D = Q^{-1}AQ$$

where D is an $n \times n$ matrix with eigenvalue entries along the diagonal. Q is the matrix with corresponding eigenvectors as column vectors matching the eigenvalues in matrix D.

Diagonalising a matrix II (20T1, Lab test 2 Q9)

The matrix $A = \begin{pmatrix} -5 & 6 & 0 \\ -3 & 4 & 0 \\ -3 & 3 & 1 \end{pmatrix}$ is diagonalisable with eigenvalues -2, 1 and 1.

An eigenvector corresponding to the eigenvalue -2 is $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

Find an invertible matrix M such that $M^{-1}AM = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Special types of matrices I

Symmetric matrices

A square matrix A is said to be **symmetric** if

$$A = A^{\mathsf{T}}.$$

- All eigenvalues of A are **real**.
- There always exists a full set of eigenvectors.
- Eigenvectors corresponding to different eigenvalues are **orthogonal**.

Special types of matrices II

Orthogonal matrices

A square matrix Q is said to be **orthogonal** if

$$Q^{\intercal}Q = QQ^{\intercal} = I \iff Q^{-1} = Q^{\intercal}.$$

- The columns of Q form an orthonormal set:
  - The columns are **orthogonal** to every other column in Q.
  - The columns are unitary: their magnitudes are 1.

- If Q is a **real** orthogonal matrix, then $\det(Q) = \pm 1$.

Quadric surfaces I

We can express a standard quadric surface as

$$\pm\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1.$$

The **shortest distance** from the origin to the surface is a straight line which can be calculated by taking the smallest denominator or largest eigenvalue and setting the other variables to be zero.

Quadric surfaces II

- $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is an **ellipse** in $\mathbb{R}^2$. $(+,+)$
- $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$ is a **hyperbola** in $\mathbb{R}^2$. $(+,-)$
- $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}$ is an **ellipsoid** in $\mathbb{R}^3$. $(+,+,+)$
- $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2}$ is a **hyperboloid of *one* sheet** in $\mathbb{R}^3$. $(+,+,-)$
- $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2}$ is a **hyperboloid of *two* sheet** in $\mathbb{R}^3$. $(+,-,-)$

Quadric surfaces II

Example: (20T1, Lab test 1)

You are given that the matrix A has eigenvalues 1444, 722 and 722. Hence the equation of the surface in terms of the principal axes X, Y and Z can be written as

$$1444X^2 + 722Y^2 + 722Z^2 = 17689.$$

Enter the shortest distance from the origin to the surface.

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- Take the variable with the largest coefficient and set the other variables to 0.

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- Take the variable with the largest coefficient and set the other variables to 0.

$$Y = 0, \ Z = 0.$$

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- The shortest distance occurs when we solve for the variable.

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Enter the shortest distance from the origin to the surface.

- The shortest distance occurs when we solve for the variable.

$$1444X^2 = 17689 \iff X = \pm\frac{7}{2}.$$

Quadric surfaces III

But if we have xy, xz and yz terms, then we have a **rotation** of axes. To account for this rotation, we can express the equation of the surface as

$$\mathbf{x}^{\mathsf{T}} A\mathbf{x} = 1,$$

where A is a **symmetric** matrix and $\mathbf{x}$ is a vector in $\mathbb{R}^n$. Then find the eigenvalues and unitary eigenvectors of A; the eigenvectors form the **principal axes** of the quadric curve. In this new coordinate system, we attain a new curve which we can find the shortest distance from the origin to the surface.

Quadric surfaces III

Example: (18S2, Q2iii)

A quadratic curve is given by the equation $7x^2 + 6xy + 7y^2 = 200$.

1. Express the curve in the form
$$\mathbf{x}^\mathsf{T} A\mathbf{x} = 200$$
where $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, and A is a 2×2 symmetric matrix.
2. Find the eigenvalues and eigenvectors of the matrix A.
3. Hence, or otherwise, find the shortest distance between the curve and the origin.

System of ODEs

Using the analysis from quadric surfaces, we can use this same process to simplify our working to find solutions to a system of ODEs.

Method of solution

1. Write the system of ODEs into the form

$$\mathbf{y}' = A\mathbf{y}$$

where A is the coefficient of the ODEs.

2. Determine the eigenvalues λ_i and eigenvectors $\mathbf{v}_i$ of A.
3. Solution is of the form

$$\mathbf{y} = \sum_{k=1}^{n} c_k \mathbf{v}_k e^{\lambda_k t}.$$

(A final example) System of ODEs

Solve the system of differential equations.

$$\begin{aligned} y_1' &= 2y_1 + y_2 \\ y_2' &= -y_1 + y_3 \\ y_3' &= y_1 + y_2 + y_3. \end{aligned}$$