MATH1081 Test 1 2008 S1 v1A

February 25, 2015

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1. Firstly, define $A := \{10k + 7 \,|\, k \in \mathbb{Z}\}$ and $B := \{5m - 8 \,|\, m \in \mathbb{Z}\}$. Let $x \in A$.
   Therefore, $x = 10k + 7$ where $k \in \mathbb{Z}$.
   Manipulating this expression,

$$
\begin{aligned}
x &= 10k + 7 \\
&= 10k + 15 - 8 \\
&= 5\,(5k + 3) - 8 \\
&= 5m - 8 && \text{(where } 2k + 3 = m \in \mathbb{Z}\text{)} \\
\therefore x &\in B \\
\therefore A &\subseteq B
\end{aligned}
$$

   To prove it is a proper subset, we need to find an element in B not in A.
   Choose $m = 0$ and so we have $-8 \in B$.
   Seeing if this element is in A, consider $10k + 7 = -8 \implies k = -\frac{2}{3} \notin \mathbb{Z}$.
   Thus, $A \subset B$.

2. Since $g \circ f$ is one-to-one, we have

$$g \circ f(x_1) = g \circ f(x_2) \implies x_1 = x_2.$$

We need to show that given the above is true, $f(x_1) = f(x_2) \implies x_1 = x_2$.
Considering the LHS,

$$\begin{aligned} & f(x_1) = f(x_2) \\ \implies & g(f(x_1)) = g(f(x_2)) \\ & g \circ f(x_1) = g \circ f(x_2) \\ \implies & x_1 = x_2 && \text{(since } g \circ f \text{ is one-to-one)} \\ \therefore\, & f(x_1) = f(x_2) \implies x_1 = x_2 \end{aligned}$$

Therefore, f is one-to-one.

3. Combining the two terms on the left and simplifying,

$$\begin{aligned} LHS &= \frac{1}{(k-1)^2} - \frac{1}{(k+1)^2} \\ &= \frac{(k+1)^2 - (k-1)^2}{(k-1)^2 (k+1)^2} \\ &= \frac{((k+1)-(k-1))((k+1)+(k-1))}{[(k-1)(k+1)]^2} && \text{(difference of two squares)} \\ &= \frac{(k+1-k+1)(k+1+k-1)}{(k^2-1)^2} \\ &= \frac{2(2k)}{(k^2-1)^2} \\ &= \frac{4k}{(k^2-1)^2} = RHS \end{aligned}$$

for $k > 1$.

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Simplifying the sum using the proven result,

$$\begin{aligned}
\sum_{k=2}^{n} \frac{k}{\left(k^2-1\right)^2} &= \frac{1}{4} \sum_{k=2}^{n} \frac{4k}{\left(k^2-1\right)^2} \\
&= \frac{1}{4} \sum_{k=2}^{n} \left[\frac{1}{(k-1)^2} - \frac{1}{(k+1)^2} \right] \\
&= \frac{1}{4} \left[\sum_{k=2}^{n} \frac{1}{(k-1)^2} - \sum_{k=2}^{n} \frac{1}{(k+1)^2} \right] \\
&= \frac{1}{4} \left[\sum_{k=0}^{n-2} \frac{1}{(k+1)^2} - \sum_{k=2}^{n} \frac{1}{(k+1)^2} \right] \\
&= \frac{1}{4} \left[\sum_{k=2}^{n-2} \frac{1}{(k+1)^2} + \frac{1}{1^2} + \frac{1}{2^2} - \sum_{k=2}^{n-2} \frac{1}{(k+1)^2} - \frac{1}{n^2} - \frac{1}{(n+1)^2} \right] \\
&= \frac{1}{4} \left(1 + \frac{1}{4} - \frac{1}{n^2} - \frac{1}{(n+1)^2} \right) \\
&= \frac{1}{4} \left(\frac{5}{4} - \frac{1}{n^2} - \frac{1}{(n+1)^2} \right).
\end{aligned}$$

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MATH1081 Test 1 2008 S2 v1B

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1. Let $A_1 = \{a\}$, $A_2 = A_1 \cup \{A_1\}$, $A_3 = A_2 \cup \{A_2\}$.

   (i)

$$\begin{aligned} A_3 &= A_2 \cup \{A_2\} \\ &= A_1 \cup \{A_1\} \cup \{A_1 \cup \{A_1\}\} \\ &= \{a\} \cup \{\{a\}\} \cup \{\{a\} \cup \{\{a\}\}\} \\ &= \{a, \{a\}\} \cup \{\{a, \{a\}\}\} \\ &= \{a, \{a\}, \{a, \{a\}\}\} \end{aligned}$$

   So the elements of A_3 are $a, \{a\}$ and $\{a, \{a\}\}$.

   (ii) (i) $\{\{a\}\} \notin A_3$. False. $\{\{a\}\}$ is not in the list of elements of A_3.

   (ii) $\{\{a\}\} \subseteq A_3$. True, as $\{a\} \in A_3$ and hence a possible subset is $\{\{a\}\}$.

2. Since $g \circ f$ is one-to-one, we have

$$g \circ f(x_1) = g \circ f(x_2) \implies x_1 = x_2.$$

We need to show that given the above is true, $f(x_1) = f(x_2) \implies x_1 = x_2$.
Considering the LHS,

$$\begin{aligned} f(x_1) &= f(x_2) \\ \implies g(f(x_1)) &= g(f(x_2)) \\ g \circ f(x_1) &= g \circ f(x_2) \\ \implies x_1 &= x_2 && \text{(since } g \circ f \text{ is one-to-one)} \\ \therefore f(x_1) = f(x_2) &\implies x_1 = x_2 \end{aligned}$$

Therefore, f is one-to-one.

3. Let $k > 1$. We have

$$\begin{aligned} LHS &= \frac{1}{k-1} + \frac{6}{k} - \frac{7}{k+1} \\ &= \frac{(k+1) - 7(k-1)}{(k-1)(k+1)} + \frac{6}{k} \\ &= \frac{8-6k}{k^2-1} + \frac{6}{k} \\ &= \frac{8k - 6k^2 + 6(k^2-1)}{k(k^2-1)} \\ &= \frac{8k-6}{k(k^2-1).} \end{aligned}$$

So

$$\begin{aligned} \sum_{k=2}^{n} \frac{4k-3}{k(k^2-1)} &= \frac{1}{2} \sum_{k=2}^{n} \frac{8k-6}{k(k^2+1)} \\ &= \frac{1}{2} \sum_{k=2}^{n} \left[\frac{1}{k-1} + \frac{6}{k} - \frac{7}{k+1} \right] && \text{(using the proven result)} \\ &= \frac{1}{2} \left(\sum_{k=2}^{n} \frac{1}{k-1} + \sum_{k=2}^{n} \frac{6}{k} - \sum_{k=2}^{n} \frac{7}{k+1} \right) \end{aligned}$$

We shift the series in order to match the denominators,

$$= \frac{1}{2} \left(\sum_{k=2}^{n} \frac{1}{k-1} + \sum_{k=3}^{n+1} \frac{6}{k-1} - \sum_{k=4}^{n+2} \frac{7}{k-1} \right)$$

Next, we add or remove terms in order to match the limits of the sums,

$$\begin{aligned}
&= \frac{1}{2}\left(\left[1+\frac{1}{2}+\sum_{k=4}^{n}\frac{1}{k-1}\right]+\left[\frac{6}{2}+\sum_{k=4}^{n}\frac{6}{k-1}+\frac{6}{n}\right]-\left[\sum_{k=4}^{n}\frac{7}{k-1}+\frac{7}{n}+\frac{7}{n+1}\right]\right)\\
&= \frac{1}{2}\left(\left[\sum_{k=4}^{n}\frac{1}{k-1}+\sum_{k=4}^{n}\frac{6}{k-1}-\sum_{k=4}^{n}\frac{7}{k-1}\right]+\left[\frac{9}{2}-\frac{1}{n}-\frac{7}{n+1}\right]\right)\\
&= \frac{1}{2}\left(\frac{9}{2}-\frac{1}{n}-\frac{7}{n+1}\right).
\end{aligned}$$

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MATH1081 Test 1 2009 S1 v1B

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1. (i) To calculate this, recall that $|P(A)| = 2^{|A|}$. So $|A_3| = 2^8$.

   (ii) (i) $A_2 \in A_3$: True. A_3 is the set of all possible subsets of of A_2, which includes A_2.

   (ii) $A_2 \subseteq A_3$: False. A set cannot be a subset of its power set. You could write down the set A_2 to clearly show that A_2 cannot be a subset of A_3.

2. $\text{Range}(f) = \mathbb{R}$ (prove this).
   f is not one-to-one since it is neither monotonically increasing or decreasing (prove this).
   f is onto since it's range is the same as its codomain.
   f is not a bijection since it is not one-to-one (it must be both one-to-one and onto to be bijective).

3. Combine all the fractions together on the LHS to obtain the RHS.
   Alternatively, split the RHS using partial fractions.
   Using the partial fractions result, and creating some telescoping sums, we obtain the answer of $\frac{20}{9}$.

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1. (i) $P(A) = \{\phi, \{n\}, \{s\}, \{w\}, \{n, s\}, \{n, w\}, \{s, w\}, A\}$

   (ii)

$$\begin{aligned} |P(A \cup P(A))| &= 2^{|A \cup P(A)|} \\ &= 2^{3+2^3} && (A \text{ and } P(A) \text{ are disjoint}) \\ &= 2^{11}. \end{aligned}$$

2. (i) It means that $f(x_1) = f(x_2)$ for $x_1, x_2 \in A$ implies that $x_1 = x_2$.
   Similarly, if g is a one-to-one function, $g(b_1) = g(b_2) \implies b_1 = b_2$.
   In other words, every value in the codomain has at most one input value (value in the domain) that maps to it.

   (ii)

$$f(x_1) = f(x_2) \implies x_1 = x_2$$

$$g(y_1) = g(y_2) \implies y_1 = y_2.$$

Consider

$$\begin{aligned}
g \circ f(x_1) &= g \circ f(x_2) \\
g(f(x_1)) &= g(f(x_2)) \\
\implies f(x_1) &= f(x_2) && \text{(since } g \text{ is one-to-one)} \\
\implies x_1 &= x_2. && \text{(since } f \text{ is one-to-one)}
\end{aligned}$$

3. Whenever simplifying these, always state whatever law you have used in your working.

$$\begin{aligned}
(A \cap B^c) \cup (A^c \cap B^c)^c &= (A \cap B^c) \cup ((A^c)^c \cup (B^c)^c) && \text{(De Morgan's Law)} \\
&= (A \cap B^c) \cup (A \cup B) && \text{(Double complement)} \\
&= [(A \cap B^c) \cup A] \cup B && \text{(Associative Law)} \\
&= A \cup B. && \text{(Absorption Law)}
\end{aligned}$$

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1. Let $A = \{a\}$, $B = \{b, c\}$, $C = \{d, e, f, g, h, i, j\}$.

   (i) $A \times B = \{(a, b), (a, c)\}$. So

   $$\mathrm{P}(A \times B) = \{\{(a, b)\}, \{(a, c\}, \emptyset, \{(a, b), (a, c)\}\}.$$

   (ii) $|B \times C| = 14$ as $|B| = 2, |C| = 7$. Then $|\mathrm{P}(B \times C)| = 2^{14}$. $\square$

2. Let $f, g : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$, $g(x) = e^x$.

   (i) $(f \circ g) = f(g(x)) = (e^x)^2 = e^{2x}$.

   $(g \circ f)(x) = g(f(x)) = e^{x^2}$.

   (ii) $f \circ g$ is one-to-one (prove this).
   $g \circ f$ is not one-to-one (prove this).

3. By inspecting the given formula and the result we must prove, we see it is reasonable to let $A = k$ and $B = k - 1$ and rearrange to obtain the answer.

Using the proved formula to simplify the sum, we see this is a telescoping sum which simplifies to

$$\sum_{k=1}^{n} \tan k \tan (k-1) = -n + \frac{\tan n}{\tan 1}.$$

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