MATH1131/1141 Calculus Test 1 2008 S1 v5a

March 19, 2017

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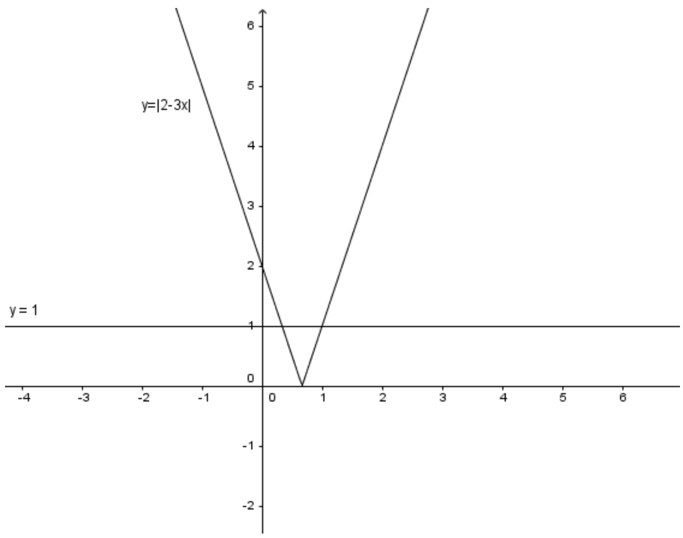
1. Graphing $y = |2 - 3x|$ and $y = 1$ on the same axes gives:



We then solve:

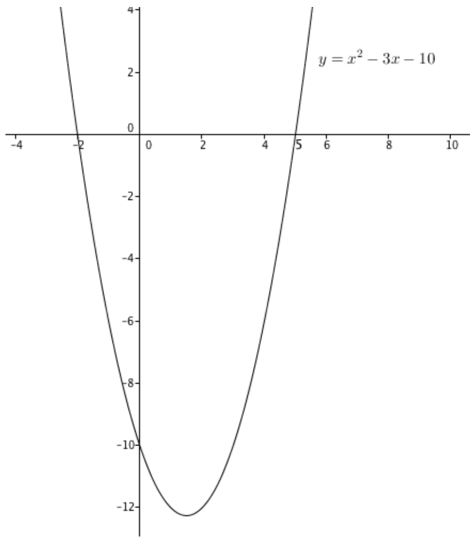
$$\begin{aligned} |2-3x| \quad &= \quad 1 \\ 2-3x=1 \quad &\textbf{or} \quad 2-3x=-1 \\ x=\frac{1}{3} \quad &\textbf{or} \quad x=1 \end{aligned}$$

Using this and by examining the graph, we find the solution is $\frac{1}{3} \leq x \leq 1$.
Alternatively, solving the inequality directly,

$$\begin{aligned} &|2-3x| \leq 1 \\ \iff &-1 \leq 2-3x \leq 1 \\ \iff &-3 \leq -3x \leq -1 \\ \iff &\frac{1}{3} \leq x \leq 1. \end{aligned}$$

2. Maximal domain refers to the largest possible set of numbers that we could plug in to the function. Range refers to the set of values that the function puts out by plugging in values from the domain.
   Maximal domain: Note that we require $3-x>0$, since the denominator must be non-zero, and terms inside the square root must be positive. Thus, the maximal domain of f is $\{x \in \mathbb{R} : x < 3\}$, which in interval notation is $(-\infty, 3)$.
   Range: The range of f is $\{y \in \mathbb{R} : y > 0\}$, or in interval notation, $(0, \infty)$. This is the case because $f(x)$ is always strictly positive (as it is 1 over a positive number), and $f(x)$ can be made to take on any positive value (given any $r > 0$, we can make $f(x) = r$ by taking $x = 3 - r^{-2}$).

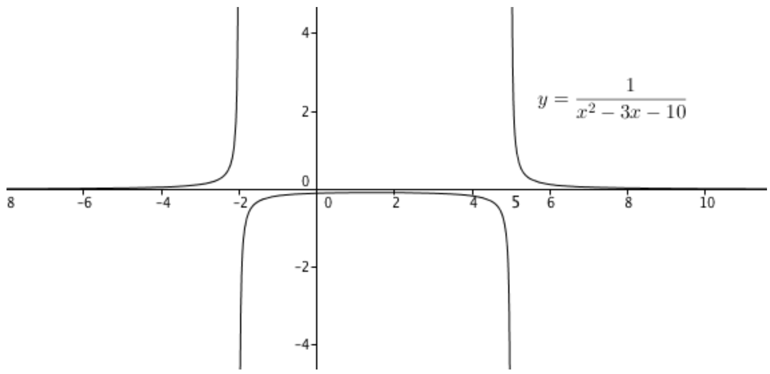
3. To sketch $y = x^2 - 3x - 10$, notice that it is a quadratic and so will take the shape of a parabola. When sketching a parabola, x-intercepts, the y-intercept and vertex must be labelled. The y-intercept is clearly -10 (the constant term of the quadratic). Since $x^2 - 3x - 10 = (x-5)(x+2)$, the x-intercepts are -2 and 5.
   The graph of the quadratic function is as follows:



To then graph $y = \dfrac{1}{x^2 - 3x - 10}$, we rewrite it as $y = \dfrac{1}{(x-5)(x+2)}$. In this form, it is easy to see where the vertical asymptotes are, namely $x = -2$ and $x = 5$ (where the denominator becomes zero). Also, we can check the limits as x approaches the right side and left side of the intercepts, since they are now points where the function is undefined. By doing this, we can determine the behaviour of the graph around these asymptotes.
We can see that

$$f(x) \to \infty \text{ as } x \to (-2)^- \quad \text{and} \quad f(x) \to -\infty \text{ as } x \to (-2)^+,$$
$$f(x) \to -\infty \text{ as } x \to 5^- \quad \text{and} \quad f(x) \to \infty \text{ as } x \to 5^+.$$

Furthermore, to find the horizontal asymptotes, simply take the limit of $f(x)$ as $x \to \infty$ and $x \to -\infty$. Since $x^2 - 3x - 10 \to \infty$ as $x \to \pm\infty$, We have that $\lim_{x \to \pm\infty} \dfrac{1}{x^2 - 3x - 10} = 0$. Hence there is just one horizontal asymptote, namely $y = 0$. This should be enough information to draw your graph. In your sketch, you must draw (in dashed or dotted lines) and label the asymptotes, as well as label the y-intercept and the point the vertex of the original parabola gets moved to.



Note the vertical asymptotes at $x = -2$ and $x = 5$.

4. Consider $f(x) = \dfrac{|x^2 - 9|}{x - 3} = \dfrac{|x-3|\,|x+3|}{x-3}$. We need to see what happens to $f(x)$ just on either side of $x = 3$. We look at two cases:
Case 1: $x > 3$ (the right-hand side),
In this case, we know that $|x - 3| = x - 3$ (as $x - 3 > 0$) and $|x + 3| = x + 3$ (as $x + 3 > 0$). Rewriting the function,

$$f(x) = \frac{(x-3)(x+3)}{x-3} = x + 3.$$

Thus, taking the limit as we approach $x = 3$ from the right,

$$\lim_{x \to 3^+} (x + 3) = 3 + 3 = 6.$$

Hence, $f(x) \to 6$ as $x \to 3^+$.
Case 2: $2.9 < x < 3$ (the left-hand side),
In this case, we know that $|x - 3| = -(x - 3)$ (as $x - 3 < 0$) and $|x + 3| = x + 3$ (as $x + 3 > 0$). Rewriting the function,

$$f(x) = \frac{-(x-3)(x+3)}{x-3} = -(x+3).$$

Thus, taking the limit as we approach $x = 3$ from the left,

$$\lim_{x \to 3^-} -(x + 3) = -(3 + 3) = -6.$$

Hence, $f(x) \to -6$ as $x \to 3^-$.

Since $\lim\limits_{x \to 3^+} f(x) \neq \lim\limits_{x \to 3^-} f(x)$, i.e. the one-sided limits do not agree, we have that $\lim\limits_{x \to 3} f(x)$ does not exist.

5. We have $f(x) = \dfrac{2x + 3x^2 + e^{-x}}{2x^2 + \cos x}$.
To help us evaluate the limit, we divide numerator and denominator by x^2:

$$\begin{aligned}
\lim_{x \to \infty} \frac{2x + 3x^2 + e^{-x}}{2x^2 + \cos x} &= \lim_{x \to \infty} \frac{\frac{2}{x} + 3 + \frac{e^{-x}}{x^2}}{2 + \frac{\cos x}{x^2}} \\
&= \frac{\lim\limits_{x \to \infty} \left(\frac{2}{x} + 3 + \frac{e^{-x}}{x^2}\right)}{\lim\limits_{x \to \infty} \left(2 + \dfrac{\cos x}{x^2}\right)} && \text{(by the Algebra of Limits)} \\
&= \frac{0 + 3 + 0}{2 + 0} && \text{(using standard limits)} \\
&= \frac{3}{2}.
\end{aligned}$$

Hence the answer is that $\lim\limits_{x \to \infty} f(x) = \dfrac{3}{2}$.

MATH1131/1141 Calculus Test 1 2008 S2 v2b

March 19, 2017

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1. Let $f(x) = x^2 + 4$ and $g(x) = \dfrac{1}{\sqrt{x+1}}$. Then:

$$(f \circ g)(x) = f(g(x)) \stackrel{\text{def}}{=} (g(x))^2 + 4 = \left(\frac{1}{\sqrt{x+1}}\right)^2 + 4 = \frac{1}{x+1} + 4 = \frac{4x+5}{x+1}$$

and

$$(g \circ f)(x) = g(f(x)) \stackrel{\text{def}}{=} \frac{1}{\sqrt{f(x)+1}} = \frac{1}{\sqrt{(x^2+4)+1}} = \frac{1}{\sqrt{x^2+5}}.$$

2. Note that for all $x \neq 2$,

$$\begin{aligned} f(x) &= \frac{x^2 - 5x + 6}{2x^2 - 5x + 2} \\ &= \frac{(x-3)(x-2)}{(2x-1)(x-2)} \\ &= \frac{x-3}{2x-1}. \end{aligned}$$

So we get $\lim\limits_{x \to 2} f(x) = \dfrac{2-3}{2 \times 2 - 1} = -\dfrac{1}{3}$.

3. Consider $f(x) = \dfrac{\left|x^2 - 4x + 3\right|}{x-1}$. We need to see what happens to $f(x)$ just on either side of $x = 1$. We consider two cases:
<u>Case 1</u>: $1 < x < 1.01$ (the right hand side),

We have $\dfrac{|x^2-4x+3|}{x-1} = \dfrac{|x-3|\,|x-1|}{x-1}$. In this case, we note that $|x-3| = -(x-3)$ (as $x-3<0$) and $|x-1| = x-1$ (as $x-1>0$). Rewriting the function,

$$\frac{|x-3|\,|x-1|}{x-1} = \frac{-(x-3)(x-1)}{x-1} = -(x-3) = 3-x$$

Now we take the limit as $x \to 1^+$:

$$\begin{aligned}\lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} (3-x) = 3-1 \\ &= 2.\end{aligned}$$

Hence, $f(x) \to 2$ as $x \to 1^+$.

<u>Case 2</u>: $0.99 < x < 1$ (the left hand side),
In this case, we note that $|x-3| = -(x-3)$ (as $x-3<0$) and $|x-1| = -(x-1)$ (as $x-1<0$). Rewriting the function,

$$\frac{|x-3|\,|x-1|}{x-1} = \frac{(-(x-3))\times(-(x-1))}{x-1} = x-3$$

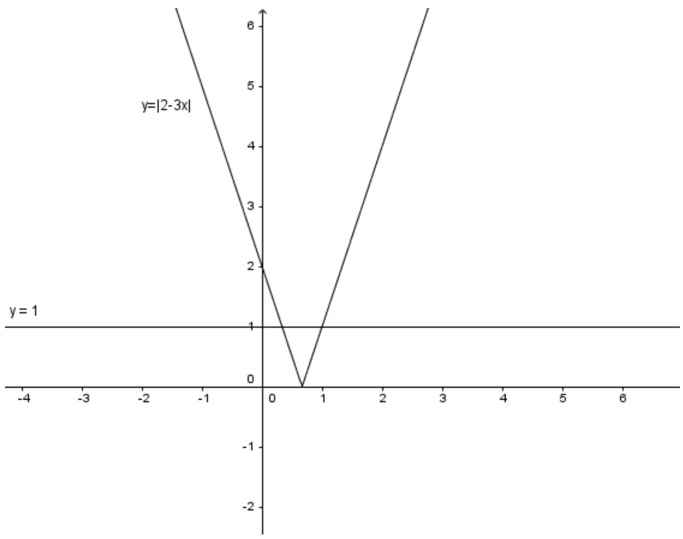
Now we take the limit as $x \to 1^-$:

$$\begin{aligned}\lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} (x-3) = 1-3 \\ &= -2.\end{aligned}$$

Hence, $f(x) \to -2$ as $x \to 1^-$.

Since $\lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x)$, i.e. the one-sided limits do not agree, we have that $\lim_{x\to 1} f(x)$ does not exist.

4. Sketching the graphs of $y = |2 - 3x|$ and $y = 1$ on the same axes gives:



We then solve:

$$\begin{aligned} |2 - 3x| &= 1 \\ 2 - 3x = 1 \quad &\textbf{or} \quad 2 - 3x = -1 \\ x = \frac{1}{3} \quad &\textbf{or} \quad x = 1 \end{aligned}$$

Using this and by examining the graph, we find the solution is $\dfrac{1}{3} \le x \le 1$.

5. For $f(x) = \ln\left(x^2 - 5\right)$:
Maximal domain: We require $x^2 - 5 > 0$ (since the argument of a log should be a positive number), i.e. $x^2 > 5 \iff x < -\sqrt{5}$ or $x > \sqrt{5}$. Thus the maximal domain of f is $\left\{x \in \mathbb{R} : x < -\sqrt{5} \text{ or } x > \sqrt{5}\right\}$.
Range: Notice that for x in the maximal domain, $\left(x^2 - 5\right)$ can take on any positive value. But the set of positive numbers is the maximal domain of the ln function, whose range is in $\mathbb{R}$. Hence, the range of f is $\mathbb{R}$.

MATH1131/1141 Calculus Test 1 2009 S1 v6a

March 19, 2017

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1. **Hints**

   Multiply both sides of the inequality by $(x+1)^2$. (The key thing to remember is that if you multiply (or divide) both sides of the inequality by a negative number, the inequality sign switches. But we avoid this issue, by simply multiplying both sides of the inequality by a positive number, $(x+1)^2$.)
   The solution is $-3 \leq x < -1$. (Remember that $x \neq -1$, because of the fraction in the inequality.)

   Worked Sample Solution

   We note $x \neq -1$. Upon multiplying both sides of the inequality by $(x+1)^2$, we obtain

$$
\begin{aligned}
x+1 &\leq -\frac{1}{2}(x+1)^2 \\
\iff -2(x+1) &\geq (x+1)^2 \\
\iff (x+1)^2 + 2(x+1) &\leq 0 \\
\iff (x+1)((x+1)+2) &\leq 0 \\
\iff (x+1)(x+3) &\leq 0.
\end{aligned}
$$

   Quickly sketching the quadratic, we find that the solution to the inequality is $-3 \leq x < -1$ (*remembering that $x \neq -1$, so we need a **strict inequality** for the right-hand inequality here*).

2. **Hints**

Maximal domain: Similarly to some questions in other tests,[1] we need the term inside the square root to be non-negative. Solving the subsequent inequality gives the maximal domain of f to be $\{x \in \mathbb{R} : x \geq -\ln 2\}$.
Range: Find the range of values $2 - e^{-x}$ takes on in the given domain, and then apply the square root function to this.

Worked Sample Solution

Maximal domain: We need the expression under the square root to be non-negative, i.e.

$$\begin{aligned} 2 - \underbrace{e^{-x}}_{=\frac{1}{e^x}} &\geq 0 \\ \iff \frac{1}{e^x} &\leq 2 \\ \iff e^x &\geq \frac{1}{2} \\ \iff x &\geq \ln\left(\frac{1}{2}\right) = -\ln 2. \end{aligned}$$

So the maximal domain is $\{x \in \mathbb{R} : x \geq -\ln 2\}$, or in interval notation, $[-\ln 2, \infty)$.
Range: For x in the domain, we have that $2 - e^{-x}$ takes on values from 0 to 2, because $2 - e^{-x}$ is a monotonic continuous function that equals 0 when $x = -\ln 2$ and approaches 2 as $x \to \infty$. Therefore (since the square root function is monotonically increasing and continuous), the square root of this function $(2 - e^{-x})$ will take on values from 0 to $\sqrt{2}$, i.e. the range of f is $\left[0, \sqrt{2}\right)$.

3. **Hints**

It can be found that[2]

$$(f \circ g)(x) = \frac{3}{\sqrt{x-2}} + 4 \qquad \text{and} \qquad (g \circ f)(x) = \frac{1}{\sqrt{3x+2}}.$$

Worked Sample Solution

[1] Refer to Question 2 in Test 1 2008 S1 v5a
[2] Refer to Question 1 in Test 1 2008 S2 v2b

We have

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &\stackrel{\text{def}}{=} 3g(x) + 4 \\ &= \frac{3}{\sqrt{x+2}} + 4.\end{aligned}$$

Also,

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &\stackrel{\text{def}}{=} \frac{1}{\sqrt{f(x) - 2}} \\ &= \frac{1}{\sqrt{3x + 4 - 2}} \\ &= \frac{1}{\sqrt{3x+2}}.\end{aligned}$$

4. **Hints**

Similar to questions in previous papers[3], factorise the expressions in the numerator and denominator. You should find that $\lim_{x \to 2} \frac{2x^2 - x - 6}{3x^2 - 2x - 8} = \frac{7}{10}$.

Worked Sample Solution

Observe that $2x^2 - x - 6 = (2x + 3)(x - 2)$ and $3x^2 - 2x - 8 = (3x + 4)(x - 2)$. Hence for all $x \neq 2$, we have $f(x) = \frac{(2x+3)(x-2)}{(3x+4)(x-2)} = \frac{2x+3}{3x+4}$. Thus

$$\begin{aligned}\lim_{x \to 2} f(x) &= \lim_{x \to 2} \frac{2x+3}{3x+4} \\ &= \frac{2 \times 2 + 3}{3 \times 2 + 4} && \text{(direct substitution)} \\ &= \frac{4+3}{6+4} \\ &= \frac{7}{10}.\end{aligned}$$

5. **Hints**

Consider the sign of the absolute value when you're approaching $a = 3$ from either sides separately in cases.
You should find that

$$\lim_{x \to 3^-} f(x) = -9 \qquad \text{and} \qquad \lim_{x \to 3^+} f(x) = 9.$$

[3] Refer to Question 2 in Test 1 2008 S2 v2b

Since the one-sided limits do not agree, the limit of $f(x)$ as $x \to a$ does not exist.

Worked Sample Solution

Observe that $x^2 + 3x - 18 = (x+6)(x-3)$. Thus for all real $x \neq 3$, we have

$$\begin{aligned} f(x) &= \frac{|(x+6)(x-3)|}{x-3} \\ &= |x+6| \frac{|x-3|}{x-3}. \end{aligned}$$

Now, for all x slightly greater than 3 (say $3 < x < 3.01$), we have $|x-3| = x-3$ (as $x-3>0$), so $\dfrac{|x-3|}{x-3} = \dfrac{x-3}{x-3} = 1$ and $|x+6| = x+6$ (as $x+6>0$), so $f(x) = x+6$ for all x just greater than 3. Hence $\lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^+} (x+6) = 9$.

For all x slightly less than 3 (say $2.99 < x < 3$), we have $|x-3| = -(x-3)$ (as $x-3<0$), so $\dfrac{|x-3|}{x-3} = \dfrac{-(x-3)}{x-3} = -1$ and $|x+6| = x+6$ (as $x+6>0$), so $f(x) = -(x+6)$ for all x just less than 3. Hence $\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^+} (-(x+6)) = -9$.

Since the one-sided limits do not agree, the limit of $f(x)$ as $x \to 3$ does not exist.

MATH1131/1141 Calculus Test 1 2009 S1 v8a

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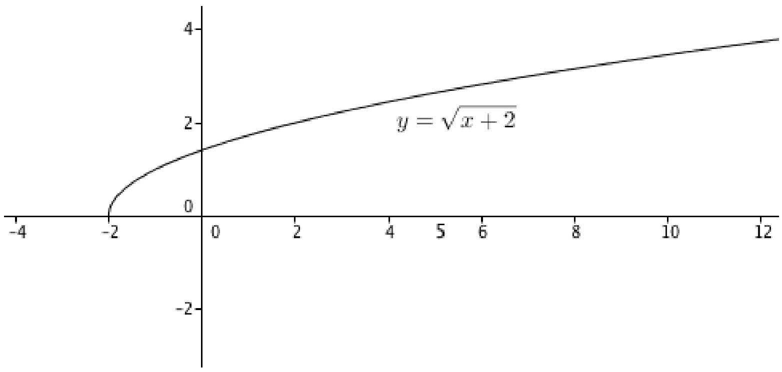
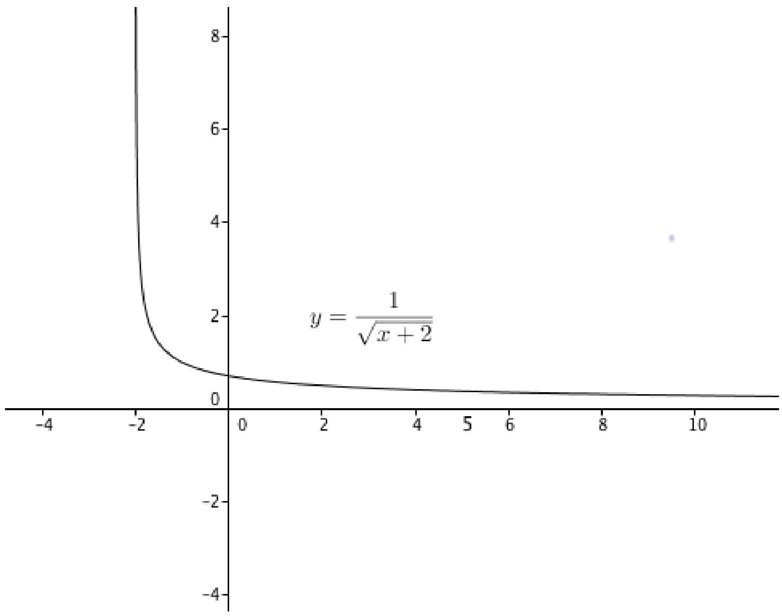
1. To graph $y = f(x) := \sqrt{x+2}$, we simply observe that it is the standard $y = \sqrt{x}$ graph, shifted 2 units to the left. The x-intercept is -2 and the y-intercept is $\sqrt{2}$. The curve is sketched below.

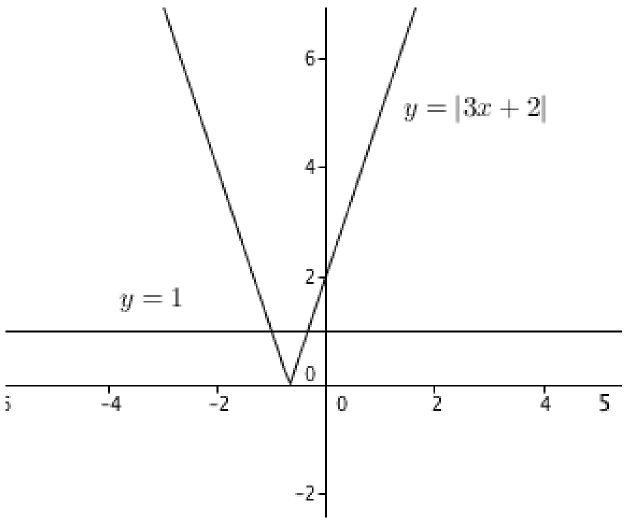


To then graph its reciprocal, $y = \dfrac{1}{\sqrt{x+2}}$, first note that there is a vertical asymptote at $x = -2$. Next, we need to consider the behaviour of $f(x)$ near $x = -2$. But, we can find that $\lim_{x \to (-2)^+} f(x) = \infty$. Furthermore, the horizontal asymptote is $y = 0$, by checking what happens to $f(x)$ when $x \to \infty$ (no need to check $-\infty$ in this case, since x must be greater than -2 for the domain of the function).
We now have enough to make a sketch:



Remember to label y-intercept and asymptotes in your exam! (The y-intercept is $\frac{1}{\sqrt{2}}$, there is a vertical asymptote at $x = -2$, and a horizontal asymptote of $y = 0$.)

2. To solve $|3x + 2| \geq 1$, we sketch $y = |3x + 2|$ and $y = 1$ on the same axes:



Solving the two equations simultaneously to find the points of intersection,

$$\begin{aligned} |3x+2| &= 1 \\ 3x + 2 = 1 \quad &\textbf{or} \quad 3x + 2 = -1 \\ x = -\frac{1}{3} \quad &\textbf{or} \quad x = -1 \end{aligned}$$

Using this and our graph, we find the solution is $x \geq -\frac{1}{3}$ or $x \leq -1$.

Alternatively, solving the inequality directly,

$$\begin{aligned} &|3x+2| \geq 1 \\ \iff\ & 3x+2 \geq 1 \text{ or } 3x+2 \leq -1 \\ \iff\ & x \geq -\frac{1}{3} \text{ or } x \leq -1. \end{aligned}$$

3. For the function $f(x) = \dfrac{1}{\sqrt{9-x^2}}$.
   Maximal domain: For the domain, we need $9 - x^2$ to be strictly positive (since the denominator cannot be 0, and $9 - x^2$ may not be negative, as it is under a square root). So we need $9 - x^2 > 0 \iff x^2 < 9 \iff -3 < x < 3$. Hence the maximal domain is $\{x \in \mathbb{R} : -3 < x < 3\}$, or in interval notation, $(-3, 3)$.

   Range: We notice that the function is minimised when the denominator is maximised, that is, when $x = 0$. Therefore, the minimum value of $f(x)$ is $f(0) = \dfrac{1}{3}$. As we allow x to approach the boundaries of the domain, we see the function increases (denominator gets smaller), and it approaches positive infinity as $x \to 3^+$ or $x \to 3^-$. Thus, the range of f is $\left\{y \in \mathbb{R} : y \geq \dfrac{1}{3}\right\}$, or in interval notation, $\left[\dfrac{1}{3}, \infty\right)$.

4. Let $f(x) = \dfrac{e^{-x} + 3x^2 - 2}{4x^2 + 3x + \sin x}$. Then:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{e^{-x} + 3x^2 - 2}{4x^2 + 3x + \sin x}.$$

   Dividing through by x^2,

$$= \lim_{x\to\infty} \frac{\dfrac{e^{-x}}{x^2} + 3 - \dfrac{2}{x^2}}{4 + \dfrac{3}{x} + \dfrac{\sin x}{x^2}}.$$

   By using the Algebra of Limits,

$$\begin{aligned} &= \frac{\lim\limits_{x\to\infty}\left(\dfrac{e^{-x}}{x^2} + 3 - \dfrac{2}{x^2}\right)}{\lim\limits_{x\to\infty}\left(4 + \dfrac{3}{x} + \dfrac{\sin x}{x^2}\right)} \\ &= \frac{0+3-0}{4+0+0} \quad \text{(using standard limits)} \\ &= \frac{3}{4}. \end{aligned}$$

5. Let $f(x) = \dfrac{x-2}{|x^2-4|} = \dfrac{x-2}{|x-2|\,|x+2|}$ and $a = 2$. We consider two cases:
   Case 1: $x > 2$, In this case, we note that $|x-2| = x-2$ (as $x-2 > 0$) and $|x+2| = x+2$

(as $x+2>0$). Rewriting the function,

$$\frac{x-2}{|x-2|\,|x+2|}=\frac{x-2}{(x-2)\,(x+2)}=\frac{1}{x+2}, \qquad \text{since } x\neq 2.$$

Hence,

$$\begin{aligned}\lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+}\frac{1}{x+2}\\ &= \frac{1}{4}.\end{aligned}$$

<u>Case 2</u>: $1.9<x<2$,
In this case, we note that $|x-2|=-(x-2)$ (as $x-2<0$) and $|x+2|=x+2$ (as $x+2>0$). Rewriting the function,

$$\frac{x-2}{|x-2|\,|x+2|}=\frac{x-2}{-(x-2)\,(x+2)}=-\frac{1}{x+2}, \qquad \text{since } x\neq 2$$

Hence,

$$\begin{aligned}\lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} -\frac{1}{x+2}\\ &= -\frac{1}{4}\end{aligned}$$

Since $\lim_{x\to 2^+} f(x)\neq\lim_{x\to 2^-} f(x)$, i.e. the one-sided limits do not agree, we have that $\lim_{x\to 2} f(x)$ does not exist.

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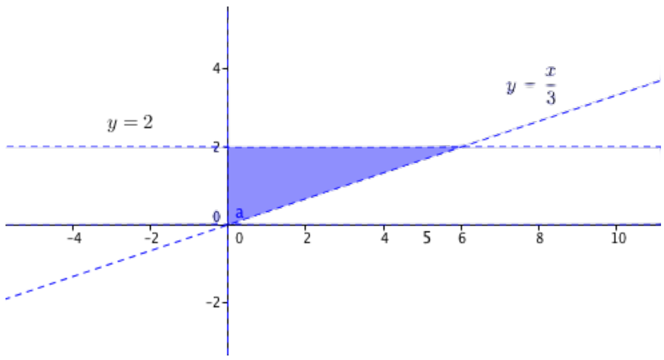
1. **Hints**

   If you're having difficulty, split the inequalities in to $0 < x$, $x < 3y$, $y > 0$ and $y < 2$ and sketch the region that satisfies all these conditions. Also, remember that the boundary should be in dotted lines (since the inequalities are all strict, so the boundary is excluded).



2. **Hints**

   The inequality is equivalent to $|3x + 1| \leq 4$. Similar to questions in previous papers, [1] it can be found that the solution is

$$-\frac{5}{3} \leq x \leq 1.$$

   Worked Sample Solution

[1] Refer to Question 1 in Test 1 2008 S1 v5a

Observe that the inequality is equivalent to $|3x+1| \leq 4$ (multiply the original inequality through by 2). Now,

$$\begin{aligned} |3x+1| \leq 4 &\iff -4 \leq 3x+1 \leq 4 \\ &\iff -5 \leq 3x \leq 3 \\ &\iff -\frac{5}{3} \leq x \leq 1, \end{aligned}$$

so this is the solution to the inequality.

3. **Hints**

This question is extremely similar to a question in a previous test paper[2] The maximal domain of f is $\{x \in \mathbb{R} : x > 1\}$.
The range of f is $\{y \in \mathbb{R} : y > 0\}$.

Worked Sample Solution

Maximal domain: We need $x - 1 > 0$ (non-negative since it's under a square root, and in fact strictly positive since we cannot have a 0 denominator). In other words, $x > 1$. So the maximal domain is $\{x \in \mathbb{R} : x > 1\}$.
Range: As x varies throughout the domain, the denominator takes on all positive real values, so the fraction does this too (since the reciprocal function $t \mapsto \dfrac{1}{t}$ attains all positive real values as t varies throughout the positive reals), i.e. the range is $\{y \in \mathbb{R} : y > 0\}$.

4. **Hints**

Consider the sign of the absolute value when you're approaching $a = 3$ from either sides separately in cases[3].
You should find that

$$\lim_{x \to 3^-} f(x) = 2 \qquad \text{and} \qquad \lim_{x \to 3^+} f(x) = -2.$$

Since the one-sided limits do not agree, the limit of $f(x)$ as $x \to a$ does not exist.

Worked Sample Solution

Observe that $x^2 - 4x + 3 = (x-1)(x-3)$. Thus for all real $x \neq 3$, we have

$$\begin{aligned} f(x) &= \frac{|(x-1)(x-3)|}{3-x} \\ &= -|x-1| \frac{|x-3|}{x-3}. \end{aligned}$$

Now, for all x slightly greater than 3 (say $3 < x < 3.01$), we have $|x-3| = x-3$ (as $x-3 > 0$), so $\dfrac{|x-3|}{x-3} = \dfrac{x-3}{x-3} = 1$ and $|x-1| = x-1$ (as $x-1 > 0$), so $f(x) = -(x-1) =$

[2] Refer to Question 2 in Test 1 2008 S1 v5a
[3] Refer to Question 4 in Test 1 2008 S1 v5a

$1-x$ for all x just greater than 3. Hence $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(1-x) = -2$.

For all x slightly less than 3 (say $2.99 < x < 3$), we have $|x-3| = -(x-3)$ (as $x-3<0$), so $\dfrac{|x-3|}{x-3} = \dfrac{-(x-3)}{x-3} = -1$ and $|x-1| = x-1$ (as $x-1>0$). Hence $f(x) = -(x-1)\times -1 = x-1$ for all x just less than 3. Thus $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+}(x-1) = 2$.

Since the one-sided limits do not agree, the limit of $f(x)$ as $x \to 3$ does not exist.

5. **Hints**

No, there is no maximum value on the interval, since there exists no $x_1 \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ such that $f(x_1) \geq f(x)$ for all $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.
To help illustrate this, draw a graph, or use a more rigorous argument, as shown below!

Worked Sample Solution

Assume that there is in fact a maximum value M on this interval, i.e. $f(x_1) = M$ for some $x_1 \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ and $f(x) \leq f(x_1)$ for all $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$. We know that $f(x)$ is bounded above by 0 on the given interval. Hence M must satisfy $M \leq 0$. But this maximum value M can't be 0, because the only way $f(x)$ can attain the value 0, is if $x = -\dfrac{\pi}{2}$ or $x = \dfrac{\pi}{2}$. But they aren't in the (open) interval! So, this means M must be strictly less than 0, yet still the maximum value of the function. But this is impossible because, no matter how "big" we make M, while still being less than 0, we can always find a point on the graph that is between M and 0. This follows from the definition of limits and the fact that $\lim_{x\to\pm\frac{\pi}{2}} f(x) = 0$, and since $f(x) \leq 0$ for all x in the given interval. Thus we have a contradiction, and so there is no maximum attained by $f(x)$ on this open interval.

Comments

Keep in mind, you can't directly use the "max-min theorem" (aka Extreme Value Theorem) in this question, since the theorem requires the function to be defined on a closed and bounded interval. The interval in this question is not closed!
Lastly, just because the interval is open, it doesn't mean we can claim straight away that there is no maximum value. There are plenty of functions defined on an open interval, that still have maximum and minimum values (for example, $\sin x$ on the interval $(0, 2\pi)$, or any constant function on an open interval). We need to use a similar argument as above to satisfactorily answer this question.

MATH1131/1141 Calculus Test 1 2010 S1 v7b

March 19, 2017

These hints were written and typed up by Brendan Trinh and edited by Henderson Koh and Vishaal Nathan, with further edits and some worked sample solutions provided by Aaron Hassan and Dominic Palanca. Please be ethical with this resource. It is for the use of MathSoc members, so do not repost it on other forums or groups without asking for permission. If you appreciate this resource, please consider supporting us by coming to our events and buying our T-shirts! Also, happy studying :).

We cannot guarantee that our answers are correct – please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are often multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

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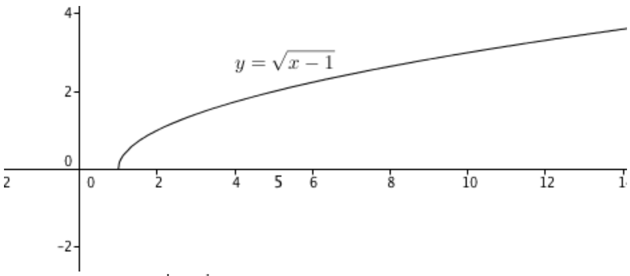
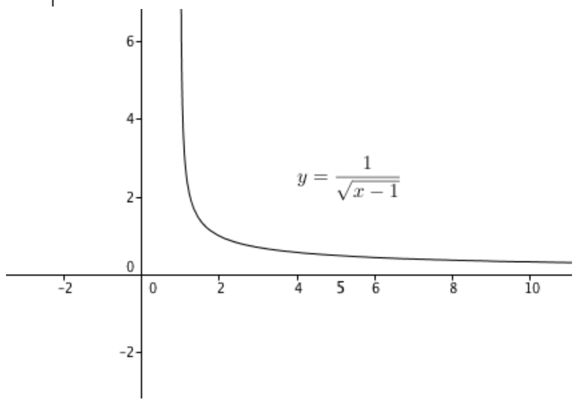
1. The graphs are given below.





In a similar way to previous papers[1], simply find the vertical and horizontal asymptotes

[1]Refer to Question 1 in Test 1 2009 S1 v8a

of $y = \dfrac{1}{\sqrt{x-1}}$ and examine the behaviour of $f(x)$ near the vertical asymptote, all by considering relevant limits of $f(x)$ as x approaches certain places.
Remember to label all relevant x- and y-intercepts, asymptotes etc. in your exam!

2. **Hints**

Consider the sign of the absolute value when you're approaching $a = 1$ from either sides separately in cases[2].
You should find that

$$\lim_{x\to 1^-} f(x) = -3 \qquad \text{and} \qquad \lim_{x\to 1^+} f(x) = 3$$

Since the one-sided limits do not agree, the limit of $f(x)$ as $x \to a$ does not exist.

Worked Sample Solution

Consider $f(x) = \dfrac{|x^2 + x - 2|}{x-1}$. Since $x^2 + x - 2 = (x+2)(x-1)$, we have for all $x \neq 1$ that $f(x) = |x+2| \dfrac{|x-1|}{x-1}$. We need to see what happens to $f(x)$ just on either side of $x = 1$. We consider two cases:
Case 1: $1 < x < 1.01$ (the right-hand side),
In this case, we note that $|x+2| = x+2$ (as $x+2 > 0$) and $|x-1| = x-1 \implies \dfrac{|x-1|}{x-1} = 1$ (as $x - 1 > 0$). Thus in this case, $f(x) = x + 2$. Hence the right-hand limit is $\lim\limits_{x\to 1^+} f(x) = 1 + 2 = 3$.
Case 2: $0.99 < x < 1$ (the left-hand side),
In this case, we note that $|x+2| = x+2$ (as $x+2 > 0$) and $|x-1| = -(x-1)$ (as $x - 1 < 0$). Hence in this case, $f(x) = -(x+2)$. Thus the left-hand limit is $\lim\limits_{x\to 1^-} f(x) = -(1+2) = -3$.

Since $\lim\limits_{x\to 1^+} f(x) \neq \lim\limits_{x\to 1^-} f(x)$, i.e. the one-sided limits do not agree, we have that $\lim\limits_{x\to 1} f(x)$ does not exist.

3. **Hints**

First note that $p(-2) = -10$ and $p(0) = 2$. Also, notice that p is continuous on the interval $[-2, 0]$. Now, from the Intermediate Value Theorem, we know that there exists $c \in (-2, 0)$ such that $p(c) = 0$, since 0 is between -10 and 2. Make sure to clearly state that the hypotheses of the Intermediate Value Theorem are satisfied in this situation.

Worked Sample Solution

Observe that $p(-2) = -8 - 12 + 8 + 2 = -10 < 0$ and $p(0) = 2 > 0$. Since 0 is between $p(-2)$ and $p(0)$ (i.e. p changes sign) and p is continuous on the closed interval $[-2, 0]$ (as

[2] Refer to Question 4 in Test 1 2008 S1 v5a

polynomial functions are continuous), it follows from the intermediate value theorem that there is a $c \in (-2, 0)$ such that $p(c) = 0$, i.e. p has a root (strictly) between -2 and 0.

4. **Hints**

   Multiply both sides by $(x+1)^2$. (The key thing to remember is that if you multiply (or divide) both sides of the inequality by a negative number, the inequality sign switches. But we avoid this issue, by simply multiplying both sides of the inequality by a positive number, $(x+1)^2$.)
   The answer is $x < -3$ or $x > -1$.

   Worked Sample Solution

   Note that $x \neq -1$ (otherwise we are dividing by 0 on the LHS). Multiplying the inequality through by $(x+1)^2$, we have

$$\begin{aligned} x+1 > -\frac{1}{2}(x+1)^2 &\iff -2(x+1) < (x+1)^2 \\ &\iff (x+1)^2 + 2(x+1) > 0 \\ &\iff (x+1)(x+1+2) > 0 \\ &\iff (x+1)(x+3) > 0. \end{aligned}$$

   By sketching the parabola described by the LHS here, we can see that the solution is $x < -3$ or $x > -1$.

5. **Hints**

   This question can be done in a similar way as previous papers[3].
   The maximal domain of f is $\{x \in \mathbb{R} : x \geq -3\}$.
   The range of f is $\{y \in \mathbb{R} : y \geq 0\}$.

   Worked Sample Solution

   We must have $3 + x \geq 0$, so the maximal domain is $\{x \in \mathbb{R} : x \geq -3\}$. The range is $[0, \infty)$, because $f(x)$ is always non-negative (being a square root), and $f(x)$ can clearly attain any given non-negative value (given any $r \geq 0$, $f(x)$ can be made to equal r by taking $x = r^2 - 3$).

[3] Refer to Question 2 in Test 1 2009 S1 v6a