2020 MathSoc Integration Bee
Qualifiers Solutions

1. Standard integral, but the bounds are tricky:

$$\begin{aligned}\int_{503}^{507} x \ \mathrm{d}x &= \frac{1}{2}\left(507^2 - 503^2\right) \\ &= \frac{1}{2}\left(507 + 503\right)\left(507 - 503\right) \\ &= \frac{1}{2} \cdot 1010 \cdot 4 \\ &= 2020.\end{aligned}$$

2. Rewrite $3^{\ln x}$ as $x^{\ln 3}$ using log laws, so now we have a standard integral:

$$\int 3^{\ln x} \ \mathrm{d}x = \int x^{\ln 3} \ \mathrm{d}x = \frac{x^{1+\ln 3}}{1+\ln 3} + C.$$

3. This is simply integrating $e^{x/2}$:

$$\int \sqrt{e^x} \ \mathrm{d}x = \int e^{x/2} \ \mathrm{d}x = 2\sqrt{e^x} + C.$$

4. Substitute $u = \sqrt{e^x - 1}$:

$$\begin{aligned}\int \sqrt{e^x - 1} \ \mathrm{d}x &= 2\int \frac{u^2}{u^2+1} \ \mathrm{d}u = 2\int 1 - \frac{1}{1+u^2} \ \mathrm{d}u \\ &= 2\left(u - \tan^{-1} u\right) + C \\ &= 2\left(\sqrt{e^x - 1} - \tan^{-1}\sqrt{e^x - 1}\right) + C.\end{aligned}$$

5. Substitute $x = e^u$ then the integral becomes

$$\int e^u \cos u \, \mathrm{d}u.$$

Apply integration by parts twice:

$$\begin{aligned}\int e^u \cos u \, \mathrm{d}u &= e^u \cos u + \int e^u \sin u \, \mathrm{d}u \\ &= e^u \cos u + e^u \sin u - \int e^u \cos u \, \mathrm{d}u.\end{aligned}$$

Rearrange to obtain

$$\int e^u \cos u \, \mathrm{d}u = \frac{1}{2}e^u\left(\cos u + \sin u\right),$$

then

$$\int \cos\left(\ln x\right) \mathrm{d}x = \frac{1}{2}x\left(\cos\left(\ln x\right) + \sin\left(\ln x\right)\right) + C.$$

6. Substitute $u = e^{2x}$, then

$$\int \frac{e^{2x}}{\sqrt{1 - e^{4x}}} \ \mathrm{d}x = \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}} \ \mathrm{d}u = \frac{1}{2}\sin^{-1} u + C = \frac{1}{2}\sin^{-1}\left(e^{2x}\right) + C.$$

7. This is an odd function from "$-a$ to a" so $\displaystyle\int_{-7\pi/4}^{7\pi/4} \frac{4x\cos x}{x^2 - \sin|x| + \cos|x|} \ \mathrm{d}x = 0.$

8. Substitute $u = \dfrac{\pi}{2} - x$, then

$$\int_0^{\pi/2} \frac{\sin^k x}{\sin^k x + \cos^k x}\,\mathrm{d}x = \int_0^{\pi/2} \frac{\cos^k u}{\cos^k u + \sin^k u}\mathrm{d}u.$$

Adding the two integrals, we have

$$2\int_0^{\pi/2} \frac{\sin^k x}{\sin^k x + \cos^k x}\,\mathrm{d}x = \int_0^{\pi/2} \frac{\sin^k x + \cos^k x}{\sin^k x + \cos^k x}\,\mathrm{d}x = \int_0^{\pi/2}\,\mathrm{d}x = \frac{\pi}{2}$$

$$\int_0^{\pi/2} \frac{\sin^k x}{\sin^k x + \cos^k x}\,\mathrm{d}x = \frac{\pi}{4}.$$

9. Multiply numerator and denominator by e^x:

$$\int \frac{e^x}{1+e^{2x}}\,\mathrm{d}x.$$

Now substitute $u = e^x$:

$$\int \frac{e^x}{1+e^{2x}}\,\mathrm{d}x = \int \frac{1}{1+u^2}\,\mathrm{d}u = \tan^{-1} u + C = \tan^{-1}(e^x) + C.$$

10. First, substitute $u = \tan x$:

$$\int \sec^2(x)\sec^2(\tan(x))\sec^2(\tan(\tan(x)))\,\mathrm{d}x = \int \sec^2(u)\sec^2(\tan(u))\,\mathrm{d}u.$$

Now substitute $v = \tan(u)$:

$$\int \sec^2(u)\sec^2(\tan(u))\,\mathrm{d}u. = \int \sec^2 v\,\mathrm{d}v = \tan v + C = \tan(\tan(\tan x)) + C.$$

11. From 0 to 1, the integrand is identically 0. From 1 to 2, $\left\lfloor \dfrac{x}{2} \right\rfloor$ is 0 and $\lfloor x \rfloor$ is 1. Hence

$$\int_0^2 \lfloor x \rfloor - 2\left\lfloor \frac{x}{2} \right\rfloor\,\mathrm{d}x = \int_1^2\,\mathrm{d}x = 1.$$

12. Substitute $u = 2x$, then

$$\int_0^{\sqrt{3}/4} \frac{2x\sin^{-1}(2x)}{\sqrt{1-4x^2}}\,\mathrm{d}x = \frac{1}{2}\int_0^{\sqrt{3}/2} \frac{u\sin^{-1} u}{\sqrt{1-u^2}}\,\mathrm{d}u.$$

Now substitute $u = \sin v$:

$$\frac{1}{2}\int_0^{\sqrt{3}/2} \frac{u\sin^{-1} u}{\sqrt{1-u^2}}\,\mathrm{d}u = \frac{1}{2}\int_0^{\pi/3} v\sin v\,\mathrm{d}v.$$

Here, we use by parts to obtain the answer,

$$\frac{1}{2}\int_0^{\pi/3} v\sin v\,\mathrm{d}v = \frac{\sqrt{3}}{4} - \frac{\pi}{12}.$$

13. Substitute $x = u^2$, then

$$\int_0^1 \sin^{-1}\sqrt{x}\,\mathrm{d}x = \int_0^1 2u\sin^{-1} u\,\mathrm{d}u.$$

Now substitute $u = \sin v$, then

$$\int_0^1 2u\sin^{-1} u\,\mathrm{d}u = \int_0^{\pi/2} v\sin 2v\,\mathrm{d}v.$$

Here, we use by parts to obtain the final answer:

$$\int_0^{\pi/2} v\sin 2v\,\mathrm{d}v = \frac{\pi}{4}.$$

14. First we multiply the numerator and denominator by $\cos^4 x$:

$$\int_0^{\pi/2} \frac{1}{1+\tan^4 x}\ \mathrm{d}x = \int_0^{\pi/2} \frac{\cos^4 x}{\cos^4 x + \sin^4 x}\ \mathrm{d}x.$$

Now we can use the answer from Q8:

$$\int_0^{\pi/2} \frac{\cos^4 x}{\cos^4 x + \sin^4 x}\ \mathrm{d}x = \frac{\pi}{4}.$$

15. From 0 to $\pi/2$ the integrand is x. From $\pi/2$ to $3\pi/2$ the integrand is $\pi - x$. Hence

$$\begin{aligned}\int_0^{3\pi/2} \sin^{-1}(\sin x)\ \mathrm{d}x &= \int_0^{\pi/2} x\ \mathrm{d}x + \int_{\pi/2}^{3\pi/2} (\pi - x)\ \mathrm{d}x \\ &= \frac{\pi^2}{8} + \pi^2 - \pi^2 = \frac{\pi^2}{8}.\end{aligned}$$

16. Separate the integrand via partial fractions:

$$\int_1^\infty \frac{\mathrm{d}x}{x(x^2+1)} = \int_1^\infty \frac{1}{x} - \frac{x}{x^2+1}\ \mathrm{d}x.$$

These can both be integrated into logarithms:

$$\int_1^\infty \frac{1}{x} - \frac{x}{x^2+1}\ \mathrm{d}x = \left[\ln x - \ln\sqrt{1+x^2}\right]_1^\infty = \left[\ln \frac{x}{\sqrt{1+x^2}}\right]_1^\infty = \frac{1}{2}\ln 2.$$

17. Multiply the numerator and denominator by $1 - \sin x$:

$$\begin{aligned}\int_0^{\pi/2} \frac{1}{1+\sin x}\ \mathrm{d}x = \int_0^{\pi/2} \frac{1-\sin x}{1-\sin^2 x}\ \mathrm{d}x &= \int_0^{\pi/2} \sec^2 x - \frac{\sin x}{\cos^2 x}\ \mathrm{d}x \\ &= [\tan x - \sec x]_0^{\pi/2} = \lim_{x\to\pi/2} \frac{\sin x - 1}{\cos x} + 1 \\ &= 1.\end{aligned}$$

18. Substitute $x = \cos u$, then

$$\int e^{\cos^{-1} x}\ \mathrm{d}x = -\int e^u \sin u \, \mathrm{d}u.$$

We can apply integration by parts here, or simply quote the result:

$$\int e^u \sin u \, \mathrm{d}u = \frac{1}{2} e^u (\sin u - \cos u).$$

Hence

$$\int e^{\cos^{-1} x}\ \mathrm{d}x = -\frac{1}{2} e^{\cos^{-1} x} \left(\sin\left(\cos^{-1} x\right) - \cos\left(\cos^{-1} x\right)\right) + C.$$

We can simplify further by observing that $\sin\left(\cos^{-1} x\right) = \sqrt{1-x^2}$ and $\cos\left(\cos^{-1} x\right) = x$ so

$$\int e^{\cos^{-1} x}\ \mathrm{d}x = \frac{1}{2} e^{\cos^{-1} x} \left(x - \sqrt{1-x^2}\right) + C.$$

19. Begin by transforming the denominator into a cosine function via auxiliary angle method:

$$3\cos x + 4\sin x = 5\cos\left(x - \sin^{-1}\left(\frac{4}{5}\right)\right).$$

Substituting this result into the integral:

$$\begin{aligned}\int_0^{\pi/2} \frac{25}{(3\cos x + 4\sin x)^2}\,\mathrm{d}x &= \int_0^{\pi/2} \sec^2\left(x - \sin^{-1}\left(\frac{4}{5}\right)\right)\,\mathrm{d}x \\ &= \left[\tan\left(x - \sin^{-1}\left(\frac{4}{5}\right)\right)\right]_0^{\pi/2} \\ &= \tan\left(\frac{\pi}{2} - \sin^{-1}\frac{4}{5}\right) + \tan\left(\sin^{-1}\frac{4}{5}\right) \\ &= \cot\left(\sin^{-1}\frac{4}{5}\right) + \tan\left(\sin^{-1}\frac{4}{5}\right) \\ &= \frac{3}{4} + \frac{4}{3} = \frac{25}{12}.\end{aligned}$$

20. Multiply the numerator and denominator by x^{-7}:

$$\int \frac{1}{x^7 + x}\,\mathrm{d}x = \int \frac{x^{-7}}{1 + x^{-6}}\,\mathrm{d}x = -\frac{1}{6}\ln\left(1 + x^{-6}\right) + C.$$

2020 MathSoc Integration Bee
Team Standoff Solutions

- Team A Question 1: Substitute $u = \ln x$, then

$$\int \frac{x-1}{x+x^2\ln x}\,\mathrm{d}x = \int \frac{e^u-1}{1+ue^u}\,\mathrm{d}u.$$

Now we multiply numerator and denominator by e^{-u}:

$$\int \frac{e^u-1}{1+ue^u}\,\mathrm{d}u = \int \frac{-e^{-u}+1}{e^{-u}+u}\,\mathrm{d}u = \ln\left|e^{-u}+u\right| + C.$$

- Team A Question 2: We have symmetry around $x = 1$ so

$$\int_0^2 \sin^2\left(\frac{\pi\,|x-1|}{2}\right)\,\mathrm{d}x = 2\int_0^1 \sin^2\left(\frac{\pi\,|x-1|}{2}\right)\,\mathrm{d}x.$$

For this interval, $|x-1| = 1-x$ so

$$\begin{aligned}2\int_0^1 \sin^2\left(\frac{\pi\,|x-1|}{2}\right)\,\mathrm{d}x &= 2\int_0^1 \sin^2\left(\frac{\pi(1-x)}{2}\right)\,\mathrm{d}x\\ &= 2\int_0^1 \cos^2\left(\frac{\pi x}{2}\right)\,\mathrm{d}x\\ &= \int_0^1 \left(1-\cos\left(\pi x\right)\right)\,\mathrm{d}x\\ &= 1.\end{aligned}$$

- Team A Question 3: Substitute $x = u^2$:

$$\int_0^{1/4} e^{\sqrt{x}}\,\mathrm{d}x = \int_0^{1/2} 2ue^u\,\mathrm{d}u.$$

Then by integration by parts,

$$\begin{aligned}\int_0^{1/2} 2ue^u\,\mathrm{d}u &= 2\left[ue^u\right]_0^{1/2} - 2\int_0^{1/2} e^u\,\mathrm{d}u\\ &= 2-\sqrt{e}.\end{aligned}$$

- Team B Question 1: First we complete the square in the square root,

$$\sqrt{x-x^2} = \sqrt{\frac{1}{4}-\left(x-\frac{1}{2}\right)^2}.$$

Hence our integral can be written as

$$\begin{aligned}\int \frac{1}{\sqrt{x-x^2}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{\frac{1}{4}-\left(x-\frac{1}{2}\right)^2}}\,\mathrm{d}x\\ &= 2\int \frac{1}{\sqrt{1-(2x-1)^2}}\,\mathrm{d}x\\ &= \sin^{-1}\left(2x-1\right)+C.\end{aligned}$$

- Team B Question 2: Using the substitution $u = \dfrac{\pi}{4} - x$,

$$\int_0^{\pi/4} \ln(1+\tan x)\ \mathrm{d}x = \int_0^{\pi/4} \ln\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)\ \mathrm{d}x.$$

However $\tan\left(\dfrac{\pi}{4}-x\right) = \dfrac{1-\tan x}{1+\tan x}$ so

$$\begin{aligned}\int_0^{\pi/4} \ln(1+\tan x)\ \mathrm{d}x &= \int_0^{\pi/4} \ln\left(1+\frac{1-\tan x}{1+\tan x}\right)\ \mathrm{d}x\\ &= \int_0^{\pi/4} \ln\left(\frac{2}{1+\tan x}\right)\ \mathrm{d}x\\ &= \int_0^{\pi/4} \ln 2\ \mathrm{d}x - \int_0^{\pi/4} \ln(1+\tan x)\ \mathrm{d}x\\ \int_0^{\pi/4} \ln(1+\tan x)\ \mathrm{d}x &= \frac{\pi\ln 2}{8}.\end{aligned}$$

- Team B Question 3: We use the substitution $u = \dfrac{\pi}{2} - x$:

$$\int_0^{\pi/2} \frac{\cos^2 x}{\sin x + \cos x}\ \mathrm{d}x = \int_0^{\pi/2} \frac{\sin^2 x}{\cos x + \sin x}.$$

Hence by adding the two integrals, we have

$$2\int_0^{\pi/2} \frac{\cos^2 x}{\sin x + \cos x}\ \mathrm{d}x = \int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x}\ \mathrm{d}x = \int_0^{\pi/2} \frac{1}{\sin x + \cos x}\ \mathrm{d}x.$$

Now use the auxiliary angle method on the denominator:

$$\sin x + \cos x = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right).$$

So the RHS integral becomes

$$\begin{aligned}\int_0^{\pi/2} \frac{1}{\sin x + \cos x}\ \mathrm{d}x &= \int_0^{\pi/2} \frac{1}{\sqrt{2}}\sec\left(x-\frac{\pi}{4}\right)\ \mathrm{d}x\\ &= \left[\frac{1}{\sqrt{2}}\ln\left(\sec\left(x-\frac{\pi}{4}\right)+\tan\left(x-\frac{\pi}{4}\right)\right)\right]_0^{\pi/2}\\ &= \frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right).\end{aligned}$$

Hence our original integral is

$$\int_0^{\pi/2} \frac{\cos^2 x}{\sin x + \cos x}\ \mathrm{d}x = \frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right).$$

- Team C Question 1: First we divide the numerator and denominator by x^2:

$$\int \frac{x^2-1}{x^4+1}\ \mathrm{d}x = \int \frac{1-x^{-2}}{x^2+x^{-2}}\ \mathrm{d}x.$$

Now by using the substitution $u = x + \dfrac{1}{x}$,

$$\int \frac{1-x^{-2}}{x^2+x^{-2}}\ \mathrm{d}x = \int \frac{1}{u^2-2}\ \mathrm{d}u.$$

Separating the integrand using partial fractions,

$$\int \frac{1}{u^2-2}\,\mathrm{d}u = \frac{1}{2\sqrt{2}}\int \frac{1}{u-\sqrt{2}} - \frac{1}{u+\sqrt{2}}\,\mathrm{d}u = \frac{1}{2\sqrt{2}}\ln\frac{u-\sqrt{2}}{u+\sqrt{2}} + C.$$

Hence our integral is

$$\int \frac{x^2-1}{x^4+1}\,\mathrm{d}x = \frac{1}{2\sqrt{2}}\ln\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1} + C.$$

- Team C Question 2: The terms of odd power integrate to 0, so we only need to consider the even powered terms. So

$$\begin{aligned}\int_{-1}^{1}\sum_{k=0}^{9} kx^k\,\mathrm{d}x &= 2\int_0^1 \left(2x^2+4x^4+6x^6+8x^8\right)\,\mathrm{d}x \\ &= 4\left(\frac{1}{3}+\frac{2}{5}+\frac{3}{7}+\frac{4}{9}\right).\end{aligned}$$

- Team C Question 3: Using the substitution $u=-x$,

$$\int_{-1}^{1}\tan^{-1}\left(2^x\right)\,\mathrm{d}x = \int_{-1}^{1}\tan^{-1}\left(\frac{1}{2^u}\right)\,\mathrm{d}u.$$

Note that $2^x > 0$, so adding this integral to the original give us

$$\begin{aligned}\int_{-1}^{1}\tan^{-1}\left(2^x\right)\,\mathrm{d}x &= \frac{1}{2}\int_{-1}^{1}\tan^{-1}\left(2^x\right)+\tan^{-1}\left(\frac{1}{2^x}\right)\,\mathrm{d}x \\ &= \frac{1}{2}\int_{-1}^{1}\frac{\pi}{2}\,\mathrm{d}x \\ &= \frac{\pi}{2}.\end{aligned}$$

- Team D Question 1: Substitute $u=x^{3/2}$, then

$$\int\sqrt{\frac{x}{1-x^3}}\,\mathrm{d}x = \frac{2}{3}\int\frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u.$$

This is a standard inverse sine integral, so

$$\int\sqrt{\frac{x}{1-x^3}}\,\mathrm{d}x = \frac{2}{3}\sin^{-1}u + C = \frac{2}{3}\sin^{-1}x^{3/2} + C.$$

- Team D Question 2: Pull a factor of x^4 out of the brackets:

$$\int\frac{1}{x^2\left(x^4+1\right)^{3/4}}\,\mathrm{d}x = \int\frac{1}{x^5\left(1+x^{-4}\right)^{3/4}}\,\mathrm{d}x.$$

Now by using the substitution $u=1+x^{-4}$,

$$\begin{aligned}\int\frac{1}{x^5\left(1+x^{-4}\right)^{3/4}}\,\mathrm{d}x &= -\frac{1}{4}\int u^{-3/4}\,\mathrm{d}u \\ &= -u^{1/4}+C \\ &= -\left(1+x^{-4}\right)^{\frac{1}{4}}+C\end{aligned}$$

- Team D Question 3: Using the substitution $u=1+\ln x$,

$$\int_1^{e^2}\frac{\ln\left(1+\ln x\right)}{x}\,\mathrm{d}x = \int_1^3 \ln u\,\mathrm{d}u.$$

Now apply integration by parts:

$$\begin{aligned}\int_1^3 \ln u\,\mathrm{d}u &= \left[u\ln u\right]_1^3 - \int_1^3\mathrm{d}u \\ &= 3\ln\left(3\right)-2.\end{aligned}$$

2020 MathSoc Integration Bee
Semi-Finals Solutions

- Round 1 Question 1: Observe that $\dfrac{\mathrm{d}}{\mathrm{d}x}(x\sin x+\cos x)=x\cos x$. We want this in the numerator:

$$\int \frac{x^2}{(x\sin x+\cos x)^2}\ \mathrm{d}x=\int \frac{x}{\cos x}\frac{x\cos x}{(x\sin x+\cos x)^2}\ \mathrm{d}x.$$

Now we can use integration by parts to evaluate the integral,

$$\begin{aligned}\int \frac{x}{\cos x}\frac{x\cos x}{(x\sin x+\cos x)^2}\ \mathrm{d}x &= \frac{-x\sec x}{x\sin x+\cos x}+\int \sec^2 x\ \mathrm{d}x\\ &= \frac{-x\sec x+\sec x\sin x\,(x\sin x+\cos x)}{x\sin x+\cos x}+C\\ &= \frac{\sin x-x\cos x}{x\sin x+\cos x}+C.\end{aligned}$$

- Round 1 Question 2: In order to evaluate this integral, we seek a reduction formula. This can be done via integration by parts. We set I_n to be the integral, then

$$\begin{aligned}I_n &= \left[\frac{x^{m+1}}{m+1}(\ln x)^n\right]_0^1-\int_0^1 \frac{x^{m+1}}{m+1}\frac{n(\ln x)^{n-1}}{x}\ \mathrm{d}x\\ &= 0-\frac{n}{m+1}\int_0^1 x^m(\ln x)^{n-1}\ \mathrm{d}x\\ &= -\frac{n}{m+1}I_{n-1}.\end{aligned}$$

Hence by taking the product of the ratios $\dfrac{I_k}{I_{k-1}}$ from $k=1$ to $k=n$:

$$\begin{aligned}\prod_{k=1}^{n}\frac{I_k}{I_{k-1}} &= \prod_{k=1}^{n}\frac{(-1)k}{m+1}\\ \frac{I_n}{I_0} &= \frac{(-1)^n n!}{(m+1)^n}.\end{aligned}$$

Now we must evaluate I_0, which is a standard integral

$$I_0=\int_0^1 x^m\ \mathrm{d}x=\frac{1}{m+1}.$$

Hence

$$I_n=\frac{(-1)^n n!}{(m+1)^{n+1}}.$$

- Round 1 Question 3: Using the substitution $u=808-x$,

$$\int_{403}^{405}\frac{\sqrt{\ln(2020-x)}}{\sqrt{\ln(2020-x)}+\sqrt{\ln(1212+x)}}\ \mathrm{d}x=\int_{403}^{405}\frac{\sqrt{\ln(1212+u)}}{\sqrt{\ln(1212+u)}+\sqrt{\ln(2020-u)}}\ \mathrm{d}u.$$

Adding this integral to the original, we have

$$\begin{aligned}\int_{403}^{405}\frac{\sqrt{\ln(2020-x)}}{\sqrt{\ln(2020-x)}+\sqrt{\ln(1212+x)}}\ \mathrm{d}x &= \frac{1}{2}\int_{403}^{405}\frac{\sqrt{\ln(2020-x)}+\sqrt{\ln(1616+x)}}{\sqrt{\ln(2020-x)}+\sqrt{\ln(1212+x)}}\ \mathrm{d}x\\ &= \frac{1}{2}\int_{403}^{405}\ \mathrm{d}x=1.\end{aligned}$$

- Round 2 Question 1: We require a reduction formula, which can be found via integration by parts. Denote the original integral as I_n, then

$$\begin{aligned}I_n &= -\frac{1}{2}\int_0^\infty x^{2n-1}\left(-2xe^{-x^2}\right)\, \mathrm{d}x \\ &= -\frac{1}{2}\left[x^{2n-1}e^{-x^2}\right]_0^\infty + \frac{1}{2}\int_0^\infty (2n-1)x^{2n-2}e^{-x^2}\, \mathrm{d}x \\ &= \frac{2n-1}{2}I_{n-1}.\end{aligned}$$

Taking the product of the ratios $\dfrac{I_k}{I_{k-1}}$ from $k=1$ to $k=n$,

$$\begin{aligned}\prod_{k=1}^n \frac{I_k}{I_{k-1}} &= \prod_{k=1}^n \frac{2k-1}{2} \\ \frac{I_n}{I_0} &= \frac{(2n-1)!!}{2^n}.\end{aligned}$$

Evaluating I_0:

$$I_0 = \int_0^\infty e^{-x^2}\, \mathrm{d}x = \frac{\sqrt{\pi}}{2}.$$

Hence

$$I_n = \frac{(2n-1)!!\sqrt{\pi}}{2^{n+1}}.$$

- Round 2 Question 2: For $\dfrac{1}{\pi} < x < \dfrac{1}{3}$, the integrand is $\ln 3$. Also for $\dfrac{1}{3} < x < \dfrac{1}{e}$, the integrand is $\ln 2$. So, we are integrating constants along two separate bounds:

$$\begin{aligned}\int_{1/\pi}^{1/e} \ln\left\lfloor \frac{1}{x}\right\rfloor\, \mathrm{d}x &= \int_{1/\pi}^{1/3} \ln 3\, \mathrm{d}x + \int_{1/3}^{1/e} \ln 2\, \mathrm{d}x. \\ &= \left(\frac{1}{3} - \frac{1}{\pi}\right)\ln 3 + \left(\frac{1}{e} - \frac{1}{3}\right)\ln 2.\end{aligned}$$

- Round 2 Question 3: We need to apply integration by parts twice:

$$\begin{aligned}\int_0^1 \sin(x)\sinh(x-1)\, \mathrm{d}x &= \left[\sin(x)\cosh(x-1)\right]_0^1 - \int_0^1 \cos(x)\cosh(x-1)\, \mathrm{d}x \\ &= \sin(1) - \left[\cos(x)\sinh(x-1)\right]_0^1 - \int_0^1 \sin(x)\sinh(x-1)\, \mathrm{d}x \\ \int_0^1 \sin(x)\sinh(x-1)\, \mathrm{d}x &= \frac{1}{2}\left(\sin(1) - \sinh(1)\right).\end{aligned}$$

2020 MathSoc Integration Bee
Runner Up Solutions

- Harder Integral: First, notice that the denominator can be simplified as

$$\left|\sin\left(\tan^{-1}\left(\sqrt{1-x^2}/x\right)\right)\right| = \sqrt{1-x^2}.$$

Then

$$\int_0^1 \frac{\sqrt{1-x^2}+\sin^{-1}\sqrt{\dfrac{1+x}{2}}}{\sqrt{1-x^2}}\,\mathrm{d}x = \int_0^1 1+\frac{\sin^{-1}\sqrt{\dfrac{1+x}{2}}}{\sqrt{1-x^2}}\,\mathrm{d}x$$

$$= 1+\frac{1}{2}\int_0^1 \frac{2\sin^{-1}\sqrt{\dfrac{1+x}{2}}}{\sqrt{1-x^2}}\,\mathrm{d}x.$$

Observe that $\dfrac{d}{\mathrm{d}x}\left(2\sin^{-1}\sqrt{\dfrac{1+x}{2}}\right) = \dfrac{1}{\sqrt{1-x^2}}$, hence

$$\int_0^1 \frac{2\sin^{-1}\sqrt{\dfrac{1+x}{2}}}{\sqrt{1-x^2}}\,\mathrm{d}x = \left[2\left(\sin^{-1}\sqrt{\frac{1+x}{2}}\right)^2\right]_0^1$$

$$= \frac{3\pi^2}{8}.$$

So our original integral evaluates to

$$\int_0^1 \frac{\sqrt{1-x^2}+\sin^{-1}\sqrt{\dfrac{1+x}{2}}}{\sqrt{1-x^2}}\,\mathrm{d}x = 1+\frac{3\pi^2}{16}.$$

- Easier Integral: Using the substitution $u=\sqrt{4x-3}$,

$$\int_1^3 3^{\sqrt{4x-3}}\,\mathrm{d}x = \frac{1}{2}\int_1^3 u\cdot 3^u\,\mathrm{d}u.$$

Now we can apply integration by parts:

$$\frac{1}{2}\int_1^3 u\cdot 3^u\,\mathrm{d}u = \frac{1}{2}\left[\frac{u\cdot 3^u}{\ln 3}\right]_1^3 - \frac{1}{2\ln 3}\int_1^3 3^u\,\mathrm{d}u$$

$$= \frac{39}{\ln 3} - \frac{12}{(\ln 3)^2}.$$

2020 MathSoc Integration Bee
Finals Solutions

- Question 1: First, notice that the integrand can be simplified into a product:

$$\frac{x}{\sqrt{x}}\frac{\sqrt[3]{x}}{\sqrt[4]{x}}\frac{\sqrt[5]{x}}{\sqrt[6]{x}}\cdots=\prod_{k=1}^{\infty}x^{\left(\frac{1}{2k-1}-\frac{1}{2k}\right)}.$$

The product can be moved into the exponent as a sum, giving us the expression

$$\prod_{k=1}^{\infty}x^{\left(\frac{1}{2k-1}-\frac{1}{2k}\right)}=x^{\sum_{k=1}^{\infty}\left(\frac{1}{2k-1}-\frac{1}{2k}\right)}.$$

The sum in the exponent is the taylor series of $\ln(1+x)$ evaluated at $x=1$, i.e. $\ln 2$. So our integral becomes

$$\begin{aligned}\int\frac{x}{\sqrt{x}}\frac{\sqrt[3]{x}}{\sqrt[4]{x}}\frac{\sqrt[5]{x}}{\sqrt[6]{x}}\cdots\,\mathrm{d}x&=\int x^{\ln 2}\,\mathrm{d}x\\&=\frac{x^{1+\ln 2}}{1+\ln 2}+C.\end{aligned}$$

- Question 2: We simplify the expression by logarithm laws:

$$\int e^{x^x}\ln\left(e^{x^{2x}}x^{x^{2x}}\right)\,\mathrm{d}x=\int e^{x^x}x^{2x}\left(1+\ln x\right)\,\mathrm{d}x.$$

By using the substitution $u=x^x$,

$$\begin{aligned}\int e^{x^x}x^{2x}\left(1+\ln x\right)\,\mathrm{d}x&=\int ue^u\,\mathrm{d}u\\&=(u-1)e^u+C\\&=(x^x-1)e^{x^x}+C.\end{aligned}$$

- Question 3: First, we must simplify the integrand:

$$\begin{aligned}\int\frac{1}{x}\prod_{k=1}^{\infty}\left(1-\tan^2\left(\frac{x}{2^k}\right)\right)\,\mathrm{d}x&=\int\frac{1}{x}\prod_{k=1}^{\infty}\left(2-\sec^2\left(\frac{x}{2^k}\right)\right)\,\mathrm{d}x\\&=\int\frac{1}{x}\prod_{k=1}^{\infty}\left(\sec^2\left(\frac{x}{2^k}\right)\left(2\cos^2\left(\frac{x}{2^k}\right)-1\right)\right)\,\mathrm{d}x\\&=\int\frac{1}{x}\prod_{k=1}^{\infty}\left(\sec^2\left(\frac{x}{2^k}\right)\cos\left(\frac{x}{2^{k-1}}\right)\right)\,\mathrm{d}x\\&=\int\frac{\cos x}{x}\prod_{k=1}^{\infty}\left(\sec\left(\frac{x}{2^k}\right)\right)\,\mathrm{d}x.\end{aligned}$$

We can simplify the product in the integrand now, by using trig identities.

$$\begin{aligned}\prod_{k=1}^{\infty}\left(\sec\left(\frac{x}{2^k}\right)\right)&=\lim_{N\to\infty}\prod_{k=1}^{N}\frac{2\sin\left(\frac{x}{2^k}\right)}{\sin\left(\frac{x}{2^{k-1}}\right)}\\&=\lim_{N\to\infty}\frac{2^N\sin\left(\frac{x}{2^N}\right)}{\sin x}\\&=\frac{x}{\sin x}.\end{aligned}$$

Hence our integral becomes

$$\int \frac{1}{x} \prod_{k=1}^{\infty} \left(1 - \tan^2\left(\frac{x}{2^k}\right)\right) \, \mathrm{d}x = \int \frac{\cos x}{\sin x} \, \mathrm{d}x$$

which is integrable into a logarithm:

$$\int \frac{\cos x}{\sin x} \, \mathrm{d}x = \ln(\sin x) + C.$$

- Question 4: Using the substitution $\tan\theta = \sqrt{e^x - 1}$,

$$\int_0^{\infty} \frac{x}{\sqrt{e^x - 1}} \, \mathrm{d}x = -4 \int_0^{\frac{\pi}{2}} \ln(\cos\theta) \, \mathrm{d}\theta.$$

Substituting $u = \frac{\pi}{2} - \theta$,

$$\int_0^{\frac{\pi}{2}} \ln(\cos\theta) \, \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \ln(\sin u) \, \mathrm{d}u.$$

We can add these two integrals together:

$$\begin{aligned} 2\int_0^{\frac{\pi}{2}} \ln(\cos\theta) \, \mathrm{d}\theta &= \int_0^{\frac{\pi}{2}} \ln(\sin\theta\cos\theta) \, \mathrm{d}\theta \\ &= \int_0^{\frac{\pi}{2}} \ln(\sin 2\theta) - \ln 2 \, \mathrm{d}\theta \\ &= \int_0^{\frac{\pi}{2}} \ln(\sin 2\theta) \, \mathrm{d}\theta - \frac{\pi \ln 2}{2}. \end{aligned}$$

For this resulting integral, we substitute $\phi = 2\theta$:

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \ln(\sin 2\theta) \, \mathrm{d}\theta &= \frac{1}{2} \int_0^{\pi} \ln(\sin\phi) \, \mathrm{d}\phi \\ &= \int_0^{\frac{\pi}{2}} \ln(\sin\phi) \, \mathrm{d}\phi \\ &= \int_0^{\frac{\pi}{2}} \ln(\cos\phi) \, \mathrm{d}\phi. \end{aligned}$$

Hence

$$\int_0^{\frac{\pi}{2}} \ln(\cos\theta) \, \mathrm{d}\theta = -\frac{\pi \ln 2}{2}.$$

So our integral evaluates to

$$\int_0^{\infty} \frac{x}{\sqrt{e^x - 1}} \, \mathrm{d}x = \pi \ln 4.$$

- Question 5: We seek a reduction formula for this integral. Denote the original integral as $I(m, n)$, then applying integration by parts we get

$$\int_0^1 x^m (1-x)^n \, \mathrm{d}x = \left[\frac{x^{m+1}(1-x)^n}{m+1}\right]_0^1 + \frac{n}{m+1} \int_0^1 x^{m+1}(1-x)^{n-1} \, \mathrm{d}x$$

$$I(m, n) = \frac{n}{m+1} I(m+1, n-1).$$

Taking the product of the ratios $\dfrac{I(m+n-k, k)}{I(m+n+1-k, k-1)}$ from $k = 1$ to $k = n$:

$$\begin{aligned} \prod_{k=1}^{n} \frac{I(m+n-k, k)}{I(m+n+1-k, k-1)} &= \prod_{k=1}^{n} \frac{k}{m+n+1-k} \\ \frac{I(m, n)}{I(m+n, 0)} &= \frac{m! \cdot n!}{(m+n)!}. \end{aligned}$$

Evaluating $I(m+n,0) = \dfrac{1}{m+n+1}$:

$$\begin{aligned} I(m+n,0) &= \int_0^1 x^{m+n}\ \mathrm{d}x \\ &= \frac{1}{m+n+1}. \end{aligned}$$

So our integral is

$$I(m,n) = \frac{m!\cdot n!}{(m+n)!}\frac{1}{m+n+1}$$

$$I(m,n) = \frac{m!\cdot n!}{(m+n+1)!}.$$