MATH1151 Calculus Test 1 2008 S1 v1b

March 19, 2017

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We cannot guarantee that our working is correct, or that it would obtain full marks – please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are often multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

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1. (i) Observe that $f(x) = \frac{x+4-7}{x+4} = 1 - \frac{7}{x+4}$. The graph is a hyperbola.

   We can see that the maximal domain $S = \mathbb{R}\setminus\{-4\}$ because you cannot divide by zero, i.e. there is a vertical asymptote at $x = -4$.

   It is clear (e.g. from a sketch or simple algebra) that $f(x)$ can take on any given real value $r \neq 1$ (just take $x = \frac{7}{1-r} - 4$). However, $f(x)$ can never take on the value 1, since $x - 3$ can never equal $x + 4$. Hence, the range is $T = \mathbb{R}\setminus\{1\}$.

   Note that another method of finding the range of a function of this form would be to divide top and bottom by the highest power of x and taking a limit as $x \to \infty$. For example, in this case, you can show that

$$f(x) = \frac{x-3}{x+4} = \frac{1-\frac{3}{x}}{1+\frac{4}{x}}.$$

   Now as $x \to \infty$, $\frac{1}{x} \to 0$ so

$$\lim_{x\to\infty} f(x) = \frac{1-0}{1+0} = 1$$

This must mean that $y = 1$ is a horizontal asymptote, and so $T = \mathbb{R}\backslash\{1\}$.
Note that for general functions with asymptotes, it is entirely possible for the curve to pass through a horizontal asymptote for small values of x. However, for hyperbolas, this method works fine.

(ii) To prove a function has an inverse function, you must prove that it is bijective, that is, it is one-to-one and onto.

To prove that it is one-to-one, we need to show that "for $x_1, x_2 \in S$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$". We have for $x_1, x_2 \in S$ that

$$\begin{aligned} f(x_1) &= f(x_2) \\ \implies 1 - \frac{7}{x_1+4} &= 1 - \frac{7}{x_2+4} \\ \implies \frac{7}{x_1+4} &= \frac{7}{x_2+4} \\ \implies x_1 + 4 &= x_2 + 4 \\ \implies x_1 &= x_2. \end{aligned}$$

And so, we have proven that $f(x_1) = f(x_2)$ does imply $x_1 = x_2$.

To show $f : S \to T$ is onto, we must show that for any $r \in T$, there is an $x \in S$ such that $f(x) = r$. To show this, let r be any element of T (i.e. any real number other than 1), and take $x = \frac{7}{1-r} - 4$. Then clearly x is well-defined (as $r \neq 1$) and $x \in S$, as x is a real number that is not equal to -4, as $\frac{7}{1-r} \neq 0$. Also,

$$\begin{aligned} x &= \frac{7}{1-r} - 4 \\ \implies x + 4 &= \frac{7}{1-r} \\ \implies \frac{1}{x+4} &= \frac{1-r}{7} \\ \implies 1 - \frac{7}{x+4} &= r \\ \implies f(x) &= r \\ \implies f(x) &\text{ is onto.} \end{aligned}$$

Hence, $f(x)$ is bijective and has an inverse.

To find the inverse function, switch x and y and make y the subject:

$$\begin{aligned} x &= 1 - \frac{7}{y+4} \\ \implies \frac{7}{y+4} &= 1 - x \\ \implies y + 4 &= \frac{7}{1-x} \\ \implies y &= -4 + \frac{7}{1-x} \end{aligned}$$

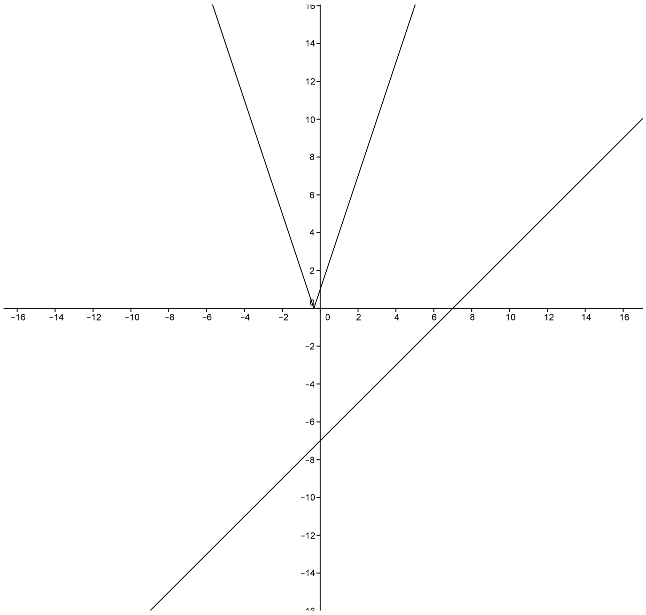
Hence the inverse function $g : T \to S$ is defined by $g(x) = f^{-1}(x) = -4 + \frac{7}{1-x}$.

(Note that to find what x should have been to make $f(x) = r$ earlier, we really did the working out to find the inverse function $g(x)$ first and used this to exhibit the earlier x in terms of r.)

2. For $x > 1$,

$$\begin{aligned} \tan\left(\sec^{-1} x\right) &= \sqrt{x^2 - 1} \\ \implies \frac{\mathrm{d}}{\mathrm{d}x}\left(\tan\left(\sec^{-1} x\right)\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x^2-1}\right) \\ \implies \sec^2\left(\sec^{-1} x\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\sec^{-1} x\right) &= \frac{x}{\sqrt{x^2-1}} \\ \implies x^2 \frac{\mathrm{d}}{\mathrm{d}x}\left(\sec^{-1} x\right) &= \frac{x}{\sqrt{x^2-1}} \quad \left(\text{as } \sec\left(\sec^{-1} x\right) = x \Rightarrow \sec^2\left(\sec^{-1} x\right) = x^2\right) \\ \implies \frac{\mathrm{d}}{\mathrm{d}x}\left(\sec^{-1} x\right) &= \frac{1}{x\sqrt{x^2-1}}. \end{aligned}$$

3. Your sketch should look similar to the following:



It is clear that there are no intersections between the two graphs. Thus, there are no solutions to the equation $|3x + 1| = x - 7$.

4. For all real x, y, we have

$$\begin{aligned} (x-y)^2 &\geq 0 && \text{(Squares are non-negative)} \\ \implies x^2 - 2xy + y^2 &\geq 0 \\ \implies x^2 + y^2 &\geq 2xy. \end{aligned}$$

Letting θ, ϑ be x^2 and y^2 respectively, we find that for any $\theta, \vartheta \geq 0$, we have $\theta + \vartheta \geq 2\sqrt{\theta\vartheta}$. Applying this inequality, we have the following for any $a, b, c \geq 0$:

$$a + b \geq 2\sqrt{ab}$$

$$a + c \geq 2\sqrt{ac}$$

$$b + c \geq 2\sqrt{bc}.$$

Multiplying these inequalities together, we have

$$(a+b)(b+c)(a+c) \geq 8abc.$$

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1. (i) We must show that f is bijective, that is, it is one-to-one and onto. We will assume it is onto because the codomain is not specified and the range is clearly $\mathbb{R}$ (assuming the codomain is $\mathbb{R}$)[1]. Note that

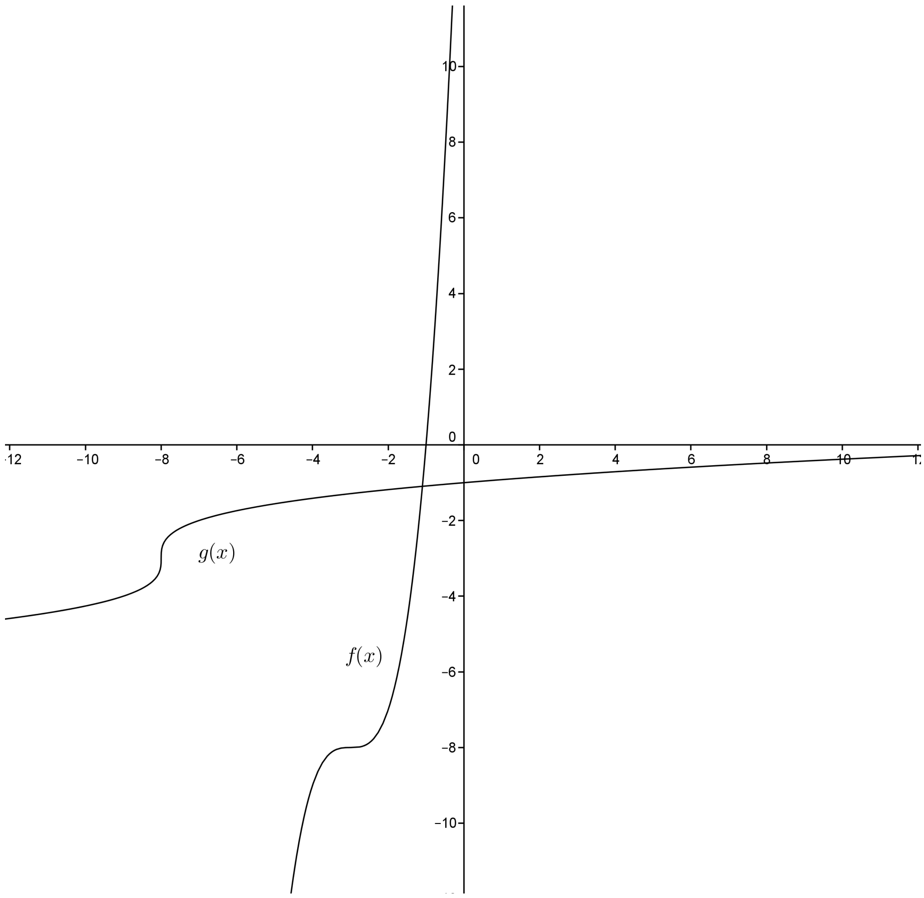
$$f'(x) = 3x^2 + 18x + 27 = 3(x+3)^2 \geq 0 \quad \forall x \in \mathbb{R}.$$

Hence, the function is always non-decreasing (in fact, strictly increasing since the derivative is only equal to zero at one point ($x = -3$)), which implies that it is one-to-one.

As f is one-to-one and onto, it is bijective and hence has an inverse function.

[1] That the range is $\mathbb{R}$ follows from the more general result that any odd-degree real polynomial function has range $\mathbb{R}$. This can be deduced by recalling the fact that any odd-degree real polynomial has a real root, which can be shown with the help of the Intermediate Value Theorem, but is not the focus of this particular test problem.

(ii) Your sketch should look something like this.



2. We have

$$
\begin{aligned}
\cos^{-1}\left(\cos\left(\frac{4\pi}{3}\right)\right) &= \cos^{-1}\left(\cos\left(\pi+\frac{\pi}{3}\right)\right)\\
&= \cos^{-1}\left(-\cos\frac{\pi}{3}\right) \quad (\text{using } \cos(\pi+\theta) = -\cos\theta)\\
&= \cos^{-1}\left(-\frac{1}{2}\right)\\
&= \frac{2\pi}{3}.
\end{aligned}
$$

(Recall that $\cos^{-1} x$ must be between 0 and π.)

3. We have

$$\begin{aligned}
\text{LHS} &= (\cosh x + \sinh x)^3 \\
&= \left(\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}\right)^3 \\
&= (e^x)^3 \\
&= e^{3x} \\
&= \text{RHS}.
\end{aligned}$$

Expanding by using the Binomial Theorem:

$$\cosh^3 x + 3\cosh^2 x \sinh x + 3\cosh x \sinh^2 x + \sinh^3 x = e^{3x} \quad (1)$$

Similarly, we can also prove that $(\cosh x - \sinh x)^3 = e^{-3x}$. And so, we can get the similar result:

$$\cosh^3 x - 3\cosh^2 x \sinh x + 3\cosh x \sinh^2 x - \sinh^3 x = e^{-3x} \quad (2)$$

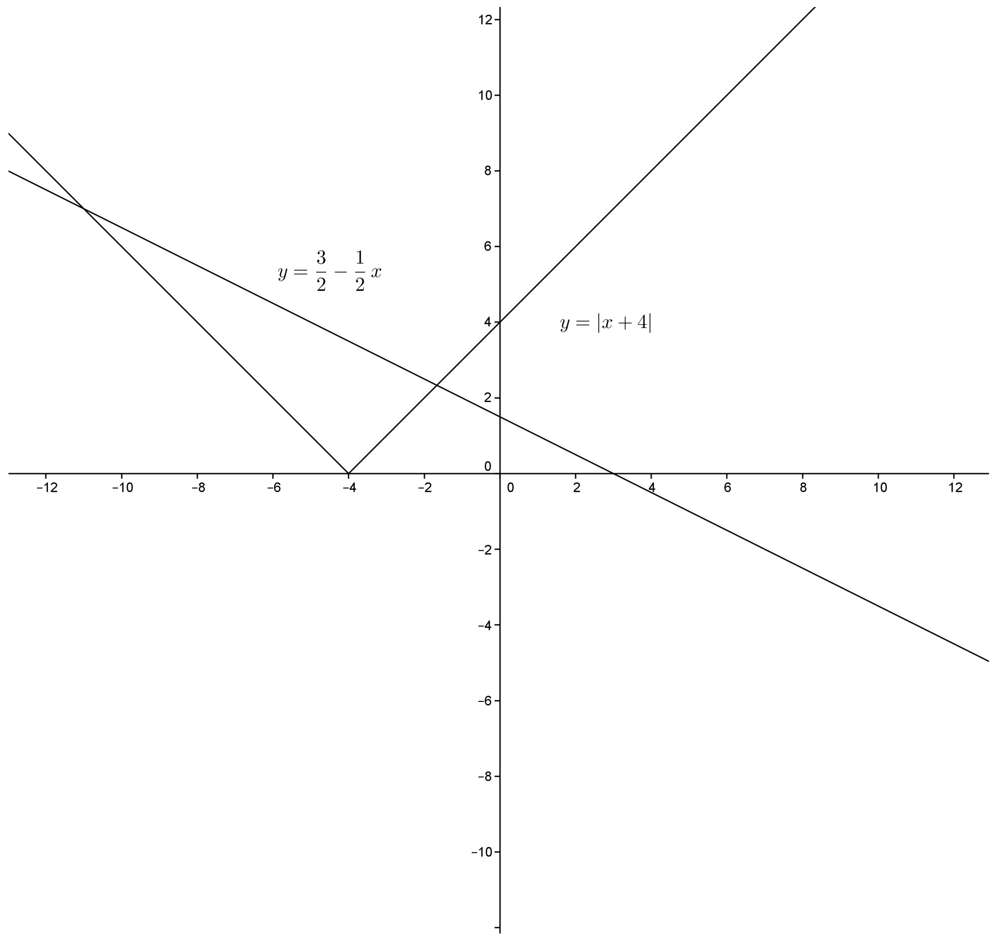
Subtracting (2) from (1), and dividing by 2:

$$\frac{e^{3x} - e^{-3x}}{2} = 3\cosh^2 x \sinh x + \sinh^3 x$$

The expression on the left hand side is simply $\sinh 3x$. We can then simplify using the identity $\cosh^2 x - \sinh^2 x = 1$.

Note that the question asks for a polynomial in $\cosh x$ but this is actually impossible, as you can show as an exercise (hint: consider odd and even functions). *The question had a typo and should have done something like asked for a polynomial in* $\sinh x$*.*

4. You should get something like this.



Using the graph, $\frac{3}{2} - \frac{1}{2}x = |x+4|$ at the intersection points (in fact there should be two solutions corresponding to the two points of intersection). Hence, we solve $\frac{3}{2} - \frac{1}{2}x = x+4$ and $\frac{3}{2} - \frac{1}{2}x = -(x+4)$. The solution to the first equation should give us the point of intersection on the right, and the second equation should give us the point of intersection on the left. These are linear equations which you can easily solve to get $x = -\frac{5}{3}$ or $x = -11$.

5. The first clue is to see that the "larger" side of the given inequality is $(b+c)(c+a)(a+b)$, which is the product of essentially three numbers. The "larger" side of the required inequality is $\alpha\beta\gamma$, which is also the product of three numbers. So, it'd be good to try and let $b+c = \alpha$, $a+c = \beta$ and $a+b = \gamma$.

   So given $(b+c)(c+a)(a+b) > 8abc$, we have $\alpha\beta\gamma > 8abc = (2a)(2b)(2c)$, after substituting the values on the left hand side.

   Now we look ahead and notice that

$$\beta + \gamma - \alpha = a + c + a + b - b - c = 2a,$$

$$\gamma + \alpha - \beta = a + b + b + c - a - c = 2b,$$

$$\alpha + \beta - \gamma = b + c + a + c - a - b = 2c.$$

So substituting these results into our expression, we have

$$\alpha\beta\gamma > (\beta + \gamma - \alpha)(\gamma + \alpha - \beta)(\alpha + \beta - \gamma).$$

Alternatively, comparing the "smaller" sides of the inequalities, we can also try and let $a = \beta + \gamma - \alpha$, $b = \gamma + \alpha - \beta$ and $c = \alpha + \beta - \gamma$.
Note the following,

$$\begin{aligned} b + c &= 2\alpha \\ c + a &= 2\beta \\ a + b &= 2\gamma \end{aligned}$$

Hence, substituting this in to the given inequality,

$$\begin{aligned} 8\left(\beta + \gamma - \alpha\right)\left(\gamma + \alpha - \beta\right)\left(\alpha + \beta - \gamma\right) &< \left(2\alpha\right)\left(2\beta\right)\left(2\gamma\right) \\ &= 8\alpha\beta\gamma \\ \therefore \left(\beta + \gamma - \alpha\right)\left(\gamma + \alpha - \beta\right)\left(\alpha + \beta - \gamma\right) &< \alpha\beta\gamma. \end{aligned}$$

(It is easy to show in each of these cases that the substitutions do indeed produce three unequal numbers, so we can use the given inequality results.)

MATH1151 Calculus Test 1 2009 S1 v2b

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1. Let $f(x) = x^3 - 2x^2 + 2x + 5$.

   (i) To show that an inverse function exists, we must prove that the function must be bijective (one-to-one and onto). We will assume it is onto because the codomain is not specified and the range is clearly $\mathbb{R}$ (assuming the codomain is $\mathbb{R}$), since in fact any odd-degree real polynomial function with codomain $\mathbb{R}$ has range $\mathbb{R}$. Note that

   $$f'(x) = 3x^2 - 4x + 2.$$

   We want to show that the derivative is always non-negative in order to show that f is strictly increasing, which would show that it is one-to-one.

   Method 1. One way we can do this is checking the discriminant. The discriminant is $\Delta = (-4)^2 - 4(3)(2) = -8 < 0$.

   Since the discriminant is negative and the leading coefficient is positive, $f'(x)$ is positive for all real x (or "positive definite"), implying that f is one-to-one.

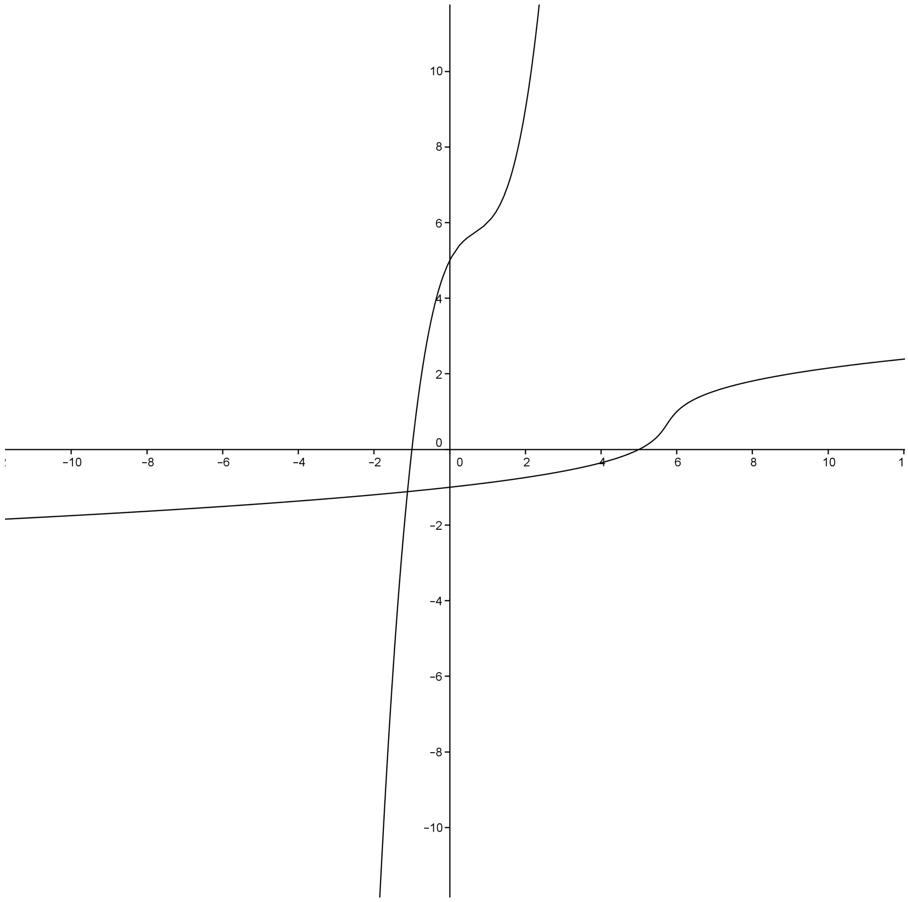
Method 2. We could also *complete the square*:

$$\begin{aligned} f'(x) &= 3\left(x^2 - \frac{4}{3}x + \frac{2}{3}\right) \\ &= 3\left(x^2 - \frac{4}{3}x + \frac{4}{9} + \frac{2}{9}\right) \\ &= 3\left(x - \frac{2}{3}\right)^2 + \frac{2}{3} \end{aligned}$$

Since the square of any real number is always non-negative, we can deduce that $f'(x) > 0$ (in fact $f'(x) \geq \frac{2}{3}$) for all real x, which implies that f is one-to-one.

Thus f is both one-to-one and onto, so is invertible.

(ii) Your sketch should look something like this. Note that you may have to find the second derivative to find the point of inflexion of $f(x)$ and then reflect the graph along the line $y = x$.

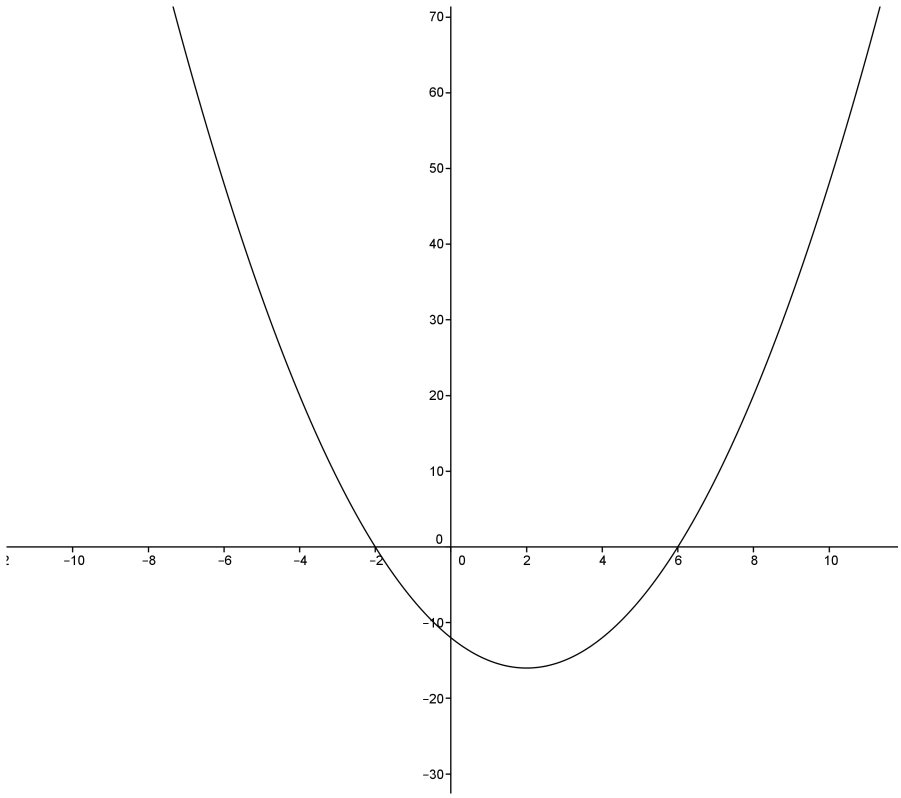


2. We have

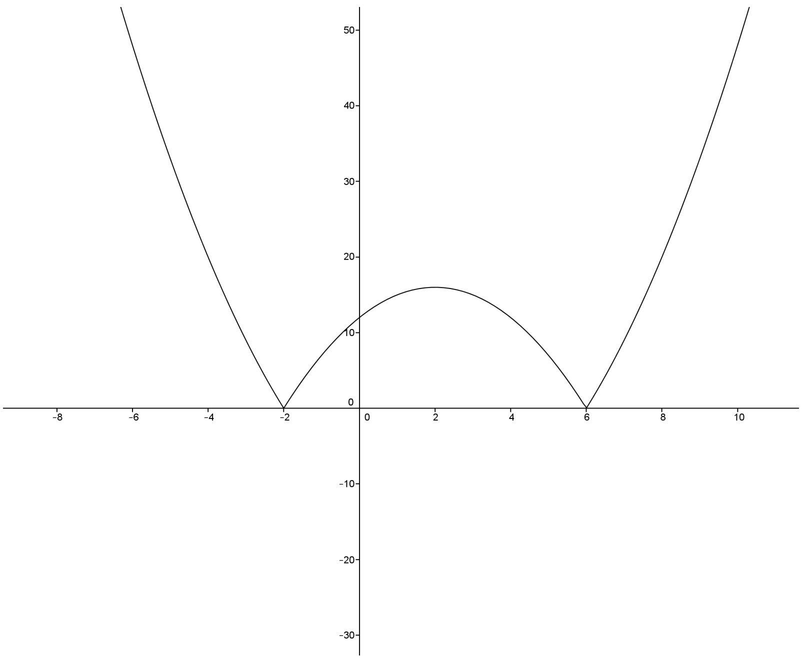
$$f'(x) = (2x)\,\frac{1}{\sqrt{1-(x^2)^2}} \qquad \text{(chain rule)}$$
$$= \frac{2x}{\sqrt{1-x^4}}.$$

3. We have $y = x^2 - 4x - 12 = (x-6)(x+2)$. Note the y-intercept is -12 (the constant term of the quadratic).

   From here we can see that the x-intercepts are -2 and 6 and that the graph is a concave up parabola (since the leading coefficient is positive). The vertex also occurs at the midpoint of the roots, i.e. at $x = 2$, where $y = 2^2 - 4 \times 2 - 12 = -16$. So the graph will look like this:



   To sketch $y = \left|x^2 - 4x - 12\right|$, we simply reflect any part of the graph that lies below the x-axis, about the x-axis like so (remember to label intercepts and the new location of the vertex, which will be $(2, 16)$):



4. Starting with the fact that the square of any real number is non-negative:

$$
\begin{aligned}
&(x-y)^2 \geq 0 \\
&x^2 - 2xy + y^2 \geq 0 \\
&x^2 + y^2 \geq 2xy
\end{aligned}
$$

Applying this on relevant values for x and y, and adding, we get:

$$
\begin{aligned}
x^4 + y^4 &\geq 2x^2y^2 \\
z^4 + t^4 &\geq 2z^2t^2 \\
x^4 + y^4 + z^4 + t^4 &\geq 2(x^2y^2 + z^2t^2)
\end{aligned}
$$

Notice that the term on the right hand side can also be used in our original inequality, that is:

$$
\begin{aligned}
2(x^2y^2 + z^2t^2) = 2\left((xy)^2 + (zt)^2\right) &\geq 4xyzt \\
\therefore x^4 + y^4 + z^4 + t^4 \geq 2(x^2y^2 + z^2t^2) &\geq 4xyzt
\end{aligned}
$$

5. Notice that this is a quadratic in 2^x:

$$\begin{aligned}
&(2^{-1})(2^x)^2 - (3)(2)(2^x) + 16 = 0 \\
\iff\ &(2^x)^2 - 12(2^x) + 32 = 0 && \text{(multiplying both sides by 2)} \\
\iff\ &(2^x - 4)(2^x - 8) = 0 && \text{(factorising)} \\
\iff\ &2^x = 4 \text{ or } 8 \\
\iff\ &x = 2 \text{ or } 3
\end{aligned}$$

So the solution is $x = 2$ or 3.

If this is confusing, you can let $u = 2^x$, solve the quadratic in u, and substitute it back in to get the x solutions using $x = \log_2 u$.