MATH1231/1241 Final Exam Revision Session – Algebra (Semester 2, 2017)

UNSW Mathematics Society

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Introduction

Today, we will cover

- Vector spaces
- Linear transformations
- Eigenvalues and eigenvectors
- Introduction to probability and statistics.

We will go through sample exam questions from these topics, showing how to approach such questions for the final exam.

Vector spaces – Problem 1

[MATH1231 2014 S2 Q1 iv)]

Let $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : z^2 = x^2 + y^2 \right\}$.

a) Prove that S is closed under scalar multiplication.

b) Prove that S is **not** a subspace of $\mathbb{R}^3$.

Vector spaces – Problem 2

[MATH1231 2016 S2 Q3 ii)]

Consider the set $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \right\}$ of vectors in $\mathbb{R}^3$.

a) Prove that S is linearly dependent.

b) Write the last vector in S as a linear combination of the other two.

Vector spaces – Problem 3

[MATH1231 2014 S2 Q3 vi)]

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a set of three non-zero vectors in $\mathbb{R}^3$.

a) State the definition for the set B to be a linearly independent set.

b) Prove that if B is an orthogonal set then B is linearly independent.

c) Hence explain why any orthogonal set of 3 non-zero vectors in $\mathbb{R}^3$ forms a basis for $\mathbb{R}^3$.

Vector spaces – Problem 4

[MATH1241 2016 S2 Q3 iii)]
The field $\mathbb{F} = \mathrm{GF}(4)$ has elements $\{0, 1, \alpha, \beta\}$, with addition and multiplication defined by the following tables.

$+$	0	1	α	β
0	0	1	α	β
1	1	0	β	α
α	α	β	0	1
β	β	α	1	0

$\times$	0	1	α	β
0	0	0	0	0
1	0	1	α	β
α	0	α	β	1
β	0	β	1	α

For the vectors $\mathbf{b}_1 = \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} \beta \\ 1 \\ 1 \end{pmatrix}, \mathbf{b}_3 = \begin{pmatrix} 1 \\ 0 \\ \alpha \end{pmatrix}, \mathbf{v} = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}$,

a) show that $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis for $\mathbb{F}^3$
b) explain without calculation why $\{\mathbf{b}_1, \mathbf{b}_1 + \mathbf{b}_2, \mathbf{b}_2 + \mathbf{b}_3, \mathbf{b}_3\}$ is a spanning set but not a basis for $\mathbb{F}^3$;
c) find the coordinate vector of $\mathbf{v}$ with respect to the ordered basis $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ of $\mathbb{F}^3$.

Linear transformations – Problem 1

[MATH1231 2012 S2 Q3 v)]

Suppose A is a 3×2 matrix and let A^T denote its transpose.

a) Prove that the column space of the matrix AA^T is a subset of the column space of A and deduce that $\operatorname{rank}\left(AA^T\right) \leq \operatorname{rank}(A)$.

b) Deduce that $\operatorname{nullity}\left(AA^T\right) \geq 1$, and explain why the matrix AA^T can never be the identity.

Linear transformations – Problem 2

[MATH1231 2013 S2 Q3 ii)]
Let

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 3 \\ 1 & 3 & 4 & 9 \end{pmatrix}.$$

a) Find a basis for $\ker(A)$.

b) Hence state the value of $\operatorname{nullity}(A)$. Give a reason.

Linear transformations – Problem 3

[MATH1241 2014 S2 Q3 iii)]
Let $\mathcal{R}[\mathbb{R}]$ denote the vector space of all real-valued functions defined on $\mathbb{R}$. Let S be the subspace of $\mathcal{R}[\mathbb{R}]$ that is spanned by the **ordered** basis $\mathcal{B} = \{\cos x, \sin x\}$. Define the linear map $T : S \to S$ by

$$T(f) = f - 2f', \quad \text{where } f' \equiv \frac{\mathrm{d}f}{\mathrm{d}x}.$$

a) Calculate the matrix C that represents T with respect to the basis $\mathcal{B}$.
b) State the rank of the matrix C found in part a).
c) From part b), what can be deduced about the solutions $y \in S$ of the differential equation

$$y - 2y' = g,$$

where g is a given function in S?
d) Using parts a) and c), or otherwise, find all solutions $y \in S$ of the differential equation

$$y - 2y' = \cos x.$$

Eigenvalues and eigenvectors – Problem 1

[MATH1231 2013 S2 Q3 iii)]
Let

$$C = \begin{pmatrix} 1 & 2 \\ -2 & 5 \end{pmatrix}.$$

a) Find the eigenvalue(s) of C and for each eigenvalue, find the corresponding eigenvectors.

b) Is C diagonalisable? Give a reason for your answer.

Eigenvalues and eigenvectors – Problem 2

[MATH1231 2014 S2 Q2 iv)]

Consider the set S consisting of the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ from $\mathbb{R}^3$ and let $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 12 \end{pmatrix}$.

a) Find scalars λ and μ such that $\mathbf{u} = \lambda \mathbf{v}_1 + \mu \mathbf{v}_2$.

b) A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ has $\mathbf{v}_1, \mathbf{v}_2$ as eigenvectors with eigenvalues 2 and -1, respectively.

   α) Find $T(\mathbf{u})$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2$.

   β) Denote $T(T(\mathbf{u}))$ by $T^2(\mathbf{u})$, $T(T(T(\mathbf{u})))$ by $T^3(\mathbf{u})$, and so on. Express $T^n(\mathbf{u})$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, where n is a positive integer.

Eigenvalues and eigenvectors – Problem 3

[MATH1241 2015 S2 Q3 iv)]
A linear transformation $P : V \to V$ is said to be **idempotent** if $P\left(P(\mathbf{v})\right) = P(\mathbf{v})$ for all $\mathbf{v} \in V$ (in other words, $P^2 = P$).

a) Show that the only possible eigenvalues for an idempotent linear transformation are 0 and 1.

b) Show that if P is idempotent and P is neither the zero nor the identity transformation on V, then both 0 and 1 are eigenvalues.

Introduction to probability and statistics – Problem 1

[MATH1231 2012 S2 Q3 iv), MATH1241 2012 S2 Q3 i)]
A six-sided die, with faces numbered 1 to 6, is suspected of being unfair, so that the number 6 will occur more frequently than should happen by chance. During 300 test rolls of the die, the number 6 occurred 68 times.

a) Write down an expression for a tail probability that measures the chance of rolling a 6 at least 68 times.
b) Use the normal approximation to the binomial to estimate this probability.
c) Is this evidence that the die is unfair?

Introduction to probability and statistics – Problem 2

[MATH1231 2013 S2 Q1 iii)]
A collection of discs consists of DVDs and Blu-ray discs. In the collection, 75% are DVDs and 25% are Blu-ray discs. Among the DVDs, 60% are movies. Among the Blu-ray discs, 90% are movies.

a) Find the probability that a disc chosen randomly from the collection is a movie.

b) Find the probability that a randomly chosen disc from the collection is a Blu-ray disc given that it is a movie.

Introduction to probability and statistics – Problem 3

The probability density function f of a continuous random variable X is given by

$$f(x) = \begin{cases} 3(1-x)^2 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

a) Sketch the graph of $y = f(x)$.

b) Find $\mathbb{E}\left[X\right]$ and $\mathrm{Var}\left(X\right)$.

c) Find $\mathbb{P}\left(\frac{1}{2} < \sin\left(\pi X\right) < \frac{1}{\sqrt{2}}\right)$.

d) The **median** of a distribution is defined to be the real number m such that $\mathbb{P}\left(X \leq m\right) = \frac{1}{2}$. Find the median of the above distribution.

Introduction to probability and statistics – Problem 4

[The log-normal distribution]
Let X be a normal random variable with mean μ and standard deviation σ. Find the probability density function of the random variable $Y := e^X$. (Y is said to be a *log-normal* random variable with parameters μ and σ.)

General tips

- Know the formal definitions of **<u>all</u>** key terms and concepts and precise statements of **<u>all</u>** key theorems/propositions. E.g. you should be able to give a precise mathematical definition of "linearly independent set".
- Past final exam questions have asked to state definitions of such terms (e.g. 2015 MATH1231/1241 Question 2 v) a)). If you know your definitions, these are essentially free marks!
- Look at pages *xiii* and *xiv* of the Algebra course pack notes (pages 13-14 of electronic PDF copy). They tell you which definitions you should know, and which theorems/propositions you should be able to state (search these up in the relevant sections of the notes).
- To show something is **not** the case, it suffices to just give a specific/numerical example showing this. E.g. to show a set S is not a subspace, it suffices to find specific vectors $\mathbf{x}, \mathbf{y} \in S$ and demonstrate that $\mathbf{x} + \mathbf{y} \notin S$. No need for any sort of general argument.

General tips (cont.)

- When finding eigenvectors, when you row-reduce $A - \lambda I$ after having found λ, the row-echelon form **must** have at least one zero row. If it doesn't, you know you've made a mistake somewhere (perhaps found the wrong value for λ), and should go back and fix it.
- For any square matrix A,

  $$\text{sum of eigenvalues of } A = \text{sum of diagonal entries of } A.$$

  This gives you a useful check on your calculations: if the sum of the eigenvalues you have found does not equal the sum of diagonal entries of A, you must have made a mistake somewhere. Always use this check when asked to find the eigenvalues of a matrix!
- Remember, for **continuous** random variables, it does not matter whether you use strict or non-strict inequalities for probabilities. E.g. $\mathbb{P}(a \leq X \leq b) = \mathbb{P}(a < X \leq b) = \mathbb{P}(a \leq X < b) = \mathbb{P}(a < X < b)$ **if X is continuous**. (This is because $\mathbb{P}(X = x) = 0$ for any x if X is continuous. I.e. the probability that X takes on any specific value is 0.)
- For **discrete** random variables however, the strictness of inequality signs <u>does</u> matter in general.

Good luck!

ANY QUESTIONS?
GOOD LUCK FOR YOUR FINAL EXAMS!
MATHSOC WISHES YOU ALL THE BEST IN YOUR FUTURE STUDY.

UNSW MATHSOC MATH1231/1241 FINAL EXAM REVISION SESSION – ALGEBRA (Semester 2, 2017)

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