MATH1231/1241 Algebra S2 2007 Test 1 v1B

Full Solutions

September 15, 2017

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1. (i) Rewrite the set as $S = \left\{\mathbf{x} \in \mathbb{R}^4 | x_1 - 5x_3 - 2x_4 = 0\right\}$.

- Clearly, $\mathbf{0} \in S$, since $0 - 5(0) - 2(0) = 0$.
- Now, suppose that $\mathbf{x}, \mathbf{y} \in S$, where $\mathbf{x} = (x_1, x_2, x_3, x_4)^T$ and $\overrightarrow{y} = (y_1, y_2, y_3, y_4)^T$. Then we have the two conditions,

$$x_1 - 5x_3 - 2x_4 = 0 \text{ and } y_1 - 5y_3 - 2y_4 = 0. \qquad (\star)$$

Now, for $\mathbf{x}+\mathbf{y}$ to lie in set S, the following condition must be true:

$$(x_1+y_1)-5(x_3+y_3)-2(x_4+y_4)=0.$$

Considering the LHS,

$$\begin{aligned}(x_1+y_1)-5(x_3+y_3)-2(x_4+y_4)&=(x_1-5x_3-2x_4)+(y_1-5y_3-2y_4)\\&=0+0\quad\text{(using the conditions }(\star))\\&=0.\end{aligned}$$

Hence, $\mathbf{x}+\mathbf{y}\in S$, and so set S is closed under addition.

- Suppose that $\lambda\in\mathbb{R}$. Now, for $\lambda\mathbf{x}$ to lie in set S, the following condition must be true,

$$(\lambda x_1)-5(\lambda x_3)-2(\lambda x_4)=0. \qquad (\dagger)$$

Considering the LHS,

$$\begin{aligned}(\lambda x_1)-5(\lambda x_3)-2(\lambda x_4)&=\lambda(x_1-5x_3-2x_4)\\&=\lambda(0)\,(\dagger)\qquad\qquad\text{(using the condition)}\\&=0.\end{aligned}$$

Hence, $\lambda\mathbf{x}\in S$, and so set S is closed under scalar multiplication.

Thus, by the Subspace Theorem, S is a subspace of $\mathbb{R}^4$.

(ii) There are an infinite possible number of choices. For example, we can pick a simple one by choosing $x_2=0, x_3=0, x_4=1$ such that $x_1=2$. Thus, one non-zero element in S is $\begin{pmatrix}2\\0\\0\\1\end{pmatrix}$.

2. For these polynomials to span $\mathbb{P}_2$, every polynomial in $\mathbb{P}_2$ in the form $b_0+b_1t+b_2t^2$ where $b_0,b_1,b_2\in\mathbb{R}$ must be expressible as a linear combination of $p_1(t),p_2(t)$ and $p_3(t)$. Let $\alpha,\beta,\gamma,\delta\in\mathbb{R}$ such that

$$\alpha p_1(t)+\beta p_2(t)+\gamma p_3(t)+\delta p_4(t)=b_0+b_1t+b_2t^2.$$

Substituting in the polynomials and grouping the terms, we obtain

$$(\alpha+2\beta-\gamma)+(-\alpha-\beta+5\gamma+\delta)\,t+(2\alpha+3\beta-5\gamma-2\delta)\,t^2=b_0+b_1t+b_2t^2.$$

By equating the coefficients, we can obtain 4 simultaneous equations which can be written in an augmented matrix,

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & b_0 \\ -1 & -1 & 5 & 1 & b_1 \\ 2 & 3 & -5 & -2 & b_2 \end{array}\right).$$

Perform the row operations $R_2 \leftarrow R_2 + R_1$ and $R_3 \leftarrow R_3 - 2R_1$,

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & b_0 \\ 0 & 1 & 4 & 1 & b_0 + b_1 \\ 0 & -1 & -3 & -2 & -2b_0 + b_2 \end{array}\right).$$

Next, $R_3 \leftarrow R_3 + R_2$,

$$\left(\begin{array}{cccc|c} \boxed{1} & 2 & -1 & 0 & b_0 \\ 0 & \boxed{1} & 4 & 1 & b_0 + b_1 \\ 0 & 0 & \boxed{1} & -1 & -b_0 + b_1 + b_2 \end{array}\right).$$

If we let $\delta = c$ where c can take any value from $\mathbb{R}$, back-substitution can be performed to find expressions for α, β and γ in terms of c, b_0, b_1, b_2. We observe that a solution always exists for the system, and hence the polynomials form a spanning set for $\mathbb{P}_2$.

Alternatively, for these polynomials to span $\mathbb{P}_2$, exactly three of these polynomials must be linearly independent. Use this fact to construct a matrix to show that three of them are linearly independent.

3. Let $\alpha, \beta, \gamma \in \mathbb{R}$. For these vectors to be linearly independent, the following must only be true for $\alpha = \beta = \gamma = 0$,

$$\alpha \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 5 \\ 5 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Rewriting this in matrix form,

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 4 & 5 & 3 & 0 \\ -2 & 5 & 1 & 0 \end{array}\right).$$

Row-reducing, we first perform the row operations $R_2 \leftarrow R_2 - 4R_1$ and $R_3 \leftarrow R_3 + 2R_1$,

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 9 & 3 & 0 \end{array}\right).$$

Next, $R_3 \leftarrow R_3 + 3R_2$,

$$\left(\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 0 \\ 0 & \boxed{-3} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

We observe that there are infinitely many solutions for α, β and γ as not all columns are leading, and hence the vectors are linearly dependent.

MATH1231/1241 Algebra S2 2009 Test 1 v2B

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1. Note that S does not contain the zero vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (because $3 \times 0 - 2 \times 0 = 0 \neq 1$). Therefore, S cannot be a subspace of $\mathbb{R}^2$.

2. We first need to establish whether the set containing these polynomials is linearly independent or not. Noticing that we are working with quadratic polynomials, if it is linearly independent, there can only be at most 3 polynomials in the set. But since we are working with 4 polynomials, the set containing these polynomials is thus linearly dependent .

   Next, to find which one of these polynomials can be written as a linear combination of the others, we let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\alpha p_1(x) + \beta p_2(x) + \gamma p_3(x) + \delta p_4(x) = 0.$$

Substituting in the polynomials, we obtain

$$\alpha\left(1 - 3x + 2x^2\right) + \beta\left(3 - 8x + 5x^2\right) + \gamma\left(1 + x - 2x^2\right) + \delta\left(-x + 8x^2\right) = 0.$$

Grouping together like terms,

$$(\alpha + 3\beta + \gamma) + (-3\alpha - 8\beta + \gamma - \delta)\,x + (2\alpha + 5\beta - 2\gamma + 8\delta)\,x^2 = 0.$$

By equating the coefficients, we see that

$$\alpha + 3\beta + \gamma = 0$$

$$-3\alpha - 8\beta + \gamma - \delta = 0$$

$$2\alpha + 5\beta - 2\gamma + 8\delta = 0.$$

We set up an augmented matrix,

$$\left(\begin{array}{cccc|c} 1 & 3 & 1 & 0 & 0 \\ -3 & -8 & 1 & -1 & 0 \\ 2 & 5 & -2 & 8 & 0 \end{array}\right). \tag{$\star$}$$

Row-reducing, we perform the operations $R_2 \leftarrow R_2 + 3R_1$ and $R_3 \leftarrow R_3 - 2R_1$,

$$\left(\begin{array}{cccc|c} 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & -1 & 0 \\ 0 & -1 & -4 & 8 & 0 \end{array}\right).$$

Next, $R_3 \leftarrow R_3 + R_2$,

$$\left(\begin{array}{cccc|c} \boxed{1} & 3 & 1 & 0 & 0 \\ 0 & \boxed{1} & 4 & -1 & 0 \\ 0 & 0 & 0 & \boxed{7} & 0 \end{array}\right).$$

As the first, second and fourth columns are leading, p_1, p_2, p_4 are linearly independent polynomials. However, the third column is non-leading. This implies that we can express p_3 as a linear combination of p_1, p_2 and p_4.

To find this linear combination, set γ to be a free parameter (as it belongs to column 3, which is non-leading). From row 3, we see $\delta = 0$. Now, row 2 implies that $\beta + 4\gamma - \delta = 0 \Rightarrow \beta = -4\gamma$ (as $\delta = 0$. Now row 1 gives us $\alpha = -3\beta - \gamma = 12\gamma - \gamma = 11\gamma$, since $\beta = -4\gamma$. Hence substituting these into the original equation $\alpha p_1(x) + \beta p_2(x) + \gamma p_3(x) + \delta p_4(x) = 0$, we have

$$\begin{aligned} 11\gamma p_1(x) - 4\gamma p_2(x) + \gamma p_3(x) + 0p_4(x) = 0, \quad \forall \gamma \in \mathbb{R} \\ \Rightarrow \gamma p_3(x) = -11\gamma p_1(x) + 4\gamma p_2(x) \\ \Rightarrow p_3(x) = -11p_1(x) + 4p_2(x) \quad \text{(setting } \gamma = 1\text{).} \end{aligned}$$

Remarks/Tips. In your working out, you can just go straight to the augmented matrix in ($\star$). We have just provided detailed explanations before that step for your own benefit (so that you may understand why we set up that augmented matrix). Also, near the end, we set $\gamma = 1$. You could also just set $\gamma = 1$ at the start of the back-substitution (rather than using an arbitrary free parameter γ).

3. As we know, the vectors will form a basis for $\mathbb{R}^3$ if and only if the matrix whose columns are $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ has every row and column leading in the row echelon form. Set this up as a matrix:

$$\begin{pmatrix} \boxed{1} & 3 & 4 \\ 1 & 5 & 3 \\ -2 & 4 & -7 \end{pmatrix}.$$

Row-reducing by performing the row operations $R_2 \leftarrow R_2 - R_1$ and $R_3 \leftarrow R_3 + 2R_1$,

$$\begin{pmatrix} \boxed{1} & 3 & 4 \\ 0 & \boxed{2} & -1 \\ 0 & 10 & 1 \end{pmatrix}.$$

Next, $R_3 \leftarrow R_3 - 5R_2$,

$$\begin{pmatrix} \boxed{1} & 3 & 4 \\ 0 & \boxed{2} & -1 \\ 0 & 0 & \boxed{6} \end{pmatrix}.$$

As we can see, every row and column in the row echelon form is leading. Hence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form a basis for $\mathbb{R}^3$.

4. (i) False. If they are linearly dependent (e.g. because one of them is the zero vector), then it cannot be a basis.

   (ii) True. A basis contains the smallest number of linearly independent vectors to span the set, that is, there can only be 5 vectors in a basis for $\mathbb{R}^5$.

   (iii) False. For example, the set $\{\mathbf{e}_1, 2\mathbf{e}_1, 3\mathbf{e}_1, \ldots, 7\mathbf{e}_1\}$ (where $\mathbf{e}_1$ is the first standard basis vector of $\mathbb{R}^5$) will not span $\mathbb{R}^5$ (its span is just span $\{\mathbf{e}_1\}$, so for example $\mathbf{e}_2$ is not in this span, so the set is not spanning).

MATH1231/1241 Algebra S2 2011 Test 1 v1A

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1. To show that S is a subspace of $\mathbb{R}^3$, we must show three things: that S is non-empty, closed under addition and closed under scalar multiplication.

   - Clearly $\mathbf{0} = (0,0,0)^T \in S$ as $7 \times 0 + 3 \times 0 = 0$ and $0 - 4\times = 0$. So S is non-empty.
   - To show that S is closed under addition, let $\mathbf{x}, \mathbf{y} \in S$ where $\mathbf{x} = (x_1, x_2, x_3)^T$ and

$\mathbf{y} = (y_1, y_2, y_3)^T$ (note $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)^T \in \mathbb{R}^3$). That is,

$$\begin{aligned} 7x_1 + 3x_2 &= 0 \qquad (1) \\ x_2 - 4x_3 &= 0 \qquad (2) \\ 7y_1 + 3y_2 &= 0 \qquad (3) \\ y_2 - 4y_3 &= 0 \qquad (4). \end{aligned}$$

Adding (1) and (2) together, and (3) and (4) together, we obtain

$$\begin{aligned} (1) + (3): \quad & 7x_1 + 3x_2 + 7y_1 + 3y_2 = 0 \\ \implies & 7(x_1 + y_1) + 3(x_2 + y_2) = 0. \qquad (*) \\ (1) + (4): \quad & x_2 - 4x_3 + y_2 - 4y_3 = 0 \\ \implies & (x_2 + y_2) - 4(x_3 + y_3) = 0. \qquad (**) \end{aligned}$$

Hence, from $(*)$ and $(**)$, we see that$\mathbf{x} + \mathbf{y} \in S$ and so S is closed under addition.

- Next, to show that S is closed under scalar multiplication, let $\lambda \in \mathbb{R}$, with $\mathbf{x} \in S$ as before. So $\lambda\mathbf{x} = (\lambda x_1, \lambda x_2, \lambda x_3)^T \in \mathbb{R}^3$ and Equations (1) and (2) (from the previous part) hold again. Now, consider the following:

$$\begin{aligned} \lambda \times (1): \quad & \lambda(7x_1 + 3x_2) = \lambda \times 0 \\ \implies & 7(\lambda x_1) + 3(\lambda x_2) = 0 \\ \lambda \times (2): \quad & \lambda(x_2 - 4x_3) = \lambda \times 0 \\ \implies & (\lambda x_2) - 4(\lambda x_3) = 0. \end{aligned}$$

Hence, $\lambda\mathbf{x} \in S$ and so S is closed under scalar multiplication.

As S is non-empty, closed under addition and closed under scalar multiplication, it follows from the Subspace Theorem that S is a subspace of $\mathbb{R}^3$.

2. For $\mathbf{b}$ to be an element of span $(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3})$, there must exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\alpha\mathbf{v_1} + \beta\mathbf{v_2} + \gamma\mathbf{v_3} = \mathbf{b}.$$

Substituting in the vectors writing this as an augmented matrix as usual,

$$\left(\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 1 & 3 & 1 & b_2 \\ 2 & 4 & 1 & b_3 \\ -1 & 1 & 6 & b_4 \end{array}\right).$$

For $\mathbf{b}$ to be in the span, we must be able to row-reduce to find solutions for α, β and γ which will be in terms of b_1, b_2, b_3 and b_4.

Row-reducing, $R_2 \rightsquigarrow R_2 - R_1$, $R_3 \rightsquigarrow R_3 - 2R_1$ and $R_4 \rightsquigarrow R_4 + R_1$,

$$\left(\begin{array}{ccc|c} \boxed{1} & 2 & -2 & b_1 \\ 0 & 1 & 3 & -b_1 + b_2 \\ 0 & 0 & 5 & -2b_1 + b_3 \\ 0 & 3 & 4 & b_1 + b_4 \end{array}\right).$$

Next, $R_4 \rightsquigarrow R_4 - 3R_2$,

$$\left(\begin{array}{ccc|c} \boxed{1} & 2 & -2 & b_1 \\ 0 & \boxed{1} & 3 & -b_1 + b_2 \\ 0 & 0 & \boxed{5} & -2b_1 + b_3 \\ 0 & 0 & -5 & 4b_1 - 3b_2 + b_4 \end{array}\right).$$

Finally, $R_4 \rightsquigarrow R_4 + R_3$,

$$\left(\begin{array}{ccc|c} \boxed{1} & 2 & -2 & b_1 \\ 0 & \boxed{1} & 3 & -b_1 + b_2 \\ 0 & 0 & \boxed{5} & -2b_1 + b_3 \\ 0 & 0 & 0 & 2b_1 - 3b_2 + b_3 + b_4 \end{array}\right).$$

Hence, from row 4, the necessary and sufficient condition for $\mathbf{b}$ to be an element of $\text{span}(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4})$ (i.e. for a solution to exist) is that

$$2b_1 - 3b_2 + b_3 + b_4 = 0.$$

Note: the reason this is the condition for $\mathbf{b}$ *to be in the span of the vectors is that once this is true, we can perform back substitution for the rest of the values of* α, β *and* γ *without any problems (thus we can find a solution). In other words, there will be a solution if and only if the right-hand column in the row-echelon form is non-leading, as you learnt in Semester 1. The right-hand column here will be non-leading if and only if* $2b_1 - 3b_2 + b_3 + b_4 = 0$.*)*

3. (i) False. A basis for P_4 requires five linearly independent polynomials in P_4. Suppose the set of five polynomials were linearly dependent (a simple counterexample would be the set $\{0, \}$, which is clearly not a basis for P_4). Then it would be untrue.

   (ii) True. We require five linearly independent polynomials in P_4 for a basis in P_4. An example is the standard basis for P_4, namely $\left\{1, x, x^2, x^3, x^4\right\}$.

4. To show that T is not a linear transformation, we simply need to show that it does not satisfy a property of linear transformations. Knowing that linear transformations satisfy the scalar multiplication condition, i.e.

$$T(\lambda x) = \lambda T(x), \text{ for } \textit{all } \lambda, x$$

(and noticing that the expression is not going to satisfy this), we try apply this to the

given transformation.
Let $\lambda = 2 \in \mathbb{R}$ and $x = \frac{\pi}{2} \in \mathbb{R}$. Consider $T(\lambda x)$:

$$\begin{aligned} T(\lambda x) &= (\lambda x)\cos(\lambda x) \\ &= 2 \cdot \frac{\pi}{2} \cos\left(2 \cdot \frac{\pi}{2}\right) \\ &= \pi \cos \pi \\ &= -\pi, \end{aligned}$$

whereas $\lambda T(x) = 0$ (since $T(x) = T\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\cos\frac{\pi}{2} = 0$). Thus, T is not a linear transformation as it does not satisfy the scalar multiplication condition, since $T\left(2 \cdot \frac{\pi}{2}\right) \neq 2T\left(\frac{\pi}{2}\right)$.

As the above solution demonstrates, to show that a function T is *not* a linear map, it suffices to come up with a specific example where a property of linear maps is not satisfied. *Note: you could have chosen many other numbers for λ and x, as long as in the end, you find that $T(\lambda x) \neq \lambda T(x)$.*

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Answers/Hints & Worked Sample Solutions

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1. (i) **Hints**

Many, many (infinitely many in fact) choices for your answer. Choose a nice and easy value for x_1 or x_2 and solve for the other. (E.g. set $x_2 = 0$ and solve for x_1, which will then obviously have to be 5 for $\mathbf{x}$ to be in S.) An example answer is $\mathbf{x} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$.

Sample Answer

An example is the vector $\mathbf{x} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \in \mathbb{R}^2$, since $5 - 3 \times (0) = 5$.

(ii) **Hints**

To prove that S is not closed under multiplication (or addition), you can do this generally by finding a counterexample, i.e. naming a specific $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in S$, for example using your answer in part (i). Take a specific $\lambda \in \mathbb{R}$ and consider $\lambda\mathbf{x}$ and go from there to show that it is not closed under scalar multiplication. Here is a sample answer.

<u>Worked Sample Solution</u>

Let $\mathbf{x} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$, so $\mathbf{x} \in S$ from part (i). Now, $2\mathbf{x} = \begin{pmatrix} 10 \\ 0 \end{pmatrix}$, but $10 - 3 \times (0) = 10 \neq 5 \Rightarrow 2\mathbf{x} \notin S$. So S is not closed under scalar multiplication.

2. (i) **<u>Hints</u>**

For q to be in span (p_1, p_2, p_3), there must exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\alpha p_1 + \beta p_2 + \gamma p_3 = q$. Form an augmented matrix and row-reduce to eventually find that q does lie in the span (because the right-hand column will be leading)! Here is the sample working.

<u>Worked Sample Solution</u>

Let $S = \{p_1, p_2, p_3\}$. We try solving the equation

$$\alpha p_1(x) + \beta p_2(x) + \gamma p_3(x) = q(x) \qquad (1)$$

for real scalars α, β and γ. As usual, we place the polynomials (as coordinate vectors with respect to the standard basis for $\mathbb{P}_2$) as columns of an augmented matrix and row-reduce. Place p_1's coordinate vector in column 1, and similarly for p_2, p_3 in column 2 and 3. We place the coordinate vector of q in the augmented part. (The reason why we do all this is that we equate coefficients of like powers of x in Equation (1) and obtain simultaneous equations represented by such an augmented matrix.) We get

$$\left(\begin{array}{ccc|c} \boxed{1} & -4 & 1 & 2 \\ 1 & -3 & 2 & 1 \\ -2 & 3 & -7 & 1 \end{array}\right) \xrightarrow[R_3 \rightsquigarrow R_3 + 2R_2]{R_2 \rightsquigarrow R_2 - R_2} \left(\begin{array}{ccc|c} \boxed{1} & -4 & 1 & 2 \\ 0 & \boxed{1} & 1 & -1 \\ 0 & -5 & -5 & 5 \end{array}\right)$$

$$\xrightarrow{R_3 \rightsquigarrow R_3 + 5R_2} \left(\begin{array}{ccc|c} \boxed{1} & -4 & 1 & 2 \\ 0 & \boxed{1} & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This is now in row-echelon form, and since the right-hand column is non-leading here, there is a solution for $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\alpha p_1 + \beta p_2 + \gamma p_3 = q$, so q is in the span of S.

(ii) **<u>Hints</u>**

Recall that the process to decide whether the given set S is a basis for $\mathbb{P}_2$ is to place the polynomials (as vectors) into the columns of a matrix, and row-reduce. Then the set S will be a basis for $\mathbb{P}_2$ if and only if this matrix has every row and column leading in the row-echelon form. (Since S has 3 members and is a subset of a three-dimensional vector space, $\mathbb{P}_2$, the matrix in question will be *square*, so it is equivalent to say that S will be a basis if and only if the row-echelon form of the matrix has no zero rows, or equivalently has every column leading.) Here is a sample answer.

Sample Solution

From the working out of the previous question, the left-hand matrix above has a zero row in the row echelon form. Hence the given set S is *not* a basis for $\mathbb{P}_2$ (as this zero row means S is not spanning, so cannot be a basis). And so, the set does not span P_2.

3. (i) **Hints**

To evaluate this, recall the property of linear transformations that for scalars $\alpha, \beta \in \mathbb{R}$, we have

$$T\begin{pmatrix}1\\0\end{pmatrix} = T\left(\alpha\begin{pmatrix}2\\3\end{pmatrix} + \beta\begin{pmatrix}1\\2\end{pmatrix}\right) = \alpha T\begin{pmatrix}2\\3\end{pmatrix} + \beta T\begin{pmatrix}1\\2\end{pmatrix} \quad (*).$$

That is, all we really need to do is find α and β such that $\alpha\begin{pmatrix}2\\3\end{pmatrix} + \beta\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$. We can find using Gaussian Elimination (or inspection) that the solution is $\alpha = 2$ and $\beta = -3$. From this, we find that $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}3\\-2\end{pmatrix}$, using Equation $(*)$. Here is a worked sample solution.

Worked Sample Solution

We solve for α and β in the equation

$$\alpha\begin{pmatrix}2\\3\end{pmatrix} + \beta\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}.$$

By inspection, the solution is $\alpha = 2$, $\beta = -3$. (The motivation for trying this as a solution comes from looking at the second elements. An obvious solution to $3\alpha + 2\beta = 0$ is $\alpha = 2, \beta = -3$, and we observe that the first entries match up too in this case, so this is the solution.) Thus

$$
\begin{aligned}
T\begin{pmatrix}1\\0\end{pmatrix} &= T\left(2\begin{pmatrix}2\\3\end{pmatrix} - 3\begin{pmatrix}1\\2\end{pmatrix}\right)\\
&= 2T\begin{pmatrix}2\\3\end{pmatrix} - 3T\begin{pmatrix}1\\2\end{pmatrix} && \text{(linearity of } T\text{)}\\
&= 2\begin{pmatrix}9\\8\end{pmatrix} - 3\begin{pmatrix}5\\6\end{pmatrix} && \text{(given data)}\\
&= \begin{pmatrix}18-15\\16-18\end{pmatrix}\\
&= \begin{pmatrix}3\\-2\end{pmatrix}.
\end{aligned}
$$

(ii) **Hints**

We want to find the matrix of T (with respect to standard bases). The standard method to do this is to apply T to the standard basis vectors and place them in as columns of the matrix. In other words, column 1 of A should be $T\begin{pmatrix}1\\0\end{pmatrix}$ and column 2 should be $T\begin{pmatrix}0\\1\end{pmatrix}$. We have already calculated $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}3\\-2\end{pmatrix}$ earlier. We just need to find this second column, which is simply $T\begin{pmatrix}0\\1\end{pmatrix}$. To calculate $T\begin{pmatrix}0\\1\end{pmatrix}$, do a similar process to the first part of the question. A sample solution is below.

Worked Sample Solution

We have already calculated $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}3\\-2\end{pmatrix}$ earlier, so this is column 1 of A, so A is of the form $\begin{pmatrix}3 & ?\\-2 & ?\end{pmatrix}$.

Observe that $\begin{pmatrix}0\\1\end{pmatrix} = -1\begin{pmatrix}2\\3\end{pmatrix} + 2\begin{pmatrix}1\\2\end{pmatrix}$ (find this either by inspection or Gaussian

Elimination etc.). So

$$
\begin{aligned}
T\begin{pmatrix}0\\1\end{pmatrix} &= T\left(-1\begin{pmatrix}2\\3\end{pmatrix}+2\begin{pmatrix}1\\2\end{pmatrix}\right)\\
&= -T\begin{pmatrix}2\\3\end{pmatrix}+2T\begin{pmatrix}1\\2\end{pmatrix} && \text{(linearity)}\\
&= -\begin{pmatrix}9\\8\end{pmatrix}+2\begin{pmatrix}5\\6\end{pmatrix} && \text{(given data)}\\
&= \begin{pmatrix}1\\4\end{pmatrix}.
\end{aligned}
$$

So this is the second column, and thus the required matrix is $A = \begin{pmatrix}3 & 1\\-2 & 4\end{pmatrix}$.

MATH1231/1241 Algebra S2 2011 Test 1 v2A

Answers/Hints & Worked Sample Solutions

September 15, 2017

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1. **Hints**

   To show that S is a subspace of $\mathbb{R}^2$, we need to show that S is non-empty, is closed under addition and closed under scalar multiplication. From there, we can use the Subspace Theorem.

   Recall matrix properties such as $A\mathbf{x} + A\mathbf{y} = A(\mathbf{x} + \mathbf{y})$ to help answer the question. Here is a sample answer.

   Worked Sample Solutions

   Note that S is a sub*set* of $\mathbb{R}^2$, which is a vector space. Now, clearly S is nonempty, as the zero vector of $\mathbb{R}^2$, $\mathbf{0}_{\mathbb{R}^2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, indeed satisfies $A\mathbf{0}_{\mathbb{R}^2} = \mathbf{0}_{\mathbb{R}^3}$, where $\mathbf{0}_{\mathbb{R}^3}$ is the zero vector of $\mathbb{R}^3$, i.e. $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ (since any matrix times the relevant zero vector results in a zero vector).

Now, suppose $\mathbf{x}, \mathbf{y}$ are vectors in S. Then $\mathbf{x}+\mathbf{y} \in \mathbb{R}^2$ and

$$\begin{aligned} A(\mathbf{x}+\mathbf{y}) &= A\mathbf{x} + A\mathbf{y} \quad \text{(Distributive Law of matrix multiplication)} \\ &= \mathbf{0}_{\mathbb{R}^3} + \mathbf{0}_{\mathbb{R}^3} \quad (\text{as } \mathbf{x}, \mathbf{y} \in S) \\ &= \mathbf{0}_{\mathbb{R}^3} \\ &\Rightarrow \mathbf{x}+\mathbf{y} \in S. \end{aligned}$$

Hence S is closed under addition.
Finally, suppose $\mathbf{x}$ is a vector in S, and let α be a scalar. Then $\alpha\mathbf{x}$ is a vector in $\mathbb{R}^2$ satisfying

$$\begin{aligned} A(\alpha\mathbf{x}) &= \alpha(A\mathbf{x}) \\ &= \alpha\mathbf{0}_{\mathbb{R}^3} \quad (\text{as } \mathbf{x} \in S) \\ &= \mathbf{0}_{\mathbb{R}^3} \\ &\Rightarrow \alpha\mathbf{x} \in S. \end{aligned}$$

Hence S is closed under scalar multiplication. It follows from the Subspace Theorem that S is a subspace of $\mathbb{R}^2$.

2. **Hints**

The column space of A is the span of the columns, i.e. if $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 4 \\ -3 \\ 2 \\ 7 \end{pmatrix}$, then we want to see if $\mathbf{b} \in \operatorname{span}(\mathbf{v_1}, \mathbf{v_2})$.
To do this, we need to check if there exists $\alpha, \beta \in \mathbb{R}$ such that $\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 = \mathbf{b}$.
You will find that no solutions exist for α and β such that $\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 = \mathbf{b}$, and so $\mathbf{b}$ is not in the column space of A. Here is a sample answer.

Worked Sample Solutions

Note that $\mathbf{b}$ is in the column space of A if and only if there exist scalars α, β such that $\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 = \mathbf{b}$, where $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 4 \\ -3 \\ 2 \\ 7 \end{pmatrix}$.

So we try and solve $\alpha \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 4 \\ -3 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 5 \\ -6 \end{pmatrix}$.

Comparing the third entry of each vector $(0\alpha + 2\beta = 5)$, it is evident that β must be 2.5. So comparing fourth entries now, we have

$$\begin{aligned} 2\alpha + 7\beta &= -6 \\ 2\alpha + 7 \times 2.5 &= -6 \\ \Rightarrow 2\alpha &= -23.5 \\ \Rightarrow \alpha &= -11.75. \end{aligned}$$

But now the first entries tell us that

$$\begin{aligned} \alpha + 4\beta &= -1 \\ \Rightarrow -11.75 + 4 \times 2.5 &= -1 \\ \Rightarrow -11.75 + 10 &= -1 \\ \Rightarrow -1.75 &= -1, \end{aligned}$$

which is a contradiction. So there is no solution for α, β, so $\mathbf{b}$ is *not* in the column space of A.

Alternatively, you can create the corresponding augmented matrix:

$$\left(\begin{array}{cc|c} 1 & 4 & -1 \\ -1 & -3 & 5 \\ 0 & 2 & 5 \\ 2 & 7 & -6 \end{array}\right)$$

and after row-reducing, realise that no solutions exist for α and β.

3. (i) **Sample Answer**

   If S is a basis for V, then it is a set of linearly independent vectors, and so $m \leq n$. But also, $\operatorname{span}(S) = V$, and so $n \leq m$. This implies that $m = n$.

   Alternative answer: Recall that the definition of the dimension of a vector space is the number of vectors in any basis for it (provided this is finite). Hence by definition, m and n are equal.

   (ii) **Sample Answer**

   If S is linearly dependent, no relationship can be inferred between m and n. S can be linearly dependent no matter what value m takes, so long as another one of the vectors in S can be written as a linear combination of the others.

Alternative answer: Note that for *any* positive integer $m \geq 2$, the following set is linearly dependent: $\{\mathbf{0}, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$, where $\mathbf{v}_2, \ldots, \mathbf{v}_m$ are any vectors from V. (Recall that any set with the zero vector is linearly dependent.)
For the case when S only has one vector, S can still be linearly dependent if its only vector is the zero vector, $\mathbf{0}$.
This shows that there is no relation between m and n, since for *any* m, there exist linearly dependent sets.

4. **Hints**

Very similar to MATH1231/1241 Algebra 2011 S2 Test 1 v1B Q3.
To check if T is a linear transformation, it would be easiest to check if the following property holds: that $T(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha T(\mathbf{x}) + \beta T(\mathbf{y})$ where $\alpha, \beta \in \mathbb{R}$. A sample answer is provided below.

Worked Sample Solutions

We will answer this question with a proof by contradiction.
Suppose that T is linear. Let $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$. Note that $\mathbf{w} = 5\mathbf{e}_1 - \mathbf{e}_2$. So we have

$$\begin{aligned} T(\mathbf{w}) &= T(5\mathbf{e}_1 - \mathbf{e}_2) \\ &= 5T(\mathbf{e}_1) - T(\mathbf{e}_2) \quad \text{(linearity)} \\ &= 5\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \quad \text{(using given values)} \\ &= \begin{pmatrix} 3 \\ 2 \\ -7 \end{pmatrix}. \end{aligned}$$

But we were told that $T(\mathbf{w}) = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$, which is not equal to $\begin{pmatrix} 3 \\ 2 \\ -7 \end{pmatrix}$. So we have a contradiction. Hence T is not linear.

MATH1231/1241 Algebra S2 2012 Test 1 v1A

Answers/Hints & Worked Sample Solutions

September 15, 2017

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1. **Hints**

   To show that S is a subspace of $\mathbb{R}^3$, you need to show that it is non-empty, is closed under addition and under scalar multiplication. A core step you'll be required to make is $\mathbf{x} = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ such that $x_1 - 2x_2 = 0$ and $5x_2 + x_3 = 0$.

   Worked Sample Solution

   Note that clearly S is a sub*set* of $\mathbb{R}^3$, which is a vector space. Since $0 - 2 \times 0 = 0$ and $5 \times 0 + 0 = 0$, the zero vector from $\mathbb{R}^3$, namely $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, is in S, so S is non-empty.

Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ be in S. Then $\mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix}$ is an element of $\mathbb{R}^3$ since

$$\begin{aligned} (x_1 + y_1) - 2(x_2 + y_2) &= (x_1 - 2x_2) + (y_1 - 2y_2) \quad \text{(rearranging terms)} \\ &= 0 + 0 \quad (\text{since } \mathbf{x}, \mathbf{y} \in S) \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} 5(x_2 + y_2) + (x_3 + y_3) &= (5x_2 + x_3) + (5y_2 + y_3) \quad \text{(rearranging terms)} \\ &= 0 + 0 \quad (\text{since } \mathbf{x}, \mathbf{y} \in S) \\ &= 0. \end{aligned}$$

Thus S is closed under addition.

Now, suppose $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in S$ and $\alpha \in \mathbb{R}$, then $\alpha\mathbf{x} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{pmatrix}$ is a vector in $\mathbb{R}^3$. Moreover,

$$\begin{aligned} \alpha x_1 - 2(\alpha x_2) &= \alpha(x_1 - 2x_2) \\ &= \alpha \times 0 \quad (\text{as } \mathbf{x} \in S) \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} 5(\alpha x_2) + \alpha x_3 &= \alpha(5x_2 + x_3) \\ &= \alpha \times 0 \quad (\text{as } \mathbf{x} \in S) \\ &= 0. \end{aligned}$$

Hence S is closed under scalar multiplication.

It follows by the Subspace Theorem that S is a subspace of $\mathbb{R}^3$.

2. **Hints**

Set up an augmented matrix and row-reduce. Your conditions on $\mathbf{b}$ to lie in the span$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ should look like

$$15b_1 + 6b_2 - b_3 + b_4 = 0.$$

Worked Sample Solution

As usual we set up the relevant augmented matrix, namely $\left(\begin{array}{ccc|c} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{b} \end{array}\right)$, and row-reduce. Note that this matrix corresponds to the equation $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{b}$. This is because (by definition of span) $\mathbf{b} \in \text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ if and only if this equation has a solution for α_1, α_2 and α_3.

We can represent $\mathbf{b}$ in the augmented matrix either as $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$ or by creating a column each for b_1, b_2, b_3, b_4. Here, we do the latter. And so, we have

$$\left[\begin{array}{ccc|cccc} \boxed{1} & 3 & -2 & 1 & 0 & 0 & 0 \\ -2 & -5 & 5 & 0 & 1 & 0 & 0 \\ 0 & 7 & 1 & 0 & 0 & 1 & 0 \\ -3 & -8 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_4 \rightsquigarrow R_4 + 3R_1]{R_2 \rightsquigarrow R_2 + 2R_1} \left[\begin{array}{ccc|cccc} \boxed{1} & 3 & -2 & 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & 1 & 2 & 1 & 0 & 0 \\ 0 & 7 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -5 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_4 \rightsquigarrow R_4 - R_2]{R_3 \rightsquigarrow R_3 - 7R_2} \left[\begin{array}{ccc|cccc} \boxed{1} & 3 & -2 & 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & \boxed{-6} & -14 & -7 & 1 & 0 \\ 0 & 0 & -6 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_4 \rightsquigarrow R_4 - R_3} \left[\begin{array}{ccc|cccc} \boxed{1} & 3 & -2 & 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & \boxed{-6} & -14 & -7 & 1 & 0 \\ 0 & 0 & 0 & 15 & 6 & -1 & 1 \end{array}\right].$$

Recall that $\mathbf{b}$ will be in the span if and only if the columns on the right-hand side of the augmented matrix in row-echelon form are all non-leading. We see that from row 4 (the only row with a zero row in the left-hand part), the necessary and sufficient condition on b_1, b_2, b_3, b_4 for $\mathbf{b}$ to be in the span of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is $15b_1 + 6b_2 - b_3 + b_4 = 0$.

3. **<u>Hints</u>**

Remember that the dimension of P_5 is 6 (not 5). (In general, the dimension of P_n is $n+1$, for any $n \in \mathbb{N}$.) Also make sure to recall the relationships between dimension, basis, and the number of elements in a set.

<u>Worked Sample Solution</u>

(i) False. A counterexample will be a set of six polynomials that are all constant polynomials. Then, this set will not span $\mathbb{P}_5$, and so will not be a basis.

(ii) True, as the dimension of $\mathbb{P}_5$ is 6 and so six linearly independent polynomials in $\mathbb{P}_5$ will form a basis for $\mathbb{P}_5$. For example, the standard basis for $\mathbb{P}_5$ $\big($the set $\{1, t, t^2, t^3, t^4, t^5\}\big)$ is a set of six polynomials in $\mathbb{P}_5$ that form a basis for $\mathbb{P}_5$.

3. **<u>Hints</u>**

We need to check which properties of linear transformations do not hold. Remember, to show that a map T is *not* linear, we just need to produce a single example that demonstrates that T does not satisfy a condition of linearity.

Worked Sample Solution

Note that $T\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)^2 \sin\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)^2$. Also, $T\left(2\left(\frac{\pi}{2}\right)\right) = T(\pi) = \pi^2 \sin \pi = 0$. Hence $T\left(2\left(\frac{\pi}{2}\right)\right) \neq 2T\left(\frac{\pi}{2}\right)$, and so T does not satisfy the scalar multiplication condition, whence T is not linear.

MATH1231/1241 Algebra S2 2012 Test 1 v1B

Answers/Hints & Worked Sample Solutions

September 15, 2017

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1. (i) There are many, many different choices. But one easy example you can try is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, since $1 - 5 \times 0 < 2$.

   (ii) **Hints**

   To show that a set is not closed under scalar multiplication, we need to find a vector in the set, such that when we multiply this vector by a scalar, the resulting vector will not be in S.

   Worked Sample Solution

   Using the vector in part (i) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we can multiply it by 2, so that the resulting vector is $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$. Clearly, $2 - 5(0) = 2$. So the resulting vector is not in S. Hence, S is not closed under scalar multiplication.

2. (i) **Hints**

To check if a particular polynomial is an element of the span of the given 3 vectors, we just need to see if we can express this particular polynomial as a linear combination of these 3 vectors, i.e. if there exist α_1, α_2 and α_3 such that $\alpha_1 p_1(x) + \alpha_2 p_2(x) + \alpha_3 p_3(x) = q(x)$.

Worked Sample Solution

We construct the corresponding augmented matrix to represent this equation.

$$\left(\begin{array}{ccc|c} \boxed{1} & 3 & -1 & 1 \\ -1 & -1 & 0 & 2 \\ 2 & 0 & 1 & -7 \end{array}\right).$$

Performing the row operations $R_2 \rightsquigarrow R_2 + R_1$ and $R_3 \rightsquigarrow R_3 - 2R_1$ gives the augmented matrix

$$\left(\begin{array}{ccc|c} \boxed{1} & 3 & -1 & 1 \\ 0 & \boxed{2} & -1 & 3 \\ 0 & -6 & 3 & -9 \end{array}\right).$$

Performing $R_3 \rightsquigarrow R_3 + 3R_2$ gives us the row-echelon form

$$\left(\begin{array}{ccc|c} \boxed{1} & 3 & -1 & 1 \\ 0 & \boxed{2} & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The right-hand column is non-leading, so q **is** an element of $\operatorname{span}(p_1, p_2, p_3)$.

(ii) **Hints**

Look at the row-echelon form of the matrix whose columns contain p_1, p_2, p_3 (as vectors), i.e. the left-hand part of the matrix in (i). Remember what the row-echelon form has to look like in order for the set to be spanning.

Worked Sample Solution

The given set would span P_2 if and only if the left-hand part of the augmented matrix in part (i) (the matrix whose columns contain the coordinate vectors of p_1, p_2, p_3 (with respect to the standard basis of P_2)) had no zero rows in its row-echelon form. But looking at its row-echelon form that we found, we see that this is not the case (since the third row is a zero row). Therefore, the given set does **not** span P_2.

3. (i) **Hints**

To evaluate this, recall the property of linear transformations that for scalars $\alpha, \beta \in$

$\mathbb{R}$, we have

$$T\begin{pmatrix}1\\0\end{pmatrix} = T\left(\alpha\begin{pmatrix}5\\3\end{pmatrix} + \beta\begin{pmatrix}2\\1\end{pmatrix}\right) = \alpha T\begin{pmatrix}5\\3\end{pmatrix} + \beta T\begin{pmatrix}2\\1\end{pmatrix} \quad (*).$$

We're already given what $T\begin{pmatrix}5\\3\end{pmatrix}$ and $T\begin{pmatrix}2\\1\end{pmatrix}$ are. So, all we really need to do is find α and β such that $\alpha\begin{pmatrix}5\\3\end{pmatrix} + \beta\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$. We can find using Gaussian Elimination (or inspection) that the solution is $\alpha = -1$ and $\beta = 3$. From this, we find that $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\-2\end{pmatrix}$, using Equation $(*)$. Here is a worked sample solution.

Worked Sample Solution

We solve for α and β in the equation

$$\alpha\begin{pmatrix}5\\3\end{pmatrix} + \beta\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}.$$

After row-reducing the augmented matrix

$$\left(\begin{array}{cc|c}5 & 2 & 1\\3 & 1 & 0\end{array}\right) \xrightarrow{R_2 \rightsquigarrow R_2 - \frac{3}{5}R_1} \left(\begin{array}{cc|c}5 & 2 & 1\\0 & -\frac{1}{5} & -\frac{3}{5}\end{array}\right)$$

we know that $-\frac{1}{5}\beta = -\frac{3}{5}$ and $5\alpha + 2\beta = 1$. Solving these simultaneously, we find that $\alpha = -1$ and $\beta = 3$.

Thus

$$\begin{aligned}
T\begin{pmatrix}1\\0\end{pmatrix} &= T\left(-\begin{pmatrix}5\\3\end{pmatrix} + 3\begin{pmatrix}2\\1\end{pmatrix}\right)\\
&= -T\begin{pmatrix}5\\3\end{pmatrix} + 3T\begin{pmatrix}2\\1\end{pmatrix} && \text{(linearity of } T\text{)}\\
&= -\begin{pmatrix}8\\-1\end{pmatrix} + 3\begin{pmatrix}3\\-1\end{pmatrix} && \text{(given data)}\\
&= \begin{pmatrix}-8+9\\1-3\end{pmatrix}\\
&= \begin{pmatrix}1\\-2\end{pmatrix}.
\end{aligned}$$

(ii) **Hints**

We want to find the matrix of T (with respect to standard bases). The standard method to do this is to apply T to the standard basis vectors and place them in as columns of the matrix. In other words, column 1 of A should be $T\begin{pmatrix}1\\0\end{pmatrix}$ and column 2 should be $T\begin{pmatrix}0\\1\end{pmatrix}$. We have already calculated $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\-2\end{pmatrix}$ earlier. We just need to find the second column, which is simply $T\begin{pmatrix}0\\1\end{pmatrix}$. To calculate $T\begin{pmatrix}0\\1\end{pmatrix}$, do a similar process to the first part of the question. A sample solution is below.

Worked Sample Solution

We have already calculated $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\-2\end{pmatrix}$ earlier, so this is column 1 of A, so A is of the form $\begin{pmatrix}1 & ?\\-2 & ?\end{pmatrix}$.

Observe that $\begin{pmatrix}0\\1\end{pmatrix} = 2\begin{pmatrix}2\\3\end{pmatrix} - 5\begin{pmatrix}1\\2\end{pmatrix}$ (find this either by inspection or Gaussian Elimination similar to the previous part). So

$$
\begin{aligned}
T\begin{pmatrix}0\\1\end{pmatrix} &= T\left(2\begin{pmatrix}5\\3\end{pmatrix} - 5\begin{pmatrix}2\\1\end{pmatrix}\right) \\
&= 2T\begin{pmatrix}5\\3\end{pmatrix} - 5T\begin{pmatrix}2\\1\end{pmatrix} && \text{(linearity)} \\
&= 2\begin{pmatrix}8\\-1\end{pmatrix} - 5\begin{pmatrix}3\\-1\end{pmatrix} && \text{(given data)} \\
&= \begin{pmatrix}1\\3\end{pmatrix}.
\end{aligned}
$$

So this is the second column, and thus the required matrix is $A = \begin{pmatrix}1 & 1\\-2 & 3\end{pmatrix}$.