2019 MathSoc Integration Bee
Qualifiers Answers

1. $\boxed{2}$ $\displaystyle\int e^x \ln x + \frac{e^x}{x}\,\mathrm{d}x = \; e^x \ln x$
   Reverse the product rule, or just do a clever integration by parts.

2. $\boxed{3}$ $\displaystyle\int_{-1}^{1} \cos^{-1} x + \sin^{-1} x\,\mathrm{d}x = \; \pi$
   The integrand is identically equal to $\frac{\pi}{2}$.

3. $\boxed{4}$ $\displaystyle\int_{1160}^{1163} 2x\,\mathrm{d}x = 6969$
   You can rewrite $1163^2 - 1160^2 = (1163 - 1160)(1163 + 1160)$ to save time.

4. $\boxed{5}$ For $n \in \mathbb{Z}_+$, $\displaystyle\int_0^\infty x^n e^{-x}\,\mathrm{d}x = \; n!$
   Standard reduction formulae.

5. $\boxed{5}$ $\displaystyle\int_0^\infty \frac{e^{2x}}{1+e^{4x}}\,\mathrm{d}x = \; \frac{\pi}{8}$
   Standard reverse chain rule, or sub $u = e^{2x}$.

6. $\boxed{5}$ $\displaystyle\int e^{2019x}\cos 2019x\,\mathrm{d}x = \; \frac{1}{4038}e^{2019x}(\cos 2019x + \sin 2019x)$
   Standard integration by parts, or use complexifying the integral. Alternatively memorise the formula.

7. $\boxed{5}$ $\displaystyle\int_{-2019}^{0} \sqrt{4076361 - x^2}\,\mathrm{d}x = \frac{4076361\pi}{4}$
   One quarter of a circle with radius 2019.

8. $\boxed{6}$ $\displaystyle\int 2019^{2019x}\,\mathrm{d}x = \; \frac{2019^{2019x-1}}{\ln 2019}$
   Standard reverse chain rule for $\int a^x\,\mathrm{d}x = a^x \ln a$. Alternatively first sub $u = 2019x$.

9. $\boxed{6}$ $\displaystyle\int \frac{\sin 4x}{\sin x}\,\mathrm{d}x = \; 4\left(\sin x - \frac{2}{3}\sin^3 x\right)$ OR $\displaystyle\frac{2}{3}(3\sin x + \sin 3x)$
   One approach is to use double angles to break the numerator.

10. $\boxed{7}$ $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{2x\sin|x|}{5+\cos 4x}\,\mathrm{d}x = 0$
    The integrand is an odd function.

11. $\boxed{7}$ $\displaystyle\int_0^{\pi/2} \frac{5\cos x}{3\sin x + 4}\,\mathrm{d}x = \frac{5}{3}\ln\left(\frac{7}{4}\right)$
One approach is just to resort to $t = \tan\frac{x}{2}$.

12. $\boxed{7}$ $\displaystyle\int \cos x\cos(\sin x)\cos(\sin(\sin x))\,\mathrm{d}x = \sin(\sin(\sin x))$
Sub $u = \sin x$, then $s = \sin u$.

13. We messed this one up... Sorry!!!

14. $\boxed{8}$ $\displaystyle\int_{-9\pi}^{2019\pi} \sin^{-1}(\sin x)\,\mathrm{d}x = 0$
A graph verifies that $\int_{k\pi}^{k\pi+2\pi}\sin^{-1}(\sin x)\,\mathrm{d}x = 0$ for $k \in \mathbb{Z}$. (Also true for $k \in \mathbb{R}$.)

15. $\boxed{9}$ $\displaystyle\int_0^1 x^3(1-x)^7\,\mathrm{d}x = \frac{1}{1320}$
Set $I_{m,n} = \int_0^1 x^m(1-x)^n\,\mathrm{d}x$ and prove that $I_{m,n} = \frac{m}{n+1}I_{m-1,n+1}$.

16. $\boxed{9}$ $\displaystyle\int_0^{\pi/2} \sin x\sin 2x\sin 3x\,\mathrm{d}x = \frac{1}{6}$
One approach is to convert $\sin 3x\sin x = \frac{1}{2}(\cos 2x - \cos 4x) = \frac{1}{2}(\cos 2x - (2\cos^2 2x - 1))$ and sub $u = \cos 2x$.

17. $\boxed{9}$ $\displaystyle\int_0^{\pi/2} \ln(\tan x)\,\mathrm{d}x = 0$
Calling the integral I, sub $u = \frac{\pi}{2} - x$ to obtain $I = \int_0^{\pi/2}\ln(\cot x)\,\mathrm{d}x$. Then add.

18. $\boxed{10}$ $\displaystyle\int_{-\pi/4}^{\pi/24} 8\cot 8x + 4\tan 4x + 2\tan 2x + \tan x - \cot x\,\mathrm{d}x = 0$
The integrand is identically equal to 0.

19. $\boxed{10}$ $\displaystyle\int \frac{\mathrm{d}x}{(x^2-2x)(x^2-2x+1)(x^2-2x+2)} = \frac{4}{x-1} - 2\arctan(1-x) + \ln\left|\frac{2-x}{x}\right|$
Nasty partial fractions.

20. $\boxed{12}$ $\displaystyle\int_{-3}^{3} \frac{x^4}{e^x+1}\,\mathrm{d}x = \frac{243}{5}$
Calling the integral I, sub $u = -x$ to obtain $I = \int_{-3}^{3}\frac{e^x x^4}{e^x+1}\,\mathrm{d}x$. Then add.

21. $\boxed{13}$ $\displaystyle\int_{-2}^{2} \frac{|x-2|+|x|+|x+2|}{|x-1|+|x+1|}\,\mathrm{d}x = \frac{11}{2} + 4\ln 2$
Tedious case-breaking with absolute values.

22. $\boxed{13}$ $\displaystyle\int_{-1}^{1} \frac{e^{2x}+1-(x+1)(e^x+e^{-x})}{x(e^x-1)}\,\mathrm{d}x = e - e^{-1}$
Rewrite numerator as $(e^x+e^{-x})(e^x-x-1)$. Then repeat method for Q20.

23. $\boxed{14}$ For $x>0$, $\displaystyle\int \sqrt{1+\frac{1}{x}}\,\mathrm{d}x = \sqrt{x^2+x} + \ln\left(\sqrt{x}+\sqrt{x+1}\right)$
Consider $\int \sqrt{\frac{x+1}{x}}\,\mathrm{d}x = \int \frac{x+1}{\sqrt{x^2+x}}\,\mathrm{d}x$.

24. $\boxed{14}$ $\displaystyle\int_0^{\pi/4} \sec^5 x\,\mathrm{d}x = \frac{3\ln(3+2\sqrt{2}) + 7\times 2^{3/2}}{16} \approx 1.56795196$
Use reduction formulae or two applications of integration by parts.

25. $\boxed{16}$ $\displaystyle\int_{-4}^{0} \frac{\sqrt{\ln(5-2x)}}{\sqrt{\ln(5-2x)}+\sqrt{\ln(2x+13)}}\,\mathrm{d}x = 2$
There is symmetry about $x=-2$, so sub $s=-x-2$. Then repeat method for Q20.

26. $\boxed{17}$ $\displaystyle\int \frac{x\cos x+1}{\sqrt{2x^3e^{\sin x}+x^2}}\,\mathrm{d}x = \ln\left|\frac{\sqrt{2xe^{\sin x}+1}-1}{\sqrt{2xe^{\sin x}+1}+1}\right|$
Sub $u = xe^{\sin x}$ and prove that $\frac{du}{u} = (x\cos x+1)\frac{dx}{x}$. Then sub $s^2 = 2u+1$.

27. $\boxed{17}$ $\displaystyle\int_0^{\pi/2} \sqrt{\tan x}\,\mathrm{d}x = \frac{\pi}{\sqrt{2}}$
Repeat method for Q17 so that $I = \int_0^{\pi/2} \sqrt{\cot x}\,\mathrm{d}x$, so that $2I = \int_0^{\pi/2} \sqrt{\tan x} + \sqrt{\cot x}\,\mathrm{d}x$. Then use trig identities, starting with $\tan x = \frac{\sin x}{\cos x}$.

28. $\boxed{19}$ $\displaystyle\int_0^{\pi} \frac{\sin\frac{2019x}{2}}{\sin\frac{x}{2}}\,\mathrm{d}x = \pi$
Let $I_n = \int_0^{\pi} \frac{\sin\frac{(2n+1)x}{2}}{\sin\frac{x}{2}}\,\mathrm{d}x$ and expand using compound angles. Use product-to-sum identities on $\int_0^{\pi} \frac{\sin nx \cos\frac{x}{2}}{\sin\frac{x}{2}}\,\mathrm{d}x$ to help prove $I_n = I_{n-1}$.

29. $\boxed{20}$ $\displaystyle\int_0^{1} \ln x \sin^{-1} x\,\mathrm{d}x = 2 - \frac{\pi}{2} - \ln 2$
Extremely tedious integration by parts

30. $\boxed{20}$ $\displaystyle\int_0^{\infty} \frac{x-1}{\sqrt{2^x-1}\,\ln(2^x-1)}\,\mathrm{d}x = \frac{\pi}{2(\ln 2)^2}$
Refer to Math Stack Exchange proofs

Ro16 Answers

- Group A Question 1:
$$\int_{-2}^{1} \sqrt{e^x}\,\mathrm{d}x = 2\left(e^{1/2} - e^{-1}\right)$$
The integrand is just $e^{x/2}$.

- Group A Question 2:
$$\int_0^1 \left(1+x^2\right)\left(1 - x^2 + x^4 - x^6 + \cdots - x^{4038}\right)\,\mathrm{d}x = \frac{4040}{4041}$$
The second bracket forms a geometric series, whose denominator cancels out with the first bracket.

- Group B Question 1:
$$\int_{13}^{27} x^2\,\mathrm{d}x = \frac{17486}{3}$$
Use the same trick as Q3 of the qualifiers

- Group B Question 2:
$$\int \frac{x}{\sqrt{x^2+2x+2}}\,\mathrm{d}x = \sqrt{x^2+2x+2} - \sinh^{-1}(x+1) \text{ OR } \sqrt{x^2+2x+2} - \ln(x+1+\sqrt{x^2+2x+2})$$
Standard $\int \frac{ax+b}{\sqrt{Ax^2+Bx+C}}\,\mathrm{d}x$ question, although trig sub may have been needed if you did not know hyperbolics.

- Group C Question 1:
$$\int_{-1}^{1} x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 6x^6\,\mathrm{d}x = \frac{488}{105}$$
$\int_{-1}^{1} x + 3x^3 + 5x^5\,\mathrm{d}x = 0$ from considering odd functions, so you only had to handle the even powers.

- Group C Question 2:
$$\int \frac{\mathrm{d}x}{\sqrt{3x(4-3x)}} = \frac{1}{3}\arcsin\left(\frac{3x-2}{2}\right)$$
Standard complete-the-square question

- Group D Question 1:
$$\int_{\pi/6}^{\pi/3} \frac{\mathrm{d}x}{\tan x + \cot x} = \frac{1}{4}$$
Start with $\tan x = \frac{\sin x}{\cos x}$ and $\cot x = \frac{\cos x}{\sin x}$ to bring it down to $\int_{\pi/6}^{\pi/3} \frac{1}{2}\sin 2x\,\mathrm{d}x$.

- Group D Question 2:
$$\int_{-5}^{6} |x|^3\,\mathrm{d}x = \frac{1921}{4}$$
Needed to compute $\frac{5^4+6^4}{4}$.

Quarter Final Answers

- Quarter Final Question 1:
$$\int_0^1 x^2\sqrt{4-x^2}\,\mathrm{d}x = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$$
Standard trig/hyperbolic sub.

- Quarter Final Question 2:
$$\int_0^\pi e^x \cos^2 x\,\mathrm{d}x = \frac{3}{5}(e^\pi - 1)$$
One method is to rewrite as $\frac{1}{2}\int_0^\pi e^x(1+\cos 2x)\,\mathrm{d}x$.

- Quarter Final Question 3:
$$\int_0^1 \left(\cos^{-1} x\right)^2\,\mathrm{d}x = \pi - 2$$
Double integration by parts.

- Quarter Final Question 4:
$$\int_0^{3\pi/2} \cos^{-1}(\cos x)\,\mathrm{d}x = \frac{7\pi^2}{8}$$
$y = \cos^{-1}(\cos x)\,\mathrm{d}x$ for $x \in [0, 2\pi]$ has the graph of a isosceles triangle with base along the x-axis and apex at the point (π, π). The boundaries required the triangle for $x \in [0, \pi]$ and the trapezium for $x \in [\pi, \frac{3\pi}{2}]$.

Semi Final Answers

- Semifinal A Question 1:

$$\int \frac{\mathrm{d}x}{\sum_{k=1}^{2019}(x+k)} = \frac{1}{2019}\ln|x+1010|$$

You could recall that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ so you could rewrite as $\int \frac{\mathrm{d}x}{2019x+\frac{2019\times 2020}{2}} = \frac{1}{2019}\int \frac{dx}{x+1010}$.

- Semifinal A Question 2:

$$\int_0^{10} x^2 + \lceil x \rceil^2 \,\mathrm{d}x = \frac{2155}{3}$$

Note that $\int_0^n \lceil x \rceil^2 = \sum_{k=1}^{n} k^2 = \frac{n}{2}(n+1)(2n+1)$ for integers n. Subbing $n = 10$ was required here.

- Semifinal B Question 1:

$$\int_0^1 \frac{\exp(-\tan(\sin^{-1} x))\sec^2(\sin^{-1} x)\tan(\sin^{-1} x)}{\sqrt{1-x^2}}\,\mathrm{d}x = 1$$

Sub $u = \sin^{-1} x$ and then $s = \tan u$ to obtain $\int_0^\infty se^{-s}\mathrm{d}s$, which can be done by parts.

- Semifinal B Question 2:

$$\int_0^{\pi/4} \sum_{k=0}^{2019} \tan(x+k\pi)\,\mathrm{d}x = 1010\ln 2$$

Because $\tan(x+\pi) = \tan x$, we have $\tan(x+k\pi) = \tan x$ for all integers k, so it reduces to $\int_0^{\pi/4} 2020\tan x\,\mathrm{d}x$.

Third Place Answers

- Third Place Question:

$$\int_{-1}^{1} \sin\left(\pi|x|\right) \sin^{-1}\left(\sqrt{|x|}\right)\, \mathrm{d}x = 1$$

  The symmetry of the absolute value breaks down to $I = 2\int_0^1 \sin \pi x \sin^{-1} \sqrt{x}\, \mathrm{d}x$. The reflection substitution $u = 1 - x$ gives $I = 2\int_0^1 \sin(\pi - \pi x) \sin^{-1} \sqrt{1-x}\, \mathrm{d}x = 2\int_0^1 \sin \pi x \cos^{-1} \sqrt{x}\, \mathrm{d}x$. Recognising $\sin^{-1}\sqrt{1-x} = \cos^{-1}\sqrt{x}$ was crucial to realise that now we can add the expressions for I.

- Third Place Easier Question:

$$\int \frac{e^{2x} + 2e^{x} + 1}{e^{2x} - 2e^{x} + 1}\, \mathrm{d}x = x - \frac{4}{e^{x} - 1}$$

  Write as $\int \frac{e^{2x}-2e^{x}+1}{e^{2x}-2e^{x}+1} + \frac{4e^{x}}{e^{2x}-2e^{x}+1}\, \mathrm{d}x$. The second integral can be handled with $u = e^{x}$.

Grand Finals Answers

- Grand Final Question 1 (Hard):

$$\int_0^{2019\pi} \sum_{k=0}^{5} \sin^{-1}(\sin kx)\,\mathrm{d}x = \frac{23\pi^2}{30}$$

From the graphs, for even k, $\int_{m\pi}^{m\pi+\pi} \sin^{-1}(\sin kx)\,\mathrm{d}x = 0$ where m is an integer. So all the even terms vanish.
As for odd k, it is still true that $\int_{m\pi}^{m\pi+2\pi} \sin^{-1}(\sin x)\,\mathrm{d}x = 0$, so we only need to focus on $\int_0^{\pi} \sin^{-1}(\sin kx)\,\mathrm{d}x$. Note that the period of the function is $\frac{2\pi}{k}$, so that may help you sketch it for $x \in [0, 2\pi]$. After you do, you will see that $\int_0^{\pi} \sin^{-1}(\sin kx)\,\mathrm{d}x = \frac{\pi^2}{2k}$ for odd k, so the final answer equals $\frac{\pi^2}{2} + \frac{\pi^2}{6} + \frac{\pi^2}{10}$.

- Grand Final Question 2 (Hard):

$$\int \frac{70\sin x + 23\cos x}{5\sin x + 8\cos x}\,\mathrm{d}x = 6x - 5\ln|5\sin x + 8\cos x|$$

Write $70\sin x + 23\cos x \equiv A(5\sin x + 8\cos x) + B(5\cos x - 8\sin x)$. The coefficients are found through subbing $x = 0$ and $x = \frac{\pi}{2}$, before doing simultaneous equations. Rewriting the numerator this way allows the integrand to be pulled apart conveniently.

- Grand Final Question 3 (Hard):

$$\int_{-\pi/4038}^{\pi/4038} \frac{\cos^{2019} 2019x}{\left(2019^{2019x} + 1\right)\left(\sin^{2019} 2019x + \cos^{2019} 2019x\right)}\,\mathrm{d}x = \frac{\pi}{8076}$$

Sub $u = 2019x$, then repeat the method of Q20 of the qualifiers, and then Q17 of the qualifiers.

- Grand Final Question 4 (Easy):

$$\int \sqrt{x}e^{\sqrt{x}}\,\mathrm{d}x = 2e^{\sqrt{x}}\left(x - 2\sqrt{x} + 2\right)$$

Sub $u = \sqrt{x}$ and then perform integration by parts.

- Grand Final Question 5 (Easy):

$$\int e^{2019x + e^{2019x}}\,\mathrm{d}x = \frac{1}{2019}e^{e^{2019x}}$$

Rewrite as $\int e^{2019x}e^{e^{2019x}}\,\mathrm{d}x$ to see the substitution $u = e^{2019x}$.

- Grand Final Question 6 (Easy):

$$\int_{20}^{89} 3x^2\,\mathrm{d}x = 696969$$

More number crunching.