MATH1131/1141 Algebra Test 1 2014 S1 v1a

March 27, 2017

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1. (i) The line AB passes through the point A and has a gradient vector which is parallel to $\overrightarrow{B} - \overrightarrow{A}$ (where $\overrightarrow{P}$ refers to the coordinate vector of a point P). Therefore, a parametric vector equation for the line is

$$\mathbf{x} = \overrightarrow{A} + \lambda\left(\overrightarrow{B} - \overrightarrow{A}\right) = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 5-4 \\ -7-2 \\ -2-3 \end{pmatrix}$$

$$\text{i.e. } \mathbf{x} = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -9 \\ -5 \end{pmatrix},$$

where $\lambda \in \mathbb{R}$.

(ii) If C lies on the line AB, then there must exist $\lambda \in \mathbb{R}$ such that $\overrightarrow{C} = \overrightarrow{A} + \lambda\left(\overrightarrow{B} - \overrightarrow{A}\right)$.

That is,

$$\begin{aligned}
&\overrightarrow{C} = \overrightarrow{A} + \lambda\left(\overrightarrow{B} - \overrightarrow{A}\right)\\
\iff &\begin{pmatrix} 7 \\ -25 \\ -10 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -9 \\ -5 \end{pmatrix}\\
\iff &\begin{pmatrix} 3 \\ -27 \\ -13 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -9 \\ -5 \end{pmatrix}.
\end{aligned}$$

However, it is clear that no $\lambda \in \mathbb{R}$ can satisfy the above equation (equating first components shows $\lambda = 3$, but if $\lambda = 3$, then the third components do not match). So C does **not** lie on AB.

2. Let $\lambda = x_2$ and $\mu = x_3$. Solving for x_1 we obtain

$$x_1 = \frac{7 + 5x_2 - x_3}{2} = \frac{7 + 5\lambda - \mu}{2}.$$

Hence,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{7+5\lambda-\mu}{2} \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} \frac{7}{2} \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{5}{2} \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix},$$

which is a parametric vector equation for the plane. So, the two direction vectors, $\begin{pmatrix} \frac{5}{2} \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$ are parallel to the plane. Furthermore, it is clear that they are non-zero and non-parallel as either is not a scalar multiple of the other (in fact, this is guaranteed to happen when doing this method in general). Hence,

$$\begin{pmatrix} \frac{5}{2} \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix},$$

are two non-parallel non-zero vectors that are parallel to the plane.

Note. Here we isolated x_1 at the start and set x_2 and x_3 to free parameters. You could instead isolate a different variable (x_2 or x_3) and set the remaining two variables to free parameters, and you would then obtain a different, but equally correct, answer (remember, the parametric vector equation of a plane is not unique). In fact, choosing x_3

as the variable to isolate at the start would allow you to completely avoid having fractions pop up, as x_3 has a coefficient of 1 in the Cartesian equation.

3. (i) Recall that the cross product of two vectors in $\mathbb{R}^3$, say $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, is given by

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}.$$

Hence,

$$\begin{aligned}
\overrightarrow{AB} \times \overrightarrow{AC} &= \left(\overrightarrow{B} - \overrightarrow{A}\right) \times \left(\overrightarrow{C} - \overrightarrow{A}\right) \\
&= \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 2 \times 1 - (-2) \times 1 \\ (-2) \times 2 - 2 \times 1 \\ 2 \times 1 - 2 \times 2 \end{pmatrix} \\
&= \begin{pmatrix} 4 \\ -6 \\ -2 \end{pmatrix}.
\end{aligned}$$

Alternatively, one can find $\begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ by evaluating the *formal determinant*

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 2 & -2 \\ 2 & 1 & 1 \end{vmatrix}$$

where $\hat{\imath} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\hat{\jmath} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\hat{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are the standard basis vectors, and arrive at the same answer. In fact, in practice most people would do it this way (as the cross product formula is hard to remember), so here is how it would be done:

$$\begin{aligned}
\begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 2 & -2 \\ 2 & 1 & 1 \end{vmatrix} \\
&= \hat{\imath} \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} + \hat{k} \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} \\
&= (2+2)\,\hat{\imath} - (2+4)\,\hat{\jmath} + (2-4)\,\hat{k} = 4\hat{\imath} - 6\hat{\jmath} - 2\hat{k} \\
&= \begin{pmatrix} 4 \\ -6 \\ -2 \end{pmatrix}. \checkmark
\end{aligned}$$

(ii) The $\triangle ABC$ has half the area of the parallelogram formed by the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. The area of the parallelogram is given by $\left\| \overrightarrow{AB} \times \overrightarrow{AC} \right\|$, hence

$$\text{Area of } \triangle ABC = \frac{1}{2} \left\| \overrightarrow{AB} \times \overrightarrow{AC} \right\| = \sqrt{14},$$

since

$$\begin{aligned}
\left\| \overrightarrow{AB} \times \overrightarrow{AC} \right\| &= \left\| \begin{pmatrix} 4 \\ -6 \\ -2 \end{pmatrix} \right\| \\
&= \sqrt{16+36+4} \\
&= \sqrt{56} \\
&= 2\sqrt{14}.
\end{aligned}$$

4. (i) The projection of a vector, $\mathbf{a}$, onto another (non-zero) vector, $\mathbf{v}$, is given by

$$\text{proj}_{\mathbf{b}}(\mathbf{a}) = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2} \mathbf{b}.$$

Substituting given data yields

$$\text{proj}_{\mathbf{v}}\left(\overrightarrow{PQ}\right) = \frac{\left(\overrightarrow{Q} - \overrightarrow{P}\right) \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v}.$$

Note that $\overrightarrow{Q} - \overrightarrow{P} = \begin{pmatrix} 1-1 \\ 4-2 \\ 4-3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$. So $\left(\overrightarrow{Q} - \overrightarrow{P}\right) \cdot \mathbf{v} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = 0 \times 2 +$

$2 \times 3 + 1 \times 1 = 7$. Moreover, $\|\mathbf{v}\|^2 = 2^2 + 3^2 + 1^2 = 4 + 9 + 1 = 14$. Hence

$$\begin{aligned}
\operatorname{proj}_{\mathbf{v}}\left(\overrightarrow{PQ}\right) &= \frac{\left(\overrightarrow{Q} - \overrightarrow{P}\right) \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{7}{14}\mathbf{v} \\
&= \frac{1}{2}\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ \frac{3}{2} \\ \frac{1}{2} \end{pmatrix}.
\end{aligned}$$

(ii) *Drawing a rough picture is a good idea for this question.*

By construction, the vector that minimises the distance from Q to ℓ is given by $\overrightarrow{PQ} - \operatorname{proj}_{\mathbf{v}}\left(\overrightarrow{PQ}\right)$. After calculating that $\overrightarrow{PQ} - \operatorname{proj}_{\mathbf{v}}\left(\overrightarrow{PQ}\right) = \begin{pmatrix} -1 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$, we find that

$$\begin{aligned}
d &= \left\| \overrightarrow{PQ} - \operatorname{proj}_{\mathbf{v}}\left(\overrightarrow{PQ}\right) \right\| \\
&= \left\| \begin{pmatrix} -1 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right\| \\
&= \sqrt{1 + \frac{1}{4} + \frac{1}{4}} \\
&= \sqrt{\frac{3}{2}}.
\end{aligned}$$

An alternative method is to use Pythagoras' Theorem, and so $d^2 = \sqrt{\left\|\overrightarrow{PQ}\right\|^2 - \left\|\operatorname{proj}_{\mathbf{v}}\left(\overrightarrow{PQ}\right)\right\|^2}$.

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1. The 3 points are collinear if and only if the line segment $\overrightarrow{AB}$ is parallel to the line segment $\overrightarrow{AC}$. But

$$\overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A} = \begin{pmatrix} 5-3 \\ -4-5 \\ 3-7 \end{pmatrix} = \begin{pmatrix} 2 \\ -9 \\ -4 \end{pmatrix} \quad \text{and} \quad \overrightarrow{AC} = \overrightarrow{C} - \overrightarrow{A} = \begin{pmatrix} -5-3 \\ 41-5 \\ 22-7 \end{pmatrix} = \begin{pmatrix} -8 \\ 36 \\ 15 \end{pmatrix}.$$

   Since neither of these vectors are a scalar multiple of the other, they are not parallel, and hence the 3 points are **not** collinear.

2. To find the parametric vector equation of a plane we need two non-parallel direction vectors and a fixed vector, which lies on the plane. The point A must be on the plane, so we shall take that to be the fixed vector. As long as $\overrightarrow{AB}$ is not parallel to $\overrightarrow{AC}$ we can take our two direction vectors to be $\overrightarrow{AB}$ and $\overrightarrow{AC}$. (Note that if these two vectors were parallel, then A, B, C would be collinear and there wouldn't be a unique plane through A, B and

C.) First we calculate $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and find that

$$\overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A} = \begin{pmatrix} 3-1 \\ 4-2 \\ 2-1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \overrightarrow{AC} = \overrightarrow{C} - \overrightarrow{A} = \begin{pmatrix} 5-1 \\ 2-2 \\ 1-1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}.$$

Since $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are clearly not parallel, we can take them as our direction vectors. Hence, for $\lambda_1, \lambda_2 \in \mathbb{R}$

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$$

is a parametric vector equation for the plane through A, B and C.

Note. Remember, there are multiple possible parametric vector equations for a given plane, so you may get a different answer to this one if you used different direction vectors.

3. (i) The distance between A and B is given by $\left\|\overrightarrow{AB}\right\|$. Hence

$$d(A, B) = \left\|\overrightarrow{AB}\right\| = \sqrt{(5-1)^2 + (6-2)^2 + (4-3)^2} = \sqrt{16+16+1} = \sqrt{33}$$

(ii) The projection of a vector, $\mathbf{a}$, onto another (non-zero) vector, $\mathbf{b}$, is given by

$$\text{proj}_{\mathbf{b}}(\mathbf{a}) = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b}.$$

Therefore, if $\mathbf{a} \cdot \mathbf{b} = 0$ (i.e. the two vectors are orthogonal), we see that $\text{proj}_{\mathbf{b}}(\mathbf{a}) = 0\mathbf{b} = \mathbf{0}$.

Now, observe that

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{B} - \overrightarrow{A} \\ &= \begin{pmatrix} 5 \\ 6 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix}. \end{aligned}$$

Also,

$$
\begin{aligned}
\overrightarrow{AC} &= \overrightarrow{C} - \overrightarrow{A} \\
&= \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}
\end{aligned}
$$

Thus

$$
\begin{aligned}
\overrightarrow{AB} \cdot \overrightarrow{AC} &= \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \\
&= 4 - 4 \\
&= 0.
\end{aligned}
$$

Hence by our earlier comment about orthogonal vectors having zero vector as the projection, we have $\text{proj}_{\overrightarrow{AC}}\left(\overrightarrow{AB}\right) = \mathbf{0}$ (the zero vector in $\mathbb{R}^3$, i.e. $(0,0,0)^T$).

4. First we derive the parametric vector equations for the lines AY and BX.

   First, the line AY. The point Y is given by $\frac{2}{3}B$, since Y is on the line we can take it to be our fixed vector. We can take $\overrightarrow{A} - \overrightarrow{Y}$ as our direction vector. Hence, for $\lambda \in \mathbb{R}$, the parametric vector equation is

$$
\begin{aligned}
\mathbf{x} &= \overrightarrow{Y} + \lambda\left(\overrightarrow{A} - \overrightarrow{Y}\right) \\
&= \frac{2}{3}\begin{pmatrix} 0 \\ -3 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} 4 - \left(\frac{2}{3}(0)\right) \\ -4 - \left(\frac{2}{3}(-3)\right) \\ 8 - \left(\frac{2}{3}(-6)\right) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ -2 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ 12 \end{pmatrix}.
\end{aligned}
$$

   Second, the line BX. The method is the same as above, hence, for $\mu \in \mathbb{R}$, the parametric vector equation is

$$\begin{aligned}\mathbf{x} &= \overrightarrow{X} + \mu\left(\overrightarrow{B} - \overrightarrow{X}\right)\\ &= \begin{pmatrix}3\\-3\\6\end{pmatrix} + \mu\begin{pmatrix}-3\\0\\-12\end{pmatrix}.\end{aligned}$$

We will show that P lies on both the lines AY and BX. This implies that P is the intersection of AY and BX as the lines only intersect once. Hence we will solve $P = X + \mu(B - X)$ and $P = Y + \lambda(A - Y)$ for μ and λ (a solution for μ and λ will exist if and only if P lies on the respective line).

$$\begin{aligned}\overrightarrow{P} &= \overrightarrow{X} + \mu\left(\overrightarrow{B} - \overrightarrow{X}\right)\\ \begin{pmatrix}2\\-3\\2\end{pmatrix} &= \begin{pmatrix}3\\-3\\6\end{pmatrix} + \mu\begin{pmatrix}-3\\0\\-12\end{pmatrix}\\ \begin{pmatrix}-1\\0\\-4\end{pmatrix} &= \mu\begin{pmatrix}-3\\0\\-12\end{pmatrix} \implies \mu = \frac{1}{3}.\end{aligned}$$

As there is a solution for μ, P lies on BX.

$$\begin{aligned}\overrightarrow{P} &= \overrightarrow{Y} + \lambda\left(\overrightarrow{A} - \overrightarrow{Y}\right)\\ \begin{pmatrix}2\\-3\\2\end{pmatrix} &= \begin{pmatrix}0\\-2\\-4\end{pmatrix} + \lambda\begin{pmatrix}4\\-2\\12\end{pmatrix}\\ \begin{pmatrix}2\\-1\\6\end{pmatrix} &= \lambda\begin{pmatrix}4\\-2\\12\end{pmatrix} \implies \lambda = \frac{1}{2}\end{aligned}$$

As there is a solution for λ, P lies on AY. Hence P is the intersection of AY and BX.

MATH1131/1141 Algebra Test 1 2014 S1 v2b

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1. **Hints**

   (i) Let $\lambda = \frac{x-2}{3}$, $\lambda = \frac{y+1}{4}$ and $\lambda = \frac{z+3}{1}$. Make x, y and z the subject for each equation$^{(*)}$ and you should be able to arrive at the following parametric equation

   $$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}, \quad \text{where } \lambda \in \mathbb{R}.$$

   Alternatively, just do what the worked sample solution does.

   (ii) One way is to find the parametric equation for Π [1], equate this equation with the equation found in part (i) and then solve the system of three linear equations.

[1] Refer to Question 2 in Test 1 2014 S1 v1a

The faster way is to substitute the equations found earlier marked (*) into the Cartesian equation of Π. You should find that

$$P = (11, 11, 0)\,.$$

Worked Sample Solution

(i) Recall that if a line has Cartesian equation

$$\frac{x-a}{v_1} = \frac{y-b}{v_2} = \frac{z-c}{v_3},$$

then it has a parametric vector form

$$\mathbf{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \lambda \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

So by inspection, a parametric vector equation for ℓ is

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}, \quad \text{where } \lambda \in \mathbb{R}.$$

(ii) On ℓ, we have $x = 2 + 3\lambda$, $y = -1 + 4\lambda$, $z = -3 + \lambda$, for $\lambda \in \mathbb{R}$. Substitute these into the plane equation to obtain the value of λ at the point of intersection (a solution for λ will exist if and only if the line has an intersection with the plane):

$$\begin{aligned} 3\,(2+3\lambda) - 2\,(-1+4\lambda) - 4\,(-3+\lambda) &= 11 \\ \implies 6 + 9\lambda + 2 - 8\lambda + 12 - 4\lambda &= 11 \\ \implies -3\lambda &= -9 \\ \implies \lambda &= 3. \end{aligned}$$

Substitute into the line's parametric equation to find the point of intersection:

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} + 3 \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 11 \\ 11 \\ 0 \end{pmatrix}. \end{aligned}$$

Hence the coordinates of the point P are $(11, 11, 0)$.

2. **Hints**

(i) It can be found that[2]

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix}.$$

(ii) Area of a parallelogram spanned by two vectors is the Euclidean norm of the cross product of them, i.e. $\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\|$. It can be found that[2]

$$A = 3\sqrt{5} \text{ units}^2.$$

Worked Sample Solutions

(i) Note that

$$\begin{aligned}
\overrightarrow{AB} &= \overrightarrow{B} - \overrightarrow{A} \\
&= \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}.
\end{aligned}$$

Also,

$$\begin{aligned}
\overrightarrow{AC} &= \overrightarrow{C} - \overrightarrow{A} \\
&= \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.
\end{aligned}$$

[2]Refer to Question 3 in Test 1 2014 S1 v1a

Thus

$$
\begin{aligned}
\overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & -2 \\ 1 & 2 & 0 \end{vmatrix} \\
&= \begin{vmatrix} -1 & -2 \\ 2 & 0 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 2 & -2 \\ 1 & 0 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} \hat{k} \\
&= 4\hat{\imath} - 2\hat{\jmath} + 5\hat{k} \\
&= \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix}.
\end{aligned}
$$

(ii) The area of the parallelogram is the norm of the relevant cross product, i.e. it is

$$
\begin{aligned}
\left\| \overrightarrow{AB} \times \overrightarrow{AC} \right\| &= \left\| \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} \right\| \\
&= \sqrt{16 + 4 + 25} \\
&= \sqrt{45} \\
&= 3\sqrt{5}.
\end{aligned}
$$

3. **<u>Hints</u>**

(i) It can be found that[3]

$$
\operatorname{proj}_{\mathbf{v}} \left(\overrightarrow{PQ} \right) = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}.
$$

(ii) The shortest distance between a point and a line is the perpendicular distance. Form a right angled triangle to find that[3]

$$
d = \frac{\sqrt{10}}{2} \text{ units.}
$$

(iii) This can most easily be done by drawing a picture and noticing that the vector $\overrightarrow{OM}$ is found by simply adding the vectors $\overrightarrow{OP}$ and $\operatorname{proj}_{\mathbf{v}} \left(\overrightarrow{PQ} \right)$. You should find that the co-ordinates of the point M are

$$
\left(\frac{5}{2}, \frac{5}{2}, 4 \right).
$$

[3]Refer to Question 4 in Test 1 2014 S1 v1a

Worked Sample Solutions

(i) Note that

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{Q} - \overrightarrow{P} \\ &= \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.\end{aligned}$$

Since $\mathbf{v} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$, we have $\overrightarrow{PQ} \cdot \mathbf{v} = 3 + 2 + 2 = 7$ and $\|\mathbf{v}\|^2 = 9 + 1 + 4 = 14$. Thus

$$\begin{aligned}\operatorname{proj}_{\mathbf{v}}\left(\overrightarrow{PQ}\right) &= \frac{\overrightarrow{PQ} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} \\ &= \frac{7}{14}\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}\end{aligned}$$

(ii) By drawing a diagram and using Pythagoras' theorem, we see that the distance d is given by

$$d = \sqrt{\left\|\overrightarrow{PQ}\right\|^2 - \left\|\operatorname{proj}_{\mathbf{v}}\left(\overrightarrow{PQ}\right)\right\|^2}$$

Now, since $\overrightarrow{PQ} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, we have $\left\|\overrightarrow{PQ}\right\|^2 = 1 + 4 + 1 = 6$. Also, from the answer to the previous question, we have that

$$\begin{aligned}\left\|\operatorname{proj}_{\mathbf{v}}\left(\overrightarrow{PQ}\right)\right\|^2 &= \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 1^2\\ &= \frac{9}{4} + \frac{1}{4} + \frac{4}{4}\\ &= \frac{14}{4}\\ &= \frac{7}{2}.\end{aligned}$$

Thus

$$\begin{aligned}d &= \sqrt{\left\|\overrightarrow{PQ}\right\|^2 - \left\|\operatorname{proj}_{\mathbf{v}}\left(\overrightarrow{PQ}\right)\right\|^2} = \sqrt{6 - \frac{7}{2}}\\ &= \sqrt{\frac{5}{2}} = \sqrt{\frac{10}{4}}\\ &= \frac{\sqrt{10}}{2}.\end{aligned}$$

MATH1131/1141 Algebra Test 1 2014 S1 v3a

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---

1. (i) The line AB passes through the point A and has a gradient vector which is parallel to $\overrightarrow{A} - \overrightarrow{B}$. Therefore

$$\mathbf{x} = \overrightarrow{A} + \lambda\left(\overrightarrow{A} - \overrightarrow{B}\right) = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -1 \\ 3 \end{pmatrix}$$

where $\lambda \in \mathbb{R}$.

(ii) Suppose that (x_1, x_2, x_3) is a point on the line AB. Then, for some real number λ we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -1 \\ 3 \end{pmatrix}.$$

Rewriting this equation coordinate-wise yields

$$x_1 = 3 - 3\lambda,$$
$$x_2 = 2 - \lambda,$$
$$x_3 = 1 + 3\lambda.$$

By solving for λ we obtain

$$\lambda = \frac{3 - x_1}{3},$$
$$\lambda = 2 - x_2,$$
$$\lambda = \frac{x_3 - 1}{3}.$$

Thus, by eliminating λ from the above equations we obtain

$$\frac{x_1 - 3}{-3} = \frac{x_2 - 2}{-1} = \frac{x_3 - 1}{3},$$

which is the Cartesian form of the line AB.

2. It can be found that[1]

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1-2\lambda+\mu}{7} \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} \frac{1}{7} \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{-2}{7} \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} \frac{1}{7} \\ 0 \\ 1 \end{pmatrix},$$

is a parametric vector equation for the plane. Also, two non-parallel, non-zero vectors which are parallel to the plane are

$$\begin{pmatrix} \frac{-2}{7} \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} \frac{1}{7} \\ 0 \\ 1 \end{pmatrix}.$$

3. (i) The cosine of the angle θ between two vectors, say $\mathbf{u}$ and $\mathbf{v}$, is given by

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}.$$

In this question, $\angle BAC$ is the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Substituting $\mathbf{u} = \vec{B} - \vec{A}$ and $\mathbf{v} = \vec{C} - \vec{A}$ yields

$$\cos(\angle BAC) = \frac{\left(\vec{B} - \vec{A}\right) \cdot \left(\vec{C} - \vec{A}\right)}{\left\|\vec{B} - \vec{A}\right\| \left\|\vec{C} - \vec{A}\right\|} = \frac{1}{\sqrt{91}}.$$

[1] Refer to Question 2 in Test 1 2014 S1 v1a

(ii) The projection of a vector, $\mathbf{u}$, onto another vector, $\mathbf{v}$, is given by

$$\mathrm{proj}_{\mathbf{v}}(\mathbf{u}) = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}.$$

Substituting $\mathbf{u} = \vec{B} - \vec{A}$ and $\mathbf{v} = \vec{C} - \vec{A}$ yields

$$\mathrm{proj}_{\overrightarrow{AC}}\left(\overrightarrow{AB}\right) = \frac{\left(\vec{B}-\vec{A}\right)\cdot\left(\vec{C}-\vec{A}\right)}{\left(\vec{C}-\vec{A}\right)\cdot\left(\vec{C}-\vec{A}\right)}\left(\vec{C}-\vec{A}\right) = \frac{1}{13}\begin{pmatrix}4\\-3\\1\end{pmatrix}.$$

4. (i) Drawing a picture makes this question much easier.

The line OC goes through the origin and has direction parallel to the line segment $\overrightarrow{OC}$. By drawing a relevant parallelogram, it is easy to see that $\overrightarrow{OC}$ is simply the vector addition of vectors $\overrightarrow{OA}$ and $\overrightarrow{AC}$. But since opposite sides of a parallelogram are equal, then it is clear that $\overrightarrow{AC} = \overrightarrow{OB}$. And so, we can conclude that $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB} = \mathbf{a} + \mathbf{b}$. Hence OC has parametric vector form

$$\mathbf{x} = \lambda(\mathbf{a}+\mathbf{b}),$$

where $\lambda \in \mathbb{R}$.

The line AB goes through the point A and has direction parallel to the line segment $\overrightarrow{AB}$. The line segment $\overrightarrow{AB}$ is parallel to the vector $\mathbf{b}-\mathbf{a}$. Hence AB has parametric vector form

$$\mathbf{x} = \mathbf{a} + \mu(\mathbf{b}-\mathbf{a}),$$

where $\mu \in \mathbb{R}$. Note that there are a number of similar parameterisations for this question, the solution listed is not the only solution.

(ii) There are a number of ways to do this question, so here are two:

1. This solution is more geometrically inclined.[1] Note that the diagonals of any parallelogram, which in this case are the vectors $\overrightarrow{OC}$ and $\overrightarrow{AB}$, bisect each other. Hence the intersection is given by the midpoint of $\overrightarrow{OC}$, which is simply $\frac{\mathbf{a}+\mathbf{b}}{2}$.

2. This solution is more algebraic and a little more involved, but it is more rigorous. The intersection of the two lines can be obtained by solving the line equations

[1]This method was probably not intended by the examiner, due to part (iii), but is still good to have an intuitive understanding of this question.

simultaneously:

$$\begin{aligned}\lambda(\mathbf{a}+\mathbf{b}) &= \mathbf{a}+\mu(\mathbf{b}-\mathbf{a})\\ (\lambda+\mu-1)\mathbf{a}+(\lambda-\mu)\mathbf{b} &= \mathbf{0}\end{aligned}$$

Since $\mathbf{a}$ is not parallel to $\mathbf{b}$, the only possible solution is one where $(\lambda+\mu-1) = (\lambda-\mu) = 0$. Solving for λ and μ yields the solutions $\lambda = \mu = \frac{1}{2}$. By substituting this into the line equations, we obtain the coordinates for P, namely $\frac{\mathbf{a}+\mathbf{b}}{2}$.

(iii) Calculating $\overrightarrow{OP}$ and $\overrightarrow{PC}$ yields:

$$\begin{aligned}\overrightarrow{OP} &= \frac{\mathbf{a}+\mathbf{b}}{2} - 0 = \frac{\mathbf{a}+\mathbf{b}}{2},\\ \overrightarrow{PC} &= \overrightarrow{OC} - \overrightarrow{OP} = \mathbf{a}+\mathbf{b} - \frac{\mathbf{a}+\mathbf{b}}{2} = \frac{\mathbf{a}+\mathbf{b}}{2}.\end{aligned}$$

Hence, as $\overrightarrow{OP} = \overrightarrow{PC}$, $\left\|\overrightarrow{OP}\right\| = \left\|\overrightarrow{PC}\right\|$.

Calculating $\overrightarrow{PA}$ and $\overrightarrow{PB}$ yields:

$$\begin{aligned}\overrightarrow{PA} &= \overrightarrow{OA} - \overrightarrow{OP} = \mathbf{a} - \frac{\mathbf{a}+\mathbf{b}}{2} = \frac{\mathbf{a}-\mathbf{b}}{2},\\ \overrightarrow{PB} &= \overrightarrow{OB} - \overrightarrow{OP} = \mathbf{b} - \frac{\mathbf{a}+\mathbf{b}}{2} = \frac{\mathbf{b}-\mathbf{a}}{2}.\end{aligned}$$

Hence, as $\overrightarrow{PA} = -\overrightarrow{PB}$, $\left\|\overrightarrow{PA}\right\| = \left\|\overrightarrow{PB}\right\|$.

This question basically showed that the diagonals of any parallelogram bisects each other.

MATH1131/1141 Algebra Test 1 2014 S1 v4a

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We cannot guarantee that our answers are correct – please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are often multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

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1. (i) The co-ordinates of $\mathbf{t}$ can be found by dividing the interval AB in to thirds. By drawing a picture, it can be seen that T lies "two-thirds of the way from A to B". So,

$$\overrightarrow{OT} = \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 4 \\ 5 \\ -5 \end{pmatrix}.$$

And so, the coordinates $\mathbf{t}$ of the point T are

$$\mathbf{t} = \left(\frac{11}{3}, \frac{16}{3}, -\frac{1}{3}\right).$$

(ii) Draw out a rough diagram to have a general idea of how the vectors look. Use the fact that the quadrilateral was named in cyclic order, so we know through simple vector addition that $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$. But since, opposite sides of a parallelogram are equal, we can write this as $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{BC}$. And so, the coordinates $\mathbf{d}$ of the point D are

$$\mathbf{d} = (4, -8, 7)\,.$$

2. It can be found that[1]

$$\mathbf{x} = \begin{pmatrix} \frac{8}{3} \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{3} \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -\frac{2}{3} \\ 0 \\ 1 \end{pmatrix}, \quad \text{where } \lambda, \mu \in \mathbb{R}$$

is a parametric vector equation for the plane. Also, two non-parallel non-zero vectors parallel to the plane are

$$\begin{pmatrix} \frac{1}{3} \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} -\frac{2}{3} \\ 0 \\ 1 \end{pmatrix}.$$

3. By using the formula or evaluating the relevant pseudo-determinant[2]

$$\begin{pmatrix} 5 \\ 8 \\ -7 \end{pmatrix}.$$

4. (i) It can be found that[3]

$$\begin{pmatrix} 1 \\ 3/2 \\ 1/2 \end{pmatrix}.$$

(ii) Draw a diagram to help see which vector you need to determine the shortest distance[3]

$$d = \sqrt{\frac{3}{2}}.$$

(iii) Drawing a diagram will help us see that $\overrightarrow{OM} = \overrightarrow{OP} + \overrightarrow{PM}$. But $\overrightarrow{PM}$ is precisely the projection of $\overrightarrow{PQ}$ onto $\mathbf{v}$, which we have just found in part (i). From this, it can be found that the co-ordinates of the point M are

$$\left(2, \frac{7}{2}, \frac{7}{2}\right).$$

[1] Refer to Question 2 in Test 1 2014 S1 v1a
[2] Refer to Question 3 in Test 1 2014 S1 v1a
[3] Refer to Question 4 in Test 1 2014 S1 v1a