UNSW Mathematics Society Presents

MATH1231/1241 Workshop: Calculus

Presented by Yvonne and Wendy (watch lectures on 0.25x speed in the dark and you'll get a HD)

Overview I

1. Functions of Several Variables
2. Integration Techniques
3. ODE's
4. Taylor Series
5. Applications of Integration

1. Functions of Several Variables

Partial Differentiation

Functions of Several Variables

- Let $F(x, y) = x^2 + xy + y^2$. Since this is a function of more than one variable, in order to find the rate of change with respect to x or y, we must use partial differentiation.
- This involves treating all variables other than the one you're differentiating with, as constants.

Example

Using the example above

$$F(x, y) = x^2 + xy + y^2$$

- $\dfrac{\partial F}{\partial x} = F_x = 2x + y,$
- $\dfrac{\partial F}{\partial y} = F_y = x + 2y.$

Tangent Planes to Surfaces

Tangent Planes

- Given a function of the form $F(x, y, z) = 0$, the normal vector at a point $P = (x_0, y_0, z_0)$ on F is given by

$$\begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} \text{ evaluated at } P.$$

Thus, using the point-normal representation of a plane, the equation of a tangent plane is given by

$$\mathbf{n} \cdot (\mathbf{x} - P) = 0,$$

where $\mathbf{x} = (x, y, z)^T$

Tangent Planes to Surface

MATH1241 November 2014

Show that the tangent plane to the paraboloid S given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

at the point $P(x_0, y_0, z_0)$ is given by

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = \frac{z + z_0}{2c}.$$

Tangent Planes to Surface

Solution

$$\mathbf{n} \cdot (\mathbf{x} - P) = \begin{pmatrix} 2x_0/a^2 \\ 2y_0/b^2 \\ -1/c \end{pmatrix} \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} = 0$$

$$\begin{aligned} \frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) - \frac{(z - z_0)}{c} &= 0 \\ \frac{2x_0 x}{a^2} + \frac{2y_0 y}{b^2} &= \frac{(z - z_0)}{c} + 2\left(\frac{{x_0}^2}{a^2} + \frac{{y_0}^2}{b^2}\right) \\ \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} &= \frac{(z - z_0)}{2c} + \left(\frac{z_0}{c}\right) \\ &= \frac{(z + z_0)}{2c} \end{aligned}$$

Total Differential Approximation

Given some $F(x, y)$

$$\Delta F \approx \frac{\partial F}{\partial x}\Delta x + \frac{\partial F}{\partial y}\Delta y.$$

Rationale

- Since F is a function of two variables, the change in F (ΔF) is dependent on Δx and Δy.
- Since partial differentiation yields the rate of change of F w.r.t x or y, we can approximate the rate of change of F through the expression above.

Total Differential Approximation

MATH1241 November 2012

b) The area A of an ellipse has the form

$$A = ab\left(\frac{\sqrt{3}}{4} + \frac{\pi}{6}\right).$$

If the ellipse is deformed in such a way that a is increased by 4% and b is decreased by 1%, use the total differential approximation to show that the approximate relative change of area $\frac{\Delta A}{A}$ is independent of a and b and calculate its value.

Total Differential Approximation

Solution

$$
\begin{aligned}
\Delta A &\approx \frac{\partial A}{\partial a}\Delta a + \frac{\partial A}{\partial b}\Delta b \\
&= b\left(\frac{\sqrt{3}}{4} + \frac{\pi}{6}\right)\Delta a + a\left(\frac{\sqrt{3}}{4} + \frac{\pi}{6}\right)\Delta b \\
\frac{\Delta A}{A} &\approx \frac{\Delta a}{a} + \frac{\Delta b}{b} \\
&= 4\% - 1\% \\
&= 3\%
\end{aligned}
$$

Error Estimation

The total differential approximation can be used to estimate the error of a variable that's dependent on other variables. For instance, if F is a function of x and y, then

$$\begin{aligned}
|\Delta F| &= \left|\frac{\partial F}{\partial x}\Delta x + \frac{\partial F}{\partial y}\Delta y\right| \\
&\leq \left|\frac{\partial F}{\partial x}\right| |\Delta x| + \left|\frac{\partial F}{\partial y}\right| |\Delta y| \quad \text{(by the Triangle Inequality)}
\end{aligned}$$

Chain Rule I

Let's say we have $F(x,y)$ such that $x = x(t)$ and $y = y(t)$. In this case, dividing the Total Differential Approximation by Δt yields,

$$\frac{\Delta F}{\Delta t} = \frac{\partial F}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial F}{\partial y}\frac{\Delta y}{\Delta t}$$

Now, as $\Delta t \to 0$, we have that

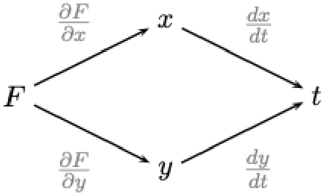
- $\dfrac{\Delta F}{\Delta t} \to \dfrac{dF}{dt}$
- $\dfrac{\Delta x}{\Delta t} \to \dfrac{dx}{dt}$
- $\dfrac{\Delta y}{\Delta t} \to \dfrac{dy}{dt}$

And so,

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt}$$

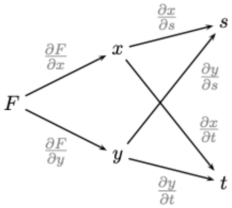
Chain Rules I

We can attempt to simplify that expression through use of a chain rule diagram, seen here.



Chain Rules II

Now, let's say we have a function $F(x, y)$ such that $x = x(s, t)$ and $y = y(s, t)$. Drawing out our Chain Rule diagram, we have,



Hence, the chain rule expression becomes

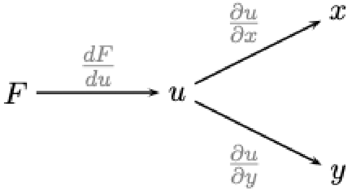
$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t}$$

and similarly

$$\frac{\partial F}{\partial s} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial s}$$

Chain Rules III

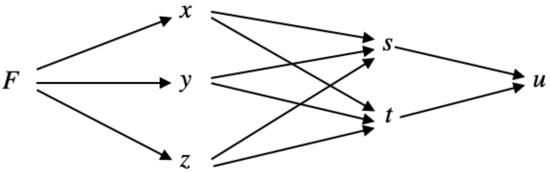
Now, let's say we have $F(u)$ where $u = u(x, y)$. In this case,



- $\dfrac{\partial F}{\partial x} = \dfrac{dF}{du}\dfrac{\partial u}{\partial x}$
- $\dfrac{\partial F}{\partial y} = \dfrac{dF}{du}\dfrac{\partial u}{\partial y}$

Functions of three or more variables

Let F be a function $F(x, y, z)$ where $x = x(s, t)$, $y = y(s, t)$ and $z = z(s, t)$, where $s = s(u)$ and $t = t(u)$.



The truly monstrous expression we end up with is

$$\begin{aligned}\frac{dF}{du} &= \frac{ds}{du}\left(\frac{\partial F}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial s}\right) \\ &\quad + \frac{dt}{du}\left(\frac{\partial F}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial t}\right)\end{aligned}$$

2. Integration Techniques

Trigonometric Integrals

$$\int \sin^m(x)\cos^n(x)\,dx$$

Case 1: n odd

$u = \sin x$ $\qquad du = \cos x\,dx$ $\qquad \cos^2 x = 1 - \sin^2 x$

Case 2: m odd

$u = \cos x$ $\qquad du = -\sin x\,dx$ $\qquad \sin^2 x = 1 - \cos^2 x$

Case 3: m, n even

Use identities

$$\cos^2 x = \frac{1}{2}(1+\cos 2x), \quad \sin^2 x = \frac{1}{2}(1-\cos 2x).$$

Trigonometric Integrals

Example

$$\int \sin^2 x \cos^5 x dx$$

Solution

$$\begin{aligned}\int \sin^2 x \cos^5 x dx &= \int \sin^2 x \cos^4 x (\cos x dx)\\ &= \int u^2(1-u^2)^2 du\\ &= \frac{\sin^3 x}{3} - \frac{2\sin^5 x}{5} + \frac{\sin^7 x}{7} + C\end{aligned}$$

Trigonometric Integrals

Example

$$\int \sin^2 x \cos^4 x dx$$

Solution

$$\begin{aligned}
\int \sin^2 x \cos^4 x dx &= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx \\
&= \frac{1}{8}\int 1 + \cos 2x - \cos^2 2x - \cos^3 2x dx \\
&= \frac{x}{8} + \frac{\sin 2x}{16} - \frac{1}{8}\int \cos^2 2x + \cos^3 2x dx \\
&= \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C
\end{aligned}$$

Trigonometric Integrals

$$\int \sin(mx)\cos(nx)\ dx, \int \cos(mx)\cos(nx)\ dx, \int \sin(mx)\sin(nx)\ dx$$

Case 1

$$\sin(mx)\cos(nx) = \frac{1}{2}\Big(\sin\big((m+n)x\big) + \sin\big((m-n)x\big)\Big)$$

Case 2

$$\cos(mx)\cos(nx) = \frac{1}{2}\Big(\cos\big((m+n)x\big) + \cos\big((m-n)x\big)\Big)$$

Case 3

$$\sin(mx)\sin(nx) = \frac{1}{2}\Big(\cos\big((m-n)x\big) - \cos\big((m+n)x\big)\Big)$$

Trigonometric Integrals

Important Identities

$$\tan^2 x + 1 = \sec^2 x \qquad \frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\sec x = \sec x \tan x$$

Example

Solve

$$\int \sec^4 x \tan x dx$$

$$\begin{aligned}\int \sec^4 x \tan x \, dx &= \int (\sec^3 x)(\sec x \tan x) \, dx \\ &= \int u^3 \, du \\ &= \frac{u^4}{4} + C = \frac{\sec^4 x}{4} + C\end{aligned}$$

Reduction Formulae

Example

Let I_n be defined as

$$I_n = \int_0^{\pi/4} \tan^n x \, dx.$$

a) Find a reduction formula in terms of I_{n-2}.

Reduction Formulae

$$
\begin{aligned}
\int_0^{\pi/4} \tan^n x \, dx &= \int_0^{\pi/4} \tan^{n-2} x \tan^2 x \, dx \\
&= \int_0^{\pi/4} \tan^{n-2} x(\sec^2 x - 1) \, dx \\
&= \int_0^{\pi/4} \tan^{n-2} x \sec^2 x \, dx - \int_0^{\pi/4} \tan^{n-2} x \, dx \\
&= \left[\frac{u^{n-1}}{n-1}\right]_0^1 - I_{n-2} \\
&= \frac{1}{n-1} - I_{n-2}.
\end{aligned}
$$

Reduction Formulae

b) Use the reduction forumula obtained on the previous slide to work out the value of
$$\int_0^{\pi/4} \tan^5 x.$$

$$\begin{aligned}
I_5 &= \frac{1}{5-1} - I_3 && (1)\\
I_3 &= \frac{1}{3-1} - I_1 \\
-I_3 &= -\frac{1}{3-1} + I_1 && (2)\\
I_5 &= \frac{1}{4} - \frac{1}{2} + \int_0^{\pi/4} \tan x && (1)+(2)\\
&= \frac{1}{2}\ln 2 - \frac{1}{4}
\end{aligned}$$

Trigonometric & Hyperbolic Substitutions

Substitutions

Integral Factor	**Trigonometric**	**Hyperbolic**
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ $dx = a\cos\theta d\theta$	$x = a\tanh\theta$ $dx = a\,\text{sech}^2\,\theta d\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$ $dx = a\sec^2\theta d\theta$	$x = a\sinh\theta$ $dx = a\cosh\theta d\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$ $dx = a\sec\theta\tan\theta d\theta$	$x = a\cosh\theta$ $dx = a\sinh\theta d\theta$

Using Trig Substitutions

MATH1241 November 2012

a) Show that the area A of the planar region between the lines $x = -\frac{a}{2}$ and $x = \frac{a}{2}$ bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a, b > 0$, has the form Kab, and find the value of the constant K.

Solution

$$y = \frac{b}{a}\sqrt{a^2 - x^2} \qquad A = \frac{b}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} \sqrt{a^2 - x^2}dx$$

$$\text{let } x = a\sin\theta \qquad dx = a\cos\theta d\theta$$

$$= \frac{b}{a}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} a^2\cos^2\theta d\theta$$

$$= \frac{ab}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}(\cos 2\theta + 1)d\theta$$

$$= \frac{ab}{2}\left(\frac{\sin 2\theta}{2} + \theta\right)\Bigg|_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$

$$= ab\left(\frac{\sqrt{3}}{4} + \frac{\pi}{6}\right)$$

3. ODE's

Types of ODE's

- Separable ODE
- First-Order Linear ODE
- Exact ODE
- Second-Order Linear ODE
    - The Homogeneous Case
    - The Non-Homogeneous Case

Separable ODE

Definition

A *separable ODE* is one where both of the variables involved in the ODE can be separated fully into two halves of the equation. In other words, where the equation can be written in the form,

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}.$$

This makes it easier to solve the differential equation, as we can integrate both sides.

Example

$$\frac{dy}{dx} = 4x^4y^2$$

Separable ODE

MATH1241 November 2011

Consider the ODE

$$\frac{dy}{dx} = \frac{xy - y^2}{x^2}.$$

Determine the general solution $y = y(x)$ to the above ODE.

Separable ODE

a)

$$\frac{dy}{dx} = \frac{xy - y^2}{x^2} = \frac{y}{x} - \left(\frac{y}{x}\right)^2$$

Using the substitution $y(x) = x \cdot u(x)$,

$$\frac{dy}{dx} = \frac{d}{dx}(xu) = u\frac{dx}{dx} + x\frac{du}{dx} = u + x\frac{du}{dx}$$

$$u + x\frac{du}{dx} = \frac{dy}{dx}$$

Separable ODE

$$
\begin{aligned}
u + x\frac{du}{dx} &= u - u^2 \\
\int -\frac{1}{u^2} du &= \int \frac{1}{x} dx \\
\frac{1}{u} &= \ln|x| + c \\
\frac{x}{y} = \ln|x| + c &\implies y = \frac{x}{\ln|x| + c}
\end{aligned}
$$

First-Order Linear ODE

Definition

A first-order linear ODE is one that can be expressed in the following form,

$$\frac{dy}{dx} + f(x)y = g(x),$$

where f and g are functions in x.

Example

$$2\frac{dy}{dx} + 4x^3y = 3x$$

First-Order Linear ODE

Method

1. Write the ODE in the form

$$\frac{dy}{dx} + f(x)y = g(x).$$

2. Calculate the integrating factor $h(x) = e^{\int f(x)\,dx}$ (ignore the constant).
3. Multiply the ODE by $h(x)$ to get

$$\frac{dy}{dx}h(x) + h(x)f(x)y = h(x)g(x).$$

4. Because of the product rule, this is equivalent to

$$\frac{d}{dx}\Big(h(x)y\Big) = g(x)h(x).$$

First-Order Linear ODE's

Example

Solve

$$(x-1)^3\frac{dy}{dx}+4(x-1)^2y=x+1, \quad y(0)=2$$

First-Order Linear ODE's

Firstly, expressing ODE in the proper form leads to

$$\frac{dy}{dx} + 4y(x-1)^{-1} = \frac{x+1}{(x-1)^3}.$$

Now,

$$e^{\int 4(x-1)^{-1}dx} = e^{4\ln(x-1)} = (x-1)^4.$$

Multiplying through the *integrating factor*

$$(x-1)^4\frac{dy}{dx} + 4y(x-1)^3 = (x+1)(x-1)$$

$$(x-1)^4\frac{dy}{dx} + 4y(x-1)^3 = x^2 - 1.$$

Now, by the product rule, this simplifies to

$$\frac{d}{dx}\Big(4y(x-1)^4\Big) = x^2 - 1.$$

First-Order Linear ODE's

Upon integrating both sides with respect to x,

$$4y(x-1)^4 = \frac{x^3}{3} - x + C$$

Since $y(0) = 2$, substituting $x = 0, y = 2$,

$$4(2)(-1)^4 = 8 = 0 + 0 + C.$$

Hence, $C = 8$, and upon dividing by $4(x-1)^4$, we get our solution,

$$y = \frac{x^3 - 3x + 24}{12(x-1)^4}$$

Exact ODE

Definition

Exact ODE's are of the form

$$F(x, y) + G(x, y)\frac{dy}{dx} = 0,$$

such that,

$$\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x}.$$

In this case, the solution to the ODE is given by $H(x, y) = C$, where

$$\frac{\partial H}{\partial x} = F \quad \text{and} \quad \frac{\partial H}{\partial y} = G,$$

and C is just a constant.

Exact ODE's

Example

Show that

$$\frac{dy}{dx} = -\frac{2x+y+1}{2y+x+1}$$

is exact, and hence find its solution.

Rearranging, we have

$$2x+y+1+(2y+x+1)\frac{dy}{dx} = 0,$$

which is in the form of an exact ODE, since

$$\frac{\partial F}{\partial y} = 1 = \frac{\partial G}{\partial x}$$

Exact ODE's

Hence, there must exist a $H(x, y)$ such that

$$\frac{\partial H}{\partial x} = 2x + y + 1 = F$$
$$\frac{\partial H}{\partial y} = 2y + x + 1 = G.$$

Now, when we integrate F with respect to x, we get

$$H(x, y) = x^2 + xy + x + C_1(y),$$

where the constant of integration is with respect to y, since it's treated as a constant w.r.t x.

Exact ODE's

Similarly, when integrating G with respect to y, we obtain

$$H(x, y) = y^2 + xy + y + C_2(x),$$

where the constant is a function of x.
Now, comparing these two forms, we can see that the final form is

$$H(x, y) = x^2 + xy + y^2 + x + y.$$

The solution to this exact ODE is

$$x^2 + xy + y^2 + x + y = C,$$

the value of C dependent on initial conditions.

Exact ODE's

MATH1231 Feburary 2012

Consider the differential equation

$$2xy + (x^2 + y^3)\frac{dy}{dx} = 0.$$

a) Show that the differential equation is exact.

b) Hence solve the differential equation.

a)

$$F(x,y) = 2xy, \quad G(x,y) = x^2 + y^3$$
$$\frac{\partial F}{\partial y} = 2x = \frac{\partial G}{\partial x}$$

Therefore the equation is exact.

Exact ODE's

b) The solution is $H(x, y) = C$ where

$$\frac{\partial H}{\partial x} = F(x, y) = 2xy \Rightarrow H = x^2 y + \phi(y)$$
$$\frac{\partial H}{\partial y} = G(x, y) = x^2 + y^3 \Rightarrow H = x^2 y + \frac{y^4}{4} + \psi(x)$$

Comparing the equations above gives the solution

$$H(x, y) = x^2 y + \frac{y^4}{4} = C$$

Second-Order Linear ODE

The Homogeneous Case

- A second order linear ODE is **homogeneous** if it is of the form:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0.$$

- If y_1 and y_2 are two solutions to this ODE, then any linear combination (i.e. $Ay_1 + By_2$) is also a solution.
- If y_1 and y_2 are two linearly independent solutions to the above ODE, then every solution can be written in the form $y = Ay_1 + By_2$.

Second-Order Linear ODE

Method

- To solve second-order linear ODE's of the form:
$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0,$$
substitute in the solution $y = Ae^{\lambda x}$ (a function that does not change too much upon differentiation and thus cancel each other out to give 0):
$$\lambda^2(Ae^{\lambda x}) + a\lambda(Ae^{\lambda x}) + b(Ae^{\lambda x}) = 0$$
- Factorising out $Ae^{\lambda x}$ produces the **characteristic equation**, which allows us to find the λ's:
$$\lambda^2 + a\lambda + b = 0$$

Second-Order Linear ODE

Solving the characteristic equation leads to one of three cases:

i) If there are two distinct, real roots (λ_1 and λ_2), then the general solution is:

$$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}.$$

ii) If there is one repeated real root (λ_1), then the general solution is:

$$y = Ae^{\lambda_1 x} + Bxe^{\lambda_1 x}.$$

iii) If there are two complex conjugate roots ($\alpha \pm \beta i$), then the general solution is:

$$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x).$$

Second-Order Linear ODE

MATH1241 November 2012

a) Write down the general solution to the second order differential equation
$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0, y(0) = 0.$$

b) For what value of x does the solution have a maximum?

Second-Order Linear ODE

a) First, solve the corresponding characteristic equation:

$$\begin{aligned} \lambda^2 + 6\lambda + 9 &= 0 \\ (\lambda + 3)^2 &= 0 \\ \lambda &= -3 \end{aligned}$$

Since there is a repeated root, the general solution will be in the form:

$$y = Ae^{-3x} + Bxe^{-3x}$$

Substituting $y(0) = 0$ yields $A = 0$. Hence the general solution is

$$y = Bxe^{-3x}.$$

Second-Order Linear ODE

b)

$$
\begin{aligned}
y &= Bxe^{-3x} \\
\frac{dy}{dx} &= Be^{-3x} - 3Bxe^{-3x} \\
&\text{at the maximum, } \frac{dy}{dx} = 0. \\
0 &= Be^{-3x}(1 - 3x) \\
1 - 3x &= 0 \\
x &= \frac{1}{3}
\end{aligned}
$$

Second-Order Linear ODE

The Non-Homogeneous Case

A non-homogeneous ODE will be in the form:

$$y'' + ay' + by'' = f(x).$$

Method

- To solve this, we first find the homogeneous solution before looking for a **particular solution**. The general solution will then be a sum of these two:

$$y = y_H + y_P.$$

- To find the particular solution (y_P), we make a “guess”, which will depend on the form of f.

Second-Order Linear ODE

- If any term for the guess for y_P is a homogeneous solution, then multiply it by x. If it is still a homogeneous solution, then multiply it by x again.
- After making the appropriate guess for the particular solution, substitute it into the ODE and equate to find the unknown coefficients.
- Add the particular solution to the homogeneous solution to get the general solution.
- If initial values are given, substitute them in at this point to find the coefficients from the homogeneous solution.

Second-Order Linear ODE

Guesses for y_P

$f(x)$	Guess for particular solution y_p
$P(x)$ (polynomial of degree n)	$Q(x)$ (polynomial of degree n)
$P(x)e^{sx}$	$Q(x)e^{sx}$
$P(x)\cos(sx)$ or $P(x)\sin(sx)$	$Q_1(x)\cos(sx) + Q_2(x)\sin(sx)$
$P(x)e^{sx}\cos(tx)$ or $P(x)e^{sx}\sin(tx)$	$Q_1(x)e^{sx}\cos(tx) + Q_2(x)e^{sx}\sin(tx)$

If $P(x)$ is a constant, then $Q(x)$ is also a constant.

Second-Order Linear ODE

MATH1241 S2 2018

a) Find the general solution of the following differential equation.

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$$

b) What form of solution would you try in order to find a particular solution to the following differential equation?

$$\frac{d^2y}{dx^2} + 4\frac{y}{x} + 4y = e^{-2x}$$

Second-Order Linear ODE

a) Solve the characteristic equation:

$$\begin{aligned} \lambda^2 + 4\lambda + 4 &= 0 \\ \lambda &= -2 \\ y_H &= Ae^{-2x} + Bxe^{-2x} \end{aligned}$$

b) As e^{-2x} and xe^{-2x} are part of the solution for the homogeneous equation, the particular solution should be in the form:

$$y = Cx^2e^{-2x}.$$

Second-Order Linear ODE

Example

Solve the ODE

$$y''(x) - 4y'(x) + 5y(x) = 20e^{-x}$$

Second-Order Linear ODE

Find the homogeneous solution by solving the characteristic equation:

$$\begin{aligned} \lambda^2 - 4\lambda + 5 &= 0 \\ \lambda &= 2 \pm i \\ y_H &= e^{2x}(A\cos x + B\sin x) \end{aligned}$$

Find the particular solution y_P:

$$\begin{aligned} y_P &= ae^{-x} \\ y'_P &= -ae^{-x} \\ y''_P &= ae^{-x} \end{aligned}$$

Substitute into ODE:

$$\begin{aligned} ae^{-x} + 4ae^{-x} + 5ae^{-x} &= 20e^{-x} \\ a &= 2 \end{aligned}$$

Hence

$$y = y_H + y_P = e^{2x}(A\cos x + B\sin x) + 2e^{-x}$$

Second-Order Linear ODE

MATH1251 S2 2017

Solve the following initial-value problem

$$y'' - 5y' + 6y = 10e^{2x}, \quad y(0) = 1, \quad y'(0) = 1.$$

Solve the characteristic equation for the homogeneous problem:

$$\lambda^2 - 5\lambda + 6\lambda = 0$$

$$\lambda = 2, 3$$

$$y_H = Ae^{2x} + Be^{3x}$$

Second-Order Linear ODE

Find the particular solution:

$$\begin{aligned} y_P &= Cxe^{2x} \\ y_P' &= C(e^{2x} + 2xe^{2x}) \\ y_P'' &= C(4e^{2x} + 4xe^{2x}) \end{aligned}$$

Substitute into the equation:

$$\begin{aligned} C\left[(4e^{2x} + 4xe^{2x}) - 5(e^{2x} + 2xe^{2x}) + 6(xe^{2x})\right] &= 10e^{2x} \\ C\left[(4-5)e^{2x} + (4 - 10 + 6)xe^{2x}\right] &= 10e^{2x} \end{aligned}$$

$$\begin{aligned} -Ce^{2x} &= 10e^{2x} \\ C &= -10 \end{aligned}$$

Second-Order Linear ODE

$$
\begin{aligned}
y &= y_H + y_P \\
&= Ae^{2x} + Be^{3x} - 10xe^{2x}
\end{aligned}
$$

Substitute initial values:

$$
\begin{gathered}
y(0) = Ae^0 + Be^0 - 10(0)e^0 = 1 \\
\implies A + B = 1 \\
y'(0) = 2Ae^0 + 3Be^0 - 10e^0 - 20(0)e^0 = 1 \\
\implies 2A + 3B - 10 = 1 \\
\implies 2A + 3B = 11
\end{gathered}
$$

$$
\begin{aligned}
A &= -8 \\
B &= 9
\end{aligned}
$$

$$
y = -8e^{2x} + 9e^{3x} - 10xe^{2x}
$$

4. Taylor Series

Applications to Stationary Points

Classifying Stationary Points

- Suppose that a function f is n times differentiable at a and that $f'(a) = 0$. If

$$f''(a) = f'''(a) = ... = f^{(k-1)}(a) = 0$$

but $f^{(k)}(a) \neq 0$, where $k \leq n$, then

i) a is a local minimum point if k is even and $f^{(k)}(a) > 0$ (e.g. $f''(a) > 0$);

ii) a is a local maximum if k is even and $f^{(k)}(a) < 0$;

iii) a is a horizontal point of inflexion if k is odd.

Sequences I

Definition of a Sequence

- A **sequence** is a function with the natural numbers as its domain and real numbers as its codomain. Sequences have their own notation:

$$\{a_n\} \text{ or } \{a_n\}_{n=0}^{\infty}$$

- Sequences are defined by a rule which tells us how to find each term. For example:

$$a_n = n^2$$

$$a_n = a_{n-1} + a_{n-2}$$

Sequences II

Convergence and Divergence

- A sequence a_n is **convergent** if it approaches some finite number L as n approaches infinity.

$$\lim_{n\to\infty} a_n = L$$

- A sequence that is not convergent is **divergent**. A divergent sequence can diverge to infinity, diverge to negative infinity, be boundedly divergent, or unboundedly divergent. An example of a boundedly divergent sequence is:

$$a_n = \sin n.$$

- If $a_n = f(n)$ for all large n and $\lim_{x\to\infty} f(x)$ exists and is finite, then

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$$

Taylor Series Example I

MATH1231 (S2, 2016) Q2 i) a)

Determine whether the sequence

$$\sqrt{n+\sqrt{n}}-\sqrt{n}$$

converges or diverges as $n \to \infty$. If it converges, find its limit.

Taylor Series Example I

$$\begin{aligned}
\sqrt{n+\sqrt{n}}-\sqrt{n} &= \frac{\sqrt{n}}{\sqrt{n+\sqrt{n}}+\sqrt{n}} \\
&= \frac{1}{\sqrt{\frac{n+\sqrt{n}}{n}}+1} \\
&= \frac{1}{\sqrt{1+\frac{1}{\sqrt{n}}}+1} \\
&\to 0.5 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

So sequence converges to 0.5.

Sequences III

Combination of Sequences

- Since sequences are a type of function with the same domain ($\mathbb{N}$), they can be added, subtracted, multiplied and divided to produce a new sequence.

$$\{a_n\} + \{b_n\} = \{a_n + b_n\}$$

- If two sequences are convergent, then the same applies to their limits.

$$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \times \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$$

Sequences IV

Order of Growth

- To determine the convergence/divergence of a sequence composed of elementary functions, it is important to know the order of growth between them.

a_n	growth rate as $n \to \infty$
1	constant
$\log n$	grows slowly
n^k, where $k > 0$	growth rate is faster for larger k
c^n, where $c > 1$	growth rate is faster for larger c
$n!$	grows rapidly
n^n	grows very rapidly

Sequences V

Pinching Theorem For Sequences

- Suppose that $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences and that for all $n > N$ for some positive integer N, the following inequality holds.

  $$a_n \leq b_n \leq c_n$$

  If $\{a_n\}$ and $\{c_n\}$ both converge to some value L, then $\{b_n\}$ converges to L.

Another Test For Convergence

- A sequence is **monotonic** if it is increasing, nondecreasing, decreasing, or non-increasing for all n.
- A sequence is **bounded above** if there exists an M such that $a_n \leq M$ for all natural numbers n.
- A sequence is **bounded below** if there exists an M such that $a_n \geq M$ for all natural numbers n.
- If $\{a_n\}_{n=0}^{\infty}$ is a bounded monotonic sequence of real numbers, then it converges to some real number L.

Sequences VII: but it's still MATH1241

Suprema and Infima

- The **supremum** M of a sequence $\{a_n\}_{n=0}^{\infty}$ is its least upper bound. It has two conditions:
  i) $a_n \leq M$ for all n.
  ii) If K is an upper bound, then $K \geq M$.
- Similarly, the **infimum** of a sequence is its greatest lower bound.
- According to the least upper bound axiom, every nonempty set of real numbers that is bounded above, has a least upper bound.

Taylor Series Example II

MATH1231/1241 Calculus Notes Q16

Find the supremum and infimum of each of the following sets.

a) $\left\{ \dfrac{n}{1+n^2} \,:\, n = 1, 2, \dots \right\}$

e) $\{x \in (0, \infty) \,:\, \sin x < 0\}$

Taylor Series Example II

MATH1231/1241 Calculus Notes Q16

Find the supremum and infimum of each of the following sets.

a) $\left\{ \dfrac{n}{1+n^2} \, : \, n = 1, 2, \dots \right\}$

e) $\{x \in (0, \infty) \, : \, \sin x < 0\}$

a) Since the sequence is strictly decreasing for $n = 1, 2, \dots$, its supremum will be $\frac{1}{2}$ (at $n = 1$).
Since the sequence converges to 0, its infimum will be 0.

Taylor Series Example II

MATH1231/1241 Calculus Notes Q16

Find the supremum and infimum of each of the following sets.

a) $\left\{\dfrac{n}{1+n^2} : n = 1, 2, \dots\right\}$

e) $\{x \in (0, \infty) : \sin x < 0\}$

a) Since the sequence is strictly decreasing for $n = 1, 2, \dots$, its supremum will be $\frac{1}{2}$ (at $n = 1$).
Since the sequence converges to 0, its infimum will be 0.

e) This sequence does not have a supremum due to the periodicity of $\sin x$. Its infimum is π.

Infinite Series I

Sums

- A **partial sum** s_n represents the sum of terms of a sequence up to n.

$$s_n = a_0 + a_1 + \cdots + a_n = \sum_{k=0}^{n} a_k$$

- If the partial sum approaches some finite L as $n \to \infty$, then the **infinite series** is **summable** and converges to L.

$$\lim_{n\to\infty} s_n = \sum_{k=0}^{\infty} a_k = L$$

- If the series does not approach some finite number, then it diverges.

Infinite Series II

Summable Series

- Since summable series can be equated to real numbers, the summations can be manipulated as regular sums:

$$\sum_{k=0}^{\infty}(a_k + b_k) = \sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} b_k$$

$$\sum_{k=0}^{\infty}(\alpha a_k) = \alpha \sum_{k=0}^{\infty} a_k \text{ for every real number } \alpha$$

- As a finite sum (of finite terms) will always be finite, the first N (where $N \in \mathbb{Z}^+$) terms are irrelevant to the convergence of a sum.

$$\sum_{k=0}^{\infty} a_k \quad \text{converges iff} \quad \sum_{k=N}^{\infty} a_k \quad \text{converges.}$$

Tests for Series Convergence I

The kth Term Test for Divergence

- If $\{a_k\} \nrightarrow 0$ as $k \to \infty$, then the series $\displaystyle\sum_{k=0}^{\infty} a_k$ diverges.
- This test is for divergence only and cannot test for convergence. Consider the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$. $\displaystyle\lim_{k\to\infty} \frac{1}{k} = 0$ but series still diverges.

Theorem

If the series $\sum_{k=0}^{\infty} a_k$ converges, then $a_k \to 0$ as $k \to \infty$.

Tests for Series Convergence II

The Comparison Test

- Suppose that $0 \le a_k \le b_k$ for every natural number k. Then
  - i) If $\sum_{k=0}^{\infty} b_k$ converges, then $\sum_{k=0}^{\infty} a_k$ also converges.
  - ii) If $\sum_{k=0}^{\infty} a_k$ diverges, then $\sum_{k=0}^{\infty} b_k$ also diverges.
- A ***p*-series** will converge if $p > 1$ and will diverge if $p \le 1$.

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

Taylor Series Example III

MATH1231 (S2, 2018) Q4 vi)

Suppose that $\sum_{n=0}^{\infty} a_n$ is a convergent series with $a_n > 0$ for all n.

a) State $\lim_{n\to\infty} a_n$.

b) Use the nth test to show that $\sum_{n=0}^{\infty} \ln(a_n)$ diverges.

c) Given that $f(x) = x - \ln(1+x)$ is positive for $x > 0$, determine whether $\sum_{n=0}^{\infty} \ln(1+a_n)$ converges or diverges. Explain your answer.

Taylor Series Example III

a) For the series to converge, we must have $\lim_{n\to\infty} a_n = 0$.

Taylor Series Example III

a) For the series to converge, we must have $\lim_{n\to\infty} a_n = 0$.

b) From a), $\lim_{n\to\infty} \ln(a_n) = -\infty$. Thus, the sequence $\{\ln(a_n)\}$ diverges, and by kth test, the infinite sum diverges.

Taylor Series Example III

a) For the series to converge, we must have $\lim_{n\to\infty} a_n = 0$.

b) From a), $\lim_{n\to\infty} \ln(a_n) = -\infty$. Thus, the sequence $\{\ln(a_n)\}$ diverges, and by kth test, the infinite sum diverges.

c) Rearranging, we know that for $x > 0$,

$$x > \ln(1+x) > 0$$

Then, by substituting $x = a_n$ (as we know $a_n > 0$ for all n), we obtain

$$a_n > \ln(1+a_n) > 0$$

By the comparison test, since $\sum_{n=0}^{\infty} a_n$ converges, then

$\sum_{n=0}^{\infty} \ln(1+a_n)$ also converges.

Taylor Series Example IV

MATH1231 (T1, 2019) Q1 d)

Giving brief reasons, state whether the following is true or false?

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \text{ diverges if } s = \frac{2}{3}.$$

Taylor Series Example IV

MATH1231 (T1, 2019) Q1 d)

Giving brief reasons, state whether the following is true or false?

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \text{ diverges if } s = \frac{2}{3}.$$

True. This is a p-series and will diverge since $s \leq 1$.

Tests for Series Convergence III: but only for the elite

The Limit Form of the Comparison Test

- Suppose that a_n, b_n are positive sequences and suppose that $\lim_{n \to \infty} \frac{a_n}{b_n} = L$, where L is some non-zero, finite number. Then $\sum^{\infty} a_n$ converges if and only if $\sum^{\infty} b_n$ converges.

Tests for Series Convergence IV

The Integral Test

- Replace the formula for a_k with $f(x)$. If $f(x)$ is a continuous, positive function that is decreasing on $[1, \infty)$, then we can use it to apply the integral test:
  i) If $\displaystyle\int_1^\infty f(x)dx$ converges, then so does $\displaystyle\sum_{k=1}^{\infty} a_k$.
  ii) If $\displaystyle\int_1^\infty f(x)dx$ diverges, then so does $\displaystyle\sum_{k=1}^{\infty} a_k$.

Tests for Series Convergence V

The Ratio Test

- Suppose that $\sum a_k$ is an infinite series with positive terms and that

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = r$$

i) If $r < 1$, then $\sum a_k$ converges.
ii) If $r > 1$, then $\sum a_k$ diverges.
iii) If $r = 1$, then the test is inconclusive.

Taylor Series Example V

MATH1231 (S2, 2016) Q2 ii)

By using an appropriate test, determine whether each of the following series converges or diverges.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

Taylor Series Example V

MATH1231 (S2, 2016) Q2 ii)

By using an appropriate test, determine whether each of the following series converges or diverges.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

a) We use the ratio test:

$$\lim_{n\to\infty} \left[\frac{(n+1)^2}{2^{n+1}} \bigg/ \frac{n^2}{2^n} \right] = \lim_{n\to\infty} \frac{n^2+2n+1}{2n^2} = \frac{1}{2}$$

Since the ratio is less than 1, the series converges.

Taylor Series Example V

b) $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k(\ln k)^2}$

Taylor Series Example V

b) $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k(\ln k)^2}$

b) We use the integral test:

$$\begin{aligned}\int_3^\infty \frac{1}{x(\ln x)^2}\,dx &= \int_{\ln 3}^\infty \frac{du}{u^2} \\ &= \left[-\frac{1}{u}\right]_{\ln 3}^\infty \\ &= \frac{1}{\ln 3}\end{aligned}$$

As the integral is finite, series converges.

Tests for Series Convergence VI

Leibniz' Test for Convergence

- An **alternating series** is whose terms have alternating signs. They exist in the form:

$$a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - \cdots$$

- An alternating series of real numbers will converge if the positive versions of its terms satisfy the following:
  i) $a_k \geq 0$;
  ii) $a_k \geq a_{k+1}$ for all k;
  iii) $\lim_{k\to\infty} a_k = 0$.

Absolute and Conditional Convergence

Absolute and Conditional Convergence

- A series is **absolutely convergent** if the following is convergent.

$$\sum_{k=0}^{\infty} |a_k|$$

- Absolute convergence implies convergence.
- If a series converges, but does not converge absolutely, then it is **conditionally convergent**.
- A series that converges absolutely will converge to a unique value. A series that converges conditionally can be rearranged to converge to any real number, or even to diverge.

Taylor Series Example VI

MATH1251 (S2, 2018) Q3 iv)

Determine whether the series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \log(n)}$$

absolutely or conditionally converges, or diverges. Provide reasons.

Taylor Series Example VI

MATH1251 (S2, 2018) Q3 iv)

Determine whether the series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n\log(n)}$$

absolutely or conditionally converges, or diverges. Provide reasons.

First, apply Leinbiz' Test.

i) $\dfrac{1}{n\log(n)}$ is non-negative for $n \geq 2$.

ii) $\dfrac{1}{(n+1)\log(n+1)} < \dfrac{1}{n\log(n+1)} < \dfrac{1}{n\log(n)}$. Therefore the terms are decreasing.

iii) $\lim\limits_{n\to\infty} \dfrac{1}{n\log(n)} = 0$ since $\lim\limits_{n\to\infty} n\log(n) \to \infty$.

Taylor Series Example VI

As it passes the Leibniz' test, the series converges. To test for absolute convergence, use the integral test.

$$\begin{aligned}\int_2^\infty \frac{1}{n\log(n)}dx &= \int_{\log(2)}^\infty \frac{du}{u} \\ &= \Big[\log(u)\Big]_{\log(2)}^\infty\end{aligned}$$

As this tends to infinity, the positive series fails the integral test and so the series converges conditionally.

Taylor Series I

Introduction to Taylor Series

- **Taylor Series** are infinite sums used to approximate "smooth" functions.

$$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

- By approximating functions as polynomials, they become easier to understand as well as to compute.

Taylor Series II

Taylor Polynomials

- The nth **taylor polynomial** for a "smooth" function f about a is defined by:

$$
\begin{aligned}
p_n(x) &= f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!} \\
&= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k
\end{aligned}
$$

Taylor Series III

Taylor's Theorem

- **Taylor's theorem** states that a function f that has $n+1$ continuous derivatives on an open interval I containing a can be approximated using a Taylor polynomial.

$$f(x) = p_n(x) + R_{n+1}(x)$$

- The remainder can be found exactly as:

$$R_{n+1}(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n dt$$

Taylor Series IV

Lagrange Form

- Since this is usually difficult to compute, a more convenient form is the **Lagrange form** of the remainder:

$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some real number c between a and x.

Taylor Series Example VII

MATH1241 (T1, 2020) Q6

Let $P(x)$ be a real polynomial of degree N and $c \in \mathbb{R}$. Using Taylor Polynomials, we can always write:

$$P(x) = \sum_{i=0}^{M} a_i (x-c)^i$$

Explain why this is true. In particular:

- state any theorem you would use to prove the equality above;
- give an expression for the largest M such that $a_M \neq 0$ in terms of N and/or $P(x)$;
- explain how the numbers a_i are obtained in terms of $P(x)$.

Taylor Series Example VII

$$P(x) = \sum_{i=0}^{M} a_i (x-c)^i$$

- Since $P(x)$ is a polynomial, it is infinitely differentiable and by Taylor's theorem, we can always approximate it using a Taylor Polynomial of any degree M.
- $a_i = \dfrac{P^{(i)}(c)}{i!}$.
- $a_M = \frac{P^{(M)}(c)}{M!}$ will equal 0 for any $M \geq N+1$, as $P(x)$ is of degree N
- The largest M such that $a_M \neq 0$ is $M = N$.

Taylor Series V

Taylor Series

- A **Taylor Series** for a function f about a is its Taylor polynomial where $n \to \infty$. For the case where $a = 0$, the series is also called the **Maclaurin Series** for f.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

- If $\lim_{n \to \infty} R_{n+1}(x) = 0$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Taylor Series VI

Some examples of convergent Taylor Series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \qquad x \in (-1, 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \qquad x \in \mathbb{R}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad x \in \mathbb{R}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad x \in \mathbb{R}$$

$$\ln(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots \qquad x \in (-1, 1]$$

Power Series I

Power Series

- A Taylor Series is a type of **power series**, which is just a sum of integer powers of x:

$$\sum_{k=0}^{\infty} a_k (x-a)^k,$$

where $\{a_k\}_{k=0}^{\infty}$ is a sequence of real coefficients.

Power Series II

Convergence/Divergence of Power Series

- As with any series, a power series may converge or diverge. However, its convergence/divergence depends on the value of x.
- If a power series of the form $\sum_{k=0}^{\infty} a_k(x-a)^k$ converges for all points in some interval $(-R+a,\ R+a)$, then R is called the **radius of convergence** and this interval is called the **interval of convergence**.

Power Series III

Radius of Convergence

- Suppose that for the sequence of coefficients $\{a_k\}_{k=0}^{\infty}$,

$$\lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right| = R$$

  for some real number R. Then R is the radius of convergence and the respective power series will:

  i) converge absolutely whenever $|x-a| < R$;
  ii) diverge whenever $|x-a| > R$.

- If the limit does not exist, the radius of convergence can still exist.
- To test at the endpoints, substitute the appropriate values for x and determine convergence/divergence using the previous methods for series (endpoint testing is reserved for the 1241 club)

Taylor Series Example VIII

MATH1251 (S2, 2018) Q4 i)

Find the interval of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{(x-3)^n}{3^n+1}$$

Make sure that you consider the behaviour at the end-points of your interval and provide reasons for your answers.

Taylor Series Example VIII

MATH1251 (S2, 2018) Q4 i)

Find the interval of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{(x-3)^n}{3^n+1}$$

Make sure that you consider the behaviour at the end-points of your interval and provide reasons for your answers.

First, we find R:

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{1}{3^n+1} \middle/ \frac{1}{3^{n+1}+1} \right| &= \lim_{n\to\infty} \frac{3^{n+1}+1}{3^n+1} \\ &= \lim_{n\to\infty} \frac{3+\frac{1}{3^n}}{1+\frac{1}{3^n}} \\ &= 3 \end{aligned}$$

Taylor Series Example VIII

So we know our interval of convergence is (0, 6).
At the endpoints, both series diverge

$$\sum_{n=0}^{\infty} \frac{(-3)^n}{3^n + 1} \quad \text{or} \quad \sum_{n=0}^{\infty} \frac{3^n}{3^n + 1}$$

since both series do not pass the kth term test.
Therefore the series does not converge at either endpoint and so the interval of convergence is (0, 6).

Power Series IV

Manipulation of Power Series

- Within their respective intervals of convergence, power series can be added or multiplied together, differentiated or integrated. For example:

$$f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$$

$$f'(x) = \sum_{k=1}^{\infty} k a_k (x-a)^{k-1}$$

$$F(x) = \sum_{k=0}^{\infty} \frac{a_k}{k+1} (x-a)^{k+1} + C$$

- Any function that can be expressed as a power series is continuous and differentiable (for all orders) within its radius of convergence.

Taylor Series Example IX

MATH1251 (S2, 2018) Q4 ii)

a) Write down the Taylor Series for $f(x) = \sin(x^2)$ about $x = 0$ and state its radius of convergence.

Taylor Series Example IX

MATH1251 (S2, 2018) Q4 ii)

a) Write down the Taylor Series for $f(x) = \sin(x^2)$ about $x = 0$ and state its radius of convergence.

a) We can use the Taylor Series for $\sin(x)$ about $x = 0$:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2(2k+1)}}{(2k+1)!}$$

As the series will always converge, the radius of convergence is infinite.

Taylor Series Example IX

b) Use part (a) to determine an infinite series for the integral

$$I = \int_0^1 \sin(x^2)\,dx.$$

Taylor Series Example IX

b) Use part (a) to determine an infinite series for the integral

$$I = \int_0^1 \sin(x^2)dx.$$

b)

$$\begin{aligned} I &= \sum_{k=0}^{\infty} \int_0^1 \frac{(-1)^k x^{2(2k+1)}}{(2k+1)!} dx \\ &= \sum_{k=0}^{\infty} \left[\frac{(-1)^k x^{4k+3}}{(4k+3)(2k+1)!} \right]_0^1 \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+3)(2k+1)!} \end{aligned}$$

5. Applications of Integration

Average Value of a Function I

Average Value of a Function

- Suppose that f is integrable on a closed interval $[a, b]$. Then the **average value** of f in this interval is defined as:

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)dx$$

Average Value of a Function II

Mean Value Theorem for Integrals

- Suppose that f is continuous on $[a, b]$. Then, there exists a $c \in (a, b)$ such that

$$\int_a^b f(t)dt = f(c)(b-a).$$

- This can be rewritten to resemble the typical mean value theorem in the following way:

$$\frac{F(b) - F(a)}{b-a} = F'(c).$$

Arc Length of a Curve I

Arc Length of a Parametrised Curve

- Curves are typically expressed in the following parametric form:

$$\mathcal{C} = \{(x(t), y(t)) \in \mathbb{R}^2 : a \leq t \leq b\}.$$

- The length of of the curve can calculated by the formula:

$$\ell = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2} dt.$$

- It is important that the path does not retrace its steps.

Arc Length of a Curve II

Arc Length of a Function

- Where the curve is expressed as a function of x, the arc length on the interval $[a, b]$ is given by:

$$\ell = \int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

Applications of Integration Example I

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Let $h(x) = \cosh(x)$ where $a \leq x \leq b$. Define L to be the arc length of the graph of h between $x = a$ and $x = b$ and define A to be the area bounded by the graph of h and the x-axis between $x = a$ and $x = b$. Prove that $L = A$ for all $a, b \in \mathbb{R}$.

Applications of Integration Example I

MATH1231 (S2, 2015) Q4 vi)

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Using the properties $\frac{d}{dx}\cosh x = \sinh x$ and $\cosh^2 x - \sinh^2 x = 1$, we know that

$$\begin{aligned} L &= \int_a^b \sqrt{1 + \sinh^2(x)}dx \\ &= \int_a^b \sqrt{\cosh^2(x)}dx \\ &= \int_a^b \cosh(x)dx \end{aligned}$$

Applications of Integration Example I

Additionally,

$$A = \int_a^b \cosh(x)\,dx$$

Therefore, $L = A$.

Arc Length of a Curve III

Arc Length of a Polar Curve

- Where the curve is expressed using polar coordinates in the form

$$r = f(\theta),$$

the length of the arc is then given by:

$$\ell = \int_{\theta_0}^{\theta_1} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

Surface Areas

Surface Area Formulae

- Suppose we have have a curve $\mathcal{C}$ that is **simple** and lies above or on the x-axis. When rotated around the x-axis, the surface area can be found using one of the following:

$$A = \int_a^b 2\pi y(t)\sqrt{x'(t)^2 + y'(t)^2}dt, \tag{1}$$

$$A = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}dx, \tag{2}$$

$$A = \int_{\theta_0}^{\theta_1} 2\pi r \sin\theta\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}d\theta. \tag{3}$$

Applications of Integration Example II

MATH1231 (S2, 2018) Q4 v)

A surface is formed by rotating the curve $y = \frac{1}{4}x^2 - 1$ for $2 \leq x \leq 3$ around the y-axis. What is its surface area?

Applications of Integration Example II

MATH1231 (S2, 2018) Q4 v)

A surface is formed by rotating the curve $y = \frac{1}{4}x^2 - 1$ for $2 \le x \le 3$ around the y-axis. What is its surface area?

Notice that the curve is rotated around the y-axis. Then the equation becomes $x = \sqrt{4(y+1)}$, with the bounds as $0 \le y \le \frac{5}{4}$.

$$\begin{aligned} f(y) &= 2\sqrt{y+1} \\ f'(y) &= \frac{2}{2\sqrt{y+1}} \\ &= \frac{1}{\sqrt{y+1}} \\ A &= \int_0^{\frac{5}{4}} 2\pi\left(2\sqrt{y+1}\right)\sqrt{1+\left(\frac{1}{\sqrt{y+1}}\right)^2}dy \end{aligned}$$

Applications of Integration Example II

$$
\begin{aligned}
A &= 4\pi \int_0^{\frac{5}{4}} \sqrt{y+1}\sqrt{1+\frac{1}{y+1}}dy \\
&= 4\pi \int_0^{\frac{5}{4}} \sqrt{y+1}\sqrt{\frac{y+2}{y+1}}dy \\
&= 4\pi \int_0^{\frac{5}{4}} \sqrt{y+2}dy \\
&= 4\pi \left[\frac{2}{3}(y+2)^{\frac{3}{2}}\right]_0^{\frac{5}{4}} \\
&= \frac{8\pi}{3}\left[\left(\frac{13}{4}\right)^{\frac{3}{2}} - 2^{\frac{3}{2}}\right] \\
&= \frac{(13\sqrt{13}-16\sqrt{2})\pi}{3}u^2
\end{aligned}
$$