



MATH1231/1241 Calculus Test 1 2008 S2

v2a

Full Solutions

August 24, 2018

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Please note quiz papers that are **NOT** in your course pack will not necessarily reflect the style & difficulty of questions in your quiz.

-
1. Partial differentiating with respect to x ,

$$\frac{\partial z}{\partial x} = ye^{xy}.$$

Since $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right)$, we have

$$\begin{aligned} \frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) \\ &= \frac{\partial}{\partial y} (ye^{xy}) \\ &= y \cdot xe^{xy} + e^{xy} \cdot 1 \\ &= yxe^{xy} + e^{xy} \\ &= e^{xy} (xy + 1). \end{aligned}$$

(THOUGHT PROCESS: *This question involves basic partial differentiation.*

For more information on this content, please refer to course notes pp.8-9.

For more practice on this type of question, go to course notes p.22 Problems 1.2.)

2. We are required to show that z satisfies the differential equation $4\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$.
Since $z = \cos(x + 2y)$, we find expressions for the partial derivatives.

$$\begin{aligned} \frac{\partial z}{\partial x} &= -\sin(x + 2y) & \frac{\partial z}{\partial y} &= -2\sin(x + 2y) \\ \frac{\partial^2 z}{\partial x^2} &= -\cos(x + 2y) & \frac{\partial^2 z}{\partial y^2} &= -4\cos(x + 2y) \end{aligned}$$

Now, calculating the following,

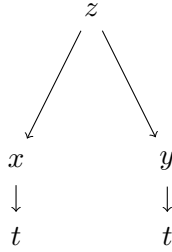
$$\begin{aligned} 4\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} &= 4(-\cos(x + 2y)) - (-4\cos(x + 2y)) \\ &= -4\cos(x + 2y) + 4\cos(x + 2y) \\ &= 0 \text{ as required.} \end{aligned}$$

(THOUGHT PROCESS: *This question involves basic partial differentiation.*

For more information on this content, please refer to course notes pp.8-9.

For more practice on this type of question, go to course notes p.22 Problems 1.2.)

3. Notice that z is dependent on x and y , x is dependent on t and y is dependent on t . We draw the corresponding chain diagram below.



Thus, when differentiating z with respect to t , we must apply the chain rule and move down the chains with the variable t . We see that to reach t , we can go from $z \rightarrow x \rightarrow t$ or $z \rightarrow y \rightarrow t$. We move down one of the possible chains, so say $z \rightarrow x$. When there are choices between chains, we use partial differentiation so we have $\frac{\partial z}{\partial x}$. Next, moving down from x , we can only reach t , and so now we can differentiate x w.r.t. t normally attaining $\frac{\partial z}{\partial x} \frac{dx}{dt}$. We add the second case where we move down the chain through y to obtain,

$$\begin{aligned}
 \frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \\
 &= y \times 3t^2 + x \times e^t \\
 &= 3yt^2 + xe^t \\
 &= 3(e^t)t^2 + (t^3)e^t && \text{(substituting in expressions for } x \text{ and } y) \\
 &= 3t^2e^t + t^3e^t \\
 &= t^2e^t(3+t).
 \end{aligned}$$

(THOUGHT PROCESS: *This question examines multivariable chain rules. Note that constructing a chain diagram usually helps for this type of questions.*

For more information on this content, please refer to course notes pp.17-20.

For more practice on this type of question, go to course notes pp.23-25 Problems 1.5.)

4. Using the hint and rearranging for the identity $\cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta)$, we can apply this to the integral to solve it,

$$\begin{aligned}\int \cos^4 \theta \, d\theta &= \int (\cos^2 \theta)^2 \, d\theta \\&= \int \left(\frac{1}{2} (1 + \cos 2\theta) \right)^2 \, d\theta \\&= \int \frac{1}{4} (1 + \cos 2\theta)^2 \, d\theta \\&= \frac{1}{4} \int 1 + 2 \cos 2\theta + \cos^2 2\theta \, d\theta \\&= \frac{1}{4} \int 1 + 2 \cos 2\theta + \frac{1}{2} + \frac{1}{2} \cos 4\theta \, d\theta \\&= \frac{1}{4} \int \frac{3}{2} + 2 \cos 2\theta + \frac{1}{2} \cos 4\theta \, d\theta \\&= \frac{1}{4} \left(\frac{3}{2} \theta + \sin 2\theta + \frac{1}{8} \sin 4\theta \right) \\&= \frac{3}{8} \theta + \frac{1}{4} \sin 2\theta + \frac{1}{32} \sin 4\theta + c.\end{aligned}$$

(THOUGHT PROCESS: *It might be helpful to review some of the strategies to solve trigonometric integrals. Make sure you are able to do this type of questions without being given hints.*

For more information on this content, please refer to course notes pp.27-31.

For more practice on this type of question, go to course notes p.50 Question 3.)



MATH1231/1241 Calculus S2 2009 Test 1

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Full Solutions

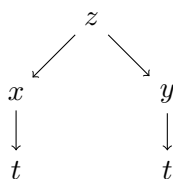
September 4, 2018

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-
1. If $z = xy$, then we can use the chain rule to differentiate it, using following the chain diagram as a mnemonic:



$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \\ &= (y)(-6 \sin 3t) + x(12 \cos 3t) \\ &= -6y \sin 3t + 12x \cos 3t \\ &= -6(4 \sin 3t) \sin 3t + 12(2 \cos 3t) \cos 3t && \text{(Substituting in } x \text{ and } y\text{)} \\ &= -24 \sin^2 3t + 24 \cos^2 3t\end{aligned}$$

(THOUGHT PROCESS: *This question examines multivariable chain rules. Note that constructing a chain diagram usually helps for this type of questions.*

For more information on this content, please refer to course notes pp.17-20.

For more practice on this type of question, go to course notes pp.23-25 Problems 1.5.)

2. One way to find the the normal vector is to treat z as $f(x, y)$ and remember that a normal is given by $\begin{pmatrix} \frac{\partial f}{\partial x}(-1, 2) \\ \frac{\partial f}{\partial y}(-1, 2) \\ -1 \end{pmatrix}$. So:

$$\frac{\partial f}{\partial x} = 6x$$

$$\frac{\partial f}{\partial y} = -2y$$

Hence a normal is $\begin{pmatrix} -6 \\ -4 \\ -1 \end{pmatrix}$. Let's make things simpler by using $\mathbf{n} = \begin{pmatrix} 6 \\ 4 \\ 1 \end{pmatrix}$ since this is also normal. Once we have found a normal vector, we can use the point-normal form of a plane, where \mathbf{n} is a normal, and $\mathbf{x} = \begin{pmatrix} x & y & z \end{pmatrix}^T$ where $x, y, z \in \mathbb{R}$:

$$\mathbf{n} \cdot \left(\mathbf{x} - \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} \right) = 0$$

$$\begin{pmatrix} 6 \\ 4 \\ 1 \end{pmatrix} \cdot \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} \right) = 0$$

$$6x + 4y + z = -1$$

3.

$$\begin{aligned} \int \sec^4 \theta \, d\theta &= \int (\tan^2 \theta + 1) \sec^2 \theta \, d\theta \\ &= \int (u^2 + 1) \, du && \text{(If we let } u = \tan \theta \text{ and } du = \sec^2 \theta \, d\theta) \\ &= \frac{u^3}{3} + u + C \\ &= \frac{\tan^3 \theta}{3} + \tan \theta + C \end{aligned}$$

(THOUGHT PROCESS: *It might be helpful to review some of the strategies to solve trigonometric integrals. Make sure you are able to do this type of questions without being given hints.*

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4. Since we have $4 - x^2$, the most fitting substitution is $x = 2 \sin \theta$ because we foresee that it simplifies most easily.

From this we get $dx = 2 \cos \theta d\theta$. So ensuring that we change the limits:

$$\int_0^{\pi/2} \sqrt{4 - 4 \sin^2 \theta} (2 \cos \theta) d\theta = 4 \int_0^{\pi/2} \cos^2 \theta d\theta$$

Now we could do this using double angle formulae (i.e. recalling that $\cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta)$), but let's have some fun and use some definite integral properties:

$$I = \int_0^{\pi/2} \cos^2 \theta d\theta = \int_0^{\pi/2} \cos^2 \left(\frac{\pi}{2} - \theta\right) d\theta = \int_0^{\pi/2} \sin^2 \theta d\theta,$$

where we used the properties $\int_0^a f(x) dx = \int_0^a f(a - x) dx$ and $\cos(\frac{\pi}{2} - x) = \sin(x)$.

Hence, we can add the two different expressions for I :

$$2I = \int_0^{\pi/2} (\cos^2 \theta + \sin^2 \theta) d\theta = \int_0^{\pi/2} d\theta = \frac{\pi}{2}$$

So

$$I = \int_0^{\pi/2} \cos^2 \theta d\theta = \frac{\pi}{4}$$

So our original integral is equal to $4 \times \frac{\pi}{4} = \pi$.

Note: We could also recognise that the function is a semicircle, and work this out geometrically – it is just the area of a quadrant of a circle of radius 2 which is equal to $\frac{\pi(2^2)}{4} = \pi$. But the question specifies to use a trigonometric substitution, so just use this fact to check your answer.



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MATH1231/1241 Calculus Test 1 2011 S2

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Full Solutions

August 24, 2015

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1.

$$\frac{\partial z}{\partial y} = 2xye^{xy^2}$$

2. We will calculate the partial derivatives $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$ then check if they satisfy the appropriate differential equation. Hence,

$$\frac{\partial z}{\partial x} = -\sin(x + 2y) \implies \frac{\partial^2 z}{\partial x^2} = -\cos(x + 2y)$$

and

$$\frac{\partial z}{\partial y} = -2 \sin(x + 2y) \implies \frac{\partial^2 z}{\partial y^2} = -4 \cos(x + 2y).$$

Substituting this into the given differential equation gives

$$4 \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 4 \times -\cos(x + 2y) - (-4 \cos(x + 2y)) = 0,$$

as required.

3. As z is a function of x and y and as x and y are both functions of t by the chain rule we have

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \\ &= 4t \frac{\partial z}{\partial x} + 3t^2 \frac{\partial z}{\partial y}. \end{aligned}$$

Now as

$$\begin{aligned} \frac{\partial z}{\partial x} &= 2x \\ &= 4t^2 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial z}{\partial y} &= 6y \\ &= 6t^3 \end{aligned}$$

we have

$$\begin{aligned} \frac{dz}{dt} &= 4t \times 4t^2 + 3t^2 \times 6t^3 \\ &= 16t^3 + 18t^5. \end{aligned}$$

4. The vector \mathbf{n} is given by $(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1)$ evaluated at $(-1, 2, 3)$. Firstly, we calculate the partial derivatives $\frac{\partial z}{\partial y}$ and $\frac{\partial z}{\partial x}$. We find that

$$\begin{aligned} \frac{\partial z}{\partial y} &= \frac{4y}{2\sqrt{x^2 + 2y^2}} \\ &= \frac{2y}{\sqrt{x^2 + 2y^2}} \end{aligned}$$

and

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{2x}{2\sqrt{x^2 + 2y^2}} \\ &= \frac{x}{\sqrt{x^2 + 2y^2}}.\end{aligned}$$

Hence, by substituting in the point $(-1, 2, 3)$ we find that the required vector is

$$\mathbf{n} = \begin{pmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{-1}{\sqrt{(-1)^2 + 2(2)^2}} \\ \frac{2 \times 2}{\sqrt{(-1)^2 + 2(2)^2}} \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ \frac{4}{3} \\ -1 \end{pmatrix}.$$

To find the tangent plane at this point, we use the point-normal form of a plane, i.e.

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_1) = 0$$

where $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$, \mathbf{x}_1 is a point on the plane, and \mathbf{n} is the normal to the plane.

We use a scalar multiple of \mathbf{n} to make things easier, so we use $\mathbf{n} = (-1, 4, -3)^T$, and let $\mathbf{x}_1 = (-1, 2, 3)^T$. Substituting these values in and simplifying,

$$\begin{aligned}\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_1) &= 0 \\ \Rightarrow \begin{pmatrix} -1 \\ 4 \\ -3 \end{pmatrix} \cdot \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \right) &= 0 \\ (-x + 4y - 3z) - (1 + 8 - 9) &= 0 \\ x - 4y + 3z &= 0.\end{aligned}$$

5. We make the substitution $x = 2 \tan \theta$. Then

$$\frac{dx}{d\theta} = 2 \sec^2 \theta$$

and

$$x = 0 \Rightarrow \theta = 0, \quad x = 2 \Rightarrow \theta = \frac{\pi}{4}.$$

Hence we have

$$\begin{aligned}\int_0^2 \frac{1}{(4+x^2)^{3/2}} dx &= \int_0^{\pi/4} \frac{2 \sec^2 \theta}{(4+4 \tan^2 \theta)^{3/2}} d\theta \\&= \int_0^{\pi/4} \frac{2 \sec^2 \theta}{8(1+\tan^2 \theta)^{3/2}} d\theta \\&= \frac{1}{4} \int_0^{\pi/4} \frac{\sec^2 \theta}{(\sec^2 \theta)^{3/2}} d\theta \\&= \frac{1}{4} \int_0^{\pi/4} \cos \theta d\theta \\&= \frac{1}{4} [\sin \theta]_0^{\pi/4} \\&= \frac{1}{4\sqrt{2}}.\end{aligned}$$





MATH1231/1241 Calculus S2 2011 Test 1

v1b

Answers & Worked Sample Solutions

September 4, 2018

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1. Answer

$$\frac{dz}{dt} = 8t^3 + 96t^5$$

Worked Sample Solution

Let $f(x, y) = 2x^2 + y^2$, so $z = f(x, y)$. Then $\frac{\partial f}{\partial x} = 4x$ and $\frac{\partial f}{\partial y} = 2y$. Also, $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 12t^2$. So by the multivariate chain rule, we have

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \\ &= 4x \times 2t + 2y \times 12t^2 \\ &= 4t^2 \times 2t + 2 \times 4t^3 \times 12t^2 \quad (\text{using the given expressions for } x, y \text{ as functions of } t) \\ &= 8t^3 + 96t^5.\end{aligned}$$

2. Answer

$$\mathbf{n} = \begin{pmatrix} 5 \\ -3 \\ -1 \end{pmatrix}, \quad 5x - 3y - z = 0$$

Worked Sample Solution

Recall that if $z = f(x, y)$ is a smooth surface, then a normal to the surface at a point \mathbf{a} on the surface is $\mathbf{n} = \begin{pmatrix} f_x \\ f_y \\ -1 \end{pmatrix}$, where the partial derivatives are evaluated at the x and y values of the

point \mathbf{a} . In this particular question, $f(x, y) = x^2y - 2y^2 + 3x$, and the point is $\mathbf{a} = (1, 1, 2)$ (so we evaluate the partials at $x = 1, y = 1$). So $f_x(x, y) = 2xy + 3 \Rightarrow f_x(1, 1) = 2 + 3 = 5$.

Also, $f_y(x, y) = x^2 - 4y \Rightarrow f_y(1, 1) = 1 - 4 = -3$. So a normal to the surface is $\mathbf{n} = \begin{pmatrix} 5 \\ -3 \\ -1 \end{pmatrix}$.

An equation for the tangent plane can be found using the point-normal form

$$\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{a},$$

where $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Since $\mathbf{n} \cdot \mathbf{a} = \begin{pmatrix} 5 \\ -3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 5 - 3 - 2 = 0$, the plane has equation $5x - 3y - z = 0$. (Remember, the equation of a plane or its normal is not unique; you could have ended up with a non-zero scalar multiple of the normal given in this answer and it would still be correct. But if the normal you obtained is *not* a scalar multiple of the one in this answer, it is not correct.)

3. Answer

$$\int_0^3 \frac{1}{(9+x^2)^{\frac{3}{2}}} dx = \frac{1}{9\sqrt{2}}$$

Worked Sample Solution

Let

$$I = \int_0^3 \frac{1}{(9+x^2)^{\frac{3}{2}}} dx.$$

Substitute $x = 3 \tan \theta$, $\theta \in [0, \frac{\pi}{4}]$. So $dx = 3 \sec^2 \theta d\theta$, and $9 + x^2 = 9(1 + \tan^2 \theta) = 9 \sec^2 \theta \Rightarrow (9 + x^2)^{\frac{3}{2}} = (9 \sec^2 \theta)^{\frac{3}{2}} = 3^3 \sec^3 \theta$. When $x = 0$, $\theta = 0$, and when $x = 3$, $\theta = \frac{\pi}{4}$.

So

$$\begin{aligned}
 I &= \int_0^{\frac{\pi}{4}} \frac{3 \sec^2 \theta}{3^3 \sec^3 \theta} d\theta \\
 &= \frac{1}{3^2} \int_0^{\frac{\pi}{4}} \frac{1}{\sec \theta} d\theta \\
 &= \frac{1}{9} \int_0^{\frac{\pi}{4}} \cos \theta d\theta \\
 &= \frac{1}{9} \left(\sin \frac{\pi}{4} - \sin 0 \right) \quad (\text{Fundamental Theorem of Calculus}) \\
 &= \frac{1}{9} \left(\frac{1}{\sqrt{2}} - 0 \right) \\
 &= \frac{1}{9\sqrt{2}}.
 \end{aligned}$$

4. By simple calculus (differentiation of a logarithm and chain rule),

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2}.$$

5. Worked Sample Solution

Since¹

$$\frac{\partial^2 z}{\partial x^2} = -\sin(x + 2y)$$

and

$$\frac{\partial^2 z}{\partial y^2} = -4 \sin(x + 2y),$$

we have

$$\frac{\partial^2 z}{\partial y^2} = 4 \frac{\partial^2 z}{\partial x^2} \Rightarrow 4 \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0,$$

and the claim is proved.

¹These second derivatives can be obtained very quickly by recalling that if $u = A \sin(\omega t + \varphi)$ for constants A, ω, φ , then

$$\frac{d^2 u}{dt^2} = -\omega^2 u.$$

This was an important result from the Simple Harmonic Motion topic in HSC.



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1. We find

$$\begin{aligned} z &= e^{x^2y^3} \\ \implies \frac{\partial z}{\partial x} &= 2xy^3e^{x^2y^3} \\ \implies \frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) \\ &= \frac{\partial}{\partial y} (2xy^3e^{x^2y^3}) = 2x \frac{\partial}{\partial y} (y^3e^{x^2y^3}) \\ &= 2x \left[y^3 (3x^2y^2e^{x^2y^3}) + 3y^2 (e^{x^2y^3}) \right] \quad (\text{product rule}) \\ &= 6xy^2e^{x^2y^3} (x^2y^3 + 1) \quad (\text{cleaning up}). \end{aligned}$$

2. Recall that using the *total differential approximation*, the maximum error is given by

$$|\Delta z|_{\max.} \approx \left| \frac{\partial z}{\partial x} \right| |\Delta x| + \left| \frac{\partial z}{\partial y} \right| |\Delta y| \quad (*)$$

where the partial derivatives are evaluated at the measured values of x and y .

Note $\frac{\partial z}{\partial x} = \frac{1}{2\sqrt{x}} \times \frac{1}{y+1}$ and $\frac{\partial z}{\partial y} = -\frac{\sqrt{x+1}}{(y+1)^2}$.

So, at $x = 9, y = 1$, we have $\frac{\partial z}{\partial x} = \frac{1}{2 \times 3} \times \frac{1}{2} = \frac{1}{12}$ and $\frac{\partial z}{\partial y} = -\frac{3+1}{2^2} = -\frac{4}{4} = -1$.

It is also given the measurement of each x and y have an error with absolute value at most 0.03 cm. This implies that $\Delta x = \Delta y = 0.03$ for the formula (*).

Hence, the maximum error in the calculated value of z is found by the following:

$$\begin{aligned} |\Delta z|_{\max.} &\approx \left| \frac{\partial z}{\partial x} \right| |\Delta x| + \left| \frac{\partial z}{\partial y} \right| |\Delta y| \\ &= \left| \frac{1}{12} \right| |0.03| + |-1| |0.03| \\ &= \frac{1}{12} \times 0.03 + 1 \times 0.03 \\ &= \frac{13}{12} \times 0.03 = \frac{13}{12} \times \frac{3}{100} \\ &= 0.0325 \text{ cm} = \frac{13}{400} \text{ cm}. \end{aligned}$$

That is, the maximum error in the calculated value of z is $0.0325 \text{ cm} = \frac{13}{400} \text{ cm}$.

Tip. Since in the class tests you are not allowed a calculator, it should be fine to leave the answer in fraction form here (which is probably slightly easier/faster to obtain without a calculator than the decimal form). Here is a way to calculate the decimal form from the fraction without a calculator if you wanted to give the answer as a decimal.

We want to express $\frac{13}{400}$ as a decimal. Let's express $\frac{13}{4}$ as a decimal, and then divide by 100 (i.e. shift the decimal point two spaces left). Note that $\frac{13}{4} = 3\frac{1}{4} = 3.25$. So shifting the decimal point two spaces left, we have $\frac{13}{400} = 0.0325$.

3. Note that $\sin^3 \theta \cos^2 \theta = \sin \theta \sin^2 \theta \cos^2 \theta = (1 - \cos^2 \theta) \cdot \cos^2 \sin \theta$. Also $\sin \theta d\theta = -d(\cos \theta)$. Since $\cos 0 = 1$ and $\cos \frac{\pi}{2} = 0$, we obtain using the substitution $u = \cos \theta$:

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sin^3 \theta \cos^2 \theta d\theta &= - \int_1^0 (1 - u^2) u^2 du && \text{(where we substituted } u = \cos \theta) \\ &= \int_0^1 u^2 - u^4 du && \text{(as } - \int_a^b f(x) dx = \int_b^a f(x) dx) \\ &= \left[\frac{u^3}{3} - \frac{u^5}{5} \right]_0^1 \\ &= \frac{1}{3} - \frac{1}{5} \\ &= \frac{2}{15}. \end{aligned}$$

Alternatively, a slightly quicker (but equivalent) method is to just skip the substitution

and do steps in your head:

$$\begin{aligned}
 \int_0^{\frac{\pi}{2}} \sin^3 \theta \cos^2 \theta \, d\theta &= - \int_0^{\frac{\pi}{2}} (1 - \cos^2 \theta) \cos^2 \theta \, d(\cos \theta) \\
 &= - \int_0^{\frac{\pi}{2}} (\cos^2 \theta - \cos^4 \theta) \, d(\cos \theta) \\
 &= \frac{1}{3} - \frac{1}{5} \quad (*) \\
 &= \frac{2}{15}.
 \end{aligned}$$

Remarks/Tips. To get line (*), we essentially did a mental substitution of $u = \cos \theta$ in the previous line and noted that this becomes integrating powers of u from 0 to 1, and used the fact that $\int_0^1 u^n \, du = \frac{1}{n+1}$ for $n > 0$ (in fact $n > -1$). Also, since the integrand of the original integral is positive throughout the domain of integration, we know the final answer must be positive, so we could say it had to be $\frac{1}{3} - \frac{1}{5}$ rather than $\frac{1}{5} - \frac{1}{3}$ without needing to think about how the minus sign and the bounds of the integral affected anything. If you feel uncomfortable doing so many steps in your head, feel free to use the first method with all steps shown.

4. Let $I_n = \int_0^1 (1 - x^2)^n \, dx$.

Then, we have $I_n = \frac{2n}{2n+1} I_{n-1}$ from the given reduction formula.

Solving the given integral,

$$\begin{aligned}
 \int_0^1 (1 - x^2)^4 \, dx &= I_4 \\
 &= \frac{2(4)}{2(4)+1} I_3 \\
 &= \frac{8}{9} \cdot \frac{2(3)}{2(3)+1} I_2 \\
 &= \frac{8}{9} \cdot \frac{6}{7} \cdot \frac{2(2)}{2(2)+1} I_1 \\
 &= \frac{8}{9} \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2(1)}{2(1)+1} \cdot I_0 \\
 &= \frac{8}{9} \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot I_0.
 \end{aligned}$$

As $I_0 = \int_0^1 1 \, dx = 1$, we have

$$\begin{aligned}\int_0^1 (1-x^2)^4 \, dx &= \frac{8}{9} \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1 \\ &= \frac{8 \times 2 \times 4 \times 2}{9 \times 7 \times 5}\end{aligned}$$

(simplifying the blue factors to help since we do not have a calculator)

$$= \frac{128}{315}.$$

(no more common factors above, so just calculate numerator and denominator)

