UNSW Mathematics Society Presents
MATH2018/2019 Workshop

Presented by Jordan Russo and John Kameas

Overview I

Overview II

1. Functions of Several Variables

Easy Question

The following is a light way to get us started.

Example

Let $f(x,y) = \dfrac{y}{x+y}$. Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

Easy Question Continued

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Taking partial derivatives is almost identical to taking a normal derivative. We just treat the variable which is not being differentiated as a constant.

So when evaluating $\dfrac{\partial f}{\partial x}$, every-time we see y we'll just imagine that it's fixed.

Hence,

$$\frac{\partial}{\partial x}\left(\frac{y}{x+y}\right) = \frac{\partial}{\partial x}(y(x+y)^{-1}) = -y(x+y)^{-2} = -\frac{y}{(x+y)^2}.$$

Easy Question

Example

Let $f(x,y) = \dfrac{y}{x+y}$. Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

For $\dfrac{\partial f}{\partial y}$ we do it similarly, giving us

$$\begin{aligned}\frac{\partial}{\partial y}\left(\frac{y}{x+y}\right) = \frac{\partial}{\partial y}\left(\frac{{\color{red}x}+y-{\color{red}x}}{x+y}\right) &= \frac{\partial}{\partial y}\left(1-\frac{x}{x+y}\right)\\ &= \frac{\partial}{\partial y}\left(1-x(x+y)^{-1}\right)\\ &= x(x+y)^{-2} = \frac{x}{(x+y)^2}.\end{aligned}$$

Easy Question

Example

Let $f(x, y) = \dfrac{y}{x+y}$. Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

Wrapping it all up, we have

$$\boxed{\frac{\partial f}{\partial x} = \frac{y}{(x+y)^2} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{x}{(x+y)^2}.}$$

Multi-variable Chain Rule

Example

Suppose that $z = x^2 + 4xy$ where $x = u^3 \ln v$ and $y = uv^2$.
Find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

Multi-variable Chain Rule

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The naive, yet sure way to solve this problem is to simply substitute $x = u^3 \ln v$ and $y = uv^2$ into z, to get $z = (u^3 \ln v)^2 + 4u^4 v^2 \ln v$. But this way you're more prone to make a mistake!

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Find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

As such, we recall a friend from single variable calculus, albeit in a new context.

Multi-variable Chain Rule

If $z = f(x, y)$, $x = x(u, v)$, $y = y(u, v)$, then:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}; \qquad (*)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}. \qquad (**)$$

Multi-variable Chain Rule

Example

Suppose that $z = x^2 + 4xy$ where $x = u^3 \ln v$ and $y = uv^2$.
Find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

So to calculate $\dfrac{\partial z}{\partial u}$ we have to calculate, $\dfrac{\partial z}{\partial x}, \dfrac{\partial x}{\partial u}, \dfrac{\partial z}{\partial y}$ and $\dfrac{\partial y}{\partial u}$ as to substitute it into $(*)$.
So doing the grunt work we get:

$$\frac{\partial z}{\partial x} = 2x + 4y \qquad \frac{\partial x}{\partial u} = 3u^2 \ln v$$

$$\frac{\partial z}{\partial y} = 4x \qquad \frac{\partial y}{\partial u} = v^2$$

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Find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

Hence, we get that

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\ &= (2x + 4y)3u^2 \ln v + 4xv^2.\end{aligned}$$

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Unfortunately, this is *not* the final answer! We need to do a little more work.

Multi-variable Chain Rule

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For reference, we currently have

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Recall that we are given that $x = u^3 \ln v$ and $y = uv^2$. So we need to substitute these values back in!

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Recall that we are given that $x = u^3 \ln v$ and $y = uv^2$. So we need to substitute these values back in! Doing so, leads to

$$\begin{aligned}\frac{\partial z}{\partial u} &= (2x + 4y)3u^2 \ln v + 4xv^2 \\ &= (2u^3 \ln v + 4uv^2)3u^2 \ln v + 4u^3 \ln v v^2 \\ &= 2u^3 \ln v(3u^2 \ln v + 8v^2).\end{aligned}$$

Multi-variable Chain Rule

Example

Suppose that $z = x^2 + 4xy$ where $x = u^3 \ln v$ and $y = uv^2$.
Find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

An almost identical computation can be done for $\frac{\partial z}{\partial v}$ to get,

$$\frac{\partial z}{\partial v} = \frac{2\left(2u^4v^2 + u^6 \ln v + 4u^4v^2 \ln v\right)}{v}.$$

Multi-variable Chain Rule

Example

Suppose that $z = x^2 + 4xy$ where $x = u^3 \ln v$ and $y = uv^2$.
Find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

Hence, our final answer will be

$$\frac{\partial z}{\partial u} = 2u^3 \ln v(3u^2 \ln v + 8v^2),$$
$$\frac{\partial z}{\partial v} = \frac{2\left(2u^4v^2 + u^6 \ln v + 4u^4v^2 \ln v\right)}{v}.$$

Taylor Series

Recall from first year maths that a function $f(x)$ at some point $(a, f(a))$ can be approximated as a polynomial by an infinite sum.

Single-Variable Taylor Series

This is,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

But, we're a little more sophisticated now, how about in the case that we have a multi-variable function $f(x, y)$?

Taylor Series

Multi-Variable Taylor Series

The Taylor Series of a multi-variable function $f(x, y)$ at the point (a, b) is given by the following behemoth of a formula,

$$f(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) + \frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}(a,b)(x-a)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(a,b)(x-a)(y-b) + \frac{\partial^2 f}{\partial y^2}(a,b)(y-b)^2\right] + \cdots .$$

- Red signifies **first derivative**; Blue signifies second derivative.
- Where are all the other terms, isn't this an infinite series?

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$$+ \frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}(a,b)(x-a)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(a,b)(x-a)(y-b)\right.$$
$$\left. + \frac{\partial^2 f}{\partial y^2}(a,b)(y-b)^2\right] + \cdots .$$

- Red signifies **first derivative**; Blue signifies second derivative.
- Where are all the other terms, isn't this an infinite series? I haven't forgotten them, it's just very rare that you'll be asked to compute a Taylor Series with a third derivative or higher!

Taylor Series: (17S2, Q1ai)

Example

Calculate the Taylor series expansion of the function $f(x, y) = \ln(x + y)$ about the point $(1, 0)$ up to and including the quadratic terms.

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I will omit the details of the computations of *all* the necessary partial derivatives as to not waste too much time. I trust that you will be able to recover the steps since we have done all that warm up in the previous slides! My only recommendation is to take these questions slowly. There is a lot of working out and so if you're reckless, then silly mistakes will happen.

Taylor Series: (17S2, Q1ai)

Example

Calculate the Taylor series expansion of the function $f(x, y) = \ln(x + y)$ about the point $(1, 0)$ up to and including the quadratic terms.

Here is what you should get for the constant term and first order derivatives,

$$f(1,0) = 0;$$
$$\frac{\partial f}{\partial x} = \frac{1}{x+y} \implies \frac{\partial f}{\partial x}(1,0)(x-1) = x-1;$$
$$\frac{\partial f}{\partial y} = \frac{1}{x+y} \implies \frac{\partial f}{\partial y}(1,0)(y-0) = y.$$

Taylor Series: (17S2, Q1ai)

Example

Calculate the Taylor series expansion of the function $f(x, y) = \ln(x + y)$ about the point $(1, 0)$ up to and including the quadratic terms.

Here is what you should get for the second order derivatives

$$\frac{\partial^2 f}{\partial x^2} = -\frac{1}{(x+y)^2} \implies \frac{\partial^2 f}{\partial x^2}(1,0)(x-1)^2 = -(x-1)^2;$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\frac{1}{(x+y)^2} \implies 2\frac{\partial^2 f}{\partial x \partial y}(1,0)(x-1)(y-0) = -2(x-1)y;$$

$$\frac{\partial^2 f}{\partial y^2} = -\frac{1}{(x+y)^2} \implies \frac{\partial^2 f}{\partial y^2}(1,0)(y-0)^2 = -y^2.$$

Taylor Series: (17S2, Q1ai)

Example

Calculate the Taylor series expansion of the function $f(x, y) = \ln(x + y)$ about the point $(1, 0)$ up to and including the quadratic terms.

Bringing all these terms together we get that,

$$f(x, y) \approx (x - 1) + y - \frac{1}{2}\left[(x - 1)^2 + 2y(x - 1) + y^2\right].$$

Application to Error Approximations

Example

The volume V of a cone with radius r and perpendicular height h is given by $V = \frac{1}{3}\pi r^2 h$. Determine the maximum absolute error and the maximum percentage error in calculating V given that $r = 5$cm and $h = 3$cm to the nearest millimetre.

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Error Approximation

The equation

$$|\Delta f| \leq \left|\frac{\partial f}{\partial x}\right| |\Delta x| + \left|\frac{\partial f}{\partial y}\right| |\Delta y|$$

gives the maximum error of f in terms of the errors in x and y.

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So in our example, we'd have the equation,

$$|\Delta V| \leq \left|\frac{\partial V}{\partial r}\right| |\Delta r| + \left|\frac{\partial V}{\partial h}\right| |\Delta h|.$$

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Since V is a function of r and h then we can easily calculate that

$$\frac{\partial V}{\partial r} = \frac{2\pi r h}{3} \qquad \text{and} \qquad \frac{\partial V}{\partial h} = \frac{\pi r^2}{3}.$$

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$\rightsquigarrow$ $\boxed{\text{But what are the values of } |\Delta r| \text{ and } |\Delta h| \text{ ?}}$

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$\rightsquigarrow$ $\boxed{\text{But what are the values of } |\Delta r| \text{ and } |\Delta h| \text{ ?}}$

An important detail given in the question is that we're finding r and h to the nearest *millimetre*. This means that r and h each have a maximum error of 0.5mm (or 0.05cm). Why?

Explanation of why the maximum error is 0.5mm

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- Here's the kicker, if $\varepsilon > 0.5$, then it must be the case that $|(k-1) - (k-\varepsilon)| = |1 - \varepsilon| < 0.5$. This means that $k - \varepsilon$ to the nearest millimetre is $k - 1$ and not k like we thought!

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- Now we know that if we have something to be $k$mm after rounding our error is less than or equal to 0.5mm. So our *maximum error* is 0.5mm.

Application to Error Approximations

Example

The volume V of a cone with radius r and perpendicular height h is given by $V = \frac{1}{3}\pi r^2 h$. Determine the maximum absolute error and the maximum percentage error in calculating V given that $r = 5\text{cm}$ and $h = 3\text{cm}$ to the nearest millimetre.

Now that we have convinced ourselves that $|\Delta r| = 0.05\text{cm}$ and $|\Delta h| = 0.05\text{cm}$, we can continue along with our question.

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Now that we have convinced ourselves that $|\Delta r| = 0.05\text{cm}$ and $|\Delta h| = 0.05\text{cm}$, we can continue along with our question.
As such,

$$\begin{aligned} |\Delta V| &\leq \left|\frac{2\pi r h}{3}\right|(5,3) \times 0.05 + \left|\frac{\pi r^2}{3}\right|(5,3) \times 0.05 \\ &= 10\pi \times 0.05 + \frac{25\pi}{3} \times 0.05 \\ &\approx 2.87. \end{aligned}$$

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Now that we have $|\Delta V| \approx 2.87$ we can calculate the percentage error by using the formula

$$\frac{|\Delta V|}{V} = \frac{2.87}{(75\pi/3)} \approx 3.65\%.$$

Leibniz Rule

Example: 18S2, Q1(iv)

You are given that

$$\int_0^\infty \frac{1}{\alpha^2 + x^2}\,\mathrm{d}x = \frac{\pi}{2}\alpha^{-1}.$$

Use Leibniz's Theorem to find the following integral in terms of α

$$\int_0^\infty \frac{1}{(\alpha^2 + x^2)^2}\,\mathrm{d}x.$$

Leibniz Rule

Leibniz Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{u(x)}^{v(x)} f(x,t)\,\mathrm{d}t = \int_{u(x)}^{v(x)} \frac{\partial f}{\partial x}\,\mathrm{d}t + f(x,v(x))\frac{\mathrm{d}v}{\mathrm{d}x} - f(x,u(x))\frac{\mathrm{d}u}{\mathrm{d}x}.$$

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How do we use this to get out an answer to our question? We let,

$$f(\alpha, x) = \frac{1}{\alpha^2 + x^2}$$

and look back at our rule. Is it a bit more obvious what route we should take? With what we are given, what should we differentiate the left hand side in respect to?

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in respect to α! Hence,

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\int_0^\infty \frac{1}{\alpha^2 + x^2}\, \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\frac{\pi}{2}\alpha^{-1}\right] = -\frac{\pi}{2\alpha^2}.$$

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We should differentiate the equation

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in respect to α! Hence,

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\alpha}\int_0^\infty \frac{1}{\alpha^2 + x^2}\,\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\frac{\pi}{2}\alpha^{-1}\right] = -\frac{\pi}{2\alpha^2}.$$

We then apply Leibniz' Rule to the boxed part of our equation.

Leibniz' Rule

Note that the bounds of our integral do *not* depend on α. So the partial derivatives of the bounds of our integral (i.e. the extra terms in Leibniz' Rule) evaluate to zero.

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We have:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{u(x)}^{v(x)} f(x,t)\,\mathrm{d}t = \int_{u(x)}^{v(x)} \frac{\partial f}{\partial x}\,\mathrm{d}t + \xcancel{f(x,v(x))\frac{\mathrm{d}v}{\mathrm{d}x}} - \xcancel{f(x,u(x))\frac{\mathrm{d}u}{\mathrm{d}x}}.$$

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As such,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\alpha}\int_0^\infty \frac{1}{\alpha^2+x^2}\,\mathrm{d}x &= \int_0^\infty \frac{\partial}{\partial\alpha}\left(\frac{1}{\alpha^2+x^2}\right)\mathrm{d}x \\ &= -2\alpha\int_0^\infty \frac{1}{(\alpha^2+x^2)^2}\,\mathrm{d}x.\end{aligned}$$

Leibniz' Rule

There are a lot of moving parts to this question so do not lose sight of our goal of finding the integral of

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There are a lot of moving parts to this question so do not lose sight of our goal of finding the integral of

$$\int_0^\infty \frac{1}{(\alpha^2 + x^2)^2} \, \mathrm{d}x.$$

From the previous slide, we can re-express our desired integral as

$$\int_0^\infty \frac{1}{(\alpha^2 + x^2)^2} \, \mathrm{d}x = -\frac{1}{2\alpha} \frac{\mathrm{d}}{\mathrm{d}\alpha} \int_0^\infty \frac{1}{\alpha^2 + x^2} \, \mathrm{d}x.$$

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But remember we calculated what the expression in the red box is; it's simply $-\dfrac{\pi}{2\alpha^2}$.

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But remember we calculated what the expression in the red box is; it's simply $-\dfrac{\pi}{2\alpha^2}$.Hence, we find that

$$\int_0^\infty \frac{1}{(\alpha^2 + x^2)^2}\,\mathrm{d}x = \left(-\frac{1}{2\alpha}\right) \times \left(-\frac{\pi}{2\alpha^2}\right) = \frac{\pi}{4\alpha^3}.$$

Leibniz' Rule Cont.

These questions are initially quite tricky because of how many things are going on at any one time. As such, it's worth going through one more example being a bit more direct with our solution. So the exposition for this question will be kept at a minimum, so that we can focus on the method, rather than all the precise details.

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17S2, Q1(e)

You are given the following integral

$$\int_0^a \frac{1}{(x^2+a^2)^{1/2}}\,\mathrm{d}x = \sinh^{-1}(1).$$

Use Leibniz' Rule to evaluate

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We will differentiate our given integral in terms of a to give,

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We then apply Leibniz' Rule to the LHS of the above equation.

Leibniz' Rule Cont.

Upper bound of our integral is dependent on a so we need to use our rule with *one* constant term giving us,

$$\frac{\mathrm{d}}{\mathrm{d}a}\int_0^a \frac{1}{(x^2+a^2)^{1/2}}\,\mathrm{d}x = \int_0^a \frac{\partial}{\partial a}\left(\frac{1}{(x^2+a^2)^{1/2}}\right)\,\mathrm{d}x + \frac{1}{(a^2+a^2)^{1/2}} \times \frac{\mathrm{d}a}{\mathrm{d}a} - 0.$$

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Simplifying and differentiating where possible gives us

$$\frac{\mathrm{d}}{\mathrm{d}a}\int_0^a \frac{1}{(x^2+a^2)^{1/2}}\,\mathrm{d}x = -a\int_0^a \frac{1}{(x^2+a^2)^{3/2}}\,\mathrm{d}x + \frac{1}{\sqrt{2}a}.$$

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However, we know

$$\frac{\mathrm{d}}{\mathrm{d}a} \int_0^a \frac{1}{(x^2 + a^2)^{1/2}} \, \mathrm{d}x = 0.$$

Which means that,

$$-a \int_0^a \frac{1}{(x^2 + a^2)^{3/2}} \, \mathrm{d}x + \frac{1}{\sqrt{2}a} = 0.$$

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Hence, by a little bit of rearranging we get that

$$\int_0^a \frac{1}{(x^2+a^2)^{3/2}}\,\mathrm{d}x = \frac{1}{\sqrt{2}a^2}$$

and we are done.

2. Extreme Values

Critical Points

15S2 Q1(d)

Find and classify the critical points of

$$h(x,y) = 2x^3 + 3x^2y + y^2 - y.$$

Provide the function value at these critical points.

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Remember when we have a single variable function, we found the critical (stationary) points by differentiating our function and then solving for the roots of the derivative.
Thankfully in the multi-variable case, it is quite similar. We find all first order partial derivatives and then solve for the roots by solving a system of simultaneous equations.

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Provide the function value at these critical points.

As such,

$$\frac{\partial h}{\partial x} = 6x^2 + 6xy = 0 \qquad (*)$$

$$\frac{\partial h}{\partial y} = 3x^2 + 2y - 1 = 0. \qquad (**)$$

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Hence, our critical points are $(0, \frac{1}{2}), (-\frac{1}{3}, \frac{1}{3})$ and $(1, -1)$.

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Hence, our critical points are $(0, \frac{1}{2}), (-\frac{1}{3}, \frac{1}{3})$ and $(1, -1)$.
But we are not quite yet done! We still have to classify our critical points.

Critical Points

Classification of Critical Points

We define the expression $\mathcal{D} := \dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial^2 f}{\partial y^2} - \dfrac{\partial^2 f}{\partial x \partial y}$.

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- If $\mathcal{D} = 0$ at crit $= (a, b)$, then the test was inconclusive. (Don't worry, it is extremely unlikely they will test you on a sitation when $\mathcal{D} = 0$.)

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- At $(0, \frac{1}{2})$ we have $\mathcal{D}(0, \frac{1}{2}) = 6 > 0$ and $\frac{\partial^2 f}{\partial x^2}(0, \frac{1}{2}) = 3 > 0$. So $(0, \frac{1}{2})$ is a local minimum.

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- At $(1, -1)$ we have $\mathcal{D}(1, -1) = 6 > 0$ and $\dfrac{\partial^2 f}{\partial x^2}(1, -1) = 6 > 0$. So $(1, -1)$ is a local minimum.

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We're now almost done. All that is left to do is find the function values at our critical points.

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- At $(-\frac{1}{3}, \frac{1}{3})$ we have $h(-\frac{1}{3}, \frac{1}{3}) = -\frac{5}{27}$.
- At $(1, -1)$ we have $h(1, -1) = 1$.

Critical Points Continued

Example: Lagrange Multipliers

Find the extreme value(s) of $z = f(x, y) = x^4 + y^4$ subject to the condition $x + y - 1 = 0$.

Critical Points Continued

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We will solve this question by the method of Lagrange multipliers. Recall that for some n-dimensional function f we have

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}.$$

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That is, ∇f is the collection of all first order partial derivatives of a function composed as a vector!

Critical Points Continued

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Let $g(x, y) = x + y$, then our constraint can be reformulated as $g(x, y) = 0$.

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Lagrange Multipliers

The critical values (local max or min) of a function f subject to the constraint g, satisfy the equation

$$\nabla f = \lambda \nabla g.$$

Critical Points Continued

Example: Lagrange Multipliers

Find the extreme value(s) of $z = f(x, y) = x^4 + y^4$ subject to the condition $x + y - 1 = 0$.

Hence, we have the following system of equations by equating components of the vectors,

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \implies 4x^3 = \lambda; \tag{1}$$

$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \implies 4y^3 = \lambda; \tag{2}$$

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We then solve for λ.

Critical Points Continued

To solve the system of equations on the previous slide, we first find what x and y are in terms of λ.

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and by solving for λ we get that $\lambda = \frac{1}{2}$.

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and by solving for λ we get that $\lambda = \frac{1}{2}$. Finally, all we need to do is sub λ back into (1) and (2) to get that our critical point, subject to our constraint, is $(\frac{1}{2}, \frac{1}{2})$.

3. Vector Field Theory

15S2 Q3c(ii)

Given a vector field

$$\mathbf{F} = 8e^{-x}\mathbf{i} + \cosh z\,\mathbf{j} - y^2\mathbf{k}$$

calculate the line integral $\displaystyle\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\boldsymbol{r}$ where $\mathcal{C}$ is the straight line path from $A = (0, 1, 0)$ to $B = (\ln(2), 1, 2)$.

Q3c(ii)

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A straight line path L from A to B can always be constructed by the formula

$$L = (1-t)A + tB,$$

where $t \in [0, 1]$.

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- $x(t) = 0 \cdot t + \ln(2)(1-t) = \ln(2) \cdot (1-t)$.

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- $z(t) = 0 \cdot t + 2(1-t) = 2(1-t)$.

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So we have:

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This means that:

- $\mathrm{d}x = -\ln(2)\,\mathrm{d}t$.

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Q3c(ii)

Line Integral Formula

$$\int_{\mathcal{C}} \mathbf{F} \cdot \, \mathrm{d}\boldsymbol{r} = \int_{\mathcal{C}} F_1(x(t)) \, \mathrm{d}x + \int_{\mathcal{C}} F_2(y(t)) \, \mathrm{d}y + \int_{\mathcal{C}} F_3(z(t)) \, \mathrm{d}z$$

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From the previous slides we have all the required information to use this formula. Hence,

$$\int_{\mathcal{C}} \mathbf{F} \cdot \, \mathrm{d}\boldsymbol{r} = \int_0^1 -\ln 2 \left(8e^{-\ln 2(1-t)} \right) \, \mathrm{d}t + \int_0^1 0(\cosh(2(1-t))) \, \mathrm{d}t + \int_0^1 (-1)^2 \cdot (-2) \, \mathrm{d}t$$

Q3c(ii)

Line Integral Formula

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which simplifies to

$$\begin{aligned}\int_0^1 \left[8\ln(2)e^{t\ln 2 - \ln 2} + 2\right] \, \mathrm{d}t &= 4\ln 2 \int_0^1 2^t \, \mathrm{d}t + 2 \\ &= 6.\end{aligned}$$

Line Integrals Continued

18S1 Q1(a)

Consider the scalar field

$$\phi(x, y, z) = xe^{z-1} + \cos y$$

and let $\mathbf{F} = \nabla\phi$.

- What is $\nabla \times \mathbf{F}$?
- Hence, or otherwise, calculate the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}r$ along the straight line path $\mathcal{C}$ from $(1, 0, 1)$ to $(5, \pi, 1)$.

Line Integrals Continued

The expression $\nabla \times \mathbf{F}$ is known as the curl of $\mathbf{F}$ and is computed by the quite painful formula:

Curl

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Line Integrals Continued

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Thankfully, for this question we don't need to use this formula. Why?

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Thankfully, for this question we don't need to use this formula. Why? We are told that $\mathbf{F} = \nabla\phi$ and that ϕ is a scalar field which means that $\mathbf{F}$ is a conservative vector field.

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Thankfully, for this question we don't need to use this formula. Why? We are told that $\mathbf{F} = \nabla \phi$ and that ϕ is a scalar field which means that $\mathbf{F}$ is a conservative vector field. This automatically implies that $\nabla \times \mathbf{F} = \mathbf{0}$. The proof of this claim isn't actually very difficult, but it requires a cumbersome calculation and use of Clairaut's Theorem, so it's best to just commit this fact to memory to avoid doing lengthy calculations.

Line Integrals Continued

Knowing that $\mathbf{F}$ is a conservative vector field will actually help us in the second part of the question as well, as we can appeal to the following theorem:

Line Integrals Continued

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Line Integrals on Conservative Fields

Line Integrals on Conservative Fields, are *path independent*. This means that,

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}r = \phi(B) - \phi(A).$$

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As such, we have

$$\int_{\mathcal{C}} \mathbf{F} \cdot \, \mathrm{d}r = (5e^{1-1} + \cos \pi) - (1 \cdot e^{1-1} + \cos 0) = 4 - 2 = 2.$$

More Line Integrals!

18S2 Q4(i)

Consider the vector field

$$\mathbf{F} = yz^2\mathbf{i} + xz^2\mathbf{j} + (2xyz + 3)\mathbf{k}$$

- Show that $\mathbf{F}$ is a conservative vector field by evaluating $\operatorname{curl}(\mathbf{F})$.
- The path $\mathcal{C}$ in $\mathbb{R}^3$ starts at the point $(3, 4, 7)$ and subsequently travels anti-clockwise four complete revolutions around the circle $x^2 + y^2 = 25$ within the plane $z = 7$, returning to the point $(3, 4, 7)$. Using the first part or otherwise, evaluate the work integral $\displaystyle\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\boldsymbol{r}$.

More Line Integrals!

Recall the formula defined on slide 60 for the curl, it is given as

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.$$

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It's a bit of a pain to work through this formula but our hands are tied as the question is forcing us to do it. So let's substitute each entry recalling that $\mathbf{F} = yz^2\mathbf{i} + xz^2\mathbf{j} + (2xyz + 3)\mathbf{k}$.

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More Line Integrals!

We can evaluate the determinant as,

$$\operatorname{curl}\mathbf{F} = \boldsymbol{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^2 & 2xyz+3 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ yz^2 & 2xyz+3 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ yz^2 & xz^2 \end{vmatrix}$$

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Computing the determinant gives us,

$$\begin{aligned}\operatorname{curl}\mathbf{F} =& \mathbf{i}\left[\frac{\partial}{\partial y}(2xyz+3) - \frac{\partial}{\partial z}(xz^2)\right] - \mathbf{j}\left[\frac{\partial}{\partial x}(2xyz+3) - \frac{\partial}{\partial z}(yz^2)\right] \\ &+ \mathbf{k}\left[\frac{\partial}{\partial x}(xz^2) - \frac{\partial}{\partial y}(yz^2)\right].\end{aligned}$$

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As such,

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$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\boldsymbol{r} = \phi(B) - \phi(A).$$

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But now a new question arises, how do we find ϕ? Recall that this is the question:

Second Part

The path $\mathcal{C}$ in $\mathbb{R}^3$ starts at the point $(3, 4, 7)$ and subsequently travels anti-clockwise four complete revolutions around the circle $x^2 + y^2 = 25$ within the plane $z = 7$, returning to the point $(3, 4, 7)$. Using the first part or otherwise, evaluate the work integral $\displaystyle\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\boldsymbol{r}$.

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What key detail do we have that will allow us to finish the question?

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$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\boldsymbol{r} = 0$$

and we are done.

Double Integrals

Example

Evaluate

$$\iint_{\Omega} x \, \mathrm{d}A$$

where Ω is the region in the first quadrant bounded by the parabola $y = 4 - x^2$ and the coordinate axis.

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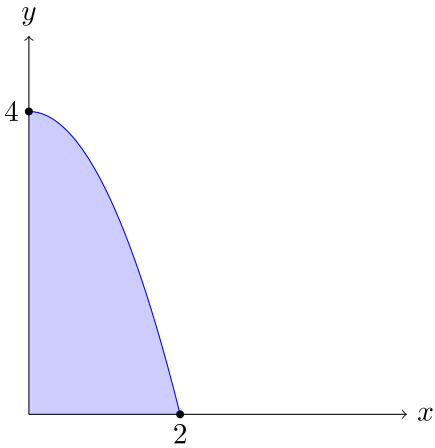
- Double integrals look a lot more daunting than they actually are.
- Whatever region is specified, it's always advisable to attempt to sketch it and see what's going on.
- After we've sketched it, it should be 'obvious' what the limits of our integral are.

Sketch of the region Ω



Limits of the Integral

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- Or, we can let y range over $[0, 4]$ then $x = \sqrt{4 - y}$.

I'll take the first route and so the required integral is

$$\begin{aligned}\iint_\Omega x \,\mathrm{d}A = \int_0^2 \int_0^{4-x^2} x \,\mathrm{d}y \,\mathrm{d}x &= \int_0^2 x \,[y]_0^{4-x^2} \,\mathrm{d}x \\ &= \int_0^2 4x - x^3 \,\mathrm{d}x \\ &= \left[2x^2 - \frac{1}{4}x^4\right]_0^2 \\ &= (8 - 4) - (0 - 0) \\ &= 4.\end{aligned}$$

4. Matrices

15S2 Q1(b)

The matrix B is given by

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

1. Show that the vector
$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$
is an eigenvalue of the matrix B and find the corresponding eigenvalue.
2. Given that the other two eigenvalues of B are -1 and -2, find the eigenvectors corresponding to these eigenvalues.

First Part

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As such,

$$\begin{aligned} B\boldsymbol{v} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} &= \begin{pmatrix} (0 \cdot 1) + (-1) \cdot (-1) + 0 \cdot 0 \\ (-1) \cdot 1 + 0 \cdot (-1) + 0 \cdot 0 \\ 0 \cdot 1 + (0) \cdot (-1) + 2 \cdot 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \\ &= 1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 1 \cdot \boldsymbol{v}. \end{aligned}$$

Second Part

Recall that to find an eigenvector given an eigenvalue λ, we row reduce the matrix $B - \lambda I$. Why?

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$$B + I = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

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Therefore, the eigenvector corresponding to $\lambda = -1$ is

$$\boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Second Part Continued

Similarly for the eigenvector $\lambda = -2$, we row-reduce the matrix $B - 2I$ to give us

$$B - 2I = \begin{pmatrix} -2 & -1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -2 & -1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

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So our eigenvector corresponding to $\lambda = 2$ is,

$$\boldsymbol{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Eigenvalues & Eigenvectors Continued

18S1, Q2b(i)

A real symmetric 3×3 matrix A has eigenvalues λ_1, λ_2 and λ_3.
A student is given the following facts about A:

- $\operatorname{trace}(A) = 0$,
- $\lambda_1 = 2$ and $\lambda_3 = 4$.

What is the value of the the remaining eigenvalue, namely λ_2?

Eigenvalues & Eigenvectors Continued

Recall the important fact the the trace of any matrix A, is the sum of all it's eigenvalues. Hence,

$$\text{trace}(A) = 0 = \lambda_1 + \lambda_2 + \lambda_3.$$

Eigenvalues & Eigenvectors Continued

Recall the important fact the the trace of any matrix A, is the sum of all it's eigenvalues. Hence,

$$\text{trace}(A) = 0 = \lambda_1 + \lambda_2 + \lambda_3.$$

But we are also told that $\lambda_1 = 2$ and $\lambda_3 = 4$. Hence,

$$2 + \lambda_2 + 4 = 0 \implies \boxed{\lambda_2 = -6}.$$

Eigenvalues & Eigenvectors Continued ... again

Example

The matrix $A = \begin{pmatrix} -5 & 6 & 0 \\ -3 & 4 & 0 \\ -3 & 3 & 1 \end{pmatrix}$ is diagonalisable with eigenvalues $-2, 1$ and 1.

An eigenvector corresponding to the eigenvalue -2 is $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

Find an invertible matrix M such that $M^{-1}AM = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Eigenvalues & Eigenvectors Continued ... again

Observe the non-zero entries of the matrix

$$M^{-1}AM = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

exactly corresponds to the eigenvalues $\lambda_1 = -2, \lambda_2 = 1$ and $\lambda_3 = 1$ for the matrix A.

Eigenvalues & Eigenvectors Continued ... again

Connection between Eigenvectors and Diagonalisation

If a $n \times n$ matrix A is diagonalisable, then there exists a matrix Q such that

$$A = QDQ^{-1} \iff \boxed{D = Q^{-1}AQ}$$

where D is an $n \times n$ matrix with eigenvalue entries along the diagonal. Q is the matrix corresponding to eigenvectors as column vectors matching the eigenvalues in matrix D.

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Hence, our required matrix is in-fact Q, that is, the matrix with eigenvectors as it's columns.

Eigenvalues & Eigenvectors Continued ... again

Since we are given that $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ corresponds to the eigenvalue -2, all we need to do is find the eigenvectors corresponding to the eigenvalue 1.

Eigenvalues & Eigenvectors Continued ... again

Since we are given that $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ corresponds to the eigenvalue -2, all we need to do is find the eigenvectors corresponding to the eigenvalue 1. So we row-reduce $A - I$, giving us

$$A - I = \begin{pmatrix} -6 & 6 & 0 \\ -3 & 3 & 0 \\ -3 & 3 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, we have two eigenvectors

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

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Eigenvalues & Eigenvectors Continued ... again

Hence, our required matrix M is

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Conics

Example

Identify and sketch the curve,

$$8x^2 + 12xy + 3y^2 = 48.$$

Find the points, if any, on the curve which are farthest from, and closest to, the origin.

How to do these kinds of questions:

Conics

Example

Identify and sketch the curve,

$$8x^2 + 12xy + 3y^2 = 48.$$

Find the points, if any, on the curve which are farthest from, and closest to, the origin.

How to do these kinds of questions:

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Conics

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- We first need to express our curve in as the form $\boldsymbol{x}^T A \boldsymbol{x} = c$.
- We then find the eigenvalues and eigenvectors of A.
- Next, we diagnoalise A into the form $A = QDQ^T$ and so our new axis is of the form $X = Q^T \boldsymbol{x}$.

First Step

Since we have the curve $8x^2 + 12xy + 3y^2 = 48$ then we can write it in the form

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with

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Why didn't we choose any of these other other candidates for A:

$$A = \begin{pmatrix} 8 & 2 \\ 10 & 3 \end{pmatrix} \text{ or } A = \begin{pmatrix} 8 & 3 \\ 9 & 3 \end{pmatrix} \text{ or } A = \begin{pmatrix} 8 & a \\ 12 - a & 3 \end{pmatrix}?$$

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In so far as getting the correct equation for the curve goes, all of the above matrices would have done. However, we need a **symmetric matrix** in order to guarantee that our procedure will work. A non–symmetric real matrix need not have real eigenvalues, and even if it does, it need not be diagonalisable: in either case our method will fail.

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This is an equation of a hyperbola, and so our curve is a hyperbola with axes

$$\frac{1}{\sqrt{13}}\begin{pmatrix} 3 \\ 2 \end{pmatrix} \text{ and } \frac{1}{\sqrt{13}}\begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

Where did the $1/\sqrt{13}$ come from?

Third Step

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Clearly (is it?) the curve has no points which are farthest from the origin. How about the points closest to the origin? In our transformed coordinates, we solve for the variable which has the largest coefficient (i.e. the largest eigenvalue) and let the other variable(s) be set to zero. Hence,

$$12X^2 - 1(0)^2 = 48 \implies X^2 = 4 \implies \boxed{X = \pm 2}.$$

Hence, the point closest to the curve is $\pm(2, 0)$. But, these are funky coordiantes, how about in the normal x, y coordinates? Recall that $\mathbf{X} = Q^T \boldsymbol{x}$ and so, $\boldsymbol{x} = Q\mathbf{X}$ and Q is our matrix of eigenvectors

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Thus,

$$\boldsymbol{x} = \pm Q \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \pm \frac{1}{\sqrt{13}} \begin{pmatrix} 6 \\ 4 \end{pmatrix}.$$

Example

Solve the system of differential equations

$$\begin{aligned} y_1' &= -3y_1 + 4y_2, & y_1(0) &= 4; \\ y_2' &= 8y_1 + y_2, & y_2(0) &= -1. \end{aligned}$$

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Recall that the goal is to find the vector $\boldsymbol{y}$. At the moment we are are given the vector $\boldsymbol{y}'$ and an initial condition $\boldsymbol{y}(0)$. To solve these kinds of problems, we represent $\boldsymbol{y}'$ into a matrix A and solve for it's eigenvalues and eigenvectors. Something for you all to ponder is the reason why we find eigenvalues and eigenvectors for DEs.

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To represent our system into a matrix, we take the coefficients of the equations as the entries. Hence,

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The eigenvalues are the roots of the equation

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -3-\lambda & 4 \\ 8 & 1-\lambda \end{vmatrix} \\ &= (\lambda - 1)(\lambda + 3) - 32 \\ &= (\lambda + 7)(\lambda - 5) \implies \lambda = -7, 5. \end{aligned}$$

Matrices & Systems of Differential Equations

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For the Eigenvectors we have

$$\begin{aligned} A + 7I &= \begin{pmatrix} 4 & 4 \\ 8 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \implies \boldsymbol{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ A - 5I &= \begin{pmatrix} -8 & 4 \\ 8 & -4 \end{pmatrix} \sim \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix} \implies \boldsymbol{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \end{aligned}$$

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Wrapping it all up

So our general solution is given by the formula

Gen Sol

$$\boldsymbol{y} = \sum_{k=0}^{n} c_k \boldsymbol{v}_k e^{\lambda k t}.$$

Thus,

$$\begin{aligned}\boldsymbol{y} = \sum_{k=0}^{n} c_k \boldsymbol{v}_k e^{\lambda k t} &= c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-7t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t} \\ &= \begin{pmatrix} c_1 e^{-7t} + c_2 e^{5t} \\ -c_1 e^{-7t} + 2c_2 e^{5t} \end{pmatrix}.\end{aligned}$$

Wrapping it all up

But we're not quite yet done! Recall that $\boldsymbol{y}(0) = (4, -1)$. Which means, substituting $t = 0$ gives

$$\begin{pmatrix} c_1 + c_2 \\ -c_1 + 2c_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}.$$

We can solve this either by Gaussian Elimination or by HS simultaneous equations. Either way, we get $c_1 = 3$ and $c_2 = 1$.

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We can solve this either by Gaussian Elimination or by HS simultaneous equations. Either way, we get $c_1 = 3$ and $c_2 = 1$. In conclusion, the final answer is

$$\boldsymbol{y} = \begin{pmatrix} 3e^{-7t} + e^{5t} \\ -3e^{-7t} + 2e^{5t} \end{pmatrix}.$$

First Half is done!

Thanks for watching everyone!

5. Ordinary Differential Equations

What is an ordinary differential equation (ODE)?

- A differential equation is an equation that relates an unknown function with one or more of its derivatives.
- When only ordinary derivatives, rather than partial derivatives, are involved, the equations are called ordinary differential equations (ODEs).

First-Order Differential Equations

Example

Solve

$$(x+y)\frac{dy}{dx} = e^{3x} - x - y \qquad \text{with} \quad y(0) = 2$$

First-Order Differential Equations

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Firstly, it's important to observe that because of the $y\dfrac{dy}{dx}$ term on the LHS, this first-order ODE is non-linear. Thus, we should try and find a way to separate the variables. Without a clear solution in mind, intuition tells us that we should try grouping the $x+y$ and $-x-y$ expressions.

First-Order Differential Equations

Example

Solve
$$(x+y)\frac{dy}{dx} = e^{3x} - x - y \qquad \text{with} \quad y(0) = 2$$

$$\begin{aligned}
(x+y)\frac{dy}{dx} &= e^{3x} - x - y \\
(x+y)\frac{dy}{dx} &= e^{3x} - (x+y) \\
(x+y) + (x+y)\frac{dy}{dx} &= e^{3x} \\
(x+y)\left(1 + \frac{dy}{dx}\right) &= e^{3x}
\end{aligned}$$

Example

Solve
$$(x+y)\frac{dy}{dx} = e^{3x} - x - y \qquad \text{with} \quad y(0) = 2$$

Notice that we have now separated the terms $x+y$ and e^{3x}. At this point, it's a good idea to use a substitution for $x+y$ so that the separated terms are distinct from one another. If we make a clever enough substitution we can get rid of the 1 within the $1 + \dfrac{dy}{dx}$ term which would allow us to directly solve the ODE.

$$(x+y)\left(1 + \frac{dy}{dx}\right) = e^{3x}$$

First-Order Differential Equations

Example

Solve
$$(x+y)\frac{dy}{dx} = e^{3x} - x - y \qquad \text{with} \quad y(0) = 2$$

A simple substitution is $v = x + y$.
Rearranging gives us $y = v - x$ which means that
$\dfrac{d}{dx}(y) = \dfrac{d}{dx}(v - x) \implies \dfrac{dy}{dx} = \dfrac{dv}{dx} - 1.$

$$(x+y)\left(1 + \frac{dy}{dx}\right) = e^{3x}$$
$$v\ \frac{dv}{dx} = e^{3x}*$$

***A common mistake to make is substituting in here as well.**

First-Order Differential Equations

Example

Solve

$$(x+y)\frac{dy}{dx} = e^{3x} - x - y \qquad \text{with} \quad y(0) = 2$$

From here we can integrate both side w.r.t x and substitute back in for the original variables $v = x + y$.

$$\begin{aligned}
\int v \, \frac{dv}{dx} \, dx &= \int e^{3x} \, dx \\
\int v \, dv &= \int e^{3x} \, dx \\
\frac{1}{2}v^2 &= \frac{1}{3}e^{3x} + C \\
3(x+y)^2 &= 2e^{3x} + C'
\end{aligned}$$

First-Order Differential Equations

Example

Solve

$$(x+y)\frac{dy}{dx} = e^{3x} - x - y \qquad \text{with} \quad y(0) = 2$$

Now, using the initial value $y(0) = 2$ to find the particular solution.

$$\begin{aligned} 3(0+2)^2 &= 2e^{3(0)} + C' \\ \text{So}, C' &= 10 \\ \therefore 3(x+y)^2 &= 2e^{3x} + 10 \end{aligned}$$

Second-order Differential Equations

Example

Use the method of undetermined coefficients to solve the second-order differential equation

$$y'' + 3y' + 2y = e^{-2t} + 4t^2 + 2 \qquad \text{with} \quad y(0) = 7,\ y'(0) = -13$$

Also describe the steady state solution and calculate the Wronskian for the system.

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Also describe the steady state solution and calculate the Wronskian for the system.

The first step to solving this ODE is finding the homogeneous solution which requires determining the characteristic polynomial.

To do this, consider $y'' + 3y' + 2y = 0$ assume $y = e^{\lambda t}$, where $\lambda \in \mathbb{C}$.

Second-order Differential Equations

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Also describe the steady state solution and calculate the Wronskian for the system.

$$\begin{aligned} \frac{d^2}{dt^2}(e^{\lambda t}) + 3\frac{d}{dt}(e^{\lambda t}) + 2(e^{\lambda t}) &= 0 \\ \lambda^2(e^{\lambda t}) + 3\lambda(e^{\lambda t}) + 2(e^{\lambda t}) &= 0 \\ (\lambda^2 + 3\lambda + 2)(e^{\lambda t}) &= 0 \end{aligned}$$

For a non-trivial solution, assume $\lambda^2 + 3\lambda + 2 = 0$, thus $\lambda_1 = -1$ and $\lambda_2 = -2$.

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Also describe the steady state solution and calculate the Wronskian for the system.

Since λ_1 and λ_2 are real and distinct, we can assume that the homogeneous solution has the following form.

$$y_h = Ae^{-t} + Be^{-2t}, \qquad \text{where } A, B \in \mathbb{R}$$

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Also describe the steady state solution and calculate the Wronskian for the system.

To find the particular solution it is first necessary to guess the form that it will take. Let $f = e^{-2t} + 4t^2 + 2$, then $f' = -2e^{-2t} + 8t$ and $f'' = 4e^{-2t} + 8$. Our particular solution y_p should contain all of these possibilities in the most general form possible.

$$\text{Thus, } y_p = Ce^{-2t} + \alpha t^2 + \beta t + \gamma, \qquad \text{where } C, \alpha, \beta, \gamma \in \mathbb{R}$$

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Also describe the steady state solution and calculate the Wronskian for the system.

$$\begin{aligned} y_p &= Ce^{-2t} + \alpha t^2 + \beta t + \gamma \\ y_p' &= -2Ce^{-2t} + 2\alpha t + \beta \\ y_p'' &= 4Ce^{-2t} + 2\alpha \end{aligned}$$

$$(4Ce^{-2t} + 2\alpha) + 3(-2Ce^{-2t} + 2\alpha t + \beta) + 2(Ce^{-2t} + \alpha t^2 + \beta t + \gamma) = e^{-2t} + 4t^2 + 2$$

Second-order Differential Equations

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Also describe the steady state solution and calculate the Wronskian for the system.

$$\begin{aligned} C &= 0 \\ 2\alpha &= 4 \implies \alpha = 2 \\ 6\alpha + 2\beta &= 0 \implies \beta = -6 \\ 2\alpha + 3\beta + 2\gamma &= 2 \implies \gamma = 8 \end{aligned}$$

$$\text{Thus, } y_p = 2t^2 - 6t + 8$$

Example

Use the method of undetermined coefficients to solve the second-order differential equation

$$y'' + 3y' + 2y = e^{-2t} + 4t^2 + 2 \qquad \text{with} \quad y(0) = 7,\ y'(0) = -13$$

Also describe the steady state solution and calculate the Wronskian for the system.

Lastly, in order to obtain the general solution all we have to do is add the homogeneous and particular solutions together.

$$\begin{aligned} y_g &= y_h + y_p \\ \therefore y_g &= Ae^{-t} + Be^{-2t} + 2t^2 - 6t + 8 \\ \text{with, } y_g' &= -Ae^{-t} - 2Be^{-2t} + 4t - 6 \end{aligned}$$

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Also describe the steady state solution and calculate the Wronskian for the system.

Using the initial conditions yields the following pair of simultaneous equations.

$$\begin{aligned} 7 &= A + B + 8 \\ -13 &= -A - 2B - 6 \end{aligned}$$

Meaning that $A = -9$ and $B = 8$.

Second-order Differential Equations

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Use the method of undetermined coefficients to solve the second-order differential equation

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Also describe the steady state solution and calculate the Wronskian for the system.

$$\therefore y = -9e^{-t} + 8e^{-2t} + 2t^2 - 6t + 8$$

.

Second-order Differential Equations

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Also describe the steady state solution and calculate the Wronskian for the system.

To find the steady state solution we shall consider what happens at $t \to \infty$. The exponential terms in y will decay, so we can say that the steady state solution is $y = 2t^2 - 6t + 8$.

Second-order Differential Equations

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Also describe the steady state solution and calculate the Wronskian for the system.

Wronskian Formula

The formula for calculating the Wronskian of y with $y_h = Ay_1 + By_2$ is:

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

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Also describe the steady state solution and calculate the Wronskian for the system.

In our case, $y_1 = e^{-t}$ and $y_2 = e^{-2t}$.

$$W(t) = \begin{vmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{vmatrix}$$

$$\therefore W(t) = (e^{-t})(-2e^{-2t}) - (e^{-2t})(-e^{-}t) = -e^{-3t}$$

Forced Oscillations and Resonance

- Every object in the world around us has an infinite set of natural frequencies that it can freely oscillate (vibrate) at.
- Forced oscillations occur when a periodic force is applied to a given system.
- Damping is the processing of removing energy from a system and consequentially reduces the amplitude of oscillation.

Forced Oscillations and Resonance

Example

A freely vibrating system is represented by

$$y'' + cy' + 4y = 0$$

What damping constants c produce overdamping, critical damping, underdamping and no damping.

Forced Oscillations and Resonance

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Types of Damping

For an oscillating mass-spring system with damping ($c \geq 0$)

$$my'' + cy' + ky = 0$$

Overdamping: $c^2 > 4mk \implies c > 2\sqrt{mk}$
Critical Damping: $c^2 = 4mk \implies c = 2\sqrt{mk}$
Underdamping: $0 < c^2 < 4mk \implies 0 < c < 2\sqrt{mk}$
No damping: $c = 0$

Forced Oscillations and Resonance

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What damping constants c produce overdamping, critical damping, underdamping and no damping.

For the system above, $2\sqrt{mk} = 2\sqrt{(1)(4)} = 4$.

Overdamping: $c > 4$
Critical Damping: $c = 4$
Underdamping: $0 < c < 4$
No damping: $c = 0$

Forced Oscillations and Resonance

Example

A forced vibrating system is represented by

$$y'' + 5y' + 4y = 6\sin(2t)$$

where $6\sin(2t)$ is the driving force and y is the displacement from the equilibrium position. Describe the y displacement of the system over time.

Forced Oscillations and Resonance

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A forced vibrating system is represented by

$$y'' \ + \ 5y' \ + \ 4y = 6\sin(2t)$$

where $6\sin(2t)$ is the driving force and y is the displacement from the equilibrium position. Describe the y displacement of the system over time.

The characteristic equation for the homogeneous solution is $\lambda^2 + 5\lambda + 4 = 0$, so $\lambda_1 = -1$ and $\lambda_2 = -4$. For real constants A and B:

$$y_h = Ae^{-t} + Be^{-4t}$$

Forced Oscillations and Resonance

Example

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$$y'' + 5y' + 4y = 6\sin(2t)$$

where $6\sin(2t)$ is the driving force and y is the displacement from the equilibrium position. Describe the y displacement of the system over time.

A particular solution can be assumed to be $y_p = \alpha\cos(2t) + \beta\sin(2t)$.

$$\begin{aligned} y_p &= \alpha\cos(2t) + \beta\sin(2t) \\ y_p' &= -2\alpha\sin(2t) + 2\beta\cos(2t) \\ y_p'' &= -4\alpha\cos(2t) - 4\beta\sin(2t) = -4y_p \end{aligned}$$

Forced Oscillations and Resonance

Example

A forced vibrating system is represented by

$$y'' + 5y' + 4y = 6\sin(2t)$$

where $6\sin(2t)$ is the driving force and y is the displacement from the equilibrium position. Describe the y displacement of the system over time.

A particular solution can be assumed to be $y_p = \alpha\cos(2t) + \beta\sin(2t)$.

$$(-4\alpha\cos(2t) - 4\beta\sin(2t)) + 5(-2\alpha\sin(2t) + 2\beta\cos(2t)) + 4(\alpha\cos(2t) + \beta\sin(2t)) = 6\sin(2t)$$

$$-10\alpha\sin(2t) + 10\beta\cos(2t) = 6\sin(2t)$$

$$\alpha = -\frac{3}{5}, \beta = 0$$

$$\therefore\ y_g = Ae^{-t} + Be^{-4t} - \frac{3}{5}\cos(2t)$$

Forced Oscillations and Resonance

Example

Now, find the motion of the system corresponding to the following initial displacement and velocity $y(0) = 1$, $y'(0) = 0$. Then find the steady state oscillations (i.e., the response of the system after a sufficiently long time).

Forced Oscillations and Resonance

Example

Now, find the motion of the system corresponding to the following initial displacement and velocity $y(0) = 1$, $y'(0) = 0$. Then find the steady state oscillations (i.e., the response of the system after a sufficiently long time).

$$\text{For, } y = Ae^{-t} + Be^{-4t} - \frac{3}{5}\cos(2t)$$
$$y' = -Ae^{-t} - 4Be^{-4t} + \frac{6}{5}\sin(2t)$$

$$y(0) = 1 = A + B - \frac{3}{5}$$
$$y'(0) = 0 = -A - 4B$$

Thus $A = \frac{32}{15}$, $B = -\frac{8}{15}$.

Example

Now, find the motion of the system corresponding to the following initial displacement and velocity $y(0) = 1$, $y'(0) = 0$. Then find the steady state oscillations (i.e., the response of the system after a sufficiently long time).

Therefore, the solution is:

$$y = \frac{32}{15}e^{-t} - \frac{8}{15}Be^{-4t} - \frac{3}{5}\cos(2t)$$

Then, as $t \to \infty$ $y = -\frac{3}{5}\cos(2t)$

Forced Oscillations and Resonance

Example

Considering $y'' + 4y = 8cos(2\pi\omega t)$, which value(s) of ω will the system achieve resonance?

The characteristic equation gives $\lambda = 2i$.
So, $y_h = A\cos(2t) + B\sin(2t)$ for A,B$\in \mathbb{R}$
Resonance is achieved when a forcing frequency matches a natural frequency.

$$2\pi\omega = 2 \implies \omega = \frac{1}{\pi}\ rads^{-1}$$

6. Laplace Transforms

Laplace Transforms

- Laplace transforms provide a mathematical tool to solve linear ODEs and systems of linear ODEs.
- Briefly speaking, the given ODE is transformed into an algebraic equation, which can be solved by algebraic operations.
- The solution to this latter equation is the transformed back to yield the solution to the ODE.

The Heaviside Function

Example

Use the integral definition to prove that

$$\mathcal{L}(u(t-a)) = \frac{e^{-as}}{s}$$

where $u(t-a)$ is the Heaviside step function.

The Heaviside Function

Example

Use the integral definition to prove that

$$\mathcal{L}(u(t-a)) = \frac{e^{-as}}{s}$$

where $u(t-a)$ is the Heaviside step function.

Integral Definition

Recall the integral definition of the Laplace transform of a function $f(t)$.

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt = F(s)$$

Example

Use the integral definition to prove that

$$\mathcal{L}(u(t-a)) = \frac{e^{-as}}{s}$$

where $u(t-a)$ is the Heaviside step function.

$$\begin{aligned}
\mathcal{L}(u(t-a)) &= \int_0^\infty e^{-st} u(t-a)dt \\
&= \int_0^a e^{-st} u(t-a)dt + \int_a^\infty e^{-st} u(t-a)dt \\
&= \int_0^a e^{-st} u(t-a)dt + \lim_{b\to\infty} \int_a^b e^{-st} u(t-a)dt
\end{aligned}$$

The Heaviside Function

Example

Use the integral definition to prove that

$$\mathcal{L}(u(t-a)) = \frac{e^{-as}}{s}$$

where $u(t-a)$ is the Heaviside step function.

The Heaviside function is defined as

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a. \end{cases}$$

$$\text{So, } \mathcal{L}(u(t-a)) = \int_0^a e^{-st} u(t-a) dt + \lim_{b \to \infty} \int_a^b e^{-st} u(t-a) dt$$

$$= \int_0^a e^{-st}(0) dt + \lim_{b \to \infty} \int_a^b e^{-st}(1) dt = \boxed{\lim_{b \to \infty} \int_a^b e^{-st} dt}$$

The Heaviside Function

Example

Use the integral definition to prove that

$$\mathcal{L}(u(t-a)) = \frac{e^{-as}}{s}$$

where $u(t-a)$ is the Heaviside step function.

$$\begin{aligned}
\lim_{b\to\infty}\int_a^b e^{-st}dt &= \lim_{b\to\infty}\left[-\frac{e^{-st}}{s}\right]_a^b \\
&= \lim_{b\to\infty}\left(-\frac{e^{-bs}}{s}+\frac{e^{-as}}{s}\right) \\
&= \left(-\frac{0}{s}+\frac{e^{-as}}{s}\right) = \boxed{\frac{e^{-as}}{s}}
\end{aligned}$$

Shifting Theorems

Example

Find $\mathcal{L}(t^2u(t-3))$.

First we need to observe that there is a cut at $t = 3$ but no shift currently present. We will need to introduce a shift so that the appropriate shifting theorem can be used.

$$
\begin{aligned}
t^2 &= a(t-3)^2 + b(t-3) + c \\
&= at^2 + (-6a + b)t + (9a + -3b + c)
\end{aligned}
$$

$$\text{Thus, } t^2 = (t-3)^2 + 6(t-3) + 9$$

Shifting Theorems

Example

Find $\mathcal{L}(t^2u(t-3))$.

Using this and applying the linearity of the Laplace transform:

$$\begin{aligned}\mathcal{L}(t^2u(t-3)) &= \mathcal{L}\left(((t-3)^2+6(t-3)+9)u(t-3)\right) \\ &= \mathcal{L}\left((t-3)^2u(t-3)+6(t-3)u(t-3)+9u(t-3)\right) \\ &= \mathcal{L}\left((t-3)^2u(t-3)\right)+\mathcal{L}\left(6(t-3)u(t-3)\right)+\mathcal{L}\left(9u(t-3)\right)\end{aligned}$$

Laplace Transform for Positive Powers of t

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

Second Shifting Theorem

$$\mathcal{L}(f(t-a)u(t-a)) = e^{-as}F(s)$$

Example

Find $\mathcal{L}(t^2u(t-3))$.

Using this transformation of t^2 and applying the linearity of the Laplace transform:

$$\begin{aligned}
\mathcal{L}(t^2u(t-3)) &= \mathcal{L}\left(((t-3)^2+6(t-3)+9)u(t-3)\right) \\
&= \mathcal{L}\left((t-3)^2u(t-3)+6(t-3)u(t-3)+9u(t-3)\right) \\
&= \mathcal{L}\left((t-3)^2u(t-3)\right)+\mathcal{L}\left(6(t-3)u(t-3)\right)+\mathcal{L}\left(9u(t-3)\right) \\
&= \mathcal{L}\left((t-3)^2u(t-3)\right)+6\mathcal{L}\left((t-3)u(t-3)\right)+9\mathcal{L}\left(u(t-3)\right) \\
&= e^{-3s}\left(\frac{2!}{s^{2+1}}\right)+6e^{-3s}\left(\frac{1!}{s^{1+1}}\right)+9e^{-3s}\left(\frac{1}{s}\right) \\
&= e^{-3s}\left(\frac{2}{s^3}+\frac{6}{s^2}+\frac{9}{s}\right)
\end{aligned}$$

Inverse Laplace Transform

Example

Find $\mathcal{L}^{-1}\left(\dfrac{7s+1}{(s+1)(s-1)}e^{-5s}\right)$.

Inverse Laplace Transform

Example

Find $\mathcal{L}^{-1}\left(\frac{7s+1}{(s+1)(s-1)}e^{-5s}\right)$.

We will begin to solve this question by first turning $\frac{7s+1}{(s+1)(s-1)}$ into partial fractions.

$$(7s+1) = a(s-1) + b(s+1)$$

$$s = 1 : 9 = 2a \implies a = \frac{9}{2}$$
$$s = -1 : -6 = -2a \implies b = 3$$

$$\therefore \frac{7s+1}{(s+1)(s-1)} = \frac{3}{s+1} + \frac{9}{2(s-1)}$$

Inverse Laplace Transform

Example

Find $\mathcal{L}^{-1}\left(\dfrac{7s+1}{(s+1)(s-1)}e^{-5s}\right)$.

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{7s+1}{(s+1)(s-1)}e^{-5s}\right) &= \mathcal{L}^{-1}\left(\left(\frac{3}{s+1}+\frac{9}{2(s-1)}\right)e^{-5s}\right)\\
&= \mathcal{L}^{-1}\left(\frac{3}{s+1}e^{-5s}+\frac{9}{2(s-1)}e^{-5s}\right)\\
&= \mathcal{L}^{-1}\left(\frac{3}{s+1}e^{-5s}\right)+\mathcal{L}^{-1}\left(\frac{9}{2(s-1)}e^{-5s}\right)\\
&= \mathcal{L}^{-1}\left(\frac{3}{s+1}e^{-5s}\right)+\mathcal{L}^{-1}\left(\frac{9}{2(s-1)}e^{-5s}\right)\\
&= 3\mathcal{L}^{-1}\left(\frac{1}{s+1}e^{-5s}\right)+\frac{9}{2}\mathcal{L}^{-1}\left(\frac{1}{s-1}e^{-5s}\right)
\end{aligned}$$

Inverse Laplace Transform

Example

Find $\mathcal{L}^{-1}\left(\dfrac{7s+1}{(s+1)(s-1)}e^{-5s}\right)$.

Noticing the e^{-5s} term in each inverse Laplace transform we can see that this question is potentially suitable for application of the Second Shifting Theorem. To ensure this, first we need to manipulate each expression to introduce an $(s-5)$ term.

$$\frac{1}{s+1} = \frac{1}{(s-5)+6}$$
$$\frac{1}{s-1} = \frac{1}{(s-5)+4}$$

Inverse Laplace Transform

Example

Find $\mathcal{L}^{-1}\left(\dfrac{7s+1}{(s+1)(s-1)}e^{-5s}\right)$.

Relevant Inverse Laplace Transform Formula

$\mathcal{L}^{-1}\left(\dfrac{1}{s-a}\right) = e^{at}$

$$\begin{aligned}3\mathcal{L}^{-1}\left(\frac{1}{s+1}e^{-5s}\right) &= 3\mathcal{L}^{-1}\left(\frac{1}{(s-5)+6}e^{-5s}\right)\\ &= 3\mathcal{L}^{-1}\left(\frac{1}{(s-5)-(-6)}e^{-5s}\right)\\ &= 3e^{-6t}u(t-5)\end{aligned}$$

Inverse Laplace Transform

Example

Find $\mathcal{L}^{-1}\left(\dfrac{7s+1}{(s+1)(s-1)}e^{-5s}\right)$.

Relevant Inverse Laplace Transform Formula

$$\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at}$$

$$\begin{aligned}
\frac{9}{2}\mathcal{L}^{-1}\left(\frac{1}{s-1}e^{-5s}\right) &= \frac{9}{2}\mathcal{L}^{-1}\left(\frac{1}{(s-5)+4}e^{-5s}\right) \\
&= \frac{9}{2}\mathcal{L}^{-1}\left(\frac{1}{(s-5)-(-4)}e^{-5s}\right) \\
&= \frac{9}{2}e^{-4t}u(t-5)
\end{aligned}$$

Inverse Laplace Transform

Example

Find $\mathcal{L}^{-1}\left(\dfrac{7s+1}{(s+1)(s-1)}e^{-5s}\right)$.

$$\therefore \mathcal{L}^{-1}\left(\frac{7s+1}{(s+1)(s-1)}e^{-5s}\right) = 3e^{-6t}u(t-5) + \frac{9}{2}e^{-4t}u(t-5)$$

Laplace Transform and Systems of ODEs

Example

Consider the system of differential equations

$$\frac{dx}{dt} + 4x + 10y = 0$$
$$\frac{dy}{dt} - 5x - 11y = 0$$

where $x(0) = -1$ and $y(0) = 0$.

Laplace Transform and Systems of ODEs

Example

Consider the system of differential equations

$$\frac{dx}{dt} + 4x + 10y = 0, \quad \frac{dy}{dt} - 5x - 11y = 0$$

where $x(0) = -1$ and $y(0) = 0$.

We will begin by taking the Laplace transform of each side of the equations, one equation at a time.

$$\begin{aligned} \mathcal{L}\left(\frac{dx}{dt} + 4x + 10y\right) &= \mathcal{L}(0) \\ \mathcal{L}\left(\frac{dx}{dt}\right) + 4\mathcal{L}(x) + 10\mathcal{L}(y) &= \mathcal{L}(0) \\ \mathcal{L}(x') + 4\mathcal{L}(x) + 10\mathcal{L}(y) &= \mathcal{L}(0) \end{aligned}$$

Laplace Transform and Systems of ODEs

Example

Consider the system of differential equations

$$\frac{dx}{dt} + 4x + 10y = 0, \quad \frac{dy}{dt} - 5x - 11y = 0$$

where $x(0) = -1$ and $y(0) = 0$.

$$\mathcal{L}(x') + 4\mathcal{L}(x) + 10\mathcal{L}(y) = \mathcal{L}(0)$$

Laplace transform of Derivatives

$$\mathcal{L}(f^n(t)) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - ... - s f^{n-2}(0) - f^{n-1}(0)$$

$$(sX - x(0)) + 4X + 10Y = 0$$

Laplace Transform and Systems of ODEs

Example

Consider the system of differential equations

$$\frac{dx}{dt} + 4x + 10y = 0, \quad \frac{dy}{dt} - 5x - 11y = 0$$

where $x(0) = -1$ and $y(0) = 0$.

$$\begin{aligned} (sX - x(0)) + 4X + 10Y &= 0 \\ (sX - (-1)) + 4X + 10Y &= 0 \\ sX + 1 + 4X + 10Y &= 0 \\ X(s+4) + 10Y &= -1 \end{aligned} \tag{1}$$

Laplace Transform and Systems of ODEs

Example

Consider the system of differential equations

$$\frac{dx}{dt} + 4x + 10y = 0, \quad \frac{dy}{dt} - 5x - 11y = 0$$

where $x(0) = -1$ and $y(0) = 0$.

Similarly we can obtain an expression for the second ODE.

$$\mathcal{L}(y') - 5\mathcal{L}(x) - 11\mathcal{L}(y) = \mathcal{L}(0)$$

Laplace transform of Derivatives

$\mathcal{L}(f^n(t)) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - ... - s f^{n-2}(0) - f^{n-1}(0)$

$$(sY - y(0)) - 5X - 11Y = 0$$

Laplace Transform and Systems of ODEs

Example

Consider the system of differential equations

$$\frac{dx}{dt} + 4x + 10y = 0, \quad \frac{dy}{dt} - 5x - 11y = 0$$

where $x(0) = -1$ and $y(0) = 0$.

$$\begin{aligned}
(sY - y(0)) - 5X - 11Y &= 0 \\
(sY - 0) - 5X - 11Y &= 0 \\
sY + 0 - 5X - 11Y &= 0 \\
Y(s - 11) - 5X &= 0 \\
Y &= \frac{5X}{s - 11} \qquad (2)
\end{aligned}$$

Laplace Transform and Systems of ODEs

Example

Consider the system of differential equations

$$\frac{dx}{dt} + 4x + 10y = 0, \quad \frac{dy}{dt} - 5x - 11y = 0$$

where $x(0) = -1$ and $y(0) = 0$.

Substituting (2) into (1) an expression for X can be found.

$$\begin{aligned} X(s+4) + 10\left(\frac{5X}{s-11}\right) &= -1 \\ X(s+4)(s-11) + 50X &= -(s-11) \\ X(s^2 - 7s + 6) &= 11 - s \\ X(s-6)(s-1) &= 11 - s \\ X = \frac{11-s}{(s-6)(s-1)} & \qquad (3) \end{aligned}$$

Laplace Transform and Systems of ODEs

Example

Consider the system of differential equations

$$\frac{dx}{dt} + 4x + 10y = 0, \quad \frac{dy}{dt} - 5x - 11y = 0$$

where $x(0) = -1$ and $y(0) = 0$.

Applying partial fractions to $X = \dfrac{11 - s}{(s-6)(s-1)}$:

$$11 - s = a(s-1) + b(s-6)$$

$$s = 1 : \; 10 = -5b \implies b = -2$$
$$s = 6 : \; 5 = 5a \implies a = 1$$

$$X = \frac{1}{s-6} + \frac{-2}{s-1}$$

Laplace Transform and Systems of ODEs

Example

Consider the system of differential equations

$$\frac{dx}{dt} + 4x + 10y = 0, \quad \frac{dy}{dt} - 5x - 11y = 0$$

where $x(0) = -1$ and $y(0) = 0$.

Applying the inverse Laplace transform:

$$\mathcal{L}^{-1}(X) = \mathcal{L}^{-1}\left(\frac{1}{s-6} + \frac{-2}{s-1}\right)$$
$$\mathcal{L}^{-1}(X) = \mathcal{L}^{-1}\left(\frac{1}{s-6}\right) + \mathcal{L}^{-1}\left(\frac{-2}{s-1}\right)$$
$$\mathcal{L}^{-1}(X) = \mathcal{L}^{-1}\left(\frac{1}{s-6}\right) - 2\mathcal{L}^{-1}\left(\frac{1}{s-1}\right)$$
$$\therefore x = e^{6t} - 2e^{t}$$

Laplace Transform and Systems of ODEs

Example

Consider the system of differential equations

$$\frac{dx}{dt} + 4x + 10y = 0, \quad \frac{dy}{dt} - 5x - 11y = 0$$

where $x(0) = -1$ and $y(0) = 0$.

Substituting (3) back into (2) an expression for Y can be found:

$$Y = \frac{5X}{s-11}$$

$$Y = \frac{5\left(\frac{11-s}{(s-6)(s-1)}\right)}{s-11}$$

$$Y = \frac{-5}{(s-6)(s-1)}$$

Laplace Transform and Systems of ODEs

Example

Consider the system of differential equations

$$\frac{dx}{dt} + 4x + 10y = 0, \quad \frac{dy}{dt} - 5x - 11y = 0$$

where $x(0) = -1$ and $y(0) = 0$.

Applying partial fractions to $Y = \dfrac{-5}{(s-6)(s-1)}$:

$$-5 = c(s-1) + d(s-6)$$

$$s = 1 : \;\; -5 = -5d \implies d = 1$$

$$s = 6 : \;\; -5 = 5c \implies c = -1$$

$$Y = \frac{-1}{s-6} + \frac{1}{s-1}$$

Laplace Transform and Systems of ODEs

Example

Consider the system of differential equations

$$\frac{dx}{dt} + 4x + 10y = 0, \quad \frac{dy}{dt} - 5x - 11y = 0$$

where $x(0) = -1$ and $y(0) = 0$.

Applying the inverse Laplace transform:

$$\mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left(\frac{-1}{s-6} + \frac{1}{s-1}\right)$$

$$\mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left(\frac{-1}{s-6}\right) + \left(\frac{1}{s-1}\right)$$

$$\mathcal{L}^{-1}(Y) = -\mathcal{L}^{-1}\left(\frac{1}{s-6}\right) + \left(\frac{1}{s-1}\right)$$

$$\therefore y = -e^{6t} + e^{t}$$

Laplace Transform and Systems of ODEs

Example

Consider the system of differential equations

$$\frac{dx}{dt} + 4x + 10y = 0, \quad \frac{dy}{dt} - 5x - 11y = 0$$

where $x(0) = -1$ and $y(0) = 0$.

Our solution can easily be checked by using $x' = 6e^{6t} - 2e^t$ and $y' = -6e^{6t} + e^t$, as well as the initial conditions.

$$x(0) = e^{6(0)} - 2e^0 = 1 - 2 = -1$$
$$y(0) = -e^{6(0)} + e^0 = -1 + 1 = 0$$

$$(6e^{6t} - 2e^t) + 4(e^{6t} - 2e^t) + 10(-e^{6t} + e^t) = 0$$
$$(-6e^{6t} + e^t) - 5(e^{6t} - 2e^t) - 11(-e^{6t} + e^t) = 0$$

7. Fourier Series

Fourier Series

- Many applications in engineering involve phenomena that have a cyclical or periodic behaviour.
- The Fourier series provides us with a tool to analyse periodic phenomena by transforming periodic signals into a series (or infinite sum) of sines and cosines.

Fourier Series with Arbitrary Period

Example

Prove that

$$\frac{1}{L}\int_{-L}^{L}(f(x))^2dx = 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

[This is known as Parseval's identity.]

Fourier Series with Arbitrary Period

Example

Prove that

$$\frac{1}{L}\int_{-L}^{L}(f(x))^2 dx = 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

[This is known as Parseval's identity.]

General form of a Fourier Series

For f with period $2L$,

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right)$$

$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx,$
$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx,$
$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx.$

Fourier Series with Arbitrary Period

Example

Prove that

$$\frac{1}{L}\int_{-L}^{L}(f(x))^2dx = 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

[This is known as Parseval's identity.]

$$\begin{aligned}
\frac{1}{L}\int_{-L}^{L}(f(x))^2dx &= \frac{1}{L}\int_{-L}^{L} f(x)f(x)dx \\
&= \frac{1}{L}\int_{-L}^{L} f(x)\left(a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi}{L}x\right) + b_n\sin\left(\frac{n\pi}{L}x\right)\right)\right)dx \\
&= a_0\frac{1}{L}\int_{-L}^{L} f(x) + \frac{1}{L}\int_{-L}^{L} f(x)\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi}{L}x\right) + \frac{1}{L}\int_{-L}^{L} f(x)b_n\sin\left(\frac{n\pi}{L}x\right)dx
\end{aligned}$$

We will now consider each term one at a time (left to right).

Fourier Series with Arbitrary Period

Example

Prove that

$$\frac{1}{L}\int_{-L}^{L}(f(x))^2dx = 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

[This is known as Parseval's identity.]

For the first term:

$$\begin{aligned} a_0\frac{1}{L}\int_{-L}^{L}f(x) &= 2a_0\frac{1}{2L}\int_{-L}^{L}f(x) \\ &= 2a_0(a_0) \\ &= 2a_0^2 \end{aligned}$$

Fourier Series with Arbitrary Period

Example

Prove that

$$\frac{1}{L}\int_{-L}^{L}(f(x))^2dx = 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

[This is known as Parseval's identity.]

For the third term:

$$\begin{aligned}\frac{1}{L}\int_{-L}^{L} f(x)\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) &= \sum_{n=1}^{\infty} a_n \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)\\ &= \sum_{n=1}^{\infty} a_n(a_n)\\ &= \sum_{n=1}^{\infty} a_n^2\end{aligned}$$

Example

Prove that

$$\frac{1}{L}\int_{-L}^{L}(f(x))^2dx = 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

[This is known as Parseval's identity.]

For the third term:

$$\begin{aligned}\frac{1}{L}\int_{-L}^{L} f(x)\sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi}{L}x\right) &= \sum_{n=1}^{\infty} b_n \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right) \\ &= \sum_{n=1}^{\infty} b_n(b_n) \\ &= \sum_{n=1}^{\infty} b_n^2\end{aligned}$$

Fourier Series with Arbitrary Period

Example

Prove that

$$\frac{1}{L}\int_{-L}^{L}(f(x))^2dx = 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

[This is known as Parseval's identity.]

Bringing the terms back together:

$$\begin{aligned}\frac{1}{L}\int_{-L}^{L}(f(x))^2dx &= 2a_0^2 + \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2 \\ &= 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\end{aligned}$$

Periodic Extensions

3 Types of Periodic Extensions

1. Periodic extension: Shift f by L. In this way we obtain a periodic function of period L.
2. Even periodic extension: Reflect f over the y-axis, then shift by 2L. In this way we obtain an even periodic function of period 2L.
3. Odd periodic extension: Reflect f over the origin, then shift by 2L. In this way we obtain an odd periodic function of period 2L.

Half-Range Expansions

Example

Define the piece wise continuous function f by

$$f(x) = \begin{cases} 1, & 0 \leq x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \leq x < \pi. \end{cases}$$

Sketch the even periodic extension of f on the interval $-2\pi \leq x \leq 2\pi$.

Half-Range Expansions

Example

Define the piece wise continuous function f by

$$f(x) = \begin{cases} 1, & 0 \le x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \le x < \pi. \end{cases}$$

Sketch the even periodic extension of f on the interval $-2\pi \le x \le 2\pi$.

Half-Range Expansions

Example

Define the piece wise continuous function f by

$$f(x) = \begin{cases} 1, & 0 \leq x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \leq x < \pi. \end{cases}$$

Calculate the Fourier cosine series of f.

Half-Range Expansions

Example

Define the piece wise continuous function f by

$$f(x) = \begin{cases} 1, & 0 \leq x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \leq x < \pi. \end{cases}$$

Calculate the Fourier cosine series of f.

By construction, the Fourier series we're looking for is even. This means that b_n will be 0.

Also, from our sketch in the previous question, we know that the period of this Fourier series is $2\pi = 2L$.

So, all we need to calculate are a_0 and a_n!

Half-Range Expansions

Example

Define the piece wise continuous function f by

$$f(x) = \begin{cases} 1, & 0 \leq x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \leq x < \pi. \end{cases}$$

Calculate the Fourier cosine series of f.

First up is a_0:

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx \\ &= \frac{2}{2\pi} \int_{0}^{\pi} f(x)dx \\ &= \frac{1}{\pi} \left(\int_{0}^{\frac{\pi}{2}} f(x)dx + \int_{\frac{\pi}{2}}^{\pi} f(x)dx \right) \end{aligned}$$

Half-Range Expansions

Example

Define the piece wise continuous function f by

$$f(x) = \begin{cases} 1, & 0 \leq x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \leq x < \pi. \end{cases}$$

Calculate the Fourier cosine series of f.

$$\begin{aligned} &= \frac{1}{\pi}\left(\int_0^{\frac{\pi}{2}} (1) dx + \int_{\frac{\pi}{2}}^{\pi} (0) dx\right) \\ &= \frac{1}{\pi}[1]_0^{\frac{\pi}{2}} \\ &= \frac{1}{\pi}\left(\frac{\pi}{2} - 0\right) \\ \therefore a_0 &= \frac{1}{2} \end{aligned}$$

Half-Range Expansions

Example

Define the piece wise continuous function f by

$$f(x) = \begin{cases} 1, & 0 \leq x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \leq x < \pi. \end{cases}$$

Calculate the Fourier cosine series of f.

Next up is a_n:

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos\left(\frac{n\pi}{\pi} x\right) dx \\ &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) dx \\ &= \frac{2}{\pi} \left(\int_0^{\frac{\pi}{2}} f(x) \cos(nx) dx + \int_{\frac{\pi}{2}}^{\pi} f(x) \cos(nx) dx \right) \end{aligned}$$

Half-Range Expansions

Example

Define the piece wise continuous function f by

$$f(x) = \begin{cases} 1, & 0 \leq x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \leq x < \pi. \end{cases}$$

Calculate the Fourier cosine series of f.

Substituting in the relevant values for $f(x)$ in each range:

$$\begin{aligned} &= \frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}} (1)\cos(nx)dx + \int_{\frac{\pi}{2}}^{\pi} (0)\cos(nx)dx\right) \\ &= \frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}} \cos(nx)dx\right) \\ &= \frac{2}{\pi}\left[\frac{\sin(nx)}{n}\right]_0^{\frac{\pi}{2}} \end{aligned}$$

Half-Range Expansions

Example

Define the piece wise continuous function f by

$$f(x) = \begin{cases} 1, & 0 \leq x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \leq x < \pi. \end{cases}$$

Calculate the Fourier cosine series of f.

Substituting in the relevant values for $f(x)$ in each range:

$$\begin{aligned} \frac{2}{\pi}\left[\frac{\sin(nx)}{n}\right]_0^{\frac{\pi}{2}} &= \frac{2}{n\pi}\left(\sin\left(n\left(\frac{\pi}{2}\right)\right) - \sin(n(0))\right) \\ &= \frac{2}{n\pi}\left(\sin\left(\frac{n\pi}{2}\right) - \sin(0)\right) \\ &= \frac{2}{\pi(2k+1)}((-1)^k - 0) \end{aligned}$$

Half-Range Expansions

Example

Define the piece wise continuous function f by

$$f(x) = \begin{cases} 1, & 0 \leq x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \leq x < \pi. \end{cases}$$

Calculate the Fourier cosine series of f.

Why is this the case?

$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & \text{for } n = 0, 2, 4, 6, \ldots \\ 1, & \text{for } n = 1, 5, 9, 13, \ldots \\ -1, & \text{for } n = 3, 7, 11, 15, \ldots \end{cases}$$

Thus, $a_{2k+1} = \frac{2(-1)^k}{\pi(2k+1)}$

Half-Range Expansions

Example

Define the piece wise continuous function f by

$$f(x) = \begin{cases} 1, & 0 \leq x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \leq x < \pi. \end{cases}$$

Calculate the Fourier cosine series of f.

Hence our Fourier cosine series of f is

$$\frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)} \cos((2k+1)x)$$

Half-Range Expansions

Example

Define the piece wise continuous function f by

$$f(x) = \begin{cases} 1, & 0 \leq x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \leq x < \pi. \end{cases}$$

To what value will the Fourier cosine series converge at $x = \frac{\pi}{2}$?

Half-Range Expansions

Example

Define the piece wise continuous function f by

$$f(x) = \begin{cases} 1, & 0 \leq x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \leq x < \pi. \end{cases}$$

To what value will the Fourier cosine series converge at $x = \frac{\pi}{2}$?

At $x = \frac{\pi}{2}$ we can see that the argument of the cosine will have an odd multiple of $\frac{\pi}{2}$ for every term in the infinite sum within the Fourier series. Thus, for all $k \in \mathbb{Z}$, $\cos\left((2k+1)\left(\frac{\pi}{2}\right)\right) = 0$.

Hence the Fourier cosine series will converge to $\frac{1}{2}\left(a_0\right)$ at $x = \frac{\pi}{2}$.
(or just look at the graph)

8. Partial Differential Equations

Partial Differential Equations

- Most problems in fluid and solid mechanics, quantum mechanics, electromagnetism, and other area of physics can be modelled as partial differential equations (PDEs).
- There are many PDEs that cannot be solved exactly but can be solved approximately by numerical methods.

The Wave Equation

Example

Consider the one-dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2}$$

where $u(x,t)$ is the displacement at position x and time t.
D'Alembert's solution to this wave equation is

$$u(x,t) = \phi(x+3t) + \psi(x-3t)$$

for arbitrary functions ϕ and ψ. If the initial displacement of the wave is $u(x,0) = g(x)$ and the initial velocity is $u_t(x,0) = 0$, prove that

$$u(x,t) = \frac{1}{2}[g(x+3t) + g(x-3t)]$$

.

The Wave Equation

Example

Prove that

$$u(x,t) = \frac{1}{2}[g(x+3t) + g(x-3t)]$$

.

Let $v = x + 3t$ and $w = x - 3t$ such that $u = \phi(v(x,t)) + \psi(w(x,t))$.
Using the chain rule

$$\frac{\partial u}{\partial t} = \frac{d\phi}{dv}\frac{\partial v}{\partial t} + \frac{d\psi}{dw}\frac{\partial w}{\partial t} = 3\phi'(v) - 3\psi'(w).$$

Applying the initial conditions yields

$$u(x,0) = g(x) = \phi(x) + \psi(x)$$
$$\frac{\partial u}{\partial t}(x,0) = 0 = 3\phi'(v) - 3\psi'(w)$$

The Wave Equation

Example

Prove that

$$u(x,t) = \frac{1}{2}[g(x+3t) + g(x-3t)]$$

.

Integrating the second equation w.r.t x:

$$\int 3\phi'(v) - 3\psi'(w)\ dx = \int 0\ dx$$

$$\phi(v) - \psi(w) = C, \quad \text{where } C \text{ is a constant}$$

Adding this to $g(x) = \phi(x) + \psi(x)$

$$\phi(x) = \frac{1}{2}(g(x) + C)$$

The Wave Equation

Example

Prove that

$$u(x,t) = \frac{1}{2}[g(x+3t) + g(x-3t)]$$

.

$\phi(x) = \frac{1}{2}(g(x) + C)$ means that

$$\psi(x) = \frac{1}{2}(g(x) - C)$$

$$\begin{aligned} \text{Hence,} \quad u &= \phi(v) + \psi(w) \\ &= \frac{1}{2}(g(v) + C) + \frac{1}{2}(g(w) - C) \\ &= \frac{1}{2}[g(x+3t) + g(x-3t)] \end{aligned}$$

The Heat Equation

Example

The temperature in a bar of length π metres satisfies the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

where $u(x,t)$ is the temperature in $^\circ C$, t is the time in seconds and x is the distance in metres from the left hand side of the bar. Both ends of the bar are maintained at a temperature of $0^\circ C$. Hence

$$u(0,t) = u(\pi,t) = 0 \text{ for all } t.$$

Assuming a solution of the form $u(x,t) = F(x)G(t)$ show that

$$\frac{1}{G}\frac{dG}{dt} = \frac{1}{F}\frac{d^2F}{dx^2} = k \quad \text{where k is a constant}$$

The Heat Equation

Example

Assuming a solution of the form $u(x,t) = F(x)G(t)$ show that

$$\frac{1}{G}\frac{dG}{dt} = \frac{1}{F}\frac{d^2F}{dx^2} = k \quad \text{where k is a constant}$$

$$\begin{aligned}
\frac{\partial}{\partial t}(F(x)G(t)) &= \frac{\partial^2}{\partial x^2}(F(x)G(y)) \\
F(x)\frac{dG}{dt} &= \frac{d^2F}{dx^2}G(t) \\
\frac{1}{G(t)}\frac{dG}{dt} &= \frac{1}{F(x)}\frac{d^2F}{dx^2} = k
\end{aligned}$$

Notice the LHS is only a function of t whereas the RHS is only a function of x. Therefore, for equality to hold, both sides must be equal to a *separation constant* k.

The Heat Equation

Example

Now assuming that only $k < 0$ (what about if this is not assumed?) yields non-trivial solutions and set $k = -p^2$ for some $p > 0$. Applying the initial conditions show that $p = n,\ n = 1, 2, 3, \ldots$ and that possible solutions for $F(x)$ are

$$F_n(x) = b_n \sin(nx)$$

where b_n are constants and $n = 1, 2, 3, \ldots$.

From the previous slide we can see that

$$F''(x) - kF(x) = 0$$

Letting $k = -p^2 < 0$

$$\begin{aligned} F''(x) + p^2 F(x) &= 0 \\ \implies F(x) &= a\cos(px) + b\sin(px) \end{aligned}$$

We can see this using our knowledge of ODEs from earlier!

The Heat Equation

Example

Now assuming that only $k < 0$ yields non-trivial solutions and set $k = -p^2$ for some $p > 0$. Applying the initial conditions show that $p = n, \ n = 1, 2, 3, ...$ and that possible solutions for $F(x)$ are

$$F_n(x) = b_n \sin(nx)$$

where b_n are constants and $n = 1, 2, 3, ...$

Applying the boundary conditions:

$$\begin{aligned} u(0,t) = \ 0 \ &= F(0)G(t) \\ u(\pi,t) = \ 0 \ &= F(\pi)G(t) \end{aligned}$$

If $G(t) = 0$ for $t > 0$ then $u(x,t) = 0$ which is a trivial solution. To obtain a non-trivial solution instead consider $F(0) = F(\pi) = 0$.

The Heat Equation

Example

Now assuming that only $k < 0$ yields non-trivial solutions and set $k = -p^2$ for some $p > 0$. Applying the initial conditions show that $p = n,\ n = 1, 2, 3, ...$ and that possible solutions for $F(x)$ are

$$F_n(x) = b_n \sin(nx)$$

where b_n are constants and $n = 1, 2, 3, ...$

Using this result obtained from the boundary conditions:

$$\begin{aligned} F(0) &= a\cos(p(0)) + b\sin(p(0)) = a = 0 \\ F(\pi) &= (0)\cos(p(\pi)) + b\sin(p(\pi)) = b\sin(p\pi) = 0 \end{aligned}$$

If $b = 0$, since $a = 0$, then $F(x) = 0$ which is a trivial solution. Again, to obtain a non-trivial solution we will assume $b \neq 0$ and instead $\sin(p\pi) = 0$.

Example

Now assuming that only $k < 0$ yields non-trivial solutions and set $k = -p^2$ for some $p > 0$. Applying the initial conditions show that $p = n,\ n = 1, 2, 3, ...$ and that possible solutions for $F(x)$ are

$$F_n(x) = b_n \sin(nx)$$

where b_n are constants and $n = 1, 2, 3, ...$

If $\sin(p\pi) = 0$, this implies that $p\pi = n\pi \implies p = n$ for $n = 1, 2, 3, ...$ So, $k = -n^2$. Therefore, the non-trivial solutions to $F''(x) + nF(x) = 0$ are

$$F_n(x) = b_n \sin(nx), \quad \text{with } n = 1, 2, 3, ...$$

Example

Using the previous result, find all possible solutions $G_n(t)$ for $G(t)$.

We now know that $k = -n^2$.

$$G'(t) + n^2 G(t) = 0$$

Again, using our knowledge about solving ODEs...

$$G_n(t) = c_n e^{-n^2} t, \quad c_n \in \mathbb{R},\ n = 1, 2, 3, ...$$

The Heat Equation

Example

Suppose that the initial temperature distribution of the bar is

$$u(x,0) = 2\sin(x) - 16\sin(2x)$$

Find the general solution u(x,t).

We can see that

$$u_n(x,t) = F_n(x)G_n(t) = d_n \sin(nx)e^{-n^2 t}, \quad \text{where } d_n = b_n c_n$$

Since the PDE is linear, we can use the Principle of Superposition to construct the general solution.

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} d_n \sin(nx)e^{-n^2 t}$$

.

The Heat Equation

Example

Suppose that the initial temperature distribution of the bar is

$$u(x,0) = 2\sin(x) - 16\sin(2x)$$

Find the general solution u(x,t).

Finally, using the initial conditions, it is possible to obtain the coefficients d_n.

$$u(x,0) = \sum_{n=1}^{\infty} d_n \sin(nx) = 2\sin(x) - 16\sin(2x)$$

Thus, $d_1 = 2, d_2 = -16$ and $d_3 = d_4 = ... = 0$.

$$\text{Hence,} \quad u(x,t) = 2\sin(x)e^{-t} - 16\sin(2x)e^{-4t}$$

The Heat Equation

Example

Determine all points x along the bar with a temperature of $0°C$ after $t = ln(2)$ seconds.

$$\begin{aligned} u(x, ln(2)) = 0 = &= 2\sin(x)e^{-ln(2)} - 16\sin(2x)e^{-4ln(2)} \\ &= 2\sin(x)\frac{1}{2} - 16\sin(2x)\frac{1}{16} \\ &= \sin(x) - \sin(2x) \\ &= \sin(x) - 2\sin(x)\cos(x) \\ &= \sin(x)(1 - 2\cos(x)) \end{aligned}$$

As the bar has a length of π metres then the only solutions are

$$x = 0, \frac{\pi}{3}, \pi \quad \text{metres}$$

The End

Thanks for watching/attending!
Best of luck with all of your exams!