MATH1081 Lab Test 1

Sample Solutions

October 8, 2019

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We cannot guarantee that our answers are correct - please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are sometimes multiple methods of solving the same question.

Note: Any text presented `like this` is presented as required in Numbas syntax.

Question 1

In a class of 37 students:

- 21 study French,
- 17 study Physics,
- 10 study both English and French,
- 8 study both English and Physics,
- 9 study both French and Physics,
- 3 study all three subjects, and
- 4 study none of these subjects

How many people study English?

Firstly, of the 37 students, 4 study none of the three subjects, so there are 33 students who study at least one subject. Let the sets F, E, P represent the students studying in French, English and Physics respectively. By the inclusion-exclusion principle:

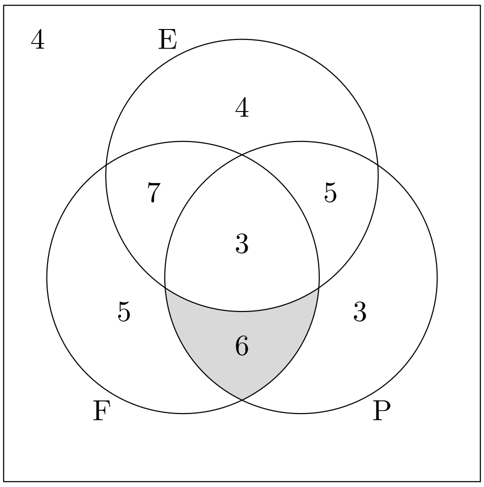
$$\begin{aligned}|F \cup E \cup P| &= |F| + |E| + |P| - (|F \cap E| + |E \cap P| + |P \cap F|) + |F \cap E \cap P| \\ 33 &= 21 + |E| + 17 - (10 + 8 + 9) + 3 \\ 19 &= |E|.\end{aligned}$$

Writing E, F, P for the sets of students studying English, French and Physics respectively, evaluate $|E^c \cap (F^c \cup P^c)^c|$

Solution 1:

$$|E^c \cap (F^c \cup P^c)^c| = |E^c \cap F \cap P| \quad \text{(De Morgan's laws)}$$

Then we manually fill out the Venn Diagram:



Solution 2:

In this question we will use the fact that $|A - B| = |A| - |A \cap B|$. For justification, we have provided the proof below, but you will not need to prove this during the lab test.

$$\begin{aligned}
|A| &= |A \cap U| && \text{(Identity laws)} \\
&= |A \cap (B \cup B^c)| && \text{(Union with complement)} \\
&= |(A \cap B) \cup (A \cap B^c)| && \text{(Distributive laws)} \\
&= |A \cap B| + |A \cap B^c| - |A \cap B \cap A \cap B^c| && \text{(Inclusion-Exclusion)} \\
&= |A \cap B| + |A \cap B^c| - |\emptyset| && \text{(Intersection with complement)} \\
|A| &= |A \cap B| + |A - B| && \text{(Difference law)} \\
|A - B| &= |A| - |A \cap B|
\end{aligned}$$

We will call this Lemma 1.1. Now we can quickly solve the problem:

$$\begin{aligned}
|E^c \cap (F^c \cup P^c)^c| &= |E^c \cap (F \cap P)| && \text{(De Morgan's laws)} \\
&= |(F \cap P) \cap E^c| && \text{(Commutative law)} \\
&= |(F \cap P) - E|, && \text{(Difference law)} \\
&= |F \cap P| - |F \cap P \cap E| && \text{(Lemma 1.1)} \\
&= 9 - 3 \\
&= 6.
\end{aligned}$$

Question 2

For any integer k, let S_k be the set defined by:

$$S_k = \left\{ n \in \mathbb{Z} \mid k + 8 \leq n \leq \frac{1}{2}k + 16 \right\}.$$

Recall that the Numbas syntax for the set $\{a, b, c\}$ is `set(a,b,c)`.

For use in the following questions we've listed out the relevant sets:

$$\begin{aligned} S_1 &= \{9, 10, 11, 12, 13, 14, 15, 16\} \\ S_5 &= \{13, 14, 15, 16, 17, 18\} \\ S_1 \cap S_5 &= \{13, 14, 15, 16\} \\ S_1 \cap (S_5)^c &= \{9, 10, 11, 12\} \end{aligned}$$

Note that the last one is determined by considering the elements in S_1, that are *not* in S_5.

What is $S_1 - S_5$?

Relatively straightforward, but make sure you answer as a set.

$$\begin{aligned} S_1 - S_5 &= S_1 \cap (S_5)^c && \text{(Difference law)} \\ &= \texttt{set(9,10,11,12)}. \end{aligned}$$

Find $|P(S_1) \times P(S_5)|$.

The set product rule is that $|A \times B| = |A| \cdot |B|$, so:

$$|P(S_1) \times P(S_5)| = |P(S_1)| \cdot |P(S_5)|.$$

The power set's cardinality is $|P(S)| = 2^{|S|}$, so:

$$\begin{aligned} |P(S_1)| \cdot |P(S_5)| &= 2^{|S_1|} \cdot 2^{|S_5|} \\ &= 2^{|\{9,10,11,12,13,14,15,16\}|} \cdot 2^{|\{13,14,15,16,17,18\}|} \\ &= 2^8 \cdot 2^6 \\ &= 16384. \end{aligned}$$

Find $|P(S_1 \times S_5)|$.

Using the same rules in the opposite order:

$$\begin{aligned}
|P(S_1 \times S_5)| &= 2^{|S_1 \times S_5|} \\
&= 2^{|S_1| \cdot |S_5|} \\
&= 2^{|\{9,10,11,12,13,14,15,16\}| \cdot |\{13,14,15,16,17,18\}|} \\
&= 2^{8 \cdot 6} \\
&= 2^{48}.
\end{aligned}$$

Find $|P(S_1) \cap P(S_5)|$

In this question we will use the fact that $P(A) \cap P(B) = P(A \cap B)$. For justification, we have provided the proof below, but you will not need to prove this during the lab test.

Note that $x \in P(A) \iff x \subseteq A$, by definition of a power set.

$$\begin{aligned}
x \in (P(A) \cap P(B)) &\iff x \in P(A) \text{ and } x \in P(B) \\
&\iff x \subseteq A \text{ and } x \subseteq B \\
&\iff x \subseteq (A \cap B) \\
&\iff x \in P(A \cap B)
\end{aligned}$$

So all the elements of $P(A) \cap P(B)$ are in $P(A \cap B)$ and vice versa. For the sake of the following questions we will call this Lemma 2.1.

$$\begin{aligned}
|P(S_1) \cap P(S_5)| &= |P(S_1 \cap S_5)| \quad \text{(Lemma 2.1)} \\
&= 2^{|S_1 \cap S_5|} \\
&= 2^{|\{13,14,15,16\}|} \\
&= 2^4 \\
&= 16.
\end{aligned}$$

Find $|P(S_1) \cup P(S_5)|$

The method for this question is similar to the previous one, but we first need to use the inclusion-exclusion principle to break up the union. We will use Lemma 2.1 from the question above:

$$P(A) \cap P(B) = P(A \cap B).$$

$$\begin{aligned}
|P(S_1) \cup P(S_5)| &= |P(S_1)| + |P(S_5)| - |P(S_1) \cap P(S_5)| && \text{(Inclusion-Exclusion)} \\
&= |P(S_1)| + |P(S_5)| - |P(S_1 \cap S_5)| && \text{(Lemma 2.1)} \\
&= 2^{|S_1|} + 2^{|S_5|} - 2^{|S_1 \cap S_5|} \\
&= 2^{|\{9,10,11,12,13,14,15,16\}|} + 2^{|\{13,14,15,16,17,18\}|} - 2^{|\{13,14,15,16\}|} \\
&= 2^8 + 2^6 - 2^4 \\
&= 304.
\end{aligned}$$

Find $|P(S_1) - P(S_5)|$

This question will combine our knowledge of sets. We will use Lemma 1.1 from Question 1:

$$|A - B| = |A| - |A \cap B|,$$

and Lemma 2.1 from the question above:

$$P(A) \cap P(B) = P(A \cap B).$$

$$\begin{aligned}
|P(S_1) - P(S_5)| &= |P(S_1)| - |(P(S_1) \cap P(S_5))| && \text{(Lemma 1.1)} \\
&= |P(S_1)| - |P(S_1 \cap S_5)| && \text{(Lemma 2.1)} \\
&= 2^{|S_1|} - 2^{|S_1 \cap S_5|} \\
&= 2^{|\{9,10,11,12,13,14,15,16\}|} - 2^{|\{13,14,15,16\}|} \\
&= 2^8 - 2^4 \\
&= 240.
\end{aligned}$$

Question 3

Suppose $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and that the function $f : S \to S$ is given by:

$$f(x) = 5x^2 + 10x + 7 \pmod{11}.$$

Let $T = \{2, 6\}$.
Recall that the Numbas syntax for the set $\{a, b, c\}$ is `set(a,b,c)`.

Constructing a reference table:

x	0	1	2	3	4	5	6	7	8	9	10
$f(x)$	7	0	3	5	6	6	5	3	0	7	2

What is $f(T)$?

From the table, $f(2) = 3$ and $f(6) = 5$. Therefore $f(\{2, 6\}) = \{3, 5\}$, giving us

`set(3, 5).`

What is $f^{-1}(T)$?

$f^{-1}(T)$ is the set of elements that map to an element in T. From the table, $f^{-1}(2) = \{10\}$ and $f^{-1}(6) = \{4, 5\}$. Therefore $f^{-1}(\{2, 6\}) = \{4, 5, 10\}$, giving us

`set(4, 5, 10).`

Complete the sentence:
f is

- neither injective nor surjective.
- injective but not surjective.
- surjective but not injective.
- bijective.

As some outputs are repeated - $f(4) = f(5) = 6$ - so the function is not injective (one-to-one). Some outputs are never reached - $f(x) = 1$ has no solutions - so the function is not surjective.

Question 4

Complete the following:

$$\frac{3}{k-2} - \frac{5}{k} + \frac{2}{k+1} = \frac{Ak+B}{(k-2)(k)(k+1)},$$

where $A = \_\_$ and $B = \_\_$.

$$\begin{aligned}\frac{3}{k-2} - \frac{5}{k} + \frac{2}{k+1} &= \frac{3(k)(k+1) - 5(k-2)(k+1) + 2(k-2)(k)}{(k-2)(k)(k+1)} \\ &= \frac{3k^2 + 3k - 5k^2 + 5k + 10 + 2k^2 - 4k}{(k-2)(k)(k+1)} \\ &= \frac{4k+10}{(k-2)(k)(k+1)},\end{aligned}$$

so $A = 4$ and $B = 10$.

Hence find a closed form for $\displaystyle\sum_{k=3}^{n} \frac{2k+5}{(k-2)(k)(k+1)}$.

In this question we change the bounds of summation frequently. Note that:

$$\sum_{k=1}^{n-1} \frac{1}{k+1} = \sum_{k=2}^{n} \frac{1}{k} = \sum_{k=3}^{n+1} \frac{1}{k-1}$$

and that

$$\sum_{k=2}^{n} \frac{1}{k} = \frac{1}{2} + \frac{1}{3} + \left(\sum_{k=4}^{n-2} \frac{1}{k}\right) + \frac{1}{n-1} + \frac{1}{n}$$

$$\begin{aligned}
&\sum_{k=3}^{n} \frac{2k+5}{(k-2)(k)(k+1)} \\
&= \frac{1}{2}\left[\sum_{k=3}^{n}\left(\frac{3}{k-2} - \frac{5}{k} + \frac{2}{k+1}\right)\right] \\
&= \frac{1}{2}\left[\sum_{k=3}^{n} \frac{3}{k-2} - \sum_{k=3}^{n} \frac{5}{k} + \sum_{k=3}^{n} \frac{2}{k+1}\right] \\
&= \frac{1}{2}\left[\sum_{k=1}^{n-2} \frac{3}{k} - \sum_{k=3}^{n} \frac{5}{k} + \sum_{k=4}^{n+1} \frac{2}{k}\right] \\
&= \frac{1}{2}\left[\left(\frac{3}{1} + \frac{3}{2} + \frac{3}{3} + \sum_{k=4}^{n-2} \frac{3}{k}\right) - \left(\frac{5}{3} + \sum_{k=4}^{n-2} \frac{5}{k} + \frac{5}{n-1} + \frac{5}{n}\right) + \left(\sum_{k=4}^{n-2} \frac{2}{k} + \frac{2}{n-1} + \frac{2}{n} + \frac{2}{n+1}\right)\right] \\
&= \frac{1}{2}\left[\frac{3}{1} + \frac{3}{2} + \frac{3}{3} + \frac{-5}{3} + \frac{-5}{n-1} + \frac{-5}{n} + \frac{2}{n-1} + \frac{2}{n} + \frac{2}{n+1} + \sum_{k=4}^{n-2} \frac{3-5+2}{k}\right] \\
&= \frac{1}{2}\left[\frac{23}{6} + \frac{-3}{n-1} + \frac{-3}{n} + \frac{2}{n+1}\right]
\end{aligned}$$

Question 5

Answer the following, given that

$$3780 = 2^2 \cdot 3^3 \cdot 5 \cdot 7, \qquad 40020750 = 2 \cdot 3^3 \cdot 5^3 \cdot 7^2 \cdot 11^2, \qquad 45 = 3^2 \cdot 5.$$

What is $\operatorname{lcm}(3780, 40020750)$?

Using their prime factorisations, we take the larger of each exponent:

$$\begin{aligned}
\operatorname{lcm}(3780, 40020750) &= 2^{\max(2,1)} \cdot 3^{\max(3,3)} \cdot 5^{\max(1,3)} \cdot 7^{\max(1,2)} \cdot 11^{\max(0,2)} \\
&= 2^2 \cdot 3^3 \cdot 5^3 \cdot 7^2 \cdot 11^2 \\
&= 80041500.
\end{aligned}$$

What is $\gcd(40020750, 45)$?

Similarly, we take the smaller of each exponent:

$$\begin{aligned}
\gcd(40020750, 45) &= 2^{\min(1,0)} \cdot 3^{\min(3,2)} \cdot 5^{\min(1,1)} \cdot 7^{\min(2,0)} \cdot 11^{\min(2,0)} \\
&= 3^2 \cdot 5^1 \\
&= 45
\end{aligned}$$

Question 6

When evaluating modulo m, give your answer in its lowest non-negative form - that is, as an element of $\{0, 1, 2, ...m - 1\}$

Evaluate $5^{108} \pmod{11}$.

Since $5^5 \equiv 1 \pmod{11}$, we can quickly reduce 5^{108} to

$$\begin{aligned}
5^{108} &\equiv (5^5)^{21} \cdot 5^3 \pmod{11} \\
&\equiv 1^{21} \cdot 5^3 \pmod{11} \\
&\equiv 125 \pmod{11} \\
&\equiv 4 \pmod{11}.
\end{aligned}$$

Evaluate $2^{178} \pmod{18}$

Since $2^4 = 16 \equiv -2 \pmod{18}$, we can quickly reduce 2^{178} to

$$\begin{aligned}
2^{178} &\equiv (2^4)^{44} \cdot 2^2 \pmod{18} \\
&\equiv (-2)^{44} \cdot 2^2 \pmod{18} \\
&\equiv 2^{44} \cdot 2^2 \pmod{18} \\
&\equiv (2^4)^{11} \cdot 2^2 \pmod{18} \\
&\equiv -2^{11} \cdot 2^2 \pmod{18} \\
&\equiv -(2^4)^3 \cdot 2 \pmod{18} \\
&\equiv -(-2)^3 \cdot 2 \pmod{18} \\
&\equiv 16 \pmod{18}.
\end{aligned}$$

Question 7

Solve each of the following modular arithmetic equations, giving your answer as a set of all possible solutions in the given modulus.

- If there are no solutions, enter `set()`.
- If there is one solution, say 1, enter `set(1)`.
- If there are multiple solutions, say 1 and 2, enter `set(1, 2)`.

When evaluating in modulo m, give each answer in its lowest non-negative form - that is, as an element of $\{0, 1, 2, ...m-1\}$.

Solve $123x \equiv 3 \pmod{217}$

This is equivalent to solving $123x + 217y = 3$.
We are going to first solve $123x + 217y = 1$, and then multiply our solution by 3.
Using the Euclidean Algorithm:

$$\begin{aligned}
\underline{217} &= 1 \cdot \underline{123} \; + \underline{94} \\
\underline{123} &= 1 \cdot \underline{94} \; + \underline{29} \\
\underline{94} &= 3 \cdot \underline{29} \; + \underline{7} \\
\underline{29} &= 4 \cdot \underline{7} \; + \underline{1} \\
\underline{7} &= 7 \cdot \underline{1} \; + \underline{0}
\end{aligned}$$

We rearrange the second last equation and work our way up again:

$$\begin{aligned}
1 &= \underline{29} - 4 \cdot \underline{7} \\
&= \underline{29} - 4(\underline{94} - 3 \cdot \underline{29}) \\
&= 13 \cdot \underline{29} - 4 \cdot \underline{94} \\
&= 13(\underline{123} - \underline{94}) - 4 \cdot \underline{94} \\
&= 13 \cdot \underline{123} - 17 \cdot \underline{94} \\
&= 13 \cdot \underline{123} - 17(\underline{217} - \underline{123}) \\
&= 30 \cdot \underline{123} - 17 \cdot \underline{217}
\end{aligned}$$

So thus far we have the solution $(x = 30, y = -17)$ to the equation $123x + 217y = 1$.
To solve for $123x + 217y = 3$, we need to multiply our solution by 3

$$\begin{aligned}
x &\equiv 30 \cdot 3 \pmod{217} \\
x &\equiv 90 \pmod{217}
\end{aligned}$$

Our final solution is:

```
set(90)
```

Solve $208x \equiv 5 \pmod{663}$

This is equivalent to solving $208x + 663y = 5$.
The GCD of 208 and 663 is 13, but that is not a factor of 5, so there are no solutions for x that can work
Our final solution is:

```
set()
```

Solve $484x \equiv 20 \pmod{1340}$.

This is equivalent to solving $484x + 1340y = 20$.
Since all three numbers are multiples of 4, we can divide them all by 4 and keep in mind that we'll end up with 4 solutions in the end.
So now we are solving $121x \equiv 5 \pmod{335}$, or $121x + 335y = 5$.
We are going to first solve $121x + 335y = 1$, and then multiply our solution by 5.
Using the Euclidean Algorithm:

$$\begin{aligned}
\underline{335} &= 2 \cdot \underline{121} \ + \underline{93} \\
\underline{121} &= 1 \cdot \underline{93} \ + \underline{28} \\
\underline{93} &= 3 \cdot \underline{28} \ + \underline{9} \\
\underline{28} &= 3 \cdot \underline{9} \ + \underline{1} \\
\underline{9} &= 9 \cdot \underline{1} \ + \underline{0}.
\end{aligned}$$

We rearrange the second last equation and work our way up again:

$$\begin{aligned}
1 &= \underline{28} - 3 \cdot \underline{9} \\
&= \underline{28} - 3(\underline{93} - 3 \cdot \underline{28}) \\
&= 10 \cdot \underline{28} - 3 \cdot \underline{93} \\
&= 10(\underline{121} - \underline{93}) - 3 \cdot \underline{93} \\
&= 10 \cdot \underline{121} - 13 \cdot \underline{93} \\
&= 10 \cdot \underline{121} - 13(\underline{335} - 2 \cdot \underline{121}) \\
&= 36 \cdot \underline{121} - 13 \cdot \underline{335}.
\end{aligned}$$

So thus far we have the solution $(x = 36, y = -13)$ to the equation $121x + 335y = 1$.

To solve for $121x + 335y = 5$, we need to multiply our solution by 5

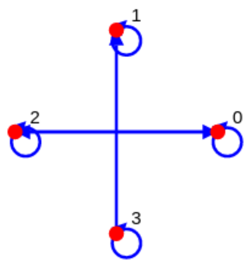
$$x \equiv 36 \cdot 5 \pmod{335}$$
$$x \equiv 180 \pmod{335}$$

In truth, there are infinite solutions to $121x+335y = 5$, of the form $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + k \begin{pmatrix} 335 \\ -121 \end{pmatrix}$ for all integers k, and where $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ is one of the solutions. Since we divided everything by 4 early on, x has 4 solutions in the range $[0, 1340)$ (the original modulus) each separated by 335 (the divided modulus).
Our final solution is:

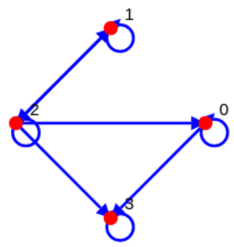
```
set(180, 515, 850, 1185)
```

Question 8

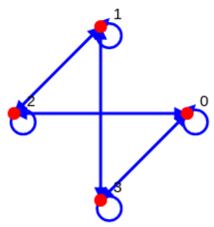
For each of the adjacency graphs below, indicate whether whether they are reflexive, symmetric and/or transitive relations.
Marks will be deducted for incorrect selection, but the minimum possible total mark for this question is 0.



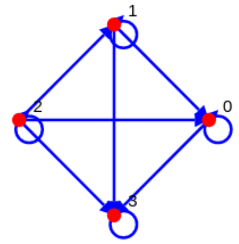
All nodes have self-loops, so the relation is **reflexive**
not all edges are bi-directional, so the relation is **not symmetric**
Every pair of nodes satisfies the transitive property, so the relation is **transitive**



All nodes have self-loops, so the relation is **reflexive**
Not all edges are bi-directional, so the relation is **not symmetric**
$1 \to 2$ and $2 \to 0$ but $1 \not\to 0$ so the relation is **not transitive**



All nodes have self-loops, so the relation is **reflexive**
All edges are bi-directional, so the relation is **symmetric**
$0 \to 2$ and $2 \to 1$ but $0 \not\to 1$, so the relation is **not transitive**



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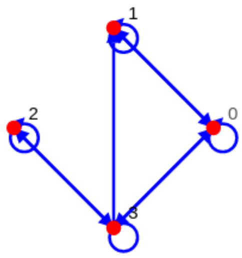

All nodes have self-loops, so the relation is **reflexive**
Not all edges are bi-directional, so the relation is **not symmetric**
$2 \to 3$ and $3 \to 1$ but $2 \not\to 1$ so the relation is **not transitive**