Weekly Question W1 Easy

Kabir Agrawal

July 2020

$$\begin{aligned}
\lim_{n\to\infty} n^3 \int_n^{2n} \frac{x}{1+x^5}\,dx &= \lim_{n\to\infty} \frac{\int_n^{2n} \frac{x}{1+x^5}\,dx}{\frac{1}{n^3}} \\
&= \lim_{n\to\infty} \frac{\frac{d}{dn}\int_n^{2n} \frac{x}{1+x^5}\,dx}{\frac{d}{dn} n^{-3}} && \text{(By L'Hôpital's Rule)} \\
&= \lim_{n\to\infty} \frac{\frac{2n}{1+(2n)^5}\frac{d}{dn}(2n) - \frac{n}{1+n^5}\frac{d}{dn}(n)}{\frac{-3}{n^4}} && \text{(By Leibniz Rule/FTC, take your pick)} \\
&= \lim_{n\to\infty} -\frac{n^4}{3}\left(\frac{4n}{1+32n^5} - \frac{n}{1+n^5}\right) \\
&= -\frac{1}{3}\lim_{n\to\infty} \frac{4n^5}{1+32n^5} - \frac{n^5}{1+n^5} \\
&= -\frac{1}{3}\left(\frac{1}{8} - 1\right) \\
&= \frac{7}{24}
\end{aligned}$$

The L'Hôpital's Rule is applicable since $\frac{1}{n^3} \to 0$ as $n \to \infty$ and:

$$0 \le \int_n^{2n} \frac{x}{1+x^5}\,dx$$

since we integrate a function that is non-negative on the interval $0 < n \le x \le 2n$ and:

$$\int_n^{2n} \frac{x}{1+x^5}\,dx \le \int_n^{2n} \frac{x}{x^5}\,dx = \frac{7}{24n^3} \to 0$$

as $n \to \infty$ and thus $\int_n^{2n} \frac{x}{1+x^5}\,dx \to 0$ as $n \to \infty$. Thus, we have an indeterminate form.