MATH1081 S1 2008 Test 2 v2A

August 18, 2017

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1. (i) The congruence is equivalent to the equation $28x - 66y = 3$. Noting that $\gcd(28, 66) = 2$ and that $2 \nmid 3$, then by the Bézout property, we can conclude that there are no integer solutions.

   (ii) The congruence is equivalent to the equation $29x - 67y = 3$. Noting that $\gcd(29, 67) = 1$ and that $1|3$, then by the Bézout property, we know that there are multiple integer solutions, in particular the solutions are congruent modulo 67. To find them, use the extended Euclidean algorithm:

$$\begin{aligned} \mathbf{67} &= 2 \times \mathbf{29} + \mathbf{9} \\ \mathbf{29} &= 3 \times \mathbf{9} + \mathbf{2} \\ \mathbf{9} &= 4 \times \mathbf{2} + \mathbf{1}. \end{aligned}$$

Reversing this,

$$
\begin{aligned}
\mathbf{1} &= \mathbf{9} - 4 \times \mathbf{2} \\
&= \mathbf{9} - 4 \times (\mathbf{29} - 3 \times \mathbf{9}) \\
&= 13 \times \mathbf{9} - 4 \times \mathbf{29} \\
&= 13 \times \mathbf{67} - 26 \times \mathbf{29} - 4 \times \mathbf{29} \\
&= 13 \times \mathbf{67} - 30 \times \mathbf{29}.
\end{aligned}
$$

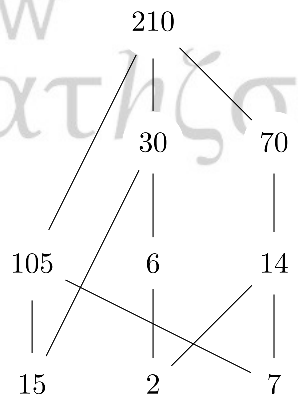
But in order to get the result in a similar form to our original equation, we must multiply both sides by 3, giving

$$-90 \times \mathbf{29} + 39 \times \mathbf{67} = \mathbf{3}.$$

Hence the solution is

$$x \equiv -90 \equiv 44 \pmod{67}.$$

2. Divisibility on the set: $S = \{2, 6, 7, 14, 15, 30, 70, 105, 210\}$



(i) The Hasse diagram is shown above.

(ii) Recall that maximal elements are those that are related to no element in S except itself (nothing else on top in the Hasse diagram) and minimal elements are those that have no elements except itself that are related to it (nothing below them in the Hasse diagram).
The maximal element of S is 210 and the minimal elements are $2, 7$ and 15.

(iii) Recall that a greatest element of S is one where every element in S is related to it (where all other elements precede it) and the least element of S is one that is related to every element in S (precedes all other elements).
So S has a greatest element 210. But, S does not have a least element as 2, 7, 15 are not comparable (divisible) to each other and do not fit the criterion above.

3. Define a relation $\sim$ on $\mathbb{Z}^+$ by

$$x \sim y \text{ iff } y = 3^k x \quad \text{for some integer } k.$$

We must show that $\sim$ is reflexive, symmetric and transitive.
Recall the following definitions for some relation $\sim$ on a set A:

- Reflexive if for all $a \in A$, $a \sim a$,
- Symmetric if for all $a, b \in A$, if $a \sim b$, then $b \sim a$,
- Transitive if for $a, b, c \in A$ where $a \sim b$ and $b \sim c$ implies that $a \sim c$.

We now show that each of these properties are true,

Reflexive: Let $a \in \mathbb{Z}^+$. We need to show that there exists an integer k such that $a = 3^k a$. But $k = 0$ works here, so we can write $a = 3^0 a$, which implies that $a \sim a$.

Symmetric: Let $a, b \in \mathbb{Z}^+$. Suppose $a \sim b$. Then, there exists an integer k such that $b = 3^k a$. Multiplying both sides by 3^{-k} gives $a = 3^{-k} b$. Noting that $-k$ is also an integer, the result implies that $b \sim a$. Hence, $a \sim b \implies b \sim a$.

Transitive: Let $a, b, c \in \mathbb{Z}^+$. Suppose $a \sim b$ and $b \sim c$. Then there exist integers k and n, such that we can write $b = 3^k a$ and $c = 3^n b$. Then through substitution, we have $c = 3^n(3^k a) = 3^{n+k} a$. Notice that $n + k$ is indeed an integer. So, we can conclude that $a \sim c$. Hence, $a \sim b$ and $b \sim c \implies a \sim c$.

Thus, $\sim$ is an equivalence relation.

MATH1081 S2 2008 Test 2 v2A

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1. (i) Dividing both sides of the congruence $20x \equiv 16 \pmod{92}$ by 4, we have

$$5x \equiv 4 \pmod{23}.$$

This congruence is equivalent to the equation $5x - 23y = 4$. Noting that $\gcd(23, 5) = 1$ and $1 \mid 4$, then by the Bézout property, we can conclude that there are multiple integer solutions, in particular the solutions are congruent modulo 23. Using the extended Euclidean algorithm, we have

$$\mathbf{23} = 4 \times \mathbf{5} + \mathbf{3}$$

$$\mathbf{5} = 1 \times \mathbf{3} + \mathbf{2}$$

$$\mathbf{3} = 1 \times \mathbf{2} + \mathbf{1}.$$

Reversing this,

$$\begin{aligned}
\mathbf{1} &= \mathbf{3} - \mathbf{2} \\
&= \mathbf{3} - (\mathbf{5} - \mathbf{3}) \\
&= 2 \times \mathbf{3} - \mathbf{5} \\
&= 2(\mathbf{23} - 4 \times \mathbf{5}) - \mathbf{5} \\
&= 2 \times \mathbf{23} - 9 \times \mathbf{5}.
\end{aligned}$$

Thus -9 is an inverse of 5 modulo 23. But, after multiplying both sides by 4, we have the solution

$$x \equiv -9 \times 4 = -36 \equiv 10 \pmod{23}.$$

Alternatively, we could do this question without the extended Euclidean algorithm by noticing that

$$\begin{aligned}
5x &\equiv 4 \pmod{23} \\
&\equiv 50 \pmod{23}.
\end{aligned}$$

Dividing by 5 gives

$$\begin{aligned}
x &\equiv 10 \left(\bmod \frac{23}{\gcd(23,5)}\right) \\
&= 10 \pmod{23}.
\end{aligned}$$

(ii) $x \equiv 10 \pmod{23}$ implies that $x \equiv 10, 33, 56, 79 \pmod{92}$.

2. Let $a, b, c \in \mathbb{Z}$ such that $a^2 \mid b$ and $b^3 \mid c$. Then we can write $b = ra^2$ and $c = sb^3$ for some integers r and s. We then have:

$$\begin{aligned}
c^3 &= s^3 b^9 \\
&= s^3 b^5 b^4 \\
&= s^3 b^5 (ra^2)^4 \\
&= s^3 r^4 a^8 b^5 \\
&= (s^3 r^4 a^4) a^4 b^5.
\end{aligned}$$

Now $s^3 r^4 a^4 \in \mathbb{Z}$, so we can conclude that $a^4 b^5 \mid c^3$.

3. A relation $\sim$ is defined on $\mathbb{R}$ by $x \sim y$ iff $\sin x = \sin y$.

   (i) We are basically finding the set of values that x can take such that $\sin x = \sin 0 = 0$. The equivalence class of 0 is then

   $$\begin{aligned}\{x \in \mathbb{R} \,|\, 0 \sim x\} &= \{k\pi, k \in \mathbb{Z}\}\\ &= \{\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots\}.\end{aligned}$$

   (ii) The equivalence class of a, $a \in \mathbb{R}$ is

   $$\{x \in \mathbb{R} \,|\, a \sim x\}.$$

   We are basically finding the set of values that x can take such that $\sin x = \sin a$. If $\sin a \neq 0$, the equivalence class is

   $$\{k\pi + (-1)^k a,\ k \in \mathbb{Z}\} = \{\ldots, a, \pi - a, 2\pi + a, \ldots\}.$$

   If $\sin a = 0$, see part (i).

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1. (i) Dividing both sides of the congruence $45x \equiv 15 \pmod{78}$ by 3 gives

$$15x \equiv 5 \pmod{26}.$$

This congruence is equivalent to $15x - 26y = 5$. Since $\gcd(15, 26) = 1$ and $1 \,|\, 5$, there is a unique solution modulo 26. We can find the solution using the extended Euclidean algorithm:

$$\begin{aligned}
\mathbf{26} &= 1 \times \mathbf{15} + \mathbf{11} \\
\mathbf{15} &= 1 \times \mathbf{11} + \mathbf{4} \\
\mathbf{11} &= 2 \times \mathbf{4} + \mathbf{3} \\
\mathbf{4} &= 1 \times \mathbf{3} + \mathbf{1}.
\end{aligned}$$

Reversing this,

$$\begin{aligned}
\mathbf{1} &= \mathbf{4} - \mathbf{3} \\
&= \mathbf{4} - (\mathbf{11} - 2 \times \mathbf{4}) \\
&= 3 \times \mathbf{4} - \mathbf{11} \\
&= 3(\mathbf{15} - \mathbf{11}) - \mathbf{11} \\
&= 3 \times \mathbf{15} - 4 \times \mathbf{11} \\
&= 3 \times \mathbf{15} - 4(\mathbf{26} - \mathbf{15}) \\
&= 7 \times \mathbf{15} - 4 \times \mathbf{26}.
\end{aligned}$$

So 7 is an inverse of 15 modulo 26. Hence, the solution is

$$x \equiv 7 \times 5 \equiv 9 \mod 26.$$

Alternatively, we could do this question without the extended Euclidean algorithm by noticing that

$$\begin{aligned}
15x &\equiv 5 \pmod{26} \\
3x &\equiv 1 \left(\text{mod } \frac{26}{\gcd(26,5)}\right) \\
&= 1 \pmod{26} \\
&\equiv 27 \pmod{26} \\
x &\equiv 9 \left(\text{mod } \frac{26}{\gcd(26,3)}\right) \\
&\equiv 9 \pmod{26}.
\end{aligned}$$

(ii) $x \equiv 9 \pmod{26}$ implies that $x \equiv 9, 35, 61 \pmod{78}$.

2. Let $a, m \in \mathbb{Z}$ and suppose $a \mid m$ and $(a+1) \mid m$. Since $a \mid m$, $(a+1) \mid m$ and $\gcd(a, a+1) = 1$, we can say that
$$(a+1) \mid \frac{m}{a} \text{ and } a | \frac{m}{a+1}.$$
Hence, it follows that $a(a+1) \mid m$. $\square$

Alternatively, Let $a, m \in \mathbb{Z}$ and suppose $a \mid m$ and $(a+1) \mid m$. Then, for some integers r and s, we can write
$$m = ra \quad \text{and} \quad m = s(a+1).$$

First, multiplying both sides of the first equation by $(a+1)$ gives

$$m = ra \implies m(a+1) = ra(a+1).$$

Next, multiplying both sides of the second equation by a gives

$$m = s(a+1) \implies ma = sa(a+1).$$

Subtracting the second line from the first, we have

$$m = (r-s)a(a+1) \text{ where } r-s \in \mathbb{Z}.$$

Hence, $a\,(a+1) \,|\, m$. $\square$

3. Define a relation $\preceq$ on $\mathbb{Z}^+$ by

$$x \preceq y \quad \text{iff} \quad y = 3^k x \quad \text{for some non-negative integer } k.$$

We must show that $\preceq$ is reflexive, antisymmetric and transitive. Recall the following defitions for some relation $\sim$ on a set A:

- Reflexive if for all $a \in A$, $a \sim a$,
- Antisymmetric if for all $a, b \in A$, if $a \sim b$ and $b \sim a$, then $a = b$,
- Transitive if for $a, b, c \in A$ where $a \sim b$ and $b \sim c$ implies that $a \sim c$.

We now show that each of these properites are true,

Reflexive: Let $a \in \mathbb{Z}^+$. Since $a = 3^0 a$, it follows that $a \preceq a$.

Antisymmetric: Let $a, b \in \mathbb{Z}^+$. Suppose that $a \preceq b$ and $b \preceq a$. Then $b = 3^k a$ and $a = 3^l b$ for some integers k and l. We then have

$$b = 3^k(3^l b) = 3^{k+l} b.$$

Since k and l must be non-negative, the above line implies $k = l = 0$. Hence $a = b$.

Transitive: Let $a, b, c \in \mathbb{Z}^+$. Suppose $a \preceq b$, $b \preceq c$. Then $b = 3^k a$ and $c = 3^l b$ for some integers k and l. So

$$c = 3^l(3^k a) = 3^{l+k} a.$$

As $l + k$ is indeed a non-negative integer, $a \preceq c$.

Since $\preceq$ is reflexive, antisymmetric and transitive, it is a partial order.

MATH1081 S2 2009 Test 2 v2B

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1. (i) The congruence is equivalent to $79x - 98y = 5$. After using the extended Euclidean algorithm[1], you should find that -31 is an inverse of 79 modulo 98. You should then find that the solution to the congruence is

$$x \equiv -31 \times 5 = -155 \equiv 41 \pmod{98}.$$

Here is a worked solution. We first find $x_0 \in \mathbb{Z}$ such that $79x_0 \equiv 1 \pmod{98} \iff 79x_0 + 98y_0 = 1$ for some integer y_0, by using the extended Euclidean algorithm. We firstly note that

$$\begin{aligned}
\mathbf{98} &= 1 \times \mathbf{79} + \mathbf{19} \\
\mathbf{79} &= 4 \times \mathbf{19} + \mathbf{3} \\
\mathbf{19} &= 6 \times \mathbf{3} + \mathbf{1}.
\end{aligned}$$

[1]Refer to Question 1. (ii) in Test 2 2008 S1 v2A

Therefore, $\gcd(79, 98) = \gcd(3, 1) = 1$, so there exists a unique solution to the original congruence modulo 98. Now,

$$\begin{aligned}
1 &= \mathbf{19} - 6 \times \mathbf{3} \\
&= \mathbf{19} - 6 \times (\mathbf{79} - 4 \times \mathbf{19}) \\
&= -6 \times \mathbf{79} + 25 \times \mathbf{19} \\
&= -6 \times \mathbf{79} + 25 \times (\mathbf{98} - 1 \times \mathbf{79}) \\
&= -31 \times \mathbf{79} + 25 \times \mathbf{98}.
\end{aligned}$$

So as we can see, the desired x_0 is -31. Thus

$$\begin{aligned}
79x_0 &\equiv 1 \pmod{98} \\
\Rightarrow 79\,(5x_0) &\equiv 5 \pmod{98},
\end{aligned}$$

so the solution to the original congruence is

$$\begin{aligned}
x &\equiv 5x_0 \pmod{98} \\
&\equiv 5 \times (-31) \pmod{98} \\
&\equiv -155 \pmod{98} \\
&\equiv 41. \pmod{98}
\end{aligned}$$

(ii) Note that $\gcd(78, 99) = 3$ and $3 \nmid 5$. So by the Bézout property, there are no integer solutions.

2. Let $x, y, m \in \mathbb{Z}$. Suppose $m \mid (4x + y)$ and $m \mid (7x + 2y)$. Then for some integers a and b,

$$4x + y = am \quad \text{and} \quad 7x + 2y = bm.$$

Solve simultaneously for x and y. You should find that $x = m(2a - b)$ and $y = m(4b - 7a)$. And so, $m \mid x$ and $m \mid y$.

3. Let F be the set of all functions $f : \mathbb{R} \to \mathbb{R}$, and define the relation $\preceq$ on F by

$$f \preceq g \quad \text{iff} \quad f(x) \leq g(x) \quad \text{for all } x \in \mathbb{R}.$$

To prove that $\preceq$ is a partial order, we need to show that it is reflexive, antisymmetric and transitive. To do this, let $a, b, c \in F$ and use these functions appropriately for each property. Here is a sample answer:

Reflexive: Let $f \in F$, then $f(x) = f(x)$ for all $x \in \mathbb{R}$, so $f(x) \leq f(x)$ for all $x \in \mathbb{R}$. Thus $f \preceq f$ for any $f \in F$, so $\preceq$ is reflexive.

Antisymmetric: Let $f, g \in F$ and suppose $f \preceq g$ and $g \preceq f$. Then $f(x) \leq g(x)$ and $g(x) \leq f(x)$ for all $x \in \mathbb{R}$. This implies that $f(x) = g(x)$ for all $x \in \mathbb{R}$ (since $\leq$ is antisymmetric). Hence $f = g$, so $\preceq$ is antisymmetric.

Transitive: Let $f, g, h \in F$ with $f \preceq g$ and $g \preceq h$. Then $f(x) \leq g(x)$ for all $x \in \mathbb{R}$ and $g(x) \leq h(x)$ for all $x \in \mathbb{R}$. This implies that $f(x) \leq h(x)$ for all $x \in \mathbb{R}$ (as $\leq$ is transitive), so $f \preceq h$. Thus $\preceq$ is transitive.

Therefore, $\preceq$ is a partial order.

MATH1081 S1 2010 Test 2 v1A

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1. (i) The congruence is equivalent to $25x - 109y = 3$. After using the extended Euclidean algorithm[1], you should find that 48 is an inverse of 25 modulo 109. You should find that the solution is

$$x \equiv 48 \times 3 = 144 \equiv 35 \pmod{109}.$$

Alternatively, we can find that

$$\begin{aligned} 25x &\equiv 3 \pmod{109} \\ &\equiv -215 \pmod{109} \\ 5x &\equiv -43 \pmod{109} \\ &\equiv 175 \pmod{109} \\ x &\equiv 35 \pmod{109}. \end{aligned}$$

(ii) Note that $\gcd(25, 110) = 5$, but $5 \nmid 3$. So by the Bézout property, there are no integer solutions.

[1]Refer to Question 1. (ii) in Test 2 2008 S1 v2A

2. (i) One checks that $6500 = 2^2 \times 5^3 \times 13$ and $1120 = 2^5 \times 5 \times 7$.

(ii) Given the prime factorisation of two positive integers, their gcd is found by taking the common prime factors and their lcm is found by taking the maximum of the prime powers present in each. That is, if $a = p_1^{m_1} p_2^{m_2} \dots p_k^{m_k}$ and $b = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$ where each p_i is a prime (i.e. these are prime factorisations of a and b), then

$$\gcd(a,b) = p_1^{\min(m_1,n_1)} \dots p_k^{\min(m_k,n_k)}$$

and

$$\operatorname{lcm}(a,b) = p_1^{\max(m_1,n_1)} \dots p_k^{\max(m_k,n_k)}.$$

So from our answer to (i), we have

$$\begin{aligned} &\gcd(6500, 1120) = 2^2 \times 5 \\ \text{and} \quad &\operatorname{lcm}(6500, 1120) = 2^5 \times 5^3 \times 7 \times 13. \end{aligned}$$

3. Define a relation $\sim$ on $\mathbb{R}^+$ by

$$x \sim y \quad \text{iff} \quad x - y \in \mathbb{Z}.$$

We must prove that $\sim$ is reflexive, symmetric and transitive.

We note that for all $x \in \mathbb{R}^+$, we have $x - x = 0 \in \mathbb{Z}$, which implies that $x \sim x$. Thus $\sim$ is reflexive.

If $x, y \in \mathbb{R}^+$ with $x \sim y$, then $x - y = n$ for some integer n, so $y - x = -n$ and $-n$ is an integer too, so $y \sim x$. Thus $\sim$ is symmetric.

Now suppose $x, y, z \in \mathbb{R}^+$ with $x \sim y$ and $y \sim z$. We show that $x \sim z$. Since $x \sim y$, we have $x - y = m$ for some integer m. Also, since $y \sim z$, we have $y - z = n$ for some integer n. Thus

$$\begin{aligned} x - z &= (x - y) + (y - z) \\ &= m + n, \end{aligned}$$

which is an integer since m and n are integers. Thus $x \sim z$, whence $\sim$ is transitive.

We have shown that $\sim$ is reflexive, symmetric and transitive. Hence $\sim$ is an equivalence relation.