MATH1081 Test 4 2008 S1 v3b

May 15, 2016

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1. 8 letter words

   (i) No letter used twice

   1. Pick the first letter of the word: 26 ways
   2. Pick a different letter for the second letter/place: 25 ways.
   3. Pick a different letter for the third letter/place. 24 ways:

   . . . etc . . .

   8. Pick a different letter for the 8th letter/place: 19 ways.

   Answer[1]*:* $26 \times 25 \times 24 \ldots \times 19$ or $P(26, 8)$.

[1] In these solutions, $P(m, n)$ is the number of permutations of n objects chosen from a set of m objects.

(ii) One letter used twice and six letters used once at most

1. Pick the letter to be used twice: 26 ways.
2. Choose two places from 8 for this double letter: $\binom{8}{2}$ ways.
3. Choose 6 different letters for the remaining spots, and note that order is important: $P(25, 6)$ ways.

Answer: $26 \times \binom{8}{2} \times P(25, 6)$.

2. An ordinary die is rolled ten times. There are 6^{10} possibilities in total.

There are $\binom{6}{5}$ ways to choose the letters of rolls of the die to appear twice each.
Then, we find the number of permutations of these 10 letters, keeping in mind that each distinct letter has two of it appearing in the word, given by

$$\frac{10!}{2! \times 2! \times 2! \times 2! \times 2!}.$$

Thus, the required probability is:

$$\frac{\binom{6}{5} \times 10!}{2! \times 2! \times 2! \times 2! \times 2! \times 6^{10}}.$$

3. Particular solution of $a_n - 8a_{n-1} + 15a_{n-2} = 21 \times 2^n$.

Guess: $a_n = c \times 2^n$. Sub into the recurrence to get

$$(c \times 2^n) - 8(c \times 2^{n-1}) + 15(c \times 2^{n-2}) = 21 \times 2^n.$$

Divide both sides by 2^{n-2} to get

$$\begin{aligned} 4c - 8(2)c + 15c &= 21(4) \\ 4c - 16c + 15c &= 84 \\ 3c &= 84 \\ \implies c &= 28. \end{aligned}$$

So $a_n = 28 \times 2^n$ is a particular solution.

3. Let

$$
\begin{aligned}
A_1 &= \{\text{hands with exactly 5 hearts}\} \\
A_2 &= \{\text{hands with exactly 5 diamonds}\} \\
A_3 &= \{\text{hands with exactly 5 spades}\} \\
A_4 &= \{\text{hands with exactly 5 clubs}\}
\end{aligned}
$$

We want $|A_1 \cup A_2 \cup A_3 \cup A_4|$.
By using the Inclusion-Exclusion principle, we see that

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = \sum_{i=1}^{4} |A_i| - \sum_{i \neq j} |A_i \cap A_j| \, .$$

We find $\sum_{i=1}^{4} |A_i|$, i.e., we want hands of 13 cards with exactly 5 cards in some suit.

1. Choose the suit to have 5 cards: 4 ways.
2. Choose 5 from that suit: $\binom{13}{5}$ ways.
3. Choose a further 8 from the remaining cards in the deck: $\binom{39}{8}$ ways.

Now we must find $\sum_{i \neq j} |A_i \cap A_j|$, i.e. we must account for hands where the remaining 8 (in step 3. above) feature 5 cards of another suit.

1. Choose the 2 suits to have 5 cards each in the 13 card hand: $\binom{4}{2}$ ways.
2. Choose 5 cards from the first of these suits: $\binom{13}{5}$ ways.
3. Repeat for the other suit: $\binom{13}{5}$ ways.
4. Choose 3 cards from the remaining 26 cards in the deck: $\binom{26}{3}$ ways.

Therefore,

$$
\begin{aligned}
|A_1 \cup A_2 \cup A_3 \cup A_4| &= \sum_{i=1}^{4} |A_i| - \sum_{i \neq j} |A_i \cap A_j| \\
&= 4\binom{13}{5}\binom{39}{8} - \binom{4}{2}\binom{13}{5}^2\binom{26}{3}.
\end{aligned}
$$

Note: Be careful with questions like this, since the big trap is accidentally overcounting/undercounting in situations like these. To overcome this issue, we must use the Inclusion-Exclusion Princple.

MATH1081 Test 4 2008 S2 v1b

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1. We want seven-letter words with at least six consonants. We break the question down in to the following 2 cases to help us find the answer:

   Case 1: 6 consonants

   1. Pick a vowel. 5 ways.
   2. Choose the position in the word for the vowel to be in. 7 ways.
   3. Pick a consonant for the first available position in the word. 21 ways.
   4. Repeat for the second available position. 21 ways.

      . . . etc . . .

   8. Pick a consonant for the last available position. 21 ways.

   Total: $5 \times 7 \times 21^6$ ways.

Case 2: 7 consonants

1. Pick a consonant for the first position. 21 ways.

   ...etc ...

7. Pick a consonant for the last position. 21 ways.

Total: 21^7 ways.

Answer: $5 \times 7 \times 21^6 + 21^7$ ways.

2. Total possibilities: $\binom{52}{8}$.

   1. Exclude the kings from the 52 and choose 5 cards from the remaining deck. $\binom{48}{5}$ ways.

   2. Choose one of the 4 kings that will be excluded. 4 ways.

   Answer:

$$\frac{4 \times \binom{48}{5}}{\binom{52}{8}}.$$

3. (i) We want the number of solutions to $x_1+x_2+x_3+x_4+x_5 = 16$, where $x_1, \dots, x_5 \geq 0$. We can imagine a solution as a series of 16 identical dots and 4 identical lines arranged in a row. For example,

$$\circ \quad \circ \quad \circ \quad \circ \quad | \quad \circ \quad \circ \quad \circ \quad \circ \;|\; \circ \quad \circ \quad \circ \quad \circ \;|\; \circ \quad \circ \quad | \quad \circ \quad \circ$$

represents the solution $x_1 = 4, x_2 = 4, x_3 = 4, x_4 = 2$ and $x_5 = 2$.

We count the number of ways we can arrange these identical dots and identical lines in a row to obtain the number of possibilities. But, since they're all identical, We just need to count how many ways we can choose 5 spots in the 20 available positions for the lines. The total number of arrangements is $\binom{20}{4}$.

(ii) We want the number of solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 16$, where $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$, $x_4 \geq 2$ and $x_5 \geq 1$.
Let

$$x_1 = y_1 + 1$$
$$x_2 = y_2 + 2$$
$$x_3 = y_3 + 3$$
$$x_4 = y_4 + 2$$
$$x_5 = y_5 + 1$$

Adding the equations above and substituting into the original equation yields $y_1 +$

$y_2 + y_3 + y_4 + y_5 = 7$, and require that $y_i \geq 0$. Note that this new equation is equivalent to the original equation.
Similar to part (i), we use the dots and lines analogy: 7 dots and 4 lines. We arrange them in a row, and choose the positions for the lines. This can be done $\binom{11}{4}$ ways.
Answer:

$$\binom{11}{4}.$$

4. We seek the general solution to $a_n - 16a_{n-2} = 9 \cdot 2^n$. To do this, we work out the homogeneous case first. The characteristic equation is

$$\lambda^2 - 16 = 0 \implies \lambda = \pm 4.$$

So $h_n = A \cdot 4^n + B \cdot (-4)^n$, where A and B are arbitrary constants.

For a particular solution, 'guess' $p_n = c \cdot 2^n$. Substituting in, we get

$$(c \cdot 2^n) - 16(c \cdot 2^{n-2}) = 9 \cdot 2^n.$$

Divide by 2^{n-2}:

$$4c - 16c = 9 \cdot 4 \implies -12c = 36 \implies c = -3.$$

Therefore a general solution is $a_n = h_n + p_n$ or

$$a_n = A \cdot 4^n + B \cdot (-4)^n - 3 \cdot 2^n.$$

MATH1081 Test 4 2009 S1 v1a

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1. We seek the solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 66$.

   (i) $x_1, \cdots, x_5$ are non-negative.

   Use the dots and lines analogy[1]. We have 66 dots and 4 lines in a row. We arrange these 4 lines in the 70 available positions in the row.

   Answer: $\binom{70}{4}$.

   Note: For brevity, the explanation of the "dots and line" analogy has been omitted here. However, you should write your explanation in full in the real test, similar to what was written in Question 3 on the previous two pages. Note what it says on the past tests: 'To obtain full marks for this test, your solutions must be fully and clearly explained!'

   (ii) $x_1, x_2, \cdots x_5$ are even numbers. Use the dots and lines analogy.

   Here, we pair the 66 dots into 33 pairs, and arrange these instead. This ensures that

[1] Refer to Question 3 in Test 4 2008 S2 v1b

all of the x_k terms are even. So we have 33 pairs of dots and 4 lines. Arrange in a row and choose positions for the lines.

Answer: $\binom{37}{4}$.

2. Number of ways to fill out form:

   1. Choose 1st favourite. 20 ways.
   2. Choose 2nd favourite. 19 ways.
   3. Choose 3rd favourite. 18 ways.
   4. For each of the 17 remaining programs, tick or do not tick. 2^{17} ways.

   Answer: $20 \times 19 \times 18 \times 2^{17}$.

3. Characteristic equation:

$$\lambda^2 - 10\lambda + 21 = 0 \implies (\lambda - 7)(\lambda - 3) = 0 \implies \lambda = 3 \text{ or } \lambda = 7.$$

Thus, $a_n = A \cdot 3^n + B \cdot 7^n$.

We must now work out A and B. Use the initial conditions:

$$\begin{aligned} a_0 &= -1 = A \cdot 3^0 + B \cdot 7^0 \implies A + B = -1 \\ a_1 &= 1 = A \cdot 3^1 + B \cdot 7^1 \implies 3A + 7B = 1. \end{aligned}$$

We have:

$$\begin{aligned} A + B = -1 \implies 3A + 3B &= -3, \text{ and} \\ 3A + 7B &= 1. \end{aligned}$$

Subtract:

$$\begin{aligned} (3A + 3B) - (3A + 7B) &= -3 - 1 \\ -4B &= -4 \\ B &= 1. \end{aligned}$$

Since $A + B = 1$, we have $A = -2$. Therefore $a_n = (-2) \cdot 3^n + 7^n$.

4. This is a pigeonhole principle problem. Let the students be pigeons to be placed in the pigeonholes

$$(1,\ 6,\ 11, \dots\ 46)$$

$$(2,\ 7,\ 12, \dots\ 47)$$

$$(3,\ 8,\ 13, \dots\ 48)$$

$$(4,\ 9,\ 14, \dots\ 49)$$

$$(5,\ 10,\ 15, \dots\ 50)$$

where each number denotes the house number on one side of the street.

There are 5 pigeonholes and 26 pigeons. As $5 \times 5 = 25 < 26$, it follows that at least one pigeonhole has at least 6 pigeons in it.

Now, each pigeonhole has 10 houses. Without loss of generality, assume the pigeonhole $(1,\ 6,\ 11, \dots\ 46)$ is the pigeonhole with at least 6 pigeons.

We split the 10 houses into pairs:

$$(1,\ 6),\ (11,\ 16),\ (21,\ 26),\ (31,\ 36),\ (41,\ 46).$$

There are at least 6 students and only 5 pairs of houses. By the pigeonhole principle again, it follows that since $6 > 5$, at least one of these pairs of houses will have 2 pigeons in it. That is, two students live exactly 5 houses away from each other.

MATH1081 Test 4 2009 S2 v2b

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1. One method is to consider cases of 8 consonants, then 9, then 10, and add them all together[1].
   The answer may take the form of $\binom{21}{8} \times \binom{5}{2} \times 10! + \binom{21}{9} \times \binom{5}{1} \times 10! + \binom{21}{10} \times 10!$.
   Although your answer may look different, ensure that they take the same value.

2. It can be found[2] that there are

$$\binom{2}{1}\binom{13}{4}\binom{39}{9} - \binom{13}{4}^2\binom{26}{5} \text{ hands.}$$

   Note: There is a slight ambiguity in the question, regarding whether we are dealing with "inclusive or" or "exclusive or". Here, we interpreted it as "inclusive or" (If it was "exclusive or", the answer would instead be $\binom{2}{1}\binom{13}{4}\binom{39}{9} - 2\binom{13}{4}^2\binom{26}{5}$*.)*
   Note 2: Because it is possible that the hand could have exactly four spades AND exactly four diamonds, we need to use the Inclusion-Exclusion Principle.

[1] Refer to Question 1 in Test 4 2008 S2 v1B
[2] Refer to Question 3 in Test 4 2008 S1 v3b

3. It can be found[3] that the homogeneous solution is

$$h_n = A \cdot (-5)^n + B \cdot 2^n,$$

for some constants A and B.
It can also be found that a particular solution is

$$p_n = \frac{2}{7} n \cdot 2^n.$$

Note: Since 2^n is a solution to the homogeneous equation, our 'guess' needs to be adjusted to $cn \cdot 2^n$, for some constant c.
The general solution should be

$$a_n = h_n + p_n = A(-5)^n + B \cdot 2^n + \frac{2}{7} n \cdot 2^n.$$

4. This is a pigeonhole principle problem. Let the pigeonholes be the different combinations of choosing three electives. There should be $\binom{7}{3}$ pigeonholes. Let the students be the pigeons.
So there are 35 pigeonholes and 200 pigeons. As $5 \times 35 = 175 < 200$, it follows from the pigeonhole principle that at least one pigeonhole has at least 6 pigeons in it. In other words, at least six of the students must have completed the same electives as each other.

[3] Refer to Question 4 in Test 4 2008 S2 v1b

MATH1081 Test 4 2010 S1 v2a

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1. (i) Note that the 4 possible time slots each day are each 1 hour long. With 4 possible timeslots in each of the five days, it can be found that there are $\binom{20}{6}$ ways.

   (ii) By first choosing which day of the week contains the two timeslots, it can be found that there are $5 \times \binom{4}{2} \times 4^4$ ways.

2. (i) Using the dots and lines analogy[1], it can be found that there are $\binom{59}{4}$ solutions.

   (ii) Let $x_k = 2y_k + 1$. This reduces the equality[1] of x to $\sum_{k=1}^{5} y_k = 25$. Note that number of solutions for the y equation is equivalent to the number of solutions to the x equation. Using the dots and lines method again, it can be found that there are $\binom{29}{4}$ ways.

3. It can be found[2] that $a_n = -3 \cdot 5^n + 8 \cdot 2^n$.

4. Let 'FRED' be considered one 'letter'. So in a word that we want, there is the string of letters 'FRED'; and 7 other letters. In order to find the answer, ensure to account for the

[1]Refer to Question 3 in Test 4 2008 S2 v1b
[2]Refer to Question 3 in Test 4 2009 S1 v1a

possibility of having two 'FRED's in the word. The answer should be

$$8 \cdot 26^7 - \binom{5}{2} \cdot 26^3.$$