UNSW Mathematics Society presents

MATH1231/1241 Revision Seminar

(Higher) Mathematics 1B

Calculus

T2, 2020

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Functions of Several Variables

Partial Differentiation

Functions of Several Variables

- Let $F(x, y) = x^2 + xy + y^2$. Since this is a function of more than one variable, in order to find the rate of change with respect to x or y, we must use partial differentiation.
- This involves treating all variables other than the one you're differentiating with, as constants.

Example

Using the example above

$$F(x, y) = x^2 + xy + y^2$$

- $\dfrac{\partial F}{\partial x} = F_x = 2x + y,$
- $\dfrac{\partial F}{\partial y} = F_y = x + 2y.$

Tangent Planes to Surfaces

Tangent Planes

Suppose that $z = F(x, y)$. Then, a tangent plane to this curve, evaluated at point $P = (x_0, y_0, z_0)$ has normal vector

$$\mathbf{n} = \begin{pmatrix} F_x \\ F_y \\ -1 \end{pmatrix} \text{ evaluated at } P.$$

Thus, using the point-normal representation of a plane, the equation of a tangent plane is given by

$$\mathbf{n} \cdot (\mathbf{x} - P) = 0,$$

where $\mathbf{x} = (x, y, z)^T$

Tangent Planes to Surface, continued

General Case

- More generally, given a function of the form $g(x, y, z) = 0$, the normal vector at a point P on g is given by

$$= \begin{pmatrix} g_x \\ g_y \\ g_z \end{pmatrix} \text{ evaluated at } P.$$

Example

The equation of the tangent plane to $z = 4x^3 + 3y^4$ at point $(1, 1, 7)$ is given by

$$\mathbf{n} \cdot (\mathbf{x} - P) = \begin{pmatrix} 12 \\ 12 \\ -1 \end{pmatrix} \begin{pmatrix} x - 1 \\ y - 1 \\ z - 7 \end{pmatrix} = 12x + 12y - z - 17 = 0$$

Total Differential Approximation

Given some $F(x, y)$

$$\Delta F \approx \frac{\partial F}{\partial x}\Delta x + \frac{\partial F}{\partial y}\Delta y.$$

Rationale

- Since F is a function of two variables, the change in F (ΔF) is dependent on Δx and Δy.
- Since partial differentiation yields the rate of change of F w.r.t x or y, we can approximate the rate of change of F through the expression above.

Error Estimation

The total differential approximation can be used to estimate the error of a variable that's dependent on other variables. For instance, if F is a function of x and y, then

$$\begin{aligned} |\Delta F| &= \left|\frac{\partial F}{\partial x}\Delta x + \frac{\partial F}{\partial y}\Delta y\right| \\ &\leq \left|\frac{\partial F}{\partial x}\right| |\Delta x| + \left|\frac{\partial F}{\partial y}\right| |\Delta y| \end{aligned}$$

Chain Rule I

Let's say we have $F(x, y)$ such that $x = x(t)$ and $y = y(t)$. In this case, dividing the Total Differential Approximation by Δt yields,

$$\frac{\Delta F}{\Delta t} = \frac{\partial F}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial F}{\partial y}\frac{\Delta y}{\Delta t}$$

Now, as $\Delta t \to 0$, we have that

- $\dfrac{\Delta F}{\Delta t} \to \dfrac{dF}{dt}$
- $\dfrac{\Delta x}{\Delta t} \to \dfrac{dx}{dt}$
- $\dfrac{\Delta y}{\Delta t} \to \dfrac{dy}{dt}$

And so,

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt}$$

Chain Rules I

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And so,

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt}$$

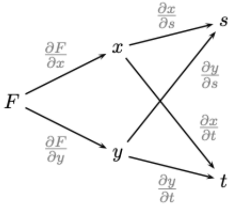
Chain Rules I

We can attempt to simplify that expression through use of a chain rule diagram, seen here.

$$
\begin{tikzcd}
 & x \arrow[dr, "\frac{dx}{dt}"] & \\
F \arrow[ur, "\frac{\partial F}{\partial x}"] \arrow[dr, "\frac{\partial F}{\partial y}"'] & & t \\
 & y \arrow[ur, "\frac{dy}{dt}"'] &
\end{tikzcd}
$$

Chain Rules II

Now, let's say we have a function $F(x, y)$ such that $x = x(s, t)$ and $y = y(s, t)$. Drawing out our Chain Rule diagram, we have,



Hence, the chain rule expression becomes

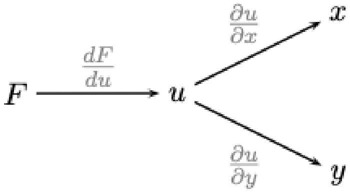
$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t}$$

and similarly

$$\frac{\partial F}{\partial s} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial s}$$

Chain Rules III

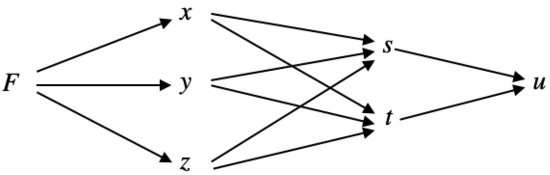
Now, let's say we have $F(u)$ where $u = u(x, y)$. In this case,



- $\dfrac{\partial F}{\partial x} = \dfrac{dF}{du}\dfrac{\partial u}{\partial x}$
- $\dfrac{\partial F}{\partial y} = \dfrac{dF}{du}\dfrac{\partial u}{\partial y}$

Functions of three or more variables

Let F be a function $F(x, y, z)$ where $x = x(s, t)$, $y = y(s, t)$ and $z = z(s, t)$, where $s = s(u)$ and $t = t(u)$.



The truly monstrous expression we end up with is

$$\frac{dF}{du} = \frac{ds}{du}\left(\frac{\partial F}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial s}\right) + \frac{dt}{du}\left(\frac{\partial F}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial t}\right)$$

Integration Techniques

Trigonometric Integrals

$$\int \sin^m(x)\cos^n(x)\ dx$$

Case 1: n odd

$$u = \sin x \qquad du = \cos x\ dx \qquad \cos^2 x = 1 - \sin^2 x$$

Case 2: m odd

$$u = \cos x \qquad du = -\sin x\ dx \qquad \sin^2 x = 1 - \cos^2 x$$

$$\int \sin^2 x \cos^5 x\ dx = \int \sin^2 x \cos^4 x(\cos x\, dx)$$

$$= \int u^2(1-u^2)^2\ du = \frac{\sin^3 x}{3} - \frac{2sin^5 x}{5} + \frac{\sin^7 x}{7} + c$$

Trigonometric Integrals

Case 4: m, n even

Pray.

Trigonometric Integrals

Case 1

$$\sin(mx)\cos(nx) = \frac{1}{2}\Big(2\sin(mx)\cos(nx)\Big)$$
$$= \frac{1}{2}\Big(\sin\left((m+n)x\right) + \sin\left((m-n)x\right)\Big)$$

Case 2

$$\cos(mx)\cos(nx) = \frac{1}{2}\Big(\cos\left((m+n)x\right) + \cos\left((m-n)x\right)\Big)$$

Case 3

$$\sin(mx)\sin(nx) = \frac{1}{2}\Big(\cos\left((m-n)x\right) - \cos\left((m+n)x\right)\Big)$$

Trigonometric Integrals

Important Identities

$$\tan^2 x + 1 = \sec^2 x \qquad \frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\sec x = \sec x \tan x$$

$$\int \tan^2 x \, dx = \int \sec^2 x - 1 \, dx = \tan x - x + C$$

$$\begin{aligned}\int \sec^4 x \tan x \, dx &= \int (\sec^3 x)(\sec x \tan x) \, dx \\ &= \int u^3 \, du \\ &= \frac{u^4}{4} + C = \frac{\sec^4 x}{4} + C\end{aligned}$$

Reduction Formulae

Let I_n be defined as

$$I_n = \int_0^{\pi/4} \tan^n x \, dx.$$

Find a reduction formula in terms of I_{n-2}.

$$\begin{aligned}
\int_0^{\pi/4} \tan^n x \, dx &= \int_0^{\pi/4} tan^{n-2} x \tan^2 x \, dx \\
&= \int_0^{\pi/4} tan^{n-2} x (\sec^2 x - 1) \, dx \\
&= \int_0^{\pi/4} tan^{n-2} \sec^2 x \, dx - \int_0^{\pi/4} \tan^{n-2} x \, dx \\
&= \left[\frac{u^{n-1}}{n-1}\right]_0^1 - I_{n-2} = \frac{1}{n-1} - I_{n-2}.
\end{aligned}$$

Reduction Formulae, continued

Use the reduction forumula obtained on the previous slide to work out the value of

$$\int_0^{\pi/4} \tan^5 x.$$

$$\begin{aligned}
\int_0^{\pi/4} \tan^5 x &= I_5 \\
&= \frac{1}{4} - I_3 \\
&= \frac{1}{4} - \frac{1}{2} + I_1 \\
&= -\frac{1}{4} + \int_0^{\pi/4} \tan x \; dx \\
&= \Big[\ln(\sec x) \Big]_0^{\pi/4} - \frac{1}{4} = \frac{1}{2}\ln 2 - \frac{1}{4}
\end{aligned}$$

Trigonometric Substitutions

Substitution 1

$$\sqrt{a^2 - x^2} \quad \Longleftrightarrow \quad \begin{aligned} x &= a\sin\theta \\ dx &= a\cos\theta \; d\theta \end{aligned}$$

Substitution 2

$$\sqrt{a^2 + x^2} \quad \Longleftrightarrow \quad \begin{aligned} x &= a\tan\theta \\ dx &= a\sec^2\theta \; d\theta \end{aligned}$$

Substitution 3

$$\sqrt{x^2 - a^2} \quad \Longleftrightarrow \quad \begin{aligned} x &= a\sec\theta \\ dx &= a\sec\theta\tan\theta \; d\theta \end{aligned}$$

Hyperbolic Substitutions

Substitution 1

$$\sqrt{a^2 - x^2} \quad \rightleftarrows \quad \begin{aligned} x &= a\tanh\theta \\ dx &= a\,\mathrm{sech}\,\theta\ d\theta \end{aligned}$$

Substitution 2

$$\sqrt{a^2 + x^2} \quad \rightleftarrows \quad \begin{aligned} x &= a\sinh\theta \\ dx &= a\cosh\theta\ d\theta \end{aligned}$$

Substitution 3

$$\sqrt{x^2 - a^2} \quad \rightleftarrows \quad \begin{aligned} x &= a\cosh\theta \\ dx &= a\sinh\theta\ d\theta \end{aligned}$$

Using Trig Substitutions

Solve the integral

$$\int \frac{dx}{\sqrt{4-x^2}}$$

through use of the substitution $x = 2\tanh\theta$.

$$\begin{aligned}\int \frac{dx}{\sqrt{4-x^2}} &= \int \frac{2\operatorname{sech}\theta \; d\theta}{\sqrt{4-4\tanh^2\theta}}\\ &= \int \frac{2\operatorname{sech}\theta}{2\operatorname{sech}\theta} d\theta\\ &= \int d\theta = \theta + C = \tanh^{-1}(x/2) + C.\end{aligned}$$

Hyperbolic Trig Identities

- $\cosh^2 x - \sinh^2 x = 1$
- $1 - \tanh^2 x = \operatorname{sech}^2 x$

ODE's

What are ODE's

Definition

An ordinary differential equation is an equation which describes a relationship between a variable, and its first- (and second-) derivatives.

Some examples

$$\frac{dP}{dt} = \pm kP \qquad \frac{d^2x}{dt^2} = -kx \qquad \frac{dT}{dt} = k(T - T_s)$$

Initial Value Problems

Definition

An initial value problem is an nth order ODE, with a set of values for the variable itself, as well as all the derivatives until $n-1$.

Example

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 4y = 0.$$

When $t = 0$, then $y = 17$ and $y' = -1$.

Solving an IVP

Solving a IVP

Solve

$$\frac{d^2y}{dx^2} = x^3,$$

given $y'(2) = 5$ and $y(0) = 1$.

We first integrate both sides of the ODE

$$y' = \frac{dy}{dx} = \frac{x^4}{4} + C.$$

Since $y'(2) = 5$,

$$\frac{2^4}{4} + C = 5$$
$$C = 1.$$

Hence, $y' = x^4/4 + 1$.

Solving an IVP, continued

Now, we integrate both sides of the equation again, ending up with

$$y = \frac{x^5}{20} + x + C.$$

Now, we know that $y(0) = 1$, so

$$\begin{gathered} 0 + 0 + C = 1 \\ C = 1. \end{gathered}$$

Hence,

$$y = \frac{x^5}{20} + x + 1.$$

Separable ODE's

Definition

A *separable ODE* is one where both of the variables involved in the ODE (e,g, y and x) can be separated fully into two halves of the equation.
This makes it easier to solve the differential equation, as we can integrate both sides.

Separating ODE's

$$\frac{dy}{dx} = 4x^4y^2$$

$$\int \frac{dy}{y^2} = 4\int x^4 \, dx$$

Solving Separable ODE's

Solve

$$\frac{dy}{dx} = yx^4.$$

$$\begin{aligned}\int \frac{dy}{y} &= \int x^4 \, dx \\ \ln y &= \frac{x^5}{5} + C \\ y &= \exp\left(\frac{x^5}{5} + C\right) \\ &= A\exp\left(\frac{x^5}{5}\right).\end{aligned}$$

First-Order Linear ODE's

Definition

A first-order linear ODE is one that can be expressed in the following form,

$$\frac{dy}{dx} + f(x)y = g(x),$$

where f and g are functions in x.

$$2\frac{dy}{dx} + 4x^3y = 3x,$$

is an example of a first order linear ODE.

Solving First-Order Linear ODE's

$$\frac{dy}{dx} + f(x)y = g(x).$$

Method

1. Write the ODE in the above form.
2. Calculate $h(x) = e^{\int f(x)\, dx}$ (ignore the constant).
3. Multiply the ODE by $h(x)$ to get

$$\frac{dy}{dx}h(x) + h(x)f(x)y = h(x)g(x).$$

4. Because of the product rule, this is equivalent to

$$\frac{d}{dx}\Big(h(x)y\Big) = g(x)h(x).$$

Solving First-Order Linear ODE's, continued

Solve

$$(x-1)^3\frac{dy}{dx} + 4(x-1)^2 y = x+1, \quad y(0) = 2$$

Firstly, expressing ODE in the proper form leads to

$$\frac{dy}{dx} + 4y(x-1)^{-1} = \frac{x+1}{(x-1)^3}.$$

Now,

$$e^{\int 4(x-1)^{-2}dx} = e^{4\ln(x-1)} = (x-1)^4.$$

Multiplying through the *integrating factor*

$$(x-1)^4\frac{dy}{dx} + 4y(x-1)^3 = (x+1)(x-1)$$

Solving First-Order Linear ODE's, continued

$$(x-1)^4\frac{dy}{dx} + 4y(x-1)^3 = x^2 - 1.$$

Now, by the product rule, this simplifies to

$$\frac{d}{dx}\Big(4y(x-1)^4\Big) = x^2 - 1.$$

Upon integrating both sides with respect to x,

$$4y(x-1)^4 = \frac{x^3}{3} - x + C$$

Since $y(0) = 2$, substituting $x = 0, y = 2$,

$$4(2)(-1)^4 = 8 = 0 - 0 + C.$$

Solving First-Order Linear ODE's, continued

$$4(2)(-1)^4 = 8 = 0 + 0 + C.$$

Hence, $C = 8$, and upon dividing by $4(x-1)^4$, we get our solution,

$$y = \frac{x^3 - 3x + 24}{12(x-1)^4}$$

Exact ODE's

Definition

Exact ODE's are ODE's of the form

$$F(x, y) + G(x, y)\frac{dy}{dx} = 0,$$

such that,

$$\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x}.$$

In this case, the solution to the ODE is given by $H(x, y) = C$, where

$$\frac{\partial H}{\partial x} = F \quad \text{and} \quad \frac{\partial H}{\partial y} = G,$$

and C is just a constant.

Solving Exact ODE's

Show that

$$\frac{dy}{dx} = -\frac{2x + y + 1}{2y + x + 1}$$

is exact, and hence find its solution.

Rearranging, we have

$$2x + y + 1 + (2y + x + 1)\frac{dy}{dx} = 0,$$

which is in the form of an exact ODE, since

$$\frac{\partial F}{\partial y} = 1 = \frac{\partial G}{\partial x}$$

Solving Exact ODE's, continued

Hence, there must exist a $H(x, y)$ such that

$$\frac{\partial H}{\partial x} = 2x + y + 1 = F$$
$$\frac{\partial H}{\partial y} = 2y + x + 1 = G.$$

Now, when we integrate F with respect to x, we get

$$H(x, y) = x^2 + xy + x + C_1(y),$$

where the constant of integration is with respect to y, since it's treated as a constant w.r.t x.

Solving Exact ODE's, continued

Similarly, when integrating G with respect to y, we obtain

$$H(x,y) = y^2 + xy + y + C_2(x),$$

where the constant is a function of x.
Now, comparing these two forms, we can see that the final form is

$$H(x,y) = x^2 + xy + y^2 + x + y.$$

The solution to this exact ODE is

$$x^2 + xy + y^2 + x + y = C,$$

the value of C dependent on initial conditions.

Second-Order Linear ODE's I

The Homogeneous Case

- A second order linear ODE is **homogeneous** if it is of the form:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0.$$

- If y_1 and y_2 are two solutions to this ODE, then any linear combination (i.e. $Ay_1 + By_2$) is also a solution.
- If y_1 and y_2 are two linearly independent solutions to the above ODE, then every solution can be written in the form $y = Ay_1 + By_2$.

Second-Order Linear ODE's II

Finding Homogeneous Solutions

- To solve second-order linear ODE's of the form:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0,$$

substitute in the solution $y = Ae^{\lambda x}$:

$$\lambda^2(Ae^{\lambda x}) + a\lambda(Ae^{\lambda x}) + b(Ae^{\lambda x}) = 0$$

- Factorising out $Ae^{\lambda x}$ produces the **characteristic equation**, which allows us to find the λ's:

$$\lambda^2 + a\lambda + b = 0$$

Second-Order Linear ODE's III

Finding Homogeneous Solutions Continued

Solving the characteristic equation leads to one of three cases:

i) If there are two distinct, real roots (λ_1 and λ_2), then the general solution is:

$$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}.$$

ii) If there is one repeated real root (λ_1), then the general solution is:

$$y = Ae^{\lambda_1 x} + Bxe^{\lambda_1 x}.$$

iii) If there are two complex conjugate roots ($\alpha \pm \beta i$), then the general solution is:

$$y = e^{\alpha x}(A \cos \beta x + B \sin \beta x).$$

Second-Order ODE Example I

MATH1251 (S2, 2018) Q3 iii)

a) If y is a function of x, find the general solution of the following differential equation for y.

$$y'' + 6y' + 9y = 0$$

Second-Order ODE Example I

MATH1251 (S2, 2018) Q3 iii)

a) If y is a function of x, find the general solution of the following differential equation for y.

$$y'' + 6y' + 9y = 0$$

a) First, solve the corresponding characteristic equation:

$$\lambda^2 + 6\lambda + 9 = 0$$
$$(\lambda + 3)^2 = 0$$
$$\lambda = -3$$

Since there is a repeated root, the general solution will be in the form:

$$y = Ae^{-3x} + Bxe^{-3x}$$

Second-Order Linear ODE's IV

Non-homogeneous ODE's

- A non-homogeneous ODE will be in the form:

$$y'' + ay' + by'' = f(x).$$

- To solve this, we first find the homogeneous solution before looking for a **particular solution**. The general solution will then be a sum of these two:

$$y = y_H + y_P.$$

- To find the particular solution (y_P), we make a "guess", which will depend on the form of f.

Second-Order Linear ODE's V

$f(x)$	Guess for particular solution y_p
$P(x)$ (polynomial of degree n)	$Q(x)$ (polynomial of degree n)
$P(x)e^{sx}$	$Q(x)e^{sx}$
$P(x)\cos(sx)$ or $P(x)\sin(sx)$	$Q_1(x)\cos(sx) + Q_2(x)\sin(sx)$
$P(x)e^{sx}\cos(tx)$ or $P(x)e^{sx}\sin(tx)$	$Q_1(x)e^{sx}\cos(tx) + Q_2(x)e^{sx}\sin(tx)$

If $P(x)$ is a constant, then $Q(x)$ is also a constant.

Second-Order Linear ODE's VI

Non-homogeneous ODE's Continued

- If any term for the guess for y_P is a homogeneous solution, then multiply it by x. If it is still a homogeneous solution, then multiply it by x again.
- After making the appropriate guess for the particular solution, substitute it into the ODE and equate to find the unknown coefficients.
- Add the particular solution to the homogeneous solution to get the general solution.
- If initial values are given, substitute them in at this point to find the coefficients from the homogeneous solution.

Second-Order ODE Example I

b) What form of the trial solution would you use to find a particular solution to the following differential equation?

$$y'' + 6y' + 9y = e^{-3x}$$

Second-Order ODE Example I

b) What form of the trial solution would you use to find a particular solution to the following differential equation?

$$y'' + 6y' + 9y = e^{-3x}$$

b) As e^{-3x} and xe^{-3x} are part of the solution for the homogeneous equation, the particular solution should be in the form:

$$y = Cx^2e^{-3x}$$

Second-Order ODE Example II

MATH1231 (T1, 2019) Q2 d)

Find the general solution the following ordinary differential equation

$$y''(x) + 4y'(x) + 4y(x) = 8x$$

Second-Order ODE Example II

MATH1231 (T1, 2019) Q2 d)

Find the general solution the following ordinary differential equation

$$y''(x) + 4y'(x) + 4y(x) = 8x$$

First, solve the homogeneous equation by solving the characteristic equation:

$$\lambda^2 + 4\lambda + 4 = 0$$

$$\lambda = -2$$

$$y_H = Ae^{-2x} + Bxe^{-2x}$$

Second-Order ODE Example II

Next, find the particular solution:

$$y_P = Cx + D$$
$$y_P' = C$$
$$y_P'' = 0$$

Substitute into ODE:

$$0 + 4C + 4(Cx + D) = 8x$$
$$4Cx + 4C + 4D = 8x$$

$$4C = 8 \implies C = 2$$
$$4C + 4D = 0 \implies D = -2$$

$$y = Ae^{-2x} + Be^{-2x} + 2x - 2$$

Second-Order ODE Example III

MATH1251 (S2, 2017) Q3 ii)

Solve the following initial-value problem

$$y'' - 5y' + 6y = 10e^{2x}, \quad y(0) = 1, \quad y'(0) = 1.$$

Second-Order ODE Example III

MATH1251 (S2, 2017) Q3 ii)

Solve the following initial-value problem

$$y'' - 5y' + 6y = 10e^{2x}, \quad y(0) = 1, \quad y'(0) = 1.$$

Solve the characteristic equation for the homogeneous problem:

$$\lambda^2 - 5\lambda + 6\lambda = 0$$

$$\lambda = 2, 3$$

$$y_H = Ae^{2x} + Be^{3x}$$

Second-Order ODE Example III

Find the particular solution:

$$\begin{aligned} y_P &= Cxe^{2x} \\ y_P' &= C(e^{2x} + 2xe^{2x}) \\ y_P'' &= C(4e^{2x} + 4xe^{2x}) \end{aligned}$$

Substitute into the equation:

$$\begin{aligned} C\left[(4e^{2x} + 4xe^{2x}) - 5(e^{2x} + 2xe^{2x}) + 6(xe^{2x})\right] &= 10e^{2x} \\ C\left[(4-5)e^{2x} + (4-10+6)xe^{2x}\right] &= 10e^{2x} \end{aligned}$$

$$\begin{aligned} -Ce^{2x} &= 10e^{2x} \\ C &= -10 \end{aligned}$$

Second-Order ODE Example III

$$\begin{aligned} y &= y_H + y_P \\ &= Ae^{2x} + Be^{3x} - 10xe^{2x} \end{aligned}$$

Substitute initial values:

$$y(0) = Ae^0 + Be^0 - 10(0)e^0 = 1$$

$$\implies A + B = 1$$

$$y'(0) = 2Ae^0 + 3Be^0 - 10e^0 - 20(0)e^0 = 1$$

$$\implies 2A + 3B - 10 = 1$$

$$\implies 2A + 3B = 11$$

$$A = -8$$

$$B = 9$$

$$y = -8e^{2x} + 9e^{3x} - 10xe^{2x}$$

Taylor Series

Applications to Stationary Points

Classifying Stationary Points

- Suppose that a function f is n times differentiable at a and that $f'(a) = 0$. Then to classify this stationary point, we can keep differentiating f at a (up to n times) until we find a non-zero value.
- Suppose that k is the least integer such that $k < n$ and $f^{(k)}(a) \neq 0$. Then

  i) a is a local minimum point if k is even and $f^{(k)}(a) > 0$ (e.g. $f''(a) > 0$);
  ii) a is a local maximum if k is even and $f^{(k)}(a) < 0$;
  iii) a is a horizontal point of inflection if k is odd.

Sequences I

Definition of a Sequence

- A **sequence** is a function with the natural numbers as its domain and real numbers as its codomain. Sequences have their own notation:

$$\{a_n\} \text{ or } \{a_n\}_0^\infty$$

- Sequences are defined by a rule which tells us how to find each term. For example:

$$a_n = n^2$$

$$a_n = a_{n-1} + a_{n-2}$$

Sequences II

Convergence and Divergence

- A sequence a_n is **convergent** if it approaches some finite number L as n approaches infinity.

$$\lim_{n \to \infty} a_n = L$$

- A sequence that is not convergent is **divergent**. A divergent sequence can either be *boundedly divergent* or *unboundedly divergent*. An example of a boundedly divergent sequence is:

$$a_n = \sin n.$$

- If $a_n = f(n)$ for all large n and $\lim_{x \to \infty} f(x)$ exists, then

$$\lim_{n \to \infty} a_n = \lim_{x \to \infty} f(x)$$

Taylor Series Example I

MATH1231 (S2, 2016) Q2 i) a)

Determine whether the sequence

$$\sqrt{n+\sqrt{n}}-\sqrt{n}$$

converges or diverges as $n \to \infty$. If it converges, find its limit.

Taylor Series Example I

MATH1231 (S2, 2016) Q2 i) a)

Determine whether the sequence

$$\sqrt{n+\sqrt{n}}-\sqrt{n}$$

converges or diverges as $n \to \infty$. If it converges, find its limit.

Substituting large numbers into your calculator will indicate that the sequence converges to 0.5.

Taylor Series Example I

For a more intuitive answer, it can be shown that,

$$\begin{aligned}
\sqrt{n+\sqrt{n}}-\sqrt{n} &= \frac{\sqrt{n}}{\sqrt{n+\sqrt{n}}+\sqrt{n}} \\
&= \frac{1}{\sqrt{\frac{n+\sqrt{n}}{n}}+1} \\
&= \frac{1}{\sqrt{1+\frac{1}{\sqrt{n}}}+1} \\
&\to 0.5 \quad \text{as} \quad n\to\infty.
\end{aligned}$$

Sequences III

Combination of Sequences

- Since sequences are a type of function with the same domain ($\mathbb{N}$), they can be added, subtracted, multiplied and divided to produce a new sequence.

$$\{a_n\} + \{b_n\} = \{a_n + b_n\}$$

- If two sequences are convergent, then the same applies to their limits.

$$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \times \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$$

Sequences IV

Order of Growth

- To determine the convergence/divergence of a sequence composed of elementary functions, it is important to know the order of growth between them.

a_n	growth rate as $n \to \infty$
1	constant
$\log n$	grows slowly
n^k, where $k > 0$	growth rate is faster for larger k
c^n, where $c > 1$	growth rate is faster for larger c
$n!$	grows rapidly
n^n	grows very rapidly

Sequences V

Pinching Theorem For Sequences

- Suppose that $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences and that for all $n > N$ for some N, the following inequality holds.

$$a_n \leq b_n \leq c_n$$

If $\{a_n\}$ and $\{c_n\}$ both converge to some value L, then $\{b_n\}$ converges to L.

Sequences VI

Another Test For Convergence

- A sequence is **monotonic** if it is either non-increasing or non-decreasing for all n.
- A sequence is **bounded** above if there exists an M such that $a_n < M$ for all natural numbers n.
- A non-decreasing (non-increasing) sequence or real numbers that is bounded above (below) will converge to some real number L.

Sequences VII

Suprema and Infima

- The **supremum** of a sequence $\{a_n\}_{n=0}^{\infty}$ M is its least upper bound. It has two conditions:
  i) $a_n \leq M$ for all n.
  ii) If K is an upper bound, then $K \geq M$.
- Similarly, the **infimum** of a sequence is its greatest lower bound.
- According to the least upper bound axiom, every nonempty set of real numbers that is bounded above, has a least upper bound.

Taylor Series Example II

MATH1231/1241 Calculus Notes Q16

Find the supremum and infimum of each of the following sets.

a) $\left\{ \dfrac{n}{1+n^2} \,:\, n = 1, 2, \ldots \right\}$

e) $\{x \in (0, \infty) \,:\, \sin x < 0\}$

Taylor Series Example II

MATH1231/1241 Calculus Notes Q16

Find the supremum and infimum of each of the following sets.

a) $\left\{ \dfrac{n}{1+n^2} \, : \, n = 1, 2, \ldots \right\}$

e) $\{x \in (0, \infty) \, : \, \sin x < 0\}$

a) Since the sequence is strictly decreasing for $n = 1, 2, \ldots$, its supremum will be $\frac{1}{2}$ (at $n = 1$).

Since the sequence converges to 0, its infimum will be 0.

Taylor Series Example II

MATH1231/1241 Calculus Notes Q16

Find the supremum and infimum of each of the following sets.

a) $\left\{ \dfrac{n}{1+n^2} \,:\, n = 1, 2, \ldots \right\}$

e) $\{x \in (0, \infty) \,:\, \sin x < 0\}$

a) Since the sequence is strictly decreasing for $n = 1, 2, \ldots$, its supremum will be $\frac{1}{2}$ (at $n = 1$).

Since the sequence converges to 0, its infimum will be 0.

e) This sequence does not have a supremum due to the periodicity of $\sin x$. Its infimum is π.

Infinite Series I

Sums

- A **partial sum** s_n represents the sum of terms of a sequence up to n.

$$s_n = a_0 + a_1 + \cdots + a_n = \sum_{k=0}^{n} a_k$$

- If the partial sum approaches some finite L as $n \to \infty$, then the **infinite series** is **summable** and converges to L.

$$\lim_{n\to\infty} s_n = \sum_{k=0}^{\infty} a_k = L$$

- If the series does not approach some finite number, then it diverges.

Infinite Series II

Summable Series

- Since summable series can be equated to real numbers, the summations can be manipulated as regular sums:

$$\sum_{k=0}^{\infty} a_k + b_k = \sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} b_k$$

$$\sum_{k=0}^{\infty} (\alpha a_k) = \alpha \sum_{k=0}^{\infty} a_k$$

- As a finite sum (of finite terms) will always be finite, the first N (where $N \in \mathbb{Z}^+$) terms are irrelevant to the convergence of a sum.

$$\sum_{k=0}^{\infty} a_k \quad \text{converges iff} \quad \sum_{k=N}^{\infty} a_k \quad \text{converges.}$$

Tests for Series Convergence I

The kth Term Test for Divergence

- If $\{a_k\}$ diverges as $k \to \infty$, then the series $\displaystyle\sum_{k=0}^{\infty} a_k$ diverges.
- This test is for divergence only.

Taylor Series Example III

MATH1231 (S2, 2018) Q4 vi)

Suppose that $\sum_{n=0}^{\infty} a_n$ is a convergent series with $a_n > 0$ for all n.

a) State $\lim_{n\to\infty} a_n$.

b) Use the nth test to show that $\sum_{n=0}^{\infty} \ln(a_n)$ diverges.

c) Given that $f(x) = x - \ln(1+x)$ is positive for $x > 0$, determine whether $\sum_{n=0}^{\infty} \ln(1+a_n)$ converges or diverges. Explain your answer.

Taylor Series Example III

a) For the series to converge, we must have $\lim_{n \to \infty} a_n = 0$.

Taylor Series Example III

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b) From a), $\lim_{n\to\infty} \ln(a_n) = -\infty$. Thus, the sequence $\{\ln(a_n)\}$ diverges, and by kth test, the infinite sum diverges.

Taylor Series Example III

a) For the series to converge, we must have $\lim_{n\to\infty} a_n = 0$.

b) From a), $\lim_{n\to\infty} \ln(a_n) = -\infty$. Thus, the sequence $\{\ln(a_n)\}$ diverges, and by kth test, the infinite sum diverges.

c) Rearranging, we know that for $x > 0$,

$$x > \ln(1+x) > 0$$

Then, by substituting $x = a_n$ (as we know $a_n > 0$ for all n), we obtain

$$a_n > \ln(1+a_n) > 0$$

By the comparison test, since $\sum_{n=0}^{\infty} a_n$ converges, then $\sum_{n=0}^{\infty} \ln(1+a_n)$ also converges.

Tests for Series Convergence II

The Comparison Test

- Suppose that $0 \leq a_k \leq b_k$ for every natural number k. Then

  i) If $\sum_{k=0}^{\infty} b_k$ converges, then $\sum_{k=0}^{\infty} a_k$ also converges.

  ii) If $\sum_{k=0}^{\infty} a_k$ diverges, then $\sum_{k=0}^{\infty} b_k$ also diverges.

- A ***p*-series** will converge if $p > 1$ and will diverge if $p \leq 1$.

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

Taylor Series Example IV

MATH1231 (T1, 2019) Q1 d)

Giving brief reasons, state whether the following is true or false?

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \text{ diverges if } s = \frac{2}{3}.$$

Taylor Series Example IV

MATH1231 (T1, 2019) Q1 d)

Giving brief reasons, state whether the following is true or false?

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \text{ diverges if } s = \frac{2}{3}.$$

True. This is a p-series and will diverge since $s \leq 1$.

Tests for Series Convergence III

The Limit Form of the Comparison Test

- Suppose that a_n, b_n are positive sequences and suppose that $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, where L is some non-zero, finite number. Then $\sum^{\infty} a_n$ converges if and only if $\sum^{\infty} b_n$ converges.

Tests for Series Convergence IV

The Integral Test

- Replace the formula for a_k with $f(x)$. If $f(x)$ is a continuous, positive function that is decreasing on $[1, \infty)$, then we can use it to apply the integral test:

  i) If $\displaystyle\int_1^{\infty} f(x)dx$ converges, then so does $\displaystyle\sum_{k=1}^{\infty} a_k$.

  ii) If $\displaystyle\int_1^{\infty} f(x)dx$ diverges, then so does $\displaystyle\sum_{k=1}^{\infty} a_k$.

Tests for Series Convergence V

The Ratio Test

- Suppose that $\sum a_k$ is an infinite series with positive terms and that

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = r$$

i) If $r < 1$, then $\sum a_k$ converges.
ii) If $r > 1$, then $\sum a_k$ diverges.
iii) If $r = 1$, then the test is inconclusive.

Taylor Series Example V

MATH1231 (S2, 2016) Q2 ii)

By using an appropriate test, determine whether each of the following series converges or diverges.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

Taylor Series Example V

MATH1231 (S2, 2016) Q2 ii)

By using an appropriate test, determine whether each of the following series converges or diverges.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

a) We use the ratio test:

$$\lim_{n\to\infty} \left[\frac{(n+1)^2}{2^{n+1}} \Big/ \frac{n^2}{2^n} \right] = \lim_{n\to\infty} \frac{n^2+2n+1}{2n^2} = \frac{1}{2}$$

Since the ratio is less than 1, the series converges.

Taylor Series Example V

b) $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k(\ln k)^2}$

Taylor Series Example V

b) $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k(\ln k)^2}$

b) We use the integral test:

$$\int_3^{\infty} \frac{1}{x(\ln x)^2}\,dx = \int_{\ln 3}^{\infty} \frac{du}{u^2} = \left[-\frac{1}{u}\right]_{\ln 3}^{\infty} = \frac{1}{\ln 3}$$

As the integral is finite, series converges.

Tests for Series Convergence VI

Leibniz' Test for Convergence

- An **alternating series** is whose terms have alternating signs. They exist in the form:

$$a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - \cdots$$

- An alternating series of real numbers will converge if the positive versions of its terms satisfy the following:
  i) $a_k \geq 0$;
  ii) $a_k \geq a_{k+1}$ for all k;
  iii) $\lim_{k \to \infty} a_k = 0$.

Absolute and Conditional Convergence

Absolute and Conditional Convergence

- A series is **absolutely convergent** if the following is convergent.

$$\sum_{k=0}^{\infty} |a_k|$$

- Absolute convergence implies convergence.
- If a series converges, but does not converge absolutely, then it is **conditionally convergent**.
- A series that converges absolutely will converge to a unique value. A series that converges conditionally can be rearranged to converge to any real number, or even to diverge.

Taylor Series Example VI

MATH1251 (S2, 2018) Q3 iv)

Determine whether the series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \log(n)}$$

absolutely or conditionally converges, or diverges. Provide reasons.

Taylor Series Example VI

MATH1251 (S2, 2018) Q3 iv)

Determine whether the series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n\log(n)}$$

absolutely or conditionally converges, or diverges. Provide reasons.

First, apply Leinbiz' Test.

i) $\dfrac{1}{n\log(n)}$ is non-negative for $n \geq 2$.

ii) $\dfrac{1}{(n+1)\log(n+1)} < \dfrac{1}{n\log(n+1)} < \dfrac{1}{n\log(n)}$. Therefore the terms are decreasing.

iii) $\displaystyle\lim_{n\to\infty} \frac{1}{n\log(n)} = 0$ since $\displaystyle\lim_{n\to\infty} n\log(n) \to \infty$.

Taylor Series Example VI

As it passes the Leibniz' test, the series converges. To test for absolute convergence, use the integral test.

$$\begin{aligned}\int_2^\infty \frac{1}{n\log(n)}dx &= \int_{\log(2)}^\infty \frac{du}{u}\\ &= \Big[\log(u)\Big]_{\log(2)}^\infty\end{aligned}$$

As this tends to infinity, the positive series fails the integral test and so the series converges conditionally.

Taylor Series I

Introduction to Taylor Series

- **Taylor Series** are infinite sums used to approximate "smooth" functions.

$$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

- By approximating functions as polynomials, they become easier to understand as well as to compute.

Taylor Series II

Taylor Polynomials

- The nth **taylor polynomial** for a "smooth" function f about a is defined by:

$$\begin{aligned} p_n(x) &= f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!} \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \end{aligned}$$

Taylor Series III

Taylor's Theorem

- **Taylor's theorem** states that a function f that has $n+1$ continuous derivatives on an open interval I containing a can be approximated using a Taylor polynomial.

$$f(x) = p_n(x) + R_{n+1}(x)$$

- The remainder can be found exactly as:

$$R_{n+1}(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)dt$$

Taylor Series IV

Lagrange Form

- Since this is usually difficult to compute, a more convenient form is the **Lagrange form** of the remainder:

$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some real number c between a and x.

Taylor Series Example VII

MATH1241 (T1, 2020) Q6

Let $P(x)$ be a real polynomial of degree N and $c \in \mathbb{R}$. Using Taylor Polynomials, we can always write:

$$P(x) = \sum_{i=0}^{M} a_i (x-c)^i$$

Explain why this is true. In particular:

- state any theorem you would use to prove the equality above;
- give an expression for the largest M such that $a_M \neq 0$ in terms of N and/or $P(x)$;
- explain how the numbers a_i are obtained in terms of $P(x)$.

Taylor Series Example VII

$$P(x) = \sum_{i=0}^{M} a_i (x-c)^i$$

- Since $P(x)$ is a polynomial, it is infinitely differentiable and by Taylor's theorem, we can always approximate it using a Taylor Polynomial of any degree M.
- As $P(x)$ is of degree N, $P^{(N+1)}(a) = 0$ (for any a).
  Therefore, for $M > N$, from the Lagrange form $R_{M+1}(x) = 0$ and so we have $P(x) = p_M(x)$.

Taylor Series Example VII

$$P(x) = \sum_{i=0}^{M} a_i (x-c)^i$$

- Since $P(x)$ is a polynomial, it is infinitely differentiable and by Taylor's theorem, we can always approximate it using a Taylor Polynomial of any degree M.
- As $P(x)$ is of degree N, $P^{(N+1)}(a) = 0$ (for any a).
  Therefore, for $M > N$, from the Lagrange form $R_{M+1}(x) = 0$ and so we have $P(x) = p_M(x)$.
- The largest M such that $a_M \neq 0$ is $M = N$.
- $a_i = \dfrac{P^{(i)}(c)}{i!}$.

Taylor Series V

Taylor Series

- A **Taylor Series** for a function f about a is its Taylor polynomial where $n \to \infty$. For the case where $a = 0$, the series is also called the **Maclaurin Series** for f.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

- If $\lim_{n\to\infty} R_{n+1}(x) = 0$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Taylor Series VI

Some examples of convergent Taylor Series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \qquad x \in (-1, 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \qquad x \in \mathbb{R}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad x \in \mathbb{R}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad x \in \mathbb{R}$$

$$\ln(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots \qquad x \in (-1, 1]$$

Power Series I

Power Series

- A Taylor Series is a type of **power series**, which is just a sum of integer powers of x:

$$\sum_{k=0}^{\infty} a_k (x-a)^k,$$

where $\{a_k\}_{k=0}^{\infty}$ is a sequence of real coefficients.

Power Series II

Convergence/Divergence of Power Series

- As with any series, a power series may converge or diverge. However, its convergence/divergence depends on the value of x.

- If a power series of the form $\sum_{k=0}^{\infty} a_k(x-a)^k$ converges for all points in some interval $(-R+a,\ R+a)$, then R is called the **radius of convergence** and this interval is called the **interval of convergence**.

Power Series III

Radius of Convergence

- Suppose that for the sequence of coefficients $\{a_k\}_{k=0}^{\infty}$,

$$\lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right| = R$$

  for some real number R. Then R is the radius of convergence and the respective power series will:

  i) converge absolutely whenever $|x-a|<R$;
  ii) diverge whenever $|x-a|>R$.

- If the limit does not exist, the radius of convergence can still exist.
- To test at the endpoints, substitute the appropriate values for x and determine convergence/divergence using the previous methods for series.

Taylor Series Example VIII

MATH1251 (S2, 2018) Q4 i)

Find the interval of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{(x-3)^n}{3^n+1}$$

Make sure that you consider the behaviour at the end-points of your interval and provide reasons for your answers.

Taylor Series Example VIII

MATH1251 (S2, 2018) Q4 i)

Find the interval of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{(x-3)^n}{3^n+1}$$

Make sure that you consider the behaviour at the end-points of your interval and provide reasons for your answers.

First, we find R:

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{1}{3^n+1} \Big/ \frac{1}{3^{n+1}+1} \right| &= \lim_{n\to\infty} \frac{3^{n+1}+1}{3^n+1} \\ &= 3 \end{aligned}$$

Taylor Series Example VIII

So we know our interval of convergence is (0, 6).
At the endpoints, both series diverge

$$\sum_{n=0}^{\infty} \frac{(-3)^n}{3^n + 1} \quad \text{or} \quad \sum_{n=0}^{\infty} \frac{3^n}{3^n + 1}$$

since the sequences approach 1 rather than 0.
Therefore the series does not converge at either endpoint and so the interval of convergence is (0, 6).

Power Series IV

Manipulation of Power Series

- Within their respective intervals of convergence, power series can be added or multiplied together, differentiated or integrated. For example:

$$f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$$

$$f'(x) = \sum_{k=1}^{\infty} k a_k (x-a)^{k-1}$$

$$F(x) = \sum_{k=0}^{\infty} \frac{a_k}{k+1} (x-a)^{k+1} + C$$

- Any function that can be expressed as a power series is continuous and differentiable (for all orders) within its radius of convergence.

Taylor Series Example IX

MATH1251 (S2, 2018) Q4 ii)

a) Write down the Taylor Series for $f(x) = \sin(x^2)$ about $x = 0$ and state its radius of convergence.

Taylor Series Example IX

MATH1251 (S2, 2018) Q4 ii)

a) Write down the Taylor Series for $f(x) = \sin(x^2)$ about $x = 0$ and state its radius of convergence.

a) We can use the Taylor Series for $\sin(x)$ about $x = 0$:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{x^{2(2k+1)}}{(2k+1)!}$$

As the series will always converge, the radius of convergence is infinite.

Taylor Series Example IX

b) Use part (a) to determine an infinite series for the integral

$$I = \int_0^1 \sin(x^2)dx.$$

Taylor Series Example IX

b) Use part (a) to determine an infinite series for the integral

$$I = \int_0^1 \sin(x^2)dx.$$

b)

$$\begin{aligned} I &= \sum_{k=0}^{\infty} \int_0^1 \frac{x^{2(2k+1)}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} \left[\frac{x^{4k+3}}{(4k+3)(2k+1)!} \right]_0^1 \\ &= \sum_{k=0}^{\infty} \frac{1}{(4k+3)(2k+1)!} \end{aligned}$$

Applications of Integration

Average Value of a Function I

Average Value of a Function

- Suppose that f is integrable on a closed interval $[a, b]$. Then the **average value** of f in this interval is defined as:

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)dx$$

Average Value of a Function II

Mean Value Theorem for Integrals

- Suppose that f is continuous on $[a, b]$. Then, there exists a $c \in (a, b)$ such that

$$\int_a^b f(t)dt = f(c)(b-a).$$

- This can be rewritten to resemble the typical mean value theorem in the following way:

$$\frac{F(b)-F(a)}{b-a} = F'(c).$$

Arc Length of a Curve I

Arc Length of a Parametrised Curve

- Curves are typically expressed in the following parametric form:

$$\mathcal{C} = \{(x(t), y(t)) \in \mathbb{R}^2 : a \leq t \leq b\}.$$

- The length of of the curve can calculated by the formula:

$$\ell = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}dt.$$

- It is important that the path does not retrace its steps.

Arc Length of a Curve II

Arc Length of a Function

- Where the curve is expressed as a function of x, the arc length on the interval $[a, b]$ is given by:

$$\ell = \int_a^b \sqrt{1 + f'(x)^2} dx.$$

Applications of Integration Example I

MATH1131 (S2, 2015) Q4 vi)

Let $h(x) = \cosh(x)$ where $a \leq x \leq b$. Define L to be the arc length of the graph of h between $x = a$ and $x = b$ and define A to be the area bounded by the graph of h and the x-axis between $x = a$ and $x = b$. Prove that $L = A$ for all $a, b \in \mathbb{R}$.

Applications of Integration Example I

MATH1131 (S2, 2015) Q4 vi)

Let $h(x) = \cosh(x)$ where $a \leq x \leq b$. Define L to be the arc length of the graph of h between $x = a$ and $x = b$ and define A to be the area bounded by the graph of h and the x-axis between $x = a$ and $x = b$. Prove that $L = A$ for all $a, b \in \mathbb{R}$.

Using the properties $\frac{d}{dx}\cosh x = \sinh x$ and $\cosh^2 x - \sinh^2 x = 1$, we know that

$$\begin{aligned} L &= \int_a^b \sqrt{1+\sinh^2(x)}dx \\ &= \int_a^b \sqrt{\cosh^2(x)}dx \\ &= \int_a^b \cosh(x)dx \end{aligned}$$

Applications of Integration Example I

Additionally,

$$A = \int_a^b \cosh(x) dx$$

Therefore, $L = A$.

Arc Length of a Curve III

Arc Length of a Polar Curve

- Where the curve is expressed using polar coordinates in the form

$$r = f(\theta),$$

the length of the arc is then given by:

$$\ell = \int_{\theta_0}^{\theta_1} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Surface Areas

Surface Area Formulae

- Suppose we have have a curve $\mathcal{C}$ that is **simple** and lies above or on the x-axis. When rotated around the x-axis, the surface area can be found using one of the following:

$$A = \int_a^b 2\pi y(t)\sqrt{x'(t)^2 + y'(t)^2}dt, \tag{1}$$

$$A = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}dx, \tag{2}$$

$$A = \int_{\theta_0}^{\theta_1} 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta. \tag{3}$$

Applications of Integration Example II

MATH1131 (S2, 2018) Q4 v)

A surface is formed by rotating the curve $y = \frac{1}{4}x^2 - 1$ for $2 \leq x \leq 3$ around the y-axis. What is its surface area?

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Notice that the curve is rotated around the y-axis. Then the equation becomes $x = \sqrt{4(y+1)}$, with the bounds as $0 \leq y \leq \frac{5}{4}$.

$$
\begin{aligned}
f(y) &= 2\sqrt{y+1} \\
f'(y) &= \frac{2}{2\sqrt{y+1}} \\
&= \frac{1}{\sqrt{y+1}} \\
A &= \int_0^{\frac{5}{4}} 2\pi \left(2\sqrt{y+1}\right) \sqrt{1 + \left(\frac{1}{\sqrt{y+1}}\right)^2} dy
\end{aligned}
$$

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$$
\begin{aligned}
A &= 4\pi \int_0^{\frac{5}{4}} \sqrt{y+1}\sqrt{1+\frac{1}{y+1}}dy \\
&= 4\pi \int_0^{\frac{5}{4}} \sqrt{y+1}\sqrt{\frac{y+2}{y+1}}dy \\
&= 4\pi \int_0^{\frac{5}{4}} \sqrt{y+2}dy \\
&= 4\pi \left[\frac{2}{3}(y+2)^{\frac{3}{2}}\right]_0^{\frac{5}{4}} \\
&= \frac{8\pi}{3}\left[\left(\frac{13}{4}\right)^{\frac{3}{2}} - 2^{\frac{3}{2}}\right] \\
&= \frac{(13\sqrt{13}-16\sqrt{2})\pi}{3}
\end{aligned}
$$