Weekly Problems

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1 Hard Question

HARD QUESTION: Let n be an odd positive integer.Show that

$$\sum_{k=1}^{n} \tan^2 \frac{k\pi}{2n+1}$$

is an odd integer.
Proof: Let's begin De Moivre's Theorem, where we consider the following,

$$\left(\cos\left(\frac{k\pi}{2n+1}\right) + i\sin\left(\frac{k\pi}{2n+1}\right)\right)^{(2n+1)} = (e^{i\pi})^k = (-1)^k. \tag{1}$$

This helps establish a constant on the LHS of the equation, and a Binomial Series on the RHS, which can be used to determine the value of the series in question.

$$\sum_{a=0}^{2n+1} \binom{2n+1}{a} \left(i\sin\left(\frac{k\pi}{2n+1}\right)\right)^{(2n+1-a)} \left(\cos\left(\frac{k\pi}{2n+1}\right)\right)^{(a)} = (-1)^k, \tag{2}$$

the above can also be written as,

$$\sum_{a=0}^{2n+1} \left(\binom{2n+1}{a} i\tan\left(\frac{k\pi}{2n+1}\right)\right)^{(2n+1-a)} = \frac{(-1)^k}{\cos(\frac{k\pi}{2n+1})^{2n+1}}$$

where $\cos(\frac{k\pi}{2n+1})^{2n+1} \neq 0$,taking the imaginary part of both sides of the equality,

$$\mathrm{Im}\left(\sum_{a=0}^{2n+1} \binom{2n+1}{a} \left(i\tan\left(\frac{k\pi}{2n+1}\right)\right)^{(2n+1-a)}\right) = 0$$

, we take the imaginary part of the above equation, because then we have the task of finding the roots of the polynomial in terms of $\tan(x)$,however as we also notice a important point,that the equation can be written as follows,

$$\sum_{a=0}^{n} \binom{2n+1}{2a} \left(i\tan\left(\frac{k\pi}{2n+1}\right)\right)^{(2n+1-2a)} = 0. \tag{3}$$

Note that the above is the re indexing of the terms containing only imaginary part . Factoring out $i\tan\left(\frac{k\pi}{2n+1}\right)$ from (3) , we get the final form of equation (3),

$$\sum_{a=0}^{n}\binom{2n+1}{2a}\left(-\tan^2\left(\frac{k\pi}{2n+1}\right)\right)^{(n-a)}=0 \tag{4}$$

, and hence through Veita's Formula we conclude that, for $k \in [0, 2n]$,

$$\sum_{k=1}^{n}\tan^2\frac{k\pi}{2n+1}=n(2n+1).$$

Using the formula derived we can see that $2n+1$ is odd for all $n \in \mathbb{R}$ and if we choose n to be an odd integer then $n(2n+1)$ will also be odd.