Theorem. *Let G be a group of even order with identity e. Then,*

(i) there is an element $g \in G$ distinct from e such that $g^2 = e$, and

(ii) there are an odd number of such elements.

Proof. Since each $g \in G$ is invertible by definition of a group, we can take $g = g^{-1}$ to be an equivalent condition to $g^2 = e$. To this end, let

$$G_1 = \{g \in G \mid g = g^{-1}\} \quad \text{and} \quad G_2 = \{g \in G \mid g \neq g^{-1}\}$$

so that $G = G_1 \cup G_2$. It then suffices to show that $|G_1| \geq 2$ for part (i), and that $|G_1 \setminus \{e\}|$ is odd for part (ii).

(i) We have that $e \in G_1$, so $|G_1| \geq 1$. We also have that $|G_2|$ is even, since each $g \in G_2$ is distinct from its inverse g^{-1}, so it is possible to partition G_2 into pairs (g, g^{-1}). But, since $|G|$ is even, it must be that $|G_1|$ is even. Hence, $|G_1| \geq 2$, meaning there is some $g \in G_1$ distinct from e.

(ii) The results of the previous part imply that we can write $|G_1| = 2k$ for some integer k, so it follows immediately that $|G_1 \setminus \{e\}| = 2k - 1$. Hence, there are an odd number of elements in G_1 that are distinct from e.

This completes the proof. $\square$