MATH2521/MATH2621

Complex Analysis

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Basic Rules and Ideas I

Basic Exponential

1. A complex number is a number of the form $z = x + iy$ and can be written in the form $z = re^{i\theta}$, r being the distance from the origin and θ the angle through which the positive real axis rotates to hit the line.
2. Normal rules of addition, multiplication and subtraction hold. As with division, "rational" complex numbers can be simplified by "real-ising" the denominator. That is, multiply numerator and denominator by the conjugate of the denominator.
3. Geometrically, multiplying complex numbers involves scaling and rotation about the origin. Addition involves shifting in the direction of the vector that you have added.

Basic Rules and Ideas II

Important Ideas

1. $|z|^2 = z\bar{z}$
2. Extended Triangle Inequality: $||z| - |w|| \leq |z \pm w| \leq |z| + |w|$

Principal Argument

The principal argument of a complex number is the argument θ of a complex number z such that $-\pi < \theta \leq \pi$

Topology and Sets

Types of Points

Consider a set S. Then an element $x \in S$ must be one of the following:

1. interior point: There exists an $\epsilon > 0$ such that $B(x, \epsilon) \subseteq S$, where $B(x, \epsilon)$ is the open ball about x or radius ϵ.
2. exterior point: There exists an $\epsilon > 0$ such that $B(x, \epsilon) \cap S = \varnothing$.
3. boundary point: None of the above. More formally, for every $\epsilon > 0$, $B(x, \epsilon)$ overlaps with S and S^C at elements EXCEPT for x.

Arcs

Types of Arcs

1. Polygonal arcs: A polygonal arc is a set of finite line segments with the end point of a line segment equal to the initial point of the next line segment.
2. Closed polygonal arc: A polygonal arc where the end point of the final line segment is the start point of the first line segment.
3. Simple: If it does not cross over itself at any point in time.

A polygonal arc always separates the plane into 2 disjoint open sets.

Topology and sets

Types of Sets I

Consider a set S. Then it can described using the following terms:

1. Open: If every $x \in S$ is an interior point.
2. Closed: If the complement of S is open.
3. Bounded: If there exists an $M > 0, x \in S$ such that $S \subset B(x, M)$.
4. Compact: For now, it'll suffice to say that a set is compact if and only if it is closed and bounded.
5. Connected: If it cannot be written as a disjoint union of 2 open sets U, V such that $U \cap V \cap S = \varnothing$. In effect, you can always find a path between any 2 points in the set, typically a line.

Types of Sets II

Types of sets

1. Simply connected: Every element contained within a closed polygonal arc $c \subseteq S$ is contained in S.
2. Region: A set S that can be written as $\text{Int}(S) \cup \partial S$, where ∂S is the boundary of S and $\text{Int}(S)$ is non-empty.
3. Domain: A set that is connected and open.

Examples

Example 1

Describe the following sets in terms of if they are open, closed, bounded, compact, connected, simply connected, regions or domains.

1. $S_1 = \{z \in \mathbb{C} : |z| < 1\}$
2. $S_2 = \{z \in \mathbb{C} : |z| \leq 1\}$
3. $S_3 = \{p\}$
4. $S_4 = \{z \in \mathbb{C} : 0.5 < |z| < 1\}$
5. $S_5 = \{z \in \mathbb{C} : |z| > 1\}$

Graphs and special sets

Typical shapes

The typical shapes you can obtain:

1. Circle with centre a and radius r: $|z-a|=r$.
2. Line (perpendicular bisector of line segment between $a, b \in \mathbb{C}$): $|z-a|=|z-b|$.
3. Line (through 2 points a, b: $z = ta + (1-t)b, t \in \mathbb{R}$. More on this later.

Complex Transforms

There are 2 ways to go about problems like these.

1. Graphical/Geometrically
2. Algebraically

Method 1 Interpret the transformation as rotations, reflections, translations, and scales and accordingly change the shape of the region or curve. Method 2 Let $z = x + iy$ and substitute into the transformation $f : S \mapsto \mathbb{C}$. We thus obtain a new complex number $u + iv$ in terms of x, y. We then solve for the relationship between u, v.

Example

Example 2

Consider the function $f(z) = (1+i)z + 2$. Find the image of the following sets:

1. $S = \{z \in \mathbb{C} : 0 \leq \mathrm{Re}(z) \leq 4, -0.5 \leq \mathrm{Im}(z) < 5\}$
2. S is the set of points on the line passing through $z = 1, z = 3 + 4i$.

Properties and Inequalities Examples

Example 3: Images under transformations

Find the image of the rectangle
$S = \{x + iy \in \mathbb{C} : 0 \leq x \leq 1, 0 \leq y \leq 1\}$.

Types of functions

The 2 main types of transformations covered are affine (fancy word for linear) and fractional linear transformations.

Affine Transformations

An affine transformation is of the form $f(z) = az + b$. It consists of first scaling and rotating a complex number z by a, followed by shifting by a complex number b.

Estimating Sizes of functions

Bounding functions by a size

This just basically involves using Extended Triangle inequality to bound function sizes given some size of z.

Example 3

Suppose that $f(z) = \frac{1}{z^4 - 1}$ for all $z \in \mathbb{C} - \{\pm 1, \pm i\}$. Show that $|f(z)| \leq \frac{1}{15}$ for $|z| > 2$.

Limit

Definition of limits

A limit of a function $f : \mathbb{C} \to \mathbb{C}$ is denoted as $\lim_{z \to z_0} f(z)$.

Existence of limits

1. A limit is said to not exist if the function attains different values along different paths.
2. A limit is said to exist, that is, there is a unique $l \in \mathbb{C}$ with $\lim_{z \to z_0} f(z) = l$ if the following statement holds true: For every $\epsilon > 0$, there is a $\delta > 0$ such that $0 < |z - z_0| < \delta \implies |f(z) - l| < \epsilon$.
   More simply put: $f(z)$ gets close to l whenever z gets close to z_0.

Example

Example 4

Prove that $\lim_{z\to 1+i} z^2 = 2i$ using the definition of limits.

Example 5

Prove that the following limit does not exist:

$$\lim_{z\to 0} \frac{\mathrm{Re}(z)}{z}$$

Limit Properties

Properties and relationships

1. $\lim_{z\to z_0} f(z) \pm g(z) = L_1 \pm L_2$
2. $\lim_{z\to z_0} f(z)g(z) = L_1 L_2$
3. $\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{L_1}{L_2}$ provided that $L_2 \neq 0$.

Note that for polynomial function $f(z) = \sum_{k=0}^{n} a_k z^k$, for positive integer n, and $a_k \in \mathbb{C}$ for each k, we have the more specific limit:

$$\lim_{z\to a} f(z) = f(a)$$

Continuity

Definition: Continuity

A function is said to be continuous if:

$$\lim_{z \to a} f(z) = f(a)$$

That is, it's function value is equal to the limit of the function as that point.

As a result, we can say that the sum and product of continuous functions are always continuous. The quotient of 2 continuous functions, provided that the denominator does not evaluate to 0, is also continuous. If a function is continuous for each value of z on it's domain S, then we say that f is continuous on S.

Differentiability

Definition: Differentiability

1. The function values get close to each other quicker than the inputs get closer to each other.
2. A function is <u>differentiable</u> if the following limit exists:

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

Note: Differentiability implies continuity.

The differentiation rules from real numbers apply as usual.

Cauchy-Riemann Equations

Cauchy-Riemann Equations

The Cauchy-Riemann Equations state that a function $f(x+iy) = u(x,y) + iv(x,y)$ is differentiable at an interior point $z = a \in \mathrm{dom}(f)$ if and only if the partial derivatives of u, v all exist and are continuous and:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at $x + iy = a$. The derivative, provided it exists, of f is given by $f'(z) = u_x(x,y) + iv_x(x,y)$. The partials also satisfy $|\nabla u| = |\nabla v|$ and $\nabla u \cdot \nabla v = 0$.

To find out where a function is differentiable, you solve the 2 equations simultaneously and solve for all possible pairs of values of x, y.

Cauchy-Riemann Polar Equations

Polar form

Using the substitutions $x = r\cos\theta, y = r\sin\theta$, we obtain the following equations:

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}, \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Examples

Example 6

Where are the following functions differentiable?

1. $f_1(z) = z|z|^2$
2. $f_2(x + iy) = x^2 + iy^2$
3. $f_3(x + iy) = |x| + i|y|$

Holomorphic

Definitions

A function is said to be holomorphic at a if the function is differentiable in some neighbourhood of a (an open disk with centre a). A function that is holomorphic everywhere is called entire.

Thus if the function is differentiable on an open set, it is holomorphic in that set.

Holomorphic-ness

A function can only ever be holomorphic on an open set. So if a function is differentiable on a closed set, it will NOT be holomorphic.

Harmonic Functions

Definition

Let D be a domain in $\mathbb{R}^3$. A function of 2 variables u is harmonic if it satisfies Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and al first and 2nd partials are continuous in D. The harmonic conjugate is a function of 2 variables v so that the Cauchy-Riemann equations are satisfied.

Harmonic Conjugates

It has absolutely nothing to do with the actual conjugate of a complex number.

Properties of Harmonic Conjugates

Corollaries

1. $-u$ is the harmonic conjugate of v.
2. If u is a harmonic on a simply connected domain, then u has a harmonic conjugate on D.
3. Harmonic conjugates of u only ever differ by a constant.
4. Let f be a function holomorphic at $z = a$. Then $f(z)$ admits a power series expansion about a (not needed for now, but is a master-key for later).
5. Let f, g be 2 holomorphic functions on D and C be a smooth curve in D. If $f(z) = g(z)$ for each $z \in C$, then $f(z) = g(z)$ for each $z \in D$.

Examples

Example 7

Show that $\cos x \cosh y$ is harmonic and find its harmonic conjugate.

Example 8

Show that $\frac{x}{x^2+y^2}$ is harmonic and find its harmonic conjugate.

Super Important functions

Definitions

The following will be important functions we will be dealing with for solving questions:

1. $f(z) = e^z = e^{x+iy} = e^x(\cos y + i \sin y)$
2. $f(z) = \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, f(z) = \cos z = \frac{e^{iz} + e^{-iz}}{2}$

Examples: Solving Equations

Example 9

Solve the following equations

1. $e^z = 2i$
2. $\cos z = 3$
3. $\cosh z = -4.$

Example 10

Show that $\tan z = i$ has no solution.

Logarithms and Powers

Invertibility

A function $f : S \mapsto T$ is invertible if it is bijective. That is, there is an element of S that maps to T for any T, and such an element is unique. So to check for bijectivity, one must test:

1. Is there a value of z such that $f(z) = w$ for any $w \in T$
2. $f(z) = f(w) \implies z = w$

With obvious reasoning, we see that exp is not bijective on $\mathbb{C}$, because we can keep rotating by 2π and so while the inputs of $\exp : \mathbb{C} \mapsto \mathbb{C}$ might be different, the output is still the same. Thus, the idea of principal value becomes super important to create these bijective functions.

Inverting the exponential

Basic multi-valued logarithm

Consider the expression $e^z = w$. The multi-valued logarithm log is the function such that $z = \log w$. In terms of a formula:

$$z = \log w = \ln|w| + i(\arg w + 2k\pi)$$

where $k \in \mathbb{Z}$.

Now obviously, we run into some problems because this is obviously not a function, so it won't be differentiable nor holomorphic and there's no point continuing the discussion.

Principal valued logarithm

Definition

We thus yield the following definition of the principal valued logarithm:

$$\mathrm{Log} z = \ln|z| + i(\mathrm{Arg} z)$$

where Arg denotes the principal value argument function.

Differentiability and properties of Log

The principal valued logarithm is indeed differentiable everywhere where it is not continuous. Now obviously, $|z|$ is always non-negative, and since we are taking the natural logarithm (in the real numbers, we automatically know that Log is not continuous at $z = 0$. The only other issue arises with the Arg. Since Arg by definition finds the argument of z over the interval $-\pi < \arg z \leq \pi$, we can figure out that it is not differentiable on $(-\infty, 0) \subseteq \mathbb{R}$. Hence, Log is differentiable everywhere except $(-\infty, 0]$.

Powers

Finding powers of numbers

$$z^a = \exp(a\mathrm{Log}(z))$$

Note that based on this, because exp is continuous and differentiable everywhere, we only really need to check the differentiability of Log whenever we are dealing with weirder functions.

Example 11: Evaluating principal value powers

1. $\mathrm{pv}\left[\left(\frac{1+\sqrt{3}i}{2}\right)^{-3}\right]^{1-i}$
2. i^i
3. $\lim_{z\to 0}(\cos z)^{\frac{1}{z^2}}$

Examples

Example 12: Differentiability of weird functions

Where are the following functions analytic:

1. $f(z) = \mathrm{Log}(iz)$
2. $g(z) = z^{-1}\mathrm{Log}(z+1)$

Example 13: More differentiability examples

Where are the principal branches of the following operations analytic:

1. $f(z) = \sqrt{z+1}$
2. $f(z) = \sqrt{z^2-1}$

Linear Fractional Transformations

Definition

A fractional linear transformation is a function $f : \mathbb{C} \mapsto \mathbb{C}$ such that:

$$f(z) = \frac{az+b}{cz+d}$$

with $ad - bc = 1$ for some complex numbers a, b, c, d.

Theorem

A fractional linear transformation maps a line or a circle to another line or circle.

That means, you really only need 3 points to work out the nature of the shape, so just pick the easiest values you can. Typically, use $z = i, z = 1, z = 0, z \to \infty$.

Examples

Example 14

Find the image of $|z-1| \leq 1$ under the mapping $w = \frac{z}{z+2}$.

Example 15

Find the image of the line $x + 2y = 2$ under the mapping $w = \frac{1}{z+i}$.

Curves

If you've done MATH2011 or MATH2111 a lot of this may look similar to what you learnt for curves in $\mathbb{R}^2$. Most of the results for curves in $\mathbb{R}^2$ can be easily transferred to $\mathbb{C}$.

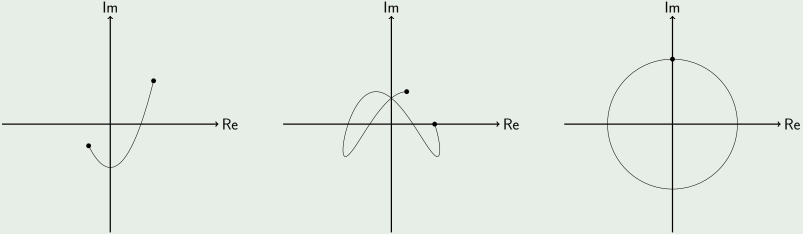
Definition 1

A **curve** in $\mathbb{C}$ is a continuous function $\gamma : [a, b] \to \mathbb{C}$.
The **initial point** of the curve is $\gamma(a)$ and the **final point** is $\gamma(b)$.
The **range** of a curve is the set $\{\gamma(t) : t \in [a, b]\}$.
A curve is **closed** if $\gamma(a) = \gamma(b)$, and **simple** if $\gamma(s) \neq \gamma(t)$ when $s < t$, except for possibly $s = a, t = b$.

Curves

Example 1

Classify the following curves and closed or simple:



Both the first and last curves are simple, as they only “cross” at the endpoints, if at all.
Only the last curve is closed, as the initial and final points are the same.

Curves

We can combine and flip curves, as you might expect.

Definition 2

Let $\alpha : [a, b] \to \mathbb{C}$ and $\beta : [c, d] \to \mathbb{C}$ be curves, with $\alpha(b) = \beta(c)$. Then the **join of α and β** is

$$(\alpha \sqcup \beta)(t) = (\alpha + \beta)(t) = \begin{cases} \alpha(t), & a \leq t \leq b; \\ \beta(t), & c \leq t \leq d. \end{cases}$$

Definition 3

Let $\gamma : [a, b] \to \mathbb{C}$ be a curve. Then the **reverse curve** $\gamma^* : [-b, -a] \to \mathbb{C}$ is

$$\gamma^*(t) = \gamma(-t).$$

Parameterisations

We could write a curve $\gamma(t) = t$ on $[0, 1]$, or $\delta(t) = t + 1$ on $[-1, 0]$. These describe the same curve in different ways, so we formalise this.

Definition 4

Suppose that $\gamma : [a, b] \to \mathbb{C}$ is a curve, and $h : [c, d] \to [a, b]$ is a continuous bijection such that $h(c) = a$ and $h(d) = b$. Then we call $\gamma \circ h : [c, d] \to \mathbb{C}$ a **reparameterisation of** γ.

Derivatives of Curves

Derivatives are used a lot in contour integration, so we define it for curves in the complex plane.

Definition 5

Suppose $\gamma : [a, b] \to \mathbb{C}$ is a curve, with $\gamma(t) = \gamma_1(t) + \gamma_2(t)i$ and γ_1, γ_2 are real-valued (real and imaginary components). Then we define the **derivative**

$$\gamma'(t) = \gamma_1' + \gamma_2'(t)i.$$

We say that γ is **continuously differentiable** if the derivative exists and is continuous on $[a, b]$.
We say that γ is **smooth** if it is continuously differentiable, and $\gamma'(t) \neq 0$ for all $t \in [a, b]$.
We say that γ is **piecewise smooth** if it is a finite join $\gamma = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, and all α_i are smooth.

Derivatives of Curves

Example 2

Is the curve $\gamma : [-1,1] \to \mathbb{C}$ given by

$$\gamma(t) = |t| + it$$

smooth? Piecewise smooth?

Since the derivative of $|t|$ doesn't exist at $t = 0$, it cannot be smooth. However, we can break it up into the curves

$$\begin{aligned} \gamma_1(t) &= t + it, \quad t \in [-1,0], \\ \gamma_2(t) &= -t + it, \quad t \in [0,1]. \end{aligned}$$

Then both γ_1 and γ_2 are smooth, and $\gamma = \gamma_1 + \gamma_2$. Thus, γ is piecewise smooth.

Curve Length

Almost every curve you'll deal with will be piecewise smooth, but keep in mind the piecewise smooth condition if you're asked to define the terms.

Definition 6

The **length** of a piecewise smooth curve $\gamma : [a, b] \to \mathbb{C}$ is

$$\text{Length}(\gamma) = \int_a^b |\gamma'(t)|\, dt.$$

This definition plays nicely with intuition in $\mathbb{R}^2$. The length of a curve in the complex plane is the same as the length of a string along the curve.

Curve Length

Example 3

Find the length of the curve

$$\gamma(t) = Re^{it},$$

for $t \in [0, 2n\pi]$, $n \in \mathbb{N}$, and $R > 0$.

$\gamma'(t) = Rie^{it}$, so the length of our curve is:

$$\begin{aligned} \text{Length}(\gamma) &= \int_0^{2n\pi} \left|Rie^{it}\right| dt \\ &= \int_0^{2n\pi} R\, dt \\ &= 2n\pi R. \end{aligned}$$

Contours

Definition 7

A **contour** is the oriented range of a piecewise smooth curve γ. In other words, it is the range $\mathrm{Range}(\gamma)$ with some orientation describing how this set should be traversed.

This is really just another word for a curve, however we don't care about how the curve is parameterised, just the direction you're meant to traverse it.
Generally these are described as a set in the complex plane, traversed in some manner. If the contour is simple (doesn't cut itself), then we traverse it **anticlockwise** or **clockwise**. If an orientation isn't defined, we traverse it anticlockwise.

Complex Integration

Definition 8

We define the **integral** of a complex-valued function $f : [a, b] \to \mathbb{C}$ where $f(t) = u(t) + v(t)i$ and u, v are both real-valued as

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$

Effectively, we treat the imaginary unit i as just another constant.

Complex Integration

Just as with real integrals, we have some familiar identities.

Theorem 1

Let $f : [a, b] \to \mathbb{C}$ and $g : [a, b] \to \mathbb{C}$. Further, let $h : [c, d] \to [a, b]$ be a differentiable with $h(c) = a$, $h(d) = b$, and $\lambda, \mu \in \mathbb{C}$. Then

- $\displaystyle\int_a^b \lambda f(t) + \mu g(t)\, dt = \lambda \int_a^b f(t)\, dt + \mu \int_a^b g(t)\, dt,$
- $\displaystyle\int_c^d f(h(t))h'(t)\, dt = \int_a^b f(t)\, dt,$
- $\displaystyle\int_a^b f'(t)h(t)\, dt = [f(t)g(t)]_a^b - \int_a^b f(t)g'(t)\, dt,$
- $\displaystyle\left|\int_a^b f(t)\, dt\right| \leq \int_a^b |f(t)|\, dt.$

Complex Integration

Example 4

Integrate $f(t) = te^{it}$ over $[0, 2\pi]$.

$$
\begin{aligned}
\int_0^{2\pi} te^{it}\,dt &= \left[t\frac{e^{it}}{i}\right]_0^{2\pi} - \int_0^{2\pi} \frac{e^{it}}{i}\,dt \\
&= -2\pi i - \left[\frac{e^{it}}{i^2}\right]_0^{2\pi} \\
&= -2\pi i.
\end{aligned}
$$

Complex Integration

Example 5

Using the previous example, deduce that

$$\int_0^{2\pi} t \cos t \, dt = 0.$$

We have

$$\begin{aligned}\int_0^{2\pi} t \cos t \, dt &= \int_0^{2\pi} \operatorname{Re}\left(te^{it}\right) \, dt \\ &= \operatorname{Re}\left(\int_0^{2\pi} te^{it} \, dt\right) \\ &= \operatorname{Re}(-2\pi i) \\ &= 0.\end{aligned}$$

Line Integration

We define line integrals in $\mathbb{C}$ much the same as $\mathbb{R}^2$.

Definition 9

Given a piecewise smooth curve $\gamma : [a, b] \to \mathbb{C}$, we define the **complex line integral**

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

You can think of this as using the substitution $z = \gamma(t)$.

Line Integration

Theorem 2

Let $\lambda, \mu \in \mathbb{C}$, α, β be piecewise smooth curves, and f, g be complex functions defined on Range(γ). Further, let $\gamma = \alpha + \beta$. Then

- $\displaystyle\int_\alpha \lambda f(z) + \mu g(z)\, dz = \lambda \int_\alpha f(z)\, dz + \mu \int_\alpha g(z)\, dz,$
- $\displaystyle\int_{\alpha^*} f(z)\, dz = -\int_\alpha f(z)\, dz,$
- $\displaystyle\int_\gamma f(z)\, dz = \int_\alpha f(z)\, dz + \int_\beta f(z)\, dz.$

Line Integration

Example 6

Show that the line integral of $f(z) = z$ along the curve $\gamma(t) = e^{it}$ for $t \in [0, 2\pi]$ is zero.

$$\begin{aligned}\int_\gamma f(z)\, dz &= \int_0^{2\pi} f(e^{it}) ie^{it}\, dt \\ &= \int_0^{2\pi} ie^{2it}\, dt \\ &= \left[\frac{e^{2it}}{2}\right]_0^{2\pi} \\ &= 0.\end{aligned}$$

Line Integration

Example 7

Evaluate the line integral of $f(z) = 1$ along the line segment from 0 to $1 + i$.

We can parameterise the segment as $\gamma(t) = (1 + i)t$ for $t \in [0, 1]$. Then

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_0^1 \gamma'(t)\,dt \\ &= [\gamma(t)]_0^1 \\ &= \gamma(1) - \gamma(0) \\ &= 1 + i.\end{aligned}$$

Line Integration

Theorem 3

Let γ, δ be piecewise smooth curves, where δ is a reparameterisation of γ, and f be complex-valued defined on Range(γ). Then

$$\int_\gamma f(z)\,dz = \int_\delta f(z)\,dz.$$

Theorem 4 (ML Lemma)

Let γ be a piecewise smooth curve and f be a complex-valued function defined on Range(γ). Then

$$\left|\int_\gamma f(z)\,dz\right| \leq ML,$$

where L is the length of γ, and M is a maximiser of $|f|$ on Range(γ).

ML Lemma

Example 8

Confirm the ML Lemma for $f(z) = \frac{1}{z^2}$ over the upper semicircle or radius $R > 0$.

We can parameterise the upper semicircle as $\gamma(t) = Re^{it}$ for $t \in [0, \pi]$. Note that Length$(\gamma) = \pi R$, and $|f(z)| = \frac{1}{R^2}$ over γ. Then

$$\begin{aligned} \left| \int_\gamma f(z)\, dz \right| &\leq \int_0^\pi \left| \frac{Rie^{it}}{R^2 e^{2it}} \right| dt \\ &= \frac{1}{R} \int_0^\pi dt \\ &= \frac{\pi}{R}. \end{aligned}$$

Here, $ML = \frac{\pi R}{R^2} = \frac{\pi}{R}$ as we expect.

Contour Integration

Definition 10

Given a contour Γ, we define the **contour integral**

$$\int_\Gamma f(z)\,dz = \int_\gamma f(z)\,dz,$$

where γ is any parameterisation of Γ. This is well-defined, as the complex line integral is independent of parameterisation.

Cauchy-Goursat Theorem

Theorem 5 (Cauchy-Goursat)

Suppose that Ω is a simply connected domain, that f is holomorphic on Ω, and that Γ is a closed contour in Ω. Then

$$\int_\Gamma f(z)\,dz = 0.$$

Theorem 6 (Cauchy-Goursat (v2.0))

Suppose that Ω is a bounded domain whose boundary consists of finitely many contours $\Gamma_1, \Gamma_2, \cdots, \Gamma_n$. Further, suppose f is holomorphic on an open set containing $\overline{\Omega}$. Then

$$\int_{\partial\Omega} f(z)\,dz = \sum_{k=1}^{n} \int_{\Gamma_k} f(z)\,dz = 0.$$

Cauchy-Goursat Theorem

Example 9

Show that the integral of $f(z) = \frac{1}{z}$ is the same along every contour $\Gamma_R = \{z \in \mathbb{C} : |z| = R\}$.

Note that f is holomorphic on $\mathbb{C} \setminus \{0\}$, so consider the bounded domain $\Omega = \{z \in \mathbb{C} : R_1 < |z| < R_2\}$ for $R_1, R_2 > 0$ (noting $\overline{\Omega} \subseteq \mathbb{C} \setminus \{0\}$). We apply Cauchy-Goursat to get

$$\begin{aligned} &\int_{\partial\Omega} f(z)\,dz = 0 \\ \implies \quad &\int_{\Gamma_{R_1}} f(z)\,dz + \int_{\Gamma^*_{R_2}} f(z)\,dz = 0 \\ \implies \quad &\int_{\Gamma_{R_1}} f(z)\,dz = \int_{\Gamma_{R_2}} f(z)\,dz. \end{aligned}$$

Cauchy-Goursat Theorem

Example 10

Find

$$\int_{-\infty}^{\infty} \frac{x^2 - 1}{(x^2+1)^2}\, dx$$

by considering an integral of

$$f(z) = \frac{1}{(z+i)^2}.$$

Let Γ_R be the upper semicircular arc or radius $R > 0$ around 0, and $\Gamma_x = [-R, R]$. $f(z)$ is holomorphic on the set $\left\{z \in \mathbb{C} : \operatorname{Im}(z) > -\frac{1}{2}\right\}$, and the join of Γ_R and Γ_x (say Γ) lies inside this domain. Thus, by Cauchy-Goursat,

$$\int_{\Gamma} f(z)\, dz = 0.$$

Cauchy-Goursat Theorem (cont.)

Now, we can evaluate each part of the contour integral separately. Note that on Γ_R, we have

$$|f(z)| = \left|\frac{1}{(z+i)^2}\right| \leq \frac{1}{(R-1)^2},$$

when $R > 1$. Since Length$(\Gamma_R) = \pi R$, by ML lemma,

$$\lim_{R\to\infty} \int_{\Gamma_R} f(z)\, dz \leq \lim_{R\to\infty} \frac{1}{(R-1)^2} \cdot \pi R = 0.$$

So, we deduce that the integral is zero as $R \to \infty$.

Cauchy-Goursat Theorem (cont.)

Along Γ_x, we have

$$\int_{\Gamma_x} f(z)\,dz = \int_{-R}^{R} \frac{1}{(x+i)^2}\,dx = \int_{-R}^{R} \frac{(x-i)^2}{(x^2+1)^2}\,dx.$$

Combining this with the integral along Γ_R, we have

$$\begin{aligned}\lim_{R\to\infty} \operatorname{Re}\left(\int_{\Gamma} f(z)\,dz\right) &= \operatorname{Re}\left(\int_{-\infty}^{\infty} \frac{(x-i)^2}{(x^2+1)^2}\,dx\right) \\ &= \int_{-\infty}^{\infty} \frac{x^2-1}{(x^2+1)^2}\,dx \\ &= 0.\end{aligned}$$

Consequences of Cauchy-Goursat

Theorem 7 (Independence of Contour)

Suppose Ω is a simply connected domain, f is holomorphic on Ω, and Γ, Δ are two contours with the same initial and final points. Then

$$\int_\Gamma f(z)\,dz = \int_\Delta f(z)\,dz.$$

Theorem 8 (Existence of Primitives)

Suppose Ω is a simply connected domain, f is holomorphic on Ω, and Γ is a contour from p to q. Then there exists some differentiable function F on Ω such that $F' = f$ and

$$\int_\Gamma f(z)\,dz = F(q) - F(p).$$

Cauchy's Integral Formula

Theorem 9 (Cauchy's Integral Formula)

Suppose that Ω is a simply connected domain, f is holomorphic on Ω, Γ is a simple closed contour in Ω, and $w \in \operatorname{Int}(\Gamma)$. Then

$$f(w) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-w}\, dz.$$

This allows us to handle integration of functions that aren't holomorphic at a point (to some extent).

Cauchy's Integral Formula

Example 11

Evaluate

$$\int_\Gamma \frac{1}{z^2+1}\,dz$$

where Γ is the circle of radius 1 centred at i.

Let $f(z) = \frac{1}{z+i}$, $w = i$, and $\Omega = B(i, 1+\varepsilon)$. Then Ω is a simply connected domain, f is holomorphic on Ω, Γ is a simple closed contour in Ω, and $w \in \operatorname{Int}(\Gamma)$. Thus, by Cauchy's Integral formula,

$$\begin{aligned}\int_\Gamma \frac{1}{z^2+1}\,dz &= \int_\Gamma \frac{f(z)}{z-w}\,dz \\ &= 2\pi i f(w) \\ &= \frac{2\pi i}{i+i} \\ &= \pi.\end{aligned}$$

Power Series

Using Cauchy's Integral Formula, we can prove that any holomorphic function can be written as a power series.

Theorem 10

Suppose that f is holomorphic on the ball $B(z_0, R)$, and Γ is a simple closed contour in $B(z_0, R)$ with $z_0 \in \mathrm{Int}(\Gamma)$. Then, for all $w \in B(z_0, R)$,

$$f(w) = \sum_{n=0}^{\infty} c_n (w - z_0)^n,$$

where

$$c_n = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$

This shows that holomorphic functions are actually infinitely differentiable.

Cauchy's Generalised Integral Formula

Theorem 11 (Cauchy's Generalised Integral Formula)

Suppose that Ω is a simply connected domain, f is holomorphic on Ω, Γ is a simple closed contour in Ω, and $w \in \mathrm{Int}(\Gamma)$. Then

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(z)}{(z-w)^{n+1}}\, dz.$$

This is one of the most useful theorems of the course.

Cauchy's Generalised Integral Formula

Example 12

Evaluate

$$\int_\Gamma \frac{1}{z^n}\, dz,$$

where Γ is the unit circle, and $n \in \mathbb{Z}$.

Let $f(z) = 1$ and $w = 0$. Then if $n > 0$, we can use Cauchy's generalised integral formula to evaluate

$$\int_\Gamma \frac{1}{z^n}\, dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(0) = \begin{cases} 2\pi i, & n = 1; \\ 0, & n > 1. \end{cases}$$

If $n \leq 0$, then $\frac{1}{z^n}$ is entire, so by Cauchy-Goursat,

$$\int_\Gamma \frac{1}{z^n}\, dz = 0.$$

Cauchy's Generalised Integral Formula

Example 13 (MATH2621 2018 Q83)

Suppose f is entire, and $|f(z)| \leq 1 + |z|$ everywhere. Show that $f(z) = az + b$ for some constants $a, b \in \mathbb{C}$.

Let Γ_R be the circle of radius $R > 0$ centred at 0. Then, by Cauchy's generalised integral formula and ML lemma,

$$|f^{(n)}(0)| = \left| \frac{n!}{2\pi i} \int_{\Gamma_R} \frac{f(z)}{z^{n+1}}\, dz \right| \leq \frac{n!}{2\pi} \frac{1+R}{R^{n+1}} \cdot 2\pi R.$$

Since this is true for all $R > 0$, taking the limit, we find that $f^{(n)}(0) = 0$ for $n > 1$. Thus, when written as a Taylor series, $f(z) = f(0) + f'(0)z$, as required.

Liouville's and Morera's Theorems

Theorem 12 (Liouville)

Suppose f is bounded and entire. Then f is constant.

Theorem 13 (Morera)

Suppose that Ω is a domain, f is continuous on Ω, and

$$\int_{\Gamma} f(z)\, dz = 0$$

for every closed contour $\Gamma \subseteq \Omega$. Then f is holomorphic on Ω.

This is, to some extent, the converse of Cauchy-Goursat.

Liouville's Theorem

Example 14

Suppose that f, g are entire functions, and $|f(z)| \leq |g(z)|$ everywhere. Prove that if g has no roots, then $f(z) = ag(z)$ for some fixed $a \in \mathbb{C}$ and all $z \in \mathbb{C}$.

Let

$$h(z) = \frac{f(z)}{g(z)}.$$

Since f and g are entire, and $g(z) \neq 0$, h is entire. Further, $|h(z)| \leq 1$ for all $z \in \mathbb{C}$. Thus, by Liouville's Theorem, $h(z) = a$ for some constant $a \in \mathbb{C}$. Simply rearranging gives us

$$f(z) = ag(z),$$

as required.

Power Series

Definition 11

A **(complex) power series** is an expression of the form

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n,$$

where z, z_0, and z are complex. The largest $R > 0$ such that the power series converges in $B(z_0, R)$ is called the **radius of convergence**. If the series converges only at z_0, we say $R = 0$. If it converges everywhere, then we say $R = \infty$.

Theorem 14

A power series can be integrated and differentiated term-by-term inside its radius of convergence.

Power Series

Example 15

Find the radius of convergence of

$$\sum_{n=2}^{\infty} \frac{2^n n}{n^2 - 1}(z-2)^n.$$

We apply the ratio test. We have

$$\lim_{n\to\infty} \left| \frac{2^{n+1}(n+1)(z-2)^{n+1}}{(n+1)^2 - 1} \cdot \frac{n^2 - 1}{2^n n(z-2)^n} \right|$$
$$= \lim_{n\to\infty} \frac{2(n+1)(n^2-1)}{n(n^2+2n)} |z-2|$$
$$= 2|z-2|.$$

Thus, we have convergence for $2|z-2| < 1$. So, our radius of convergence is $R = \frac{1}{2}$.

Taylor Series

Taylor series can be defined for complex functions exactly like real functions.

Definition 12

The **Taylor series** of a holomorphic function f "around" or "with centre" z_0 is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n.$$

A Taylor series around 0 ($z_0 = 0$) is called a **Maclaurin series**.

You are expected to know common Maclaurin series, like e^x and $\sin x$ from first year.
Their complex analogues are identical.

Taylor Series

Example 16

Find the Taylor series expansion for $\sin z$ around π.

First, we compute the derivatives of $\sin z$

$$
\begin{aligned}
f(z) &= \sin z & \implies && f(\pi) &= 0, \\
f'(z) &= \cos z & \implies && f'(\pi) &= -1, \\
f''(z) &= -\sin z & \implies && f''(\pi) &= 0, \\
f^{(3)}(z) &= -\cos z & \implies && f^{(3)}(\pi) &= 1, \\
&& \vdots &&&
\end{aligned}
$$

Then,

$$\sin z = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!}(z-\pi)^{2n-1}.$$

Taylor Series

Example 17

Find a series representation of $f(z) = \int_\Gamma \frac{\sin x}{x}\, dx$, where Γ is the line segment from 0 to z.

We know the Maclaurin series for $\sin z$:

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}.$$

From this, we find

$$\frac{\sin z}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n}.$$

Taylor Series (cont.)

Parameterising Γ as $\gamma(t) = tz$ for $t \in [0, 1]$, and noting that we can swap integration and summation inside the domain of convergence (which is all $\mathbb{C}$ in this case), we have

$$\begin{aligned}\int_\Gamma \frac{\sin x}{x}\,dx &= \sum_{n=0}^{\infty}\int_\Gamma \frac{(-1)^n}{(2n+1)!}x^{2n}\,dx\\ &= \sum_{n=0}^{\infty}\int_0^1 \frac{(-1)^n}{(2n+1)!}(tz)^{2n}z\,dt\\ &= \sum_{n=0}^{\infty} z^{2n+1}\int_0^1 \frac{(-1)^n}{(2n+1)!}t^{2n}\,dt\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(2n+1)!}z^{2n+1}.\end{aligned}$$

Laurent Series

If a function isn't holomorphic at a point, then to get a power series near that point, you'd need to find several around it. In this case, it can be useful to discuss series defined on annuli.

Theorem 15 (Laurent's Theorem)

Let A be the annulus $A = B(z_0, R_2) \setminus \overline{B(z_0, R_1)}$, and $R_1 < r < R_2$. If f is holomorphic on A, then, for every $w \in A$,

$$f(w) = \sum_{n=-\infty}^{\infty} c_n (w - z_0)^n,$$

where

$$c_n = \frac{1}{2\pi i} \int_{\partial B(z_0, r)} \frac{f(z)}{(z - z_0)^{n+1}} \, dz.$$

This is called the **Laurent series** of f on the annulus A.

Laurent Series

Example 18

Find the Laurent series of

$$f(z) = \frac{1}{z(z-1)(z-2)}$$

in the "annulus" $\{z \in \mathbb{C} : |z-1| > 1\}$.

First, we expand into partial fractions:

$$f(z) = \frac{1}{z(z-1)(z-2)} = \frac{1}{2}\frac{1}{z} - \frac{1}{z-1} + \frac{1}{2}\frac{1}{z-2}.$$

Laurent Series (cont.)

Now, we expand it out into convergent geometric series, only in terms of $(z-1)$:

$$\begin{aligned} f(z) &= \frac{1}{2}\frac{1}{z-1}\frac{1}{1+\frac{1}{z-1}} - \frac{1}{z-1} + \frac{1}{2}\frac{1}{z-1}\frac{1}{1-\frac{1}{z-1}} \\ &= \frac{1}{2}\frac{1}{z-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{(z-1)^n} - \frac{1}{z-1} + \frac{1}{2}\frac{1}{z-1}\sum_{n=0}^{\infty}\frac{1}{(z-1)^n} \\ &= \sum_{n=-\infty}^{\infty} c_n (z-1)^n, \end{aligned}$$

where

$$c_n = \begin{cases} 1, & n = -3, -5, -7, \cdots; \\ 0, & \text{otherwise.} \end{cases}$$

Laurent Series

Example 19

Find the Laurent series of

$$f(z) = \frac{z}{(z-2)(z+1)}$$

in the largest annulus containing 0 around 2.

First, we expand into partial fractions:

$$f(z) = \frac{2}{3}\frac{1}{z-2} + \frac{1}{3}\frac{1}{z+1}.$$

Now, we look for a solution on $\{z \in \mathbb{C} : 0 < |z-2| < 3\}$.

Laurent Series (cont.)

As previously, expand it using geometric series

$$
\begin{aligned}
f(z) &= \frac{2}{3}\frac{1}{z-2} + \frac{1}{3}\frac{1}{3+(z-2)} \\
&= \frac{2}{3}\frac{1}{z-2} + \frac{1}{9}\frac{1}{1+\frac{z-2}{3}} \\
&= \frac{2}{3}\frac{1}{z-2} + \frac{1}{9}\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n}(z-2)^n \\
&= \sum_{n=-\infty}^{\infty} c_n (z-2)^n,
\end{aligned}
$$

where

$$
c_n = \begin{cases} \frac{2}{3}, & n=-1; \\ \frac{(-1)^n}{3^{n+2}}, & n \geq 0; \\ 0, & \text{otherwise.} \end{cases}
$$

Singularities

Definition 13

An **isolated singularity** of f is a point z_0 for which f is holomorphic on $B^\circ(z_0, r)$ for some $r > 0$, but is not differentiable at z_0.

Definition 14

Suppose f has an isolated singularity at z_0, and has Laurent coefficients c_n. Assume $f \not\equiv 0$ so that there is at least one non-zero c_n. Then we have three exclusive and exhaustive possibilities:

1. No $n < 0$ have $c_n \neq 0$. We say that f has a **removable singularity** at z_0.
2. Some non-zero, finite number of $n < 0$ have $c_n \neq 0$. We say that f has a **pole** at z_0.
3. Infinitely many $n < 0$ have $c_n \neq 0$. We say that f has an **essential singularity** at z_0.

Singularities

Rather than using Laurent series, it can be easier to evaluate a limit, in some cases.

Theorem 16

Suppose f has an isolated singularity at z_0, and $f \not\equiv 0$. Then

1. If $\lim_{z\to z_0} f(z)$ exists, we have a removable singularity.
2. If $\lim_{z\to z_0}(z-z_0)^k f(z)$ exists for $k=n$, but not for $k=0,\cdots,n-1$, we have a pole (of order n).
3. If $\lim_{z\to z_0}(z-z_0)^k f(z)$ doesn't exist for any k, we have an essential singularity.

Poles and Zeroes

Definition 15

Suppose f has a pole at z_0. Then there is an $M < 0$ such that $c_M \neq 0$ and $c_n = 0$ for all $n < M$. We say that f has a **pole of order** $-M$ at z_0, or that the pole has order $-M$. A **simple pole** is a pole of order 1.

Definition 16

Suppose a non-constant function f has a removable singularity at z_0. If there is an $M > 0$ such that $c_M \neq 0$ and $c_n = 0$ for all $n < M$, then we say that f has a **zero of order** M at z_0. A **simple zero** is a zero of order 1.

Singularities

Example 20

Classify all singularities of $f(z) = \tan z$.

tan is undefined for $z = \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$. At these points,

$$\begin{aligned}
\lim_{z \to \frac{\pi}{2} + n\pi} \left(z - \frac{\pi}{2} - n\pi\right) \tan z &= \lim_{z \to 0} z \frac{\sin\left(z + \frac{\pi}{2} + n\pi\right)}{\cos\left(z + \frac{\pi}{2} + n\pi\right)} \\
&= \lim_{z \to 0} z \frac{(-1)^n \cos(-z)}{(-1)^n \sin(-z)} \\
&= -1.
\end{aligned}$$

Thus, at $z = \frac{\pi}{2} + n\pi$, we have a pole of order 1.
Another way to see this, is that $\cos z$ has simple zeroes at these points, so since it's in the denominator, it contributes simple poles. Thus, $\tan z$ has simple poles.

Singularities

Example 21

Show that the singularity at $z = 0$ of $f(z) = e^{-1/z}$ is essential.

We can very easily construct a Laurent series. Since

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n,$$

simply substituting, we find

$$f(z) = e^{-1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-1}{z} \right)^n = \sum_{n=-\infty}^{0} \frac{(-1)^n}{(-n)!} z^n.$$

Now, there are infinitely many non-zero negative order terms, so the singularity must be essential.

Singularities

Example 22

Classify the singularity at $z = 0$ of $f(z) = \frac{\sin^3 z}{z}$.

Note that

$$\lim_{z \to 0} \frac{\sin^3 z}{z} = 0.$$

Thus, we have a removable singularity. Since it's a zero, we find its order.

$$\lim_{z \to 0} \frac{\sin^3 z}{z^2} = 0,$$
$$\lim_{z \to 0} \frac{\sin^3 z}{z^3} = 1.$$

Since we have to force in a pole of order 2, to get a non-zero limit, we have a zero of order 2.

Residues

Residues allow us to extend our methods beyond holomorphic functions.

Definition 17

Suppose f is holomorphic on $B^\circ(z_0, r)$ for some $r > 0$, with Laurent coefficients c_n in $B^\circ(z_0, r)$. The **residue** of f at z_0 is

$$\mathrm{Res}(f, z_0) = \mathrm{Res}(f(z); z = z_0) = c_{-1}.$$

Theorem 17

If f has a pole of order N at z_0, then

$$\mathrm{Res}(f, z_0) = \frac{1}{(N-1)!} \lim_{z \to z_0} \frac{d^{N-1}}{dz^{N-1}} (z - z_0)^N f(z).$$

Residues

Example 23

Find the residues of $f(z) = \frac{\sin z}{z(z-1)(z-3)^2}$ at its singularities.

We have poles of order 1 and 2 at $z = 1$ and $z = 3$ respectively. There is a removable singularity at $z = 0$. Thus,

$$\begin{aligned}
\operatorname{Res}(f, 3) &= \frac{1}{(2-1)!} \lim_{z \to 3} \frac{d}{dz}(z-3)^2 f(z) = \frac{6\cos 3 - 5\sin 3}{36}, \\
\operatorname{Res}(f, 1) &= \frac{1}{(1-1)!} \lim_{z \to 1} (z-1) f(z) \qquad = \frac{\sin 1}{4}, \\
\operatorname{Res}(f, 0) &= 0.
\end{aligned}$$

Residues

Example 24 (MATH2621 2018 Q102c)

Find the residues of $f(z) = \exp\left(z + \frac{1}{z}\right)$ at its singularities.

Note that the only singularity is at $z = 0$. So,

$$\begin{aligned} e^{z+\frac{1}{z}} &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(z + \frac{1}{z}\right)^n \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{z^{2k-n}}{k!(n-k)!}. \end{aligned}$$

Now, we need the coefficient of z^{-1}, which occurs when we have $2k - n = -1$. Adding the relevant terms, we have

$$\operatorname{Res}(f, 0) = \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!}.$$

Cauchy's Residue Theorem

Theorem 18 (Cauchy's Residue Theorem)

Suppose Ω is a domain, and that Γ is a simple closed contour with standard (anticlockwise) orientation in Ω. Further, let f be holomorphic on Ω, and $\text{Int}(\Gamma) \cap \Omega = \text{Int}(\Gamma) \setminus \{z_1, z_2, \cdots, z_K\}$. Then

$$\int_\Gamma f(z)\, dz = 2\pi i \sum_{k=1}^{K} \text{Res}(f, z_k).$$

This theorem is used mostly in evaluating integrals around singularities, and expands the kinds of integrals we can now evaluate using complex analysis methods.

Cauchy's Residue Theorem

Example 25

Evaluate

$$\int_\Gamma \frac{1}{z(z-1)(z+2)}\,dz$$

where Γ is the circle centred at 1 or radius 2 traversed anticlockwise.

The poles at $z = 1$ and $z = 0$ lie in the contour, so we calculate (letting f be the integrand)

$$\begin{aligned} \operatorname{Res}(f,0) &= \lim_{z\to 0} \frac{1}{(z-1)(z+2)} &&= -\frac{1}{2}, \\ \operatorname{Res}(f,1) &= \lim_{z\to 1} \frac{1}{z(z+2)} &&= \frac{1}{3}. \end{aligned}$$

Thus, by Cauchy's residue theorem.

$$\int_\Gamma \frac{1}{z(z-1)(z+2)}\,dz = 2\pi i \left(-\frac{1}{2} + \frac{1}{3}\right) = -\frac{\pi i}{3}.$$

Cauchy's Residue Theorem

Example 26

Using complex methods, evaluate

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\, dx$$

Let

$$f(z) = \frac{e^{iz}}{z^2+1}.$$

Define the contours Γ_R and Γ_x to be the upper semicircle centred at zero and radius $R > 1$, and $[-R, R]$ respectively, and let Γ be the join of these traversed anticlockwise. Since we have a simple pole at $z = i$ inside our contour, by Cauchy's residue theorem,

$$\int_\Gamma f(z)\, dz = 2\pi i \operatorname{Res}(f, i) = 2\pi i \lim_{z \to i} \frac{e^{iz}}{z+i} = \pi e^{-1}.$$

Cauchy's Residue Theorem (cont.)

Now, using ML lemma on Γ_R, we see

$$\lim_{R\to\infty}\left|\int_{\Gamma_R} f(z)\,dz\right| \leq \lim_{R\to\infty}\frac{1}{R^2-1}\cdot\pi R = 0.$$

So, we conclude that the integral is zero. Then

$$\begin{aligned}
&& \int_{\Gamma} f(z)\,dz &= \pi e^{-1} \\
&\implies & \lim_{R\to\infty}\int_{\Gamma_x}\frac{e^{iz}}{z^2+1}\,dz &= \pi e^{-1} \\
&\implies & \mathrm{Re}\left(\int_{-\infty}^{\infty}\frac{e^{ix}}{x^2+1}\,dx\right) &= \pi e^{-1} \\
&\implies & \int_{-\infty}^{\infty}\frac{\cos x}{x^2+1}\,dx &= \pi e^{-1}.
\end{aligned}$$

Jordan's Lemma

Theorem 19 (Jordan's Lemma)

Let Γ_R be the upper half of the circle of radius R about 0. Suppose that f is continuous on $\{z \in \mathbb{C} : \operatorname{Im}(z) \geq 0, |z| \geq S\}$, where $S > 0$, and $|f(z)| \leq M_R$ for all $z \in \Gamma_R$ where $\lim_{R\to\infty} M_R = 0$. Then

$$\lim_{R\to\infty} \left| \int_{\Gamma_R} e^{i\xi z} f(z)\, dz \right| = 0,$$

for any $\xi > 0$.

This is particularly useful when combined with Cauchy's residue theorem or Cauchy-Goursat theorem for functions involving e^{iz} or similar. In some cases, ML Lemma isn't strong enough, and Jordan's Lemma is required.

Jordan's Lemma

Example 27

Show that

$$\int_{\infty}^{\infty} \frac{x \sin 2x}{x^2 + 4}\, dx = \pi e^{-4}.$$

We first set up the contour $\Gamma = \Gamma_R + \Gamma_x$, where $\Gamma_x = [-R, R]$ and Γ_R is the supper semicircle of radius R around 0, both traversed with standard orientation. Let

$$f(z) = \frac{z}{z^2 + 4}.$$

This functions has simple poles at $\pm 2i$, of which only $z = 2i$ lies within our contour (for $R > 2$).

Jordan's Lemma (cont.)

Clearly, f is continuous on the set $\{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0, |z| \geq 3\}$, and on Γ_R,

$$|f(z)| = \frac{|z|}{|z^2+4|} \leq \frac{R}{R^2-4} = M_R,$$

for $R > 2$. Then

$$\lim_{R\to\infty} M_R = 0,$$

so we can apply Jordan's lemma, and conclude that

$$\lim_{R\to\infty} \int_{\Gamma_R} \frac{ze^{2iz}}{z^2+4}\, dz = 0.$$

Jordan's Lemma (cont.)

Now, we find the residues in our contour. There's only one, so

$$\operatorname{Res}(f(z)e^{2iz};\, z = 2i) = \lim_{z\to 2i} \frac{ze^{2iz}}{z+2i} = \frac{1}{2}e^{-4}.$$

Thus, by Cauchy's residue theorem,

$$\int_{\Gamma} \frac{ze^{2iz}}{z^2+4}\, dz = \pi i e^{-4}.$$

Taking limits, we have

$$\lim_{R\to\infty} \int_{\Gamma} \frac{ze^{2iz}}{z^2+4}\, dz = \lim_{R\to\infty} \int_{\Gamma_x} \frac{ze^{2iz}}{z^2+4}\, dz = \pi i e^{-4}.$$

Finally, taking imaginary components,

$$\int_{-\infty}^{\infty} \frac{x\sin 2x}{x^2+4} = \operatorname{Im}\left(\lim_{R\to\infty} \int_{\Gamma_x} \frac{ze^{2iz}}{z^2+4}\, dz\right) = \pi e^{-4}.$$

Winding Numbers

Definition 18

The **winding number** of a closed curve γ around w is

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z-w}\,dz.$$

Intuitively, this becomes the change in logarithm of $z-w$, which is how much the angle changes. Effectively, we count how many times we "wind" around w.

Winding Numbers

Example 28

Find the winding number of $\gamma(t) = (t^2+1)e^{it}$ for $t \in [-\pi, \pi]$ around 0.

First, we find $\gamma'(t) = 2te^{it} + i(t^2+1)e^{it}$. Then the winding number is

$$\begin{aligned}
\frac{1}{2\pi i}\int_\gamma \frac{1}{z}\,dz &= \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{1}{(t^2+1)e^{it}}(2te^{it} + i(t^2+1)e^{it})\,dt \\
&= \frac{1}{2\pi i}\int_{-\pi}^{\pi} i + \frac{2t}{t^2+1}\,dt \\
&= \frac{1}{2\pi i}[it]_{-\pi}^{\pi} \\
&= 1.
\end{aligned}$$

Meromorphisms

Definition 19

A function f is **meromorphic** on an open set Ω if it is holomorphic on $\Omega \setminus \Delta$, where Δ is a discrete set, and the singularities at each point of Δ are poles.

Cauchy's Argument Principle

Theorem 20 (Cauchy's Argument Principle)

Suppose that f is meromorphic on a simply connected domain Ω, and has zeroes of order m_i at a_i, and poles of order n_i at b_i. Further, suppose that Γ is a simple closed contour in Ω that doesn't pass through any zero or pole of f. Then

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}\,dz = \sum_{a_i\in \mathrm{Int}(\Gamma)} m_i - \sum_{b_i\in \mathrm{Int}(\Gamma)} n_i.$$

Theorem 21

Let $f \in H(\Omega)$, where Ω is a domain, and suppose $\gamma : [a, b] \to \Omega$ be a simple closed contour such that $f(\gamma(t)) \neq 0$ for all $t \in \Omega$. Then the number of zeroes of f in $\mathrm{Int}(\gamma)$ is the same as the number of times $f \circ \gamma$ winds around 0, counting multiplicities.

Counting Zeroes

Example 29 (MATH2621 2018 Q119)

Find the number of zeroes of $f(z) = z^5 + z^4 + 2z^3 - 8z - 1$ with positive real part.

We begin by taking γ to be the right semicircle around 0 of radius R, where R is large. and break it up into two parts; the arc γ_1, and the imaginary axis γ_2.
We parameterise γ_1 as $\gamma_1(t) = Re^{it}$ where t varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.
For sufficiently large R, $f(Re^{it})$ behaves like $R^5 e^{5it}$. This winds around 0 by an angle of 5π.

Counting Zeroes (cont.)

We then parameterise γ_2 as $\gamma_2(t) = it$, varying t from R to $-R$. Since the endpoints approach the imaginary axis for $t \to \pm\infty$, we can determine how the curve winds around 0 by finding the intercepts with the real axis of $f(it)$. So,

$$f(it) = it^5 + t^4 - 2it^3 - 8it - 1.$$

For real intercepts, we then require

$$t^5 - 2t^3 - 8t = t(t^4 - 2t^2 - 8) = t(t^2 - 4)(t^2 + 2) = 0.$$

Our intercepts are thus at $t = 0, \pm 2$, for which:

$$f(0) = -1, \qquad\qquad f(\pm 2) = 15.$$

So the curve winds around 0 by an angle of -3π, which gives us a total change of 2π. Thus, there is one zero of f with positive real part.

Rouché's Theorem

Theorem 22 (Rouché's Theorem)

Suppose that $\gamma : [a, b] \to \Omega$ is a closed curve in a simply connected domain Ω. Let f, g be holomorphic on Ω, and that $|f(z)| < |g(z)|$ on γ. Then the number of zeroes of $f + g$ in γ is the same as the number of zeroes of g inside γ.

Rouché's theorem can simplify the process of counting roots significantly, by breaking more complicated functions into parts for which the roots are obvious.

Rouché's Theorem

Example 30

Show that all roots of $p(z) = 7z^5 - 2z^4 - z + 1$ lie within the unit circle.

Let $f(z) = 2z^4 + z$, and $g(z) = 7z^5 + 1$. Then along the unit circle $|z| = 1$, we have

$$|f(z)| \leq 2|z|^4 + |z| = 3 < 6 = 7|z|^5 - 1 \leq |7z^5 + 1| = |g(z)|.$$

Thus, by Rouché's theorem, the number of zeroes of p within the unit circle is the same as of g. Now, roots of g satisfy

$$7z^5 + 1 = 0 \implies z^5 = -\frac{1}{7} \implies |z| = \frac{1}{\sqrt[5]{7}} < 1.$$

So, p has five roots inside the unit circle. However, p has exactly five roots, so all of them lie within the unit circle.