MATH1251 S2 2018 Quiz Solutions

June 29, 2019

These solutions were written by our lord and saviour Rui Tong and typed by James Gao. Please be ethical with this resource. It is for the use of MathSoc members - do not repost it on other forums or groups without asking for permission. If you appreciate our resources, please consider supporting us by coming to our events! Also, happy studying :)

We cannot guarantee that our answers are correct - please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are sometimes multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

Test 1 Version 1a

1. Let $u = \cos\theta$. Then $du = -\sin\theta$.

θ	0	$\frac{\pi}{2}$
u	1	0

$$\int_0^{\frac{\pi}{2}} \sin^3(\theta)\cos^2(\theta)d\theta$$
$$= \int_0^{\frac{\pi}{2}} \sin(\theta)(1-\cos^2(\theta))\cos^2(\theta)d\theta$$
$$= \int_0^1 (1-u^2)u^2 du$$
$$= \int_0^1 u^2 - u^4 du$$
$$= \frac{1}{3} - \frac{1}{5}$$
$$= \frac{2}{15}$$

2. We first find the partial derivatives:

$$\frac{\partial}{\partial y} 9x^2y^2 = 18x^2y; \qquad \frac{\partial}{\partial x}(6x^5y + 3y^2) = 18x^2y$$

the expressions equate, the ODE is exact. Therefore there exists $F(x, y)$ such that:

$$\frac{\partial F}{\partial x} = 9x^2y^2 \tag{1}$$
$$\frac{\partial F}{\partial y} = 6x^3y + 3y^2 \tag{2}$$

Integration (1) with respect to x:

$$F(x, y) = 3x^2y^2 + C_1(y)$$

$$\frac{\partial F}{\partial y} = 6x^3y + C_1'(y)$$

Comparing with (2):

$$\begin{aligned} & C_1'(y) = 3y^2 \\ \Rightarrow \quad & C_1(y) = y^3 + C_2 \end{aligned}$$

Hence $F(x, y) = 3x^2y^2 + y^3$, and the solution is:

$$3x^3y^2 + y^3 = A$$

3. We write:

$$\frac{10x^2}{(x+1)(x^2+9)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2+9}$$

$$\therefore 10x^2 \equiv A(x^2+9) + (Bx+C)(x+1)$$

Solving for A, B and C:

$$\begin{aligned} \text{Put } x = -1 \qquad & 10 = 10A \Rightarrow A = 1 \\ \text{Put } x = 0 \qquad & 0 = 9A + C \Rightarrow C = -9 \\ \text{Coeff. } x^2 \qquad & 10 = A + B \Rightarrow B = 9 \end{aligned}$$

Finally:

$$\begin{aligned} \int \frac{10x^2}{(x+1)(x^2+9)} dx &= \int \frac{1}{x+1} + \frac{9x}{x^2+9} - \frac{9}{x^2+9} dx \\ &= \ln|x+1| + \frac{9}{2}\ln(x^2+9) - 3\tan^{-1}\left(\frac{x}{3}\right) + C \end{aligned}$$

4. The DE is linear with integrating factor:

$$e^{\int -\frac{2}{x} dx} = e^{-2} = e^{\ln(x^{-2})} = \frac{1}{x^2}$$

Hence, it may be expressed as:

$$\frac{d}{dx}\left(\frac{y}{x^2}\right) = 6x^4 \cdot \frac{1}{x^2}$$

$$\begin{aligned} \Rightarrow \frac{y}{x^2} &= \int 6x^2 dx \\ &= 2x^3 + C \end{aligned}$$

Therefore:

$$y = 2x^5 + Cx^2$$

Test 1 Version 1b

1. We write:

$$\frac{2x+9}{(x-2)(x^2+9)} \equiv \frac{A}{x-2} + \frac{Bx+C}{x^2+9}$$

$$\therefore 2x+9 \equiv A(x^2+9) + (Bx+C)(x-2)$$

Solving for A, B and C:

$$\begin{aligned}
&\text{Put } x = 2 && 13 = 13A \Rightarrow A = 1 \\
&\text{Put } x = 0 && 9 = 9A - 2C \Rightarrow C = 0 \\
&\text{Coeff. } x^2 && 0 = A + B \Rightarrow B = -1
\end{aligned}$$

Finally:

$$\begin{aligned}
\int \frac{2x+9}{(x-2)(x^2+9)} dx &= \int \frac{1}{x-2} + \frac{x}{x^2+9} dx \\
&= \ln|x-2| - \frac{1}{2}\ln(x^2+9) + C
\end{aligned}$$

2. We first find the partial derivaties:

$$\frac{\partial}{\partial y}e^y = e^y; \qquad \frac{\partial}{\partial x}(xe^y + 1) = e^y$$

the expressions equate, the ODE is exact. Therefore there exists $F(x, y)$ such that:

$$\frac{\partial F}{\partial x} = e^y \tag{3}$$

$$\frac{\partial F}{\partial y} = xe^y + 1 \tag{4}$$

Integration (1) with respect to x:

$$F(x, y) = xe^y + C_1(y)$$

$$\frac{\partial F}{\partial y} = xe^y + C_1'(y)$$

Comparing with (2):

$$\begin{aligned} & C_1'(y) = 1 \\ \Rightarrow \quad & C_1(y) = y + C_2 \end{aligned}$$

Hence $F(x, y) = xe^y + y$, and the solution is:

$$xe^y + y = A$$

3.

$$\begin{aligned}\int \sec^4(\theta)d\theta &= \int \sec^2(\theta)(1+\tan^2(\theta))d\theta \\ &= \tan(\theta) + \frac{\tan^3(\theta)}{3} + C\end{aligned}$$

The above is done by reverthe chain rule. Otherwise, we can sub $u = \tan(\theta)$, where $du = \sec^2(\theta)d\theta$. Therefore:

$$\begin{aligned}\int \sec^4(\theta)d\theta &= \int \sec^2(\theta)(1+\tan^2(\theta))d\theta \\ &= \int \sec^2(\theta)(1+\tan^2(\theta)) \cdot \left(\frac{1}{\sec^2(\theta)}du\right) \\ &= \int (1+u^2)du \\ &= u + \frac{u^3}{3} + C \\ &= \tan(\theta) + \frac{\tan^3(\theta)}{3} + C\end{aligned}$$

4. This question is identical to question 4 of Test 1 Version 1a.

Test 1 Version 2a.

1.

$$
\begin{aligned}
I_n &= \int\limits_0^{\frac{\pi}{4}} \tan^n(\theta)\sec(\theta)d\theta \\
&= \int\limits_0^{\frac{\pi}{4}} \tan^{n-1}(\theta)[\sec(\theta)\tan(\theta)]d\theta \\
&= \tan^{n-1}(\theta)\sec(\theta)|_0^{\frac{\pi}{4}} - \int\limits_0^{\frac{\pi}{4}} [\sec^2(\theta)(n-1)\tan^{n-2}(\theta)]\sec(\theta)d\theta \\
&= 1^{n-1}\cdot\sqrt{2} - 0^{n-2}\cdot 1 - (n-2)\int\limits_0^{\frac{\pi}{4}} (1+\tan^2(\theta))\tan^{n-2}(\theta)\sec(\theta)d\theta \\
&= \sqrt{2} - (n-1)\int\limits_0^{\frac{\pi}{4}} \tan^{n-2}(\theta)\sec(\theta)d\theta) - (n-1)\int\limits_0^{\frac{\pi}{4}} \tan^n(\theta)\sec(\theta)d\theta
\end{aligned}
$$

Hence:

$$I_n = \sqrt{2} - (n-1)I_{n-2} - (n-1)I_n$$

$$
\begin{aligned}
nI_n &= \sqrt{2} - (n-1)I_{n-2} \\
I_n &= \frac{1}{n}\left(\sqrt{2} - (n-1)I_{n-2}\right)
\end{aligned}
$$

2. The DE is linear with integrating factor:

$$e^{\int -\frac{1}{2x}dx} = e^{-\frac{1}{2}} = (e)^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}$$

Hence, it may be expressed as:

$$\frac{d}{dx}\left(\frac{y}{\sqrt{x}}\right) = 6x^4 * \frac{x}{\sqrt{x}}$$

$$\begin{aligned} \Rightarrow \frac{y}{\sqrt{x}} &= \int \sqrt{x}dx \\ &= \frac{2x^{\frac{3}{2}}}{3} + C \end{aligned}$$

Therefore:

$$y = \frac{2x^2}{3} + C\sqrt{x}$$

3. Let $x = 2\sin(\theta)$, where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Then $dx = 2\cos(\theta)d\theta$.

x	0	1
θ	0	$\frac{\pi}{6}$

$$\begin{aligned} \int_0^1 \frac{1}{(4-x^2)^{\frac{3}{2}}}dx &= \int_0^{\frac{\pi}{6}} \frac{2\cos(\theta)}{(4\cos^2(\theta))^{\frac{3}{2}}}d\theta \\ &= \frac{2}{*}\int_0^{\frac{\pi}{6}} \frac{\cos(\theta)}{\cos^3(\theta)}d\theta \\ &= \frac{1}{4}\int_0^{\frac{\pi}{6}} \sec^2(\theta)d\theta \\ &= \frac{1}{4}\tan(\theta)|_0^{\frac{\pi}{6}} \\ &= \frac{1}{4\sqrt{3}} \end{aligned}$$

4. We first find the partial derivatives:

$$\frac{\partial}{\partial y} 2xy = 2x; \qquad \frac{\partial}{\partial x}(x^2 + y^2) = 2x$$

the expressions equate, the ODE is exact. Therefore there exists $F(x, y)$ such that:

$$\frac{\partial F}{\partial x} = 2xy \tag{5}$$

$$\frac{\partial F}{\partial y} = x^2 + y^2 \tag{6}$$

Integration (1) with respect to x:

$$F(x, y) = x^2 y + C_1(y)$$

$$\frac{\partial F}{\partial y} = x^2 + C_1'(y)$$

Comparing with (2):

$$\begin{aligned} C_1'(y) &= y^2 \\ \Rightarrow \quad C_1(y) &= \frac{y^3}{3} + C_2 \end{aligned}$$

Hence $F(x, y) = x^2 y + \frac{y^3}{3}$, and the solution is:

$$x^2 y + \frac{y^3}{3} = A$$

Test 1 Version 2b

1. Let $x = \tan(\theta)$, where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Then $dx = \sec^2(\theta)d\theta$.

$$\begin{aligned}
\int \frac{1}{x^2\sqrt{1+x^2}} &= \int \frac{\sec^2(\theta}{\tan^2(\theta)\sqrt{\sec^2(\theta)}}d\theta \\
&= \int \frac{\sec(\theta)}{\tan^2(\theta)}d\theta \\
&= \int \frac{\cos(\theta)}{(\sin^2(\theta)}d\theta \\
&= -\frac{1}{\sin(\theta)} + C
\end{aligned}$$

Note that the last line above was obtained by multiplying both top and bottom by $\cos(\theta)$.

2. This question is identical to question 2 of Test 1 Version 1b.

3. Here, we use IBP, but integrate I into x:

$$\begin{aligned}
I_n &= \int_0^1 (1-x^2)^n dx \\
&= x(1-x^2)^n|_0^1 - \int_0^1 x[-2x \cdot n(1-x^2)^{n-1}]dx \\
&= 0 - 0 - 2n \int_0^1 -x^2(1-x^2)^{n-1}dx \\
&= -2n \int_0^1 (1-x^2)^n dx + 2n \int_0^1 (1-x^2)^{n-1}dx
\end{aligned}$$

Converting to the I_n form:

$$\begin{aligned}
I_n &= -2nI_n + 2nI_{n-1} \\
&= \frac{2n}{2n+1}I_{n-1}
\end{aligned}$$

4. The DE is linear with integration factor:

$$e^{\int -\sin(x)dx} = e^{\cos(x)}$$

Hence, we may express it as:

$$\frac{d}{dx}\left(ye^{\cos(x)}\right) = e^{\cos(x)}\sin(x)$$

$$\begin{aligned} ye^{\cos(x)} &= \int e^{\cos(x)}\sin(x) \\ &= -e^{\cos(x)} + C \end{aligned}$$

Therefore $y = -1 + Ce^{-\cos(x)}$ as required.