MATH1081 – Discrete Mathematics
Revision Handout

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UNSW Mathematics Society, Term 3

These solutions were written by the 2020 UNSW Mathematics Society Education team as preparation for the final exam workshop. Please use these solutions ethically and if you spot any mistakes in the solutions, please mail us here!

Happy studying, and best of luck with your endeavours with MATH1081.

Contents

1 Set theory

Term 1, 2019 – Q1(i)

Let $A = \{2, 0, 1, 9\}$ and $B = \{1, 0, 8, 1\}$. State

(a) $P(A - B)$;

(b) $|A \times B|$;

(c) $|P(A \cup B)|$.

(a) **Answer**: $P(A - B) = \{\varnothing, \{2\}, \{9\}, \{2, 9\}\}$.

$A - B$ defines the set difference of A and B. In other words, the set $A - B$ is the set of elements in A but not in B. So

$$A - B = \{2, 9\}.$$

Finally, the power set simply becomes the set

$$P(A - B) = \{\varnothing, \{2\}, \{9\}, \{2, 9\}\}.$$

(b) **Answer**: $|A \times B| = 12$.

Observe that B contains a duplicated element. So, simplifying B gives us

$$B = \{1, 0, 8\}$$

since sets are equivalent under duplicate elements.

Now, the *Cartesian product* is a set of ordered pairs where the first element comes from A and the second element comes from B. So, for every element in A, it must be paired with every element in B. And so, we have $|A \times B| = |A| \times |B|$ number of elements in the Cartesian product. So

$$|A \times B| = 4 \times 3 = 12.$$

(c) **Answer**: $|P(A \cup B)| = 32$.

Using the B from the previous part, the union of A and B is the set of elements that are in either A or B (or both). Hence,

$$A \cup B = \{2, 0, 1, 9, 8\}.$$

Now, the cardinality of a power set C is simply

$$|P(C)| = 2^{|C|},$$

where $|C|$ is the number of elements in C. Here, our set C is the union of A and B. So we have

$$|P(A \cup B)| = 2^{|A \cup B|} = 2^5 = 32.$$

Semester 1, 2017 – Q1 (ii)

Use the laws of set algebra to simplify

$$A^c - (A \cap B^c).$$

Show your working and give reason for each step.

$$
\begin{aligned}
A^c - (A \cap B^c) &= A^c \cap \left(A \cap B^c\right)^c && \text{(defn of set difference)} \\
&= A^c \cap \left(A^c \cup (B^c)^c\right) && \text{(de Morgan's law)} \\
&= A^c \cap \left(A^c \cup B\right) && \text{(double negation law)} \\
&= A^c && \text{(absorption law)}
\end{aligned}
$$

Semester 2, 2016 – Q1 (iii)

For each $i = 1, 2, 3, \dots$, let $A_i = [1/i, 1 + 1/i]$, the closed interval on the real line from $1/i$ to $1 + 1/i$. Determine $\bigcap_{i=1}^{\infty} A_i$, giving reasons for your answer.

Writing out the first few intersection gives us

$$\bigcap_{i=1}^{\infty} A_i = [1, 2] \cap [1/2, 3/2] \cap [1/3, 4/3] \cap \dots$$

As $i \longrightarrow \infty$, we see that the interval partitioning converges to $[0, 1]$. Taking the strictest interval that is common to all intervals gives us the interval $[1, 1]$. And so we have

$$\bigcap_{i=1}^{\infty} A_i = [1, 1].$$

2 Modular arithmetic, integers and functions

Semester 2, 2016 – Q1 (vii)

Suppose $\gcd(a, m) = d$. Show that if x is a solution of $ax \equiv b \pmod{m}$, then so is $x + m/d$.

Suppose that x is a solution to the congruence $ax \equiv b \pmod{m}$. Then there exists an integer n such that

$$ax - b = mn.$$

Since $\gcd(a, m) = d$, then there exist an integer k such that $a = kd$. In writing a this way, we see that

$$\begin{aligned} a(x + m/d) - b &= ax + am/d - b \\ &= (ax - b) + m\,(kd/d) \\ &= mn + km \\ &= m(n + k). \end{aligned}$$

So $a(x + m/d) - b$ is divisible by m and hence,

$$a(x + m/d) \equiv b \pmod{m}.$$

Semester 2, 2016 – Q1 (vi)

Let m be the product of all primes between 10 and 20. Let n be the product of all numbers between 20 and 30 (inclusive).

(a) What is $\gcd(m, n)$?

(b) What is the highest power of 10 that divides n?

(a) We observe that the prime factorisation of m is

$$m = 11 \times 13 \times 17 \times 19,$$

while the prime factorisation of n is

$$n = \left(2^2 \times 5\right) \times (3 \times 7) \times (2 \times 11) \times (1 \times 23) \times \left(2^3 \times 3\right) \times 5^2 \times (2 \times 13) \times 3^3 \times \left(2^2 \times 7\right) \times (1 \times 29) \times (2 \times 3 \times 5).$$

Observe that this number (whatever it turns out to be) has a factor of 11×13, so the greatest common divisor between m and n is simply 11×13.

(b) We need to find the highest power of 10 in this prime factorisation. Notice that 10 can be factored into 2×5. So grouping up 2's with 5's, we have

$$n = (2 \times 5)^4 \left(2^6 \times 3^6 \times 7^2 \times 11 \times 13 \times 23 \times 29\right).$$

So, the highest power of 10 is 4.

Semester 2, 2017 – Q2 (i)

Solve the following congruences, or explain why they have no solution:

(a) $15x \equiv 7 \pmod{21}$

(b) $15x \equiv 9 \pmod{21}$

Suppose that we have a general congruence equation

$$ax \equiv b \pmod{m}.$$

Recall that such a solution exists if and only if

$$\gcd(a, m) \mid b.$$

With this result in mind, the congruence equation in part (a) clearly doesn't contain any solutions. But there will be solutions to part (b). In fact, there are 3 unique solutions in $\mod 21$.

Recall that

$$15x \equiv 9 \pmod{21} \iff 15x - 21y = 9.$$

To find the solutions, we are going to reduce every term to lowest terms. Since 15, 9 and 21 have a factor of 3, the entire congruence reduces down to

$$5x \equiv 3 \pmod{7} \iff 5x - 7y = 3,$$

with 3 unique solutions to the original congruence expression.

By using the Euclidean algorithm, we can find $\gcd(5, 7)$ since

$$\begin{aligned} 7 &= 5 \times 1 + 2 \\ 5 &= 2 \times 2 + \boxed{1} \\ 2 &= 1 \times 2 + 0. \end{aligned}$$

By the reverse Euclidean Algorithm, we have

$$\begin{aligned} 1 &= 5 - 2 \times 2 \\ &= 5 - 2\,(7 - 5 \times 1) \\ &= 5 \times 3 - 2 \times 7. \end{aligned}$$

So

$$1 = 5 \times 3 - 2 \times 7 \iff 3 = 5 \times 9 - 7 \times 6.$$

By comparing the expression to the modulo expression written earlier, we obtain

$$x = 9 + 7k,$$

for integer k. In other words, $x = 2, 9, 16$.

Term 1, 2019 – Q1 (iii)

(a) Find the smallest positive integer n such that $7^n \equiv 1 \pmod{57}$.

(b) Hence or otherwise, calculate

$$\left(\sum_{n=0}^{1000} 7^n\right) \text{ mod } 57.$$

(a) To do these problems, calculate $7^n \pmod{57}$ by multiplying the result from $7^{n-1} \pmod{57}$ by 7 and reducing in modulo 57. In doing so, we have

$$\begin{aligned} 7^1 &\pmod{57} = 7, \\ 7^2 &\pmod{57} = 49, \\ 7^3 &\pmod{57} = 1. \end{aligned}$$

(b) In doing part (a), we observe that there is a loop every fourth power. That is, $7^4 \equiv 7^1 \mod 57$, $7^5 \equiv 7^2 \mod 57$, and so on. We can use this fact to then simplify the expression for any power beyond 3.

Now, the result we want to simplify is the sum reduced in mod 57. This reduces down to finding each element reduced in modulo 57 and then reducing the entire sum in modulo 57. With the previous results in mind, the sum simply becomes

$$\sum_{n=0}^{1000} 7^n = 7^0 + \left(7^1 + 7^2 + 7^3\right) + \left(7^4 + 7^5 + 7^6\right) + \cdots + 7^{998} + 7^{999} + 7^{1000}.$$

Since $7^4 \equiv 7^1, 7^5 \equiv 7^2, 7^6 \equiv 7^3$, then we have a specific product of $(7^1 + 7^2 + 7^3)$ with some powers left over. To figure out the leftover powers, we observe that

$$1000 = 3 \times 333 + 1.$$

So there must be 1 leftover power from 1 to 1000. So 7^{1000} must correspond to 7^1. Thus, we have

$$\sum_{n=0}^{1000} 7^n \equiv 7^0 + 333\,(7 + 49 + 1) + 7^1 \equiv 8 + 333 \times 57.$$

So, when reduced in mod 57, the expression becomes

$$\left(\sum_{n=0}^{1000} 7^n\right) \mod 57 = 8.$$

Term 2, 2019 – Q2 (ii)

Let $S = \mathbb{R}^2$, the set of all ordered pairs of real numbers. Define a relation $\sim$ on S as follows:

$(x_1, y_1) \sim (x_2, y_2)$ if and only if there exists a positive number λ such that $x_2 = \lambda x_1$ and $y_2 = \lambda y_1$.

(a) Prove that $\sim$ is an equivalence relation on S.

(b) Geometrically describe the equivalence classes of this relation.

(a) To show that a relation is an equivalence relation, we need to show the three properties.

Reflexivity. Let $\lambda = 1$. Then clearly, $(x_1, y_1) \sim (x_1, y_1)$ since $x_1 = x_1$ and $y_1 = y_1$.

Symmetric. Suppose that $(x_1, y_1) \sim (x_2, y_2)$. Then, for some positive number λ, define $\mu = \dfrac{1}{\lambda}$ such that $x_2 = \mu x_1$ and $y_2 = \mu y_1$ also imply that $x_1 = \lambda x_2$ and $y_1 = \lambda y_2$. So $(x_2, y_2) \sim (x_1, y_1)$.

Transitivity. Suppose that $(x_1, y_1) \sim (x_2, y_2)$ and $(x_2, y_2) \sim (x_3, y_3)$. Then there exist positive numbers λ and μ such that

$$\begin{aligned} x_2 &= \lambda x_1, & y_2 &= \lambda y_1 \\ x_3 &= \mu x_2, & y_3 &= \mu y_2. \end{aligned}$$

Then we have

$$x_3 = \mu(\lambda x_1) = (\lambda\mu)\, x_1, \qquad y_3 = \mu(\lambda y_1) = (\lambda\mu)\, y_1.$$

And so, we have the result

$$(x_1, y_1) \sim (x_3, y_3).$$

(a) If (x_1, y_1) and (x_2, y_2) share the same equivalence class, then they must be multiples of one another. Hence, we can describe the equivalence classes as vectors being multiples of each other.

Semester 2, 2017 – Q2 (ii) [modified]

Let $\{S, |\}$ be the set of all factors of 20 with the relation *divides*.

1. Prove that $\{S, |\}$ is a partially ordered set.
2. Draw a Hasse diagram of $\{S, |\}$.

1. To show that $\{S, |\}$ is a partially ordered set, we need to show that it is (a) *reflexive*, (b) *anti-symmetric*, and (c) *transitive*. Clearly $a \mid a$ since $a = 1 \times a$ so the relation is reflexive.

   Now, suppose that $a \mid b$ and $b \mid a$. By definition, there exist integers M, N such that

   $$a = M \cdot b, \quad b = N \cdot a.$$

   Substituting b into the expression for a gives us

   $$a = M(N \cdot a).$$

   So, $MN = 1$. However, since M and N are integers, it must be that $M = N = 1$ and thus, $a = b$ and so the relation is anti-symmetric.

   Finally, suppose that $a \mid b$ and $b \mid c$. Again, there exist integers M, N such that

   $$b = M \cdot a, \quad c = N \cdot b.$$

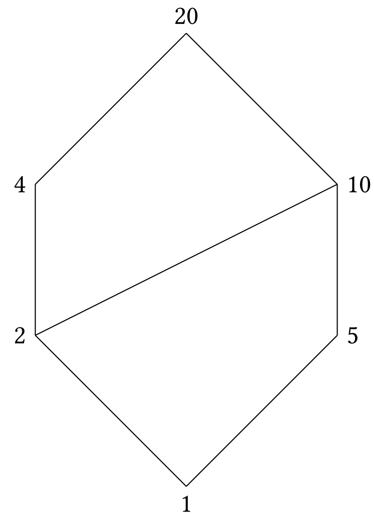
   By a similar method to the anti-symmetric, we have

   $$c = N(M \cdot a) = (NM) \cdot a.$$

   Since N and M are integers, its product is also an integer, and thus $a \mid c$. So the relation is transitive.

   Thus, the relation is a partially ordered set (notice that we don't require S).

2. Some things to note: we assume that the relations are reflexive and transitive so there's no reason to include them in our Hasse diagram. That is, we don't place arrows from x to itself. We don't place arrows from x to z if xRy and yRz since transitivity is assumed.



Term 2, 2019 – Q3 (iv)

> Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be functions, and consider the statement
> "if g is one-to-one (injective) and $g \circ f$ is onto (surjective), then f is onto (surjective)".
> Prove that this statement is true.

Let $x \in A$, $y \in B$ and $z \in C$. Firstly, clearly $g \circ f$ is a mapping from A to C. Since $g \circ f$ is onto, then for every element in C, there exists an element in A such that

$$g(f(x)) = z = g(y).$$

Now, since g is one-to-one, that means that $g(x_1) = g(x_2)$ implies that $x_1 = x_2$. By a similar nature, this means that $g(f(x)) = g(y)$ implies that $f(x) = y$.

Hence, for every $y \in B$, we can express it as a mapping f from A. Thus, f is onto (surjective).

Semester 2, 2016 – Q1 (v)

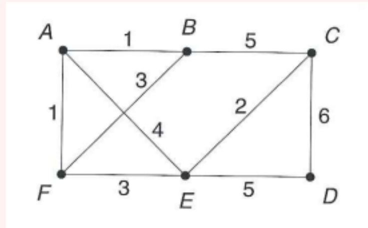
> If X and Y are sets with $|X| = 2$ and $|Y| = n$, how many injective functions are there from X to Y? Explain your answer.

Let $f : X \longrightarrow Y$ be an injective function. For every $x \in X$, there can only be one pairing in Y. Thus, if there are 2 elements in X, then $f(x_1)$ has n choices with $f(x_2)$ having $(n - 1)$ number of choices. So the total number of injective functions is simply $n(n - 1)$.

3 Graph theory

Term 2 2019 Q1(iii)

Find a minimal spanning tree in the following weighted graph. Show all your working out.

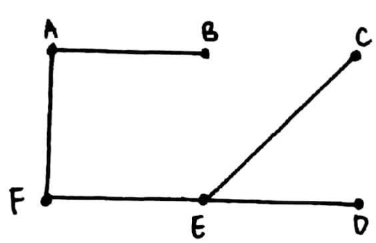


Kruskal's Algorithm is used to produce a minimal spanning tree for a given weighted graph G:

1. Start with a graph T with the same vertices as G but no edges.
2. Sort the edges into increasing order of weight.
3. Select the smallest weighted edge. Add this edge to T if it **doesn't create a circuit**.
4. Continue to the next smallest weighted edge and repeat step 3.
5. When all the vertices of T are connected, you should have a minimal spanning tree.

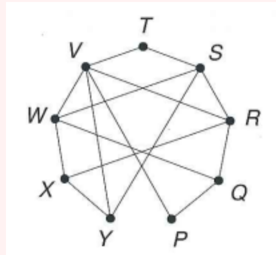
Step	Edge	Weight	Used?
1	AB	1	Yes
2	AF	1	Yes
3	EC	2	Yes
4	FB	3	No
5	FE	3	Yes
6	AE	4	No
7	BC	5	No
8	ED	5	Yes
9	CD	6	No

Total minimal weight = 1 + 1 + 2 + 3 + 5 = 12.



Term 2 2019 Q1(ii)

Consider the following graph G.



1. Does G contain an Euler path? Give reasons.
2. Show that G is bipartite.
3. Prove that G does not contain a Hamilton circuit.

1. *Euler Path*: a path containing **every edge** of G **exactly once**.

   Existence of a Euler Path: Let a and b be distinct vertices of G. A Euler path from a to b exists if and only if **a**, **b are of odd degree** and **every other vertex of G is of even degree**.

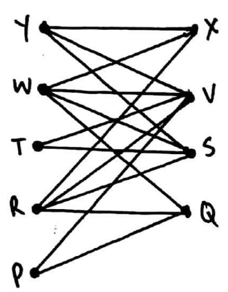
   Since

   $$deg(Q) = deg(Y) = deg(X) = 3$$
   $$deg(V) = 5,$$

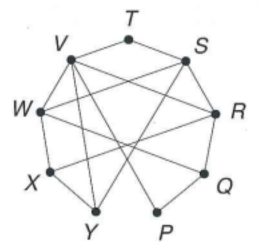
   G does not contain an Euler path.

2. By redrawing an isomorphism of G, we can show that G is bipartite.

   Bipartite Graph: A simple graph where the vertices can be partitioned into **two disjoint, non-empty sets** and no two vertices in the same set are adjacent.



3. *Hamilton Circuit*: A Hamilton circuit in G is a circuit containing **every vertex** of G **exactly once**.



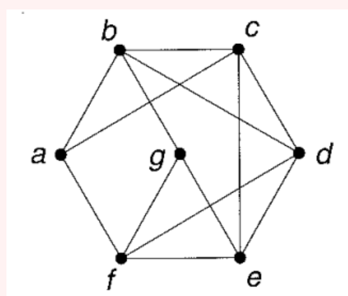
If we start at an arbitrary vertex Q, the circuit will be as follows: $Q \longrightarrow P \longrightarrow V \longrightarrow T \longrightarrow S \longrightarrow Y$(or R/W) $\longrightarrow X \longrightarrow W$(or Y/R). From here, it is impossible to complete the circuit without visiting any vertex again. Hence, G does not contain a Hamilton circuit.

Another method you could use is the *Sufficient Condition for a Hamilton circuit (Dirac's Theorem)*: G has a Hamilton circuit if $deg(v) \geq \frac{n}{2}$, where n is the number of vertices in G. So,

$$deg(v) \geq \frac{9}{2} = 4.5.$$

We can see that majority of the vertices in G are less than 5. Hence, G does not contain a Hamilton circuit.

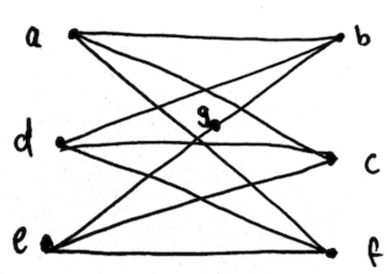
Term 2 2019 Q1(iv)



Show that the graph is NOT planar.

Kuratowski's Theorem: A graph is planar iff it has no subgraph K_5, $K_{3,3}$ or any graph homeomorphic to K_5 or $K_{3,3}$.

First, obtain a subgraph of the above graph by deleting edges bc, de and fg. Next, redrawing the subgraph gives a graph homeomorphic to $K_{3,3}$. Hence, by Kuratowki's Theorem, the graph is not planar.



Tip: notice that the given graph has 7 vertices, however vertex g looks like it was added onto edge be, hinting at a homeomorphic subgraph. The remaining 6 vertices is a hint that the subgraph could be $K_{3,3}$.

Term 1 2019 Q3(ii)

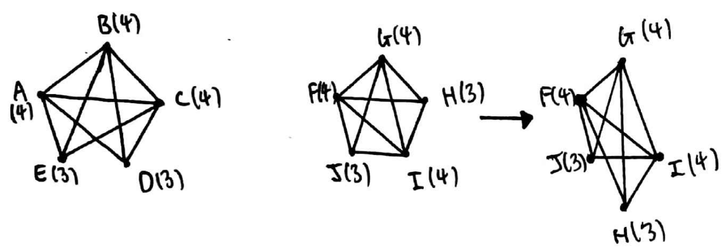
Graphs G_1 and G_2 are each simple and each have vertex degree sequence $4, 4, 4, 3, 3$. Explain why G_1 and G_2 are isomorphic.

Draw two graphs with vertex sequence $4, 4, 4, 3, 3$. By redrawing the graphs, we can prove that G_1 and G_2 are isomorphic.

Isomorphism: Let G_1 and G_2 be two graphs with vertex sets V_1 and V_2, and edge sets E_1 and E_2 respectively. G_1 and G_2 are isomorphic if there exist bijections

$$f : V_1 \longrightarrow V_2 \text{ and } g : E_1 \longrightarrow E_2$$

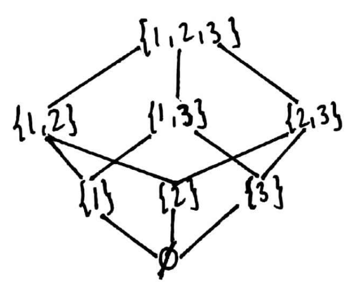
where $e \in E_1$ is incident on $v \in V_1$ iff $g(e)$ is incident on $f(v)$.



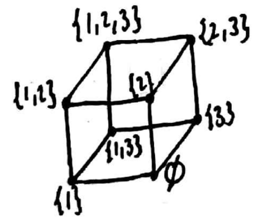
Semester 2 2018 Q2(ii)

1. Draw the Hasse diagram of the set of all the subsets of the set $S = \{1, 2, 3\}$ under the relation $\subseteq$.
2. Regarding the diagram in (a) as a graph (with the subsets of S as the vertices), is it isomorphic to the 3−cube Q_3? Prove your answer.
3. Is the graph in (a) planar? Prove your answer.

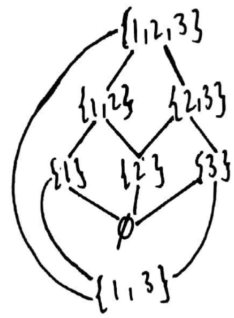
1. Hasse Diagram



2. Yes, it is isomorphic to 3−cube Q_3.



3. Yes, it is planar.

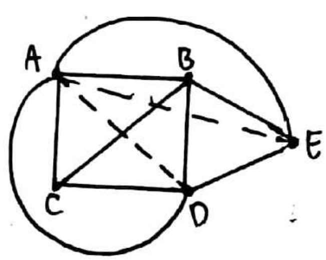


Semester 2 2017 Q2(iii)

Draw K_4, the complete graph on 4 vertices, then draw another vertex and connect it to 3 of the 4 vertices of K_4. Is the resulting graph planar? Prove your answer.

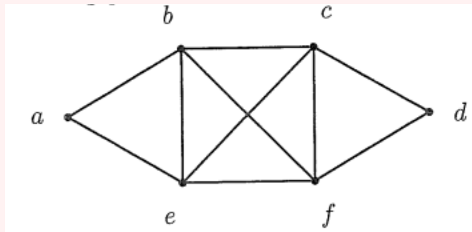
Yes, the resulting graph is planar.

Planar: A graph G is planar iff it can be drawn with no intersecting edges. This is called a **planar map/planar representation**.



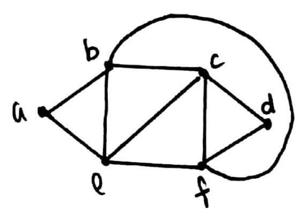
Semester 2 2017 Q2(iv)

Consider the following graph:



1. Is the graph bipartite?
2. Is the graph planar?
3. Does the graph contain an Euler circuit?
4. Does the graph contain a Hamilton circuit?

1. No, the graph is not bipartite since there are multiple odd cycles e.g. a, b, e.
2. Yes, the graph is planar.

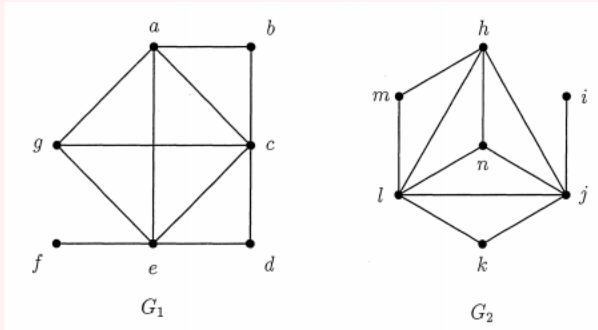


3. *Existence of an Euler circuit*: If every vertex in G has an **even degree**, then G has an Euler circuit.

   Since all vertices are of even degree, the graph contains an Euler circuit E.g. $e, a, b, c, d, f, b, e, c, f, e$.
4. Yes, a, b, c, d, f, e is an example of a Hamilton circuit. Note that Dirac's Theorem does not hold in this case (be careful when using this theorem).

Semester 2 2016 Q2(iv)

The graphs G_1 and G_2 are given in the diagram below.



1. Are these graphs G_1 and G_2 isomorphic? Prove your answer.
2. Does G_2 have a Hamilton path?
3. If the weights of all the edges in G_2 are 1, find the shortest paths from h to all the other vertices, using Dijkstra's algorithm.

1. Yes, we can map every vertex in G_1 to a distinct vertex in G_2, while preserving incidence.

$$\boxed{\begin{array}{c} a \longrightarrow h \\ b \longrightarrow m \\ c \longrightarrow l \\ d \longrightarrow k \\ e \longrightarrow j \\ f \longrightarrow i \\ g \longrightarrow n \end{array}}$$

2. Yes, i, j, k, l, n, h, m is an example of a Hamilton path.

3. *Djikstra's Algorithm* is used to produce the shortest v_0-path spanning tree for a given weighted graph G:

   (a) Start with a graph T with vertex v_0 only and no edges.

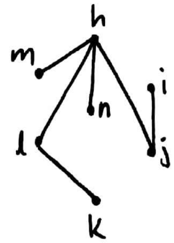
   (b) Consider all edges with one vertex in T and one vertex v NOT in T.

   (c) Choose the edge that gives a shortest path from v_0 to v.

   (d) Add this edge and v to T, provided it **doesn't create a circuit.**

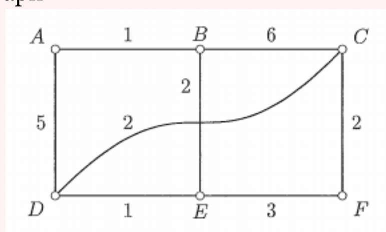
   (e) Repeat steps 2 – 4 until T contains all vertices of G.

Step	Available edges (total path weights)	Choice	Distance from h
1	$hm(1), hl(1), hn(1), hj(1)$	hm	$d(h, m) = 1$
2	$hl(1), hn(1), hj(1), ml(2)$	hl	$d(h, l) = 1$
3	$hn(1), hj(1), ln(2), lj(2), lk(2)$	hn	$d(h, n) = 1$
4	$hj(1), lj(2), lk(2), nj(2)$	hj	$d(h, j) = 1$
5	$lk(2), ji(2), jk(2)$	lk	$d(h, k) = 2$
6	$ji(2)$	ji	$d(h, i) = 2$



Semester 2 2016 Q2(v)

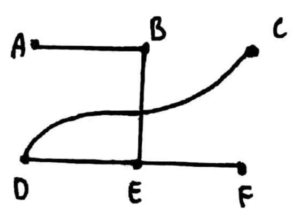
Consider the following weighted graph:



1. Use Dijkstra's algorithm to find a spanning tree that given the shortest paths from A to every other vertex of the graph. Make a table showing the details of at least the first **three** steps in your application of the algorithm.
2. Is the spanning tree found in part (a) a minimal spanning tree? Explain your answer.

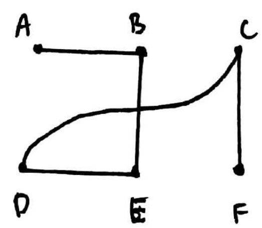
1. Djikstra's Algorithm

Step	Available edges (total path weights)	Choice	Distance from A
1	$AD(5), AB(1)$	AB	$d(A, B) = 1$
2	$AD(5), BE(3), BC(7)$	BE	$d(A, E) = 3$
3	$AD(5), BC(7), ED(4), EF(6)$	ED	$d(A, D) = 4$
4	$BC(7), EF(6), DC(6)$	EF	$d(A, F) = 6$
5	$BC(7), DC(6), FC(8)$	DC	$d(A, C) = 6$



2. Kruskal's Algorithm:

Step	Edge	Weight	Used?
1	AB	1	Yes
2	DE	1	Yes
3	BE	2	Yes
4	DC	2	Yes
5	CF	2	Yes
6	EF	3	No
7	AD	5	No
8	BC	6	No



Total minimal weight $= 1+1+2+2+2 = 8$, while the weight of the spanning tree in part (1) $= 1+1+2+2+3 = 9$. Hence, the spanning tree found in part (1) is NOT a minimal spanning tree.

Semester 1 2015 Q2(v)

Prove that the average vertex degree

$$\frac{1}{n}\sum_{v\in V(T)} d(v)$$

of a tree T on $|V(T)| = n$ vertices is strictly less than 2.

The Handshaking Lemma states that

$$\sum_{v\in V(T)} d(v) = 2e$$

where e is the number of edges in the given graph. For a tree T with n vertices, we know that there are $n-1$ edges. Substituting this into the Handshaking Lemma gives

$$\begin{aligned}\sum_{v\in V(T)} d(v) &= 2(n-1)\\ &= 2n-2.\end{aligned}$$

Dividing both sides by n where $n \geq 2$, we have

$$\frac{1}{n}\sum_{v\in V(T)} d(v) = 2 - \frac{2}{n} < 2.$$

Therefore, the average vertex degree of a tree T is strictly less than 2.

4 Proofs and Logic

Semester 1 2014 Q1(v)

(a) Prove that for all real $k > 1$,
$$\frac{1}{k^2} < \frac{1}{2}\left(\frac{1}{k-1} - \frac{1}{k+1}\right).$$

(b) Hence show that for all positive integers $n \geq 2$,
$$\sum_{k=2}^{n} \frac{1}{k^2} < \frac{3}{4}.$$

(a) We start with the right-hand side
$$\frac{1}{2}\left(\frac{1}{k-1} - \frac{1}{k+1}\right) = \frac{1}{k^2-1}.$$
Since $k > 1$, $k^2 - 1$ and k^2 are positive. Clearly, $k^2 > k^2 - 1$ and so $\dfrac{1}{k^2} < \dfrac{1}{k^2-1}$ and hence
$$\frac{1}{k^2} < \frac{1}{2}\left(\frac{1}{k-1} - \frac{1}{k+1}\right),$$
as required.

(b) From the result in (a), we know that
$$\begin{aligned}\sum_{k=2}^{n} \frac{1}{k^2} &< \sum_{k=2}^{n} \frac{1}{2}\left(\frac{1}{k-1} - \frac{1}{k+1}\right)\\ &= \frac{1}{2}\left(\sum_{k=2}^{n} \frac{1}{k-1} - \sum_{k=2}^{n} \frac{1}{k+1}\right). \qquad (1)\end{aligned}$$
By substituting $j + 1 = k - 1$ into the rightmost summation of (1), we obtain
$$\begin{aligned}\sum_{k=2}^{n} \frac{1}{k^2} &< \frac{1}{2}\left(\sum_{j=0}^{n-2} \frac{1}{j+1} - \sum_{k=2}^{n} \frac{1}{k+1}\right)\\ &= \frac{1}{2}\left(\frac{1}{0+1} + \frac{1}{1+1} + \sum_{j=2}^{n-2} \frac{1}{j+1} - \sum_{k=2}^{n-2} \frac{1}{k+1} - \frac{1}{n-1} - \frac{1}{n}\right)\\ &= \frac{1}{2}\left(\frac{3}{2} - \frac{1}{n-1} - \frac{1}{n}\right)\\ &= \frac{3}{4} - \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n-1}\right).\end{aligned}$$
We can thus conclude that
$$\sum_{k=2}^{n} \frac{1}{k^2} < \frac{3}{4}.$$

Semester 1 2014 Q3(iii) (Modified)

Prove that, if p is a prime, then $\sqrt{p}$ is irrational.

We first assume by way of contradiction that $\sqrt{p}$ is rational. We also note that $\sqrt{p}$ must be positive. Then, we can express $\sqrt{p} = \dfrac{r}{s}$ where r and s are positive integers, and $\gcd(r, s) = 1$.
Now, we can write

$$ps^2 = r^2.$$

Since s and r have no common factor other than 1, s^2 and r^2 must also have no common factors. Hence, p must divide r^2. Since p is prime, p must appear in the prime factorisation of r^2, and since r^2 and r have the same prime factors ignoring multiplicity, p must also divide r. Hence, we can write $r = pn$, where n is an integer. Substituting this into the previous equation, we get

$$ps^2 = p^2 n^2$$

and so

$$s^2 = pn^2.$$

For the same reasons as stated above, p must also divide s. However, this means that r and s have a common factor p which is greater than 1, which is a contradiction.
Hence, our assumption that $\sqrt{p}$ is rational must be incorrect. That is $\sqrt{p}$ is irrational.

Semester 1 2014 Q3(ii)

Prove by induction:
For all $n \geq 8$, the equation $3x + 5y = n$ has a solution x, y where x and y are non-negative integers.

Let $P(n)$ be the statement that $3x + 5y = n$ has a non-negative integer solution of x and y, for $n \geq 8$.
Base case(s). First, consider $P(8)$. The equation $3x + 5y = 8$ has the solution $x = 1$, $y = 1$. Hence, $P(8)$ is true.
Now, consider $P(9)$. The equation $3x + 5y = 9$ has the solution $x = 3$, $y = 0$. Hence, $P(9)$ is true.
Finally, consider $P(10)$. The equation $3x + 5y = 10$ has the solution $x = 0$, $y = 2$. Hence, $P(10)$ is true.
Inductive hypothesis. Now, we assume that the statement $P(n)$ is true for $n = 8, \dots, k$ for some $k \geq 10$ where k is an integer.
Inductive step. We will now use the inductive hypothesis to prove that $P(n)$ is true when $n = k + 1$. That is,

$$3x + 5y = k + 1 \tag{1}$$

has a solution x, y where x and y are non-negative integers.
We can rewrite (1) as

$$3x + 5y = (k - 2) + 3. \tag{2}$$

Using the assumption, (2) can be rewritten as

$$3x + 5y = 3x_1 + 5y_1 + 3 = 3(x_1 + 1) + 5y_1,$$

where x_1 and y_1 are non-negative integers. Hence, (1) has the solution $x = x_1 + 1$, $y = y_1$, where x and y are non-negative.
We can conclude by strong induction that the result is true for $n \geq 8$.

Semester 1 2014 Q3(i) (Modified)

For all integers x and y, prove that $5 | (x^2 + 2y^2)$ if and only if $5 | x$ and $5 | y$.

We consider the possible values of $(x^2 + 2y^2) \bmod 5$ indicated by the cells of the following table:

y \ x	0	1	2	3	4
0	0	1	4	4	1
1	2	3	1	1	3
2	4	4	2	2	4
3	4	4	2	2	4
4	2	3	1	1	3

First we prove that if $5|x^2 + 2y^2$, then $5|x$ and $5|y$.
In the table above, we see that the only case when $x^2 + 2y^2 \equiv 0 \bmod 5$ (that is, when $5|x^2 + 2y^2$) is when $x \equiv 0 \bmod 5$ and $y \equiv 0 \bmod 5$ (that is, when $5|x$ and $5|y$). Hence, we can conclude that if $5|x^2 + 2y^2$, then $5|x$ and $5|y$.
Now, we prove that if $5|x$ and $5|y$, then $5|x^2 + 2y^2$.
In the table above, we see that, in the case where $x \equiv 0 \bmod 5$ and $y \equiv 0 \bmod 5$ (that is, when $5|x$ and $5|x$), $x^2 + 2y^2 \equiv 0 \bmod 5$ (that is, $5|x^2 + 2y^2$), as required. Hence, $5|x^2 + 2y^2$ if and only if $5|x$ and $5|y$.

Semester 1 2014 Q3(iv)

Show that $2^n + 1$ is prime only if n is a power of 2.

(Though not stated in the question, I am assuming here that n is a positive integer and we are only considering non-negative integer powers of 2).

We are trying to prove the statement 'if $2^n + 1$ is prime, then n is a power of 2'. To do this, we will prove the contrapositive: if n is not a power of 2, then $2^n + 1$ is not prime.

Suppose that n is an integer that is not a power of two. Then, n must have a prime factor that is greater than 2 – an odd prime factor. We can express n as pm, where p is an odd prime and m is a positive integer.

We can thus write:

$$2^n + 1 = (2^m)^p + 1.$$

By factorising the sum of odd powers,

$$\begin{aligned} 2^n + 1 &= \left(2^m + 1\right)\left(1 - 2^m + 2^{2m} - \ldots + 2^{m(p-1)}\right). \\ &= xy \text{ where } x, y \in \mathbb{Z} \text{ such that } x = 2^m + 1 \text{ and } y = 1 - 2^m + 2^{2m} - \ldots + 2^{m(p-1)}. \end{aligned}$$

Note that $x = 2^m + 1 > 1$. Also, since $n = mp > m$ we know that $2^n + 1 > 2^m + 1$ and so $y > 1$. Hence, $2^n + 1$ is a composite number.

We can conclude that if n is not a power of 2, then $2^n + 1$ is not prime. That is, if $2^n + 1$ is prime, then n is a power of 2, as required.

Semester 1 2012 Q3(v) (Modified)

Let $\mathbb{R}$ be the set of real numbers, let a be an element of $\mathbb{R}$ and let S be a subset of $\mathbb{R}$. We say that a is a **limit point** of S if

$$\forall \epsilon > 0 \ \ \exists x \in S \ \ (x \neq a \wedge |x - a| < \epsilon).$$

Consider the set

$$S = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots \right\}.$$

(a) Prove that 0 is a limit point of S.

(b) Write in symbols the statement that a is **not** a limit point of S. Simplify your answer so that the symbol ~ is not used.

(c) Show that if y is an element of the set S, listed above, then y is not a limit point of S.

(a) We can represent set S as:

$$S = \left\{ \frac{1}{n} \ : \ n \in \mathbb{N}^+ \right\}.$$

Suppose $x \in S$, where $x = \frac{1}{n}$ for some $n \in \mathbb{N}^+$. Then,

$$|x - 0| = x = \frac{1}{n}.$$

If we select a positive natural n such that $n > \frac{1}{\epsilon}$ for some positive ϵ, then we can guarantee that $|x - 0| < \epsilon$.

We also note that since $\frac{1}{n} > 0$ for all positive n, and so any element $x \in S$ is non-zero.

Hence, for any positive ϵ, there exists an element $x \in S$ such that $x \neq 0$ and $|x - 0| < \epsilon$. Thus, 0 must be the limit point of S.

(b) We can express the statement that a is not a limit point of S as:

$$\sim (\forall \epsilon > 0 \ \ \exists x \in S \ \ (x \neq a \wedge |x - a| < \epsilon)).$$

Using logic laws,

$$\begin{aligned} \sim (\forall \epsilon > 0 \ \ \exists x \in S \ \ (x \neq a \wedge |x - a| < \epsilon)) &= \exists \epsilon > 0 \ \ \forall x \in S \ \ \sim (x \neq a \wedge |x - a| < \epsilon) \\ &= \exists \epsilon > 0 \ \ \forall x \in S \ \ (\sim (x \neq a) \vee \sim (|x - a| < \epsilon)) \\ &= \exists \epsilon > 0 \ \ \forall x \in S \ \ (x = a \vee |x - a| \geq \epsilon). \end{aligned}$$

(c) To prove that $y \in S$ is not a limit point of S, we must prove that there exists a positive ϵ for all $x \in S$ such that either $x = y$ or $|x - y| \geq \epsilon$.

We first consider the element y in S. Clearly, $y = y$, satisfying one of the conditions above.

Now, we consider some element $x \in S$ such that $x \neq y$. We can express y as $\frac{1}{n}$ and x as $\frac{1}{m+n}$, where n is a positive natural number and m is an integer where $m > -n$ and $m \neq 0$. Observing that

$$|x - y| = \left| \frac{1}{m+n} - \frac{1}{n} \right| = \left| \frac{-m}{n(m+n)} \right|,$$

we can choose $\epsilon = 0.1 \times \left|\frac{-m}{n(m+n)}\right|$, which is less than $|x - y|$ regardless of the value of m. That is, for every $x \in S$ except y, there exists a positive ϵ such that $\epsilon \leq |x - y|$.

Hence no $y \in S$ can be a limit point for S.

Semester 1 2015 Q3(i)

Construct truth tables for the propositional calculus formulae

$$(\sim p) \longrightarrow (q \longrightarrow (\sim r)) \text{ and } r \longrightarrow (p \wedge (\sim q)).$$

Does the first formula logically imply the second? Does the second logically imply the first? Give reasons for your answers.

We consider the following truth table, where A is the proposition $(\sim p) \longrightarrow (q \longrightarrow (\sim r))$ and B is the proposition $r \longrightarrow (p \wedge (\sim q))$.

p	q	r	$q \longrightarrow (\sim r)$	A	$p \wedge (\sim q)$	B	$A \longrightarrow B$	$B \longrightarrow A$
T	T	T	F	T	F	F	F	T
T	T	F	T	T	F	T	T	T
T	F	T	T	T	T	T	T	T
T	F	F	T	T	T	T	T	T
F	T	T	F	F	F	F	F	T
F	T	F	T	T	F	T	T	T
F	F	T	T	T	F	F	F	T
F	F	F	T	T	F	T	T	T

Since $A \longrightarrow B$ is not a tautology, A does not logically imply B.
Since $B \longrightarrow A$ is a tautology B, logically implies A.

Tips for filling the table:

1. For the $x \wedge y$ column, mark F in all the rows where x is F, then mark F in all the rows where y is F. Then, mark the rest T.
2. For the $x \vee y$ column, mark T in all the rows where x is T, then mark T in all the rows where y is T. Then, mark the rest F.
3. For the $x \longrightarrow y$ column, mark T in all the rows where x is F, then mark T in all the rows where y is T. Then, mark the rest F.

T1 2019 Q1(v)

Consider the following propositions:

(1) If the weather was nice and I was not tired then I went for a walk.

(2) If the weather was not nice and I had enough money then I went to the cinema.

(3) I did not go to the cinema.

(4) I did not go for a walk.

(5) I had enough money.

(a) Let

$$\begin{aligned} n &= \text{“The weather was nice"}, \\ t &= \text{“I was tired"}, \\ w &= \text{“I went for a walk"}, \\ c &= \text{“I went to the cinema"}, \\ m &= \text{“I had enough money"} \end{aligned}$$

Write propositions (1) – (5) in symbolic form using logical connectives.

(b) Use the rules of inference to deduce whether you were tired. Show your working.

(a) The propositions can be expressed symbolically as:

(1) $(n \wedge (\sim t)) \longrightarrow w$

(2) $((\sim n) \wedge m) \longrightarrow c$

(3) $\sim c$

(4) $\sim w$

(5) m

(b) Using logical equivalences

$$\begin{aligned}
&(6)\ \sim(\sim n \wedge m) && \text{(modus tollens (2), (3))} \\
&(7)\ (\sim\sim n) \vee (\sim m) && \text{(De Morgan's law (6))} \\
&(*\,8)\ n \vee (\sim m) && \text{(double negation law (7))} \\
&(9)\ n && \text{(using (5) and (8))} \\
&(10)\ \sim(n \wedge \sim t) && \text{(modus tollens (1), (4))} \\
&(11)\ (\sim n) \vee (\sim\sim t) && \text{(De Morgan's law (10))} \\
&(*\,12)\ (\sim n) \vee t && \text{(double negation law (11))} \\
&(13)\ t && \text{(using (9) and (12))}
\end{aligned}$$

Hence, I was tired. **Recall that* $((\sim x) \vee y) \iff (x \longrightarrow y)$, *so we could've written* (8) *and* (12) *in the* $(x \longrightarrow y)$ *form, then used modus ponens to deduce* (9) *and* (13) *respectively.*

5 Enumeration and recurrence relations

Semester 2, 2017 – Q4(iv)

Six people of different heights are in line to buy some tasty ice cream.

(a) Compute the number of ways the six people can be arranged so that **first three** are ordered according to height, tallest to shortest.

(b) Compute the number of ways the six people can be arranged so that **first four** are ordered according to height, tallest to shortest.

(c) Compute the number of ways the six people can be arranged so that **first three** are ordered according to height, tallest to shortest, and the **last three** are ordered according to height, tallest to shortest.

(d) Compute the number of ways the six people can be arranged so that **no three** consecutive people are arranged in order of height.

(a) Ways to choose the first three:

$$C(6,3).$$

Ways to arrange the remaining three:

$$3!.$$

Total ways:

$$C(6,3) \times 3! = \frac{6!}{3!}.$$

(b) Using a similar approach to (a), solution is

$$C(6,4) \times 2! = \frac{6!}{4!}.$$

(c) Ways to choose the first three:

$$C(6,3).$$

Once the first three are chosen, there is only one way to arrange the last three in order of height. Therefore the solution is just $C(6,3)$.

(d) To compute this, consider the complementary case- the number of ways the six people can be arranged such that there are three consecutive people arranged in order of height. This is requires the inclusion-exclusion principle as there is a lot of double counting.

Number of arrangements with three people in order (with double counting):

$$\frac{6!}{3!} \times 4.$$

Number of arrangements with four people in order:

$$\frac{6!}{4!} \times 3.$$

Number of arrangements with five people in order:

$$\frac{6!}{5!} \times 2.$$

Number of arrangements with six people in order:

$$1.$$

Number of ways with first three and last three in order (another case that is double counted):

$$C(6,3).$$

Total ways of arranging six people:

$$6!.$$

Solution:

$$6! - \left[\left(\frac{6!}{3!} \times 4\right) - \left(\frac{6!}{4!} \times 3\right) + \left(\frac{6!}{5!} \times 2\right) - 1 - C(6,3)\right]$$

Semester 2, 2016 – Q4(iii)

Consider a standard 52 card pack.

(a) How many seven-card hands contain exactly 3 hearts, 2 spades and 2 clubs?

(b) How many seven-card hands are there which contain either no hearts, no spades or no clubs?

(c) How many seven-card hands are there which contain at least one heart, at least one spade, and at least one club?

(a) Ways to choose 3 hearts:

$$C(13,3).$$

Ways to choose 2 spades (or 2 clubs):

$$C(13,2).$$

Total hands:

$$C(13,3) \times (C(13,2))^2 .$$

(b) The hand must satisfy one of three conditions: no hearts, no spades or no clubs.

Since there is overlap between the conditions (for example, there are hands with no hearts AND no spades), the inclusion-exclusion principle should be used.

Hands satisfying one condition (e.g. no hearts) with double counting:

$$C(39,7) \times 3.$$

Hands satisfying two conditions (e.g. no hearts and no spades) with double counting:

$$C(26,7) \times C(3,2) = C(26,7) \times 3.$$

Hands satisfying all three (i.e. only diamonds):

$$C(13,7).$$

Solution:

$$C(39,7) \times 3 - C(26,7) \times 3 + C(13,7).$$

(c) The complement would be seven-card hands which contain no hearts, no spades OR no clubs, which was counted in part (b).

The total number of seven-card hands with no restrictions is $C(52,7)$. Therefore, the solution is

$$C(52,7) - (C(39,7)\times 3 - C(26,7)\times 3 + C(13,7)).$$

Semester 1, 2016 – Q4(vi)

Consider the 10 letter word PARRAMATTA and all the words formed by rearranging its letters. How many of these words contain the subword MAP but do not contain the subword RAT?

Firstly, consider the words which contain the subword MAP. Using the grouping technique, there is 1 subword and 7 remaining letters with 2 R's, 3 A's and 2 T's. Therefore the number of such words is

$$\frac{8!}{2!\,3!\,2!}.$$

Secondly, consider the words containing both the subwords MAP and RAT. Now there are 2 subwords and 4 remaining letters with 2 A's. Therefore the number of such words is

$$\frac{6!}{2!}.$$

The solution is then

$$\frac{8!}{2!\,3!\,2!} - \frac{6!}{2!} = 1320.$$

Semester 2, 2018 – Q4(ii)

How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 83$$

if $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ are non-negative integers

(a) with no further restrictions?

(b) with x_k being an odd number for every k?

(a) Since $x_1, \dots, x_7$ are non-negative integers, consider the total 83 as 83×1's to distribute between 7 x_i's.

This then becomes a "stars and bars" with 6 bars and 83 stars to arrange. The number of solutions is

$$C(83+6,6) = C(89,6).$$

(b) To satisfy the condition without eliminating solutions, substitute $x_k = 2z_k + 1$, where z_k is a non-negative integer. The equation can then be simplified:

$$(2z_1+1)+(2z_2+1)+(2z_3+1)+(2z_4+1)+(2z_5+1)+(2z_6+1)+(2z_7+1) = 83$$

$$2z_1 + 2z_2 + 2z_3 + 2z_4 + 2z_5 + 2z_6 + 2z_7 = 76$$

$$z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + z_7 = 38$$

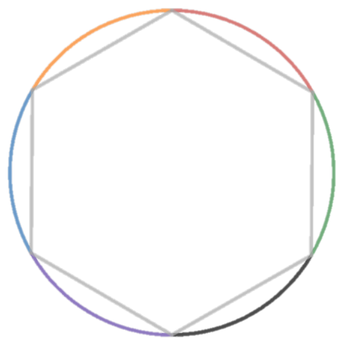
This is now back to a "stars and bars" problem with 6 bars and 38 stars. The number of solutions is

$$C(44, 6).$$

Semester 2, 2017 – Q4(iii)

Prove that given any 7 points on a circle of radius 1 there exist at least two that are less than 1 unit away from each other.

Observe that a regular hexagon inscribed inside a unit circle has side lengths of 1. Therefore if we partition the unit circle into six equal arcs (as shown below), any two points on the same arc will be less than 1 unit away from each other.



With seven points to place on six arcs, by the pigeonhole principle, there must be at least two points on the same arc and thus, there exists at least two points that are less than 1 unit away from each other.

Semester 1, 2016 – Q4(i)

When $(a + 2b + 3c)^{2016}$ is expanded and like terms collected, what is the coefficient of $a^{671}b^{672}c^{673}$?

Firstly, count the number of ways to choose 671 a's, 672b's and the 673 c's from

$$\underbrace{(a + 2b + 3c)(a + 2b + 3c)(a + 2b + 3c) \cdots (a + 2b + 3c)}_{\text{2016 terms}}.$$

The number of ways is

$$C(2016, 671) \times C(1345, 672) \times C(673, 673) = \frac{2016!}{671!\,672!\,673!}.$$

Then, after accounting for the "2" and "3" in $(a + 2b + 3c)$, the coefficient is

$$\frac{2016!}{671!\,672!\,673!} 2^{672} 3^{673}.$$

Semester 2, 2017 – Q4(ii)

(a) Compute the coefficient of the term x^{70} in the expansion of the polynomial $(1 + x + x^2 + x^3 + \cdots + x^{70})^6$.
(b) Compute the coefficient of the term x^{70} in the expansion of the polynomial $(1 + x + x^2 + x^3 + \ldots + x^{20})^6$.

(a) Another way of interpreting the problem is how many ways can 70 x's be distributed into 6 baskets. This becomes a "stars and bars" problem with 70 stars and 5 bars. The solution is

$$C(75, 5).$$

(b) This can be interpreted as how many ways there are to distribute 70 x's into 6 baskets such that the maximum in each basket is 20, making it a special case of (a). To do this, count the complementary case (ways where at least one basket has more than 20) by using the inclusion-exclusion principle.

To count the number of ways to distribute 70 x's into 6 baskets such that one basket has at least 21, first distribute 21 to one of the baskets. Then distribute the remaining 49 x's using the "stars and bars" method.

Ways with at least one basket > 21:

$$C(6, 1) \times C(54, 5).$$

Ways with at least two baskets > 21:

$$C(6, 2) \times C(33, 5).$$

Ways with at least three baskets > 21:

$$C(6, 3) \times C(12, 5).$$

Coefficient:

$$C(75, 5) - [C(6, 1) \times C(54, 5) - C(6, 2) \times C(33, 5) + C(6, 3) \times C(12, 5)] .$$

Semester 2, 2016 – Q4(i)

(a) Find the general solution to the recurrence relation

$$a_n - 4a_{n-1} + 4a_{n-2} = n.$$

(b) What *form* of solution would you try if the right-hand side were replaced by $5 \cdot 2^n$?

(a) First, solve the homogeneous problem. The characteristic equation is

$$\lambda^2 - 4\lambda + 4 = 0.$$

The general solution to the homogeneous problem is

$$h_n = A2^n + Bn2^n.$$

Next, look for the particular solution. Since the right-hand side is n, try $p_n = Cn + D$:

$$Cn + D - 4(C(n - 1) + D) + 4(C(n - 2) + D) = n$$

$$Cn + D - 4C = n$$

$$C = 1$$

$$D = 4$$

The full general solution is

$$a_n = A2^n + Bn2^n + n + 4$$

where A and B are constants.

(b) Since the 2^n and $n2^n$ are already in the solution to the homogeneous problem, the form of the solution to guess would be

$$En^2 2^n$$

where E is a constant.

Semester 2, 2013 – Q4(v)

When ascending a flight of stairs, an elf can take 1 stair in one stride or 3 stairs in one stride.
Let a_n be the number of different ways for the elf to ascend an n-stair staircase.

(a) Find a_1, a_2 and a_3.

(b) Obtain a recurrence relation for a_n. Give a brief reason. (You do NOT need to solve this recurrence relation.)

(c) Find the value of a_6.

(a) There is only one way for the elf to climb one stair so

$$a_1 = 1.$$

The only way for the elf to climb two stairs is to take them one stair at a time:

$$a_2 = 1.$$

For a 3-stair staircase, the elf can either climb it in 3 small strides or 1 long stride:

$$a_3 = 2.$$

(b) To reach the the top of an n-stair staircase, the elf must have either climbed $n-1$ stairs before taking the final stair or climbed $n-3$ stairs before taking the final three stairs in one stride. As there are a_{n-1} ways of climbing an $n-1$ stairs and a_{n-3} ways of climbing an $n-3$ stairs, the recurrence relation is

$$a_n = a_{n-1} + a_{n-3}.$$

(c) To find a_6, use the recurrence relation derived in part (b).

$$\begin{aligned} a_4 &= 2 + 1 = 3 \\ a_5 &= 3 + 1 = 4 \\ a_6 &= 4 + 2 = 6 \end{aligned}$$