MATH1231/1241 Lab Test 1
Calculus Solutions to Samples

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We cannot guarantee that our answers are correct - please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are sometimes multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

Question 1

The following integral can be evaluated using a trigonometric or hyperbolic trigonometric substitution.

$$\int \frac{dx}{(36 + x^2)^{3/2}}$$

In order to remove the fractional power, which substitutions should you try? Choose at least one circular trigonometric substitution and one hyperbolic trigonometric substitution.

1. $x = 6 \sin \theta$
2. $x = 6 \tanh u$
3. $x = 6 \sec \theta$
4. $x = 6 \cosh u$
5. $x = 6 \sinh u$
6. $x = 6 \tan \theta$

Using the fact that $\sec^2 x = 1 + \tan^2 x$ and $\cosh^2 x = 1 + \sinh^2 x$, we see that substitutions (5.) and (6.) result in the denominator becoming $(6 \cosh u)^3$ and $(6 \sec \theta)^3$ respectively.

Question 2

Find $\int \cos(8x)\cos(4x)dx$ *and enter your answer below using Maple syntax.*

Using the double angle identity for cosine, we have $\cos(8x) = 1 - 2\sin^2(4x)$. Thus,

$$\begin{aligned}\int \cos(8x)\cos(4x)dx &= \int (1 - 2\sin^2(4x))\cos(4x)dx \\ &= \int \cos(4x) - 2\sin^2(4x)\cos(4x)dx \\ &= \frac{1}{4}\sin(4x) - \frac{2}{3}\sin^3(4x) + c\end{aligned}$$

Using Maple syntax, this would be `(1/4)*sin(4*x) - (2/3)*(sin(4*x))^3`. Note that if we had $\sin(4x)$ on the right in the original product, we would have instead used this variation of the double angle identity for cosine: $\cos(8x) = 2\cos^2(4x) - 1$.

Question 3

Using the standard method, the integral $\int \sin^8 x \cos^5 x dx$ *has the form*

$$\begin{aligned}\int \sin^8 x \cos^5 x dx = a_{17}\sin^{17} x + a_{16}\sin^{16} x + a_{15}\sin^{15} x + a_{14}\sin^{14} x + a_{13}\sin^{13} x + a_{12}\sin^{12} x \\ + a_{11}\sin^{11} x + a_{10}\sin^{10} x + a_9\sin^9 x + C.\end{aligned}$$

Find this integral and enter the coefficients requested below.

- $a_9 =$
- $a_{10} =$
- $a_{11} =$
- $a_{12} =$
- $a_{13} =$

Using the Pythagorean identity $\cos^2 x = 1 - \sin^2 x$ and letting $u = \sin x$, we have

$$\begin{aligned}\int \sin^8 x \cos^5 x dx &= \int \sin^8 x(1-\sin^2 x)^2 \cos x dx \\ &= \int \sin^8 x(1 - 2\sin^2 x + \sin^4 x)\cos x dx \\ &= \int u^8 - 2u^{10} + u^{12} du \\ &= \frac{u^9}{9} - 2 \cdot \frac{u^{11}}{11} + \frac{u^{13}}{13} + C \\ &= \frac{\sin^9 x}{9} - 2 \cdot \frac{\sin^{11} x}{11} + \frac{\sin^{13} x}{13} + C\end{aligned}$$

Thus, we have $a_{10} = a_{12} = 0$ and

$$a_9 = \frac{1}{9}, \qquad a_{11} = \frac{-2}{11}, \qquad a_{13} = \frac{1}{13}.$$

Question 5

You are given that

$$\int_0^1 x^n e^{-x} dx = -e^{-1} + n\int_0^1 x^{n-1}e^{-x}dx \text{ for } n \geq 1.$$

Use this recurrence relation to calculate $\int_0^1 x^7 e^{-x} dx$ *and enter your answer in the box below.*

Let $I_n = \int_0^1 x^n e^{-x} dx$. Then our recurrence relation becomes $I_n = -e^{-1} + nI_{n-1}$. Hence, we have

$$\begin{aligned}I_7 &= -e^{-1} + 7I_6 \\ &= -e^{-1} + 7(-e^{-1} + 6I_5) \\ &= -8e^{-1} + 42(-e^{-1} + 5I_4) \\ &= \quad \vdots \\ &= -8660e^{-1} + 5040I_0.\end{aligned}$$

Now $I_0 = \int_0^1 e^{-x} dx = 1 - e^{-1}$ and thus, $I_7 = 5040 - 13700e^{-1}$.

Question 9

Let z be a function of x and y and you are given the following information. The measured values of x and y are -4 and 2 and at $(x, y) = (-4, 2)$,

$$\frac{\partial z}{\partial x} = -1 \textit{ and } \frac{\partial z}{\partial y} = 1.$$

If x is decreased by 0.02 and y is increased by 0.05, use the total differentiation approximation to estimate the change in z and enter your answer in the box below.

$$\Delta z \approx$$

Let $z = F(x, y)$. Then by total differentiation approximation, we have

$$\begin{aligned}\Delta z &= F(-4.02, 2.05) - F(-4, 2) \\ &\approx \frac{\partial z}{\partial x}(-4, 2)\Delta x + \frac{\partial z}{\partial y}(-4, 2)\Delta y \\ &= (-1)(-0.02) + (1)(0.05) \\ &= 0.07.\end{aligned}$$

Question 10

Let z be a function of x and y and you are given the following information. The measured values of x and y are -1cm and 1cm and each measurement is made with an error whose absolute value is at most 0.07cm. Furthermore, you are given that at $(x, y) = (-1, 1)$,

$$\frac{\partial z}{\partial x} = -2 \textit{ and } \frac{\partial z}{\partial y} = 3$$

Use the total differential approximation to estimate the maximum possible error in the calculated value of z and enter your answer below.

$$|\Delta z| \lessapprox$$

An upper bound for the absolute error is given by

$$\begin{aligned}|\Delta z| &\approx \left|\frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y\right| \\ &\leq \left|\frac{\partial z}{\partial x}\right||\Delta x| + \left|\frac{\partial z}{\partial y}\right||\Delta y|\end{aligned}$$

where the last line is obtained through the triangle inequality. Here, we have $|\Delta x| \leq 0.07$, $|\Delta y| \leq 0.07$, $\frac{\partial z}{\partial x} = -2$ and $\frac{\partial z}{\partial y} = 3$. Hence, $|\Delta z| \lessapprox 2 \cdot 0.07 + 3 \cdot 0.07 = 0.35$.

Question 11

Let $z = 2x^2 + 4y^2$ where x and y are functions of t. When $t = 4$, you are given that

$$x = -3, y = 1, \frac{dx}{dt} = -4, \frac{dy}{dt} = -1.$$

Find $\frac{dz}{dt}$ when $t = 4$ and enter your answer in the box below.

$$\frac{dz}{dt}(4) =$$

Applying the chain rule, we have

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Here, $\frac{\partial z}{\partial x} = 4x$ and $\frac{\partial z}{\partial y} = 8y$. Substituting the corresponding values when $t = 4$, we have

$$\begin{aligned}\frac{dz}{dt}(4) &= (4 \cdot -3)(-4) + (8 \cdot 1)(-1) \\ &= 40.\end{aligned}$$

Question 12

Find a normal vector $\boldsymbol{n}$ *to the tangent plane to the surface*

$$z = 4x^3 + 2y^3$$

at the point $(x, y, z) = (4, 3, 310)$ *and enter your answer in the box below using Maple syntax.*

For example, to enter the vector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ *you would enter* $< 1, 2, 3 >$.

Suppose that (x_0, y_0, z_0) is a point on the surface $z = F(x, y)$. Then if the surface has a tangent plane at that point, the normal vector to the tangent plane at that point is given by $\boldsymbol{n} = \begin{pmatrix} F_x(x_0, y_0) \\ F_y(x_0, y_0) \\ -1 \end{pmatrix}$. Here, $F(x, y) = 4x^3 + 2y^3$ and $(x_0, y_0, z_0) = (4, 3, 310)$. As such, we have $F_x(x, y) = 12x^2$ and $F_y(x, y) = 6y^2$, so $F_x(4, 3) = 192$ and $F_y(4, 3) = 54$. Hence,

$$\boldsymbol{n} = \begin{pmatrix} 192 \\ 54 \\ -1 \end{pmatrix}$$

Question 13

Given

$$F = yx^3 + 3xy^2 + 6x^4y^4 + 4e^x,$$

find $\frac{\partial^2 F}{\partial x \partial y}$.

We first take the partial derivative of F with respect to y before taking the partial derivative of that with respect to x. We have

$$\frac{\partial F}{\partial y} = x^3 + 6xy + 24x^4y^3.$$

Hence,

$$\frac{\partial^2 F}{\partial x \partial y} = 3x^2 + 6y + 96x^3y^3.$$

Note that in this case, it did not actually matter which order we took the partial derivatives in. To be more technical, the second order partial derivatives commute if all second order partial derivatives exist and are continuous (see Clairaut's Theorem).

Question 14

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with derivative $g : \mathbb{R} \to \mathbb{R}$, that is $g(x) = f'(x)$. Define $F : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x, y) = f(6x^5 - 2y^3).$$

Find the partial derivative of F in terms of x, y and g and enter your answer in the boxes below using Maple notation.

$$\frac{\partial F}{\partial x} =$$
$$\frac{\partial F}{\partial y} =$$

Let $u = 6x^5 - 2y^3$. Then

$$\begin{aligned}\frac{\partial F}{\partial x} &= \frac{d(f(u))}{dx} \\ &= \frac{df(u)}{du} \cdot \frac{du}{dx} \\ &= g(u) \cdot \frac{d(6x^5 - 2y^3)}{dx} \\ &= g(6x^5 - 2y^3) \cdot 30x^4\end{aligned}$$

and

$$\begin{aligned}\frac{\partial F}{\partial y} &= \frac{d(f(u))}{dy} \\ &= \frac{df(u)}{du} \cdot \frac{du}{dy} \\ &= g(u) \cdot \frac{d(6x^5 - 2y^3)}{dy} \\ &= g(6x^5 - 2y^3) \cdot -6y^2.\end{aligned}$$

Writing them in maple notation, our two answers would be `g(6*x^5 - 2*y^3)*30*x^4` and `-g(6*x^5 - 2*y^3)*6*y^2` respectively.