MathSoc Second Year Linear Algebra Solutions to Part 1 (MATH2[56]01)

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Example 1.

Converting this to a matrix problem, we get:

$$\begin{pmatrix} 1 & 2 & \lambda \\ -1 & \lambda & -1 \\ \lambda & -4 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Constructing the augmented matrix, we get:

$$\left[\begin{array}{ccc|c} 1 & 2 & \lambda & 1 \\ -1 & \lambda & -1 & 0 \\ \lambda & -4 & \lambda & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & \lambda & 1 \\ 0 & \lambda+2 & \lambda-1 & 1 \\ 0 & 0 & -\lambda^2+3\lambda-2 & -\lambda+1 \end{array}\right]$$

Comparing the last row, we see that there is no solution when $\lambda = 2$, since the last row would follow the pattern $\left.\begin{array}{ccc|c} 0 & 0 & 0 & 1 \end{array}\right.$ which is clearly invalid. When $\lambda = 1$, the last row will be of the form $\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \end{array}\right)$, and therefore there will be an infinite number of solutions. When $\lambda \neq 2, -1$, there will be a unique solution.

Example 2.

For each of the following, we only need to check the determinant of the matrix:

(a) $\det(A) = 2 \cdot 4 - 1 \cdot 7 = 1$ hence it is invertible. To compute the inverse, we shall row reduce:

$$\left[\begin{array}{cc|cc} 2 & 7 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 4 & -7 \\ 0 & 1 & -1 & 4 \end{array}\right]$$

(b) $\det(B) = 4 \cdot 5 - 2 \cdot 10 = 0$ hence it is not invertible.

Example 3.

(a) This is clearly a vector space. Run through all the axioms. You will see that there is an easier way of going through this.

(b) This is NOT a vector space. It does not satisfy addition of vectors. Consider $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in S$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \in S$, then the sum is $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ which is not in S.

(c) This is also not a vector space. The set S is equivalent to $S = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = k\pi, k \in \mathbb{Z}\}$. Suppose that $(x, y, z) = (\pi, 0, 0)$. Taking $\frac{1}{2}(x, y, z) = (\frac{\pi}{2}, 0, 0)$, we see that this is not in S.

Example 4.

(a) This is not a vector space. Easy check is $(x, y, z) = (1, 1, 0)$. But $2(x, y, z) = (2, 2, 0)$ is not in the space because $2^2 \neq 2^3$.

(b) $(0, 0, 0)$ is not in the space.

(c) $t_1 = t_2 = 0 \implies \mathbf{x} = \mathbf{0}$ for fixed $\mathbf{u}_1, \mathbf{u}_2$, so S is non-empty. Of course, if we can write $\mathbf{x} = \lambda_1\mathbf{u}_1 + \lambda_2\mathbf{u}_2$, then we have $\mu\mathbf{x} = \mu\lambda_1\mathbf{u}_1 + \mu\lambda_2\mathbf{u}_2 = \theta_1\mathbf{u}_1 + \theta_2\mathbf{u}_2$ with $\theta_1, \theta_2 \in \mathbb{R}$. So a scalar multiple is in S. Testing addition, we can write $\mathbf{x} = t_1\mathbf{u}_1 + t_2\mathbf{u}_2$, and $\mathbf{y} = s_1\mathbf{u}_1 + s_2\mathbf{u}_2$. Then $\mathbf{x} + \mathbf{y} = (s_1 + t_1)\mathbf{u}_1 + (s_2 + t_2)\mathbf{u}_2$ and since $s_1 + t_1, s_2 + t_2 \in \mathbb{R}$, so the sum of the 2 vectors is also in the set. Therefore, S is a subspace of $\mathbb{R}^3$.

Example 5.

We shall solve for $\alpha_1(t^2 + t^3) + \alpha_2(1 + t + t^2 + t^3) + \alpha_3(1 + t^2 + 2t^3) = 0 + 0t + 0t^2 + 0t^3$.

Upon equating the coefficients, we obtain the following simultaneous equations:

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

From the 2nd row, we immediately see $\alpha_3 = 0$. Substituting into Row 1, we obtain $\alpha_2 = 0$ and so $\alpha_1 = 0$ from Row 3. Since this also satisfies Row 4, we have obtained our solution.

Example 6.

Consider any real symmetric matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$. We wish to be able to solve for $\alpha_1, \alpha_2, \alpha_3$ in the following equation:

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} + \alpha_3 \begin{pmatrix} 4 & -1 \\ -1 & 5 \end{pmatrix}$$

This yields the following system of equations:

$$a = \alpha_1 + 2\alpha_2 + 4\alpha_3$$

$$b = -2\alpha_1 + \alpha_2 - \alpha_3$$

$$c = \alpha_1 + 3\alpha_2 + 5\alpha_3$$

Constructing and solving the augmented matrix, we have:

$$\left(\begin{array}{rrr|c} 1 & 2 & 4 & a \\ -2 & 1 & -1 & b \\ 1 & 3 & 5 & c \end{array}\right) \sim \left(\begin{array}{ccr|r} 1 & 0 & \frac{6}{5} & \frac{a}{5} - \frac{2b}{5} \\ 0 & 1 & \frac{7}{5} & \frac{2a}{5} + \frac{b}{5} \\ 0 & 0 & \frac{-2}{5} & -\frac{7a}{5} - \frac{b}{5} + c \end{array}\right).$$

Clearly, there are no leading columns, so such a matrix always has a unique solution regardless of the values of a, b, c, so any symmetric matrix can be constructed from a linear combination of the matrices in $\mathcal{B}$. Also note when $a, b, c = 0$, the coefficients much each be 0, which can be shown through back substitution, so the only solution to a linear combinationi numerically equivalent to 0 is the trivial solution. This completes the proof.

Example 7.

(a) We shall solve the augmented matrix system:

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -3 & -5 \\ 4 & 3 & 6 & 8 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & -5 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 19 & 43 \end{array}\right)$$

This yields the solutions for the coefficients to be $-\frac{52}{19}, \frac{34}{19}, \frac{43}{19}$ in that order with respect to the basis. Hence the co-ordinate vector for $\mathbf{b}$ with respect to the ordered basis $\mathcal{B}$ is $(-\frac{52}{19}, \frac{34}{19}, \frac{43}{19})$.

(b) We employ a similar process to obtain:

$$\begin{array}{cccc|c} -1 & 1 & 0 & 1 & 8 \end{array}$$

Example 8.

From above, we just substitute $a = 4, b = -11, c = -7$ and this yields $\alpha_3 =$

Example 9.

(a) We first have $\dim(W) + \dim(X) = \dim(R^8) + \dim(W \cap X)$. Since we have $3+5 = 8+\dim(W\cap X)$, it follows that $\dim(W\cap X) = 0$. The only 0 dimensional subspace is $\{\mathbf{0}\}$, hence $W \cap X = \{\mathbf{0}\}$.

(b) Similarly, we obtain $\dim(W) + \dim(X) = \dim(W + X) + \dim(W \cap X) \implies \dim(W \cap X) = 2$. Hence $W \cap X$ must have a basis of 2 elements since it's dimension is 2. Since a basis is the smallest linearly independent set that spans $W \cap X$, $W \cap X$ can contain more elements, with it's dimension still being 2. Since a basis is also automatically linearly independent, $W \cap X$ must contain at least 2 elements that are linearly independent.

(c) No such example can exist. $W \cap X$ can contain other elements in the set, but it's dimension must be 2.

Example 10.

(a) We must solve the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & -3 & 3 & b_1 \\ 2 & -5 & 4 & b_2 \\ 2 & -9 & 12 & b_3 \end{array}\right) \sim \left(\begin{array}{ccc|r} 1 & 0 & -3 & -5b_1+3b_2 \\ 0 & 1 & -2 & -2b_1+b_2 \\ 0 & 0 & 0 & -8b_1+3b_2+b_3 \end{array}\right)$$

Hence, the condition on b_1, b_2, b_3 is $-8b_1 + 3b_2 + b_3 = 0$. To compute the basis of the image, we have any vector (b_1, b_2, b_3) such that $b_3 = 8b_1 - 3b_2$, and so we can rewrite the vector $(b_1, b_2, b_3) = (b_1, b_2, 8b_1 - 3b_2) = (1, 0, 8)b_1 + (0, 1, -3)b_2$ for scalars b_1, b_2. Therefore, the basis for the image is $\{(1, 0, 8), (0, 1, -3)\}$.

(b) i. The kernel is given by all vectors (x_1, x_2, x_3) such that $A\mathbf{x} = \mathbf{0}$. So we have $2x_1 + 3x_2 + x_3 = 0$ and $x_1 - 2x_2 - 3x_3 = 0$. This gets $x_1 = 2x_2 + 3x_3 \implies 4x_2 + 6x_3 + 3x_2 + x_3 = 0 \implies x_2 = -x_3$. Therefore, we get $x_1 = -x_2 \implies x_2 = -x_1$. Hence, the kernel is spanned by the vector $(-1, 1, -1)$. The image is the set of all possible (b_1, b_2) for which there is an $\mathbf{x}$ such that $A\mathbf{x} = (b_1, b_2)$. Observe that the matrix $\begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix}$ is an invertible matrix, so the first 2 columns are linearly indepedent. This means that the image is $\mathbb{R}^2$ since any linear combination of the first 2 columns of A will get a vector in $\mathbb{R}^2$.

ii. The kernel is given by all vectors (x_1, x_2, x_3) such that $B\mathbf{x} = 0\mathbf{0}$. We have the following augmented matrix:

$$\left(\begin{array}{rrr|c} 1 & 3 & 1 & 0 \\ 2 & 5 & 0 & 0 \\ 1 & -2 & -9 & 0 \\ -1 & 1 & 7 & 0 \end{array}\right) \sim \left(\begin{array}{ccr|c} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Hence, we have the following: $x_1 = 5x_3, x_2 = -2x_3$. Therefore, the kernel is spanned by $(5, -2, 1)$.

The image is the set of all possible outputs. So we get the following augmented matrix and it's reduced form:

$$\left(\begin{array}{rrr|r} 1 & 3 & 1 & b_1 \\ 2 & 5 & 0 & b_2 \\ 1 & -2 & -9 & b_3 \\ -1 & 1 & 7 & b_4 \end{array}\right) \sim \left(\begin{array}{rrr|r} 1 & 0 & -5 & -5b_1+3b_2 \\ 0 & 1 & 2 & 2b_1-b_2 \\ 0 & 0 & 0 & 9b_1-5b_2+b_3 \\ 0 & 0 & 0 & -7b_1+4b_2 \end{array}\right)$$

From the last 2 rows, we deduce that $x_3 = -9x_1 + 5x_2, x_4 = 7x_1 - 4x_2$. Therefore, the image is spanned by the 2 vectors $\{(1,0,-9,7),(0,1,5,-4)\}$.

Example 11.

Constructing a commutative diagram for this, we have a map $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ with respect to the standard basis as $\begin{pmatrix} 4 & 9 \\ 1 & 1 \end{pmatrix}$. We require to find the matrix from $\mathbb{R}^2 \mapsto \mathbb{R}^2$ with respect to the basis $\mathcal{B}$. This would imply that the new matrix is $P = B^{-1}AB = \begin{pmatrix} 1 & -3 \\ -1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 4 & 9 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ -1 & 2 \end{pmatrix}$.

Example 12.

(a) All we must show is that any vector $V \cap W = \{\mathbf{0}\}$. That is, suppose we have $\mathbf{v} \in V \cap W$, then $\mathbf{v} = 0$. This is equivalent to solving a 3×3 augmented matrix system:

$$\left(\begin{array}{rrr|r} 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right) \implies \alpha_1 = \alpha_2 = \alpha_3 = 0$$

(b) Upon substitution, we find that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are all eigenvectors of Q and are hence T-invariant.

(c) Q is a map from $\mathbb{R}^3$ to $\mathbb{R}^3$ with respect to the standard basis. To compute with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, we rewrite the matrix $P = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix}$, so that the new linear map T with respect to the basis is $P^{-1}QP$.

Example 13.

(a) We see that $T(1,0)=(3,4)$ and $T(0,1)=(4,9)$ which means that $T(x_1,x_2)=\begin{pmatrix}3 & 4\\ 4 & 9\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$.

(b)

Example 14.

(a) Let $B=\begin{pmatrix}1 & 1\\ 5 & 6\end{pmatrix}$. Then the matrix representation will be $B^{-1}AB$.

(b) Same as before in terms of the formula.

(c) Same as before in terms of the formula.

Example 15.

Here, consider the matrices $A=(a_{ij})$ with $i=1,2,...,p$, $j=1,2,...,q$ and $B=(b_{kl})$ with $k=1,2,...,q$ and $l=1,2,...,p$. Then upon expanding AB and BA, we have:

$$\text{trace}(AB)=\sum_{n=1}^{n=p}\sum_{m=1}^{m=q}a_{nm}b_{mn}$$

Doing a similar thing for BA, we obtain exactly the same formula and hence the 2 are equal. Note that this can be for any 2 matrices of compatible sizes, so letting B be an invertible matrix, we obtain $\text{trace}(B^{-1}AB)=\text{trace}(A)$.

Example 16.

(a) To compute the dual basis of the vector space span $\left\{\begin{pmatrix}1\\ 4\end{pmatrix},\begin{pmatrix}1\\ 3\end{pmatrix}\right\}$, we solve the following system of equations:

$$w_1\begin{pmatrix}1\\ 4\end{pmatrix}=\alpha_1+4\alpha_2=1$$

$$w_1\begin{pmatrix}1\\ 3\end{pmatrix}=\alpha_1+3\alpha_2=0$$

$$w_2\begin{pmatrix}1\\ 4\end{pmatrix}=\beta_1+4\beta_2=0$$

$$w_2\begin{pmatrix}1\\3\end{pmatrix} = \beta 1 + 3\beta_2 = 1$$

This yields $\alpha_2 = 1, \alpha_1 = -3$, and $\beta_2 = -1, \beta_1 = 4$. Hence, The required 2 functions are $w_1 = -3x_1 + x_2$, and $w_2 = 4x_1 - x_2$. Incidentally, writing such an expression in matrix form is the same as computing the inverse of the matrix describing the vector space as the product of the 2 matrix mappings must yield the identity (we wish to obtain a 1 for each functional to its corresponding vector).

(b) The matrix describing $\mathcal{P}_2$ is given by:

$$\begin{pmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}.$$

Therefore, the dual basis operation is given by:

$$\begin{pmatrix}1 & 1 & 0\\0 & 1 & -1\\0 & 0 & 1\end{pmatrix}.$$

Example 17.

By definition, we require the set of vectors such that $\mathbf{x}\cdot(1,-1,2,0) = 0, \mathbf{x}\cdot(-2,1,0,1) = 0$. Suppose that $\mathbf{x} = (x_1, x_2, x_3, x_4)$ [*This creates a collection of scalars to help work out the basis for the orthogonal complement*]. Then we have the following:

$$x_1 - x_2 + 2x_3 = 0, -2x_1 + x_2 + x_4 = 0$$

$$\begin{pmatrix}1 & -1 & 2 & 0\\-2 & 1 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & -1 & 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since the 3rd and 4th columns are non-leading, we set them as free parameters $x_3 = s, x_4 = t$, therefore:

$$x_1 - x_2 = -2s, -x_2 = -4s - t \implies x_2 = 4s + t, x_1 = 2s + t$$

Hence: $\mathbf{x} = \begin{pmatrix} 2s+t \\ 4s+t \\ s \\ t \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} 2 \\ 4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

Example 18.

In order to find the projection, we must project onto the span of the vector space given.

(a) $\mathbf{v} = \alpha_1 \mathbf{w}_1 + \alpha_2 \mathbf{w}_2 + \mathbf{w}$ where $\mathbf{w}$ is the vector orthogonal to the span of $\mathbf{w}_1, \mathbf{w}_2$. Taking the dot product with $\mathbf{w}_1$ and $\mathbf{w}_2$, we can solve for α_1, α_2. Thus, we obtain:

$$\langle \mathbf{v}, \mathbf{w}_1 \rangle = \alpha_1 \langle \mathbf{w}_1, \mathbf{w}_1 \rangle + \alpha_2 \langle \mathbf{w}_1, \mathbf{w}_2 \rangle$$

$$\langle \mathbf{v}, \mathbf{w}_2 \rangle = \alpha_1 \langle \mathbf{w}_1, \mathbf{w}_2 \rangle + \alpha_2 \langle \mathbf{w}_2, \mathbf{w}_2 \rangle$$

Therefore, we have the following equations:

$$3 = 6\alpha_1 + \alpha_2$$

$$-5 = \alpha_1 + 2\alpha_2$$

This yields $\alpha_2 = -3$, $\alpha_1 = 1$. Therefore, the projection of $\mathbf{v} = (6, 1, -5)$ onto W is given by $\mathbf{w}_1 - 3\mathbf{w}_2$

(b) Same idea as above:

$$9 = 3\alpha_1 - 2\alpha_2$$

$$11 = -2\alpha_1 + 7\alpha_2$$

This yields $\alpha_1 = 5, \alpha_2 = 3$. Therefore, the projection of $\mathbf{v}$ onto W determined by the 2 vectors is given by $5(1,1,0,-1)+3(-1,1,1,2)=(2,8,3,1)$.

Example 19.

Evaluating each pair of matrices with respect to the inner product, we have:

$$\langle M_1, M_2\rangle = \frac{1}{4}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix} = 0$$

Repeat for each pair with non-equal indices. For each of the same indices, you should obtain a result of 1.

Example 20.

Start with $\mathbf{w}_1 = \frac{(1,3)}{\sqrt{1[2+3^2}} = \frac{1}{\sqrt{10}}(1,3)$. Then by the Gram-Schmidt procedure, we projection the vector $(-1,2)$. Then we compute $\mathbf{v} - \text{proj}_{\mathbf{w}_1}\mathbf{v} = (-1,2) - \frac{\mathbf{v}\cdot\mathbf{w}_1}{\mathbf{w}_1\cdot\mathbf{w}_1} = (-1,2) - \frac{-5}{\sqrt{10}}\frac{1}{\sqrt{10}}(1,3) = (-1,2)+\frac{1}{2}(1,3) = (-\frac{1}{2},\frac{7}{2})$.

Example 21.

Start with the function $p(x)=1$. Obviously it has norm 1, so no need to divide. To project $q(x)=x$ onto $\{1\}$, we have: $q^*(x) = x - \frac{\int_0^1 1\cdot x dx}{1} = x - \frac{1}{2}$. We still need to normalise this polynomial, so upon computation, we obtain:

$$\int_0^1 (x-\frac{1}{2})^2 dx = \frac{1}{12}$$

Therefore, our second element in the orthonormal basis is $\sqrt{12}(x-\frac{1}{2})$.

Hence, the orthonormal set is $\{1, \frac{1}{2\sqrt{3}}(x-\frac{1}{2})\}$.

Example 22.

(a) $\mathbf{0}$ is clearly in $W^\perp$ since the inner product of $\langle \mathbf{0}, \mathbf{w}\rangle = 0$ for every $\mathbf{w}\in W$.

(b) Suppose we have $\mathbf{w}\in W\cap W^\perp$. Then $\langle \mathbf{w},\mathbf{w}\rangle = 0 \implies \mathbf{w}=\mathbf{0}$.

(c) Suppose we can write $\mathbf{v} = \alpha_1\mathbf{w}_1' + \alpha_2\mathbf{w}_2'$. Then we obtain $\alpha_1 = \frac{\langle \mathbf{v},\mathbf{w}_1'\rangle}{\langle \mathbf{w}_1',\mathbf{w}_1'\rangle}, \alpha_2 = \frac{\langle \mathbf{v},\mathbf{w}_2'\rangle}{\langle \mathbf{w}_2',\mathbf{w}_2'\rangle}$. Therefore, we can write $\mathbf{v} = \frac{\langle \mathbf{v},\mathbf{w}_1'}{\langle \mathbf{w}_1',\mathbf{w}_1'\rangle}\mathbf{w}_1' + \frac{\langle \mathbf{v},\mathbf{w}_2'}{\langle \mathbf{w}_2',\mathbf{w}_2'\rangle}\mathbf{w}_2' = \mathbf{w}_1 + \mathbf{w}_2$.

(d) We are required to prove that $W^{\perp\perp} = W$. That is, both are subsets of one another. Consider a $\mathbf{w}\in W^{\perp\perp}$. Then $\langle \mathbf{w},\mathbf{v}\rangle = 0$ for every $\mathbf{v}\in W^\perp$. But by definition, $\langle \mathbf{v}, \mathbf{w'} \rangle = 0$ for every $\mathbf{w'} \in W$. Therefore, $\mathbf{w} \in W$ and so $\mathbf{W}^{\perp\perp} \subseteq W$. Similarly, $W \subseteq W^{\perp\perp}$, and therefore the 2 subspaces must be equal.

(e) $\dim(W) + \dim(W^{\perp}) = \dim(V) + \dim(W \cap W^{\perp}) = \dim(V)$.

Example 23.

Consider the inner product $\langle T(v_1, v_2), q_0 + q_1 t \rangle$. This simplifies to being $\langle v_1 + v_2 + 2v_2 t, q_0 + q_1 t \rangle = (v_1 + 3v_2)(q_0 + q_1) + (v_1 + 5v_2)(q_0 + 2q_1)$. Now suppose that $T^*(q_0 + q_1(t)) = (w_1, w_2)$. Then we also obtain $\langle (v_1, v_2), (w_1, w_2) \rangle = v_1 w_1 + v_2 w_2$. Taking the above expression and equating coefficients of v_1, v_2, we obtain $2q_0 + 5q_1 = w_1$ for the v_1 component, and $8q_0 + 13q_1 = w_2$. Defining this as a linear map, we obtain $T^*(q_0 + q_1 t) = \begin{pmatrix} 2 & 5 \\ 8 & 13 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \end{pmatrix}$.

Example 24.

(a) Let $\mathbf{v}_1 = (5, 12)$ and $\mathbf{v}_2 = (-4, 6)$. By the Gram-Schmidt procedure, we have $\mathbf{q}_1 = \frac{1}{13}(5, 12)$, and $\mathbf{q}_2' = \mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{q}_1 \rangle \mathbf{q}_1$. Substituting the numbers in, $\mathbf{q}_2' = (-4, 6) - \frac{1}{13}(52)\mathbf{q}_1 = (-\frac{72}{13}, \frac{30}{13})$. Therefore $\mathbf{q}_2 = (-\frac{12}{13}, \frac{5}{13})$. This yields the required orthogonal matrix. For the right triangular matrix, we evaluate $\langle \mathbf{q}_1, \mathbf{v}_1 \rangle = 13, \langle \mathbf{q}_1, \mathbf{v}_2 \rangle = 4, \langle \mathbf{q}_2, \mathbf{v}_2 \rangle = 6$. This yields:

$$A = \begin{pmatrix} \frac{5}{13} & -\frac{12}{13} \\ \frac{12}{13} & \frac{5}{13} \end{pmatrix} \begin{pmatrix} 13 & 4 \\ 0 & 6 \end{pmatrix}$$

(b) Begin with the vector $\mathbf{v}_1 = (-2, 1, -2) \implies \mathbf{q}_1 = \frac{1}{3}(-2, 1, -2)$. Letting $\mathbf{v}_2 = (1, 4, -8)$, we find that the projection will be

$$\mathbf{q}_2' = \mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{q}_1 \rangle \mathbf{q}_1 = (1, 4, -8) - 2(-2, 1, -2) = (5, 2, -4)$$

. Normalising, we obtain $\mathbf{q}_2 = \frac{1}{3\sqrt{5}}(5, 2, -4)$. For the right triangular matrix, we obtain:

$$B = \begin{pmatrix} -\frac{2}{3} & \frac{\sqrt{5}}{3} \\ \frac{1}{3} & \frac{2}{3\sqrt{5}} \\ -\frac{2}{3} & -\frac{4}{3\sqrt{5}} \end{pmatrix} \begin{pmatrix} 3 & 6 \\ 0 & \sqrt{45} \end{pmatrix}$$

Example 25.

Multiplying by the adjoint, we obtain:

$$\begin{pmatrix}1 & -1 & 1\\ 1 & 1 & 2\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 1\\ 1 & 2\end{pmatrix}\mathbf{x} = \begin{pmatrix}1 & -1 & 1\\ 1 & 1 & 2\end{pmatrix}\begin{pmatrix}11\\ 3\\ 4\end{pmatrix}$$

This simplies to having normal equations:

$$\begin{pmatrix}3 & 2\\ 2 & 4\end{pmatrix}\mathbf{x} = \begin{pmatrix}12\\ 22\end{pmatrix} \implies \mathbf{x} = \frac{1}{8}\begin{pmatrix}4 & -2\\ -2 & 3\end{pmatrix}\begin{pmatrix}12 & 22\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\ \frac{15}{4}\end{pmatrix}$$

Therefore, the least squares solution to the model is $\mathbf{x} = (\frac{1}{2}, \frac{15}{4})$.

Example 26.

Upon substituting each of the points into the model, we obtain the following system of equations:

$$\begin{pmatrix}1 & -1 & 1\\ 1 & 0 & 0\\ 1 & 1 & 1\\ 1 & 2 & 4\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} = \begin{pmatrix}7\\ 4\\ -2\\ -6\end{pmatrix}$$

Upon multiplying by the adjoint of the matrix on the left hand side, we obtain:

$$\begin{pmatrix}1 & 2 & 6\\ 2 & 6 & 8\\ 6 & 8 & 18\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} = \begin{pmatrix}3\\ -21\\ -19\end{pmatrix}$$

Thus, upon solving the augmented matrix system, we obtain:

$$\left(\begin{array}{ccc|c}1 & 2 & 6 & 3\\ 2 & 6 & 8 & -21\\ 6 & 8 & 18 & -19\end{array}\right) \sim \left(\begin{array}{ccc|c}1 & 2 & 6 & 3\\ 0 & 1 & -2 & -\frac{27}{2}\\ 0 & 0 & 1 & \frac{7}{2}\end{array}\right)$$

This yields coefficients $a = -5, b = -\frac{13}{2}, c = \frac{7}{2}$ and so the quadratic of best fit is $y = -5 - \frac{13}{2}x + \frac{7}{2}x^2$.

Example 27.

We are only required to show that it is an angle bisector. By definition, we have $\langle \mathbf{w}, \mathbf{v}\rangle = ||\mathbf{w}||||\mathbf{v}|| \cos\theta_1$. Similarly, $\langle \mathbf{w}, \mathbf{u}\rangle = ||\mathbf{w}||||\mathbf{u}|| \cos\theta_2$. Therefore, for $\mathbf{w}$ to be an angle bisector, we must have $\theta_1 = \theta_2$. Therefore, we must show that $\langle \mathbf{w}, \mathbf{v} \rangle = \langle \mathbf{w}, \mathbf{u} \rangle$.

$$\begin{aligned}
\text{LHS} &= \langle \mathbf{w}, \mathbf{v} \rangle \\
&= \langle \frac{1}{||\mathbf{u}|| + ||\mathbf{v}||}(||\mathbf{u}||\mathbf{v} + ||\mathbf{v}||\mathbf{u}), \mathbf{v} \rangle \\
&= \frac{1}{||\mathbf{u}|| + ||\mathbf{v}||}(||\mathbf{u}||\langle \mathbf{v}, \mathbf{v} \rangle + ||\mathbf{v}||\langle \mathbf{u}, \mathbf{v} \rangle)
\end{aligned}$$

We obtain exactly the same result when computing $\langle \mathbf{w}, \mathbf{u} \rangle$, and since $\mathbf{w}$ is a linear combination of $\mathbf{v}, \mathbf{u}$, it is in the same plane and is therefore an angle bisector of $\mathbf{v}, \mathbf{u}$.

Example 28.

Note that in any matrix, the sum of the eigenvalues is the trace of the matrix, and the product of eigenvalues is the determinant.

(a) Let the eigenvalues be λ_1, λ_2. Then $\lambda_1 + \lambda_2 = 5, \lambda_1\lambda_2 = 6$. Therefore, $\lambda_1, \lambda_2 = 2, 3$ by inspection. The eigenvectors are given by the kernel of $A - \lambda I$ for each eigenvalue. So the eigenvectors are given by: For $\lambda = 2$:

$$\ker \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix} = \text{span} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For $\lambda = 3$:

$$\ker \begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix} = \text{span} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

So for $\lambda_1 = 2, \mathbf{v}_1 = (2, 1)$, and $\lambda_2 = 3, \mathbf{v}_2 = (1, 1)$.

(b) Similarly, we obtain the eigenvalues to be $\lambda_1 = 1, \lambda_2 = 6$ upon using the same idea. The eigenvectors are then given by the following kernels respectively:

$$\ker \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} = \text{span} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

(c) Similarly, we obtain the eigenvalues to be $\lambda_1 = 1-2i, \lambda_2 = 1+2i$. The eigenvectors are of the form:

$$\ker\begin{pmatrix} 2i & 4 \\ -1 & 2i \end{pmatrix} = \text{span}\begin{pmatrix} 2i \\ 1 \end{pmatrix}$$

$$\ker\begin{pmatrix} -2i & 4 \\ -1 & -2i \end{pmatrix} = \text{span}\begin{pmatrix} -2i \\ 1 \end{pmatrix}$$

(d) The characteristic polynomial of this matrix is going to be $\det(A-\lambda I) = (2-\lambda)\det\begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix} = (2-\lambda)((3-\lambda_2)^2-1)$. Solving for λ, we obtain $\lambda = 2, 2, 4$. The corresponding eigenvectors are $\lambda_1 = (1,0,0), (0,-1,1)$ for $\lambda = 2$, and $\lambda = 4$ means that the eigenvector is just $(0,1,1)$.

Example 29.

(a) $\text{AM}(2) = 1$, $\text{AM}(3) = 1$. The corresponding GMs are $\text{GM}(2) = 1$, $\text{GM}(3) = 1$.

(b) Same as above.

(c) Same as above.

(d) $\text{AM}(2) = 2$, $\text{AM}(4) = 1$. $\text{GM}(2) = 2$, $\text{GM}(4) = 1$.

Example 30.

(a) $\lambda_1 = 2, \lambda_2 = -3$. Since trace is equal to the sum of eigenvalues, we have $\lambda_3 = (2+8+-7)-(2+-3) = 4$. The arithmetic multiplicity of each eigenvalue is 1, and since geometric multiplicity is positive and less than or equal to 1, the GM of each eigenvalue must be 1.

(b) The eigenvalue associated with $(1,0,1)$ is given by $\lambda = 3$, and for $(2,-1,2)$ is given by 1. Therefore, the last eigenvalue must be 4 because of the trace of the matrix. Taking the kernel of 4, we obtain:

$$\begin{pmatrix} -3 & 4 & 2 \\ 2 & -3 & 2 \\ -3 & 4 & 2 \end{pmatrix}$$

which has a kernel of $(-2,-2,1)$.

Example 31.

With respect to the basis, the linear map T can be written as:

$$T\begin{pmatrix}a\\b\\c\end{pmatrix} = (2a)\mathbf{v}_1 + (a+3b+c)\mathbf{v}_2 + (a+c)\mathbf{v}_3 = \begin{pmatrix}2 & 1 & 1\\0 & 2 & 0\\0 & 1 & 1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}$$

The characteristic polynomial is given by $(2-\lambda)(2-\lambda)(1-\lambda) = 0$. Thus the eigenvalues are given by $\lambda = 1, 2, 2$. Considering the kernel of $A - 2I$, we get:

$$\begin{pmatrix}0 & 1 & 1\\0 & 0 & 0\\0 & 1 & -1\end{pmatrix}$$

which has a dimension of 2. Therefore, it is diagonalisable.

Example 32.

The eigenvalues are given by $\lambda = -1, 3, 1$. Finding the eigenvectors for each of the eigenvalues, we have: For $\lambda = -1$:

$$\begin{pmatrix}0 & -12 & 0\\2 & 6 & 4\\0 & 4 & 0\end{pmatrix}$$

which has a kernel spanned by the vector $(-2, 0, 1)$. For the eigenvalue $\lambda = 3$, we obtain:

$$\begin{pmatrix}-4 & -12 & 0\\2 & 2 & 4\\0 & 4 & -4\end{pmatrix}$$

which has a kernel of $(-3, 1, 1)$. For the eigenvalue $\lambda = 1$:

$$\begin{pmatrix}-2 & -12 & 0\\2 & 4 & 4\\0 & 4 & -2\end{pmatrix}$$

which has a kernel of $(-3, \frac{1}{2}, 1)$.

So the matrix can be diagonlised as follows:

$$\begin{pmatrix} -2 & -3 & -3 \\ 0 & 1 & \frac{1}{2} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & -3 & -3 \\ 0 & 1 & \frac{1}{2} \\ 1 & 1 & 1 \end{pmatrix}^{-1}$$

Example 33.

Rewriting the equation as the following expression:

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 36$$

Computing the eigenvalues of the matrix $\begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix}$, we have $\lambda = 4, 9$. The eigenspaces are given by the span $(-2, 1)$ (for $\lambda = 4$) and the span of $(1, 2)$ (for $\lambda = 9$). Therefore, the new matrix form of the equation will be:

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = 36$$

Creating an orthogonal matrix out of this, we write:

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 36$$

Letting $X = \frac{1}{\sqrt{5}}(-2x+y)$ and $Y = \frac{1}{\sqrt{5}}(x+2y)$, we obtain the equation $4X^2+9Y^2 = 36$. So we take the graph of this ellipse with intercepts at $(X = \pm 3, 0)$ and $(0, Y = \pm 2)$. Then we rotate the axes X, Y until they match the new axes given by $X = 0$ and $Y = 0$. So the axes of the ellipse are $y = 2x$ (along which we go 3 units) and $y = -\frac{1}{2}x$ (along which we go 2 units). Note that these will also give the closest and furthest points along the ellipse.

Example 34.

The spectrum of A is given by $\lambda = 1, -i, i$ with associated eigenvectors $(2, 2, 1)$. This since one of the eigenvalues is 1, the matrix describes a rotation of α (given according to the formula $e^{i\alpha} = i \implies \alpha = \frac{\pi}{2}$), about the axis $(2, 2, 1)$. The angle of rotation is given by $2\cos\alpha + 1 = 0 \implies \alpha = \frac{\pi}{2}$ using the idea of trace.

The spectrum of B is given by $\lambda = -1, \frac{5\pm 2\sqrt{14}i}{9}$, with the eigenvector corresponding to the reflection as $(-1,-3,2)$. A reflection occurs about a plane, so the plane of reflection will be $(-1,-3,2)\cdot\mathbf{x} = 0 \implies -x_1 - 3x_2 + 2x_3 = 0$. The angle of rotation about the axis is given by $2\cos\alpha - 1 = \frac{1}{9} \implies \alpha = \cos^{-1}\frac{5}{9}$.