Weekly Problems

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1 Hard Question

HARD QUESTION: Let f be a continuous function.Prove that if

$$\lim_{n\to\infty}(f(x+1)-f(x))=\alpha,$$

then

$$\lim_{x\to\infty}\frac{f(x)}{x}=\alpha.$$

Is the converse true?

Proof: Lets start with writing the limit given to us in a desirable way;let f be a continuous function, such that

$$\lim_{n\to\infty}\frac{f(x+1)-f(x)}{(x+1)-x}=\alpha,$$

this is of the form $\frac{f(x+1)-f(x)}{g(x+1)-g(x)}$, note that $g(x)$ is an strictly monotone continuous function, and is divergent to infinity as $x\to\infty$.Then we can write the above limit as an inequality.
For $\varepsilon>0$, and $x>M$ where $M\in\mathbb{Z}^+$($\mathbb{Z}^+$,means positive integer) , we have

$$\left|\frac{f(x+1)-f(x)}{(x+1)-x}-\alpha\right|<\varepsilon, \tag{1}$$

$$(\alpha-\varepsilon)<f(x+1)-f(x)<(\alpha+\varepsilon), \tag{2}$$

note that I have also modified the definition of the limit slightly,We'll see how this effects the proof a little later .We also notice that $f(x+1)-f(x)$ in the above inequality is a difference(where arguments of the function differ by 1), hence by, recognition of a telescoping series ,and the use of the summation operator, we can hope to reduce the inequality as something much nicer and closer to the result that we want to prove.

$$(\alpha-\varepsilon)\sum_{x=M}^{x_0}1<\sum_{x=M}^{x_0}f(x+1)-f(x)<(\alpha+\varepsilon)\sum_{x=M}^{x_0}1, \tag{3}$$

note the summation is justified across the inequality, since the summation bounds are from M to x_0, where $x_0 > M$.Also note that since the bounds of the summation operator are integers, I have tweaked the definition of the limit before. Then finally we get to write (3) as,

$$(\alpha - \varepsilon)((x_0 + 1) - M) < f(x_0 + 1) - f(M) < (\alpha + \varepsilon)((x_0 + 1) - M), \tag{4}$$

$$(\alpha - \varepsilon)\left(1 - \frac{M}{(x_0+1)}\right) < \frac{f(x_0+1)}{(x_0+1)} - \frac{f(M)}{(x_0+1)} < (\alpha+\varepsilon)\left(1 - \frac{M}{(x_0+1)}\right),$$

$$(\alpha - \varepsilon)\left(1 - \frac{M}{(x_0+1)}\right) + \frac{f(M)}{(x_0+1)} < \left(\frac{f(x_0+1)}{(x_0+1)}\right) < \frac{f(M)}{(x_0+1)} + (\alpha+\varepsilon)\left(1 - \frac{M}{(x_0+1)}\right). \tag{5}$$

This means there is some K such that for $x_0 > K$ we have:

$$(\alpha - \varepsilon) < \frac{f(x_0+1)}{(x_0+1)} < (\alpha + \varepsilon). \tag{6}$$

And by the inequality definition of limits, we know that the above inequality is equivalent to saying,

$$\lim_{x \to \infty} \frac{f(x)}{x} = \alpha.$$

*A key note in the proof is notice, that the transition between (5) and (6) was possible because $(x_0 + 1)$ is monotonic increasing, and diverges to infinity, which was in turn possible because $g(x) = x$ was strictly monotonic divergent. And for the second question, the converse will be true since you can flip the argument, that we have made in the proof.