MATH1231/1241 MathSoc Calculus Revision Session 2019 T1 Solutions

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Example 1: 1231 2015 Q1.v

Prove that

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_1 - 2x_2 + 4x_3 = 0 \right\}$$

is a subspace of $\mathbb{R}^3$.

Clearly, $S \subseteq \mathbb{R}^3$ where $\mathbb{R}^3$ is a known vector space. Since $0 - 2(0) + 4(0) = 0$, $\mathbf{0} \in S$. So S contains a zero element.

Now suppose that $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in S$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in S$. Then

$$x_1 - 2x_2 + 4x_3 = 0, \tag{1}$$
$$y_1 - 2y_2 + 4y_3 = 0. \tag{2}$$

$(1)+(2)$ gives us $(x_1+y_1)-2(x_2+y_2)+4(x_3+y_3) = 0$, i.e. $\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{pmatrix} \in S$. Hence $(\mathbf{x}+\mathbf{y}) \in S$ and so S is closed under vector addition.

If $\lambda \in \mathbb{R}$ and $\mathbf{x} \in S$ then $\lambda \times (1)$ gives us $\lambda x_1 - 2\lambda x_2 + 4\lambda x_3 = 0$, i.e. $\begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{pmatrix} \in S$. Hence $\lambda\mathbf{x} \in S$ so S is closed under scalar multiplication.

Since S is a subset of $\mathbb{R}^3$ and contains a zero element, is closed under vector addition, and is closed under scalar multiplication, then by the Subspace Theorem S is a subspace of $\mathbb{R}^3$.

Example 2: 1231 2013 Q1.i

Let

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_1^3 + x_2^3 + x_3^3 = 0 \right\}.$$

a) Prove that S is closed under scalar multiplication.

*b) Show that S is **not** a subspace of $\mathbb{R}^3$.*

Suppose that $\mathbf{x} \in S$ and $\lambda \in \mathbb{R}$. Then we have

$$x_1^3 + x_2^3 + x_3^3 = 0.$$

Multiplying by λ^3,

$$(\lambda x_1)^3 + (\lambda x_2)^3 + (\lambda x_3)^3 = 0.$$

Hence $\lambda\mathbf{x} \in S$, i.e. S is closed under scalar multiplication.

Note that $\mathbf{0}$ is an element of S, so to prove that S is not a subspace we will show that S is not closed under vector addition. Take $\mathbf{x} = (1, -1, 0)^T$ and $\mathbf{y} = (-2, 0, 2)^T$. Clearly $\mathbf{x}, \mathbf{y} \in S$, but

$$\mathbf{x} + \mathbf{y} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \notin S.$$

Hence S is not closed under vector addition. By the Subspace Theorem, S is not a subspace of $\mathbb{R}^3$.

Example 3: 1231 2015 Q1.vi

Consider the vectors in $\mathbb{R}^3$,

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix}.$$

Prove that $\boldsymbol{b} \in span(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)$.

We want examine the nature of solutions $(x_1, x_2, x_3)^T$ to

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}.$$

$\mathbf{b} \in \text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ if there is at least one solution. Notice that our equation can be written in the form $A\mathbf{x} = \mathbf{b}$:

$$\begin{pmatrix} 1 & 1 & 2 \\ -1 & 2 & -3 \\ 2 & 5 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix}.$$

Applying Gaussian elimination, we have

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ -1 & 2 & -3 & 6 \\ 2 & 5 & 3 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 3 & -1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Clearly there are infinitely many solutions, so $\mathbf{b} \in \text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

Example 4

Let $\mathbb{P}_2$ be the vector space of all real polynomials of degree at most 2. Find three polynomials f_1, f_2, f_3 in $\mathbb{P}_2$ such that $f_i(0) = 1$ for $i = 1, 2, 3$ and $\{f_1, f_2, f_3\}$ is linearly independent.

We want three polynomials such that each has a different span. The easiest way to do this is to consider the set of functions

$$\begin{aligned} f_1 &= a_1, \\ f_2 &= a_2 + b_2 x, \\ f_3 &= a_3 + b_3 x + c_3 x^2. \end{aligned}$$

Since $f_i(0) = 1$ then $a_i = 1$. The other coefficients are arbitrary constants, so set all other constants to 1:

$$\begin{aligned} f_1 &= 1, \\ f_2 &= 1 + x, \\ f_3 &= 1 + x + x^2. \end{aligned}$$

Example 5: 1241 2016 Q3.iii

The field $\mathbb{F} = GF(4)$ has elements $\{0, 1, \alpha, \beta\}$ with addition and multiplication defined by the following tables. For the vectors

$+$	0	1	α	β
0	0	1	α	β
1	1	0	β	α
α	α	β	0	1
β	β	α	1	0

$\times$	0	1	α	β
0	0	0	0	0
1	0	1	α	β
α	0	α	β	1
β	0	β	1	α

$$\boldsymbol{b}_1 = \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix}, \boldsymbol{b}_2 = \begin{pmatrix} \beta \\ 1 \\ 1 \end{pmatrix}, \boldsymbol{b}_3 = \begin{pmatrix} 1 \\ 0 \\ \alpha \end{pmatrix},$$

a) show that $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ is a basis for $\mathbb{F}^3$;
b) explain why $\{\boldsymbol{b}_1, \boldsymbol{b}_1 + \boldsymbol{b}_2, \boldsymbol{b}_2 + \boldsymbol{b}_3, \boldsymbol{b}_3\}$ is a spanning set but not a basis for $\mathbb{F}^3$.

First we prove two important results. Suppose $\mathbf{x} \in \mathbb{F}^3$. Since $a + a = 0\ \forall a \in \mathbb{F}$, then

$$\mathbf{x} + \mathbf{x} = \mathbf{0} \quad \forall \mathbf{x} \in \mathbb{F}^3. \tag{*}$$

Also, since $a + 0 = a\ \forall a \in \mathbb{F}$, then

$$\mathbf{x} + \mathbf{0} = \mathbf{x} \quad \forall \mathbf{x} \in \mathbb{F}^3. \tag{**}$$

Now, note that $\dim \mathbb{F}^3 = 3$. If we can show that $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a linearly independent set then we can apply the Dimension Theorem. For $x, y, z \in \mathbb{F}$, if

$$x\mathbf{b}_1 + y\mathbf{b}_2 + z\mathbf{b}_3 = \mathbf{0}$$

then we can represent this in the form $A\mathbf{x} = \mathbf{b}$:

$$\begin{pmatrix} 1 & \beta & 1 \\ \alpha & 1 & 0 \\ \beta & 1 & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Carefully applying Gaussian elimination:

$$\left(\begin{array}{ccc|c} 1 & \beta & 1 & 0 \\ \alpha & 1 & 0 & 0 \\ \beta & 1 & \alpha & 0 \end{array}\right) \xrightarrow{R1=\alpha R1} \left(\begin{array}{ccc|c} \alpha & 1 & \alpha & 0 \\ \alpha & 1 & 0 & 0 \\ \beta & 1 & \alpha & 0 \end{array}\right) \xrightarrow{R2=R2+R1} \left(\begin{array}{ccc|c} \alpha & 1 & \alpha & 0 \\ 0 & 0 & \alpha & 0 \\ \beta & 1 & \alpha & 0 \end{array}\right)$$

$$\xrightarrow{R3=\beta R3} \left(\begin{array}{ccc|c} \alpha & 1 & \alpha & 0 \\ 0 & 0 & \alpha & 0 \\ \alpha & \beta & 1 & 0 \end{array}\right) \xrightarrow{R2\leftrightarrow R3} \left(\begin{array}{ccc|c} \alpha & 1 & \alpha & 0 \\ \alpha & \beta & 1 & 0 \\ 0 & 0 & \alpha & 0 \end{array}\right)$$

$$\xrightarrow{R2=R2+R1} \left(\begin{array}{ccc|c} \alpha & 1 & \alpha & 0 \\ 0 & \alpha & \beta & 0 \\ 0 & 0 & \alpha & 0 \end{array}\right).$$

Clearly, our solution $(x, y, z)^T = (0, 0, 0)^T$. So the only solution is the trivial solution, hence $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a linearly independent set. Therefore $\dim(\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}) = \dim(\mathbb{F}^3) = 3$, and so by the Dimension Theorem $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis for $\mathbb{F}^3$.

Now consider the second set of vectors in $\mathbb{F}^3$.

$$\begin{aligned} (\mathbf{b}_1 + \mathbf{b}_2) + (\mathbf{b}_2 + \mathbf{b}_3) + (\mathbf{b}_3) &= \mathbf{b}_1 + (\mathbf{b}_2 + \mathbf{b}_2) + (\mathbf{b}_3 + \mathbf{b}_3) && \text{(associative law)} \\ &= \mathbf{b}_1 + \mathbf{0} + \mathbf{0} && \text{(Using (*))} \\ &= \mathbf{b}_1. && \text{(Using (**))} \end{aligned}$$

Since we have written $\mathbf{b}_1$ as a linear combination of $\mathbf{b}_1 + \mathbf{b}_2$, $\mathbf{b}_2 + \mathbf{b}_3$ and $\mathbf{b}_3$, then

$$\{\mathbf{b}_1, \mathbf{b}_1 + \mathbf{b}_2, \mathbf{b}_2 + \mathbf{b}_3, \mathbf{b}_3\}$$

is a linearly dependent set. Hence the set cannot be a basis for $\mathbb{F}^3$. However since

$$\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \in \operatorname{span}(\{\mathbf{b}_1, \mathbf{b}_1 + \mathbf{b}_2, \mathbf{b}_2 + \mathbf{b}_3, \mathbf{b}_3\})$$

and $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a spanning set for $\mathbb{F}^3$, then $\{\mathbf{b}_1, \mathbf{b}_1 + \mathbf{b}_2, \mathbf{b}_2 + \mathbf{b}_3, \mathbf{b}_3\}$ is a spanning set for $\mathbb{F}^3$.

Example 6: 1241 2016 Q3.iii

Consider the field $\mathbb{F} = GF(4)$, as defined in the previous example. Let $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3$ be the vectors from the previous example. Set

$$\boldsymbol{v} = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}.$$

c) Find the coordinate vector of $\boldsymbol{v}$ with respect to the ordered basis $B = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$.

The coordinate vector $\mathbf{x} = (x, y, z)^T$ of $\mathbf{v}$ will satisfy the equation

$$x\mathbf{b}_1 + y\mathbf{b}_2 + z\mathbf{b}_3 = \mathbf{z}.$$

Writing this in the form $A\mathbf{x} = \mathbf{b}$, we have

$$\begin{pmatrix} 1 & \beta & 1 \\ \alpha & 1 & 0 \\ \beta & 1 & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}.$$

Carefully applying Gaussian elimination:

$$\left(\begin{array}{ccc|c} 1 & \beta & 1 & \alpha \\ \alpha & 1 & 0 & 0 \\ \beta & 1 & \alpha & 0 \end{array}\right) \xrightarrow{R1=\alpha R1} \left(\begin{array}{ccc|c} \alpha & 1 & \alpha & \beta \\ \alpha & 1 & 0 & 0 \\ \beta & 1 & \alpha & 0 \end{array}\right) \xrightarrow{R2=R2+R1} \left(\begin{array}{ccc|c} \alpha & 1 & \alpha & \beta \\ 0 & 0 & \alpha & \beta \\ \beta & 1 & \alpha & 0 \end{array}\right)$$

$$\xrightarrow{R3=\beta R3} \left(\begin{array}{ccc|c} \alpha & 1 & \alpha & \beta \\ 0 & 0 & \alpha & \beta \\ \alpha & \beta & 1 & 0 \end{array}\right) \xrightarrow{R2 \leftrightarrow R3} \left(\begin{array}{ccc|c} \alpha & 1 & \alpha & \beta \\ \alpha & \beta & 1 & 0 \\ 0 & 0 & \alpha & \beta \end{array}\right)$$

$$\xrightarrow{R2=R2+R1} \left(\begin{array}{ccc|c} \alpha & 1 & \alpha & \beta \\ 0 & \alpha & \beta & \beta \\ 0 & 0 & \alpha & \beta \end{array}\right).$$

From $R3$ we have $z = \alpha$ since $\alpha \times \alpha = \beta$. In $R2$ we have

$$\begin{aligned} \alpha y + \beta \times \alpha &= \beta \\ \alpha y + 1 &= \beta && (\beta \times \alpha = 1) \\ \alpha y + 1 + 1 &= \beta + 1 && (\text{Adding 1 to both sides}) \\ \alpha y &= \alpha && (1 + 1 = 0 \text{ and } \beta + 1 = \alpha) \\ y &= 1. && (\text{Since } \alpha \times 1 = \alpha) \end{aligned}$$

In $R1$ we have

$$
\begin{aligned}
\alpha x + 1 \times 1 + \alpha \times \alpha &= \beta \\
\alpha x + 1 + \beta &= \beta && (1 \times 1 = 1 \text{ and } \alpha \times \alpha = \beta) \\
\alpha x + (1+\beta) + (1+\beta) &= \beta + 1 + \beta && (\text{Adding } (1+\beta) \text{ to both sides}) \\
\alpha x &= \alpha + \beta && ((1+\beta)+(1+\beta) = 0 \text{ and } \beta + 1 = \alpha) \\
\alpha x &= 1 && (\alpha + \beta = 1) \\
x &= \beta && (\text{Since } \alpha \times \beta = 1)
\end{aligned}
$$

Hence the coordinate vector of $\boldsymbol{v}$ with respect to the basis B is

$$\mathbf{x} = \begin{pmatrix} \beta \\ 1 \\ \alpha \end{pmatrix}.$$

Example 7: 1241 2014 S2 Q3.i

Prove that the function $T : \mathbb{P}(\mathbb{R}) \to \mathbb{R}^2$ defined by

$$T(p) = \begin{pmatrix} p(0) \\ p(1) \end{pmatrix}, \textit{ for all polynomials } p \in \mathbb{P}(\mathbb{R}),$$

is a linear transformation.

For the map T to be linear, we need to show that T preserves addition and scalar multiplication. First consider addition. For any $p, q \in \mathbb{P}(\mathbb{R})$,

$$
\begin{aligned}
T(p+q) &= \begin{pmatrix} (p+q)(0) \\ (p+q)(1) \end{pmatrix} = \begin{pmatrix} p(0)+q(0) \\ p(1)+q(1) \end{pmatrix} = \begin{pmatrix} p(0) \\ p(1) \end{pmatrix} + \begin{pmatrix} q(0) \\ q(1) \end{pmatrix} \\
&= T(p) + T(q).
\end{aligned}
$$

So T preserves addition. Now consider scalar multiplication. For any $p \in \mathbb{P}(\mathbb{R})$ and $\lambda \in \mathbb{R}$,

$$
\begin{aligned}
T(\lambda p) &= \begin{pmatrix} (\lambda p)(0) \\ (\lambda p)(1) \end{pmatrix} = \begin{pmatrix} \lambda p(0) \\ \lambda p(1) \end{pmatrix} = \lambda \begin{pmatrix} p(0) \\ p(1) \end{pmatrix} \\
&= \lambda T(p).
\end{aligned}
$$

Hence T preserves scalar multiplication. Therefore since T preserves addition and scalar multiplication, then T is linear.

Example 8: 1241 2016 Q3.ii

Let V and W be vector spaces, let $T : V \to W$ be a linear transformation, and let $\boldsymbol{v}_1, \boldsymbol{v}_2 ..., \boldsymbol{v}_m$ be vectors in V.
a) Prove that if $T(\boldsymbol{v}_1), T(\boldsymbol{v}_2), ..., T(\boldsymbol{v}_m)$ are linearly independent, then $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_m$ are linearly independent.
b) Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_m$ are linearly independent. Is $T(\boldsymbol{v}_1), T(\boldsymbol{v}_2), ..., T(\boldsymbol{v}_m)$ linearly independent?

Suppose that $T(\mathbf{v}_1), T(\mathbf{v}_2), ..., T(\mathbf{v}_m)$ are linearly independent, and assume $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m$ are linearly dependent. Then for some $i \in \{1, 2, ..., m\}$ and constants λ_j, we have

$$\mathbf{v}_i = \sum_{j \neq i} \lambda_j \mathbf{v}_j.$$

So then

$$\begin{aligned} T(\mathbf{v}_i) &= T\left(\sum_{j \neq i} \lambda_j \mathbf{v}_j\right) \\ &= \sum_{j \neq i} T(\lambda j \mathbf{v}_j) && \text{(Since } T \text{ preserves addition)} \\ &= \sum_{j \neq i} \lambda_j T(\mathbf{v}_j). && \text{(Since } T \text{ preserves scalar multiplication)} \end{aligned}$$

Hence we have a contradiction, since $T(\mathbf{v}_1), T(\mathbf{v}_2), ..., T(\mathbf{v}_m)$ are linearly independent. Hence our assumption is incorrect, i.e. $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m$ are not linearly dependent. So we have proven, by contradiction, that $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m$ are linearly independent.

However, linear independence of $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m$ does not imply linear independence of $T(\mathbf{v}_1), T(\mathbf{v}_2), ..., T(\mathbf{v}_m)$. Consider, for example, the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(\mathbf{x}) = \mathbf{0} \quad \forall \mathbf{x} \in \mathbb{R}^2.$$

The standard basis vectors $\mathbf{e}_1, \mathbf{e}_2$ are linearly independent, however $xT(\mathbf{e}_1) + yT(\mathbf{e}_2) = 0$ for any choice of $x, y \in \mathbb{R}$. So $T(\mathbf{e}_1), T(\mathbf{e}_2)$ are not linearly independent. Interestingly enough, part b would be true if T were injective.

Example 9: 1231 2013 Q2,iv

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map which rotates a vector $\boldsymbol{x}$ about the origin through $\dfrac{\pi}{6}$ anti-clockwise and doubles its length.

a) Show that $T(\boldsymbol{e}_1) = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$, *where* $\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

b) Find the matrix A *such that* $T(\boldsymbol{x}) = A\boldsymbol{x}$*. for all* $\boldsymbol{x} \in \mathbb{R}^2$.

Since T rotates a vector $(x, y)^T$ anticlockwise by $\dfrac{\pi}{6}$, we know that for $x > 0$ and $y \geq 0$,

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L\cos\left(\dfrac{\pi}{6} + \tan^{-1}\left(\dfrac{y}{x}\right)\right) \\ \\ L\sin\left(\dfrac{\pi}{6} + \tan^{-1}\left(\dfrac{y}{x}\right)\right) \end{pmatrix}.$$

Since T also doubles the length of a vector $(x, y)^T$, then $L = 2\sqrt{x^2 + y^2}$. Hence

$$T\begin{pmatrix} x \\ y \end{pmatrix} = 2\sqrt{x^2 + y^2}\begin{pmatrix} \cos\left(\dfrac{\pi}{6} + \tan^{-1}\left(\dfrac{y}{x}\right)\right) \\ \\ \sin\left(\dfrac{\pi}{6} + \tan^{-1}\left(\dfrac{y}{x}\right)\right) \end{pmatrix}.$$

So

$$\begin{aligned} T\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= 2\sqrt{1+0}\begin{pmatrix} \cos\left(\dfrac{\pi}{6} + \tan^{-1} 0\right) \\ \\ \sin\left(\dfrac{\pi}{6} + \tan^{-1} 0\right) \end{pmatrix} = 2\begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}. \end{aligned}$$

Also,

$$\begin{aligned} T\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= 2\sqrt{0+1}\begin{pmatrix} \cos\left(\dfrac{\pi}{6} + \dfrac{\pi}{2}\right) \\ \\ \sin\left(\dfrac{\pi}{6} + \dfrac{\pi}{2}\right) \end{pmatrix} = 2\begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}. \end{aligned}$$

Using the Matrix Representation Theorem, $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}.$$

Example 10: 1231 2018 Q1.iv

Consider the matrix $M = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$.

a) Find a basis for $\ker(M)$.

b) Find a basis for $im(M^T)$.

c) Give a geometric description of $\ker(M)$ *and* $im(M)$ *as subspaces of* $\mathbb{R}^2$.

If $\mathbf{x} \in \ker(M)$ then $M\mathbf{x} = \mathbf{0}$. Hence

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} x+y \\ 2x+2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So $x + y = 0$, i.e. $y = -x$. So

$$\mathbf{x} = x \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Therefore

$$\ker(M) = \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\},$$

so a basis for $\ker(M)$ is

$$\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

Now, consider $\mathbf{y} \in \text{im}(M^T)$. Then

$$\mathbf{y} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ x+2y \end{pmatrix} = (x+2y) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore

$$\text{im}(M^T) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\},$$

so a basis for $\text{im}(M^T)$ is

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

The kernel of M is the line in $\mathbb{R}^2$, in the direction $(1, -1)^T$. The image of M is the line in $\mathbb{R}^2$, in the direction $(1, 2)^T$.

Example 11: 1241 2015 Q3.ii

Consider the mapping $T : \mathbb{P}_2 \to \mathbb{P}_3$ defined by

$$T(p)(x) = (x^2 + 1)p'(x) - 2xp(x).$$

Assuming T is linear, find the rank and nullity of T.

Let $p(x) = ax^2 + bx + c$. Then $p'(x) = 2ax + b$, and so

$$\begin{aligned} T(p)(x) &= (x^2 + 1)(2ax + b) - 2x(ax^2 + bx + c) \\ &= -bx^2 + 2(a - c)x + b. \end{aligned}$$

Hence

$$\begin{aligned} T(p) = \begin{pmatrix} -b \\ 2a - 2c \\ b \end{pmatrix} &= a\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + b\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c\begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} \\ &= 2(a - c)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

So if $q \in \operatorname{im}(T)$ then

$$q = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}.$$

So $\operatorname{rank}(T) = 2$. Since $\dim(\mathbb{P}_2) = 3$ (standard basis is $\{1, x, x^2\}$), then by the Rank-Nullity Theorem, $\operatorname{nullity}(T) = 1$.