MATH1231/41 Revision

Calculus Part 1

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Functions of Several Variables

Partial Differentiation

Formal Definition:

$$F_x(x,y) = \lim_{h \to 0} \frac{F(x+h,y) - F(x,y)}{h}$$

$$F_y(x,y) = \lim_{h \to 0} \frac{F(x,y+h) - F(x,y)}{h}$$

- In partial differentiation, we seek to define the gradient as a combination multiple 'partial derivatives'.
- When differentiating with respect to each variable, treat all other variables as constants

Functions of Several Variables

Partial Differentiation

Example: Suppose $F(x, y) = x^2y + 2y + 4$. Find the respective partial derivatives of F.

- F_x: Treat y as constant and differentiate F with respect to x

$$F_x = 2xy$$

- F_x: Treat x as constant and differentiate F with respect to y

$$F_y = x^2 + 2$$

Functions of Several Variables

Tangential planes

- In $\mathbb{R}^2$ the tangent line describes the gradient of a function evaluated at a specific point.
- Analogously in $\mathbb{R}^3$ we say that the tangent *plane* inherits these same properties
- Since the tangent plane is a surface of $\mathbb{R}^3$, its gradient is described by the linear combination of 2 partial derivatives.

Functions of Several Variables

Tangential planes

If the surface has a tangent plane at the point (x_0, y_0, z_0) then the tangent plane is given by the equation:

$$z = z_0 + F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0)$$

and a normal vector to the surface at(x_0, y_0, z_0) is given by:

$$\begin{pmatrix} F_x(x_0, y_0) \\ F_y(x_0, y_0) \\ -1 \end{pmatrix}$$

Functions of Several Variables

Total Differential Approximation

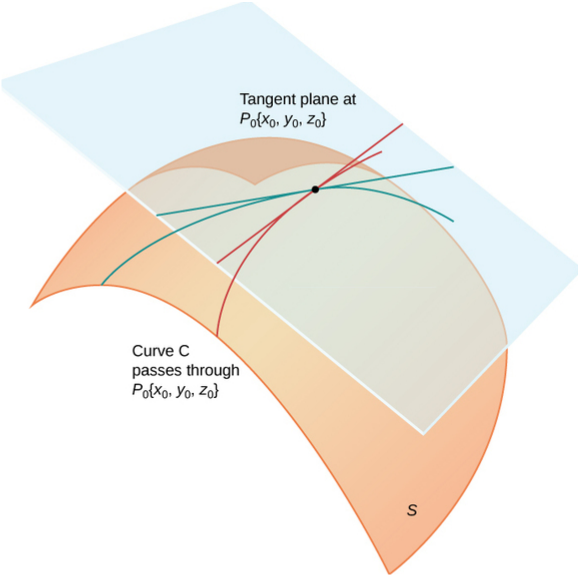
The total differential approximation is given by:

$$\Delta F \approx F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0)$$

Where ΔF denotes the approximate change in output of a function given its inputs

- This formula is only effective for small changes in x and y
- Intuitively, this means the point of approximation is very close to the point of contact between the tangential plane and surface
- As $\Delta x = x - x_0$ and $\Delta y = y - y_0$ gets smaller, we obtain a more precise approximation.

Functions of Several Variables



Multivariable Chain rule

Multivariable Chain rule

Suppose that F is a function of 2 variables and that x and y are both functions themselves of one variable. Then

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt}$$

- This is an example of a chain rule for a function of two variables.
- The idea here is that we compute each partial derivative F followed by the derivative of the composite functions.

Multivariable Chain rule

$$\begin{tikzcd}
 & x \arrow[dr, "\frac{dx}{dt}"] & \\
F \arrow[ur, "\frac{\partial F}{\partial x}"] \arrow[dr, "\frac{\partial F}{\partial y}"'] & & t \\
 & y \arrow[ur, "\frac{dy}{dt}"'] &
\end{tikzcd}$$

Multivariable Chain rule

- A chain diagram can help visualise how the multivarable chain rule operates
- Each arrow in the chain diagram points the function to each of its variable
- We then sum up all the paths that connect our desired variables.

Multivariable Chain rule

Multivariable chain rule

Suppose that $x(t) = 2t$ and $y(t) = 3t + 1$ are functions which represent the displacement of a particle on the $X - Y$ plane. The height of the particle is given by H where

$$H(x, y) = 2x + 5y.$$

Find an equation to model the rate at which the particles height changes.

Multivariable Chain rule

Multivariable Chain rule

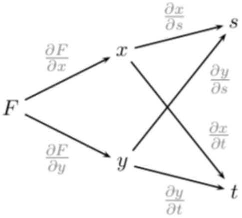
- Now suppose F is a function of 1 variable, u, which however is instead a function of 2 variables x and y
- Observe how the behaviour in the chain diagram changes.
- We can see that although there 2 paths in total, there is only a single path from $F \to x$ and $F \to y$

$$\begin{tikzcd}
 & & x \\
F \arrow[r, "\frac{dF}{du}"] & u \arrow[ur, "\frac{\partial u}{\partial x}"] \arrow[dr, "\frac{\partial u}{\partial y}"'] & \\
 & & y
\end{tikzcd}$$

Multivariable Chain rule

Multivariable Chain rule

- Yet again suppose F is a function of 2 variables, x and y, which themselves are also functions of 2 variables s and t
- We can see there are 2 possible paths to sum from $F \to s$ and likewise 2 for $F \to t$



Multivariable Chain rule

Multivariable Chain rule

- We consider layer by layer, the variables of the outer function. And for each variable that happens to also be a function in and of itself, we repeat the procedure.
- In doing so, we obtain an intuitive map that links all the derivatives can be represented diagrammatically
- Finally we sum all possible paths that link our desired variables.

Integration Techniques

Powers of Sine and Cosine

Integrating of the form: $\int \sin^m x \cos^n x dx$

Split into 2 cases

- (i) Either m is odd, n is even (and vice versa). Or they are both odd.
- (ii) Both m and n are even.

Integration Techniques

(i) Even-Odd (Odd-Odd) case

Overall Strategy: Convert as much of the odd powered function using the Pythagorean identity: $\sin^2 x + \cos^2 x = 1$. (Choose one if both are odd).

$$\begin{aligned}
&\int \sin^3 x \cos^2 x dx \\
&= \int \sin x(1 - \cos^2 x) \cos^2 x dx \\
&= \int \sin x(\cos^2 x - \cos^4 x) dx \\
&= -\frac{1}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C
\end{aligned}$$

Integration Techniques

(ii) Even-Even case

Overall Strategy: Reduce to linear form using trig identities:

- $1 + \cos 2x = 2\cos^2 x$
- $1 - \cos 2x = 2\sin^2 x$

See separate solutions:

$$\int \sin^2 x \cos^2 x dx$$

Integration Techniques

Product to Sum

When integrating multiple angles of sine and cosine we use the following set of formulas.

1. $\sin A \cos B = \frac{1}{2}\left[\sin\left(A+B\right)+\sin\left(A-B\right)\right]$
2. $\cos A \cos B = \frac{1}{2}\left[\cos\left(A-B\right)+\cos\left(A+B\right)\right]$
3. $\sin A \sin B = \frac{1}{2}\left[\cos\left(A-B\right)-\cos\left(A+B\right)\right]$

This enables us to express a product as a sum instead. Thus, now we can integrate them separately.
One may derive the formula's through **Euler's identity** or simply through **Trig expansion** identities of Sine and Cosine

Integration Techniques

Product to Sum

Evaluate:

$$\int \cos 5x \cos 3x dx$$

Identity 2 implies that:

$$\begin{aligned}\int \cos 5x \cos 3x dx &= \frac{1}{2}\int \cos(5x-3x)+\cos(5x+3x)dx\\ &= \frac{1}{2}\int \cos(2x)+\cos(8x)dx\\ &= \frac{\sin 2x}{4}+\frac{\sin 8x}{16}+C.\end{aligned}$$

Integration Techniques

Reduction Formulae

- The concept of Reduction Formulae arises from recursively expressing an integral in terms of itself.
- Effectively this *reduces* the problem into a trivial case, from which we can backtrack to the initial integral
- A nice trick is to follow the **LIATE** order

Integration Techniques

Reduction Formulae

- **L** - Logarithm
- **I** - Inverse Trig
- **A** - Algebra
- **T** - Trig
- **E** - Exponential

Notice that at the top of the list, we have Logarithmic functions which are the most difficult to integrate.
At the bottom we have, exponential functions which are the easiest to integrate.

Integration Techniques

Example

Obtain a recursive relation for:

$$\int x^n e^{x^2} \, dx$$

Integration Techniques

Expression in integrand	Trigonometric substitution	Hyperbolic substitution
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$x = a\tanh\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$x = a\sinh\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$x = a\cosh\theta$

Trig and hyperbolic substitutions

The table summarises which substitutions to apply, depending on the expression to be integrated.
Intuitively, we can verify these choices as the expression in integrand simplifies after applying the associated substitution.

Integration Techniques

Example

Evaluate:

$$\int \sqrt{1-x^2}dx$$

Integration Techniques

Partial Fraction decomposition

Generally speaking, a rational function is of the form

$$f(x) = \frac{p(x)}{q(x)}$$

- We say that f is *proper* if the **degree** of the denominator q is *greater* than that of the numerator p.
- We say that f is *improper* if the **degree** of the denominator q is *less* than that of the numerator p.

Integration Techniques

Partial Fraction decomposition

- If the rational function is improper. We use long division to bring it into proper form.
- When in proper rational form, the denominator can either be **reducible** or **irreducible**.
- If the denominator is **irreducible**, that means it has no real linear factors.
- In the next slide we will **Partial fraction decomposition** for integrating such rational functions of proper form.

Integration Techniques

Partial Fraction decomposition

It can be shown that every proper rational function f can be written as a unique sum of functions of the form:

$$\frac{A}{(x-a)^k}$$

$$\frac{Bx+C}{(x^2+bx+c)^k}$$

Where x^2+bx+c is *irreducible*

Integration Techniques

Partial Fraction decomposition

Case 1: The denominator splits into distinct linear factors.

Find the partial fraction decomposition of

$$\frac{7x-1}{(x-3)(x+1)}$$

1. Since the denominator only has linear factors, we know the decomposition only takes form 1.
2. $\frac{7x-1}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$
3. We then proceed to bring both sides to the same form by common denominating and comparing numerators.
4. $7x - 1 \equiv A(x+1) + B(x-3)$
5. Congruent sign since the statement is true for all $x \in \mathbb{R}$

Integration Techniques

Comparing Numerators: $7x - 1 \equiv A(x+1) + B(x-3)$

1. We can now evaluate the constants A and B by substituting certain values of x.
2. $x = -1 \implies -7 - 1 = A(0) + B(-1-3)$
$$\therefore B = 2$$
3. $x = 3 \implies 21 - 1 = A(3+1) + B(0)$
$$\therefore A = 5$$

$$\frac{7x-1}{(x-3)(x+1)} = \frac{5}{x-3} + \frac{2}{x+1}$$

Integration Techniques

Partial Fraction decomposition

Case 2: The denominator has a repeated linear factor. Examples of two such rational functions and the form of their partial fraction decompositions are given below

$$\frac{x^2+1}{(x+4)^3} + \frac{A}{x+4} + \frac{B}{(x+4)^2} + \frac{C}{(x+4)^3}$$

$$\frac{x^2-2}{(x-1)(x-2)^2} + \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

The important stage is identifying the decomposition form. After that we simply follow the exact same strategy as indicated in the previous slide.

Integration Techniques

Partial Fraction decomposition

Case 3: The denominator has an irreducible quadratic factor.
Examples of two such rational functions and the form of their partial fraction decompositions are given below

$$\frac{x^2+x}{(x^2+9)^3} + \frac{Ax+B}{x^2+9} + \frac{Cx+D}{(x^2+9)^2} + \frac{Ex+F}{(x^2+9)^3}$$

$$\frac{x^3-2x+4}{(x-2)(x^2+x+1)^2} + \frac{A}{x-2} + \frac{Bx+C}{x^2+x+1} + \frac{Dx+E}{(x^2+x+1)^2}$$

As before, the constants appearing in each example can be determined by algebra.

Ordinary Differential Equations

Separable ODE's

As the name suggests, in separable ODE's we aim to solve the ODE by separating the variables

1. This is generally the simplest kind of ODE you will encounter. All we need to do is use algebraic manipulation to bring all like variables to a common side of the equation.
2. We then integrate both sides with respect to each variable
3. Solve an IVP, if necessary
4. Further algebraic manipulation may be required to obtain the final solution.

Ordinary Differential Equations

Separable ODE's

Given $y(0) = 1$ solve,

$$\frac{dy}{dx} = y^2(1 + x^2)$$

Ordinary Differential Equations

1) First we separate the variables for integrating

$$\frac{1}{y^2}dy = (1+x^2)dx$$

$$\int \frac{1}{y^2}dy = \int (1+x^2)dx$$

$$-\frac{1}{y} = x + \frac{x^3}{3} + C$$

Ordinary Differential Equations

2) Now solve the IVP by imposing initial conditions: $y(0) = 1$ that is when $x = 0$, $y = 1$

$$-\frac{1}{1} = 0 + \frac{0}{3} + C \implies C = -1$$
$$-\frac{1}{y} = x + \frac{x^3}{3} - 1$$

Ordinary Differential Equations

3) Manipulate to make desired variable the subject and obtain the explicit solution.

$$\begin{aligned} y &= \frac{-1}{x + \frac{x^3}{3} - 1} \\ &= \frac{-3}{3x + x^3 - 3} \end{aligned}$$

Ordinary Differential Equations

First order linear ODE's

These ODE's can always be written in the form:

$$\frac{dy}{dx} + f(x)y = g(x)$$

1. Calculate the integrating factor $h(x) = e^{f(x)dx}$
2. Multiply by integrating factor to obtain

$$h(x)\frac{dy}{dx} + h(x)f(x)y = g(x)h(x).$$

Ordinary Differential Equations

First order linear ODE's

1. Using the product rule we can rewrite the LHS:

$$\frac{d}{dx}(h(x)y) = g(x)h(x)$$

2. Integrate both sides and then rearrange to make y the subject

Ordinary Differential Equations

First order linear ODE”s

Solve

$$\frac{dy}{dx} + 3y = e^{-x}$$

Ordinary Differential Equations

1) Since the ODE is already in the appropriate form the integrating factor $h(x)$ is given by

$$h(x) = e^{\int 3dx} = e^{3x}$$

2) Multiplying the ODE by the integrating factor gives,

$$e^{3x}\frac{dy}{dx} + 3ye^{3x} = e^{2x}$$

3) Using the product rule backwards we obtain,

$$\frac{d}{dx}(e^{3x}y) = e^{2x}$$

Ordinary Differential Equations

4) Integrating both sides yields,

$$e^{3x}y = \frac{1}{2}e^{2x} + C$$

5) Dividing by e^{3x} to make y the subject,

$$y = \frac{1}{2}e^{-x} + Ce^{-3x}$$

Ordinary Differential Equations

Exact ODE's

ODE's of the form:

$$F(x,y)dx + G(x,y)dy = 0$$

are **exact** if,

$$\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x}$$

The solution is given by:

$$H(x,y) = C$$

where H is a function satisfying the equations

$$\frac{\partial H}{\partial x} = F \qquad and \qquad \frac{\partial H}{\partial y} = G$$

Ordinary Differential Equations

Exact ODE's

Show the differential equation

$$\frac{dy}{dx} = -\frac{2x + y + 1}{2y + x + 1}$$

Ordinary Differential Equations

1) Rewrite the differential equation in the appropriate form

$$(2x + y + 1)dx + (2y + x + 1)dy = 0$$

2) Writing $F = 2x + y + 1$ and $G = 2y + x + 1$. Then

$$\frac{\partial F}{\partial y} = 1 = \frac{\partial G}{\partial x}$$

3) Therefore, the ODE is exact and there exists a function H satisfying

$$\frac{\partial H}{\partial x} = F(x, y) = 2x + y + 1$$
$$\frac{\partial H}{\partial y} = G(x, y) = 2y + x + 1.$$

Ordinary Differential Equations

4) Now, to Find H we integrate both Functions. Remember when integrating with respect to a certain variable, treat all other variable's as constants

$$H(x,y) = x^2 + xy + x + C_1(y)$$
$$H(x,y) = y^2 + xy + y + C_2(x)$$

5) We know that integrating either of the partial derivative functions F and G should yield the same primitive function H

$$H(x,y) = x^2 + xy + y^2 + x + y.$$

6) Hence the solution to the differential equation is given by

$$x^2 + xy + y^2 + x + y = C$$

Ordinary Differential Equations

ODE's by Substitution 1241

Use the substitution $y(x) = xv(x)$ to solve the differential equation

$$\frac{dy}{dx} = \frac{xy - y^2}{x^2}$$

NOTE: The most crucial step is to substitute out the original differential expression. After that it is just a matter of algebraic manipulation to bring it into a familiar ODE form.

Ordinary Differential Equations

ODE's by Substitution 1241

To substitute out $\frac{dy}{dx}$ we need to use to product rule,

$$\frac{dy}{dx} = \frac{d}{dx}(xv) = v + x\frac{dv}{dx}$$

Hence, our differential equation becomes,

$$\begin{aligned} v + x\frac{dv}{dx} &= \frac{x(xv) - (xv)^2}{x^2} \\ v + x\frac{dv}{dx} &= v - v^2 \end{aligned}$$

Thus we obtain the separable ODE

$$-\frac{dv}{v^2} = \frac{dx}{x}$$

Ordinary Differential Equations

Modelling first order ODE's

Some pointers when building a mathematical model:

1. Deduce what information needs to be extracted
2. Consider which variables are independent and dependant in the system
3. Analyse the relationship and behaviour between theses variables, which may be described by a differential equation.

Ordinary Differential Equations

Modelling first order ODE's

A tank can hold 100 litres. Initially it holds 50 litres of pure water. Brine, which contains 2 grams of salt per litre, is run in at the rate of 3 litres per minute. The mixture, which is stirred continuously, is run off at 1 litre per minute. Let $x(t)$ denote the mass of salt (in grams) present in the tank after t minutes. Set up a differential equation in x and t to model the system

Ordinary Differential Equations

Modelling first order ODE's

1) Identify what we want to find. In this case we want to derive a formula for the mass of the salt present in the tank.
2) We know that the rate at which the salt's concentration changes is given by

$$\frac{dx}{dt} = (rate\,of\,inflow) - (rate\,of\,outflow)$$

3) The rate of inflow is given as $2 \times 3 = 6$g/min. First note the total volume of liquid in the container is given by

$$50 + 3t - t = 50 + 2t$$

Ordinary Differential Equations

Modelling first order ODE's

4) Therefore, the *proportion* of salt in the container is

$$\frac{x(t)}{50+2t}$$

5) Since, the rate of outflow of the liquid is 1L/min it follows that the rate of outflow of salt is

$$\frac{x(t)}{50+2t} \times 1$$

6) Therefore our separable ODE is governed by

$$\frac{dx}{dt} = 6 - \frac{x(t)}{50+2t}, \quad x(0) = 0$$

MATH1231/1241 Revision

Calculus Part 2

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Today's plan

1. Chapter 3.5: Second order ODEs
   - Homogeneous solutions
   - Non-homogeneous solutions
2. Chapter 4: Taylor polynomials
   - Taylor series and theorem
   - Sequences and series
3. Chapter 5: Averages, arc length, speed, surface area
   - Average value of a function
   - Arc length of curves
   - Speed of a particle
   - Surface area

Characteristic polynomial

Characteristic polynomial of second order ODEs

Suppose we have a second order differential equation:

$$ay'' + by' + cy = 0.$$

Then the characteristic polynomial takes the form:

$$a\lambda^2 + b\lambda + c = 0.$$

Notice the coefficients and degree of the λs. They match the degree and coefficient of the differential equation.

Characteristic polynomial

Characteristic polynomial - Distinct roots

When solving for λ in our characteristic polynomial and we have two different **real** values for λ, then the homogeneous solution takes the form:

$$y_H(x) = c_1 \cdot e^{\lambda_1 x} + c_2 \cdot e^{\lambda_2 x}.$$

Characteristic polynomial

Distinct roots

Solve the differential equation:

$$y'' + 5y' - 6y = 0.$$

Solution for reference

We write the characteristic polynomial for the differential equation:

$$\lambda^2 + 5\lambda - 6 = 0.$$

Solving for λ, we can factorise it as:

$$(\lambda + 6)(\lambda - 1) = 0.$$

Since we have distinct roots, our solution takes the form:

$$y_H(x) = c_1 \cdot e^{-6x} + c_2 \cdot e^{x}.$$

Characteristic polynomial

Characteristic polynomial - Non distinct roots

When solving for λ in our characteristic polynomial and we have only **one real** value for λ, then the homogeneous solution takes the form:

$$y_H(x) = c_1 \cdot e^{\lambda x} + c_2 \cdot xe^{\lambda x}.$$

Characteristic polynomial

Non distinct roots

Solve the differential equation:

$$y'' - 2y' + y = 0.$$

Solution for reference

We write the characteristic polynomial for the differential equation:

$$\lambda^2 - 2\lambda + 1 = 0.$$

Solving for λ, we can factorise it as:

$$(\lambda - 1)^2 = 0.$$

Since we have only one real root, our solution takes the form:

$$y_H(x) = c_1 \cdot e^x + c_2 \cdot xe^x.$$

Characteristic polynomial

Characteristic polynomial - Complex roots

When solving for λ in our characteristic polynomial and we have **complex** values for λ, then the homogeneous solution takes the form:

$$y_H(x) = e^{\mathrm{Re}(\lambda)x}\left(c_1 \cos\left(\mathrm{Im}(\lambda)x\right) + c_2 \sin\left(\mathrm{Im}(\lambda)x\right)\right).$$

Characteristic polynomial

Complex roots

Solve the differential equation:

$$y'' + y = 0.$$

Solution for reference

We write the characteristic polynomial for the differential equation:

$$\lambda^2 + 1 = 0.$$

Solving for λ, we have:

$$\lambda = \pm i.$$

Since we have only complex roots, our solution takes the form:

$$y_H(x) = e^{0x}\left(c_1 \cos(x) + c_2 \sin(x)\right)$$

which simplifies to

$$y_H(x) = c_1 \cos(x) + c_2 \sin(x).$$

Non-homogeneous equation

What if the RHS was not 0? This is a case of a **non-homogeneous** differential equation. We shall learn techniques for finding solutions to a non-homogeneous equation.

Non-homogeneous equation

To solve a non-homogeneous equation, we first solve the homogeneous equation by setting the RHS to 0 and find two solutions using one of the three cases we saw before. We then look for a particular solution.

Non-homogeneous equation

Particular solution: exponential solutions

Suppose the RHS was of the form e^{ax}. Then we look for the particular solution of the form:

$$y_p = Ae^{ax}.$$

If y_p is a solution to the homogeneous equation, then we multiply our particular solution by x. We repeat this process until we have a particular solution that's not in the homogeneous solution.

Non-homogeneous equation

Exponential solution

Solve the non-homogeneous second order differential equation:

$$y'' - 2y' + y = e^x.$$

Solution for reference - part 1

We solve the homogeneous case by finding the characteristic polynomial. I'll leave that to you to do. Assuming everything worked out, you should end up with the homogeneous solution being:

$$y_H(x) = c_1 \cdot e^x + c_2 \cdot xe^x.$$

To find the particular solution, we look at the RHS. Since it's an exponential form, we guess:

$$y_p = Ae^x.$$

But this is a solution in the homogeneous case. Multiplying by x gives us:

$$y_p = Axe^x.$$

But this is also a solution in the homogeneous case. Multiplying by x gives us:

$$y_p = Ax^2e^x.$$

Solution for reference - part 2

The next step is to plug our guess for y_p into the differential equation to find A. Doing so gives us:

$$Ae^x(x^2 + 4x + 2 - 2x(x+2) + x^2) = e^x.$$

Simplifying the expression gives:

$$2A = 1 \implies A = 1/2.$$

So the particular solution is:

$$y_p = \frac{1}{2}x^2e^x.$$

So the general solution is:

$$\begin{aligned} y(x) &= y_H(x) + y_p(x) \\ &= c_1 \cdot e^x + c_2 \cdot xe^x + \frac{x^2e^x}{2}. \end{aligned}$$

Non-homogeneous equation

Particular solution: trigonometric solutions

Suppose the RHS was of the form $\cos(ax)$ or $\sin(ax)$. Then we look for the particular solution of the form:

$$y_p = A\cos(ax) + B\sin(ax).$$

Note that we look for both cos and sin, not just either.

Non-homogeneous equation

Trigonometric solution

Solve the second order differential equation:

$$y'' + y = \cos(x).$$

Solution for reference

Solving the homogeneous case gives:

$$y_H(x) = c_1 \cdot \cos(x) + c_2 \cdot \sin(x).$$

To find the particular solution, we look at the RHS. Since it's a trigonometric form, we guess:

$$y_p = A\cos(x) + B\sin(x).$$

But this is a solution in the homogeneous case. Multiplying by x gives us:

$$y_p = x\left(A\cos(x) + B\sin(x)\right).$$

To find our solution, we substitute y_p into our differential equation and solve for A and B. Here, we end up with $A = 0$ and $B = 1/2$ after all that work, which gives us:

$$y_p(x) = \frac{1}{2}x\sin(x).$$

Non-homogeneous equation

Particular solution: polynomial solutions

Suppose the RHS was of a polynomial form with degree r. Then we look for the particular solution of a polynomial form with degree r:

$$y_p = a_0 + a_1 x + \cdots + a_r x^r.$$

Non-homogeneous equation: summary

To solve any non-homogeneous second order ODE:

1. Find the homogeneous solution, $y_H(x)$, by setting the RHS $= 0$ and building a characteristic polynomial.
2. Check the RHS and use the appropriate form of your particular solution based on the RHS.
3. Check if the particular solution is already a homogeneous solution. If it is, then multiply by x and check again. If not, we move on.
4. Differentiate twice and substitute into the differential equation to find any constants. This is $y_p(x)$.
5. The general solution is just the sum of the homogeneous and the particular solution:

$$y(x) = y_H(x) + y_p(x).$$

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Taylor series

Definition 4.1.1: Taylor series

Suppose that f is differentiable n times at a. Then the **Taylor polynomial** p_n about $x = a$ is defined as:

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

1. Note that $f^{(k)}(x)$ simply means k^{th} derivative of f.
2. We can also write it succinctly as:

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

3. If $a = 0$, we call this the **Maclaurin series**.

Taylor series

Example 4.1.1

Suppose that $f(x) = \ln x$. Find the second order Taylor polynomial for f about $x = 1$.

Since it's second order, we only need to find its first and second derivative. So:

$$
\begin{aligned}
f(x) &= \ln x, & f(1) &= 0. \\
f'(x) &= \frac{1}{x}, & f'(1) &= 1. \\
f''(x) &= -\frac{1}{x^2}, & f''(1) &= -1.
\end{aligned}
$$

So, by the Taylor series expansion, we have

$$
\begin{aligned}
p_2(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 \\
&= (x-1) - \frac{1}{2}(x-1)^2 .
\end{aligned}
$$

Taylor's theorem

Theorem: Taylor's theorem

Suppose that f has $n+1$ continuous derivatives on an open interval I containing a. Then, for each $x \in I$,

$$f(x) = p_n(x) + R_{n+1}(x).$$

For the course, we shall use **Lagrange's formula** for the remainder, which looks suspiciously like the next term of a Taylor expansion:

$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

where c is just some real number between a and x.

Sequences

A sequence is a collection of numbers that may or may not follow a simple rule. For example:

$$\{1, 1, 2, 3, 5, \text{etc.}\}$$

follows the rule $a_n = a_{n-1} + a_{n-2}$ with $a_0 = 1$ and $a_1 = 1$. But

$$\{1, 3, 2, 4, 6, -1, 0\}$$

has no obvious rule.

Sequences

Techniques for solving limits of sequences

Suppose that $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist. Then

1. $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.
2. $\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \times \lim_{n\to\infty} b_n$.
3. $\lim_{n\to\infty} (\alpha a_n) = \alpha \lim_{n\to\infty} a_n$ for every real number α.

Sequences

Increasing and decreasing sequences

1. If $a_{n+1} > a_n$ for all n, then the sequence $\{a_n\}$ is **monotonically increasing**.
2. If $a_{n+1} < a_n$ for all n, then the sequence $\{a_n\}$ is **monotonically decreasing**.
3. If $a_{n+1} \geq a_n$ for all n, then the sequence $\{a_n\}$ is **nondecreasing**.
4. If $a_{n+1} \leq a_n$ for all n, then the sequence $\{a_n\}$ is **nonincreasing**.

Sequences

(MATH1241) 17S2 - Question 4iii)

Consider the sequence $\{a_n\}$ defined recursively by

$$a_{n+1} = \frac{a_n^2 + \pi^2}{a_n + \pi}, \quad a_0 = 1.$$

The following Maple session may be useful.

```
> factor(Pi - (a[n]^2 + Pi^2)/(a[n] + Pi));
```

$$\frac{a_n\,(\pi - a_n)}{a_n + \pi}$$

a) Show by induction that $a_n < \pi$.

Part a) - solution for reference

We begin by showing that $a_1 < \pi$. The trick here is to see that $1 + \pi^2$ is less than $\pi + \pi^2$ which can be factored into $\pi(1 + \pi)$. This will complete the proof for the base case.
Now, assume that this statement holds for some integer k. Then

$$a_k < \pi$$

and

$$\begin{aligned} a_{k+1} &= \frac{(a_k)^2 + \pi^2}{a_k + \pi} \\ &= \frac{(a_k)^2 + \pi a_k - \pi a_k + \pi^2}{a_k + \pi}. \end{aligned}$$

We can split this up into two separate fractions.

Part a) - solution for reference

Splitting it up into two separate fractions gives:

$$\begin{aligned} a_{k+1} &= \frac{(a_k)^2 - \pi a_k}{a_k + \pi} + \frac{\pi^2 + \pi a_k}{a_k + \pi} \\ &= \frac{a_k(a_k - \pi)}{a_k + \pi} + \frac{\pi(\pi + a_k)}{a_k + \pi} \\ &= \pi + \frac{a_k(a_k - \pi)}{a_k + \pi}. \end{aligned}$$

Now, by our inductive hypothesis, we stated that $a_k < \pi$ so $a_k - \pi < 0$. Furthermore, by inspection, we see that $a_k > 0$ so that $a_k + \pi > 0$. So the entire fraction is negative since we have a positive $\times$ negative $\times$ positive. So we deduce that:

$$\frac{a_k(a_k - \pi)}{a_k + \pi} < 0.$$

Part a) - solution for reference

So we deduce that

$$a_{k+1} < \pi + 0 = \pi.$$

By induction, we've successfully shown that, for all integers n, $a_n < \pi$.

Sequences

(MATH1241) 17S2 - Question 4iii)

Consider the sequence $\{a_n\}$ defined recursively by

$$a_{n+1} = \frac{a_n^2 + \pi^2}{a_n + \pi}, \quad a_0 = 1.$$

The following Maple session may be useful.

```
> factor(Pi - (a[n]^2 + Pi^2)/(a[n] + Pi));
```

$$\frac{a_n\,(\pi - a_n)}{a_n + \pi}$$

b) By considering $a_{n+1} - a_n$, show that the sequence is monotonically increasing.

Part b) - solution for reference

Considering $a_{n+1} - a_n$, we aim to show that $a_{n+1} > a_n$ or $a_{n+1} - a_n > 0$ for all n. Notice that

$$\begin{aligned}
a_{n+1} - a_n &= \frac{a_n^2 + \pi^2}{a_n + \pi} - a_n \\
&= \frac{a_n^2 + \pi^2 - a_n(a_n + \pi)}{a_n + \pi} \\
&= \frac{\pi(\pi - a_n)}{a_n + \pi} \\
&> \frac{a_n(\pi - a_n)}{a_n + \pi} \\
&= \pi - a_{n+1} \qquad\qquad \text{(from the Maple session)} \\
&> 0.
\end{aligned}$$

Thus, we have shown that $a_{n+1} - a_n > 0$ so $a_{n+1} > a_n$, and thus $\{a_n\}$ is a monotonically increasing sequence.

Series

Finite series (partial sum)

A **partial sum** is a finite sum of a sequence of real numbers.

For example, a partial sum may look like this:

$$S_n = 1 + 2 + \cdots + 10$$

which we can write as

$$S_n = \sum_{k=1}^{10} k.$$

Series

More generally, we can write a partial sum using sequence notation. Suppose we have a sequence $\{a_k\}$, where a_k denotes the kth term of a sequence. Then the series can be expressed as

$$S_n = a_0 + a_1 + \cdots + a_n = \sum_{k=0}^{n} a_k.$$

But what happens when $n \to \infty$?

Series

Infinite series

If we let n approach ∞, we end up with an infinite series. The limiting behaviour ultimately depends on the behaviour of the partial sums. If the partial sums converge to a number L, then the infinite series converges to L and we write

$$\sum_{k=0}^{\infty} a_k = L.$$

If the partial sums diverge, then the infinite series diverges.

We shall look at some examples of infinite series

Infinite series

Geometric series

Suppose that $r \in \mathbb{R}$. An infinite series is **geometric** if each consecutive term shares a common ratio. We write geometric series as

$$\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + \cdots .$$

This series will

1. **converge** if and only if $|r| < 1$.
2. **diverge** elsewhere.

Infinite series

Harmonic series

The **harmonic series** is a special series. It holds the form

$$\sum_{k=0}^{\infty} \frac{1}{k}.$$

We've seen that if a sequence of a partial sum converges, then the series also converges. However, we will see later that, while the sequence of partial sum is convergent, this is actually divergent.

Tests for convergence

Test 1: The kth term test (test for divergence)

If $a_k \not\to 0$, then $\sum_{k=0}^{\infty} a_k$ diverges.

Note: this says **nothing** about whether a series converges; all it says is that sequences in every convergent series will approach 0. If the sequence does NOT approach 0, then the series diverges. If the sequence approaches 0, we can't tell whether or not the series converges.

Tests for convergence

(MATH1231) 18S2 - Question 4vi)

Suppose that $\sum_{n=0}^{\infty} a_n$ is a convergent series with $a_n > 0$ for all n.

a) State $\lim_{n\to\infty} a_n$.

Tests for convergence

(MATH1231) 18S2 - Question 4vi)

Suppose that $\sum_{n=0}^{\infty} a_n$ is a convergent series with $a_n > 0$ for all n.

a) State $\lim_{n\to\infty} a_n$. 0

Tests for convergence

Test 2: Integral test

Suppose that $f(x)$ is a positive integrable function with $f(k) = a_k$ for every positive k. Then one of two things may occur:

1. If $\int_1^\infty f(x)\,dx$ converges, then so does $\displaystyle\sum_{k=1}^{\infty} a_k$.
2. If $\int_1^\infty f(x)\,dx$ diverges, then so does $\displaystyle\sum_{k=1}^{\infty} a_k$.

Basically, if you THINK you can integrate a_k, then you may want to use the integral test.
We shall use this test to show that the harmonic series (from before) actually diverges.

Tests for convergence

Test 3: The comparison test

Suppose a_k and b_k are positive sequences such that $a_k \leq b_k$ for every k. Then one of two things may occur:

1. If $\displaystyle\sum_{k=0}^{\infty} b_k$ converges, then $\displaystyle\sum_{k=0}^{\infty} a_k$ also converges.
2. If $\displaystyle\sum_{k=0}^{\infty} a_k$ diverges, then $\displaystyle\sum_{k=0}^{\infty} b_k$ diverges.

Note: You will most likely use this test in conjunction with test 4.

Tests for convergence

Test 4: The p-series test

The series
$$\sum_{k=0}^{\infty} \frac{1}{k^p}$$
converges if $p > 1$ and diverges if $p \leq 1$.

This is akin to the p-integral test that you met back in 1A.

Tests for convergence

(MATH1231/1241) 18S2 - Question 2iib)

Use appropriate tests to determine whether each of the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{\sin^2(2n)}{n^2}$$

Solution for reference

Here, we use the fact that $\sin^2(2n) \leq 1$ for all n. So we shall compare the series to $1/n^2$. So we shall observe that:

$$\frac{\sin^2(2n)}{n^2} \leq \frac{1}{n^2}$$

and by the p-series test, we see that $1/n^2$ converges. So by part 1 of the comparison test, we conclude that the series

$$\sum_{n=1}^{\infty} \frac{\sin^2(2n)}{n^2}$$

converges.

Tests for convergence

(MATH1241) Test 5: Limit form of the comparison test

Suppose that a_n and b_n are positive sequences and suppose that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L, \quad L \neq 0.$$

The series $\sum a_n$ converges if and only if $\sum b_n$ converges.

Tests for convergence

(MATH1241) 17S2 - Question 4iv)

Consider the sequence $\{a_n\}$ given by

$$a_n = \frac{\cos n + n}{n^3 - e^{-n}}.$$

Does the series

$$\sum_{n=1}^{\infty} a_n$$

converge? Give reasons for your answer.

Solution for reference

We shall use the limit comparison test. Now, we note that as $n \to \infty$, $a_n \to 1/n^2$. So we set $b_n = 1/n^2$. By the *p*-series test, we know that b_n converges. So since the limit of $a_n/b_n = 1$, and b_n converges, then it follows that the series:

$$\sum_{n=1}^{\infty} a_n$$

converges.

Tests for convergence

Test 6: Ratio test

Suppose that a_k is an infinite series with **positive terms** and that

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = r.$$

1. If $r < 1$, then $\sum a_k$ converges.
2. If $r > 1$, then $\sum a_k$ diverges.

TIP: If you see anything like a^n or $n!$, it may be worth considering the ratio test because things simplify **very** nicely.

Tests for convergence

Test 7: Alternating series

A series **alternates** if each successive term switches sign. An alternating series of the form

$$\sum_{k=1}^{\infty}(-1)^k a_k$$

converges if it meets all of the criteria:

1. $a_k \geq 0$.
2. $a_k \geq a_{k+1}$.
3. $\lim_{k\to\infty} a_k = 0$.

Tests for convergence

(MATH1231/1241) 18S2 - Question 2ii)

Use the appropriate tests to determine whether each of the following series converges or diverges.

$$\sum_{n=2}^{\infty}(-1)^n\frac{n}{4n^2-3}$$

Solution for reference

Since this series is of the form, we naturally try the alternating series test. We shall check to see if each of the dot points are satisfied. It follows that parts 1 and 3 are satisfied, so we only need to check part 2.
To check part 2, consider the function

$$f(x) = \frac{x}{4x^2 - 3}.$$

Taking the derivative, we get:

$$\begin{aligned} f'(x) &= \frac{4x^2 - 3 - 8x^2}{(4x^2 - 3} \\ &= \frac{-3 - 8x^2}{(4x^2 - 3)^2}. \end{aligned}$$

This is clearly decreasing for all x, so the series is decreasing which implies that $a_k \geq a_{k+1}$. We conclude that the series converges.

Tests for convergence

QUIZ TIME!

Which test would you use for the following series?

$$\sum_{k=1}^{\infty} \frac{k}{k^4+1}$$

Tests for convergence

QUIZ TIME!

Which test would you use for the following series?

$$\sum_{k=1}^{\infty} \frac{k}{k^4+1} \quad \textbf{(COMPARISON + P-SERIES)}$$

Tests for convergence

QUIZ TIME!

Which test would you use for the following series?

$$\sum_{k=3}^{\infty} \frac{k+1}{k-2}$$

Tests for convergence

QUIZ TIME!

Which test would you use for the following series?

$$\sum_{k=3}^{\infty} \frac{k+1}{k-2} \quad \textbf{(K-TH TERM TEST)}$$

In fact, this series diverges since the limit is 1 and not 0!

Tests for convergence

QUIZ TIME!

Which test would you use for the following series?

$$\sum_{k=3}^{\infty} \frac{1}{k(\ln k)^2}$$

Tests for convergence

QUIZ TIME!

Which test would you use for the following series?

$$\sum_{k=3}^{\infty} \frac{1}{k(\ln k)^2} \quad \textbf{(INTEGRAL TEST)}$$

Let $f(x) = 1/(x(\ln x)^2)$. If we use a u-substitution, where $u = \ln x$, then we can turn the series into an integral we can compute!

Absolute and conditionally convergent

We shall now divert (hehe) our attention to convergent series. We can class convergence into two classes.

Absolute and conditionally convergent

Absolute convergence

A series $\sum a_k$ is said to be **absolute convergent** if the series

$$\sum |a_k|$$

is convergent.

Absolute and conditionally convergent

Conditionally convergence

A series $\sum a_k$ is said to be **conditionally convergent** if the series converges but

$$\sum |a_k|$$

is divergent.

For the rest of the chapter, we shall consider absolute convergence more than conditionally convergent.

Interval of convergence

We shall consider the interval where the series is **absolute convergent**, meaning

$$\sum |a_k|$$

converges. What follows is a systematic method to calculating the "interval of convergence". In other words, we find the possible values for x so that the series will DEFINITELY converge.

Interval of convergence

Method to finding an interval of convergence

Suppose we have some power series of the form:

$$a_k(x-\alpha)^k$$

We shall apply the **ratio test** on the absolute value of the series.

The ratio test tells us that whatever is inside the absolute value SHOULD be less than 1 for convergence to exist. We shall see this in action.

Interval of convergence

(MATH1231/1241) 16S2 - Question 1vi)

Determine the open interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{2^n}{(n^2-n+1)3^n}(x-3)^n.$$

Solution for reference

We shall use the ratio test, but we take the limit as $n \to \infty$. Doing so gives us:

$$
\begin{aligned}
r = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{2^{n+1}(x-3)^{n+1}}{((n+1)^2-(n+1)+1)3^{n+1}} \cdot \frac{(n^2-n+1)3^n}{2^n(x-3)^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{2(x-3)(n^2-n+1)}{3((n+1)^2-n)} \right| \\
&= \frac{2}{3}\left|x-3\right|.
\end{aligned}
$$

By the ratio test, we need $r < 1$. So we aim to solve

$$\frac{2}{3}\left|x-3\right| < 1$$

to find the interval of convergence. The previous equation is also called the **radius of convergence**, where the radius is $3/2$.

(MATH1241) Interval of convergence at end points

We shall now extend this to a (closed) interval of convergence. We solve the **open** interval of convergence by applying the ratio test and taking the limit of the power.

To find the convergence at the end points, we **substitute** the end points into the power series and determine whether it converges or diverges.

(MATH1241) Interval of convergence at end points

(MATH1241) 17S2 - Question 4iv)

Consider the sequence $\{a_n\}$ given by

$$a_n = \frac{\cos n + n}{n^3 - e^{-n}}.$$

b) Determine the interval of convergence of the power series

$$\sum_{n=1}^{\infty} a_n (x+2)^n.$$

You may use the fact that the sequence $\{a_n\}$ is monotonically decreasing.

Solution for reference - part 1

We use the ratio test taking $n \to \infty$ to find the OPEN interval of convergence. I will let you do it at your own accord. If you do it correctly, you should end up with the following radius of convergence:

$$|x+2| < 1.$$

So the end points are $x = -3$ and $x = -1$. We now determine whether the series converges at the end points. Plugging in $x = -3$ into the power series gives:

$$\sum_{n=1}^{\infty} a_n (-1)^n.$$

This is an alternating series, so we shall apply the alternating series test. Part 2 is already given to us, so we only need to check the first and third criteria. The third criteria can easily be checked to be 0, so that is satisfied. It follows that part 1 is also satisfied, so it converges at $x = -3$.

Solution for reference - part 2

At $x = -1$, we have the series:

$$\sum_{n=1}^{\infty} a_n$$

which converges from a previous question. So the interval of convergence is the closed $[-3, -1]$.

Today's plan

1. Chapter 3.5: Second order ODEs
   - Homogeneous solutions
   - Non-homogeneous solutions
2. Chapter 4: Taylor polynomials
   - Taylor series and theorem
   - Sequences and series
3. Chapter 5: Averages, arc length, speed, surface area
   - Average value of a function
   - Arc length of curves
   - Speed of a particle
   - Surface area

Average value of a function

Definition

Suppose that f is integrable on a closed interval $[a, b]$. Then the **average value** $\bar{f}$ of f on the interval $[a, b]$ is:

$$\boxed{\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx.}$$

Arc length for a parameterised curve

Suppose that $\mathcal{C}$ can be expressed with a parameter t:

$$\mathcal{C} = \{(x(t), y(t)) \in \mathbb{R}^2 : a \leq t \leq b\}.$$

Then the arc length ℓ is expressed as:

Arc length of parameterised curve

$$\ell = \int_a^b \left(\sqrt{(x'(t))^2 + (y'(t))^2} \right) dt.$$

Arc length for a graph

Suppose that a function is expressed in terms of one variable. Then the arc length ℓ can be expressed as one of the following:

Arc length of function $y = f(x)$

$$\ell = \int_a^b \left(\sqrt{1 + (f'(x))^2} \right) \, dx.$$

Arc length of function $x = f(y)$

$$\ell = \int_c^d \left(\sqrt{1 + (f'(y))^2} \right) \, dy.$$

Speed of a particle

Speed of particle formula

The speed $v(t)$ of a particle at any time t is given by

$$v(t) = \sqrt{(x'(t))^2 + (y'(t))^2}.$$

Surface area

Surface area when rotated about x axis

The surface area of revolution of $y = f(x)$ when rotated about the x axis is given by

$$SA = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx.$$

Surface area when rotated about x axis

The surface area of revolution of $x = f(y)$ when rotated about the y axis is given by

$$SA = \int_c^d 2\pi f(y)\sqrt{1 + (f'(y))^2}\, dy.$$