MATH1231/1241 Lab Test 1
Algebra Solutions to Samples

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Question 1

Select the correct definition for span$\{S\}$, the set of vectors $S = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ in a vector space V with a scalar field $\mathbb{F}$ from the choices below:

1. $\text{span}\{S\} = \{\mathbf{x} \in V : \mathbf{x} = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + ... + \lambda_n\mathbf{v}_n \text{ for } \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{F}\}$
2. $\text{span}\{S\} = \{\mathbf{x} \in V : \text{if } \mathbf{x} = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + ... + \lambda_n\mathbf{v}_n \text{ for } \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{F}, \text{ then } \lambda_1 = \lambda = ... = \lambda_n = 0\}$
3. $\text{span}\{S\} = \{\mathbf{x} \in V : \mathbf{x} = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + ... + \lambda_n\mathbf{v}_n\}$
4. $\text{span}\{S\} = \{\mathbf{x} \in V : \mathbf{x} = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + ... + \lambda_n\mathbf{v}_n \text{ for } \lambda_1, \lambda_2, ..., \lambda_n \in V\}$
5. $\text{span}\{S\} = \{\mathbf{x} \in V : \text{if } \mathbf{x} = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + ... + \lambda_n\mathbf{v}_n \text{ for } \lambda_1, \lambda_2, ..., \lambda_n \in V, \text{ then } \lambda_1 = \lambda = ... = \lambda_n = 0\}$

Recall that the span of a set refers to the set of all possible **linear combinations** of its elements. More specifically, each element in the span is formed by multiplying each vector in the set by a scalar (that is, an element from the scalar field $\mathbb{F}$) and then adding the results. As such, the correct definition here is (1.). We require each λ_i to be a scalar and thus, have to be taken from the scalar field $\mathbb{F}$ (and NOT the vector space V). Moreover, they do not (and should not) all equal to 0.

Question 2

Select all of the statements below that are true.

1. *There exists a set of 4 vectors in* $\mathbb{R}^3$ *that is linearly independent.*
2. *All sets of 4 vectors in* $\mathbb{R}^3$ *must span* $\mathbb{R}^3$*.*
3. *There exists a set of 4 vectors in* $\mathbb{R}^3$ *that is linearly independent.*
4. *All sets of 3 vectors in* $\mathbb{R}^4$ *are linearly independent.*
5. *All sets of 4 vectors in* $\mathbb{R}^3$ *are linearly dependent.*
6. *All sets of 3 vectors in* $\mathbb{R}^4$ *do not span* $\mathbb{R}^4$*.*
7. *A set can have at most* n *mutually orthogonal non-zero vectors in* $\mathbb{R}^n$*.*

In general, $\mathbb{R}^n$ can have at most n linearly independent vectors and any spanning set of $\mathbb{R}^n$ must have n **linearly independent** vectors. Expanding on the latter, it means that any spanning set of $\mathbb{R}^n$ must have *at least* n vectors (however, simply having n or more vectors in a set does not imply that it spans $\mathbb{R}^n$). Hence, (5.), (6.) and (7.) are true.

Question 3

Select the correct completion of the definition of ***linear independence*** *from the choices below.*
The vectors $\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_n$ *in a vector space* V *over a scalar field* $\mathbb{F}$ *are* ***linearly independent*** *means*

1. *if there are scalars* $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{F}$ *such that* $\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + ... + \lambda_n\mathbf{v}_n = \mathbf{0}$ *then* $\mathbf{v}_1 = \mathbf{v}_2 = ... = \mathbf{v}_n = \mathbf{0}$
2. *there are scalars* $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{F}$ *such that* $\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + ... + \lambda_n\mathbf{v}_n = \mathbf{0}$ *and* $\lambda_1 = \lambda_2 = ... = \lambda_n = 0$
3. *if there are scalars* $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{F}$ *such that* $\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + ... + \lambda_n\mathbf{v}_n = \mathbf{0}$ *then* $\lambda_1 = \lambda_2 = ... = \lambda_n = 0$
4. *there are scalars* $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{F}$ *such that* $\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + ... + \lambda_n\mathbf{v}_n = \mathbf{0}$

The correct answer is (3.). It is important to note that $\lambda_1 = \lambda_2 = ... = \lambda_n = \mathbf{0}$ has to be the ONLY solution to the equation $\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + ... + \lambda_n\mathbf{v}_n = \mathbf{0}$ for the vectors to be linearly independent. The significance of this result is that no vector in $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n\}$ can be written as a linear combination of other vectors in S. If $\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + ... + \lambda_n\mathbf{v}_n = \mathbf{0}$ had a non-trivial solution, we could just move any vector $\lambda_i\boldsymbol{v}_i$ to the other side (assuming $\lambda_i \neq 0$) and divide both sides by λ_i to write $\boldsymbol{v}_i$ as a linear combination of the other vectors in S – this would imply that S is not linearly independent.

Question 4

*Select from the choices below the correct definition from a set $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ in a vector space V over a scalar field $\mathbf{F}$ to **span** V.*

1. *If there are scalars $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{F}$ such that $\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + ... + \lambda_n\mathbf{v}_n = \mathbf{0}$ then $\lambda_1 = \lambda_2 = ... = \lambda_n = 0$.*
2. *For any vector $\mathbf{x} \in V$, if there are scalars $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{F}$ such that $\mathbf{x} = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + ... + \lambda_n\mathbf{v}_n$ then $\lambda_1 = \lambda_2 = ... = \lambda_n = 0$.*
3. *For every vector $\mathbf{x} \in V$, there are scalars $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{F}$ such that $\mathbf{x} = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + ... + \lambda_n\mathbf{v}_n$.*
4. *For every vector $\mathbf{x} \in V$, there are **unique** scalars $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{F}$ such that $\mathbf{x} = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + ... + \lambda_n\mathbf{v}_n$.*

For S to span V, we need to be able to express any vector in V as a linear combination of the vectors in S. Note that the linear combination does not have to be unique - this is especially true when the number of elements in S is greater than the dimension of V (if each linear combination was unique, then S would be a **basis** for V). As such, the correct answer is (3.). The condition that $\lambda_1 = \lambda_2 = ... = \lambda_n = 0$ pertains to linear independence and is irrelevant here.

Question 5

Let $\mathbb{R}^3$ *have the usual componentwise vector space operations.*
Let S *be the subset of* $\mathbb{R}^3$ *defined by*

$$S = \left\{ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_3^3 - x_1 x_2^2 = 0 \right\}$$

Note that S *contains the zero vector* $\mathbf{0}$.
1. EITHER state S *is closed under vector addition by writing the word CLOSED in the answer box below OR show* S *is NOT closed under vector addition by specifying two different non-zero vectors* $\mathbf{x}, \mathbf{y}$ *in* S *with integer coefficients which are not scalar multiples of each other such that* $\mathbf{x} + \mathbf{y}$ *is* ***not*** *in* S.
Enter your answer in the box below using Maple notation, e.g. enter $< 1, 2, 3 >, < 7, 8, 9 >$ *for* $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$.
2. EITHER state S *is closed under scalar multiplication by entering the word CLOSED in the answer box below OR show* S *is not closed under scalar multiplication by specifying (in this order) a non-zero vector* $\mathbf{x}$ *in* S *with integer coefficients and an integer scalar* λ *such that the vector* $\lambda\mathbf{x}$ *is* ***not*** *in* S.
Enter your answer in the box below using Maple notation, e.g. enter $< 1, 2, 3 >, 4$ *for* $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \lambda = 4$.

1. We have $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in S$ but $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \notin S$. Thus, we would enter $< 1, 1, -1 >, < 1, 1, 1 >$. As a general tip, when there is a coefficient being squared in the equation (x_2 in this case), consider two vectors which are the same EXCEPT with opposite signs for the squared coefficient.
2. CLOSED. As a general tip, ensure that the degree of each term is the same in the equation - here, both terms are of degree three.

Question 13

Let

$$\mathbf{v}_1 = \begin{pmatrix}1\\2\\4\end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix}2\\3\\5\end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix}3\\2\\0\end{pmatrix}$$

Find A, an augmented matrix (with no bar), for the problem of when deciding when

$$\mathbf{x} \in \mathit{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$$

Enter your matrix in the box below.
Now use this matrix to determine what condition(s), if any, are required on x, y, z so that

$$\mathbf{x} = \begin{pmatrix}x\\y\\z\end{pmatrix} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$$

*A condition will be an equation in x, y, z. Use * for multiplication (i.e. enter expression in Maple format). For example, a typical answer would be* `2*x-y+3*z=0`
If there is more than one condition give answer as a set of equations separated by commas and enclosed in braces ,, e.g. `2*x-y+3*x=0, 4*y-z=0`
If there are NO conditions, give the answer as: `None`

We consider the matrix whose columns are formed by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{x}$. That is,

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & x \\ 2 & 3 & 2 & y \\ 4 & 5 & 0 & z \end{array}\right)$$

Reducing this to row echelon form, we have

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & x \\ 0 & 1 & 4 & 2x - y \\ 0 & 0 & 0 & 2x - 3y + z \end{array}\right)$$

This will only have a solution if $2x - 3y + z = 0$.

Question 17

1. Let $p_1(t) = 1 + 3t + 5t^2$, $p_2(t) = 2 + 3t + t^2$, $p_3(t) = 5 + 8t + 4t^2$ *and* $q(t) = 1 + t + t^2$. *Give an augmented matrix which represents the linear system*

$$\mu_1 p_1 + \mu_2 p_2 + \mu_3 p_3 = q$$

with real scalars μ_1, μ_2, μ_3.
2. Suppose four polynomials p_1, p_2, p_3, q *in* $\mathbb{P}_2$ *have augmented matrix* A *corresponding to the equation above has echelon form*

$$U = \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & 7 & 6 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

Are the three polynomials p_1, p_2, p_3 ***linearly independent****? (Yes/No)*
Give the correct reasoning from

1. *U has an all zero row in its first 4 columns*
2. *The first 3 columns of U are leading columns*
3. *There are non-leading columns in the first 3 columns of U*

If they are ***linearly dependent****, then give a linear combination of p_1 and p_2 which equals p_3.*
If they are ***linearly independent****, then give a linear combination of p_1, p_2 and p_3 which equals q.*
1. The corresponding augmented matrix A is

$$A = \left(\begin{array}{ccc|c} 1 & 2 & 5 & 1 \\ 3 & 3 & 8 & 1 \\ 5 & 1 & 4 & 1 \end{array}\right)$$

Note that the columns of the matrix are formed by the coefficients of each polynomial.
2. The polynomials p_1, p_2, p_3 are NOT linearly independent since there are non-leading columns in the first 3 columns of U (i.e. (3.)). (2.) would imply linear independence and (1.) would imply linear dependence. From U, we deduce that $-17p_1 + 7p_2 = p_3$ – ignore the 4th column and imagine that there was a bar between the 2nd and 3rd column.

Question 23

Consider the following sets

$$A = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : -9x - 8y + 5z = 0 \textit{ where } -5 < x < 5 \textit{ and } -4 < y < 4 \right\}$$

$$B = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -8 \\ 8 \end{pmatrix} \textit{ and } 6x + 9y - 2z = 0 \right\}$$

$$C = \left\{ p \in \mathbb{P}_3 : p(x) = x^3 + ax^2 + bx + c \textit{ where } a, b, c \textit{ are real numbers} \right\}$$

Select all of the following which are true.

1. *A is a subspace of $\mathbb{R}^3$*
2. *B is a subspace of $\mathbb{R}^3$*
3. *C is a subspace of $\mathbb{P}^3$*

Only (2.) is true. Neither A nor C are closed under scalar multiplication.