MATH1151 Algebra Test 2 2008 S1 v1A

January 28, 2015

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1. A matrix is invertible if and only if the determinant is not zero. So let's find the determinant.

$$\begin{aligned}
\det(A) &= 1 \times \det \begin{pmatrix} -2 & -1 \\ 3 & 0 \end{pmatrix} - 1 \times \begin{pmatrix} 2 & -1 \\ a & 0 \end{pmatrix} + a \times \begin{pmatrix} 2 & -2 \\ a & 3 \end{pmatrix} \\
&= (-2 \times 0 - (-1) \times 3) - (2 \times 0 - (-1) \times a) + (a \times (2 \times 3 - (-2) \times a) \\
&= 3 - a + a(6 + 2a) \\
&= 3 + 5a + 2a^2
\end{aligned}$$

Hence, the matrix is invertible if:

$$\begin{aligned}3 + 5a + 2a^2 &\neq 0\\(2a+3)(a+1) &\neq 0\\a &\neq -1, -\frac{3}{2}\end{aligned}$$

2. (i) We should know that the cross product of two vectors is perpendicular to the two vectors themselves. Hence, we want to find $\boldsymbol{a} \times \boldsymbol{b}$.

   To make it easy for ourselves, we should remember that

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

   So

$$\begin{aligned}\boldsymbol{a} \times \boldsymbol{b} &= \begin{vmatrix} \boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} \\ -1 & 4 & 2 \\ -2 & -1 & 3 \end{vmatrix}\\&= \boldsymbol{e_1}(4 \times 3 - 2 \times (-1)) - \boldsymbol{e_2}((-1) \times 3 - 2 \times (-2)) + \boldsymbol{e_3}((-1) \times (-1) - 4 \times (-2))\\&= \begin{pmatrix} 14 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix}\\&= \begin{pmatrix} 14 \\ -1 \\ 9 \end{pmatrix}\end{aligned}$$

   (ii) We know that the projection of $\boldsymbol{c}$ on to $\boldsymbol{a}$ is given by

$$\text{proj}_{\boldsymbol{a}}\boldsymbol{c} = \left(\frac{\boldsymbol{a} \cdot \boldsymbol{c}}{|\boldsymbol{a}|^2}\right)\boldsymbol{a}$$

.

Hence,

$$\operatorname{proj}_{\boldsymbol{a}}\boldsymbol{c} = \frac{\left(\begin{smallmatrix}-1\\4\\2\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}8\\-5\\-7\end{smallmatrix}\right)}{1^2+4^2+2^2} \times \begin{pmatrix}-1\\4\\2\end{pmatrix}$$
$$= -2\begin{pmatrix}-1\\4\\2\end{pmatrix}$$

3. If the points were all on the line, we would have:

$$\begin{aligned} \alpha + \beta &= 3 \\ \alpha + 2\beta &= -1 \\ \alpha + 4\beta &= 2 \\ \alpha + 5\beta &= 0 \end{aligned}$$

or:
$$\begin{pmatrix}1&1\\1&2\\1&4\\1&5\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \begin{pmatrix}3\\-1\\2\\0\end{pmatrix}$$

Let $A = \begin{pmatrix}1&1\\1&2\\1&4\\1&5\end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$ and $\boldsymbol{y} = \begin{pmatrix}3\\-1\\2\\0\end{pmatrix}$.

We should know that the least squares approximate solution $\boldsymbol{x_0}$ to $A\boldsymbol{x} = \boldsymbol{y}$ is the solution to the normal equations
$$A^T A\boldsymbol{x} = A^T\boldsymbol{y}$$

Hence we want to solve

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \\ 1 & 5 \end{pmatrix} \boldsymbol{x} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 12 \\ 12 & 46 \end{pmatrix} \boldsymbol{x} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$$

If we solve this by row reduction and back substitution (hopefully you know how to do this!), you should get $\alpha = \frac{19}{10}$ and $\beta = -\frac{3}{10}$.

4. By reading the coefficients, we know that a vector normal to the plane is $\left(\begin{smallmatrix} 1 \\ -2 \\ 1 \end{smallmatrix}\right)$. From this, we can tell that the line going through the point A perpendicular to the plane is represented by

$$\boldsymbol{x} = \begin{pmatrix} 2 \\ 7 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2+\lambda \\ 7-2\lambda \\ -5+\lambda \end{pmatrix}, \lambda \in \mathbb{R}$$

Hence, point P lies on this line, and it also lies on the plane, so we can substitute the equation of this line into the equation of the plane and solve for λ in order to find the intersection between the line and plane. Substituting it in:

$$\begin{aligned} (2+\lambda) - 2 \times (7 - 2\lambda) + (-5+\lambda) &= 4 \\ 2 + \lambda - 14 + 4\lambda - 5 + \lambda &= 4 \\ 6\lambda &= 21 \\ \lambda &= \frac{7}{2} \end{aligned}$$

Hence, point P is represented by

$$\begin{pmatrix} 2 + \frac{7}{2} \\ 7 - 2(\frac{7}{2}) \\ -5 + \frac{7}{2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 11 \\ 0 \\ -3 \end{pmatrix}$$

MATH1151 Algebra Test 2 2009 S1 v1B

June 1, 2015

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1. For the line & plane to not intersect, solving them simultaneously should yield no solutions (which does happen). So they do not intersect.
   Alternatively, we can consider the cross product of the gradient vectors of the plane to determine if the line is parallel to the plane.

2. (i)
   $$\begin{pmatrix} -8 & 4 \\ -3 & -11 \end{pmatrix}.$$
   For two matrices to be equal, their elements must equal. Using this, we find that $u = 1$ and $v = -10$.

   (ii) Find $\left(A^2\right)^2$ first, and use this to find A^5. We find that $x = 71$ and $y = 190$.

3. Row/column operations will help with calculating the determinant. The answer is 28.

4. $\|\overrightarrow{AB}\| = \sqrt{41}$, $\cos \angle CAB = \sqrt{\frac{35}{41}}$.