MATH2121/MATH2221 2019 T2 Revision Seminar Solutions

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We cannot guarantee that our answers are correct - please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are sometimes multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

MATH2221 2016 T2 (1i)

To find solutions to the ODE: $u'' + 4u' + 3u = 9x^2$, we first find the homogeneous solution for $u'' + 4u' + 3u = 0$. This has a characteristic equation of:

$$\lambda^2 + 4\lambda + 3 = 0$$
$$(\lambda + 3)(\lambda + 1) = 0$$

This implies that the roots of the characteristic equation are $\lambda_1 = -3$ and $\lambda_2 = -1$. Thus:

$$u_h(x) = c_1 e^{-3x} + c_2 e^{-x} \qquad \text{where } c_1, c_2 \in \mathbb{R}.$$

For the particular solution, since $f(x) = 9x^2$ is a polynomial of degree 2, we take the guess $u_p(X) = ax^2 + bx + c$ with $a, b, c \in \mathbb{R}$. Thus:

$$u_p(x) = ax^2 + bx + c$$
$$u_p(x)' = 2ax + b$$
$$u_p(x)'' = 2a.$$

Substituting back into the original ODE, we obtain:

$$\begin{aligned} u'' + 4u' + 3u &= 2a + 4(2ax + b) + 3(ax^2 + bx_c) \\ &= 2a + 8ax + 4b + 3ax^2 + 3bx + 3c \\ &= (3a)x^2 + (8a + 3b)x + (3a + 4b + 3c) \end{aligned}$$

Equating, we obtain:

$$\begin{aligned} 3a &= 9 \\ 8a + 3b &= 0 \\ 2a + 4b + 3c &= 0 \end{aligned}$$

Thus, we simultaneously solve to obtain $a = 3, b = -8, c = \frac{26}{3}$. Thus:

$$u_p(x) = 3x^2 - 8x + \frac{26}{3}$$

Since $u(x) = u_h(x) + u_p(x)$, we obtain the general solution:

$$u(x) = c_1 e^{-3x} + c_2 e^{-x} + 3x^2 - 8x + \frac{26}{3} \qquad \text{where } c_1, c_2 \in \mathbb{R}.$$

MATH2221 2014 T2 (2iiib)

When $m = 2$, we have $W = u_1u_2' - u_1'u_2$ by definition. Thus, by the product rule:

$$\begin{aligned} W' &= (u_1'u_2' + u_1u_2'') - (u_1''u_2 + u_1'u_2') \\ &= u_1u_2'' - u_1''u_2 \end{aligned}$$

Therefore:

$$\begin{aligned} a_2W' + a_1W &= a_2(u_1u_2'' - u_1''u_2) + a_1(u_1u_2' - u_1'u_2) \\ &= u_1(a_2u_2'' + a_1u_1') - u_2(a_2u_2'' + a_1u_1') \end{aligned}$$

After adding and subtracting $a_0u_1u_2$, we obtain:

$$a_2W' + a_1W = u_1(a_2u_2'' + a_1u_1' + a_0u_2) - (a_2u_2'' + a_1u_1' + a_0u_1)u_2$$

This equals 0 because u_1, u_2 are solutions to the homogeneous equation $a_2(x)u''' + a_1(x)u' + a_0(x)u = 0$.

MATH2221 2018 T2 (2ii)

By inspection, we obtain:

$$u_H = c_1e^x + c_2e^{-2x} + c_3xe^{-2x} + c_4\cos(x) + c_5\sin(x) \quad \text{where } c_i \in \mathbb{R}.$$

$$u_p = c_6(x^2 + e^{-2x}) + (c_7x^2 + c_8x + c_9) + x(c_{10}\cos(x) + c_{11}\sin(x)) \quad \text{where } c_i \in \mathbb{R}.$$

MATH2121 2018 T2 (1i)

We solve this with variation of parameters. First we find the homogeneous solutions. The character equation is $\lambda^2 - 2\lambda + 1$, which results in the solution of $\lambda = -1$ with multiplicity 2. This means that:

$$\begin{aligned} u_1(x) &= e^x \\ u_2(x) &= xe^x \end{aligned}$$

Next, we find the Wronskian, which is:

$$W(x) = e^x(xe^x + e^x) - e^x xe^x = e^{2x}.$$

Therefore:

$$\begin{aligned} v_1'(x) &= \frac{-u_2(x)f(x)}{W(x)} \\ &= \frac{-(xe^x) \cdot (e^x \cos(x))}{e^{2x}} \\ &= -x\cos(x) \\ v_2'(x) &= \frac{u_1(x)f(x)}{W(x)} \\ &= \frac{(e^x) \cdot (e^x \cos(x)}{e^{2x}} \\ &= \cos(x) \end{aligned}$$

Integrating, we get $v_1(x) = -x\sin(x) + \cos(x); \quad v_2(x) = \sin(x)$. Since $u(x) = v_1(x)u_1(x) + v_2(x)u_2(x)$, we obtain:

$$u(x) = -(x\sin(x) + \cos(x))e^x + (\sin(x))xe^x.$$

MATH2221 2015 T2 (1iii)

We note that:

$$y(x) = \sum_{n=0}^{\infty} A_n x^n$$
$$y'(x) = \sum_{n=0}^{\infty} n A_n x^{n-1}$$
$$y''(x) = \sum_{n=0}^{\infty} n(n-1) A_n x^{n-2}$$

Thus, expanding the ODE, we obtain:

$$\begin{aligned}(1+x^2)y'' - 2xy' + 20y &= y'' + (-x^2y'' - 2xy' + 20y) \\ &= \sum_{n=0}^{\infty} n(n-1)A_n x^{n-2} + \sum_{n=0}^{\infty}(-n(n-1) - n - 3)A_n x^n - 2nA_n x^n + 20A_n x^n \\ &= \sum_{n=0}^{\infty}(n+2)(n+1)A_{n+2}x^n + \sum_{n=0}^{\infty}(-n^2 - n + 20)A_n x^n \\ &= \sum_{n=0}^{\infty}[(n+2)(n+1)A_{n+2} - (n+5)(n-4)A_n]x^n\end{aligned}$$

Since coefficients must satisfy $(n+2)(n+1)A_{n+2} - (n+5)(n-4)A_n = 0$, we obtain:

$$A_{n+2} = \frac{(n+5)(n-4)}{(n+2)(n+1)} A_n$$

Thus, from $A_0 = 1, A_1 = 0$, we obtain the polynomial solution:

$$y(x) = A_0 + A_2x^2 + A_4x^4 + ...$$

where A_{2n} can be found through the recurrence relation. Note that the series starting from A_1 is the terminating one.

MATH2121 2018 T2 (1iii)

We note that:

$$
\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} A_n x^n \\
y'(x) &= \sum_{n=0}^{\infty} n A_n x^{n-1} \\
y''(x) &= \sum_{n=0}^{\infty} n(n-1) A_n x^{n-2}
\end{aligned}
$$

Thus, expanding the ODE, we obtain:

$$
\begin{aligned}
(1+z^2)u'' - zu' - 3u = u'' + (z^2u'' - zu' - 3u) \\
&= \sum_{n=0}^{\infty} n(n-1)A_n x^{n-2} + \sum_{n=0}^{\infty} (n(n-1) - n - 3)A_n z^n \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)A_{n+2} z^n + \sum_{n=0}^{\infty} (n^2 - 2n - 3)A_n z^n \\
&= \sum_{n=0}^{\infty} [(n+2)(n+1)A_{n+2} + (n+1)(n-3)A_n] z^n \\
&= \sum_{n=0}^{\infty} [(n+2)A_{n+2} + (n-3)A_k] z^n (n+1)
\end{aligned}
$$

Since coefficients must satisfy $(n+2)A_{n+2} + (n-3)A_n = 0$. This implies:

$$
A_{n+2} = \frac{3-n}{n+2} A_n \quad \forall n \geq 0.
$$

Thus:

$$
\begin{aligned}
A_5 &= \frac{3-3}{3+2} A_3 = 0 \times A_3 = 0 \\
A_7 &= \frac{3-5}{5+2} A_5 = -\frac{2}{7} \times 0 = 0 \\
A_9 &= 0 \ldots
\end{aligned}
$$

MATH2121 2016 T2 (2i)

This is a Cauchy-Euler ODE with the form $L(y) = ax^2y'' + bxy' + cy = f(x)$. Substituting $y = x^r$, we obtain:

$$\begin{aligned} L(x^r) &= ax^2(r)(r-1)x^{r-2} + bxrx^{r-1} + cx^r = f(x) \\ &= a(r)(r-1)x^r + brx^r + cx^r = f(x) \\ &= x^r[ar(r-1) + br + c] = f(x). \end{aligned}$$

Homogeneous solution if $L(y) = 0$, which for the given ODE with $a = 2, b = 7, c = 3$, we obtain:

$$\begin{aligned} 2r(r-1) + 7r + 3 = 0 \\ (r+1)(2r+3) = 0 \end{aligned}$$

Thus, $r = -1, -\frac{3}{2}$, and ODE has general homogeneous solution of the form:

$$y(x) = c_1x^{-1} + c_2x^{-\frac{3}{2}} \quad x > 0; c_1, c_2 \in \mathbb{R}.$$

The particular equation can be found through variation of parameters which produces $y_p(h) = -8x^{1/4}$. The math will be left as an exercise to the readers.

MATH2221 2015 T2 (2iii)

To prove the identity, we first note that:

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+\nu}}{k!\Gamma(k+1+\nu)}$$

Thus, keeping in mind that $\Gamma(z+1) = z\Gamma(z)$:

$$\begin{aligned}
\frac{d}{dx}[x^\nu J_\nu(x)] &= \frac{d}{dx} \sum_{k=0}^{\infty} \frac{(-1)^k (z)^{2k+2\nu}}{2^{2k+\nu} k!\Gamma(k+1+\nu)} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k (2k+2\nu)(z)^{2k+2\nu-1}}{2^{2k+\nu} k!(k+\nu)\Gamma(k+\nu)} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k 2(k+\nu)(z)^{2k+2\nu-1}}{2^{2k+\nu-1} k!(k+\nu)\Gamma(k+\nu)} \\
&= x^\nu \sum_{k=0}^{\infty} \frac{(-1)^k (k+\nu)(z)^{2k+\nu-1}}{2^{2k+\nu-1} k!(k+\nu)\Gamma(k+\nu)} \\
&= x^\nu \sum_{k=0}^{\infty} \frac{(-1)^k (z)^{2k+\nu-1}}{2^{2k+\nu-1} k!\Gamma(k+\nu)} \\
&= x^\nu \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+\nu-1}}{k!\Gamma(k+\nu)} \\
&= x^\nu J_{v-1}(x)
\end{aligned}$$

To find the integral $\int\limits_0^1 x^{\frac{7}{2}} J_{\frac{1}{2}}(x)dx$, we separate and use integration by parts:

$$\int\limits_0^1 x^2 \cdot x^{\frac{3}{2}} J_{\frac{1}{2}}(x)dx$$

where $u' = x^{\frac{3}{2}} J_{\frac{1}{2}}(x)$ and $v = x^2$. Thus:

$$
\begin{aligned}
I &= [x^2.x^{\frac{3}{2}}J_{\frac{3}{2}}(x)]_0^1 - \int\limits_0^1 2x \cdot x^{\frac{3}{2}}J_{\frac{3}{2}}(x)dx \\
&= J_{\frac{3}{2}}(1) - 2\int\limits_0^1 x^{\frac{5}{2}}J_{\frac{3}{2}}(x)dx \\
&= J_{\frac{3}{2}}(1) - 2[x^{\frac{5}{2}}J_{\frac{5}{2}}(x)]_0^1 \\
&= J_{\frac{3}{2}}(1) - 2J_{\frac{5}{2}}(1)
\end{aligned}
$$

MATH3201 Q1

Since $f'(x,t) = \frac{1}{\sqrt{x}}$, f differentiable and Lipschitz for $x > 0$. We are thus concerned where $x = 0$ is Lipschitz. Thus, we require L such that:

$$\begin{aligned}
|2\sqrt{x} - 2\sqrt{y}| &\leq L|x-y| \\
2|\sqrt{x} - \sqrt{y}| &\leq L|x-y| = L|(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y}) \\
\frac{2}{L} &\leq |\sqrt{x}+\sqrt{y}|
\end{aligned}$$

At $x = 0$, we note that $L \geq \frac{2}{|\sqrt{y}|}$. This condition cannot hold for finite L at $y \to 0$, thus $f(x)$ is NOT Lipschitz in a domain including 0.

MATH3201 Q2

$$f(x,t) = 2\sqrt{|x|} \qquad x \in \mathbb{R}^{\nvDash}$$

Lipschitz iff:

$$|f(\tilde{x}) - f(\tilde{y})| \leq L|\tilde{x} - \tilde{y}|.$$

Suppose that $|\tilde{x}| \geq |\tilde{y}| > 0$. Then:

$$\sqrt{|\tilde{x}||\tilde{y}|} \geq |\tilde{y}|$$

$$\begin{aligned}
|f(\tilde{x}) - f(\tilde{y})| &= 2|\sqrt{|x|} - \sqrt{|y|}| \\
|f(\tilde{x} - f(\tilde{y})|\sqrt{|\tilde{x}} &= 2|\sqrt{\tilde{x}} - \sqrt{\tilde{y}}|\sqrt{\tilde{x}} \\
&= 2||\tilde{x}| - \sqrt{|\tilde{x}||\tilde{y}|}| \\
&\geq 2||\tilde{x}| - |\tilde{y}||
\end{aligned}$$

This implies $|f(\tilde{x}) - f(\tilde{y})| \leq \frac{2}{\sqrt{|\tilde{x}}}|\tilde{x} - \tilde{y}|$. So f is Lipschitz for $|\tilde{x}| > 0$ with $L = \frac{2}{\sqrt{|\tilde{x}|}}$.

MATH2221 2016 T2 (2iii)

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + ...$$

Since $A^k v = \lambda^k v$, we have:

$$e^A v = \sum_{k=0}^{\infty} \frac{A^k}{k} v = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} v = e^{\lambda} v.$$

MATH2221 2018 T2 (2iii)

We are given the system of equations:

$$\frac{dx}{dt} = x + y$$
$$\frac{dy}{dt} = 2x$$

For the equilibrium point, we want x, y such that $x + y = 0; 2x = 0$ simultaneously. The only solution is the point $(0, 0)$. Furthermore, we note that:

$$\tilde{x}' = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \tilde{x}; \qquad \tilde{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Thus, $A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$ with eigenvalues of $\lambda_1 = -1, \lambda_2 = 2$. Since $\lambda_1 < 0 < \lambda_2$, and $\Re(\lambda_2) > 0$, the equilibrium point $(0, 0)$ is an unstable saddle as required.