MATH1031 Mastery Lab Test 2
All Questions Sample Solutions

October 23, 2019

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NOTE: Any text presented `like this` is presented as required in Maple syntax.

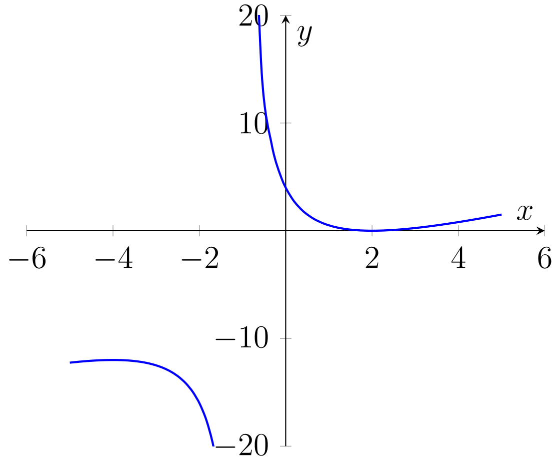
Question 2

The function

$$f(x) = x - 5 + \frac{9}{x+1}$$

has a local maximum and a local minimum.
Find the coordinates of the local maximum.
Find the coordinates of the local minimum.
Give the expression in x (in Maple syntax) for $f''(x)$.
What are the values of x for which the graph of $y = f(x)$ is concave upwards?

Solution



To find the local extrema, we begin by taking the derivative of $f(x)$:

$$f'(x) = 1 - \frac{9}{(x+1)^2}.$$

By the shape of the graph of $y = f(x)$ we deduce that the local maximum will occur to the left of -1, and the local minimum will occur to the right of -1. To find these points we solve $f'(x) = 0$:

$$\begin{aligned} 1 - \frac{9}{(x+1)^2} &= 0 \\ (x+1)^2 &= 9 \\ x &= -1 \pm 3 \\ &= -4, 2. \end{aligned}$$

$f(-4) = -12$ and $f(2) = 0$ so the coordinates of the local maximum are

$$x = -4, \quad y = -12$$

and the coordinates of the local minimum are

$$x = 2, \quad y = 0.$$

To find the second derivative of $f(x)$ we differentiate $f'(x)$:

$$f''(x) = \frac{18}{(x+1)^3}$$

or in Maple syntax, `18/(x+1)^3`. Observe that $f''(x)$ is not defined at $x = -1$, it is positive for $x > -1$, and it is negative for $x < -1$. Hence the graph of $y = f(x)$ is concave up for $x > -1$. We can also note that since it is concave up for $x > -1$, the extreme at $x = 2$ is a local minimum, if you can't visualise the graph's shape.

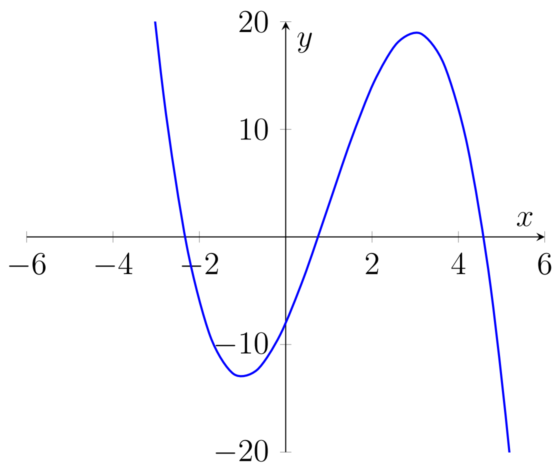
Question 3

The function

$$f(x) = -x^3 + 3x^2 + 9x - 8$$

has a local maximum and a local minimum.
Find the coordinates of the local maximum.
Find the coordinates of the local minimum.
Give the expression in x (in Maple syntax) for $f''(x)$.
What are the values of x for which the graph of $y = f(x)$ is concave upwards?

Solution



To find the local extrema, we begin by taking the derivative of $f(x)$:

$$f'(x) = -3x^2 + 6x + 9.$$

Then by considering the shape of the graph of $y = f(x)$, we know that the local maximum will occur at the larger zero of $f'(x)$ and the local minimum will occur at the smaller zero of $f'(x)$. To find these points, we solve $f'(x) = 0$:

$$\begin{aligned} -3x^2 + 6x + 9 &= 0 \\ x^2 - 2x - 3 &= 0 \\ (x+1)(x-3) &= 0. \end{aligned}$$

Hence $x = -1$ or $x = 3$. $f(-1) = -13$ and $f(3) = 19$ so the coordinates of the local maximum are

$$x = 3, \quad y = 19,$$

and the coordinates of the local minimum are

$$x = -1, \quad y = -13.$$

To find the second derivative of $f(x)$ we differentiate $f'(x)$:

$$f''(x) = -6x + 6$$

or in Maple syntax, `-6*x+6`. Here, $f''(x) > 0$ for $x < 1$ so the graph of $y = f(x)$ is concave up for $x < 1$. Again, this also means that the extreme at $x = -1$ is a local minimum.

Question 4

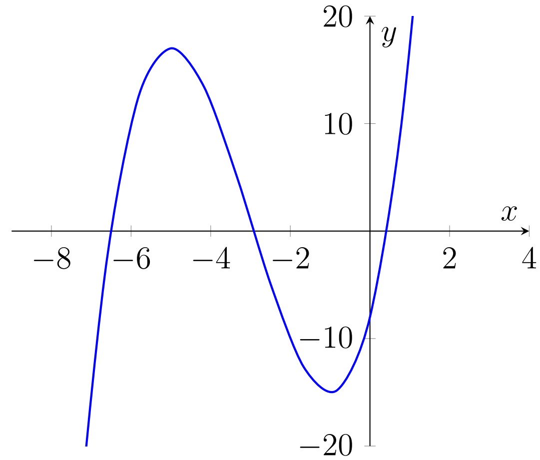
The function

$$f(x) = x^3 + 9x^2 + 15x - 8$$

has a local maximum and a local minimum.
Find the coordinates of the local maximum.
Find the coordinates of the local minimum.
Give the expression in x (in Maple syntax) for $f''(x)$.

What are the values of x for which the graph of $y = f(x)$ is concave upwards?

Solution



To find the local extrema, we begin by taking the derivative of $f(x)$:

$$f'(x) = 3x^2 + 18x + 15.$$

Then by considering the shape of the graph of $y = f(x)$, we know that the local maximum will occur at the smallest zero of $f'(x)$ and the local minimum will occur at the largest zero of $f'(x)$. To find these points, we solve $f'(x) = 0$:

$$\begin{aligned} 3x^2 + 18x + 15 &= 0 \\ x^2 + 6x + 5 &= 0 \\ (x+5)(x+1) &= 0. \end{aligned}$$

Hence $x = -5$ or $x = -1$. $f(-5) = 17$ and $f(-1) = -15$ so the coordinates of the local maximum are

$$x = -5, \quad y = 17,$$

and the coordinates of the local minimum are

$$x = -1, \quad y = -15.$$

To find the second derivative of $f(x)$ we differentiate $f'(x)$:

$$f''(x) = 6x + 18$$

or in Maple syntax, `6*x+18`. Observe that $f''(-3) = 0$. Then $f''(x)$ is positive for $x > -3$. Hence the graph of $y = f(x)$ is concave up for $x > -3$. Again, this also means that the extreme at $x = -1$ is a local minimum.

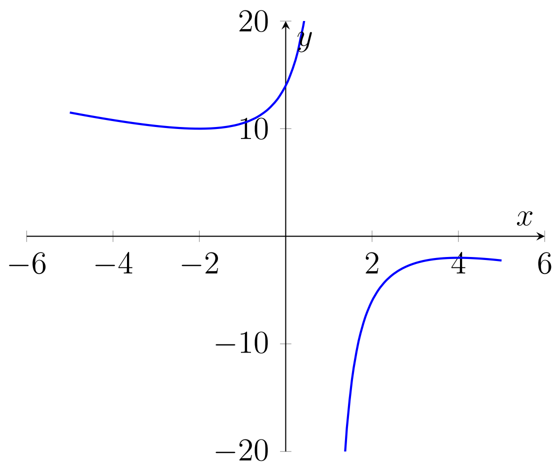
Question 5

The function

$$f(x) = -x + 5 - \frac{9}{x-1}$$

has a local maximum and a local minimum.
Find the coordinates of the local maximum.
Find the coordinates of the local minimum.
Give the expression in x (in Maple syntax) for $f''(x)$.
What are the values of x for which the graph of $y = f(x)$ is concave upwards?

Solution



To find the local extrema, we begin by taking the derivative of $f(x)$:

$$f'(x) = -1 + \frac{9}{(x-1)^2}.$$

By the shape of the graph of $y = f(x)$ we deduce that the local maximum will occur to the right of 1, and the local minimum will occur to the left of 1. To find these points we solve $f'(x) = 0$:

$$\begin{aligned}
-1 + \frac{9}{(x-1)^2} &= 0 \\
(x-1)^2 &= 9 \\
x &= -2, 4.
\end{aligned}$$

$f(-2) = 10$ and $f(4) = -2$ so the coordinates of the local maximum are

$$x = 4, \quad y = -2$$

and the coordinates of the local minimum are

$$x = -2, \quad y = 10.$$

To find the second derivative of $f(x)$ we differentiate $f'(x)$:

$$f''(x) = -\frac{18}{(x-1)^3}$$

or in Maple syntax, `-18/(x-1)^3`. Observe that $f''(x)$ is not defined at $x = 1$, it is positive for $x < 1$, and it is negative for $x > 1$. Hence the graph of $y = f(x)$ is concave up for $x < 1$. Again, this also means that the extreme at $x = -2$ is a local minimum.

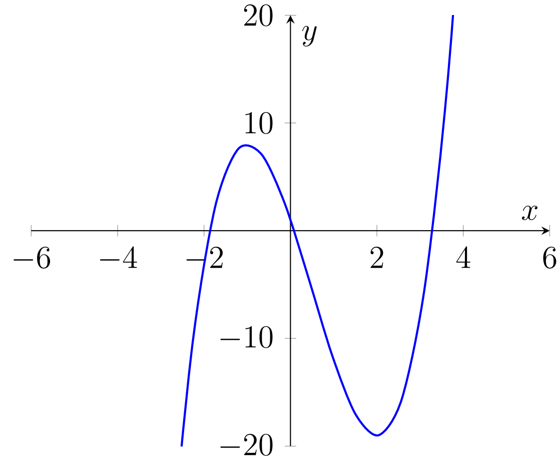
Question 6

The function

$$f(x) = 2x^3 - 3x^2 - 12x + 1$$

has a local maximum and local minimum.
Find the coordinates of the local maximum.
Find the coordinates of the local minimum.
Find the absolute maximum and minimum values for $f(x)$ on the interval $[-2, 3]$.

Solution



To find the local extrema, we begin by taking the derivative of $f(x)$ as before:

$$f'(x) = 6x^2 - 6x - 12.$$

By the cubic shape of the graph of $y = f(x)$ we deduce that the local maximum will occur at the smallest zero of $f'(x)$, and the local minimum will occur at the largest zero of $f'(x)$. To find these points we solve $f'(x) = 0$:

$$\begin{aligned} 6x^2 - 6x - 12 &= 0 \\ x^2 - x - 2 &= 0 \\ (x+1)(x-2) &= 0. \end{aligned}$$

Hence $x = -1$ or $x = 2$. $f(-1) = 8$ and $f(2) = -19$ so the coordinates of the local maximum are

$$x = -1, \quad y = 8$$

and the coordinates of the local minimum are

$$x = 2, \quad y = -19.$$

Now we consider our function on the interval $[-2, 3]$. Both extrema are in this interval, so we need only to find the values of f at the endpoints of the interval. $f(-2) = -3$ and $f(3) = -8$, so our absolute minimum value on $[-2, 3]$ is -19 and the absolute maximum value on $[-2, 3]$ is 8. In this case, the endpoints don't change our answer. However, if we were asked to consider the interval $[-2, 4]$, then the endpoint $x = 4$ would give $f(4) = 33$, which is larger than the local maximum, so the absolute maximum value on $[2, 4]$ would be 33.

Question 7

Assume that the rate at which a cup of coffee cools to room temperature is proportional to the difference between its temperature $T\,°C$ and the ambient room temperature $R\,°C$. A cup of coffee is placed in a $25\,°C$ room, cools from $97\,°C$ to $82\,°C$ in 3 minutes. The temperature varies over time t (in minutes) according to

$$T = R + Ae^{-kt}, \quad t \geq 0.$$

(a) Enter the values of R and A.

(b) Enter the exact value of k in Maple syntax.

(c) At what time t will the temperature of the cup of coffee be $60\,°C$?

(d) Find the limiting value of the temperature.

Solution

(a) Given that the ambient room temperature is $25\,°C$, so $R = 25$. Then at time $t = 0$ (initially), $T = 97$. Solving for A,

$$\begin{aligned} R + Ae^0 &= 97 \\ 25 + A &= 97 \\ A &= 72. \end{aligned}$$

Hence

$$T = 25 + 72e^{-kt}.$$

(b) We are given that at $t = 3$, $T = 82$. Hence

$$\begin{aligned} 25 + 72e^{-3k} &= 82 \\ e^{-3k} &= \frac{19}{24} \\ k &= \frac{1}{3}\ln\left(\frac{24}{19}\right), \end{aligned}$$

or in Maple syntax `(1/3)*ln(24/19)`.

(c) We want $T = 60$, so solving for t:

$$\begin{aligned} 25 + 72e^{-kt} &= 60 \\ e^{-kt} &= \frac{35}{72} \\ -kt &= \ln\left(\frac{35}{72}\right) \\ kt &= \ln\left(\frac{72}{35}\right) \\ t &= 3\frac{\ln(72/35)}{\ln(24/19)} \\ &\approx 9.262913559... \end{aligned}$$

I.e $t = 9.3$ to one decimal place.

(d) As $t \to \infty$, $e^{-kt} \to 0$ so $T \to 25$.

Question 8

During a fishing trip Alex notices that the height h of the tide (in metres) is given by

$$h = 1 - \frac{1}{4}\cos\left(\frac{\pi t}{3}\right)$$

where t is measured in hours from the start of the trip.

(a) Enter the exact value of h at the start of the trip.

(b) It is low tide at the start of the trip. Enter the value of t at which next low tide occurs.

(c) *Enter the exact value, in Maple syntax, of the rate of change of the height 3/2 hours after the start of the trip.*

Solution

(a) At the start of the trip, $t = 0$. So $h = 1 - \frac{1}{4} = \frac{3}{4}$ at the start of the trip.

(b) When $h = \frac{3}{4}$, we can solve for t:

$$\begin{aligned} 1 - \frac{1}{4}\cos\left(\frac{\pi t}{3}\right) &= \frac{3}{4} \\ \cos\left(\frac{\pi t}{3}\right) &= 1 \\ \frac{\pi t}{3} &= 2k\pi, \quad k \in \mathbb{Z}_+ \\ t &= 6k. \end{aligned}$$

Hence if the first low tide is at $t = 0$ then the next low tide is at $t = 6$.

(c) First we need to calculate the derivative:

$$\frac{dh}{dt} = \frac{\pi}{12}\sin\left(\frac{\pi t}{3}\right).$$

Then at $t = \frac{3}{2}$,

$$\frac{dh}{dt} = \frac{\pi}{12}\sin\left(\frac{\pi}{2}\right) = \frac{\pi}{12}.$$

Question 9

Two wave fronts off the coast of Sydney meet at the coast. The height at the coast (in m) of the two wave forms are given by

$$h_1 = 10 + 6\sin(6t) \qquad \textit{and} \qquad h_2 = 11 + 8\cos(6t).$$

(a) *Enter the exact value, in Maple Syntax, of the maximum height of the combined wave form $h = h_1 + h_2$.*

(b) *Enter the exact value of the initial rate of change of h.*

(c) Enter the period of the periodic function h in Maple syntax.

Solution

(a) $h = 21 + 6\sin(6t) + 8\cos(6t)$, but using auxilliary angles

$$\begin{aligned} 6\sin(6t) + 8\cos(6t) &= R\sin\left(6t + \tan^{-1}\left(\frac{b}{a}\right)\right) \\ &= 10\sin\left(6t + \tan^{-1}\left(\frac{4}{3}\right)\right) \end{aligned}$$

where $a = 6$, $b = 8$, and $R = \sqrt{a^2 + b^2} = 10$. So we can simplify h into one sinusoidal wave:

$$h = 21 + 10\sin\left(6t + \tan^{-1}\left(\frac{4}{3}\right)\right).$$

So the maximum value of h is 31. This is because sin varies from -1 to 1, so h varies from 11 to 31.

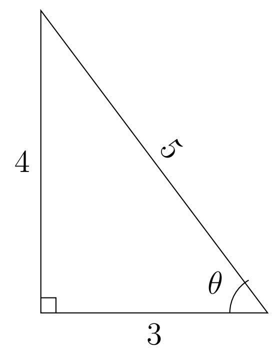
(b) The initial change of h can be found as the value of $\frac{dh}{dt}$ at $t = 0$. First we calculate the derivative:

$$\frac{dh}{dt} = 60\cos\left(6t + \tan^{-1}\left(\frac{4}{3}\right)\right).$$

Hence at $t = 0$,

$$\begin{aligned} \frac{dh}{dt} &= 60\cos\left(\tan^{-1}\left(\frac{4}{3}\right)\right) \\ &= 60\frac{3}{5} \\ &= 36. \end{aligned}$$

To evaluate $\tan^{-1}\left(\frac{4}{3}\right)$ in the above, we consider the following right-angled triangle:



Here, $\theta = \tan^{-1}\left(\frac{4}{3}\right)$, so $\cos\left(\tan^{-1}\left(\frac{4}{3}\right)\right) = \cos(\theta) = \frac{3}{5}$.

(c) Period $T = \frac{2\pi}{n}$. Here, $n = 6$ so $T = \frac{\pi}{3}$.

Question 10

Two wave fronts off the coast of Sydney meet at the coast. The height at the coast (in m) of the two wave forms are given by

$$h_1 = 9 + 9\sin(8t) \qquad \textit{and} \qquad h_2 = 17 + 2\cos(8t).$$

(a) Enter the exact value, in Maple Syntax, of the maximum height of the combined wave form $h = h_1 + h_2$.

(b) Enter the exact value of the initial rate of change of h.

(c) Enter the period of the periodic function h in Maple syntax.

Solution

(a) $h = 26 + 9\sin(8t) + 2\cos(8t)$, but

$$9\sin(8t) + 2\cos(8t) = \sqrt{85}\sin\left(8t + \tan^{-1}\left(\frac{2}{9}\right)\right).$$

So we can simplify h into one sinusoidal wave:

$$h = 26 + \sqrt{85}\sin\left(8t + \tan^{-1}\left(\frac{2}{9}\right)\right).$$

Hence maximum h is $26 + \sqrt{85}$, just like the previous question.

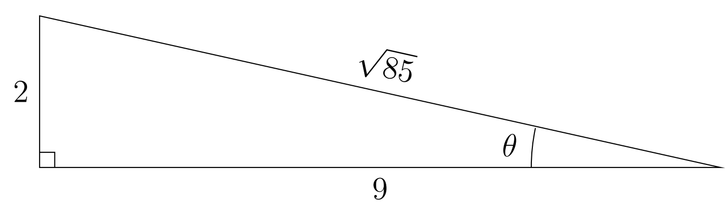
(b) The initial change of h can be found as the value of $\dfrac{dh}{dt}$ at $t = 0$. First we calculate the derivative:

$$\frac{dh}{dt} = 8\sqrt{85}\cos\left(8t + \tan^{-1}\left(\frac{2}{9}\right)\right).$$

Hence at $t = 0$,

$$\begin{aligned}\frac{dh}{dt} &= 8\sqrt{85}\cos\left(\tan^{-1}\left(\frac{2}{9}\right)\right)\\ &= 8\sqrt{85}\frac{9}{\sqrt{85}}\\ &= 72.\end{aligned}$$

Similar to the last question, to find $\cos(\tan^{-1}\left(\frac{2}{9}\right))$, we use the following triangle:



So $\cos\left(\tan^{-1}\left(\frac{2}{9}\right)\right) = \cos(\theta) = \frac{9}{\sqrt{85}}$.

(c) Period $T = \frac{2\pi}{n}$. Here, $n = 8$ so $T = \frac{\pi}{4}$.

Question 11

During a fishing trip Alex notices that the height h of the tide (in metres) is given by

$$h = 2 + \frac{1}{4}\cos\left(\frac{\pi t}{6}\right)$$

where t is measured in hours from the start of the trip.

(a) *Enter the exact value of h at the start of the trip.*

(b) *It is high tide at the start of the trip. Enter the value of t at which next high tide occurs.*

(c) *Enter the exact value, in Maple syntax, of the rate of change of the height 3 hours after* *the start of the trip.*

Solution

(a) At the start of the trip, $t = 0$. So $h = 2 + \frac{1}{4} = \frac{9}{4}$ at the start of the trip.

(b) When $h = \frac{9}{4}$, we can solve for t:

$$\begin{aligned} 2 + \frac{1}{4}\cos\left(\frac{\pi t}{6}\right) &= \frac{9}{4} \\ \cos\left(\frac{\pi t}{6}\right) &= 1 \\ \frac{\pi t}{6} &= 2k\pi, \quad k \in \mathbb{Z}_+ \\ t &= 12k. \end{aligned}$$

Hence if the first high tide is at $t = 0$ then the next high tide is at $t = 12$.

(c) First we need to calculate the derivative:

$$\frac{dh}{dt} = -\frac{\pi}{24}\sin\left(\frac{\pi t}{6}\right).$$

Then at $t = 3$,

$$\frac{dh}{dt} = -\frac{\pi}{24}\sin\left(\frac{\pi}{2}\right) = -\frac{\pi}{24}.$$

Question 12

Let A,B be points represented by $\begin{pmatrix} -9 \\ 4 \\ -4 \end{pmatrix}, \begin{pmatrix} -9 \\ -4 \\ -4 \end{pmatrix}$ *respectively.*

(a) *Enter the vector representing the midpoint of AB.*

(b) *Enter the exact value of the distance between A and B.*

Solution

(a) The midpoint of AB is given by the sum of the two vectors divided by 2:

$$Midpoint(AB) = \frac{1}{2}\left[\begin{pmatrix}-9\\4\\-4\end{pmatrix} + \begin{pmatrix}-9\\-4\\-4\end{pmatrix}\right] = \begin{pmatrix}-9\\0\\-4\end{pmatrix}.$$

(b) The distance between A and B can be found by the norm of the difference of the vectors:

$$|AB| = \left|\begin{pmatrix}-9\\4\\-4\end{pmatrix} - \begin{pmatrix}-9\\-4\\-4\end{pmatrix}\right| = \left|\begin{pmatrix}0\\8\\0\end{pmatrix}\right| = 8.$$

Question 13

Given that $Y = 5x + 0.4$, find values k and A such that

$$y = Ae^{kx}$$

where $Y = \ln(y)$.

Solution

Since $Y = 5x + 0.4$ then

$$\begin{aligned}\ln(y) &= 5x + 0.4\\ y &= e^{5x+0.4}\\ y &= e^{0.4}e^{5x}.\end{aligned}$$

So $k = 5$ and $A = e^{0.4}$.

Question 14

Given that $y = 7e^{9x}$, *find values* m *and* b *such that*

$$Y = mx + b$$

where $Y = \ln(y)$.

Solution

Since $y = 7e^{9x}$ then

$$\begin{aligned}
\ln(y) &= \ln(7e^{9x}) \\
Y &= \ln(e^{9x}) + \ln 7 \\
Y &= 9x + \ln 7.
\end{aligned}$$

So $m = 9$ and $b = \ln 7$.

Question 15

Given that $y = 2x^{-5}$, *find values* m *and* b *such that*

$$Y = mX + b$$

where $X = \ln(x)$ *and* $Y = \ln(y)$.

Solution

Since $y = 2x^{-5}$ then

$$\begin{aligned}
\ln(y) &= \ln(2x^{-5}) \\
Y &= \ln(x^{-5}) + \ln 2 \\
Y &= -5\ln(x) + \ln 2 \\
Y &= -5X + \ln 2.
\end{aligned}$$

So $m = -5$ and $b = \ln 2$.

Question 16

Given that $Y = 2X + 3$, find values k and A such that

$$y = Ax^k$$

where $X = \ln(x)$ and $Y = \ln(y)$.

Solution

Since $Y = 2X + 3$ then

$$\begin{aligned} \ln(y) &= 2\ln(x) + 3 \\ y &= e^{2\ln(x)+3} \\ y &= e^3 e^{\ln(x^2)} \\ y &= e^3 x^2. \end{aligned}$$

So $k = 2$ and $A = e^3$.

Question 17

Each of the following matrices is the augmented matrix of a system of linear equations. Select all augmented matrices which represent systems of equations with infinitely many solutions.

1. $\left(\begin{array}{ccc|c} -3 & -5 & 4 & -2 \\ 0 & 4 & -4 & 0 \\ 0 & 0 & 5 & -5 \end{array}\right)$

2. $\left(\begin{array}{ccc|c} 4 & -5 & -3 & -2 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right)$

3. $\left(\begin{array}{cccc|c} -3 & -4 & -5 & 4 & 5 \\ 0 & 5 & 0 & -2 & 0 \\ 0 & 0 & 0 & 4 & -5 \end{array}\right)$

4. $\left(\begin{array}{cc|c} 4 & -5 & 5 \\ 0 & 0 & 0 \end{array}\right)$

5. $\left(\begin{array}{ccc|c} 4 & -4 & 5 & -2 \\ 0 & -3 & 0 & -5 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

6. $\left(\begin{array}{ccc|c} 4 & -5 & -3 & -2 \\ 0 & 5 & -2 & 0 \\ 0 & 0 & 0 & 4 \end{array}\right)$

7. $\left(\begin{array}{ccc|c} 4 & -4 & 5 & -2 \\ 0 & -3 & 0 & -5 \\ 0 & 0 & 4 & 0 \end{array}\right)$

8. $\left(\begin{array}{ccccc|c} 4 & -2 & 0 & 0 & -5 & -2 \\ 0 & -5 & -3 & -2 & 0 & -5 \\ 0 & 0 & 0 & -5 & 5 & 0 \end{array}\right)$

Solution

1. Exactly one solution since each column is a leading column except for the right most column.
2. No solutions since the right most column has a leading term.
3. System of 3 linear equations with 4 variables. Hence there are infinitely many solutions.
4. A single linear equation of two variables, so there are infinitely many solutions.
5. Exactly one solution since every column is a leading column, except for the right most one.
6. No solution since the right most column is a leading column.
7. Exactly one solution.

8. Infinitely many solutions since there are 5 variables but 3 linear equations.

So our choices are 3, 4, and 8.

Question 18

Each of the following matrices is the augmented matrix of a system of linear equations. Select all augmented matrices which represent systems of equations with unique solution.

1. $\left(\begin{array}{ccc|c} 3 & -5 & -3 & -5 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right)$

2. $\left(\begin{array}{ccc|c} 3 & 5 & -5 & -5 \\ 0 & -3 & 0 & -5 \\ 0 & 0 & -5 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$

3. $\left(\begin{array}{ccc|c} 3 & 5 & -5 & -5 \\ 0 & -3 & 0 & -5 \\ 0 & 0 & -5 & 0 \end{array}\right)$

4. $\left(\begin{array}{ccc|c} -3 & -5 & -5 & -5 \\ 0 & 3 & 5 & 3 \\ 0 & 0 & -5 & -5 \end{array}\right)$

5. $\left(\begin{array}{cccc|c} -3 & 5 & -5 & 3 & -5 \\ 0 & -5 & 3 & -5 & 0 \\ 0 & 0 & 0 & 3 & -5 \end{array}\right)$

6. $\left(\begin{array}{cc|c} 3 & -5 & -5 \\ 0 & 0 & 0 \end{array}\right)$

7. $\left(\begin{array}{ccc|c} -5 & -5 & -3 & -5 \\ 0 & -5 & -5 & 3 \\ 0 & 0 & 0 & 3 \end{array}\right)$

8. $\left(\begin{array}{ccccc|c} 3 & -5 & 3 & 3 & -5 & -5 \\ 0 & -5 & -3 & -5 & 0 & -5 \\ 0 & 0 & 0 & -5 & -5 & 3 \end{array}\right)$

Solution

1. No solution.
2. Exactly one solution.
3. Exactly one solution.
4. Exactly one solution.
5. Infinitely many solutions.
6. Infinitely many solutions.
7. No solution.
8. Infinitely many solutions.

So our choices are 2, 3, and 4.

Question 19

Select all the matrices which are in row echelon from the list below.

1. $\begin{pmatrix} 0 & 2 & 6 & -2 & 0 \\ 0 & 0 & -6 & -2 & 7 \\ 0 & 0 & 0 & 8 & 0 \end{pmatrix}$

2. $\begin{pmatrix} -6 & -1 & -4 & 4 \\ 0 & 0 & -2 & -7 \\ 0 & 0 & 0 & 9 \end{pmatrix}$

3. $\begin{pmatrix} -8 & 6 & -2 \\ 1 & -9 & 0 \\ 0 & 0 & 8 \end{pmatrix}$

4. $\begin{pmatrix} 6 & -4 & -4 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2 & -9 \\ 0 & 0 & 0 & 8 & -9 \end{pmatrix}$

5. $\begin{pmatrix} 0 & -1 & -9 & -2 & -4 \end{pmatrix}$

6. $\begin{pmatrix} 0 & -8 & 6 & -2 \\ -1 & -7 & -8 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix}$

7. $\begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & -8 & 6 & -2 \\ -1 & -7 & -8 & 0 \end{pmatrix}$

8. $\begin{pmatrix} 6 & 6 & -2 \\ 0 & -9 & 2 \\ 0 & 0 & 2 \end{pmatrix}$

Solution

Recall that a matrix in row-echelon form will have all non-zero rows above the completely zero rows, and each rows leading term will be strictly to the left of the next rows leading term.

Hence matrices 1, 2, 5, and 8 are in row-echelon form.

Matrix 3 is not in row-echelon form since row 1 has a leading term not strictly to the left of the leading term of row 2.

Matrix 4 is not in row-echelon form since row 2 is a completely zero row which is above non-zero rows.

Matrix 6 is not in row-echelon form since the leading term of row 1 is to the right of the leading term of row 2.

Matrix 7 is not in row-echelon form since the leading term of row 1 is to the right of the leading term of row 2.

Question 20

Each of the following matrices is the augmented matrix of a system of linear equations. Select all augmented matrices which represent systems of equations with no solution.

1. $\left(\begin{array}{ccccc|c} -4 & -3 & 1 & 1 & 1 & -3 \\ 0 & 1 & -3 & -5 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 & 1 \end{array}\right)$

2. $\left(\begin{array}{ccc|c} -4 & 2 & 4 & -3 \\ 0 & -5 & 0 & 1 \\ 0 & 0 & -4 & 0 \end{array}\right)$

3. $\left(\begin{array}{cc|c} -4 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right)$

4. $\left(\begin{array}{ccc|c} -4 & 1 & -5 & -3 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$

5. $\left(\begin{array}{ccc|c} -5 & 1 & -4 & -3 \\ 0 & -4 & 2 & 1 \\ 0 & 0 & 4 & 1 \end{array}\right)$

6. $\left(\begin{array}{cccc|c} -5 & 2 & 1 & -4 & 4 \\ 0 & 4 & 1 & -3 & 0 \\ 0 & 0 & 0 & -4 & 1 \end{array}\right)$

7. $\left(\begin{array}{ccc|c} -4 & 1 & -5 & -3 \\ 0 & 4 & -3 & 1 \\ 0 & 0 & 0 & -4 \end{array}\right)$

8. $\left(\begin{array}{ccc|c} -4 & 2 & 4 & -3 \\ 0 & -5 & 0 & 1 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Solution

1. Infinitely many solutions.

2. Exactly one solution.
3. Infinitely many solutions.
4. No solution.
5. Exactly one solution.
6. Infinitely many solutions.
7. No solution.
8. Exactly one solution.

So our choices are 4 and 7.

Question 21

Perform the following row operations

$$R_2 = R_2 - 2R_1, \quad R_3 = R_3 + 3R_1$$

on the matrix $\begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & 2 & 3 & 5 \\ -3 & -3 & 2 & 3 \end{pmatrix}$.

Solution

The first operation changes the second row by subtracting off twice the first row:

$$\begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & 2 & 3 & 5 \\ -3 & -3 & 2 & 3 \end{pmatrix} \xrightarrow{R_2 = R_2 - 2R_1} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & 5 & 1 \\ -3 & -3 & 2 & 3 \end{pmatrix}.$$

Now we take this new matrix and apply the second row operation, which changes the third row by adding three times the first row:

$$\begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & 5 & 1 \\ -3 & -3 & 2 & 3 \end{pmatrix} \xrightarrow{R_3 = R_3 + 3R_1} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & -1 & 9 \end{pmatrix}.$$

Question 22

Perform the following row operations

$$R_1 \leftrightarrow R_2, \quad R_2 = 2R_2$$

on the matrix $\begin{pmatrix} -1 & 0 & -3 & -3 \\ -2 & -5 & 3 & -2 \\ -1 & 3 & 5 & -3 \end{pmatrix}$.

Solution

The first operation swaps the first and second row:

$$\begin{pmatrix} -1 & 0 & -3 & -3 \\ -2 & -5 & 3 & -2 \\ -1 & 3 & 5 & -3 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} -2 & -5 & 3 & -2 \\ -1 & 0 & -3 & -3 \\ -1 & 3 & 5 & -3 \end{pmatrix}.$$

Now we take this new matrix and apply the second row operation, which doubles the second row:

$$\begin{pmatrix} -2 & -5 & 3 & -2 \\ -1 & 0 & -3 & -3 \\ -1 & 3 & 5 & -3 \end{pmatrix} \xrightarrow{R_2 = 2R_2} \begin{pmatrix} -2 & -5 & 3 & -2 \\ -2 & 0 & -6 & -6 \\ -1 & 3 & 5 & -3 \end{pmatrix}.$$

Question 23

Given that A *is a* 3×3 *matrix,* $\boldsymbol{b}$ *is a* 3×1 *column vector and* $\boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

The augmented matrix of the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$ *is reduced to a row-echelon form:*

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -3 \\ 0 & -3 & 5 & 9 \\ 0 & 0 & 2 & 0 \end{array}\right).$$

Find the exact values for variables x, y *and* z.

Solution

From row 3, $2z = 0$ so $z = 0$.
In row 2,

$$\begin{aligned} -3y + 5z &= 9 \\ -3y &= 9 \\ y &= -3. \end{aligned}$$

And finally in row 1,

$$\begin{aligned} x + 2y - 3z &= -3 \\ x + 2(-3) - 3(0) &= -3 \\ x - 6 &= -3 \\ x &= 3. \end{aligned}$$

Question 24

Find the least squares line of best fit through the following points

x	*0*	*2*	*3*
y	*2*	*-2*	*-2*

as $y = ax + b$.

Solution

By plugging the given points into $y = ax + b$ we have

$$b = 2, \quad 2a + b = -2, \quad 3a + b = -2.$$

Hence we can write this as a matrix equation:

$$\begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix}.$$

Now we multiply both sides on the left by the transpose of the 3×2 matrix and solve for a and b:

$$\begin{aligned}
\begin{pmatrix} 0 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 0 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix} \\
\begin{pmatrix} 13 & 5 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} -10 \\ -2 \end{pmatrix} \\
\begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 13 & 5 \\ 5 & 3 \end{pmatrix}^{-1} \begin{pmatrix} -10 \\ -2 \end{pmatrix} \\
\begin{pmatrix} a \\ b \end{pmatrix} &= \frac{1}{14} \begin{pmatrix} 3 & -5 \\ -5 & 13 \end{pmatrix} \begin{pmatrix} -10 \\ -2 \end{pmatrix} \\
\begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} \frac{-10}{7} \\ \frac{12}{7} \end{pmatrix}.
\end{aligned}$$

Hence $a = \frac{-10}{7}$ and $b = \frac{12}{7}$, and so

$$y = \frac{-10}{7}x + \frac{12}{7}.$$

Question 25

The following system of linear equations has infinitely many solutions:

$$\begin{aligned}
x + 5y + 5z &= 12 \\
3x + 14y - z &= 34 \\
x + 6y + 21z &= 14
\end{aligned}$$

Find the particular solution with the property that $x + 2y = 6$.

Solution

First we solve the system of linear equations:

$$\left(\begin{array}{ccc|c} 1 & 5 & 5 & 12 \\ 3 & 14 & -1 & 34 \\ 1 & 6 & 21 & 14 \end{array}\right) \xrightarrow{R_2=R_2-3R_1} \left(\begin{array}{ccc|c} 1 & 5 & 5 & 12 \\ 0 & -1 & -16 & -2 \\ 1 & 6 & 21 & 14 \end{array}\right)$$
$$\xrightarrow{R_3=R_3-R_1} \left(\begin{array}{ccc|c} 1 & 5 & 5 & 12 \\ 0 & -1 & -16 & -2 \\ 0 & 1 & 16 & 2 \end{array}\right)$$
$$\xrightarrow{R_3=R_3+R_2} \left(\begin{array}{ccc|c} 1 & 5 & 5 & 12 \\ 0 & -1 & -16 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Hence we have $y = 2 - 16z$, $x + 5y + 5z = 12$, so

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= \begin{pmatrix} 2 + 75z \\ 2 - 16z \\ z \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + z \begin{pmatrix} 75 \\ -16 \\ 1 \end{pmatrix}. \end{aligned}$$

We want $x + 2y = 6$ so

$$\begin{aligned} (2 + 75z) + 2(2 - 16z) &= 6 \\ 6 + 43z &= 6 \\ z &= 0. \end{aligned}$$

Hence we have

$$\mathbf{x} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}.$$