MATH1231/1241 Revision

Algebra Part 1

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Introduction

Content reviewed today:

1. Vector Spaces
    - Vector Space
    - Subspace
    - Spans
    - Linear Independence
    - Basis & Dimension
    - Coordinate Vectors
2. Linear Transformations
    - Linear Maps
    - Associated Matrix
    - Geometry of Transformations
    - Image & Kernel
    - Rank-Nullity Theorem

Vector Space

Definition

A Vector Space is a set along with an addition and multiplication operator, such that the following 10 axioms are satisfied:

- Closure under addition
- Associativity of addition
- Commutativity of addition
- Identity element of addition
- Inverse elements of addition
- Closure under scalar multiplication
- Associativity of scalar multiplication
- Identity element of scalar multiplication
- Scalar distributivity
- Vector distributivity

Subspace

Introducing Subspace

That's a lot to remember, but you wont need it in the exam! The Subspace Theorem makes proving a set to be a vector space much easier.

Subspace Theorem

Suppose a subset S of a vector space V satisfies the following axioms:

- Identity element of addition
- Closure under vector addition
- Closure under scalar multiplication

Then S is a subspace of V, i.e. S is also a vector space.

Subspace

NOTES

- You MUST state the Subspace Theorem when you use it!
- When proving a set S is a subspace, you must show that S is a subset of a vector space before applying the Subspace Theorem.

Subspace

Example

Prove that

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_1 - 2x_2 + 4x_3 = 0 \right\}$$

is a subspace of $\mathbb{R}^3$.

(1231 2015 Q1.v)

Subspace

Example

Let

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_1^3 + x_2^3 + x_3^3 = 0 \right\}.$$

a) Prove that S is closed under scalar multiplication.
b) Show that S is **not** a subspace of $\mathbb{R}^3$.

(1231 2013 Q1.i)

Linear Combinations

Definition

Let $S = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ be a finite set of vectors in a vector space V over a field $\mathbb{F}$. Then a linear combination of S is

$$\lambda_1\mathbf{v}_1 + ... + \lambda_n\mathbf{v}_n$$

with $\lambda_1, ..., \lambda_n \in \mathbb{F}$.

NOTE

Since V is a vector field, any linear combination of a finite set S is also an element of V. In fact, the set of all linear combinations of S forms a subspace of V.

Span

Definition

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a finite set of vectors in a vector space V over a field $\mathbb{F}$. Then the span of the set S is the set of all linear combinations of S:

$$\begin{aligned}\text{span}(S) &= \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)\\ &= \{\mathbf{v} \in V : \mathbf{v} = \lambda_1\mathbf{v}_1 + \ldots + \lambda_n\mathbf{v}_n, \lambda_1, \ldots, \lambda_n \in \mathbb{F}\}.\end{aligned}$$

Also, $\text{span}(S)$ is a subspace of V.

Spanning Set

We say that a finite set of vectors S spans V if $\text{span}(S) = V$, i.e. every vector in V can be expressed as a linear combination of vectors in S.

Span

Example

Consider the vectors in $\mathbb{R}^3$,

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix}.$$

Prove that $\mathbf{b} \in \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

(1231 2015 Q1.vi)

Span

Elements of the Span

For a finite set of vectors $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ in $\mathbb{R}^m$, define an $m \times n$ matrix $A = (\mathbf{v}_1 \mid \ldots \mid \mathbf{v}_n)$. If a vector $\mathbf{b} \in \text{span}(S)$, then $A\mathbf{x} = \mathbf{b}$ for some $\mathbf{x} \in \mathbb{R}^n$.

Column Space

Observe that if $\mathbf{x} = \mathbf{e}_j$ then $\mathbf{b} = \mathbf{v}_j$. In general, a matrix A with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^m$ has the property $A\mathbf{e}_j = \mathbf{a}_j$. Here, we can see that the columns of A form a subspace of $\mathbb{R}^m$. We give this subspace a special name.

For an $m \times n$ matrix A, the subspace spanned by the columns of A is called the column space of A, $\text{col}(A)$.

Linear Independence

Definition

Suppose that $S = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ is a subset of a vector space. If the only solution to

$$x_1\mathbf{v}_1 + ... + x_n\mathbf{v}_n = \mathbf{0}$$

is the trivial solution $x_1, ..., x_n = 0$, then S is a linearly independent set. $\mathbf{v}_1, ..., \mathbf{v}_n$ are said to be linearly independent.

Linear Dependence

If there is a non-trivial solution, then S is a linearly dependent set. In other words, if $\mathbf{v}_j \in S$ and $\mathbf{v}_j \in \text{span}(S \setminus \{\mathbf{v}_j\})$, then $\mathbf{v}_1, ..., \mathbf{v}_n$ are linearly dependent.

Linear Independence

Example

Let $\mathbb{P}_2$ be the vector space of all real polynomials of degree at most 2. Find three polynomials f_1, f_2, f_3 in $\mathbb{P}_2$ such that $f_i(0) = 1$ for $i = 1, 2, 3$ and $\{f_1, f_2, f_3\}$ is linearly independent.

Basis

Basis

A set of vectors B in a vector space V is called a basis for V if B is a linearly independent set, and B is a spanning set for V.

Dimension Theorem

If $B \subseteq V$ is a basis for the vector space V, i.e. $\text{span}(B) = V$, then all linearly independent subsets $S \subseteq V$ have cardinality $|S| \leq |B|$. In the case of equality, $\text{span}(S) = V$ and S is a basis for V.

Basis

Example

The field $\mathbb{F} = GF(4)$ has elements $\{0, 1, \alpha, \beta\}$ with addition and multiplication defined by the following tables.

$+$	0	1	α	β
0	0	1	α	β
1	1	0	β	α
α	α	β	0	1
β	β	α	1	0

$\times$	0	1	α	β
0	0	0	0	0
1	0	1	α	β
α	0	α	β	1
β	0	β	1	α

Basis

Example (Cont.)

For the vectors

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} \beta \\ 1 \\ 1 \end{pmatrix}, \mathbf{b}_3 = \begin{pmatrix} 1 \\ 0 \\ \alpha \end{pmatrix},$$

a) show that $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis for $\mathbb{F}^3$;
b) explain why $\{\mathbf{b}_1, \mathbf{b}_1 + \mathbf{b}_2, \mathbf{b}_2 + \mathbf{b}_3, \mathbf{b}_3\}$ is a spanning set but not a basis for $\mathbb{F}^3$.

(1241 2016 Q3.iii)

Coordinate Vectors

Definition

Let V be a vector space and let the ordered set of vectors $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for V. If

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \textit{and} \quad \mathbf{v} = x_1\mathbf{v}_1 + \ldots + x_n\mathbf{v}_n,$$

then $[\mathbf{v}]_B = \mathbf{x}$ is called the coordinate vector of $\mathbf{v}$ with respect to B.

NOTE

If we set $A = (\mathbf{v}_1 \mid \ldots \mid \mathbf{v}_n)$ then

$$\mathbf{v} = A[\mathbf{v}]_B.$$

Coordinate Vectors

Example

Consider the field $\mathbb{F} = GF(4)$, as defined in the previous example. Let $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ be the vectors from the previous example. Set

$$\mathbf{v} = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}.$$

c) Find the coordinate vector of $\mathbf{v}$ with respect to the ordered basis $B = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$.

(1241 2016 Q3.iii)

Dimension

Definition

Recall that any two bases for a vector space V have the same number of elements. The dimension of V, $\dim(V)$, is the number of vectors in any basis for V.

Theorem

Suppose that V is a finite dimensional vector space.

1. If $\text{span}(S) = V$ then $|S| \geq \dim(V)$.
2. If $S \subseteq V$ is a linearly independent set, then $|S| \leq \dim(V)$.
3. If $\text{span}(S) = V$ and $|S| = \dim(V)$ then S is linearly independent, hence S is a basis for V.
4. If $S \subseteq V$ is a linearly independent set and $|S| = \dim(V)$, then $\text{span}(S) = V$ and hence S is a basis for V.

Linear Transformations

Definition

A linear transformation is a function from one vector space to another, such that addition and scalar multiplication are preserved.

More formally, for two vector spaces V and W over the same field $\mathbb{F}$ the linear map $T : V \to W$ satisfies the following conditions:

- $T(\mathbf{v} + \mathbf{v'}) = T(\mathbf{v}) + T(\mathbf{v'}) \qquad \forall\ \mathbf{v}, \mathbf{v'} \in V,$
- $T(\lambda \mathbf{v}) = \lambda T(\mathbf{v}) \qquad \forall\ \lambda \in \mathbb{F}, \mathbf{v} \in V.$

WARNING

Do NOT say that 'T is closed under addition' or 'T satisfies closure under scalar multiplication'. You will lose marks! Instead say that 'T preserves addition and scalar multiplication'.

Linear Transformations

Example

Prove that the function $T : \mathbb{P}(\mathbb{R}) \to \mathbb{R}^2$ defined by

$$T(p) = \begin{pmatrix} p(0) \\ p(1) \end{pmatrix}, \text{ for all polynomials } p \in \mathbb{P}(\mathbb{R}),$$

is a linear transformation.

(1241 2014 Q3.i)

Linear Transformations

Example

Let V and W be vector spaces, let $T : V \to W$ be a linear transformation, and let $\mathbf{v}_1, \mathbf{v}_2..., \mathbf{v}_m$ be vectors in V.
a) Prove that if $T(\mathbf{v}_1), T(\mathbf{v}_2), ..., T(\mathbf{v}_m)$ are linearly independent, then $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m$ are linearly independent.
b) Suppose that $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m$ are linearly independent. Is $T(\mathbf{v}_1), T(\mathbf{v}_2), ..., T(\mathbf{v}_m)$ linearly independent?

(1241 2016 Q3.ii)

Matrix of a Linear Map

Matrix Representation Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and let $\mathbf{e}_j$ for $1 \leq j \leq n$ be the standard basis vectors for $\mathbb{R}^n$. Setting

$$A = (\mathbf{a}_1 \mid \dots \mid \mathbf{a}_n)$$

where $\mathbf{a}_j = T(\mathbf{e}_j)$, then

$$T(\mathbf{x}) = A\mathbf{x} \qquad \forall \mathbf{x} \in \mathbb{R}^n.$$

Geometric Representation

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map which rotates a vector $\mathbf{x}$ about the origin through $\frac{\pi}{6}$ anticlockwise and doubles its length.

a) Show that $T(\mathbf{e}_1) = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$, where $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

b) Find the matrix A such that $T(\mathbf{x}) = A\mathbf{x}$. for all $\mathbf{x} \in \mathbb{R}^2$.

(1231 2013 Q2.iv)

Image

Definition

Let $T : V \to W$ be a linear map. Then the image of T, $\text{im}(T)$, is the set of all function values i.e.

$$\text{im}(T) = \{\mathbf{w} \in W : \mathbf{w} = T(\mathbf{v})\} \text{ for some } \mathbf{v} \in V.$$

NOTE

If you can find the associated matrix A of a linear map T, then finding the image is done simply by finding the set of vectors $\mathbf{b}$ from the matrix equation $A\mathbf{x} = \mathbf{b}$. It follows that $\text{im}(T)$ is a subspace of W.

Kernel

Definition

Let $T : V \to W$ be a linear map. Then the kernel of T, $\ker(T)$, is the set of all zeroes of T i.e.

$$\ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}.$$

NOTE

Again if we can find the associated matrix A of a linear map T, then the kernel of T is the set of all solutions of the matrix equation $A\mathbf{x} = \mathbf{0}$. It follows that $\ker(T)$ is a subspace of V.

Image and Kernel

Example

Consider the matrix $M = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$.

a) Find a basis for $\ker(M)$.

b) Find a basis for $\mathrm{im}(M^T)$.

c) Give a geometric description of $\ker(M)$ and $\mathrm{im}(M)$ as subspaces of $\mathbb{R}^2$.

(1231 2018 Q1.iv)

Rank and Nullity

Rank

The rank of a linear map T is the dimension of the image of T, i.e.

$$\operatorname{rank}(T) = \dim\left(\operatorname{im}(T)\right).$$

Nullity

The nullity of a linear map T is the dimension of the kernel of T, i.e.

$$\operatorname{nullity}(T) = \dim\left(\ker\left(T\right)\right).$$

Find a basis

If you can find a basis for the image or kernel of T, then finding the rank/nullity is easily done by counting the number of basis vectors. To make this task easier, we also have the Rank-Nullity Theorem.

Rank and Nullity

Rank-Nullity Theorem

Suppose that, for vector spaces V and W, $T : V \to W$ is a linear map. Then

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(V).$$

NOTE

For an $n \times m$ matrix A, $\operatorname{rank}(A) + \operatorname{nullity}(A) = m$.

Rank and Nullity

Example

Consider the mapping $T : \mathbb{P}_2 \to \mathbb{P}_3$ defined by

$$T(p)(x) = (x^2 + 1)p'(x) - 2xp(x).$$

Assuming T is linear, find the rank and nullity of T.

(1241 2015 Q3.ii)

MATH1231/41 Revision

Algebra Part 2

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Eigenvalues and Eigenvectors - Definition

Definition

Let A be a n x n square matrix. If a scalar $\lambda \in \mathbb{F}$ and non-zero vector $\vec{x} \in \mathbb{F}^n$ satisfy $A\vec{x} = \lambda x$ then λ is called an eigenvalue of A, and $\vec{x}$ is called the eigenvector of A for the eigenvalue λ.

Example 1 (MATH1241 2015 S2 q3 iv))

A linear transformation $P : V \to V$ is said to be idempotent if $P(P(\mathbf{v})) = P(\mathbf{v})$ for all $\mathbf{v} \in V$ (in other words $P^2 = P$).

(a) Show that the only possible eigenvalues for an idempotent linear transformation are 0 and 1.

(b) Show that if P is idempotent and P is neither the zero nor the identity transformation on V, then both 0 and 1 are eigenvalues.

Eigenvalues and Eigenvectors

Important Points

1. $\vec{0}$ cannot be an eigenvector
2. When the set of scalars is $\mathbb{C}^n$, there may be non-real eigenvalues.
3. Any non-zero vector in $\ker(A)$ is an eigenvector with eigenvalue 0.
4. There are infinitely many eigenvectors with the same eigenvalue, but there is only 1 eigenvalue for an eigenvector.
5. Only square matrices have eigenvalues and eigenvectors.

How to Find Eigenvalues and Eigenvectors

How to Find Eigenvalues and Eigenvectors

1. A scalar λ is an eigenvalue of a square matrix A iff $\det(A - \lambda I) = 0$
2. $\vec{v}$ is an eigenvector of A for the eigenvalue λ iff $\vec{v}$ is a non-zero solution of the equation $(A - \lambda I)\vec{v} = 0$ (i.e. if $\vec{v} \in \ker(A - \lambda I)$ and $\vec{v}$ is non-zero)
3. $\det(A - \lambda I)$ is called the characteristic polynomial for A.

Example 2

Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

Diagonalisation

1. If an $n \times n$ matrix A has n distinct eigenvalues then it has n linearly independent eigenvectors.
2. If an $n \times n$ matrix A has n linearly independent eigenvectors, then there exists an invertible matrix M and a diagonal matrix D such that $A = MDM^{-1}$.
3. The diagonal elements of D are the eigenvalues of A, and the jth column of M is an eigenvector of A with the jth element of the diagonal of D as eigenvalue.

Definition

A square matrix A is said to be diagonalisable if there exists an invertible matrix M and a diagonal matrix D such that $M^{-1}AM = D$.

Diagonalisation

Example 3 (Class Test 2 v1b S2 2018)

(a) Find all eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} -4 & 5 \\ 1 & 0 \end{pmatrix}$$

(b) Is A diagonalisable? Give reasons.

Powers of a Matrix

1. When a matrix A is diagonalisable, we can find an invertible matrix M and a diagonal matrix D such that $M^{-1}AM = D$.
2. Hence, we can find A^n by $A^n = (MDM^{-1})^n = MDM^{-1} \times MDM^{-1} \times ... \times MDM^{-1} = MD^nM^{-1}$

Example 4 (MATH1241 Final Exam, November 2010, q3i)

Evaluate A^8 if $A = \begin{pmatrix} 5 & -8 \\ 1 & -1 \end{pmatrix}$

Finding Solutions to First Order Linear ODEs

Definition

Let A be a $n \times n$ matrix. Then $\mathbf{y}(t) = \mathbf{v}e^{\lambda t}$ is a solution of $\mathbf{y}' = A\mathbf{y}$ iff λ is an eigenvalue of A and $\mathbf{v}$ is an eigenvector for the eigenvalue λ.

So if $\lambda_1, \lambda_2, \ldots \lambda_n$ are n distinct eigenvalues and $\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_n}$ are the n corresponding eigenvectors then the general solution to $\mathbf{y}' = A\mathbf{y}$ is $\mathbf{y}(t) = \alpha_1 e^{\lambda_1 t}\mathbf{v}_1 + \ldots + \alpha_n e^{\lambda_n t}\mathbf{v}_n$.

Finding Solutions to First Order Linear ODEs

Example 5

Solve the following system of differential equations:

$$\frac{dx}{dt} = x + 2y$$
$$\frac{dy}{dt} = 3x + 2y.$$

Finding Solutions to Second Order Linear ODEs

We can turn a 2nd order linear ODE with constant coefficients into a a system of 1st order equations, and then solve it using the method from the previous slide.

Example 6

Solve the following 2nd order ODE: $y'' + 4y' - 5y = 0$.

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Probability

Definition (Probability)

A probability $\mathbb{P}$ on a sample space S is any real function that satisfies the following:

1. $O \leq \mathbb{P}(A) \leq 1$
2. $\mathbb{P}(\emptyset) = 0$
3. $\mathbb{P}(S) = 1$
4. If A and B are mutually exclusive, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$.

Probability

Rules for Probabilities

Let A and B be events of a sample space S.

1. $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$
2. $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$
3. $\displaystyle\sum_{a \in S} \mathbb{P}(a) = 1$

Example 7

Show that the sequence defined by $p_k = \frac{7}{10}\left(\frac{3}{10}\right)^k$ for $k = 0, 1, 2, \ldots$ is a probability distribution.

Conditional Probability

Definition (Conditional Probability)

The conditional probability of A given B is
$\mathbb{P}(A \mid B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$ (provided that $\mathbb{P}(B) \neq 0$).

Definition (Multiplication Rule)

$\mathbb{P}(A \cap B) = \mathbb{P}(A \mid B)\mathbb{P}(B) = \mathbb{P}(B \mid A)\mathbb{P}(A)$

Definition (Independent Events)

2 events A and B are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.
If they are independent, the occurrence of one does not affect the probability of the other, i.e. $\mathbb{P}(A \mid B) = \mathbb{P}(A)$ and $\mathbb{P}(B \mid A) = \mathbb{P}(B)$.

Conditional Probability

Definition (Bayes' Rule)

$$\mathbb{P}(A_k \mid B) = \frac{\mathbb{P}(B \mid A_k)\mathbb{P}(A_k)}{\sum_{i=1}^{n} \mathbb{P}(B \mid A_i)\mathbb{P}(A_i)}$$

Example 8 (From Catherine Greenhill's lecture notes)

A certain diagnostic test for a disease is 99% sure of correctly indicating that a person has the disease when they actually do and 98% sure of correctly indicating that a person does not have a disease when they actually do not.
Supoose 2% of the population actually have this disease.

(a) What is the probability that a person doesn't have the disease when they test positive (false positive)?

(b) What is the probability that a person has the disease when they test negative (false negative)?

Random Variables

Some definitions for discrete random variables:

1. $\mathbb{E}(X) = \sum_{k=0}^{\infty} x_k \mathbb{P}_k$
2. $\text{Var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$
3. $\text{SD}(X) = \sqrt{\text{Var}(X)}$

Binomial Distribution

Definition (Binomial Distribution)

A Bernoulli trial is an experiment with 2 outcomes - success and failure. Suppose the chance of success is p. Let X be the random variable counting the number of successes in n independent identical Bernoulli trials. The probability distribution of X is called a Binomial Distribution, and
$X \sim B(n, p) = \binom{n}{k} p^k (1-p)^{n-k}$ where $k = 0, 1, ..., n$.
$\mathbb{E}(X) = np$ and $Var(X) = np(1-p)$.

Example 9 (MATH1241 Final Exam, November 2010, q3ii)

Suppose Sydney experienced rain on 103 out of 365 days in 2009. To estimate the probability that k out of 10 given days are rainy in Sydney, we use $B(10, \frac{103}{365})$.

Geometric Distribution

Definition (Geometric Distribution)

We perform independent identical Bernoulli trials until success. If X is the number of trials until the first success, the probability distribution of X is a geometric distribution, and
$G(p,k) = (1-p)^{k-1}(p)$, $k = 1, 2, 3, \ldots$
If $X \sim G(p)$ then $\mathbb{E}(X) = \frac{1}{p}$ and $\mathrm{Var}(X) = \frac{1-p}{p^2}$

Probability Density Functions

We can find probabilities by finding the area under the density function.

Definition (Probability Density Function)

If $F(x)$ is the cumulative distribution function, the probability density function $f_X(x)$ satisfies $f_X(x) = \frac{d}{dx}F(x)$.
Also, $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$.

So, $F(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f_X(x)\, dx$.

Probability Density Functions

Mean, Variance and SD for a continuous random variable X

1. $\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x)\,dx$ (where $f(x)$ is the probability density function)
2. $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$
3. $\text{SD}(X) = \sqrt{\text{Var}(X)}$

Linear Change of Variables

For both the discrete and continuous case:

1. $\mathbb{E}(aX + b) = a\mathbb{E}(X) + b$
2. $\text{Var}(aX + b) = a^2\text{Var}(\mathrm{X})$
3. $\text{SD}(aX + b) = |a|\text{SD}(X)$

Probability Density Functions

Example 10

The probability density function f of a continuous random variable X is given by

$$f(x) = \begin{cases} 3(1-x)^2, & \text{for } 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Sketch the graph of $y = f(x)$.

(b) Find $\mathbb{E}(X)$ and $\mathrm{Var}(X)$.

(c) Find $\mathbb{P}(\frac{1}{2} < \sin(\pi X) < \frac{1}{\sqrt{2}})$.

(d) The median of a distribution is defined to be the real number m such that $\mathbb{P}(X \le m) = \frac{1}{2}$. Find the median of the above distribution.

Normal Distribution

Definition (z score)

If X is a random variable, then $z = \frac{X - \mathbb{E}(X)}{\mathrm{SD}(X)}$.

Definition (Normal Distribution)

A continuous random variable X has normal distribution $N(\mu, \sigma^2)$ if it has probability density $\phi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$.
The integral of this can be written as $\mathbb{P}(X \leq x) = \mathbb{P}(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx$ and the value of this integral for various z can be found on the table.

Normal Distribution

Example 11 (MATH1231 2012 S2 final exam q3iv)

A 6-sided die, with faces numbered 1 to 6, is suspected of being unfair so that the number 6 will occur more frequently than should happen by chance. During 300 test rolls of the die, the number 6 occurred 68 times.

(a) Write down an expression for a tail probability that measures the chance of rolling a 6 at least 68 times.

(b) Use the normal approximation to the binomial to estimate this probability.

(c) Is this evidence that the die is unfair?

General Tips

1. Make sure to learn the definitions of all the key terms and concepts - see page *xiii* of the algebra course pack for a list of definitions and statements that you need to know.
2. To prove a statement is false, it suffices to give a specific counterexample.
3. When finding eigenvectors, when you row reduce $A - \lambda I$, there must be a zero row in your matrix. If this is not the case, then you know you have made a mistake (perhaps your eigenvalue is incorrect).

Good luck!

Any questions?
Good luck for your final exams!