MATH1151 Algebra Test 2 2008 S1 v1A

January 28, 2015

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1. A matrix is invertible if and only if the determinant is not zero. So let's find the determinant.

$$\begin{aligned}
\det(A) &= 1 \times \det \begin{pmatrix} -2 & -1 \\ 3 & 0 \end{pmatrix} - 1 \times \begin{pmatrix} 2 & -1 \\ a & 0 \end{pmatrix} + a \times \begin{pmatrix} 2 & -2 \\ a & 3 \end{pmatrix} \\
&= (-2 \times 0 - (-1) \times 3) - (2 \times 0 - (-1) \times a) + (a \times (2 \times 3 - (-2) \times a) \\
&= 3 - a + a(6 + 2a) \\
&= 3 + 5a + 2a^2
\end{aligned}$$

Hence, the matrix is invertible if:

$$
\begin{aligned}
3 + 5a + 2a^2 &\neq 0 \\
(2a+3)(a+1) &\neq 0 \\
a &\neq -1, -\frac{3}{2}
\end{aligned}
$$

2. (i) We should know that the cross product of two vectors is perpendicular to the two vectors themselves. Hence, we want to find $\boldsymbol{a} \times \boldsymbol{b}$.

   To make it easy for ourselves, we should remember that

$$
\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}
$$

   So

$$
\begin{aligned}
\boldsymbol{a} \times \boldsymbol{b} &= \begin{vmatrix} \boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} \\ -1 & 4 & 2 \\ -2 & -1 & 3 \end{vmatrix} \\
&= \boldsymbol{e_1}(4 \times 3 - 2 \times (-1)) - \boldsymbol{e_2}((-1) \times 3 - 2 \times (-2)) + \boldsymbol{e_3}((-1) \times (-1) - 4 \times (-2)) \\
&= \begin{pmatrix} 14 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix} \\
&= \begin{pmatrix} 14 \\ -1 \\ 9 \end{pmatrix}
\end{aligned}
$$

   (ii) We know that the projection of $\boldsymbol{c}$ on to $\boldsymbol{a}$ is given by

$$
\text{proj}_{\boldsymbol{a}}\boldsymbol{c} = \left(\frac{\boldsymbol{a} \cdot \boldsymbol{c}}{|\boldsymbol{a}|^2} \right) \boldsymbol{a}
$$

   .

Hence,

$$\text{proj}_{\boldsymbol{a}}\boldsymbol{c} = \frac{\left(\begin{smallmatrix}-1\\4\\2\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}8\\-5\\-7\end{smallmatrix}\right)}{1^2+4^2+2^2} \times \begin{pmatrix}-1\\4\\2\end{pmatrix}$$
$$= -2\begin{pmatrix}-1\\4\\2\end{pmatrix}$$

3. If the points were all on the line, we would have:

$$\begin{aligned}\alpha+\beta &= 3\\ \alpha+2\beta &= -1\\ \alpha+4\beta &= 2\\ \alpha+5\beta &= 0\end{aligned}$$

or:
$$\begin{pmatrix}1&1\\1&2\\1&4\\1&5\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \begin{pmatrix}3\\-1\\2\\0\end{pmatrix}$$

Let $A = \begin{pmatrix}1&1\\1&2\\1&4\\1&5\end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$ and $\boldsymbol{y} = \begin{pmatrix}3\\-1\\2\\0\end{pmatrix}$.

We should know that the least squares approximate solution $\boldsymbol{x_0}$ to $A\boldsymbol{x} = \boldsymbol{y}$ is the solution to the normal equations
$$A^TA\boldsymbol{x} = A^T\boldsymbol{y}$$

Hence we want to solve

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \\ 1 & 5 \end{pmatrix} \boldsymbol{x} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 12 \\ 12 & 46 \end{pmatrix} \boldsymbol{x} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$$

If we solve this by row reduction and back substitution (hopefully you know how to do this!), you should get $\alpha = \frac{19}{10}$ and $\beta = -\frac{3}{10}$.

4. By reading the coefficients, we know that a vector normal to the plane is $\left(\begin{smallmatrix} 1 \\ -2 \\ 1 \end{smallmatrix}\right)$. From this, we can tell that the line going through the point A perpendicular to the plane is represented by

$$\boldsymbol{x} = \begin{pmatrix} 2 \\ 7 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2+\lambda \\ 7-2\lambda \\ -5+\lambda \end{pmatrix}, \lambda \in \mathbb{R}$$

Hence, point P lies on this line, and it also lies on the plane, so we can substitute the equation of this line into the equation of the plane and solve for λ in order to find the intersection between the line and plane. Substituting it in:

$$\begin{aligned} (2+\lambda) - 2 \times (7-2\lambda) + (-5+\lambda) &= 4 \\ 2 + \lambda - 14 + 4\lambda - 5 + \lambda &= 4 \\ 6\lambda &= 21 \\ \lambda &= \frac{7}{2} \end{aligned}$$

Hence, point P is represented by

$$\begin{pmatrix} 2 + \frac{7}{2} \\ 7 - 2(\frac{7}{2}) \\ -5 + \frac{7}{2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 11 \\ 0 \\ -3 \end{pmatrix}$$

MATH1151 Algebra Test 2 2009 v1b

April 17, 2019

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Question 1

Since we do not know if there will be points of intersection, *and* we are asked to find them if they exist, we may as well attempt Gaussian elimination. Equating the two expressions gives

$$\begin{pmatrix}1\\2\\-5\end{pmatrix}+\lambda\begin{pmatrix}1\\8\\-1\end{pmatrix}=\begin{pmatrix}3\\-1\\0\end{pmatrix}+\mu\begin{pmatrix}-1\\2\\3\end{pmatrix}+\nu\begin{pmatrix}2\\1\\-5\end{pmatrix}$$

$$\implies \begin{pmatrix}-2\\3\\5\end{pmatrix}=\mu\begin{pmatrix}-1\\2\\3\end{pmatrix}+\nu\begin{pmatrix}2\\1\\-5\end{pmatrix}+\lambda\begin{pmatrix}-1\\-8\\1\end{pmatrix}.$$

Hence consider

$$\left(\begin{array}{ccc|c}-1&2&-1&-2\\2&1&-8&3\\3&-5&1&5\end{array}\right)\xrightarrow[R_3\leftarrow R_3+3R_1]{R_2\leftarrow R_2+2R_1}\left(\begin{array}{ccc|c}-1&2&-1&-2\\0&5&-10&-1\\0&1&-2&-1\end{array}\right)$$

$$\xrightarrow{R_3\leftarrow R_3-\frac{1}{5}R_2}\left(\begin{array}{ccc|c}-1&2&-1&-2\\0&5&-10&-1\\0&0&0&-\frac{4}{5}\end{array}\right).$$

We see that in a row-echelon form of the augmented matrix, the right-hand column is leading. Hence this system has no solutions, so there are no points of intersection between the plane and the line.

Question 2

Part i)

Note that you can jump straight to the final answer if you wish. In the test, however, the non-calculator restriction can get annoying, so I prefer to do the computations slowly.

$$\begin{aligned} A^2 &= \begin{pmatrix} 2 & 4 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ -3 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 4-12 & 8-4 \\ -6+3 & -12+1 \end{pmatrix} \\ &= \begin{pmatrix} -8 & 4 \\ -3 & -11 \end{pmatrix}. \end{aligned}$$

So we require constants u and v that satisfy

$$\begin{pmatrix} -8 & 4 \\ -3 & -11 \end{pmatrix} = u \begin{pmatrix} 2 & 4 \\ -3 & -1 \end{pmatrix} + v \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If you can see by inspection that $u = 1$ and $v = -10$, well done. Otherwise, the most instructive approach is just to equate components. It may be worth noting that the equality can be expressed as

$$\begin{pmatrix} -8 & 4 \\ -3 & -11 \end{pmatrix} = \begin{pmatrix} 2u+v & 4u \\ -3u & -u+v \end{pmatrix}$$

to make life easier, but this step is optional. We can equate the a_{21} or a_{12} components to get $\boxed{u = 1}$, and then either a_{11} or a_{22} components to get $\boxed{v = -10}$.

Part ii)

Essentially what we do now is recursively use the fact that $\boxed{A^2 = A - 10I}$. This allows us to gradually build our powers up. To reduce the mess, we first consider

$$\begin{aligned} A^4 &= A^2A^2 \\ &= (A-10I)(A-10I) && (*) \\ &= A^2 - 10A - 10A + 100I^2 \\ &= A^2 - 20A + 100I && (\text{because } I^2 = I) \\ &= A - 10I - 20A + 100I && (*) \\ &= -19A + 90I. \end{aligned}$$

Then

$$
\begin{aligned}
A^5 &= AA^4 \\
&= A(-19A + 90I) && (*) \\
&= -19A^2 + 90A \\
&= -19(A - 10I) + 90A \\
&= 71A + 190I
\end{aligned}
$$

and hence $x = 71$, $y = 190$. Note that at every instance of (*), I used the boxed identity above.

Part iii) Usually, if nothing stands obvious, row reductions are a good way of commencing determinant problems.

$$
\begin{aligned}
\det \begin{pmatrix} -1 & 3 & 2 \\ 3 & -2 & -7 \\ -5 & 1 & 8 \end{pmatrix} &= \det \begin{pmatrix} -1 & 3 & 2 \\ 0 & 7 & -1 \\ 0 & -14 & -2 \end{pmatrix} && (R_2 \leftarrow R_2 + 3R_1 \ \& \ R_3 \leftarrow R_3 - 5R_1) \\
&= \det \begin{pmatrix} -1 & 3 & 2 \\ 0 & 7 & -1 \\ 0 & 0 & -4 \end{pmatrix} && (R_3 \leftarrow R_3 + 2R_2)
\end{aligned}
$$

The nice thing about these is that ultimately we will end up with an upper triangular matrix, at which point we can use the shortcut of multiplying the elements on the main diagonal to obtain

$$
\det \begin{pmatrix} -1 & 3 & 2 \\ 3 & -2 & -7 \\ -5 & 1 & 8 \end{pmatrix} = 28.
$$

Note: Later problems will see alterations. Row reductions are more or less a go-to option when there's nothing else that's convenient. Sometimes there are though!

Question 4

We have

$$
\left|\overrightarrow{AB}\right| = \left| \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix} \right| = \left| \begin{pmatrix} -4 \\ -4 \\ 3 \end{pmatrix} \right| = \sqrt{41}
$$

and

$$
\left|\overrightarrow{AC}\right| = \left| \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix} \right| = \left| \begin{pmatrix} -3 \\ -5 \\ 1 \end{pmatrix} \right| = \sqrt{35}
$$

Also

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = \begin{pmatrix} -4 \\ -4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -5 \\ 1 \end{pmatrix} = 35$$

and hence $\cos\theta = \frac{35}{\sqrt{41}\sqrt{35}} = \sqrt{\frac{35}{41}}$. Note that $\overrightarrow{CA}$ was used in favour of $\overrightarrow{AC}$ because we required the angle at A. Because we already found $\overrightarrow{AB}$, which starts at A and ends at B, we choose the similar one for C.

MATH1151 Algebra Test 2 2018 v3a

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Question 1
We present two valid approaches.

Method 1
This method relies on Gaussian elimination. If $\mathbf{a}$ is parallel to the plane, then $\mathbf{a}$ should be a linear combination of the vectors that span the plane. That is, the equation

$$\lambda \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 7 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$$

should have a solution for λ and μ. This is now a system of equations, so we consider the row reductions

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ -3 & -1 & -2 \\ 1 & 7 & 5 \end{array}\right) \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 + 3R_i} \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 5 & 7 \\ 0 & 5 & 2 \end{array}\right)$$
$$\xrightarrow{R_3 \leftarrow R_3 - R_2} \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -5 \end{array}\right).$$

So we see that in the row-echelon form of the augmented matrix, the right column is leading. Hence there is no solution, and thus $\mathbf{a}$ is not parallel to the plane.

Method 2

This method relies on vector geometry techniques. We start by finding a vector normal (i.e. perpendicular) to the plane, which can of course be done so with the cross product.

$$\mathbf{n} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 7 \end{pmatrix} = \begin{pmatrix} -20 \\ -5 \\ 5 \end{pmatrix}.$$

The idea is, $\mathbf{a}$ can only be parallel to the plane, if it is *also* perpendicular to $\mathbf{n}$. This would require $\mathbf{a} \cdot \mathbf{n} = 0$, but in fact we see that

$$\mathbf{a} \cdot \mathbf{n} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -20 \\ -5 \\ 5 \end{pmatrix} = -25 \neq 0.$$

So by contradiction, $\mathbf{a}$ is not parallel to the plane.

Remark: Any scalar multiple of $\mathbf{n}$ *would have worked as well. So we could've considered dotting against* $\begin{pmatrix} -4 \\ -1 \\ 1 \end{pmatrix}$ *to make the numbers smaller, and hence our lives easier.*

Question 2

We take the augmented matrix $(A \mid I)$ and find its reduced row-echelon form, which will be of the form $(I \mid B)$. Then B is our required matrix, i.e. $B = A^{-1}$. Note that this process is fairly tedious.

$$
\begin{aligned}
&\left(\begin{array}{ccc|ccc} -1 & -2 & 2 & 1 & 0 & 0 \\ -1 & 1 & 3 & 0 & 1 & 0 \\ 2 & 1 & -4 & 0 & 0 & 1 \end{array}\right) \\
\xrightarrow[R_3 \leftarrow R_3 + 2R_1]{R_2 \leftarrow R_2 - R_1} &\left(\begin{array}{ccc|ccc} -1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 3 & 1 & -1 & 1 & 0 \\ 0 & -3 & 0 & 2 & 0 & 1 \end{array}\right) \\
\xrightarrow{R_3 \leftarrow R_3 + R_2} &\left(\begin{array}{ccc|ccc} -1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 3 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right) \\
\xrightarrow[R_2 \leftarrow R_2 - R_3]{R_1 \leftarrow R_1 - 2R_3} &\left(\begin{array}{ccc|ccc} -1 & -2 & 0 & -1 & -2 & -2 \\ 0 & 3 & 0 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right) \\
\xrightarrow[R_1 \leftarrow R_1 + 2R_2]{R_2 \leftarrow \frac{1}{3} R_2} &\left(\begin{array}{ccc|ccc} -1 & 0 & 0 & -\frac{7}{3} & -2 & -\frac{8}{3} \\ 0 & 1 & 0 & -\frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right) \\
\xrightarrow{R_1 \leftarrow -R_1} &\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{3} & -2 & \frac{8}{3} \\ 0 & 1 & 0 & -\frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right).
\end{aligned}
$$

Therefore $A^{-1} = \begin{pmatrix} \frac{7}{3} & 2 & \frac{8}{3} \\ -\frac{2}{3} & 0 & -\frac{1}{3} \\ 1 & 1 & 1 \end{pmatrix}$.

Question 3

Since the equation of the plane is given in Cartesian form, we can read off the coefficients to get a normal vector

$$\mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}.$$

Note that this is how the Cartesian and point-normal equations of a plane are linked.

For a point on the plane, I believe the easiest choice to consider is $(0, 5, 0)$. A vector joining this to $(8, -5, 4)$ is

$$\begin{pmatrix} 8 \\ -5 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ -10 \\ 4 \end{pmatrix}.$$

A vector with the shortest distance between $(8, -5, 4)$ and the plane can then be found by projecting the above vector onto $\mathbf{n}$. Note that since we just require the length $\left|\text{proj}_{\mathbf{n}}\begin{pmatrix}8\\-10\\4\end{pmatrix}\right|$,

it suffices to consider

$$\begin{aligned}\left|\text{proj}_{\mathbf{n}}\begin{pmatrix}8\\-10\\4\end{pmatrix}\right| &= \frac{\left|\begin{pmatrix}8\\-10\\4\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\-4\end{pmatrix}\right|}{\left|\begin{pmatrix}2\\-1\\-4\end{pmatrix}\right|}\\ &= \frac{|16+10+16|}{\sqrt{4+1+16}}\\ &= \frac{42}{\sqrt{21}}\\ &= 2\sqrt{21}.\end{aligned}$$

Question 4

Recall that the area of the triangle is half that of the parallelogram, spanned by the vectors joining the vertices. For two possible vectors, take

$$\begin{pmatrix}5\\-2\\0\end{pmatrix} - \begin{pmatrix}3\\1\\-2\end{pmatrix} = \begin{pmatrix}2\\-3\\2\end{pmatrix}$$

and

$$\begin{pmatrix}6\\0\\-1\end{pmatrix} - \begin{pmatrix}3\\1\\-2\end{pmatrix} = \begin{pmatrix}3\\-1\\1\end{pmatrix}.$$

Recall that the area of the parallelogram spanned is found by taking the magnitude of the cross products. Here, the cross product is

$$\begin{pmatrix}2\\-3\\2\end{pmatrix} \times \begin{pmatrix}3\\-1\\3\end{pmatrix} = \begin{pmatrix}-1\\4\\7\end{pmatrix}.$$

So the area of the *triangle* is $\dfrac{1}{2}\sqrt{1+16+49} = \dfrac{\sqrt{66}}{2}$.

MATH1151 Algebra Test 2 2018 v3b

April 17, 2019

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Question 1

For any points of intersection, upon equating,

$$\begin{aligned}\begin{pmatrix}3\\-6\\7\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\-5\end{pmatrix}&=\begin{pmatrix}1\\8\\5\end{pmatrix}+\mu\begin{pmatrix}-1\\3\\2\end{pmatrix}\\ \implies \mu\begin{pmatrix}-1\\3\\2\end{pmatrix}+\lambda\begin{pmatrix}-1\\-1\\5\end{pmatrix}&=\begin{pmatrix}2\\-14\\2\end{pmatrix}\end{aligned}$$

If you can see by inspection that $\boxed{\mu=-4}$ and $\boxed{\lambda=2}$, well done - you may skip the Gaussian elimination. Otherwise, consider

$$\begin{aligned}\left(\begin{array}{cc|c}-1&-1&2\\3&-1&-14\\2&5&2\end{array}\right)&\xrightarrow[R_3\leftarrow R_3+2R_1]{R_2\leftarrow R_2+3R_1}\left(\begin{array}{cc|c}-1&-1&2\\0&-4&-8\\0&3&6\end{array}\right)\\ &\xrightarrow[R_2\leftarrow -\frac{1}{4}R_2]{R_2\leftarrow R_3-3R_2}\left(\begin{array}{cc|c}-1&-1&2\\0&1&2\\0&0&0\end{array}\right).\end{aligned}$$

So back subbing, R_2 immediately gives us $\lambda=2$. So subbing back, the point corresponds to

$$\begin{pmatrix}3\\-6\\7\end{pmatrix}+2\lambda\begin{pmatrix}1\\1\\-5\end{pmatrix}=\begin{pmatrix}5\\-4\\-3\end{pmatrix}.$$

Note: If you have time, you probably don't want to stop here. Back-sub further into R_1 to get $\mu = -4$, and check that if you sub this value into the equation of the second line, you also get the point $(5, -4, -3)$.

Question 2

Once again, we find the required reduced row-echelon form.

$$
\begin{aligned}
&\left(\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ -3 & 1 & -3 & 0 & 1 & 0 \\ 1 & 5 & -8 & 0 & 0 & 1 \end{array}\right) \\
\xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 + 3R_1} &\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -2 & 3 & 3 & 1 & 0 \\ 0 & 6 & -10 & -1 & 0 & 1 \end{array}\right) \\
\xrightarrow{R_3 \leftarrow R_3 + 3R_2} &\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -2 & 3 & 3 & 1 & 0 \\ 0 & 0 & -1 & 8 & 3 & 1 \end{array}\right) \\
\xrightarrow[R_2 \leftarrow R_2 + 3R_3]{R_1 \leftarrow R_1 + 2R_3} &\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 17 & 6 & 2 \\ 0 & -2 & 0 & 27 & 10 & 3 \\ 0 & 0 & -1 & 8 & 3 & 1 \end{array}\right) \\
\xrightarrow[R_3 \leftarrow -R_3]{R_2 \leftarrow -\frac{1}{2}R_2} &\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 17 & 6 & 2 \\ 0 & 1 & 0 & -\frac{27}{2} & -5 & -\frac{3}{2} \\ 0 & 0 & 1 & -8 & -3 & -1 \end{array}\right) \\
\xrightarrow{R_1 \leftarrow R_1 + R_2} &\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{2} & 1 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{27}{2} & -5 & -\frac{3}{2} \\ 0 & 0 & 1 & -8 & -3 & -1 \end{array}\right).
\end{aligned}
$$

So $A^{-1} = \begin{pmatrix} \frac{7}{2} & 1 & \frac{1}{2} \\ -\frac{27}{2} & -5 & -\frac{3}{2} \\ -8 & -3 & -1 \end{pmatrix}$.

Question 3

We approach this similar to the other version. A normal vector is $\mathbf{n} = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$ and by inspection $(11, 0, 0)$ lies on the plane. Hence for a vector from $(-7, 8, -9)$ to the plane, take

$$\begin{pmatrix} 11 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} -7 \\ 8 \\ -9 \end{pmatrix} = \begin{pmatrix} 18 \\ -8 \\ 9 \end{pmatrix}.$$

The shortest distance is

$$\left|\operatorname{proj}_{\mathbf{n}}\begin{pmatrix}18\\-8\\9\end{pmatrix}\right| = \frac{\left|\begin{pmatrix}18\\-8\\9\end{pmatrix}\cdot\begin{pmatrix}1\\-3\\4\end{pmatrix}\right|}{\left|\begin{pmatrix}1\\-3\\4\end{pmatrix}\right|} = \frac{78}{\sqrt{26}} = 3\sqrt{26}.$$

Question 4

We approach this similar to the other version. For two vectors that form the sides of the triangle, consider

$$\begin{pmatrix}3\\-3\\8\end{pmatrix} - \begin{pmatrix}1\\-4\\7\end{pmatrix} = \begin{pmatrix}2\\1\\1\end{pmatrix}$$

and

$$\begin{pmatrix}3\\-3\\8\end{pmatrix} - \begin{pmatrix}0\\1\\5\end{pmatrix} = \begin{pmatrix}3\\-4\\3\end{pmatrix}.$$

The area of the triangle is then

$$\frac{1}{2}\left|\begin{pmatrix}2\\1\\1\end{pmatrix}\times\begin{pmatrix}3\\-4\\3\end{pmatrix}\right| = \frac{1}{2}\left|\begin{pmatrix}7\\-3\\-11\end{pmatrix}\right| = \frac{\sqrt{179}}{2}.$$

MATH1151 Algebra Test 2 2018 v4a

April 17, 2019

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Question 1

For the matrix to be invertible, we require $\det(A) \neq 0$. So we first find

$$\begin{aligned}
\det(A) &= \begin{vmatrix} a & 0 & 3 \\ -2 & a & 2 \\ -1 & 1 & 2 \end{vmatrix} \\
&= a\begin{vmatrix} a & 2 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} -2 & a \\ -1 & 1 \end{vmatrix} && \text{(across first row)} \\
&= a(2a-2) + 3(-2+a) && (2 \times 2 \text{ formula}) \\
&= 2a^2 + a - 6 \\
&= (2a-3)(a+2).
\end{aligned}$$

This expression is non-zero for all real a excluding $a = \dfrac{3}{2}$ and $a = -2$, so those are the values for which it is invertible.

Note: The decision to evaluate across the first row was more or less because of the 0 in the first row. We could've also evaluated down the second column instead to take time, but then you have more signs to keep track of.

Question 2

We're essentially interested in the formula $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$. Take note that since we're dealing with lines, we're interested in using their respective *direction* vectors. We have

$$\begin{pmatrix}-1\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}4\\1\\5\end{pmatrix}=-7,$$
$$\left|\begin{pmatrix}-1\\2\\-1\end{pmatrix}\right|=\sqrt{6},$$
$$\left|\begin{pmatrix}4\\1\\5\end{pmatrix}\right|=\sqrt{42}.$$

So

$$\begin{aligned}\cos\theta&=\frac{-7}{\sqrt{6}\times\sqrt{42}}\\&=-\frac{7}{\sqrt{6}\sqrt{6}\sqrt{7}}\\&=-\frac{\sqrt{7}}{6}.\end{aligned}$$

Note therefore that the *acute* angle is $\arccos\left(\dfrac{\sqrt{7}}{6}\right)$. If we took inverse cosine of the negative angle, we would've arrived at the obtuse angle instead. (This more or less ties back to the identity $\arccos(-x)=\pi-\arccos x$.)

Question 3

Part a)

In general, a Cartesian form will fall out immediately by starting with the point normal form, and then expanding the dot product. We've been given a point it passes through *and* the normal vector, so we just consider

$$\begin{aligned}\begin{pmatrix}2\\0\\-4\end{pmatrix}\cdot\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}&=\begin{pmatrix}2\\0\\-4\end{pmatrix}\cdot\begin{pmatrix}4\\2\\3\end{pmatrix}\\\implies 2x_1-4x_3&=8-12\\\implies x_1-2x_3&=-2.\end{aligned}$$

Part b)

Now that we have the Cartesian form, we can set some parameters. Since x_2 is missing, set

$\lambda = x_2$ and take $\mu = x_3$ to avoid fractions. Subbing in gives

$$x_1 - 2\mu - -2 \implies x_1 = -2 + 2\mu.$$

So the Cartesian form is

$$\begin{aligned}\mathbf{x} &= \begin{pmatrix} -2+2\mu \\ \lambda \\ \mu \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.\end{aligned}$$

where $\lambda, \mu \in \mathbb{R}$. (Note that the expression on the RHS is optional - you should be safe to jump to the answer.)

Question 4

The following method you've learnt is essentially designed to work well in all cases (i.e. if the lines intersect, are parallel or are skew). Here, we spend some time re-discussing how it works.

We start by observing that a vector normal (perpendicular) to both lines is given by

$$\mathbf{n} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \times \begin{pmatrix} -2 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}.$$

We then pluck out *any* vector that represents a line segment, such that it joins any pair of points (one on each line) together. Since the lines are given in Cartesian form, we may as well use the points they've provided to find that line segment:

$$\begin{pmatrix} 8 \\ 1 \\ -4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -6 \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \\ 2 \end{pmatrix}.$$

The idea is, since we have a vector that is perpendicular to both lines, and the shortest distance is chosen to be perpendicular to both, the vector $\text{proj}_{\mathbf{n}} \begin{pmatrix} 7 \\ -1 \\ 2 \end{pmatrix}$ will ultimately have length equal to the distance we require! (You may need to draw a picture to visualise this.) Our answer is hence

$$\begin{aligned}
\left|\operatorname{proj}_{\mathbf{n}}\begin{pmatrix}7\\-1\\2\end{pmatrix}\right| &= \frac{\left|\begin{pmatrix}7\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}3\\1\\1\end{pmatrix}\right|}{\left|\begin{pmatrix}3\\1\\1\end{pmatrix}\right|}\\
&= \frac{22}{\sqrt{11}}\\
&= 2\sqrt{11}.
\end{aligned}$$

MATH1151 Algebra Test 2 2018 v4b

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We cannot guarantee that our answers are correct - please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are sometimes multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

Question 1

This question demonstrates how evaluating determinants is sometimes an art. Finding the right technique at the right time makes life *significantly* easier, surprisingly. Recall that the row operation $R_i \leftarrow R_i + \alpha R_j$ does not change the determinant, so we do a row reduction first.

$$\begin{aligned}
\begin{vmatrix} 1 & 2 & -3 & 1 \\ -1 & 1 & 4 & -2 \\ -3 & -6 & 9 & 2 \\ 0 & 6 & 4 & -5 \end{vmatrix} &= \begin{vmatrix} 1 & 2 & -3 & 1 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 6 & 4 & -5 \end{vmatrix} && (R_2 \leftarrow R_2 + R_1 \ \&\ R_3 \leftarrow R_3 + 3R_1) \\
&= \begin{vmatrix} 3 & 1 & -1 \\ 0 & 0 & 5 \\ 6 & 4 & -5 \end{vmatrix} && (\text{first column, note all 0's vanish}) \\
&= -5 \begin{vmatrix} 3 & 1 \\ 6 & 4 \end{vmatrix} && (\text{second row!}) \\
&= -5(12-6) \\
&= -30.
\end{aligned}$$

Note: When evaluating down rows/columns that aren't the first one, make sure to keep track of the alternating signs:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Question 2

If we wish to use vectors, to find the length AB we first compute

$$\overrightarrow{AB} = \begin{pmatrix} -2 \\ 7 \\ 3 \end{pmatrix} - \begin{pmatrix} -3 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

and then compute its length

$$\left|\overrightarrow{AB}\right| = \sqrt{1+4+16} = \sqrt{21}.$$

Take note that when using the dot-product-cosine-rule formula, since we require the angle at B, we need both vectors to point towards B or both away from B! Since I've found $\overrightarrow{AB}$, I now require $\overrightarrow{CB}$.

$$\begin{aligned} \overrightarrow{CB} &= \begin{pmatrix} -2 \\ 7 \\ 3 \end{pmatrix} - \begin{pmatrix} -4 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \\ \implies \left|\overrightarrow{CB}\right| &= \sqrt{4+1+4} = \sqrt{17}. \end{aligned}$$

Then,

$$\begin{aligned} \overrightarrow{AB} \cdot \overrightarrow{CB} &= \left|\overrightarrow{AB}\right| \left|\overrightarrow{CB}\right| \cos\theta \\ \implies 2+2+8 &= \sqrt{21}\sqrt{17}\cos\theta \\ \implies \cos\theta &= \frac{12}{\sqrt{357}}. \end{aligned}$$

Question 3

We follow the same method as in the similar question in the other version.

Part a)

Considering the point-normal form,

$$\begin{aligned} \begin{pmatrix} 6 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 6 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix} \\ \implies 6x_1 + x_2 + x_3 &= 45. \end{aligned}$$

Part b)

To avoid fractions, one possible choice of parameters is $x_1 = \lambda$ and $x_2 = \mu$. Then

$$6\lambda + \mu + x_3 = 45 \implies x_3 = -6\lambda - \mu + 45.$$

Hence a parametric vector form is

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 45 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -6 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Question 4

We follow the same method as in the similar question in the other version. For a vector normal to both lines,

$$\mathbf{n} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

For one particular vector that represents an arbitrary line segment joining the lines, consider

$$\begin{pmatrix} -1 \\ 2 \\ 8 \end{pmatrix} - \begin{pmatrix} -2 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 7 \end{pmatrix}.$$

Then the shortest distance is

$$\begin{aligned} \left| \operatorname{proj}_{\mathbf{n}} \begin{pmatrix} 1 \\ -3 \\ 7 \end{pmatrix} \right| &= \frac{\left| \begin{pmatrix} 1 \\ -3 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \right|}{\left| \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right|} \\ &= \frac{18}{\sqrt{6}} \\ &= 3\sqrt{6}. \end{aligned}$$