MATH1251 CALCULUS S2 2010 TEST 1 VERSION 1A

Sample Solutions

August 20, 2017

These answers were written and typed up by Aaron Hassan. Please be ethical with this resource. It is for the use of MathSoc members, so do not repost it on other forums or groups without asking for permission. If you appreciate this resource, please consider supporting us by coming to our events and buying our T-shirts! Also, happy studying :)!

We cannot guarantee that our answers are correct – please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are often multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

Please note quiz papers that are NOT in your course pack will not necessarily reflect the style or difficulty of questions in your quiz.

---

1. Use partial fractions. Write

$$\frac{4x^2}{(x-2)(x^2+4)} \equiv \frac{A}{x-2} + \frac{Bx+C}{x^2+4}. \qquad (\star)$$

By Heaviside Cover-up, we find

$$A = \frac{4\times 2^2}{2^2+4} = \frac{4\times 4}{8} \Rightarrow \boxed{A=2}.$$

Now multiply $(\star)$ through by the denominator of the left-hand side (and remember $A=2$), so

$$4x^2 \equiv 2\left(x^2+4\right) + (Bx+C)(x-2).$$

Equating coefficients of x^2 on both sides gives

$$4 = 2 + B \Rightarrow \boxed{B=2}.$$

Equating constant terms on both sides gives

$$0 = 8 - 2C \Rightarrow \boxed{C=4}.$$

Thus

$$\frac{4x^2}{(x-2)(x^2+4)} \equiv \frac{2}{x-2} + \frac{2x+4}{x^2+4} = \frac{2}{x-2} + \frac{2x}{x^2+4} + \frac{4}{x^2+4}.$$

This results in only standard integrals being present, so

$$\begin{aligned}\int \frac{4x^2}{(x-2)(x^2+4)}\,\mathrm{d}x &= \int \left(\frac{2}{x-2} + \frac{2x}{x^2+4} + \frac{4}{x^2+4}\right)\mathrm{d}x \\ &= 2\ln|x-2| + \ln\left(x^2+4\right) + 2\tan^{-1}\left(\frac{x}{2}\right) + c.\end{aligned}$$

2. We use integration by parts. Integrating the e^{kx} and differentiating the x^n, we find that for every positive integer n, we have

$$\begin{aligned} I_n &= \int_0^1 x^n e^{kx}\,\mathrm{d}x \\ &= \left[\frac{e^{kx}}{k}x^n\right]_{x=0}^{x=1} - \int_0^1 nx^{n-1}\frac{e^{kx}}{k}\,\mathrm{d}x \\ &= \frac{e^k}{k} - 0 - \frac{n}{k}\int_0^1 x^{n-1}e^{kx}\,\mathrm{d}x \quad \text{(note } 0^n = 0 \text{ since } n \text{ is positive. Wouldn't be true if } n \text{ were negative!)} \\ &= \frac{e^k}{k} - \frac{n}{k}I_{n-1}. \end{aligned}$$

3. The ODE is first-order linear, so can be solved with an integrating factor. The integrating factor is

$$e^{\int -\frac{2}{x}\,\mathrm{d}x} = e^{-2\ln x} = x^{-2} = \frac{1}{x^2}.$$

(Note that since we are told $x > 0$, we could write $\ln x$ above without needing to write $\ln|x|$.) So multiplying the ODE through by this we have as usual

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x^2}y\right) &= 6x^4 \times \frac{1}{x^2} = 6x^2 \\ \Rightarrow \frac{1}{x^2}y &= 2x^3 + C \quad \text{for some constant } C \quad \text{(integrating both sides)} \\ \Rightarrow y &= 2x^5 + Cx^2 \quad \text{(this is the general solution).} \end{aligned}$$

4. The ODE is

$$I\,\mathrm{d}x + J\,\mathrm{d}y = 0,$$

where

$$I = 9x^2y^2 \quad \text{and} \quad J = 6x^3y + 3y^2.$$

We see that

$$\frac{\partial I}{\partial y} = 18x^2y$$

and

$$\frac{\partial J}{\partial x} = 18x^2y + 0 = 18x^2y.$$

Hence $\frac{\partial I}{\partial y} = \frac{\partial J}{\partial x}$, so the ODE is exact. To find the general solution, we search for a function $F \equiv F(x, y)$ that satisfies

$$\frac{\partial F}{\partial x} = 9x^2y^2 \tag{1}$$

$$\text{and} \quad \frac{\partial F}{\partial y} = 6x^3y + 3y^2. \tag{2}$$

Once we find such an F, the general solution will be given by $F(x, y) = C$ (for some constant C). Partially integrating (1) with respect to x, we find that

$$F = 3x^3y^2 + c(y),$$

for some function $c(y)$ that does not depend on x. Now partially differentiating this with respect to y, we find that

$$\frac{\partial F}{\partial y} = 6x^3y + c'(y).$$

Comparing with (2), we see that we must have

$$c'(y) = 3y^2,$$

and so (integrating this with respect to y) it suffices to take $c(y) = y^3$. Thus

$$F = 3x^3y^2 + c(y) = 3x^3y^2 + y^3,$$

and the general solution to the ODE is

$$3x^3y^2 + y^3 = C.$$

MATH1251 CALCULUS S2 2010 TEST 1 VERSION 1B

Sample Solutions

August 20, 2017

These answers were written and typed up by Aaron Hassan. Please be ethical with this resource. It is for the use of MathSoc members, so do not repost it on other forums or groups without asking for permission. If you appreciate this resource, please consider supporting us by coming to our events and buying our T-shirts! Also, happy studying :)!

We cannot guarantee that our answers are correct – please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are often multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

Please note quiz papers that are NOT in your course pack will not necessarily reflect the style or difficulty of questions in your quiz.

---

1. The ODE is first-order linear. The integrating factor is (noting $x > 0$)

$$\exp\left(\int -\frac{1}{2x}\,\mathrm{d}x\right) = \left(\exp\left(\ln x\right)\right)^{-1/2} = \frac{1}{\sqrt{x}}.$$

(Remember the index law $\exp(ab) = \exp(a)^b$.)

So multiplying through the ODE by the integrating factor, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sqrt{x}}y\right) = \sqrt{x}.$$

Integrating, we have

$$\frac{1}{\sqrt{x}}y = \frac{2}{3}x\sqrt{x} + C,$$

where C is an arbitrary constant (don't forget the $+C$ at this step!). Thus the general solution is

$$y = \frac{2}{3}x^2 + C\sqrt{x}.$$

Remember, when finding the integrating factor, we don't need to add a $+C$, because we just need a particular integrating factor that works. But towards the end when solving for y, you must remember the $+C$ to get the general solution.

2. The ODE is

$$I \, \mathrm{d}x + J \, \mathrm{d}y = 0,$$

where

$$I = 2x + y + 1 \quad \text{and} \quad J = 2y + x + 1.$$

Note that

$$\frac{\partial I}{\partial y} = 1 = \frac{\partial J}{\partial x}.$$

Hence $\frac{\partial I}{\partial y} = \frac{\partial J}{\partial x}$, so the ODE is exact. To find the general solution, we search for a function $F \equiv F(x, y)$ that satisfies

$$\frac{\partial F}{\partial x} = 2x + y + 1 \tag{1}$$

$$\text{and} \quad \frac{\partial F}{\partial y} = 2y + x + 1. \tag{2}$$

Once we find such an F, the general solution will be given by $F(x, y) = C$ (for some constant C). Partially integrating (1) with respect to x, we find that

$$F = x^2 + yx + x + c(y),$$

for some function $c(y)$ that does not depend on x. Now partially differentiating this with respect to y, we find that

$$\frac{\partial F}{\partial y} = x + c'(y).$$

Comparing with (2), we see that we must have

$$c'(y) = 2y + 1,$$

and so (integrating this with respect to y) it suffices to take $c(y) = y^2 + y$. Thus

$$F = x^2 + yx + x + c(y) = x^2 + yx + x + y^2 + y,$$

and the general solution to the ODE is

$$x^2 + yx + x + y^2 + y = C.$$

Tip. A check on your answer on this one: the original exact ODE is symmetric in x and y (i.e. if you interchange x and y, the ODE is unchanged), so our final solution should be symmetric in x and y too (proof: exercise), and it is.

3. Use partial fraction decomposition. As usual for when we have repeated factors, write

$$\frac{x^2+3}{(x-1)(x+1)^2} \equiv \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}. \qquad (\star)$$

As we know, A and C here can be found using Heaviside Cover-up (in general, the *highest* powers of each linear factor present can be found using this method). We find

$$A = \frac{1^2+3}{(1+1)^2} = \frac{4}{2^2} \Rightarrow \boxed{A=1}.$$

Also,

$$C = \frac{(-1)^2+3}{(-1-1)} = \frac{4}{(-2)} \Rightarrow \boxed{C=-2}.$$

Now multiplying $(\star)$ through by the denominator of the left-hand side and using the values obtained for A and C, we have

$$x^2+3 \equiv (x+1)^2 + B\left(x^2-1\right) - 2(x-1).$$

Equating coefficients of x^2 gives

$$1 = 1 + B \Rightarrow \boxed{B=0}.$$

So we have from $(\star)$ using our values of A, B and C

$$\frac{x^2+3}{(x-1)(x+1)^2} \equiv \frac{1}{x-1} - \frac{2}{(x+1)^2}.$$

Therefore,

$$\begin{aligned}\int \frac{x^2+3}{(x-1)(x+1)^2}\,\mathrm{d}x &= \int \left(\frac{1}{x-1} - \frac{2}{(x+1)^2}\right)\mathrm{d}x \\ &= \ln|x-1| + \frac{2}{x+1} + c.\end{aligned}$$

4. We use integration by parts. Integrating the e^{2x} and differentiating the x^n, we find that for every positive integer n, we have

$$\begin{aligned}I_n &= \int_0^1 x^n e^{2x}\,\mathrm{d}x \\ &= \left[\frac{e^{2x}}{2}x^n\right]_{x=0}^{x=1} - \int_0^1 nx^{n-1}\frac{e^{2x}}{2}\,\mathrm{d}x \\ &= \frac{e^2}{2} - 0 - \frac{n}{2}\int_0^1 x^{n-1}e^{2x}\,\mathrm{d}x \quad \text{(note } 0^n = 0 \text{ since } n \text{ is positive. Wouldn't be true if } n \text{ were negative!)} \\ &= \frac{e^2}{2} - \frac{n}{2}I_{n-1}.\end{aligned}$$

MATH1251 CALCULUS S2 2010 TEST 1 VERSION 2A

Sample Solutions

August 20, 2017

These answers were written and typed up by Aaron Hassan. Please be ethical with this resource. It is for the use of MathSoc members, so do not repost it on other forums or groups without asking for permission. If you appreciate this resource, please consider supporting us by coming to our events and buying our T-shirts! Also, happy studying :)!

We cannot guarantee that our answers are correct – please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are often multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

Please note quiz papers that are NOT in your course pack will not necessarily reflect the style or difficulty of questions in your quiz.

---

1. Use partial fraction decomposition. We write

$$\frac{3x-1}{(x+1)\left(x^2+1\right)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2+1}. \qquad (\star)$$

By Heaviside Cover-up, we have

$$A = \frac{3(-1)-1}{(-1)^2+1} = \frac{-4}{2} \Rightarrow \boxed{A=-2}.$$

Now, multiplying both sides of $(\star)$ by the denominator of the left-hand side (and using $A=-2$), we have

$$3x-1 \equiv -2\left(x^2+1\right) + (Bx+C)(x+1).$$

Equating constant terms, we have

$$-1 = -2 + C \Rightarrow \boxed{C=1}.$$

Now, equating coefficients of x^2, we have

$$0 = -2 + B \Rightarrow \boxed{B=2}.$$

Thus

$$\frac{3x-1}{(x+1)\left(x^2+1\right)} \equiv \frac{-2}{x+1}+\frac{2x+1}{x^2+1} = -\frac{2}{x+1}+\frac{2x}{x^2+1}+\frac{1}{x^2+1}.$$

Therefore,

$$\begin{aligned}\int \frac{3x-1}{(x+1)\left(x^2+1\right)}\,\mathrm{d}x &= \int\left(-\frac{2}{x+1}+\frac{2x}{x^2+1}+\frac{1}{x^2+1}\right)\mathrm{d}x \\ &= -2\ln|x+1|+\ln\left(x^2+1\right)+\tan^{-1}(x)+c.\end{aligned}$$

2. Note that $x=\frac{1}{2}(2x+2)-1$. Also, $x^2+2x+5=(x+1)^2+4$. It follows that

$$\begin{aligned}\int \frac{x}{x^2+2x+5}\,\mathrm{d}x &= \frac{1}{2}\int\frac{2x+2}{x^2+2x+5}\,\mathrm{d}x - \int\frac{1}{(x+1)^2+4}\,\mathrm{d}x \\ &= \frac{1}{2}\ln\left(x^2+2x+5\right)-\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+c.\end{aligned}$$

3. The ODE is first-order linear. The integrating factor is

$$e^{\int -2x\,\mathrm{d}x}=e^{-x^2}.$$

Thus multiplying the ODE through by the integrating factor yields

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-x^2}y\right)=3xe^{-x^2}.$$

Integrating both sides with respect to x (and using the Reverse Chain Rule on the right-hand side) yields

$$e^{-x^2}y=-\frac{3}{2}e^{-x^2}+C.$$

Solving for y, the general solution is

$$y=-\frac{3}{2}+Ce^{x^2}.$$

4. The ODE is

$$I\,\mathrm{d}x+J\,\mathrm{d}y=0,$$

where

$$I=x+2y+3 \quad \text{and} \quad J=2x+3y+4.$$

Note that

$$\frac{\partial I}{\partial y}=2=\frac{\partial J}{\partial x}.$$

Hence $\frac{\partial I}{\partial y}=\frac{\partial J}{\partial x}$, so the ODE is exact. To find the general solution, we search for a function

$F \equiv F(x, y)$ that satisfies

$$\frac{\partial F}{\partial x} = x + 2y + 3 \tag{1}$$

$$\text{and} \quad \frac{\partial F}{\partial y} = 2x + 3y + 4. \tag{2}$$

Once we find such an F, the general solution will be given by $F(x, y) = C$ (for some constant C). Partially integrating (1) with respect to x, we find that

$$F = \frac{x^2}{2} + 2yx + 3x + c(y),$$

for some function $c(y)$ that does not depend on x. Now partially differentiating this with respect to y, we find that

$$\frac{\partial F}{\partial y} = 2x + c'(y).$$

Comparing with (2), we see that we must have

$$c'(y) = 3y + 4,$$

and so (integrating this with respect to y) it suffices to take $c(y) = \frac{3y^2}{2} + 4y$. Thus

$$F = \frac{x^2}{2} + 2yx + 3x + c(y) = \frac{x^2}{2} + 2yx + 3x + \frac{3y^2}{2} + 4y,$$

and the general solution to the ODE is

$$\frac{x^2}{2} + 2yx + 3x + \frac{3y^2}{2} + 4y = C.$$

MATH1251 CALCULUS S2 2010 TEST 2 VERSION 1A

Sample Solutions

August 20, 2017

These answers were written and typed up by Aaron Hassan. Please be ethical with this resource. It is for the use of MathSoc members, so do not repost it on other forums or groups without asking for permission. If you appreciate this resource, please consider supporting us by coming to our events and buying our T-shirts! Also, happy studying :)!

We cannot guarantee that our answers are correct – please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are often multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

Please note quiz papers that are NOT in your course pack will not necessarily reflect the style or difficulty of questions in your quiz.

---

1. The characteristic equation is

$$\lambda^2 - 10\lambda + 25 = 0 \Rightarrow (\lambda - 5)^2 = 0.$$

Thus 5 is a repeated root, so the general solution to this ODE is

$$y = (A + Bt)e^{5t},$$

where A and B are arbitrary constants.

2. Note that it is easy to show that the terms of the sequence are well-defined for all positive integers n, e.g. we can show by induction that a_n is real and non-negative for all n. Proof of such a fact is given in (c).

(a) This result follows easily from (b). Assuming the result of (b) (which we prove in part (b)), we have that for all integers $n \geq 1$,

$$\begin{aligned} a_{n+1} &< a_n \\ \Rightarrow \sqrt{1 + a_n} &< a_n \\ \Rightarrow 1 + a_n &< a_n^2 \quad \text{(since } a_n > 0, \text{ as proved later)} \\ \Rightarrow a_n^2 - a_n - 1 &> 0. \end{aligned}$$

(b) Proof by induction. We have $a_2 = \sqrt{1+3} = 2 < a_1 = 3$. So the result holds for $n = 1$. Assume that $a_n < a_{n-1}$ for some positive integer $n-1$. We show that $a_{n+1} < a_n$. We have

$$\begin{aligned} a_{n+1} &= f\left(a_n\right) \\ &< f\left(a_{n-1}\right) \quad \text{since } a_n < a_{n-1} \text{ (hypothesis) and } f \text{ is clearly strictly increasing} \\ &= a_n. \end{aligned}$$

This proves the claim by induction.

(c) We show first that the sequence $\{a_n\}$ is bounded below by 0. Clearly $a_1 > 0$ (as $a_1 = 3$). Assume that $a_n > 0$ for some positive integer n. Then $a_{n+1} = \sqrt{1+a_n} > 0$ as $a_n > 0$ (hypothesis) and $\sqrt{1+x}$ is well-defined and positive for all $x > 0$. Hence by induction a_n is real and $a_n > 0$ for all positive integers n.

Thus the sequence of real numbers $\{a_n\}$ is bounded below and by part (b) it is decreasing. The result now follows from the Monotone Convergence Theorem.

(d) Since $a_{n+1} = f\left(a_n\right)$ for all positive integers n, we can tend $n \to \infty$ on both sides (since we know $L := \lim_{n\to\infty} a_n$ exists) to obtain

$$\begin{aligned} L &= \lim_{n\to\infty} f\left(a_n\right) \\ &= f\left(\lim_{n\to\infty} a_n\right) \quad \text{(as } f \text{ is continuous)} \\ &= f(L) = \sqrt{1+L}. \end{aligned}$$

Hence $L = \sqrt{1+L} \Rightarrow L^2 - L - 1 = 0$. Hence $L = \frac{1+\sqrt{5}}{2}$, taking the positive root since the sequence $\{a_n\}$ is a positive sequence, so its limit cannot be negative.

Remark. This question has shown that $\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}} = \frac{1+\sqrt{5}}{2}$. The number $\frac{1+\sqrt{5}}{2}$ is known as the *Golden Ratio*, commonly denoted φ. It is a famous number in mathematics and appears in many places in nature, architecture, paintings, etc., and has many interesting mathematical properties. You can find out more about the Golden Ratio at its Wikipedia page: `https://en.wikipedia.org/wiki/Golden_ratio`.

3. (a) The general formula is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k,$$

where as usual $f^{(k)}$ is the k-th derivative of f.

(b) We have $f(x) = \sqrt{x} = x^{1/2}, a = 1$ and $n = 3$ here. We compute

$$\begin{aligned}
f\left(1\right) &= 1^{1/2} = 1 \\
f'\left(1\right) &= \frac{1}{2} \times 1^{-1/2} = \frac{1}{2} \\
f''\left(1\right) &= \frac{1}{2} \times \left(-\frac{1}{2}\right) \times 1^{-3/2} = -\frac{1}{4} \\
f^{(3)}\left(1\right) &= \frac{1}{2} \times \left(-\frac{1}{2}\right) \times \left(-\frac{3}{2}\right) 1^{-5/2} = \frac{3}{8}.
\end{aligned}$$

Hence the third order Taylor polynomial of f about 1 is

$$\begin{aligned}
P_3(x) &= f\left(1\right) + f'\left(1\right)\left(x-1\right) + \frac{1}{2!} f''\left(1\right)\left(x-1\right)^2 + \frac{1}{3!} f^{(3)}\left(1\right)\left(x-1\right)^3 \\
&= 1 + \frac{1}{2}\left(x-1\right) + \frac{1}{2} \times \left(-\frac{1}{4}\right)\left(x-1\right)^2 + \frac{1}{6} \times \left(\frac{3}{8}\right)\left(x-1\right)^3 \\
&= 1 + \frac{1}{2}\left(x-1\right) - \frac{1}{8}\left(x-1\right)^2 + \frac{1}{16}\left(x-1\right)^3.
\end{aligned}$$

MATH1251 CALCULUS S2 2010 TEST 2 VERSION 1B

Sample Solutions

August 20, 2017

These answers were written and typed up by Aaron Hassan. Please be ethical with this resource. It is for the use of MathSoc members, so do not repost it on other forums or groups without asking for permission. If you appreciate this resource, please consider supporting us by coming to our events and buying our T-shirts! Also, happy studying :)!

We cannot guarantee that our answers are correct – please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are often multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

Please note quiz papers that are NOT in your course pack will not necessarily reflect the style or difficulty of questions in your quiz.

---

1. (a) Note $f(x) = \sqrt{1+3x}$ is defined in the reals for and only for $1+3x \geq 0$, i.e. $x \geq -\frac{1}{3}$. If $-\frac{1}{3} \leq x < 0$, then clearly we do *not* have $f(x) \leq x$, since $f(x)$ is always non-negative but x is negative here. *So the desired inequality can only hold if $x \geq 0$.*

   For $x \geq 0$, we have

$$f(x) \leq x \quad \text{if and only if} \quad \sqrt{1+3x} \leq x$$
$$\Leftrightarrow 1+3x \leq x^2 \quad (\text{since } t \mapsto t^2 \text{ is strictly increasing on the non-negative reals})$$
$$\Leftrightarrow x^2 - 3x - 1 \geq 0.$$

   This quadratic (x^2-3x-1) has its roots at $\frac{3\pm\sqrt{13}}{2}$. Only the root $\frac{3+\sqrt{13}}{2}$ is non-negative, and so for non-negative x, the inequality $x^2-3x-1 \geq 0$ holds if and only if $x \geq \frac{3+\sqrt{13}}{2}$, as we can see by sketching the quadratic. Hence $f(x) \leq x$ if and only if $x \geq \frac{3+\sqrt{13}}{2}$. Note that equality occurs if and only if $x = \frac{3+\sqrt{13}}{2}$.

   (b) Note that $a_{n+1} = f(a_n)$ for every integer $n \geq 1$. We prove the claim by induction. When $n = 1$, we have $a_1 = 4$ and $a_2 = f(a_1) = f(4) = \sqrt{1+3\times 4} = \sqrt{13}$. Since $\sqrt{13} < 4$ (as $13 < 4^2 = 16$), we have $a_2 < a_1$, so the claim is true for $n = 1$. Now assume that the claim is true for some positive integer $n-1$, i.e. $a_n < a_{n-1}$ for some positive integer $n-1$. We show that $a_{n+1} < a_n$. Noting that the function $f(x)$ is strictly increasing on its domain and that the sequence $\{a_k\}$ is well-defined for all positive integers k, we have

$$\begin{aligned} a_{n+1} &= f\left(a_n\right) \\ &< f\left(a_{n-1}\right) \quad \left(a_n < a_{n-1} \text{ by inductive hypothesis and } f \text{ is strictly increasing}\right) \\ &= a_n, \end{aligned}$$

and so the claim is proved by induction.

Remarks. To see that the sequence $\{a_k\}$ really is well-defined for all positive integers k, it suffices to check that $a_k \geq 0$ for all positive integers k. This can again be proved by induction and is left as an exercise for the reader.

Also, note that using the same induction method, we have essentially shown that if $\{b_k\}_{k=1}^{\infty}$ is any sequence of real numbers defined by $b_{k+1} = \phi\left(b_k\right)$ for some strictly increasing function ϕ, then this sequence will be a decreasing sequence provided that $b_2 < b_1$. So there is really nothing special about $f(x) = \sqrt{1+3x}$ other than that it is a strictly increasing function, in terms of proving this question's result. Many quiz questions ask you to show such things with a specific function f, so you can essentially use a similar inductive proof to show them.

(c) We first show that $a_n \geq 0$ for all integers $n \geq 1$. We show this by induction. Since $a_1 = 4 > 0$, it is true for $n = 1$. Now assuming $a_n \geq 0$ for some integer $n \geq 1$, we have $a_{n+1} = \sqrt{1+3a_n} \geq \sqrt{1+3\times 0} = 1 \geq 0$, since $a_n \geq 0$ by the inductive hypothesis and $x \mapsto \sqrt{1+3x}$ is an increasing function. This shows that $a_n \geq 0$ for all positive integers n.

Now that we have established that the sequence of real numbers $\{a_n\}$ has a lower bound (e.g. 0), and from part (b) we know the sequence is decreasing, it follows from the Monotone Convergence Theorem that there exists a real number L such that $a_n \to L$ as $n \to \infty$.

Remark. You may be wondering why we had to show that $a_n \geq 0$ like we did and could not just state it as "obvious" due to the fact that a_n is equal to the square root of something. The reason is that *a priori*, it could have been the case that the sequence would not even be well-defined after some point. For example, if a_1 were equal to -1 instead of 4, we would have $a_2 = \sqrt{1-3} = \sqrt{-2}$, and then our sequence would no longer be well-defined in the reals.

(d) Since $a_{n+1} = f\left(a_n\right)$, we obtain by sending $n \to \infty$ on both sides (*which we can only do since we know* $L := \lim\limits_{n\to\infty} a_n$ *exists*)

$$\begin{aligned} \lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} f\left(a_n\right) \\ \Rightarrow L &= f\left(\lim_{n\to\infty} a_n\right) \quad (\text{since } f \text{ is continuous, we can move the limit inside}) \\ \Rightarrow L &= f(L). \end{aligned}$$

Thus

$$L = \sqrt{1+3L}.$$

As we know from part (a), the solutions to this equation are $L = \frac{3\pm\sqrt{13}}{2}$. But the limit of the sequence must be non-negative since all the terms of the sequence are non-negative. So we take the positive root, and $L = \frac{3+\sqrt{13}}{2}$.

2. (a) The general formula is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k ,$$

where as usual $f^{(k)}$ is the k-th derivative of f.

(b) We have $f(x) = \sin x, a = \frac{\pi}{3}$ and $n = 3$ here. We compute

$$\begin{aligned} f\left(\frac{\pi}{3}\right) &= \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \\ f'\left(\frac{\pi}{3}\right) &= \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \\ f''\left(\frac{\pi}{3}\right) &= -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} \\ f^{(3)}\left(\frac{\pi}{3}\right) &= -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2}. \end{aligned}$$

Hence the third order Taylor polynomial of f about $\frac{\pi}{3}$ is

$$\begin{aligned} P_3(x) &= f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(x-\frac{\pi}{3}\right) + \frac{1}{2!}f''\left(\frac{\pi}{3}\right)\left(x-\frac{\pi}{3}\right)^2 + \frac{1}{3!}f^{(3)}\left(\frac{\pi}{3}\right)\left(x-\frac{\pi}{3}\right)^3 \\ &= \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x-\frac{\pi}{3}\right) + \frac{1}{2}\times\left(-\frac{\sqrt{3}}{2}\right)\left(x-\frac{\pi}{3}\right)^2 + \frac{1}{6}\times\left(-\frac{1}{2}\right)\left(x-\frac{\pi}{3}\right)^3 \\ &= \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x-\frac{\pi}{3}\right) - \frac{\sqrt{3}}{4}\left(x-\frac{\pi}{3}\right)^2 - \frac{1}{12}\left(x-\frac{\pi}{3}\right)^3. \end{aligned}$$

3. The characteristic equation is

$$\lambda^2 - 12\lambda + 36 = 0 \Rightarrow (\lambda-6)^2 = 0.$$

Thus 6 is a repeated root, so the general solution to this ODE is

$$y = (A+Bt)\, e^{6t},$$

where A and B are arbitrary constants.

MATH1251 CALCULUS S2 2010 TEST 2 VERSION 2A

Sample Solutions

August 20, 2017

These answers were written and typed up by Aaron Hassan. Please be ethical with this resource. It is for the use of MathSoc members, so do not repost it on other forums or groups without asking for permission. If you appreciate this resource, please consider supporting us by coming to our events and buying our T-shirts! Also, happy studying :)!

We cannot guarantee that our answers are correct – please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are often multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

Please note quiz papers that are NOT in your course pack will not necessarily reflect the style or difficulty of questions in your quiz.

---

1. The characteristic equation is
$$\lambda^2 + 2\lambda + 5 = 0.$$
The roots are thus
$$\begin{aligned}\lambda &= \frac{-2 \pm \sqrt{4 - 4 \times 5}}{2} \\ &= -1 \pm \sqrt{1-5} \\ &= -1 \pm 2i.\end{aligned}$$
Therefore, the general solution to this ODE is
$$y = e^{-t}\left(A\cos 2t + B\sin 2t\right),$$
where A and B are arbitrary constants.

2. (a) We prove this by induction. Note that $a_1^2 = 2^2 > 2$, so the result holds for $n = 1$. Now assume that the result holds for some positive n, i.e. $a_n^2 > 2$ for some positive integer n. We show that the result also holds for $n+1$, i.e. that $a_{n+1}^2 > 2$. We have
$$4a_{n+1}^2 - 8 = 4\left(f\left(a_n\right)\right)^2 - 8$$

$$
\begin{aligned}
&= 4\left(\frac{1}{2}a_n + \frac{1}{a_n}\right)^2 - 8 \quad \text{(note } a_n \neq 0 \text{ by the inductive hypothesis)}\\
&= 4\left(\frac{1}{4}a_n^2 + 1 + \frac{1}{a_n^2}\right) - 8\\
&= a_n^2 - 4 + \frac{4}{a_n^2}\\
&= \left(a_n - \frac{2}{a_n}\right)^2\\
&> 0.
\end{aligned}
$$

We have strict inequality here because $a_n - \frac{2}{a_n} \neq 0$, as $a_n \neq \frac{2}{a_n}$, since a_n^2 is strictly greater than 2 (by the inductive hypothesis).

Thus $4a_{n+1}^2 - 8 > 0 \Rightarrow a_{n+1}^2 > 2$, and the proof is complete by induction.

(b) Note that for all x such that $x^2 > 2$, we have

$$f'(x) = \frac{1}{2} - \frac{1}{x^2} > 0.$$

Thus f is strictly increasing on $\left[\sqrt{2}, \infty\right)$.

We can now prove the result by induction. We have $a_2 = \frac{1}{2}a_1 + \frac{1}{a_1} = \frac{1}{2} \times 2 + \frac{1}{2} = \frac{3}{2} < a_1$, so the result holds for $n = 1$. Now assume that $a_n < a_{n-1}$ for some positive integer $n - 1$. We show that $a_{n+1} < a_n$. We have

$$
\begin{aligned}
a_{n+1} &= f\left(a_n\right)\\
&< f\left(a_{n-1}\right) \quad \text{since } a_n < a_{n-1} \text{ by hypothesis and } f \text{ is strictly increasing on } \left[\sqrt{2}, \infty\right)\\
&= a_n,
\end{aligned}
$$

and the proof is complete by induction. Note that the sequence $\{a_n\}$ is a positive sequence and thus a subset of $\left[\sqrt{2}, \infty\right)$, from part (a). The fact that the sequence is positive can be shown by induction and is shown in part (c).

(c) We will show that $a_n > 0$ for all integers $n \geq 1$. We know $a_1 = 2 > 0$, and assuming $a_n > 0$ for some positive integer n, we have

$$
\begin{aligned}
a_{n+1} &= f\left(a_n\right)\\
&> 0,
\end{aligned}
$$

because $f(x) = \frac{1}{2}x + \frac{1}{x} > 0$ for all $x > 0$, and using the inductive hypothesis that $a_n > 0$. Thus $a_n > 0$ for all positive integers n by induction.

We know from (b) that the sequence $\{a_n\}$ is decreasing, and we just showed that it is bounded below (e.g. by 0), so the result follows from the Monotone Convergence Theorem.

(d) From the equation $a_{n+1} = f(a_n)$, tending $n \to \infty$ on both sides and using the fact that $L := \lim_{n\to\infty} a_n$ exists, we have

$$\begin{aligned} L &= \lim_{n\to\infty} f(a_n) \\ &= f\left(\lim_{n\to\infty} a_n\right) \quad \left(\text{as } f \text{ is continuous on } \left[\sqrt{2},\infty\right) \text{ and } \{a_n\} \subseteq \left[\sqrt{2},\infty\right)\right) \\ &= f(L). \end{aligned}$$

Thus

$$L = f(L) = \frac{1}{2}L + \frac{1}{L}.$$

Therefore, we have $\frac{1}{2}L = \frac{1}{L}$, whence $L^2 = 2$. Thus $L = \sqrt{2}$, taking the positive root since the sequence is positive and hence must have a non-negative limit.

3. (a) The general formula is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k,$$

where as usual $f^{(k)}$ is the k-th derivative of f.

(b) We have $f(x) = \sin x, a = \frac{\pi}{4}$ and $n = 3$ here. We compute

$$\begin{aligned} f\left(\frac{\pi}{4}\right) &= \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \\ f'\left(\frac{\pi}{4}\right) &= \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \\ f''\left(\frac{\pi}{4}\right) &= -\sin\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} \\ f^{(3)}\left(\frac{\pi}{4}\right) &= -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}. \end{aligned}$$

Hence the third order Taylor polynomial of f about $\frac{\pi}{4}$ is

$$\begin{aligned} P_3(x) &= f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right) + \frac{1}{2!}f''\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)^2 + \frac{1}{3!}f^{(3)}\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)^3 \\ &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) + \frac{1}{2}\times\left(-\frac{\sqrt{2}}{2}\right)\left(x-\frac{\pi}{4}\right)^2 + \frac{1}{6}\times\left(-\frac{\sqrt{2}}{2}\right)\left(x-\frac{\pi}{4}\right)^3 \\ &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x-\frac{\pi}{4}\right)^3. \end{aligned}$$