Second Year Mathematics Revision

Linear Algebra - Part 2

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Today's plan

1. Eigenvalues and Eigenvectors
   - Singular Value Decomposition (MATH2601)
2. The Jordan Canonical Form
   - Finding Jordan Forms
   - The Cayley-Hamilton Theorem
3. Matrix Exponentials
   - Computing Matrix Exponentials
   - Application to Systems of Differential Equations

Singular Values (MATH2601 only section)

Definition 1: Singular Values

A singular value of a $m \times n$ matrix A is the square root of an eigenvalue of A^*A.

Recall: A^*A denotes the adjoint of A.

Definition 2: Singular Value Decomposition

A SVD for an $m \times n$ matrix A is of the form $A = U\Sigma V^*$ where

- U is an $m \times m$ unitary matrix.
- V is an $n \times n$ unitary matrix.
- Σ has entries
  - $\sigma_{ii} > 0$. (These are determined by the singular values.)
  - $\sigma_{ij} = 0$ for all $i \neq j$.

Nice properties of A^*A

Lemma 1: Properties of A^*A

1. All eigenvalues of A^*A are real and non-negative (even if A has complex entries!)
2. $\ker(A^*A) = \ker(A)$
3. $\text{rank}(A^*A) = \text{rank}(A)$

The first one is pretty much why everything works.

SVD Algorithm

Algorithm 1: Finding a SVD

1. Find all eigenvalues λ_i of A^*A and write in descending order. Also find their associated eigenvectors of unit length $\mathbf{v}_i$.
2. Find an orthonormal set of eigenvectors for A^*A.
   - Automatically occurs when all eigenvalues are distinct, which will usually be the case. Otherwise require Gram-Schmidt for any eigenspace with dimension strictly greater than 1..
3. Compute $\mathbf{u}_i = \frac{1}{\sqrt{\lambda_i}} A\mathbf{v}_i$ for each non-zero eigenvalue.
4. State U and V from the vectors you found, and Σ from the singular values.

Lemma 2: Used to speed up step 1

- A^*A and AA^* share the same non-zero eigenvalues.
- If $\text{rank}(A) = r$, then A^*A has r non-zero eigenvalues. All other eigenvalues are 0.

SVD Example

Example 1: MATH2601 2017 Q2 c)

For the matrix $A = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix}$

1. Find the eigenvalues of AA^*.
2. Explain why the eigenvalues in part 1 are also eigenvalues of A^*A, and state any other eigenvalues of A^*A.
3. Find all eigenvectors of A^*A.
4. Find a singular value decomposition for A.

SVD Example

Part 1: We compute

$$AA^* = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 11 & 1 \\ 1 & 11 \end{pmatrix}.$$

This matrix has two eigenvalues, and they sum to $\mathrm{tr}(AA^*) = 22$ and multiply to $\det(AA^*) = 120$. By inspection, $\lambda_1 = 12$ and $\lambda_2 = 10$.

SVD Example

Part 2: Quoted word for word from the answers...

"We know that A^*A and AA^* have the same nonzero eigenvalues, so 12 and 10 are eigenvalues of A^*A.

Also, all eigenvalues of A^*A are real and nonnegative, so its third eigenvalue is 0."

SVD Example

Part 3: We compute

$$A^*A = \begin{pmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{pmatrix}$$

For $\lambda = 12$:

$$A^*A - 12I = \begin{pmatrix} -2 & 0 & 2 \\ 0 & -2 & 4 \\ 2 & 4 & -10 \end{pmatrix}.$$

Looking at row 1, arbitrarily set first component to 1, and then the third component is 1. Equating row 2, the second component is 2.

$$\therefore \boxed{\mathbf{v}_1 = t \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}.$$

SVD Example

Part 3: We compute

$$A^*A = \begin{pmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{pmatrix}$$

For $\lambda = 10$:

$$A^*A - 10I = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 4 \\ 2 & 4 & -8 \end{pmatrix}.$$

Rows 1 and 2 force the third component to be 0. Looking at row 3, it is easier to set the second component to 1, and then the first component will be -2.

$$\therefore \boxed{\mathbf{v}_2 = t \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}.$$

SVD Example

Part 3: We compute

$$A^*A = \begin{pmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{pmatrix}$$

For $\lambda = 0$, looking at A^*A itself, there really are many possibilities we can go about it. But I follow the answers, which arbitrarily set the first component to -1. See if you can then show that

$$\boxed{\mathbf{v}_3 = t \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix}}.$$

Note: In each case, $t \in \mathbb{R}$.

SVD Example

Part 4: In each case, choose the value of t that normalises the eigenvectors:

$$\mathbf{v}_1 = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\1\end{pmatrix} \qquad (\lambda_1 = 12)$$

$$\mathbf{v}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix}-2\\1\\0\end{pmatrix} \qquad (\lambda_2 = 10)$$

$$\mathbf{v}_3 = \frac{1}{\sqrt{30}}\begin{pmatrix}-1\\-2\\5\end{pmatrix} \qquad (\lambda_3 = 0)$$

SVD Example

Part 4: Compute $\mathbf{u}_i = \frac{1}{\sqrt{\lambda_i}} A\mathbf{v}_i$ for each non-zero eigenvector:

$$\mathbf{u}_1 = \frac{1}{\sqrt{12}} \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix} \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

SVD Example

Part 4: We conclude that a SVD for A is $A = U\Sigma V^*$, where

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{pmatrix}$$

$$V = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{pmatrix}$$

Remark

For $A = U\Sigma V^*$, where $A \in \mathrm{M}_{n\times n}(\mathbb{C})$:

- The columns of $U = \begin{pmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_m \end{pmatrix}$ are called the left singular vectors.
- The columns of $V = \begin{pmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{pmatrix}$ are called the right singular vectors.

Word of advice: Write some of these numbers **very quickly**! SVDs are instructive when you know the method, but it always takes forever to do.

Reduced SVD

I don't see these examined, but I should still mention them.

1. Obtain $\hat{\Sigma}$ by removing any zero columns in Σ
2. Obtain $\hat{V}$ by removing the corresponding *columns* in V.
3. Then, $A = U\hat{\Sigma}\hat{V}^*$.

For the earlier example:

$$\hat{\Sigma} = \begin{pmatrix} \sqrt{12} & 0 \\ 0 & \sqrt{10} \end{pmatrix}$$

$$\hat{V} = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{6}} & 0 \end{pmatrix}$$

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Jordan Blocks

Definition 3: Jordan blocks

The $k \times k$ Jordan block for λ is the matrix

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ 0 & 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix} \in \mathrm{M}_{k\times k}(\mathbb{C}).$$

That is, put λ on every entry along the main diagonal, and a 1 immediately above each λ wherever possible.

Jordan Blocks

Quick examples:

$$J_3(-4) = \begin{pmatrix} -4 & 1 & 0 \\ 0 & -4 & 1 \\ 0 & 0 & -4 \end{pmatrix}$$

$$J_4(0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Powers of Jordan Forms

Find the pattern.

$$J_1(\lambda)^n = \begin{pmatrix} \lambda^n \end{pmatrix}$$
$$J_2(\lambda)^n = \begin{pmatrix} \lambda^n & \binom{n}{1}\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$$
$$J_3(\lambda)^n = \begin{pmatrix} \lambda^n & \binom{n}{1}\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} \\ 0 & \lambda^n & \binom{n}{1}\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}$$
$$J_4(\lambda)^n = \begin{pmatrix} \lambda^n & \binom{n}{1}\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} & \binom{n}{3}\lambda^{n-3} \\ 0 & \lambda^n & \binom{n}{1}\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} \\ 0 & 0 & \lambda^n & \binom{n}{1}\lambda^{n-1} \\ 0 & 0 & 0 & \lambda^n \end{pmatrix}$$

Powers of Jordan Forms

Lemma 3: Computing powers of Jordan forms

1. Start with λ^n on every diagonal entry.
2. Put $\binom{n}{1}\lambda^{n-1}$ wherever you can immediately above λ^n
3. Put $\binom{n}{2}\lambda^{n-2}$ wherever you can immediately above $\binom{n}{1}\lambda^{n-1}$
4. Keep doing this, increasing the binomial coefficient and decreasing the power on λ.

Note: Not *quite* the above. If you ever bump into $\binom{n}{n}$, that's the last diagonal you fill. Just put 0's everywhere else above.

Matrix Direct Sums

Definition 4: Direct sums of matrices

The direct sum of matrices $A_1, A_2, \dots, A_n$ is the matrix formed by putting these matrices on the diagonals and zeroes everywhere else.

$$A_1 \oplus A_2 \oplus \cdots \oplus A_n = \begin{pmatrix} A_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & A_2 & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & A_n \end{pmatrix}$$

In MATH2501 and MATH2601, we only worry about this with Jordan blocks.

Matrix Direct Sums

Quick example:

$$J_2(3) \oplus J_4(5) = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

Matrix Direct Sums

Lemma 4: Powers on direct sums of Jordan blocks

The power of a Jordan form is the direct sum of powers on each individual block. I.e.,

$$(J_1 \oplus \cdots \oplus J_m)^n = J_1^n \oplus \cdots \oplus J_m^n.$$

Example:

$$[J_2(3) \oplus J_1(2)]^n = \begin{pmatrix} 3^n & \binom{n}{1}3^{n-1} & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 2^n \end{pmatrix}$$

The Generalised Eigenvector

Definition 5: Generalised Eigenvector

A generalised eigenvector corresponding to eigenvalue λ is a non-zero vector $\mathbf{v}$ satisfying the property $\boxed{(A - \lambda I)^k \mathbf{v} = \mathbf{0}}$, for some $k \geq 1$.

This differs from the (usual) eigenvector in the sense that those must satisfy $(A - \lambda I)\mathbf{v} = \mathbf{0}$, i.e. we *must* take $k = 1$.

The Generalised Eigenvector

Example 2: MATH2601 2016 Q4 c)

Let $C = \begin{pmatrix} 9 & 7 & -3 \\ -2 & 2 & 1 \\ 2 & 5 & 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$.

Show that for the matrix C, $\mathbf{v}$ is a generalised eigenvector corresponding to $\lambda = 5$.

The Generalised Eigenvector

We compute that

$$C - 5I = \begin{pmatrix} 4 & 7 & -3 \\ -2 & -3 & 1 \\ 2 & 5 & -3 \end{pmatrix}$$

and continuously left-multiplying to $\mathbf{v}$,

$$(C - 5I)\mathbf{v} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$(C - 5I)^2\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

so $\mathbf{v} \in GE_5$.

Generalised Eigenspaces

Definition 6: Generalised Eigenspace

The generalised eigenspace of λ, denoted GE_λ, is the set of all *generalised* eigenvectors corresponding to λ.

$$GE_\lambda = \{\mathbf{v} \in \mathbb{C}^n \mid (A - \lambda I)^k \mathbf{v} = \mathbf{0} \text{ for some } k \geq 1\}$$

Lemma 4: Alternate representation of GE_λ

$$GE_\lambda = \ker(A - \lambda I) \cup \ker(A - \lambda I)^2 \cup \ker(A - \lambda I)^3 \cup \dots$$

Computing Jordan Forms

Definition 7: Jordan matrix

A Jordan matrix J is a direct sum of Jordan blocks.

Lemma 5: Uniqueness

Every matrix A has one unique Jordan matrix, up to some permutation (arrangement) of the Jordan blocks.

Computing Jordan Forms

Theorem 1: Useful properties in computing Jordan forms

Let $\dim\ker(A-\lambda I)^k$, i.e. nullity$(A-\lambda I)^k = d_k$. Set $d_0 = 0$. Then

1. $\ker(A-\lambda I) \subseteq \ker(A-\lambda I)^2 \subseteq \dots$
2. $d_0 \leq d_1 \leq d_2 \leq d_3 \leq \dots$
3. $d_1 - d_0 \geq d_2 - d_1 \geq d_3 - d_2 \geq \dots$

That is to say, the nullities must *not decrease*, but the *difference* in nullities must *not INcrease*.

Remark: Multiplicity

As a corollary, the algebraic multiplicity of an eigenvalue λ equals to $\dim GE_\lambda$. This allows us to not compute $(A-\lambda I)^k$ forever - we stop when nullity$(A-\lambda I)^k = \text{AM}$.

Computing Jordan Forms

We use Jordan chains to find the matrices P and J, such that $A = PJP^{-1}$. For an eigenvalue λ with algebraic multiplicity k, we need to start with some vector $\mathbf{v}_1$ such that on multiplication, we have

$$\mathbf{v}_1 \xrightarrow{A-\lambda I} \mathbf{v}_2 \xrightarrow{A-\lambda I} \dots \xrightarrow{A-\lambda I} \mathbf{v}_k \xrightarrow{A-\lambda I} \mathbf{0}.$$

We then include (note the reverse order!)

$$\begin{pmatrix} \mathbf{v}_k & \dots & \mathbf{v}_2 & \mathbf{v}_1 \end{pmatrix}$$

to P. This corresponds to *one* Jordan block $J_k(\lambda)$ in the direct sum for the Jordan matrix J of A.

Computing Jordan Forms

(Or maybe your lecturer taught things the other way around.) We consider this chain

$$\mathbf{v}_k \xrightarrow{A-\lambda I} \mathbf{v}_{k-1} \xrightarrow{A-\lambda I} \dots \xrightarrow{A-\lambda I} \mathbf{v}_1 \xrightarrow{A-\lambda I} \mathbf{0}.$$

and we include this to P instead.

$$\begin{pmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_{k-1} & \mathbf{v}_k \end{pmatrix}$$

We still use the Jordan block $J_k(\lambda)$.

Computing Jordan Forms: Example 1

Example 3: MATH2601 2016 Q4 c)

Let $C = \begin{pmatrix} 9 & 7 & -3 \\ -2 & 2 & 1 \\ 2 & 5 & 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$.

1. Calculate $(C - 5I)\mathbf{v}$ and $(C - 5I)^2\mathbf{v}$. (Done earlier)
2. Without using any matrix calculations, write down all the eigenvalues of C and their algebraic and geometric multiplicities. Give reasons for your answers.
3. (Not originally in the exam:) Find an invertible matrix P and a Jordan matrix J such that $C = PJP^{-1}$.

Computing Jordan Forms: Example 1

Part 2: The trace is usually helpful, because it is the sum of the eigenvalues.

$$\mathrm{tr}(C) = 9 + 2 + 2 = 13$$

From part 1, 5 is an eigenvalue of C with algebraic multiplicity *at least 2*. The third eigenvalue λ_3 satisfies

$$5 + 5 + \lambda_3 = 13 \implies \lambda_3 = 3.$$

Which is, of course, the only remaining eigenvalue and hence must have AM $= 1$. So we have:

- Eigenvalue 5: AM $= 2$, GM $= 1$
- Eigenvalue 3: AM $= 1$, GM $= 1$

Note: I haven't justified the GM's! Try doing that yourself!

Computing Jordan Forms: Example 1

Part 3: Row reducing $C - 3I$,

$$C - 3I = \begin{pmatrix} 6 & 7 & -3 \\ -2 & -1 & 1 \\ 2 & 5 & -1 \end{pmatrix}$$
$$\sim \begin{pmatrix} 0 & -12 & 0 \\ -2 & -1 & 1 \\ 0 & 4 & 0 \end{pmatrix}$$
$$\sim \begin{pmatrix} -2 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so we can take a corresponding eigenvector $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$.

Computing Jordan Forms: Example 1

So our chains are:

$$\begin{pmatrix}1\\-2\\-3\end{pmatrix} \xrightarrow{C-5I} \begin{pmatrix}-1\\1\\1\end{pmatrix} \xrightarrow{C-5I} \mathbf{0}$$

$$\begin{pmatrix}1\\0\\2\end{pmatrix} \xrightarrow{C-3I} \mathbf{0}$$

and hence $A = PJP^{-1}$ where

$$J = \begin{pmatrix}5 & 1 & 0\\0 & 5 & 0\\0 & 0 & 3\end{pmatrix} \text{ and } P = \begin{pmatrix}-1 & 1 & 1\\1 & -2 & 0\\1 & -3 & 2\end{pmatrix}$$

Computing Jordan Forms: Example 2 (time permitting...)

Example 4: MATH2601 2017 Q3 a)

Let $A = \begin{pmatrix} 3 & 1 & -2 \\ 2 & 6 & -7 \\ 2 & 2 & -2 \end{pmatrix}$. We are **given** that

$$GE_2 = \text{span}\left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \text{ and } GE_3 = \text{span}\left\{ \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \right\}.$$

1. Find the Jordan chain for $\lambda = 2$ starting with $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.
2. Without any calculation, write down the geometric multiplicity of $\lambda = 2$, giving reasons for your answer.
3. Find a Jordan form J and invertible matrix P for A, such that $A = PJP^{-1}$.

Computing Jordan Forms: Example 2 (time permitting...)

Part 1:

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \xrightarrow{A-2I} \begin{pmatrix} -1 \\ -3 \\ -2 \end{pmatrix} \xrightarrow{A-2I} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Part 2: The algebraic multiplicity is 2, since we have another distinct eigenvalue, so GM ≤ 2. But GM $\neq 2$ since we have a chain of length 2, so GM $= 1$.

Computing Jordan Forms: Example 2 (time permitting...)

Part 3: $A = PJP^{-1}$ where

$$P = \begin{pmatrix} -1 & 0 & 1 \\ -3 & 1 & 4 \\ -2 & 1 & 2 \end{pmatrix}$$

$$J = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Computing Jordan Forms: Example 2 (time permitting...)

Example 4: MATH2601 2017 Q3 a)

4. Find $\mathbf{v} \in GE_2$ and $\mathbf{w} \in GE_3$ such that $\mathbf{v} + \mathbf{w} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

Theorem 2: $\mathbb{C}^n$ and the generalised eigenspaces

The direct sum of generalised eigenspaces of any $A \in \mathrm{M}_{n \times n}$ span $\mathbb{C}^n$.

Hence we just need to express $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ as a linear combination of

$\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$.

Computing Jordan Forms: Example 2 (time permitting...)

You can have fun with the row reduction... I'll just state the final answer:

$$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} - 4\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$$
$$= \underbrace{\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}}_{\mathbf{v}} + \underbrace{\begin{pmatrix} -1 \\ -4 \\ -2 \end{pmatrix}}_{\mathbf{w}}$$

Remark: Similarity Invariants

Theorem 3: Jordan forms are the complete similarity invariant

Two matrices A and B are similar, i.e. $A = PBP^{-1}$ for some invertible matrix P, if and only if they have the same Jordan forms. (At least, to within a different arrangement of direct sums.)

Jordan forms given nullities

The Jordan matrix J can sometimes be found with less information if we don't need to find P.

Example 5: MATH2601 2016 Q4 b)

Let B be a 10×10 matrix and let λ be a scalar. Suppose it is known that

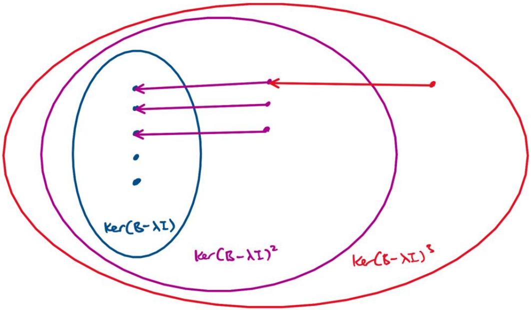
$$\begin{aligned} \text{nullity}(B - \lambda I) &= 5, \\ \text{nullity}(B - \lambda I)^2 &= 8, \\ \text{nullity}(B - \lambda I)^3 &= 9. \end{aligned}$$

Find all possible Jordan forms of B.

Idea: Our Jordan chains can be drawn on an onion diagram.

Jordan forms given nullities

There are 5 eigenvectors in $\ker(B - \lambda I)$. The idea is that there are $8 - 5 = 3$ *more* generalised eigenvectors in $\ker(B - \lambda I)^2$. This is because we know that $\ker(B - \lambda I) \subseteq \ker(B - \lambda I)^2$.



Similarly, there is another $9 - 8 = 1$ in $\ker(B - \lambda I)^3$.

Jordan forms given nullities

We've addressed 9 of the 10 eigenvalues for B. There is only one more left to go.

Case 1: The tenth eigenvalue is NOT λ.
Then it must be some other value $\mu \neq \lambda$. It can only correspond to one eigenvector, so we include $J_1(\mu)$ to the direct sum.

The Jordan chains for λ have lengths 3, 2, 2, 1 and 1, so therefore (up to some permutation),

$$J = J_3(\lambda) \oplus J_2(\lambda) \oplus J_2(\lambda) \oplus J_1(\lambda) \oplus J_1(\lambda) \oplus J_1(\mu).$$

(again, up to some permutation of the Jordan blocks).

Jordan forms given nullities

We've addressed 9 of the 10 eigenvalues for B. There is only one more left to go.

Case 2: The tenth eigenvalue IS also λ.
Problem: We cannot add it in $\ker(B-\lambda I)$, $\ker(B-\lambda I)^2$ or $\ker(B-\lambda I)^3$ without screwing up the nullities!

Recall that the difference is nullities is non-increasing. This means that the last generalised eigenvector must be in $\ker(B-\lambda I)^4$. Our original chain of length 3 also becomes chain of length 4. So we get

$$J = J_4(\lambda) \oplus J_2(\lambda) \oplus J_2(\lambda) \oplus J_1(\lambda) \oplus J_1(\lambda)$$

(again, up to some permutation of the Jordan blocks).

Jordan forms given nullities

Remark: Why $\ker(B - \lambda I)^4$? For completeness sake, here's a quick contradiction.

Suppose, say, the remaining generalised eigenvector was in $\ker(B - \lambda I)^5$ but *not* in $\ker(B - \lambda I)^4$. Then $\ker(B - \lambda I)^4$ must in fact be equal to $\ker(B - \lambda I)^3$, so $d_4 = d_3$, i.e. $d_4 - d_3 = 0$. Yet $d_5 - d_4 = 1$. Therefore $d_5 - d_4 > d_4 - d_3$, which cannot happen.

Invalid nullities

The property $d_1 - d_0 \geq d_2 - d_1 \geq d_3 - d_2 \geq \dots$ helps determine things that are impossible.

Example 6: David Angell's MATH2601 notes

Let A be a matrix with eigenvalue λ. Explain why this is not possible:

$$\begin{aligned} \text{nullity}(A - \lambda I) &= 5, \\ \text{nullity}(A - \lambda I)^2 &= 8, \\ \text{nullity}(A - \lambda I)^3 &= 9, \\ \text{nullity}(A - \lambda I)^4 &= 12, \\ \text{nullity}(A - \lambda I)^k &= 12 \text{ for all } k > 4. \end{aligned}$$

Answer: $d_4 - d_3 = 3 > 1 = d_3 - d_2$, which can't happen.

From Jordan forms back to nullities

Example 7: Peter Brown's MATH2501 notes

If A is similar to $J = J_2(-4) \oplus J_2(-4) \oplus J_2(-4) \oplus J_3(5) \oplus J_1(5)$. find

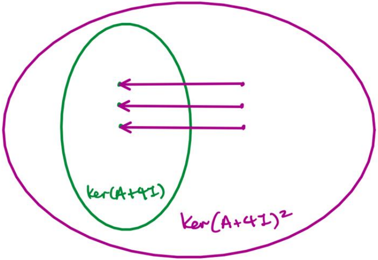
$$\text{nullity}(A+4I)^k \text{ and } \text{nullity}(A-5I)^k$$

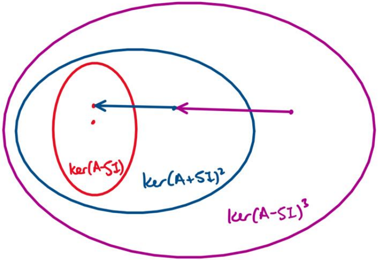
for all $k \geq 1$.

Solution: Go backwards!

From Jordan forms back to nullities

We know the lengths of the chains...





From Jordan forms back to nullities

So we see that:

- $\text{nullity}(A + 4I) = 3$
- $\text{nullity}(A + 4I)^k = 6$ for all $k \geq 2$
- $\text{nullity}(A - 5I) = 2$
- $\text{nullity}(A - 5I)^2 = 3$
- $\text{nullity}(A - 5I)^k = 4$ for all $k \geq 3$

Remark: T-invariance (MATH2601)

I've seen questions on this pop up in tutorials and exams, so I'll give an example involving proving this result. I won't have time to go over it in class though.

Definition 8: Invariance under T

A subspace U of V is said to be invariant under a transformation T if $T(U) \subseteq U$.

Example 8: MATH2601 2018 Q3 b)

Let V be a vector space, let S and T be linear transformations from V to V, and write $W = \ker(S - T)$. Show that if $ST = TS$ then W is invariant under T.

Remark: T-invariance (MATH2601)

Let V be a vector space and let S and T be linear transformations from V to V. Let $W = \ker(S - T)$ and suppose that $ST = TS$.

Let $\mathbf{v} \in T(W)$. Then $\mathbf{v} = T(\mathbf{w})$ for some $\mathbf{w} \in W$.
Goal: Show that $\mathbf{v} \in W = \ker(S - T)$, i.e. $\boxed{(S - T)(\mathbf{v}) = \mathbf{0}}$.

Then,

$$\begin{aligned}(S - T)(\mathbf{v}) &= S(\mathbf{v}) - T(\mathbf{v}) \\ &= S(T(\mathbf{w})) - T(T(\mathbf{w})) \\ &= T(S(\mathbf{w})) - T(T(\mathbf{w}))\end{aligned}$$

since $ST = TS$.

Remark: T-invariance (MATH2601)

Further, since T is linear,

$$\begin{aligned}(S - T)(\mathbf{v}) &= T\left(S(\mathbf{w}) - T(\mathbf{w})\right) \\ &= T\left((S - T)(\mathbf{w})\right).\end{aligned}$$

But since $\mathbf{w} \in W = \ker(S - T)$, we know that $(S - T)(\mathbf{w}) = \mathbf{0}$. Hence

$$\begin{aligned}(S - T)(\mathbf{v}) &= T(\mathbf{0}) \\ &= \mathbf{0}.\end{aligned}$$

Therefore $\mathbf{v} \in W$, so $T(W) \subseteq W$ and hence W is invariant under T.

The Companion Matrix (MATH2501)

The companion matrix allows us to go backwards from a characteristic polynomial to a matrix. (Or at least, one such matrix.)

Definition 9: Companion matrix

Consider the polynomial

$$f(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0.$$

A matrix C whose characteristic polynomial is $f(\lambda)$ is

$$C = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & -a_0 \\ 1 & 0 & \dots & 0 & 0 & -a_1 \\ 0 & 1 & \dots & 0 & 0 & -a_2 \\ \vdots & & \ddots & & & \vdots \\ 0 & 0 & \dots & 1 & 0 & -a_{n-2} \\ 0 & 0 & \dots & 0 & 1 & -a_{n-1} \end{pmatrix}$$

The Companion Matrix (MATH2501)

Example: The companion matrix corresponding to $p(\lambda) = \lambda^3 - 3\lambda^2 + 2\lambda - 5$ is

$$C = \begin{pmatrix} 0 & 0 & 5 \\ 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}.$$

Here, we set $a_0 = -5$, $a_1 = -2$ and $a_2 = 3$.

Recursively finding matrix powers

Theorem 4: The Cayley-Hamilton Theorem

Let A be an $n \times n$ matrix and $f(z)$ be its characteristic polynomial. Then $\boxed{f(A) = \mathbf{0}}$, the zero matrix.

Example 9: Peter Brown's MATH2501 notes

1. Verify the Cayley-Hamilton Theorem for $A = \begin{pmatrix} 1 & 3 \\ 4 & -2 \end{pmatrix}$
2. Use the Cayley-Hamilton Theorem to express A^4 and A^{-1} in terms of A and I, where I is the 2×2 identity matrix.

Recursively finding matrix powers

Part 1: Begin by computing

$$\begin{aligned}\mathrm{cp}_A(z) &= \begin{vmatrix} 1-z & 3 \\ 4 & -2-z \end{vmatrix} \\ &= (z-1)(z+2)-12 \\ &= z^2+z-14\end{aligned}$$

Then observe that

$$\mathrm{cp}_A(A) = \begin{pmatrix} 13 & -3 \\ -4 & 16 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 4 & -2 \end{pmatrix} - 14 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

where we note that the constant term gets multiplied to the identity matrix.

Recursively finding matrix powers

Part 2: Using the Cayley-Hamilton Theorem, we know that $A^2 = -A + 14I$. Hence

$$\begin{aligned} A^3 &= -A^2 + 14A \\ &= -(-A + 14I) + 14A \\ &= 15A - 14I \\ A^4 &= 15A^2 - 14A \\ &= 15(-A + 14I) - 14A \\ &= -29A + 210I \end{aligned}$$

Also $A = -I + 14A^{-1}$, so $A^{-1} = \frac{1}{14}A + \frac{1}{14}I$.

Minimal polynomials (MATH2501)

Note: This has been taken *out* of the higher syllabus.

Definition 10: Minimal polynomial of a matrix

Let A be an $n \times n$ matrix. The minimal polynomial m of A is the polynomial:

- of smallest degree possible
- and monic (i.e. the leading coefficient is 1)

such that $m(A) = \mathbf{0}$.

Lemma 6: Minimal polynomials and characteristic polynomials

The minimal polynomial is a *factor* of the characteristic polynomial. (Not really useful for computations, but it can be a nice sanity check.)

We won't delve much into the theory, we just illustrate how to find it.

Minimal polynomials (MATH2501)

Theorem 5: Explicit form for the minimal polynomial

Let A be an $n \times n$ matrix and denote the **distinct** eigenvalues of A as $\lambda_1, \lambda_2, \ldots, \lambda_r$.

For the i-th eigenvalue λ_i, let b_i be the size of the **largest** Jordan block corresponding to λ_i.

Then the minimal polynomial of A is

$$m(z) = (z - \lambda_1)^{b_1}(z - \lambda_2)^{b_2} \ldots (z - \lambda_r)^{b_r}.$$

Minimal polynomials (MATH2501)

Example 10: Peter Brown's MATH2501 notes

The Jordan form of $A \in \mathrm{M}_{15\times 15}$ is

$$J_5(2) \oplus J_2(2) \oplus J_3(-2) \oplus J_3(-2) \oplus J_2(-2).$$

What is its minimal polynomial?

The largest block for $\lambda = 2$ has size 5, and the largest block for $\lambda = -2$ has size 3. Therefore

$$m(z) = (z-2)^5(z+2)^3.$$

Minimal polynomials (MATH2501)

Example 11: Peter Brown's MATH2501 notes

The matrix $A = \begin{pmatrix} 3 & 5 & -4 \\ -2 & -4 & 4 \\ -1 & -3 & 4 \end{pmatrix}$ has characteristic polynomial

$$\mathrm{cp}_A(z) = z(z-1)(z-2).$$

What is its minimal polynomial?

The characteristic polynomial shows that the 3×3 matrix A has three *distinct* eigenvalues, so it must be *diagonalisable*. Hence $J = J_1(0) \oplus J_1(1) \oplus J_1(2)$, so for this matrix,

$$m(z) = \mathrm{cp}_A(z) = z(z-1)(z-2).$$

Today's plan

Matrix Exponential

Definition 11: Exponential of a matrix

The matrix exponential $\exp(tA)$ is defined as

$$\exp(tA) = e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k.$$

Computing matrix exponentials

We will illustrate the ideas...

Lemma 7: Properties of matrix exponentials

1. If $A = PBP^{-1}$, then $\exp(A) = P\exp(B)P^{-1}$.
2. If $A = A_1 \oplus \cdots \oplus A_n$, then $\exp(A) = \exp(A_1) \oplus \cdots \oplus \exp(A_n)$
3. $$\exp\left(tJ_k(\lambda)\right) = e^{\lambda t}\begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{k-1}}{(k-1)!} \\ 0 & 1 & t & & \\ 0 & 0 & 1 & \ddots & \\ & \vdots & \ddots & & \\ 0 & 0 & 0 & \dots & t \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

Also nice to note is that if $AB = BA$, then $\exp(A)\exp(B) = \exp(A+B)$.

Computing matrix exponentials

Example for a Jordan block:

$$\exp\left(tJ_5(2)\right) = e^{2t}\begin{pmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} & \frac{t^4}{4!} \\ 0 & 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} \\ 0 & 0 & 1 & t & \frac{t^2}{2!} \\ 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We're really filling the matrix in with terms from the power series of e^t, but then leaving a usual exponential in front.

Computing matrix exponentials

Example 12: Not really an example...

Consider $C = \begin{pmatrix} 9 & 7 & -3 \\ -2 & 2 & 1 \\ 2 & 5 & 2 \end{pmatrix}$ from earlier. We want $\exp(tC)$.

We have

$$P = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & -3 & 2 \end{pmatrix} \text{ and } J = \begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Computing matrix exponentials

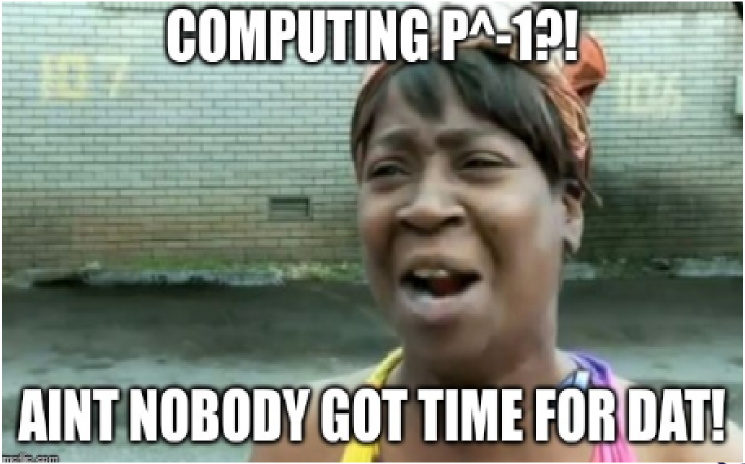
The earlier results show that we can do powers of Jordan blocks *one at a time*. So we obtain

$$\exp(J) = \begin{pmatrix} e^{5t} & te^{5t} & 0 \\ 0 & e^{5t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}$$

and hence

$$\exp(C) = P \begin{pmatrix} e^{5t} & te^{5t} & 0 \\ 0 & e^{5t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix} P^{-1}$$

A huge pain, as you can see.



So you probably won't be asked to do *that* in an exam. But you may be asked something else.

The 'Columns' technique

Theorem 6: Matrix Exponential times Generalised Eigenvector

If we have the Jordan chain

$$\mathbf{v}_1 \xrightarrow{A-\lambda I} \mathbf{v}_2 \xrightarrow{A-\lambda I} \mathbf{v}_3 \xrightarrow{A-\lambda I} \dots \xrightarrow{A-\lambda I} \mathbf{v}_k$$

then

$$\exp(tA)\mathbf{v}_1 = e^{\lambda t}\left(\mathbf{v}_1 + t\mathbf{v}_2 + \frac{t^2}{2}\mathbf{v}_3 + \cdots + \frac{t^{k-1}}{(k-1)!}\mathbf{v}_k\right)$$

The ‘Columns’ technique

This does come with a caveat in that $\mathbf{v}_1$ must be a generalised eigenvector corresponding to λ.

(Otherwise, we have to decompose it into a sum of generalised eigenvectors first.)

Solving Homogeneous systems of DEs

More often than not, we just need to compute $\exp(tA)\mathbf{v}$ for some vector $\mathbf{v}$, instead of the actual matrix exponential itself.

Theorem 7: Solution to a homogeneous system

The solution to $\frac{d\mathbf{y}}{dt} = A\mathbf{y}$ with initial condition $\mathbf{y}(0) = \mathbf{c}$ is

$$\mathbf{y} = \exp(tA)\mathbf{c}.$$

Solving Homogeneous systems of DEs

Example 13: MATH2601 2016 Q4 c)

Recall for $C = \begin{pmatrix} 9 & 7 & -3 \\ -2 & 2 & 1 \\ 2 & 5 & 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$ we have the chain

$$\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} \xrightarrow{C-5I} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \xrightarrow{C-5I} \mathbf{0}.$$

Use this to solve the initial value problem $\mathbf{y}' = C\mathbf{y}$, $\mathbf{y}(0) = \mathbf{v}$.

Solving Homogeneous systems of DEs

The solution is

$$\mathbf{y} = \exp(tA)\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}.$$

So using the columns method,

$$\begin{aligned} \mathbf{y} &= e^{5t}\left[\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right] \\ &= e^{5t}\begin{pmatrix} 1-t \\ -2+t \\ -3+t \end{pmatrix} \end{aligned}$$

The more general case (if time permits)

In general, if we can decompose **c** into a sum of generalised eigenvectors, we work our way around this issue.

Example 14: MATH2601 2017 Q3 a)

For $A = \begin{pmatrix} 3 & 1 & -2 \\ 2 & 6 & -7 \\ 2 & 2 & -2 \end{pmatrix}$, solve $\mathbf{y}' = A\mathbf{y}$ with initial condition $\mathbf{y}(0) = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, given that $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}}_{\in GE_2} + \underbrace{\begin{pmatrix} -1 \\ -4 \\ -2 \end{pmatrix}}_{\in GE_3}$.

The more general case (if time permits)

Construct the chains:

$$\begin{pmatrix}3\\5\\2\end{pmatrix} \xrightarrow{A-2I} \begin{pmatrix}4\\12\\8\end{pmatrix} \xrightarrow{A-2I} \mathbf{0}$$

$$\begin{pmatrix}-1\\-4\\-2\end{pmatrix} \xrightarrow{A-3I} \mathbf{0}$$

Our solution will thus be

$$\begin{aligned}\mathbf{y} &= e^{tA}\begin{pmatrix}3\\5\\2\end{pmatrix} + e^{tA}\begin{pmatrix}-1\\-4\\-2\end{pmatrix}\\ &= e^{2t}\begin{pmatrix}3+4t\\5+12t\\2+8t\end{pmatrix} + e^{3t}\begin{pmatrix}-1\\-4\\-2\end{pmatrix}.\end{aligned}$$

Solving Non-homogeneous systems of DEs

Lemma 8: Solution to non-homogeneous systems

The general solution to $\mathbf{y}' = A\mathbf{y}$ can be expressed as $\mathbf{y} = \mathbf{y}_H + \mathbf{y}_P$ where

- $\mathbf{y}_H$ is the general solution to $\mathbf{y}' = A\mathbf{y}$
- $\mathbf{y}_P$ is any particular solution to $\mathbf{y}' = A\mathbf{y} + \mathbf{b}$

Lemma 9: Variation of parameters

We can approach the particular solution by subbing $\mathbf{y} = e^{tA}\mathbf{z}$.

Lemma 10: Derivative of matrix exponential

$$\frac{d}{dt} e^{tA} = Ae^{tA}$$

Variation of parameters

In general, upon substituting in $\mathbf{y} = e^{tA}\mathbf{z}$ into $\frac{d\mathbf{y}}{dt} = A\mathbf{y} + \mathbf{b}$, we have

$$\begin{aligned} \frac{d(e^{tA}\mathbf{z})}{dt} &= Ae^{tA}\mathbf{z} + \mathbf{b} \\ e^{tA}\mathbf{z}' + Ae^{tA}\mathbf{z} &= Ae^{tA}\mathbf{z} + \mathbf{b} \\ e^{tA}\mathbf{z}' &= \mathbf{b} \\ \mathbf{z}' &= e^{-tA}\mathbf{b} \end{aligned}$$

So what we can do is:

1. Find $\mathbf{z}'$, probably using the columns technique again.
2. Integrate out to find $\mathbf{z}$.
3. Recompute $\mathbf{y} = e^{tA}\mathbf{z}$ for our particular solution.

Solving Non-homogeneous systems of DEs

Example 15: MATH2601 2016 Q4 c)

Find a particular solution of $\mathbf{y}' = C\mathbf{y} + te^{5t}\mathbf{w}$, where $C = \begin{pmatrix} 9 & 7 & -3 \\ -2 & 2 & 1 \\ 2 & 5 & 2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} -8 \\ 2 \\ -4 \end{pmatrix}$, given that $\mathbf{w}$ is a generalised eigenvector of C.

Subbing $\mathbf{y} = e^{tC}\mathbf{z}$ gives

$$\begin{aligned} Ce^{tC}\mathbf{z} + e^{tC}\mathbf{z}' &= Ce^{tC}\mathbf{z} + te^{5t}\mathbf{w} \\ \mathbf{z}' &= te^{5t}e^{-tC}\mathbf{w} \end{aligned}$$

Solving Non-homogeneous systems of DEs

We need to construct a Jordan chain starting at **w** first:

$$\begin{pmatrix}-8\\2\\-4\end{pmatrix} \xrightarrow{C-5I} \begin{pmatrix}-6\\6\\6\end{pmatrix} \xrightarrow{C-5I} \begin{pmatrix}0\\0\\0\end{pmatrix}$$

so therefore

$$e^{tC}\mathbf{w} = e^{5t}\left[\begin{pmatrix}-8\\2\\-4\end{pmatrix} + t\begin{pmatrix}-6\\6\\6\end{pmatrix}\right].$$

But observe how we want the negative exponent e^{-tC}! This means what we're really interested in is

$$e^{-tC}\mathbf{w} = e^{-5t}\left[\begin{pmatrix}-8\\2\\-4\end{pmatrix} - t\begin{pmatrix}-6\\6\\6\end{pmatrix}\right]$$

Solving Non-homogeneous systems of DEs

Therefore

$$\mathbf{z}' = te^{5t}e^{-tC}\mathbf{w} = t\begin{pmatrix}-8\\2\\-4\end{pmatrix} - t^2\begin{pmatrix}-6\\6\\6\end{pmatrix}$$

so upon integrating,

$$\mathbf{z} = \frac{t^2}{2}\begin{pmatrix}-8\\2\\-4\end{pmatrix} - \frac{t^3}{3}\begin{pmatrix}-6\\6\\6\end{pmatrix} + \mathbf{c}$$

Question: How to deal with that constant of integration?

Solving Non-homogeneous systems of DEs

In general, you can only deal with it when you know what $\mathbf{y}(0)$ is, i.e. you have an initial value for the original systems of DEs. When that's the case, you let $\mathbf{z}(0) = \mathbf{y}(0)$ to find it.

Here we don't, so we just proceed as usual.

$$\mathbf{y}_P = e^{tC}\mathbf{z} = \frac{t^2}{2}e^{tC}\begin{pmatrix}-8\\2\\-4\end{pmatrix} - \frac{t^3}{3}e^{tC}\begin{pmatrix}-6\\6\\6\end{pmatrix} + e^{tC}\mathbf{c}.$$

Solving Non-homogeneous systems of DEs

To finish this off, we can recycle our Jordan chain from earlier:

$$\begin{pmatrix} -8 \\ 2 \\ -4 \end{pmatrix} \xrightarrow{C-5I} \begin{pmatrix} -6 \\ 6 \\ 6 \end{pmatrix} \xrightarrow{C-5I} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This gives us

$$\mathbf{y}_P = \frac{t^2}{2}e^{5t}\left[\begin{pmatrix} -8 \\ 2 \\ -4 \end{pmatrix} - t\begin{pmatrix} -6 \\ 6 \\ 6 \end{pmatrix}\right] - \frac{t^3}{3}e^{5t}\begin{pmatrix} -6 \\ 6 \\ 6 \end{pmatrix} + e^{tC}\mathbf{c}$$

Solving Non-homogeneous systems of DEs

To finish this off, we can recycle our Jordan chain from earlier:

$$\begin{pmatrix} -8 \\ 2 \\ -4 \end{pmatrix} \xrightarrow{C-5I} \begin{pmatrix} -6 \\ 6 \\ 6 \end{pmatrix} \xrightarrow{C-5I} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This gives us

$$\mathbf{y}_P = \frac{t^2}{2} e^{5t} \left[\begin{pmatrix} -8 \\ 2 \\ -4 \end{pmatrix} - t \begin{pmatrix} -6 \\ 6 \\ 6 \end{pmatrix} \right] - \frac{t^3}{3} e^{5t} \begin{pmatrix} -6 \\ 6 \\ 6 \end{pmatrix} + e^{tC}\mathbf{c}$$

The remainder is trivial and is left as an exercise to the audience.

Solving Non-homogeneous systems of DEs

Note: The harsh reality is that if we knew what $\mathbf{c}$ was, that would potentially be *another* Jordan chain we need to deal with. Fingers crossed you don't have to do that in your exam.

Final remark: One single eigenvalue

When you're told that the matrix A only has *one* eigenvalue, you can take care of things more easily. The following comments assume $n = 3$, but the analogy can be adapted for all $n \times n$ matrices.

- Use the trace to find that eigenvalue λ.
- You automatically know that $GE_\lambda = \mathbb{C}^3$, so it's less difficult to construct a Jordan chain. Find $\ker(A - \lambda I)$, and only $\ker(A - \lambda I)^2$ if you don't already have two eigenvectors.
- Then, just pick a third vector out of thin air, not linearly indepedent to the other two. Construct a chain using that vector.
- Done!

You'll see this in all the examples in your tutorials...