Second Year Mathematics Revision

Calculus - Part 1

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Integration

Definition

$$\int_{x=a}^{x=b} f(x)dx = \lim_{\Delta x \to 0} \sum_{x=a}^{x=b} f(x_i^*)\Delta x$$

By definition, this means that to compute an area under a curve, we construct rectangles of height $f(x_i^*)$ and width Δx over the interval $a \leq x \leq b$, and add the area of the rectangles and take the limit as the rectangles become super tiny.

Double Integration

Double integration is no different, except we integrate over a 2 dimensional region.

Definition

$$\int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x,y)dxdy = \lim_{\Delta y \to 0} \lim_{\Delta x \to 0} \sum_{y=c}^{y=d} \sum_{x=a}^{x=b} f(x_i^*, y_i^*)\Delta x \Delta y$$

When integrating over a more complicated region, the process is precisely the same. The notation is more generally described as:

$$V = \iint_R f(x,y)dxdy = \iint_R f(x,y)dA$$

Evaluating integrals

Fubini's Theorem

Suppose $f(x, y)$ is a continuous function on $R = [a, b] \times [c, d]$, then

$$\int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x, y) dxdy = \int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x, y) dydx$$

Thus, it is possible to reorder the integral signs to make life easier.

Double Integration Examples

Examples

Evaluate the following integrals:

1. $\iint_R (2x - 4y^3)\,dA$, where $R = [-5, 4] \times [0, 3]$
2. $\iint_R xe^{xy}\,dA$, where $R = [-1, 2] \times [0, 1]$
3. $\iint_D e^{\frac{x}{y}}\,dA$, where $D = \{(x, y) | 1 \leq y \leq 2, y \leq x \leq y^3\}$

Solution Q1

$$\int_{x=-5}^{x=4}\int_{y=0}^{y=3}(2x-4y^3)dydx = \int_{x=-5}^{y=4}\left[2xy-y^4\right]_{y=0}^{y=3}dx$$

$$= \int_{x=-5}^{x=4}\big((6x-81)-(0-0)\big)dx$$

$$= \int_{x=-5}^{x=4}(6x-81)dx$$

$$= \left[3x^2-81x\right]_{x=-5}^{x=4}$$

$$= -756$$

Solution Q2

$$
\begin{aligned}
\int_{y=0}^{y=1}\int_{x=-1}^{x=2} xe^{xy}\,dxdy &= \int_{x=-1}^{x=2}\int_{y=0}^{y=1} xe^{xy}\,dydx \\
&= \int_{x=-1}^{x=2}\left[e^{xy}\right]_{y=0}^{y=1} dx \\
&= \int_{x=-1}^{x=2}(e^{x}-e^{0})dx \\
&= \left[e^{x}-x\right]_{x=-1}^{x=2} \\
&= e^{2}-2-(e^{-1}-(-1)) \\
&= e^{2}-e^{-1}-3
\end{aligned}
$$

Solution Q3

$$\int_{y=1}^{y=2}\int_{x=y}^{x=y^3} e^{\frac{x}{y}}\,dxdy = \int_{y=1}^{y=2}\left[ye^{\frac{x}{y}}\right]_{x=y}^{x=y^3} dy$$

$$= \int_{y=1}^{y=2} ye^{y^2} - ye^1 dy$$

$$= \int_{y=1}^{y=2} (ye^{y^2} - ey) dy$$

$$= \left[\frac{1}{2}e^{y^2} - \frac{e}{2}y^2\right]_{y=1}^{y=2}$$

$$= \frac{1}{2}e^4 - \frac{e}{2}\cdot 4 - \left(\frac{1}{2}e - \frac{e}{2}\right)$$

$$= \frac{1}{2}e^4 - 2e$$

Generalised Fubini's Theorem

To change the order of integration, first draw R and then reconstruct the limit.

General rules

1. **Draw the region before working out the new region**
2. The outer most integral should be bounded by constants, while the inner integral will typically be dependent on the other variable.

Examples

Changing order of integration [MATH2011 Q102]

Evaluate the following integrals by changing the order of integration:

1. $\int_0^1 \int_{y^2}^1 2\sqrt{x}e^{x^2}\,dx\,dy$
2. $\int_0^1 \int_y^1 \sin(x^2)dx\,dy$

Solution Q1

$$
\begin{aligned}
\int_0^1 \int_{y^2}^1 2\sqrt{x}e^{x^2}\,dx\,dy &= \int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{x}} 2\sqrt{x}e^{x^2}\,dydx \\
&= \int_{x=0}^{x=1} 2\sqrt{x}e^{x^2}\left[y\right]_{y=0}^{y=\sqrt{x}} dx \\
&= \int_0^1 2xe^{x^2}\,dx \\
&= \left[e^{x^2}\right]_0^1 \\
&= e-1
\end{aligned}
$$

Solution Q2

$$\begin{aligned}
\int_0^1 \int_y^1 \sin(x^2)\,dx\,dy &= \int_{x=0}^{x=1} \int_{y=0}^{y=x} \sin(x^2)\,dydx \\
&= \int_{x=0}^{x=1} \sin(x^2)\left[y\right]_0^x dx \\
&= \int_0^1 x\sin(x^2)\,dx \\
&= \frac{1}{2}\sin 1
\end{aligned}$$

Triple Integration

The intuition for triple integrals is precisely the same, except there needs to be 3 sets of limits, one for the x direction, one for the y and one for the z directions.

Examples

Examples

1. Find the volume of a tetrahedron bounded by $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ in the first octant.
2. Find the column common to the 2 cylinders $x^2 + y^2 \leq a^2$ and $y^2 + z^2 \leq a^2$
3. Evaluate $\iiint_S x^2 dxdydz$ where S is the region bounded by $4x^2 + y^2 = 4, z + x = 2, z = 0$

Solution Q1 [MATH2011 Q115]

$$\iiint_S dV = \int_{x=0}^{x=a} \int_{y=0}^{y=b\left(1-\frac{x}{a}\right)} \int_{z=0}^{z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz\, dy\, dx$$

$$= c \int_{x=0}^{x=a} \int_{y=0}^{y=b\left(1-\frac{x}{a}\right)} 1 - \frac{x}{a} - \frac{y}{b} dy\, dx$$

$$= c \int_{x=0}^{x=a} \left[\left(1 - \frac{x}{a}\right) y - \frac{1}{2b} y^2 \right]_{y=0}^{y=b\left(1-\frac{x}{a}\right)} dx$$

$$= c \int_{x=0}^{x=a} b\left(1 - \frac{x}{a}\right)^2 - \frac{b}{2}\left(1 - \frac{x}{a}\right)^2 dx$$

$$= \frac{1}{2} bc \int_{x=0}^{x=a} \left(1 - \frac{x}{a}\right)^2 dx$$

$$= \frac{abc}{6}$$

Solution Q2 [MATH2011 Q119]

The region in consideration is the circle $x^2 + y^2 = a^2$ in the xy-plane. So we are measuring the integral:

$$\iiint_S dV = \int_{x=-a}^{x=a} \int_{y=-\sqrt{a^2-x^2}}^{y=\sqrt{a^2-x^2}} \int_{z=-\sqrt{a^2-y^2}}^{z=\sqrt{a^2-y^2}} dz\,dy\,dx$$

$$= \int_{y=-a}^{y=a} \int_{x=-\sqrt{a^2-y^2}}^{x=\sqrt{a^2-y^2}} 2\sqrt{a^2-y^2}\,dx\,dy$$

(changing order of integration)

$$= \int_{y=-a}^{y=a} 2\sqrt{a^2-y^2}\left[x\right]_{x=-\sqrt{a^2-y^2}}^{x=\sqrt{a^2-y^2}} dy$$

$$= \int_{y=-a}^{y=a} 4(a^2-y^2)\,dy$$

$$= \frac{16}{3}a^3$$

Solution Q3 [MATH2011 Q117]

$$
\begin{aligned}
\iiint_S x^2 dV &= \int_{x=-1}^{x=1} \int_{y=-\sqrt{4-4x^2}}^{y=\sqrt{4-4x^2}} \int_{z=0}^{z=2-x} x^2 dz\, dy\, dx \\
&= \int_{x=-1}^{x=1} \int_{y=-\sqrt{4-4x^2}}^{y=\sqrt{4-4x^2}} x^2(2-x) dy\, dx \\
&= \int_{-1}^{1} x^2(2-x) \cdot 2\sqrt{4-4x^2} dx \\
&= 4\int_{-1}^{1} x^2(2-x)\sqrt{1-x^2} dx \\
&= 4\int_{-1}^{1} 2x^2\sqrt{1-x^2} dx - 4\int_{-1}^{1} x^3\sqrt{1-x^2} dx
\end{aligned}
$$

The first integral can be solved using $u = \sin(x)$, the second integral is just 0 because the function is odd.

Integration by substitution

There are 3 main types of substitutions:

1. Polar co-ordinates
2. Cylindrical co-ordinates
3. Spherical co-ordinates

Each are basic changes of variable, the more general case will be covered in time.

Jacobians

In single variable integration:

$$\int_{x=a}^{x=b} f(x)dx = \int_{u=a'}^{u=b'} f(u)\frac{du}{dx}dx$$

So there is an extra factor that influences the rate at which the function changes. Similarly, when employing any change of variable, we use the determinant of the Jacobian to adjust the infinitesimal quantities.

Jacobians for polar co-ordinates

Polar co-ordinates

In polar system, we set $x = r\cos\theta, y = r\sin\theta$ based on the geometry of the space.

$$
\begin{aligned}
J &= \det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} \\
&= \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \\
&= r
\end{aligned}
$$

Jacobian for cylindrical co-ordinates

Cylindrical co-ordinates

The system is exactly the same, except we add a 3rd dimension:
$x = r\cos\theta, y = r\sin\theta, z = z$

$$
\begin{aligned}
J &= \det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{pmatrix} \\
&= \det\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= r
\end{aligned}
$$

Jacobian for Spherical co-ordinates

Spherical co-ordinates

We let the following change of variable occur:
$x = \rho \sin\varphi \cos\theta, y = \rho \sin\varphi \sin\theta, z = \rho \cos\varphi$

$$
\begin{aligned}
J &= \det \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \theta} \end{pmatrix} \\
&= \det \begin{pmatrix} \sin\varphi\cos\theta & \sin\varphi\sin\theta & \cos\varphi \\ \rho\cos\varphi\cos\theta & \rho\cos\varphi\sin\theta & -\rho\sin\varphi \\ -\rho\sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & 0 \end{pmatrix} \\
&= \rho^2 \sin\varphi
\end{aligned}
$$

General Change of Variables

Integration by Substitution

Consider the substitutions $x = g(u, v), y = h(u, v)$. Then the following holds:

$$\iint_{\Omega} f(x, y) dx\, dy = \iint_{\Omega'} f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

Examples for polar co-ordinates

Examples

1. Use polar co-ordinates to find $\iint_\Omega x^2y^3 dR$ where $\Omega = \{(x,y)|x^2+y^2 \leq 1, x \geq 0, y \geq 0\}$
2. Find the area of the shape defined by the inequalities $y \geq 0, y \geq -x, x^2+y^2 \leq 3\sqrt{x^2+y^2} - 3x$

Solution Q1 [Paul's Online Maths Notes]

$$\begin{aligned}
\iint_R dA &= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=1} (r\cos(\theta))^2 (r\sin(\theta))^3 r dr\, d\theta \\
&= \int_{\theta=0}^{\theta=\frac{\pi}{2}} r^6 \cos^2(\theta) \sin^3(\theta) dr\, d\theta \\
&= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{1}{7} \cos^2(\theta) \sin^2\theta \sin(\theta) d\theta \\
&= \frac{1}{7}\int_1^0 u^2(1-u^2)\cdot -du \\
&= \frac{1}{7}\int_0^1 u^2 - u^4 du \\
&= \frac{2}{105}
\end{aligned}$$

Solution Q2 [MATH2011 Q109/MATH2111 Q149]

$$\begin{aligned}
\iint_R dA &= \int_{\theta=0}^{\theta=\frac{3\pi}{4}} \int_{r=0}^{r=3-3\cos\theta} r dr\, d\theta \\
&= \int_0^{\frac{3\pi}{4}} \frac{1}{2}(3-3\cos\theta)^2 d\theta \\
&= \text{la-da-daa-da-da-de-di-da-di-day} \\
&= \frac{81\pi}{16} - \frac{9}{8}(4\sqrt{2}+1)
\end{aligned}$$

Examples on Cylindrical co-ordinates

Examples from homework

1. Find the volume of the solid enclosed between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 4z$
2. Find the volume inside the cone $z + 2 = \sqrt{x^2 + y^2}$ between the planes $z = 0, z = 1$

Solution Q1 [MATH2111 Q152]

$$\iiint_S dV = \int_{r=0}^{r=\sqrt{3}} \int_{\theta=0}^{\theta=2\pi} \int_{z=2-\sqrt{4-r^2}}^{z=2+\sqrt{4-r^2}} r dz\, d\theta\, dr$$
$$= \int_{r=0}^{r=\sqrt{3}} \int_{\theta=0}^{\theta=2\pi} \int_{z=2-\sqrt{4-r^2}}^{z=2+\sqrt{4-r^2}} r dz\, d\theta\, dr$$

The outer integral for the bounds on r are because we only consider how far away from the original in the xy plane we travel, θ ranges from 0 to 2π because we can go around in a full circle in the valid region for r, and z ranges from the lower sphere to the upper sphere.

Solution Q2 [MATH2111 Q153]

$$\iiint_S dV = \int_{r=2}^{r=3} \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=r-2} r dz \, d\theta \, dr$$

Examples of Spherical co-ordinates

Examples from homework

1. Find the volume of the region above the cone $z = \sqrt{x^2 + y^2}$ and inside the sphere $x^2 + y^2 + z^2 = 2az$
2. Use spherical coordinates to find the volume enclosed by the surface $(\sqrt{x^2 + y^2} - 1)^2 + z^2 = 1$

Solution Q1 [MATH2111 Q158]

$$\iiint_S dV = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\frac{\pi}{4}} \int_{\rho=0}^{\rho=2a\cos\varphi} \rho^2 \sin(\varphi)\, d\rho\, d\varphi\, d\theta$$

Solution Q2 [MATH2011 Q128/MATH2111 Q156 iii)]

$$\iiint_S dV = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi} \int_{\rho=0}^{\rho=\frac{1}{\sin(\varphi)}} \rho^2 \sin(\varphi) d\rho \, d\varphi \, d\theta$$

Examples on change of variables

Examples from homework

1. Let Ω be the region in the first quadrant bounded by the hyperbolas $x^2 - 2y^2 = 1, x^2 - 2y^2 = 3, xy = 1, xy = 2$. Let $u = x^2 - 2y^2, v = xy$. Sketch the region Ω in the $x - y$ plane and the region Ω' in the u, v plane that corresponds to Ω. Hence evaluate $\iint_\Omega (x^2 - 2y^2)x^2y^2(2x^2 + 4y^2)dx\,dy$
2. Integrate the function $\frac{1}{xy}$ over the region Ω' bounded by the 4 circles
$x^2 + y^2 = ax, x^2 + y^2 = a'x, x^2 + y^2 = by, x^2 + y^2 = b'y$.

Solution Q1 [MATH2011 Q137/MATH2111 Q160]

We shall consider the substitution $u = x^2 - 2y^2, v = xy$. Then the Jacobian of this substitution is:

$$\det \frac{\partial(u,v)}{\partial(x,y)} = \det \begin{bmatrix} 2x & y \\ -4y & x \end{bmatrix} = 2x \cdot x - (-4y) \cdot y = 2x^2 + 4y^2$$

$$\iint_{\Omega} (x^2 - 2y^2)x^2y^2(2x^2+4y^2)\,dx\,dy = \iint_{\Omega'} uv^2(2x^2+4y^2) \cdot \frac{1}{2x^2+4y^2}\,du\,dv$$

$$= \int_{u=1}^{u=3} \int_{v=1}^{v=2} uv^2\,du\,dv$$

Solution Q2 [MATH2111 Q161] I

Suppose we re-arrange the given equations: $\frac{x^2+y^2}{x} = a, a'$ and $\frac{x^2+y^2}{y} = b, b'$. Then we use the subtitutions $u = \frac{x^2+y^2}{x}, v = \frac{x^2+y^2}{y}$. Thus, we have:

$$\begin{aligned}
\det \begin{bmatrix} 1 - \frac{y^2}{x^2} & \frac{2x}{y} \\ \frac{2y}{x} & -\frac{x^2}{y^2} + 1 \end{bmatrix} &= \left(1 - \frac{y^2}{x^2}\right)\left(1 - \frac{x^2}{y^2}\right) - \left(\frac{2y}{x}\right)\left(\frac{2x}{y}\right) \\
&= 2 - \frac{x^2}{y^2} - \frac{y^2}{x^2} - (4) \\
&= -\frac{x^2}{y^2} - 2 - \frac{y^2}{x^2} \\
&= -\left(\frac{x}{y} + \frac{y}{x}\right)^2 \\
&= -\left(\frac{x^2+y^2}{xy}\right)^2
\end{aligned}$$

Solution Q2 [MATH2111 Q161] II

Hence we have: $-\left(\frac{x^2+y^2}{xy}\right)^2 = \frac{uv}{xy}$

Thus the integral becomes:

$$\iint_{\Omega'} \frac{1}{xy} dx\, dy = \iint_{\Omega} \frac{1}{xy}\frac{xy}{uv} du\, dv = \int_{v=b}^{v=b'} \int_{u=a}^{u=a'} \frac{1}{uv} du\, dv$$

Centre of Mass

Consider a solid (either a plane or 3D solid) where every infinitesimally small part of the solid has a specific **mass density** denoted $\rho(\mathbf{x})$. This means that the overall mass of the object will be given by $\iint_\Omega \rho(\mathbf{x})dA$ or $\iiint_\Omega \rho(\mathbf{x})dV$ depending on how many dimensions the solid has.

Mass of a solid

The mass of a solid is given by the formula:

$$M(S) = \int \dots \int_\Omega \rho(\mathbf{x})dR$$

Centre of Mass (MATH2011 ONLY)

Centre of Mass

The i^{th} coordinate for the centre of mass is given according to the formula:

$$c_i = \frac{\iint_\Omega x_i \rho(\mathbf{x})}{M(S)} dR$$

Here, $M(S)$ describes the mass of the solid, and $\rho(\mathbf{x})$ describes the density of the solid as a function of $\mathbf{x}$

To compute the centre of mass can be done using whatever rules you want to, provided that are valid under double and triple integration.

Leibniz Rule (MATH2111 ONLY)

Commonly known as Differentiation under the integral sign, Leibniz Rule helps evaluate weird looking integrals based on related functions.

Leibniz Rule

$$\frac{d}{dx}\int_{t=a}^{t=b} f(x,t)dt = \int_{t=a}^{t=b} \frac{\partial}{\partial x} f(x,t)dt$$

For a more general integral, we have:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt + f(x,b(x))\cdot b'(x) - f(x,a(x))\cdot a'(x)$$

Examples

2018 Q1iii)

You are given that:

$$\int_{-\infty}^{\infty} \frac{\cos(3x)}{x^2 + a^2} dx = \frac{\pi}{a} e^{-3a}$$

Evaluate the following integral:

$$\int_{-\infty}^{\infty} \frac{\cos(3x)}{(x^2 + 4)^2} dx$$

Solution 2018 Q1iii)

Beginning with the given integral, we differentiate both sides with respect to a

$$\frac{d}{da}\int_{-\infty}^{\infty}\frac{\cos(3x)}{x^2+a^2}dx = \frac{d}{da}\frac{\pi}{a}e^{-3a}$$

$$\int_{-\infty}^{\infty}\frac{\partial}{\partial a}\frac{\cos(3x)}{x^2+a^2}dx = -\frac{-\pi e^{-3x}(3x+1)}{x^2}$$

$$\int_{-\infty}^{\infty}-\frac{2a\cos(3x)}{(x^2+a^2)^2}dx = -\frac{\pi e^{-3a}(3a+1)}{a^2}$$

Substituting $a=2$

$$\int_{-\infty}^{\infty}-4\frac{\cos(3x)}{(x^2+4)^2}dx = -\frac{7\pi e^{-6}}{4}$$

$$\int_{-\infty}^{\infty}\frac{\cos(3x)}{(x^2+4)^2}dx = \frac{7\pi e^{-6}}{16}$$

Examples

2014 Q2i)

Suppose that $f : [0,1] \times [0,1] \to \mathbb{R}$ has continuous partial derivatives. Define $F : [0,1] \times [0,1] \to \mathbb{R}$ by:

$$F(u,v) = \int_0^u f(v,y)dy$$

1. If $u = u(x), v = v(x)$, find an expression for $\frac{dF}{dx}$
2. Hence, or otherwise, compute:

$$\frac{d}{dx}\int_0^x \frac{\sin(xy)}{y}dy$$

Solution 2014 Q2i) a)

By Leibniz Rule:

$$\begin{aligned}
\frac{dF}{dx} &= \frac{d}{dx}\int_0^{u(x)} f(v(x),y)dy \\
&= \int_0^{u(x)} \frac{\partial}{\partial x} f(v(x),y)dy + f(v(x),u(x))\frac{d}{dx}u(x) - f(v(x),0)\frac{d}{dx}(0) \\
&= \int_0^{u(x)} v'(x)\frac{\partial f(v(x),y)}{\partial x}dy + f(v(x),u(x))u'(x)
\end{aligned}$$

Solution 2014 Q2i) b)

Let $u(x) = x, v(x) = x$. Then upon substitution into the expression from a), we have:

$$\frac{d}{dx}\int_0^x \frac{\sin(xy)}{y}dy = \int_0^x 1 \cdot y \cdot \frac{\cos(xy)}{y} + \frac{\sin(x^2)}{x} \cdot 1$$
$$= \int_0^x \cos(xy)dy + \frac{\sin(x^2)}{x}$$
$$= \left[\frac{\sin(xy)}{x}\right]_0^x + \frac{\sin(x^2)}{x}$$
$$= 2\frac{\sin(x^2)}{x}$$