MATH2621 Homework Worked Solutions

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We cannot guarantee that our working is correct, or that it would obtain full marks - please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are quite a lot of questions and as a new initiative we're unable to provide solutions for every question, so we've picked out some of the important ones. There are sometimes multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

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11. For the first part of the question, we need to show that any root z of $z^4 + z + 3 = 0$ satisfies $|z| > 1$. We prove by contradiction.
    Assume $|z| \leq 1$,

$$\begin{aligned}
|z^4 + z + 3| &\geq ||z|^4 - |z+3|| && \text{(Circle Inequality)} \\
&= |z+3| - |z|^4 && (|z| \leq 1) \\
&\geq ||z| - 3| - |z|^4 \\
&= 3 - |z| - |z|^4 \\
&\geq 3 - 1 - 1 = 1
\end{aligned}$$

Therefore, we can see that $z^4 + z + 3 \neq 0$. Thus, by proof of contradiction $|z| > 1$.

22. We first make z the subject of the transformation $w = iz + 2$.

$$z = \frac{w-2}{i} = i(2-w)$$

Now, we consider the image of the region.

$$Re(z) = Im(w)$$
$$\therefore 0 < Re(z) = Im(w) < \frac{\pi}{2}$$

$$Im(z) = 2 - Re(w)$$
$$\implies Im(z) = 2 - Re(w) > 0$$
$$\implies Re(w) < 2$$

24. (a) Our goal is to find the image of the region $|z-1| \leq 1$ under the mapping $w = \frac{z}{z+2}$. First we observe that the value of $z = -2$ does not need to be considered since it is outside of our domain.

Thus rearranging we see,

$$\begin{aligned} w &= \frac{z}{z+2} \\ w(z+2) &= z \\ zw + 2w &= z \\ z(w-1) &= -2w \\ z &= \frac{-2w}{w-1} \end{aligned}$$

noting the $w \neq 1$ since $1 = \frac{z}{z+2} \Rightarrow 2 = 0$ which is impossible. Now,

$$\begin{aligned} |z-1| &= \left|\frac{-2w}{w-1} - 1\right| \\ &= \left|\frac{-3w+1}{w-1}\right| \\ &= \left|\frac{3w-1}{w-1}\right| \leq 1 \quad \text{via our considered region.} \end{aligned}$$

$\Rightarrow |3w-1| \leq |w-1|$. Finally, let $w = u + iv$,

$$\Rightarrow (3u-1)^2 + (3v)^2 \leq (u-1)^2 + v^2$$
$$\Rightarrow \left(u - \frac{1}{4}\right)^2 + v^2 \leq \frac{1}{16}$$

and therefore the image of the region under the mapping is a circle with centre $(\frac{1}{4}, 0)$

with radius $\frac{1}{4}$.

41. Part (f)

We have $f(x+iy) = |x| + i|y|$. Writing $z = x+iy$, let $f(z) = u + iv$. We need to find $z \in \mathbb{C}$ such that $u, v \in C'$ and the Cauchy-Riemann equations hold.

Now, u and v are differentiable functions on $\mathbb{R}^2 \setminus \{x = 0 \vee y = 0\}$, such that

$$\begin{aligned} u(x,y) &= |x| \\ v(x,y) &= |y|. \end{aligned}$$

Then

$$\begin{aligned} u_x &= \frac{x}{|x|} & \qquad u_y &= 0 \\ v_x &= 0 & \qquad v_y &= \frac{y}{|y|} \end{aligned}$$

Setting $u_x = v_y$ and $u_y = -v_x$, we find that

$$\begin{aligned} &\frac{x}{|x|} = \frac{y}{|y|} \qquad \text{and} \qquad 0 = 0 \\ \iff\ & xy > 0. \qquad \text{(i.e. } x,\ y \text{ both positive or both negative.)} \end{aligned}$$

So f is differentiable on $\{z = x + iy \mid xy > 0\}$.