MATH2018/2019 – Seminar solutions

[Seminar I / II]

Written by: Gerald Huang, Regina Tang Updated on *May 4, 2020*

The following document are full worked solutions to the questions that was discussed in the revision seminar on April 22, 2020. The solutions were written by Gerald Huang and Regina Tang and can be used as supplement resources while preparing for final exams. Please use this resource ethically. The following document is **NOT** endorsed by the School of Mathematics and Statistics and may be prone to errors; if you spot an error, please message us here. Happy studying!

Contents

Part I: Partial differentiation

Let $f(x,y) = \dfrac{y}{x+y}$. Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

Solution.

To differentiate f with respect to x, we treat y as a constant. So rewriting f as $f(x,y) = y(x+y)^{-1}$, we can apply the power rule to see that

$$\frac{\partial f}{\partial x} = -\frac{y}{(x+y)^2}.$$

To differentiate f with respect to y, we apply the quotient rule (keeping in mind that x is a constant). Applying the quotient rule, we have

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\frac{dy}{dy}(x+y) - y\frac{d}{dy}(x+y)}{(x+y)^2} \\ &= \frac{(x+y)-y}{(x+y)^2} \\ &= \frac{x}{(x+y)^2}.\end{aligned}$$

Suppose that $z = x^2 + 4xy$ where $x = u^3 \ln v$ and $y = uv^2$. Find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

Solution.

By the multivariable chain rule, we have

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial u}. \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial v}.\end{aligned}$$

Computing each partial derivative, we have

$$\begin{aligned}\frac{\partial z}{\partial x} &= 2x + 4y & \qquad \frac{\partial z}{\partial y} &= 4x \\ \frac{\partial x}{\partial u} &= 3u^2 \ln v & \qquad \frac{\partial x}{\partial v} &= \frac{u^3}{v} \\ \frac{\partial y}{\partial u} &= v^2 & \qquad \frac{\partial y}{\partial v} &= 2uv\end{aligned}$$

So we have

$$\begin{aligned}
\frac{\partial z}{\partial u} &= (2x+4y)3u^2\ln v + 4xv^2 \\
&= (2u^3\ln v + 4uv^2)3u^2\ln v + 4u^3v^2\ln v \\
&= 6u^5(\ln v)^2 + 12u^3v^2\ln v + 4u^3v^2\ln v \\
&= 6u^5(\ln v)^2 + 16u^3v^2\ln v \\
&= 2u^3\ln v(3u^5\ln v + 8v^2).
\end{aligned}$$

$$\begin{aligned}
\frac{\partial z}{\partial v} &= (2x+4y)\frac{u^3}{v} + 8xuv \\
&= \frac{2xu^3 + 4yu^3 + 8xuv^2}{v} \\
&= \frac{2u(xu^2 + 2yu^2 + 4xv^2)}{v} \\
&= \frac{2u(u^5\ln v + 2u^3v^2 + 4u^3v^2\ln v)}{v}.
\end{aligned}$$

Calculate the Taylor series expansion of the function $f(x,y) = \ln(x+y)$ about the point $(1,0)$ up to and including quadratic terms.

SOLUTION.
For a function $f(x,y)$, the general Taylor series about the point (a,b) is given by

$$\begin{aligned}
f(x,y) \approx f(a,b) &+ \left[\frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)\right] \\
&+ \frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}(a,b)(x-a)^2 + 2\frac{\partial^2 f}{\partial x\partial y}(a,b)(x-a)(y-b) + \frac{\partial^2 f}{\partial y^2}(a,b)(y-b)^2\right] \\
&+ \cdots
\end{aligned}$$

Here, our point (a,b) is $(1,0)$ and our function is $f(x,y) = \ln(x+y)$. Calculating the respective derivatives at the point $(1,0)$, we have

$$\begin{aligned}
f(x,y) &= \ln(x+y) & f(1,0) &= 0. \\
\frac{\partial f}{\partial x} &= \frac{1}{x+y} & \frac{\partial f}{\partial x}(1,0) &= 1. \\
\frac{\partial f}{\partial y} &= \frac{1}{x+y} & \frac{\partial f}{\partial y}(1,0) &= 1. \\
\frac{\partial^2 f}{\partial x^2} &= -\frac{1}{(x+y)^2} & \frac{\partial^2 f}{\partial x^2}(1,0) &= -1. \\
\frac{\partial^2 f}{\partial x \partial y} &= -\frac{1}{(x+y)^2} & \frac{\partial^2 f}{\partial x \partial y}(1,0) &= -1. \\
\frac{\partial^2 f}{\partial y^2} &= -\frac{1}{(x+y)^2} & \frac{\partial^2 f}{\partial y^2}(1,0) &= -1.
\end{aligned}$$

So the Taylor series of $f(x,y)=\ln(x+y)$ at the point $(1,0)$ is

$$\begin{aligned}
f(x,y) \approx 0 &+ [1\times(x-1)+1\times(y-0)] \\
&+ \frac{1}{2!}\left[-1\times(x-1)^2+2(-1)\times(x-1)(y-0)+(-1)\times(y-0)^2\right] \\
&+ \cdots \\
&= [(x-1)+y] - \frac{1}{2!}\left[(x-1)^2+2y(x-1)+y^2\right] \\
&= (x-1)+y-\frac{1}{2}\left[(x-1)^2+2y(x-1)+y^2\right].
\end{aligned}$$

> The volume V of a cone with radius r and perpendicular height h is given by $V=\frac{1}{3}\pi r^2 h$. Determine the maximum absolute error and the maximum percentage error in calculating V given that $r=5$ cm and $h=3$ cm to the nearest millimetre.

SOLUTION.
Since we're approximating r and h to the nearest millimetre, then the maximal errors for r and h are both 0.05 respectively. Furthermore, the **maximum absolute error** is given by the linear Taylor seriers approximation for $f(x)$; that is

$$|\Delta V| \le \left|\frac{\partial V}{\partial r}\right| |\Delta r| + \left|\frac{\partial V}{\partial h}\right| |\Delta h|,$$

evaluated at the point $(r,h)=(5,3)$.

Computing each term, we have

$$\begin{aligned}
\frac{\partial V}{\partial h} &= \frac{\pi r^2}{3} & \frac{\partial V}{\partial h}(5,3) &= \frac{25\pi}{3}. \\
\frac{\partial V}{\partial r} &= \frac{2\pi r h}{3} & \frac{\partial V}{\partial r}(5,3) &= 10\pi.
\end{aligned}$$

Thus, we have that

$$|\Delta V| \le \frac{25\pi}{3} \times 0.05 + 10\pi \times 0.05 \approx 2.87.$$

Hence, the maximum absolute error is 2.87.

The **percentage error** is found by taking $\dfrac{|\Delta V|}{V}$. This gives us

$$\begin{aligned}\frac{|\Delta V|}{V} &= \frac{2.87}{(75\pi/3)} \\ &= \frac{2.87 \times 3}{75\pi} \\ &\approx 3.65\%.\end{aligned}$$

> You are given that
>
> $$\int_0^\infty \frac{1}{\alpha^2 + x^2}\,dx = \frac{\pi}{2}\alpha^{-1}.$$
>
> Use Leibniz' theorem to find the following integral in terms of α
>
> $$\int_0^\infty \frac{1}{(\alpha^2 + x^2)^2}\,dx.$$

Solution.

Differentiate both sides with respect to α. So we have

$$\frac{d}{d\alpha}\int_0^\infty \frac{1}{\alpha^2 + x^2}\,dx = \frac{d}{d\alpha}\left[\frac{\pi}{2}\alpha^{-1}\right].$$

On the right hand side, we have

$$\frac{d}{d\alpha}\left[\frac{\pi}{2}\alpha^{-1}\right] = -\frac{\pi}{2\alpha^2}.$$

On the left hand side, we use Leibniz' rule and turn the ordinary derivative into a partial derivative; since the bounds do not depend on α, we don't have to worry about any of the extra terms.

So we have

$$\begin{aligned}\frac{d}{d\alpha}\int_0^\infty \frac{1}{\alpha^2 + x^2}\,dx &= \int_0^\infty \frac{\partial}{\partial\alpha}\left(\frac{1}{\alpha^2 + x^2}\right)\,dx \\ &= \int_0^\infty \left[-\frac{2\alpha}{(\alpha^2 + x^2)^2}\right]\,dx.\end{aligned}$$

Since α doesn't depend on the value of x, then α can be treated as a **constant**. So we have

$$\frac{d}{d\alpha}\int_0^\infty \frac{1}{\alpha^2+x^2}\,dx = -2\alpha\int_0^\infty \frac{1}{(\alpha^2+x^2)^2}\,dx$$
$$\int_0^\infty \frac{1}{(\alpha^2+x^2)^2}\,dx = -\frac{1}{2\alpha}\frac{d}{d\alpha}\int_0^\infty \frac{1}{\alpha^2+x^2}\,dx.$$

But the value of $\dfrac{d}{d\alpha}\displaystyle\int_0^\infty \frac{1}{\alpha^2+x^2}\,dx$ is $-\dfrac{\pi}{2\alpha^2}$. So we have

$$\int_0^\infty \frac{1}{(\alpha^2+x^2)^2}\,dx = -\frac{1}{2\alpha}\cdot -\frac{\pi}{2\alpha^2} = \frac{\pi}{4\alpha^3}.$$

You are given the following integral

$$\int_0^a \frac{1}{(x^2+a^2)^{1/2}}\,dx = \sinh^{-1}(1).$$

Use Leibniz' rule to evaluate

$$\int_0^a \frac{1}{(x^2+a^2)^{3/2}}\,dx.$$

SOLUTION.
Differentiate both sides with respect to a. So we have

$$\frac{d}{da}\int_0^a \frac{1}{(x^2+a^2)^{1/2}}\,dx = \frac{d}{da}\left(\sinh^{-1}(1)\right).$$

On the right hand side, we are differentiating a constant with respect to a. So the derivative simply evaluates to 0.

On the left side, we need to use Leibniz' rule. Note that the upper bound depends on a, so we need to use the generalised rule:

$$\frac{d}{da}\int_0^a \frac{1}{(x^2+a^2)^{1/2}}\,dx = \int_0^a \frac{\partial}{\partial a}\left(\frac{1}{(x^2+a^2)^{1/2}}\right)dx + \frac{1}{(a^2+a^2)^{1/2}}\times\frac{da}{da} - 0$$
$$= \int_0^a -\frac{1}{2}\left(\frac{2a}{(x^2+a^2)^{3/2}}\right)dx + \frac{1}{\sqrt{2}a}.$$

Since a doesn't depend on x, then a can be treated as a **constant**. So we have

$$\frac{d}{da}\int_0^a \frac{1}{(x^2+a^2)^{1/2}}\,dx = -a\int_0^a \frac{1}{(x^2+a^2)^{3/2}}\,dx + \frac{1}{\sqrt{2}a}.$$

But we found the value of $\dfrac{d}{da}\displaystyle\int_0^a \frac{1}{(x^2+a^2)^{1/2}}\,dx = 0$. Hence, we have

$$\begin{aligned} -a\int_0^a \frac{1}{(x^2+a^2)^{3/2}}\,dx + \frac{1}{\sqrt{2}a} &= 0 \\ a\int_0^a \frac{1}{(x^2+a^2)^{3/2}}\,dx &= \frac{1}{\sqrt{2}a} \\ \int_0^a \frac{1}{(x^2+a^2)^{3/2}}\,dx &= \frac{1}{\sqrt{2}a^2}. \end{aligned}$$

Part II: Extrema values

Find the critical points of $h(x,y) = 2x^3 + 3x^2y + y^2 - y$.

SOLUTION.
We aim to find the partial derivative of h with respect to x and y. Differentiating with respect to x, we get

$$\frac{\partial h}{\partial x} = 6x^2 + 6xy. \tag{1}$$

Likewise, differentiating with respect to y, we get

$$\frac{\partial h}{\partial y} = 3x^2 + 2y - 1. \tag{2}$$

Setting (1) to 0, we get

$$6x^2 + 6xy = 0 \iff x = 0, \quad x = -y.$$

When $x = 0$, we get

$$3(0)^2 + 2y - 1 = 0 \implies y = \frac{1}{2}.$$

When $x = -y$, we get

$$3x^2 - 2x - 1 = 0 \implies x = -\frac{1}{3}, \quad x = 1.$$

Hence, the critical points of h are $\left(0, \dfrac{1}{2}\right)$, $\left(-\dfrac{1}{3}, \dfrac{1}{3}\right)$, $(1,-1)$.

Define $D = \dfrac{\partial^2 h}{\partial x^2}\dfrac{\partial^2 h}{\partial y^2} - \dfrac{\partial^2 h}{\partial x \partial y}$. Recall that if $D(a,b) < 0$, we have a **saddle point**. If $D(a,b) > 0$ and $\dfrac{\partial^2 h}{\partial x^2} < 0$, we have a **local maximum**, and if $D(a,b) > 0$ and $\dfrac{\partial^2 h}{\partial x^2} > 0$, we have a **local minimum**.

Computing the second order partial derivatives, we have

$$\frac{\partial^2 h}{\partial x^2} = 12x + 6y, \quad \frac{\partial^2 h}{\partial y^2} = 2, \quad \frac{\partial h^2}{\partial x \partial y} = 6x.$$

Hence, we have

$$D(x,y) = 2(12x + 6y) - 6x = 18x + 12y.$$

At $\left(0, \frac{1}{2}\right)$, we have $D\left(0, \frac{1}{2}\right) = 6 > 0$. Computing $\frac{\partial^2 h}{\partial x^2}\left(0, \frac{1}{2}\right)$, we have

$$\frac{\partial^2 h}{\partial x^2}\left(0, \frac{1}{2}\right) = 3 > 0.$$

So $\left(0, \frac{1}{2}\right)$ is a **local minimum**.

At $\left(-\frac{1}{3}, \frac{1}{3}\right)$, we have $D\left(-\frac{1}{3}, \frac{1}{3}\right) = -2 < 0$. So $\left(-\frac{1}{3}, \frac{1}{3}\right)$ is a **saddle point**.

At $(1,-1)$, we have $D(1,-1) = 6 > 0$. Computing $\frac{\partial^2 h}{\partial x^2}(1,-1)$, we have

$$\frac{\partial^2 h}{\partial x^2}(1,-1) = 6 > 0.$$

So $(1,-1)$ is a **local minimum**.

Find the extreme value(s) of $z = f(x,y) = x^4 + y^4$ subject to the condition $x + y - 1 = 0$.

SOLUTION.
We aim to find the critical points that satisfy the equation $\nabla f = \lambda \nabla g$ as well as $g(x,y) = 0$. This gives us the following system of equations.

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \implies 4x^3 = \lambda. \tag{1}$$
$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \implies 4y^3 = \lambda. \tag{2}$$
$$x + y = 1. \tag{3}$$

Since our critical point **must** satisfy the constraint, then we need to find suitable values for λ that gives us a critical point in f. The constraint is in terms of x and y, so it's naturally to find an expression in (1) and (2) in terms of x and y. Solving for x and y yields the following equations

$$x = \sqrt[3]{\frac{\lambda}{4}}, \quad y = \sqrt[3]{\frac{\lambda}{4}}. \tag{4}$$

Substituting (4) into (3) and solving for λ, we have

$$\begin{aligned} \sqrt[3]{\frac{\lambda}{4}} + \sqrt[3]{\frac{\lambda}{4}} &= 1 \\ \sqrt[3]{\frac{\lambda}{4}} &= \frac{1}{2} \\ \frac{\lambda}{4} &= \frac{1}{8} \\ \lambda &= \frac{1}{2}. \end{aligned}$$

Hence, our extreme value of f occurs at $\lambda = \frac{1}{2}$, which gives us the point

$$x = \frac{1}{2}, \quad y = \frac{1}{2}.$$

So the point $\left(\frac{1}{2}, \frac{1}{2}\right)$ is a critical point of the function $f(x, y)$ and satisfies the constraint.

Part III: Vector field theory

Given a vector field

$$\mathbf{F} = 8e^{-x}\mathbf{i} + \cosh z\mathbf{j} - y^2\mathbf{k}$$

calculate the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ where $\mathcal{C}$ is the straight line path from $A(0, 1, 0)$ to $B(\ln(2), 1, 2)$.

Solution.

Begin by **parameterising the curve**. Since the curve $\mathcal{C}$ is a line, we parameterise it as

$$\mathcal{C}: \quad t(0, 1, 0) + (1 - t)(\ln(2), 1, 2), \quad t \in [0, 1].$$

Splitting up each of these into its respective components gives us

$$\begin{aligned} x(t) &= 0 \cdot t + \ln(2)(1 - t) = \ln(2) \cdot (1 - t). \\ y(t) &= 1 \cdot t + (1 - t) = 1. \\ z(t) &= 0 \cdot t + 2(1 - t) = 2(1 - t). \end{aligned}$$

This gives us expressions for dx, dy and dz in terms of dt

$$\begin{aligned} dx &= -\ln(2)\ dt. \\ dy &= 0\ dt. \\ dz &= -2\ dt. \end{aligned}$$

So the line integral can be expressed as

$$\begin{aligned}
\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= \int_{\mathcal{C}} F_1\ dx + F_2\ dy + F_3\ dz \\
&= \int_0^1 \left[\left(8e^{-\ln 2(1-t)} \cdot -\ln 2\ dt\right) + (\cosh(2(1-t)) \cdot 0) + \left(-1^2 \cdot (-2)\ dt\right)\right] \\
&= \int_0^1 \left[8\ln(2)e^{t\ln 2 - \ln 2} + 2\right]\ dt \\
&= 8\ln(2) \cdot e^{-\ln 2} \int_0^1 e^{t\ln 2}\ dt + \int_0^1 2\ dt \\
&= 4\ln 2 \int_0^1 e^{\ln(2^t)}\ dt + 2 \\
&= 4\ln 2 \int_0^1 2^t\ dt + 2 \\
&= 4\ln 2 \cdot \left.\frac{2^t}{\ln 2}\right|_0^1 + 2 \\
&= 4\ln 2 \left(\frac{2-1}{\ln 2}\right) + 2 \\
&= 4 + 2 \\
&= 6.
\end{aligned}$$

Consider the scalar field

$$\phi(x, y, z) = xe^{z-1} + \cos y$$

and let $\mathbf{F} = \nabla\phi$.

i) What is $\nabla \times \mathbf{F}$?

ii) Hence, or otherwise, calculate the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ along the straight line path $\mathcal{C}$ from $(1, 0, 1)$ to $(5, \pi, 1)$.

Solution.

i) Since $\mathbf{F}$ can be written as the product of the gradient of a scalar field, then immediately we conclude that $\nabla \times \mathbf{F} = 0$. This is a consequence of conservative fields.

ii) The line integral can be computed by taking

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \phi|_B - \phi|_A$$

where A and B are the starting and end points respectively. So we have

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \left(5e^{1-1} + \cos(\pi)\right) - \left(1 \cdot e^{1-1} + \cos(0)\right) = 4 - 2 = 2.$$

> Consider the vector field
>
> $$\mathbf{F} = yz^2\mathbf{i} + xz^2\mathbf{j} + (2xyz + 3)\mathbf{k}.$$
>
> i) Show that $\mathbf{F}$ is conservative by evaluating $\operatorname{curl}(\mathbf{F})$.
>
> ii) The path $\mathcal{C}$ in $\mathbb{R}^3$ starts at the point $(3, 4, 7)$ and subsequently travels anti-clockwise four complete revolutions around the circle $x^2 + y^2 = 25$ within the plane $z = 7$, returning to the starting point $(3, 4, 7)$. Using the first part or otherwise, evaluate the work integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

Solution.

i) Computing the curl of $\mathbf{F}$, we have

$$\begin{aligned}
\operatorname{curl}(\mathbf{F}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 & xz^2 & 2xyz + 3 \end{vmatrix} \\
&= \mathbf{i}\left[\frac{\partial}{\partial y}(2xyz + 3) - \frac{\partial}{\partial z}(xz^2)\right] - \mathbf{j}\left[\frac{\partial}{\partial x}(2xyz + 3) - \frac{\partial}{\partial z}\left(yz^2\right)\right] + \mathbf{k}\left[\frac{\partial}{\partial x}\left(xz^2\right) - \frac{\partial}{\partial y}\left(yz^2\right)\right] \\
&= \mathbf{i}\left[2xz - 2xz\right] - \mathbf{j}\left[2yz - 2yz\right] + \mathbf{k}\left[z^2 - z^2\right] \\
&= 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k}.
\end{aligned}$$

Hence, the curl of $\mathbf{F}$ is the 0 vector which implies that $\mathbf{F}$ is a conservative vector field.

ii) The result from part i) is useful because it means that line integrals are path-independent on $\mathbf{F}$. In other words, the evaluation of the line integral can be simplified to simply finding a scalar potential for $\mathbf{F}$ and evaluating it at the end and starting positions. In essence,

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \phi|_B - \phi|_A.$$

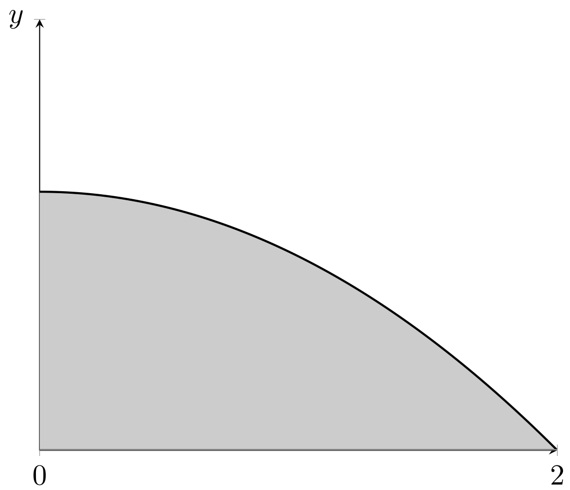
Furthermore, we see that the point A is the same as the point B since the path of interest begins at $(3, 4, 7)$ and returns to $(3, 4, 7)$. This means that the line integral evaluates to

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \phi|_B - \phi|_A = \phi|_B - \phi_B = 0.$$

In fact, for any closed and simple (non self-intersecting) path, the line integral over this path in a **conservative** vector field is always 0.

Evaluate $\iint_\Omega x dA$ where Ω is the region in the first quadrant bounded by the parabola $y = 4 - x^2$ and the coordinate axes.

SOLUTION.
Let Ω be the region in the first quadrant bounded by $y = 4 - x^2$ and the coordinate axes.

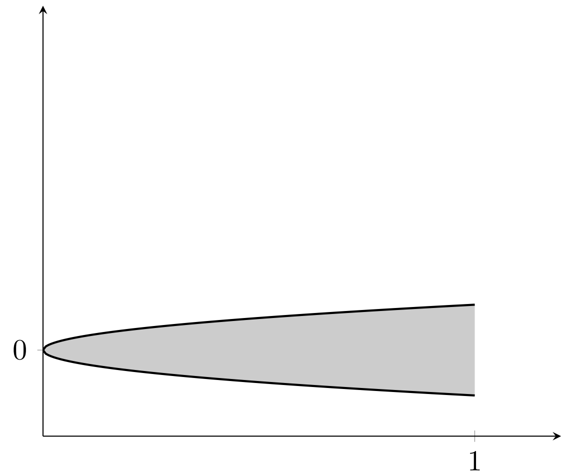


By sketching Ω, we can see that the y value depends on x. By considering vertical strips, we can see that the lower bound of y is 0 while the upper bound is the value at $4 - x^2$. On the other hand, x goes from 0 to 2. So we can see that the double integral can be written as

$$\begin{aligned}\iint_\Omega x\ dA &= \int_0^2 \int_0^{4-x^2} x\ dy\ dx \\ &= \int_0^2 x(4-x^2)\ dx \\ &= \int_0^2 (4x - x^3)\ dx \\ &= 2x^2 - \frac{1}{4}x^4 \Big|_0^2 \\ &= (8-4) - (0-0) \\ &= 4.\end{aligned}$$

Evaluate $\int_{-1}^1 \int_{y^2}^1 2\sqrt{x} e^{x^2} dx dy$ by first changing the order of integration.

Solution.
Let Ω be the region bounded by $x = y^2$ to $x = 1$ and $y = -1$ to $y = 1$. This looks like the following region.



Now consider vertical strips along this region since we're considering values for y first. We can see that the lower and upper bounds of y depend on the values of x; more specifically, the lower bound occurs on the function $y = -\sqrt{x}$ while the upper bound occurs on the function $y = \sqrt{x}$. The x bounds go from $x = 0$ to $x = 1$. Together, we arrive at the double integral

$$\begin{aligned}\int_{-1}^{1}\int_{y^2}^{1} 2\sqrt{x}e^{x^2}\ dx\ dy &= \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} 2\sqrt{x}e^{x^2}\ dy\ dx \\ &= \int_0^1 2\sqrt{x}e^{x^2} y\Big|_{-\sqrt{x}}^{\sqrt{x}}\ dx \\ &= \int_0^1 2\sqrt{x}e^{x^2}\left(\sqrt{x} - (-\sqrt{x})\right)\ dx \\ &= \int_0^1 4xe^{x^2}\ dx.\end{aligned}$$

This becomes an integration by substitution question by letting $u = x^2$. We then have $du = 2x\,dx$. In other words, $2\,du = 4x\,dx$. The bounds change from $x = 0 \to 1$ to $u = 0 \to 1$. In other words, we have the following integral

$$\begin{aligned}\int_0^1 4xe^{x^2}\ dx &= \int_0^1 2e^u\ du \\ &= 2e^u\Big|_0^1 \\ &= 2e - 2 \\ &= 2(e-1).\end{aligned}$$

Consider the double integral

$$I = \int_0^4 \int_{\sqrt{x}}^2 10x \, dy dx.$$

Evaluate I with the order of integration reversed.

SOLUTION.
Sketch the region of integration and consider vertical strips since the order of the integration deals with x variables first instead of the y variable.

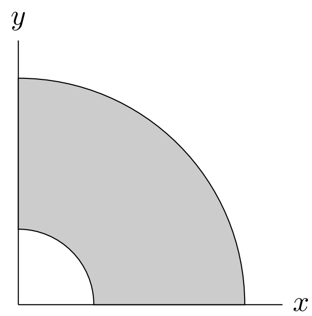
You'll end up seeing that the bounds of x become $x = 0 \to y^2$ while the bounds of y become $y = 0 \to 2$.

This gives us the integral

$$\begin{aligned} I &= \int_0^2 \int_0^{y^2} 10x \ dx \ dy \\ &= \int_0^2 5x^2 \Big|_0^{y^2} \ dy \\ &= \int_0^2 5y^4 \ dy \\ &= y^5 \Big|_0^2 \\ &= 2^5 - 0 \\ &= 32. \end{aligned}$$

Evaluate $\iint_\Omega 2xy \, dy dx$ where Ω is the region in the first quadrant between the circles of radius 2 and radius 5 centred at the origin.

SOLUTION.
The region of integration Ω is the set of points where the radius is between 2 and 5; in other words, Ω is an annulus with $r = 2 \to 5$ and $\theta = 0 \to \dfrac{\pi}{2}$, represented as below.



Doing this using Cartesian coordinates will prove to be quite difficult, so instead, we'll look towards converting everything into its polar form counterpart.

Before continuing with the solution, we shall consider how an integral changes in polar coordinates.

- The strip dA changes from $dxdy$ or $dydx$ to $rdrd\theta$.
- x becomes $r\cos\theta$, while y becomes $r\sin\theta$.

Using these two ideas, we can begin to convert everything into polar form. The bounds we are considering are in terms of r and θ as discussed earlier. So we have

$$\iint_\Omega \to \int_0^{\frac{\pi}{2}} \int_2^5 .$$

The function itself becomes

$$2xy \to 2\underbrace{(r\cos\theta)}_{x}\underbrace{(r\sin\theta)}_{y}.$$

Finally, the dA becomes

$$dA = dxdy = dydx \to r\ dr\ d\theta.$$

So, in the end, our integral becomes

$$\begin{aligned}
\iint_\Omega 2xy\ dy\ dx &= \int_0^{\frac{\pi}{2}} \int_2^5 2\,(r\cos\theta)\,(r\sin\theta)\,r\ dr\ d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_2^5 2r^3\cos\theta\sin\theta\ dr\ d\theta \\
&= \int_0^{\frac{\pi}{2}} 2\cos\theta\sin\theta\left(\int_2^5 r^3\ dr\right)\ d\theta \\
&= \int_0^{\frac{\pi}{2}} \sin(2\theta)\left(\frac{1}{4}r^4\Big|_2^5\ d\theta\right. \\
&= \int_0^{\frac{\pi}{2}} \left(\frac{625-16}{4}\right)\sin(2\theta)\ d\theta \\
&= \frac{609}{4}\int_0^{\frac{\pi}{2}} \sin(2\theta)\ d\theta \\
&= \frac{609}{4}\left(-\frac{1}{2}\cos(2\theta)\Big|_0^{\frac{\pi}{2}}\right. \\
&= \frac{609}{4}\left[-\frac{\cos(\pi)}{2}+\frac{\cos(0)}{2}\right] \\
&= \frac{609}{4}\left[\frac{1+1}{2}\right] \\
&= \frac{609}{4}.
\end{aligned}$$

Part IV: Matrices

The matrix B is given by

$$B = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

i) Show that the vector

$$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

is an eigenvector of the matrix B and find the corresponding eigenvalue.

ii) Given that the other two eigenvalues of B are -1 and 2, find the eigenvectors corresponding to these two eigenvalues.

SOLUTION.

i) We aim to show that $B\mathbf{v} = \lambda\mathbf{v}$ and find the corresponding value for λ.

By matrix multiplication, we have

$$\begin{aligned} B\mathbf{v} &= \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \cdot 1 + (-1) \cdot (-1) + 0 \cdot 0 \\ (-1) \cdot 1 + 0 \cdot (-1) + 0 \cdot 0 \\ 0 \cdot 1 + 0 \cdot (-1) + 2 \cdot 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \\ &= 1 \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \\ &= 1 \cdot \mathbf{v}. \end{aligned}$$

So $B\mathbf{v} = \mathbf{v}$ and so $\mathbf{v}$ is an **eigenvector** with eigenvalue of 1.

ii) To find the eigenvector for eigenvalue -1, we row reduce the matrix $B-(-1)I = B+I$ which is the matrix

$$B + I = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Row reducing the matrix, we have the matrix

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

The solutions are of the form

$$\begin{aligned} v_1 - v_2 &= 0. \\ v_3 &= 0. \end{aligned}$$

So a possible eigenvector is

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Similarly, to find the eigenvector for eigenvalue 2, we row reduce the matrix $B - 2I$ which is the matrix

$$B - 2I = \begin{pmatrix} -2 & -1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Row reducing the matrix, we end up with the matrix

$$\begin{pmatrix} -2 & -1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So the solutions are of the form

$$\begin{aligned} -2v_1 - v_2 &= 0 \\ v_2 &= 0. \end{aligned}$$

So our eigenvector requires that $v_1 = v_2 = 0$. Since we don't want a zero eigenvector, then choose v_3 to be something nonzero (e.g. $v_3 = 1$). Hence, a possible eigenvector for the eigenvalue 2 is

$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

A **real symmetric** 3×3 matrix A has eigenvalues denoted by λ_1, λ_2 and λ_3.
A student is given the following information about A:

- $\mathrm{trace}(A) = 0$,
- $\lambda_1 = 2$ and $\lambda_3 = 4$.

What is the value of the remaining eigenvalue, namely λ_2?

SOLUTION.
Recall that the trace of a matrix is equal to the sum of the eigenvalues of the matrix. The sum of the eigenvalues is given by

$$\lambda_1 + \lambda_2 + \lambda_3 = 2 + \lambda_2 + 4 = 6 + \lambda_2.$$

We're additionally given information about the trace of A. Hence, equating the two pieces of information gives us

$$6 + \lambda_2 = 0 \implies \lambda_2 = -6.$$

The matrix $A = \begin{pmatrix} -5 & 6 & 0 \\ -3 & 4 & 0 \\ -3 & 3 & 1 \end{pmatrix}$ is diagonalisable with eigenvalues -2, 1 and 1.

An eigenvector corresponding to the eigenvalue -2 is $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

Find an invertible matrix M such that $M^{-1}AM = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

SOLUTION.
A lot of the work is done for us; we need to finish it off by finding the eigenvectors for 1. Note that we should end up with two distinct non-zero eigenvectors for 1. To do this, we need to row reduce $A - I$. This gives us

$$A - I = \begin{pmatrix} -6 & 6 & 0 \\ -3 & 3 & 0 \\ -3 & 3 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We see that there are 2 zero rows, which tells us that there are two eigenvectors to find, all of which satisfy the condition that

$$-v_1 + v_2 = 0.$$

Setting $v_1 = v_2 = 1$ and $v_3 = 0$ gives us the eigenvector

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$

and setting $v_1 = v_2 = 0$ and $v_3 = 1$ gives us the eigenvector

$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Our construction of M must also agree with the diagonal matrix provided. The construction of M directly corresponds to the construction of the diagonal matrix; that is, we need to place the corresponding eigenvectors in the correct position based on the diagonal matrix. The first column vector of M must, therefore, be the eigenvector corresponding to 2. This gives us the matrix

$$M = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Note that the order of our last two columns don't matter and either ordering is perfectly okay.

> You are given that the matrix A has eigenvalues 1444, 722 and 722. Hence the equation of the surface in terms of the principal axes X, Y and Z can be written as
>
> $$1444X^2 + 722Y^2 + 722Z^2 = 17689.$$
>
> Enter the shortest distance from the origin to the surface.

Solution.
The shortest distance is a straight line which can be found by taking the component with the **largest eigenvalue** and setting the other components to equal to zero. Here, we take $Y = Z = 0$. Solving for X, we get

$$1444X^2 = 17689 \implies X = \pm\frac{7}{2}.$$

So the shortest distance from the origin to the surface is $\frac{7}{2}$.

> A quadratic curve is given by the equation $7x^2 + 6xy + 7y^2 = 200$.
>
> i) Express the curve in the form
>
> $$\mathbf{x}^\intercal A\mathbf{x} = 200$$
>
> where $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, and A is a 2×2 symmetric matrix.
>
> ii) Find the eigenvalues and eigenvectors of the matrix A.
>
> iii) Hence, or otherwise, find the shortest distance between the curve and the origin.

Solution.

i) Let A be the symmetric matrix defined by the coefficients of the quadratic curve. Note that the xy terms need to be evenly distributed into xy and yx in the matrix itself.

Doing so, we should have the matrix

$$A = \begin{bmatrix} 7 & 3 \\ 3 & 7 \end{bmatrix}.$$

To check that this matrix is **symmetric**, we just verify that $A = A^{\intercal}$.

So the curve is of the form

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 7 & 3 \\ 3 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 200.$$

ii) We want to find the eigenvalues and corresponding eigenvectors of A from part (i). To do this, we solve for the roots to the characteristic polynomial

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 7-\lambda & 3 \\ 3 & 7-\lambda \end{vmatrix} \\ &= (7-\lambda)^2 - 9. \end{aligned}$$

Solving for λ, we arrive at $\lambda = 4$ and $\lambda = 10$.

Now, when $\lambda = 4$, we have

$$A - 4I = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

So, an eigenvector for $\lambda = 4$ is

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Similarly, when $\lambda = 10$, we have

$$A - 10I = \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}.$$

So, an eigenvector for $\lambda = 10$ is

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

So we have the eigenvalue-eigenvector pair set

$$\lambda_n = \{4, 10\}, \quad \mathbf{v}_n = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

iii) Now, because we have an "xy" term in our original curve, then it means that the x-y axes are no longer the principal axes to the curve; that is, we have some sort of rotation happening. To account for this, we need to develop a new coordinate system where the principal axes no longer have the rotation. This principal axes is found to be the **unitary eigenvectors** of A. That is, the principal axes are

$$\hat{\mathbf{v}}_1, \quad \hat{\mathbf{v}}_2.$$

We found the eigenvectors in part (ii), but they are not unitary. So we need to divide by the length of $\mathbf{v}_1$ and $\mathbf{v}_2$. Doing this, we get

$$\hat{\mathbf{v}}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}, \quad \hat{\mathbf{v}}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}.$$

With these principal axes, the curve is transformed into a new curve that does not have the rotation; these are found by the *eigenvalues*. So the eigenvalues found are 4 and 10 respectively, which creates the following curve in the new coordinate system

$$4X^2 + 10Y^2 = 200.$$

The shortest distance can be found by taking $X = 0$ and solving for Y to give us

$$Y = \pm\sqrt{20} = \pm 2\sqrt{5}.$$

Hence, the shortest distance from the origin to the original curve is $2\sqrt{5}$ at the points

$$\frac{2\sqrt{5}}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad \text{and} \quad -\frac{2\sqrt{5}}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}.$$

Simplifying the expression gets us to the points $\begin{pmatrix}\sqrt{10}\\\sqrt{10}\end{pmatrix}$ and $\begin{pmatrix}-\sqrt{10}\\-\sqrt{10}\end{pmatrix}$.

Solve the system of differential equations.

$$\begin{aligned} y_1' &= 2y_1 + y_2 \\ y_2' &= -y_1 + y_3 \\ y_3' &= y_1 + y_2 + y_3. \end{aligned}$$

Solution.

Let $\mathbf{y} = \begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$ so that $\mathbf{y}' = \begin{pmatrix}y_1'\\y_2'\\y_3'\end{pmatrix}$. Then, we can express the following system of differential equations as

$$\mathbf{y}' = A\mathbf{y}$$

where A is the coefficient matrix of the system of differential equations; that is,

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Our goal here is to find the eigenvalues and corresponding eigenvectors of A. To find the eigenvalues, we solve the roots to the characteristic polynomial

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & 1 & 0 \\ -1 & -\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} \\
&= (2-\lambda)\left[-\lambda(1-\lambda) - 1\right] - \left[(\lambda - 1) - 1\right] \\
&= (2-\lambda)\left(-\lambda + \lambda^2 - 1\right) + (2-\lambda) \\
&= (2-\lambda)(-\lambda + \lambda^2 - 1 + 1) \\
&= (2-\lambda)(\lambda^2 - \lambda) \\
&= \lambda(\lambda - 1)(2-\lambda).
\end{aligned}$$

So the roots occur at $\lambda = 0, 1, 2$. These form the $e^{\lambda_n t}$ component of the solution. We need to find the corresponding eigenvectors to complete the set of solutions.

Consider $\lambda = 0$. This means we need to row reduce A, which becomes

$$\begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

This means our eigenvector needs to satisfy the following conditions:

$$\begin{aligned}
2v_1 + v_2 &= 0. \\
v_2 + 2v_3 &= 0.
\end{aligned}$$

Take $v_1 = v_3 = 1$ and $v_2 = -2$. Then we have the following eigenvector

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

Now consider $\lambda = 1$. This means we need to row reduce $A - I$, which becomes

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

This means our eigenvector needs to satisfy the following conditions:

$$\begin{aligned} v_1 + v_2 &= 0 \\ v_3 &= 0. \end{aligned}$$

Take $v_1 = -1$, $v_2 = 1$, and $v_3 = 0$. Then we have the following eigenvector

$$\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Finally, consider $\lambda = 2$. This means we need to row reduce $A - 2I$, which becomes

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

Note that we begin with a 0 in the first entry, so instead consider swapping R_1 and R_2 to get

$$\begin{bmatrix} -1 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

which can be row reduced to become

$$\begin{bmatrix} -1 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This means our eigenvector needs to satisfy the following conditions:

$$\begin{aligned} -v_1 - 2v_2 + v_3 &= 0 \\ v_2 &= 0. \end{aligned}$$

Take $v_1 = v_3 = 1$ and $v_2 = 0$. Then we have the following eigenvector

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

So we have the following eigenvalues with associated eigenvectors

$$\lambda_n = \{0, 1, 2\}, \quad \mathbf{v}_n = \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

So our general solution becomes

$$\begin{aligned}
\mathbf{y} &= \sum_{k=0}^{n} c_k \mathbf{v}_k e^{\lambda_k t} \\
&= c_0 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{0t} + c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t} \\
&= c_0 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t}.
\end{aligned}$$

The individual solutions can be extracted component-wise:

$$\begin{aligned}
y_1(t) &= c_0 - c_1 e^{t} + c_2 e^{2t}. \\
y_2(t) &= -2c_0 + c_1 e^{t}. \\
y_3(t) &= c_0 + c_2 e^{2t}.
\end{aligned}$$