# MATH1251 ALGEBRA S2 2010 TEST 1 VERSION 1A

*Sample Solutions*

September 13, 2017

These answers were written and typed up by Allan Loi and edited by Aaron Hassan. Please be ethical with this resource. It is for the use of MathSoc members, so do not repost it on other forums or groups without asking for permission. If you appreciate this resource, please consider supporting us by coming to our events and buying our T-shirts! Also, happy studying !

We cannot guarantee that our answers are error-free or would obtain full marks – please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are often multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

Please note quiz papers that are NOT in your course pack will not necessarily reflect the style or difficulty of questions in your quiz.

---

## Question 1

Firstly, we write $-2+2i\sqrt{3}$ in polar form. Noting that the modulus $-2+2i\sqrt{3}$ is 4 and its argument is $2\pi/3$, we have,

$$-2+2i\sqrt{3} = 4e^{\frac{2i\pi}{3}}.$$

Let the fourth roots of $4e^{\frac{2i\pi}{3}}$ be $re^{i\theta}$. This gives us the expression,

$$(re^{i\theta})^4 = 4e^{\frac{2i\pi}{3}},$$

Expanding the left side,

$$r^4e^{4i\theta} = 4e^{\frac{2i\pi}{3}},$$

Equating the modulus and argument in the above expression,

$$r^4 = 4, \quad 4\theta = \frac{2\pi}{3}.$$

Finally, we solve for $r$ and $\theta$, noting that $r$ is a non-negative real value and $\theta$ lies between $-\pi$ and $\pi$. This $r=\sqrt{2}$ and we find that $\theta$ can be $\frac{-5\pi}{6}$, $\frac{-\pi}{3}$, $\frac{\pi}{6}$ or $\frac{2\pi}{3}$. Hence, the roots are

$$\sqrt{2}e^{\frac{\pi}{6}},$$

$$\sqrt{2}e^{\frac{2\pi}{3}},$$
$$\sqrt{2}e^{-\frac{\pi}{3}},$$
$$\sqrt{2}e^{-\frac{5\pi}{6}}.$$

## Question 2

We aim to prove that $S = \{p \in \mathbb{P}_3 \mid p(5) = 0\}$ is a subspace of $\mathbb{P}_3$ using the subspace theorem.

Consider a polynomial $p$ that performs the mapping $x \mapsto (x-5)^3$. We note that this polynomial is in $\mathbb{P}_3$ and $p(5) = 0$. Hence, $p$ is in the set $S$, which implies that $S$ is not empty. (An alternative example you can use to show that $S$ is non-empty is to use the zero polynomial, since this also satisfies $p(5) = 0$, and belongs to $\mathbb{P}_3$.)

Let $p$ and $q$ be elements in $S$. Consider the polynomial $p + q$. Since $\mathbb{P}_3$ is a vector space, it is closed under addition and hence $p + q$ is in $\mathbb{P}_3$. By using the fact that $p(5) = q(5) = 0$, since they are in $S$, we have $(p + q)(5) = p(5) + q(5) = 0 + 0 = 0$. Hence, $p + q$ is also in $S$, which implies that $S$ is closed under addition.

Let $\lambda$ be real and suppose $p$ is in $S$. Since $\mathbb{P}_3$ is a vector space, it is closed under scalar multiplication, so $\lambda p$ is in $\mathbb{P}_3$. By using the fact that $p(5) = 0$, since it is in $S$, we have $(\lambda p)(5) = \lambda p(5) = \lambda(0) = 0$. Hence, $\lambda p$ is also in $S$, which implies that $S$ is closed under scalar multiplication.

Since $S$ is a non-empty subset of $\mathbb{P}_3$ that is closed under addition and scalar multiplication, it is a subspace of $\mathbb{P}_3$.

## Question 3

**(i)** Let $z = x + iy$ and suppose that $|z+1| = |z-1|$. By squaring both sides of this expression and substituting $z$, we have,

$$|(x+1) + iy|^2 = |(x-1) + iy|^2.$$

Using the fact that $|w|^2 = w\overline{w}$,

$$((x+1) + iy)((x+1) - iy) = ((x-1) + iy)(x - 1 - iy).$$

Expanding,

$$(x+1)^2 + y^2 = (x-1)^2 + y^2,$$

Further expanding and cancelling terms gives us,

$$
\begin{aligned}
(x^2 + 2x + 1) + y^2 &= (x^2 - 2x + 1) + y^2, \\
\Rightarrow 2x &= -2x, \\
\Rightarrow 4x &= 0, \\
\Rightarrow x &= 0.
\end{aligned}
$$

Therefore, $x = 0$ and $z = iy$. We have two cases to consider now, $y \geq 0$ and $y < 0$.

Suppose $y \geq 0$. Then $\text{Arg}(z+1) = \text{Arg}(1+iy) = \tan^{-1}(y)$ and $\text{Arg}(z-1) = \text{Arg}(-1+iy) = \tan^{-1}(-y) + \pi$. Thus,

$$
\begin{aligned}
\text{Arg}(z+1) + \text{Arg}(z-1) &= \tan^{-1}(y) + \tan^{-1}(-y) + \pi, \\
&= \tan^{-1}(y) - \tan^{-1}(y) + \pi, \\
&= \pi.
\end{aligned}
$$

Suppose $y < 0$. Then $\text{Arg}(z+1) = \text{Arg}(1+iy) = \tan^{-1}(y)$ and $\text{Arg}(z-1) = \text{Arg}(-1+iy) = \tan^{-1}(-y) - \pi$. Thus,



$$
\begin{aligned}
\text{Arg}(z+1) + \text{Arg}(z-1) &= \tan^{-1}(y) + \tan^{-1}(-y) - \pi, \\
&= \tan^{-1}(y) - \tan^{-1}(y) - \pi, \\
&= -\pi.
\end{aligned}
$$

Therefore, we have $\text{Arg}(z+1) + \text{Arg}(z-1) = \pm\pi$.

**(ii)** Angles of isosceles triangles opposite to the sides of equal length are equal. To draw the diagram for this question, draw two point on the $y$-axis (one above the $x$-axis and one below; these correspond to the point $z$) and the triangle joining this point to the points $(-1, 0)$ and $(1, 0)$, and label the base angles of this (isosceles) triangle as equal.

# MATH1251 ALGEBRA S2 2010 TEST 1 VERSION 1B

*Sample Solutions*

September 13, 2017

These answers were written and typed up by Allan Loi and edited by Aaron Hassan. Please be ethical with this resource. It is for the use of MathSoc members, so do not repost it on other forums or groups without asking for permission. If you appreciate this resource, please consider supporting us by coming to our events and buying our T-shirts! Also, happy studying !

We cannot guarantee that our answers are error-free or would obtain full marks – please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are often multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

Please note quiz papers that are NOT in your course pack will not necessarily reflect the style or difficulty of questions in your quiz.

---

## Question 1

Let $z = \cos(\theta) + i\sin(\theta)$. We can write $z^4$ in two ways. Firstly, by using de Moivre's formula, we have

$$\begin{aligned} z^4 &= (\cos(\theta) + i\sin(\theta))^4, \\ &= \cos(4\theta) + i\sin(4\theta). \end{aligned}$$

Secondly, by using a binomial expansion, we have

$$\begin{aligned} z^4 &= \cos^4(\theta) + 4i\cos^3(\theta)\sin(\theta) - 6\cos^2(\theta)\sin^2(\theta) - 4i\cos(\theta)\sin^3(\theta) + \sin^4(\theta), \\ &= \cos^4(\theta) - 6\cos^2(\theta)\sin^2(\theta) + \sin^4(\theta) + i(4\cos^3(\theta)\sin(\theta)) - 4\cos(\theta)\sin^3(\theta)), \end{aligned}$$

Equating the real components of these two expressions gives,

$$\cos(4\theta) = \cos^4(\theta) - 6\cos^2(\theta)\sin^2(\theta) + \sin^4(\theta).$$

Using the fact that $\sin^2\theta = 1 - \cos^2(\theta)$,

$$\begin{aligned}
\cos(4\theta) &= \cos^4(\theta) - 6\cos^2(\theta)(1-\cos^2(\theta)) + (1-\cos^2(\theta))^2, \\
&= \cos^4(\theta) - 6\cos^2(\theta) + 6\cos^4(\theta) + (\cos^4(\theta) - 2\cos^2(\theta) + 1), \\
&= 8\cos^4(\theta) - 8\cos^2(\theta) + 1.
\end{aligned}$$

## Question 2

We aim to prove that $S = \{\mathbf{x} \in \mathbb{R}^3 \mid x_1 - 2x_2 = 0 \text{ and } 4x_1 + x_3 = 0\}$ is a subspace of $\mathbb{R}^3$ using the subspace theorem. We start by verifying the necessary condition that $S$ is a subset of $\mathbb{R}^3$. This can be done by noting that all elements of $S$ are required to be in $\mathbb{R}^3$, from the way $S$ is defined.

Consider the zero vector $(0,0,0) \in \mathbb{R}^3$. Since $0 - 2(0) = 0$ and $4(0) + 0 = 0$, we see that the zero vector is in $S$, so $S$ is non-empty.

Suppose $\mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$ and $\mathbf{v} = (u_1, u_2, u_3) \in \mathbb{R}^3$ are in $S$. This means that the following expressions are true:

$$\begin{aligned}
u_1 - 2u_2 &= 0, \\
4u_1 + u_3 &= 0, \\
v_1 - 2v_2 &= 0, \\
4v_1 + v_3 &= 0.
\end{aligned}$$

Let $\mathbf{w} = \mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3) \in \mathbb{R}^3$. Observe that,

$$\begin{aligned}
w_1 + 2w_2 &= (u_1 + v_1) + 2(u_2 + v_2), \\
&= (u_1 + 2u_2) + (v_1 + 2v_2), \\
&= 0 + 0, \\
&= 0,
\end{aligned}$$

and,

$$\begin{aligned}
4w_1 + w_3 &= 4(u_1 + v_1) + (u_3 + v_3), \\
&= (4u_1 + u_3) + (4v_1 + v_3), \\
&= 0 + 0, \\
&= 0.
\end{aligned}$$

Hence, $\mathbf{w}$ is an element of $S$, which implies that $S$ is closed under addition.

Let $\lambda \in \mathbb{R}$ and let $\mathbf{y} = \lambda\mathbf{u} = (\lambda u_1, \lambda u_2, \lambda u_3)$. Observe that,

$$\begin{aligned} y_1 + 2y_2 &= \lambda u_1 + 2\lambda u_2, \\ &= \lambda(u_1 + 2u_2), \\ &= \lambda(0), \\ &= 0. \end{aligned}$$

and,

$$\begin{aligned} 4y_1 + y_3 &= 4\lambda u_1 + \lambda u_3, \\ &= \lambda(4u_1 + u_3), \\ &= \lambda(0), \\ &= 0. \end{aligned}$$

Hence, $\mathbf{y}$ is an element of $S$, which implies that $S$ is closed under scalar multiplication.

We can therefore conclude that $S$ is a subspace of $\mathbb{R}^3$.

## Question 3



**(i)** Let $\alpha = e^{2\pi i/5}$. Raising $\alpha$ to the fifth power gives,

$$\begin{aligned} \alpha^5 &= \left(e^{2\pi i/5}\right)^5, \\ &= e^{10\pi i/5}, \\ &= e^{2\pi i}, \\ &= \cos 2\pi + i \sin 2\pi, \\ &= 1. \end{aligned}$$

Hence, we have $\alpha^5 = 1$. Rearranging this gives us

$$\alpha^5 - 1 = 0,$$

which can be factorised to obtain

$$(\alpha - 1)\left(1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4\right) = 0.$$

(You may expand this expression to confirm this factorisation.)

Since $\alpha$ is not equal to 1, $\alpha - 1$ is not equal to zero. Therefore, we can divide both sides by $\alpha - 1$ to get

$$1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0.$$

**(ii)** We can expand the required expression as follows

$$\begin{aligned}\left(z - \alpha - \alpha^{-1}\right)\left(z - \alpha^2 - \alpha^{-2}\right) &= \left(z^2 - \alpha^2 z - \alpha^{-2} z\right) - \left(\alpha z - \alpha^3 - \alpha^{-1}\right) - \left(\alpha^{-1} z - \alpha - \alpha^{-3}\right), \\ &= z^2 - \left(\alpha^2 + \alpha^{-2} + \alpha + \alpha^{-1}\right) z + \left(\alpha + \alpha^{-1} + \alpha^3 + \alpha^{-3}\right).\end{aligned}$$

This can be simplified to

$$z^2 + z - 1,$$

as is shown in the next part.

**(iii)** Consider putting $\alpha = e^{2\pi i/5}$ into the quadratic $p(z) = (z - \alpha - \alpha^{-1})(z - \alpha^2 - \alpha^{-2})$. By noting that for any integer $n$,

$$\begin{aligned}\alpha^n + \alpha^{-n} &= e^{2n\pi i/5} + e^{-2n\pi i/5}, \\ &= (\cos(2n\pi/5) + i\sin(2n\pi/5)) + (\cos(-2n\pi/5) + i\sin(-2n\pi/5)), \\ &= (\cos(2n\pi/5) + i\sin(2n\pi/5)) + (\cos(2n\pi/5) - i\sin(2n\pi/5)), \\ &= 2\cos(2n\pi/5),\end{aligned}$$



we can write the factorised form of $p(z)$ as $(z - 2\cos(2\pi/5))(z - 2\cos(4\pi/5))$, which implies that $2\cos(2\pi/5)$ is a root of $p(z)$. Using part (ii), we can write the expanded form of $p(z)$ as

$$\begin{aligned}p(z) &= z^2 - (\alpha^2 + \alpha^{-2} + \alpha + \alpha^{-1})z + (\alpha + \alpha^{-1} + \alpha^3 + \alpha^{-3}), \\ &= z^2 - \alpha^{-5}(\alpha^7 + \alpha^3 + \alpha^6 + \alpha^4)z + \alpha^{-5}(\alpha^6 + \alpha^4 + \alpha^8 + \alpha^2).\end{aligned}$$

Using the fact that $\alpha^5 = 1$ and $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0$, we can simplify $p(z)$ as follows,

$$\begin{aligned}p(z) &= z^2 - \alpha^{-5}(\alpha^5\alpha^2 + \alpha^3 + \alpha^5\alpha^1 + \alpha^4)z + \alpha^{-5}(\alpha^5\alpha^1 + \alpha^4 + \alpha^3\alpha^5 + \alpha^2), \\ &= z^2 - (\alpha^2 + \alpha^3 + \alpha^1 + \alpha^4)z + (\alpha^1 + \alpha^4 + \alpha^3 + \alpha^2), \\ &= z^2 - (-1)z + (-1), \\ &= z^2 + z - 1,\end{aligned}$$

which is the quadratic polynomial we seek.

**(iii)** From the factorised form of $p(z)$, which is $(z - 2\cos(2\pi/5))(z - 2\cos(4\pi/5))$, we can see that $2\cos(4\pi/5)$ is the other root of $p(z)$.

# MATH1251 ALGEBRA S2 2010 TEST 1 VERSION 2A

*Sample Solutions*

September 13, 2017

These answers were written and typed up by Allan Loi and edited by Aaron Hassan. Please be ethical with this resource. It is for the use of MathSoc members, so do not repost it on other forums or groups without asking for permission. If you appreciate this resource, please consider supporting us by coming to our events and buying our T-shirts! Also, happy studying !

We cannot guarantee that our answers are error-free or would obtain full marks – please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are often multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

Please note quiz papers that are NOT in your course pack will not necessarily reflect the style or difficulty of questions in your quiz.

---

## Question 1

Let $z = \cos(\theta) + i\sin(\theta)$. We can write $z^5$ in two ways. Firstly, by using de Moivre's formula, we have

$$\begin{aligned} z^5 &= (\cos(\theta) + i\sin(\theta))^5, \\ &= \cos(5\theta) + i\sin(5\theta). \end{aligned}$$

Secondly, by using a binomial expansion, we have

$$\begin{aligned} z^5 &= \cos^5(\theta) + 5i\cos^4(\theta)\sin(\theta)) - 10\cos^3(\theta)\sin^2(\theta) - 10i\cos^2(\theta)\sin^3(\theta) + 5\cos(\theta)\sin^4(\theta) + i\sin^5(\theta), \\ &= (\cos^5(\theta) - 10\cos^3(\theta)\sin^2(\theta) + 5\cos(\theta)\sin^4(\theta)) + i(5\cos^4(\theta)\sin(\theta)) - 10\cos^2(\theta)\sin^3(\theta) + \sin^5(\theta)). \end{aligned}$$

Equating the imaginary components of these two expressions gives,

$$\sin(5\theta) = 5\cos^4(\theta)\sin(\theta)) - 10\cos^2(\theta)\sin^3(\theta) + \sin^5(\theta)$$

Using the fact that $\cos^2\theta = 1 - \sin^2(\theta)$,

$$\sin(5\theta) = 5(1 - \sin^2(\theta))^2\sin(\theta) - 10(1 - \sin^2(\theta))\sin^3(\theta) + \sin^5(\theta),$$

$$
\begin{aligned}
&= (5\sin^5(\theta) - 10\sin^3(\theta) + 5\sin(\theta)) - (10\sin^3(\theta) - 10\sin^5(\theta)) + \sin^5(\theta), \\
&= 16\sin^5(\theta) - 20\sin^3(\theta) + 5\sin(\theta)
\end{aligned}
$$

## Question 2

We aim to prove that $S = \{\mathbf{x} \in \mathbb{R}^3 \mid 5x_1 - 2x_2 + x_3 = 0\}$ is a subspace of $\mathbb{R}^3$ using the subspace theorem. Since all elements of $S$ are in $\mathbb{R}^3$, $S$ is a subset of $\mathbb{R}^3$.

Consider the zero vector $(0, 0, 0)$. Since $5(0) - 2(0) + (0) = 0$, we see that the zero vector is in $S$, so $S$ is non-empty.

Suppose $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (u_1, u_2, u_3)$ are in $S$. This means that the following expressions are true,

$$
\begin{aligned}
5u_1 - 2u_2 + u_3 &= 0, \\
5v_1 - 2v_2 + v_3 &= 0.
\end{aligned}
$$

Let $\mathbf{w} = \mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$. Observe that,

$$
\begin{aligned}
5w_1 - 2w_2 + w_3 &= 5(u_1 + v_1) - 2(u_2 + v_2) + (u_3 + v_3), \\
&= (5u_1 - 2u_2 + u_3) + (5v_1 - 2v_2 + v_3), \\
&= 0 + 0, \\
&= 0.
\end{aligned}
$$

Hence, $\mathbf{w}$ is an element of $S$, which implies that $S$ is closed under addition.

Let $\lambda \in \mathbb{R}$ and let $\mathbf{y} = \lambda\mathbf{u} = (\lambda u_1, \lambda u_2, \lambda u_3)$. Observe that,

$$
\begin{aligned}
5y_1 - 2y_2 + y_3 &= 5(\lambda u_1) - 2(\lambda u_2) + (\lambda u_3), \\
&= \lambda(5u_1 - 2u_2 + u_3), \\
&= \lambda(0), \\
&= 0.
\end{aligned}
$$

Hence, $\mathbf{y}$ is an element of $S$, which implies that $S$ is closed under scalar multiplication.

We can therefore conclude that $S$ is a subspace of $\mathbb{R}^3$.

## Question 3

**(i)** We can simplify the required expression as follows:

$$\begin{aligned}\frac{e^{i\theta}-1}{e^{i\theta}+1} &= \frac{e^{i\theta/2}}{e^{i\theta/2}}\frac{e^{i\theta/2}-e^{-i\theta/2}}{e^{i\theta/2}+e^{-i\theta/2}},\\ &= \frac{e^{i\theta/2}-e^{-i\theta/2}}{e^{i\theta/2}+e^{-i\theta/2}},\\ &= \frac{2i\sin(\theta/2)}{2\cos(\theta/2)},\\ &= i\tan\left(\frac{\theta}{2}\right).\end{aligned}$$

**(ii)** Since $z$ is on the unit circle, we can write $z$ as $e^{i\theta}$ for some real $\theta$. Conversely, since the modulus of $e^{i\theta}$ is 1, we know that $e^{i\theta}$ represents a point on the unit circle for any real $\theta$. From part (i), we have

$$e^{i\theta}-1 = i\tan\left(\frac{\theta}{2}\right)(e^{i\theta}+1).$$

Substituting $z = e^{i\theta}$, where $z$ is a point on the unit circle, and noting that $i = e^{i\pi/2}$,

$$z-1 = e^{i\pi/2}\tan\left(\frac{\theta}{2}\right)(z+1)$$

We have two cases, $\tan(\theta/2) \geq 0$ and $\tan(\theta/2) < 0$.

If $\tan(\theta/2) \geq 0$, then $\tan(\theta/2)$ is a non-negative real that is scaling the complex number $z+1$, while $e^{i\pi/2}$ rotates the number by $\pi/2$. Therefore, we can see that $z-1$ is a scaling of $z+1$ by $\tan(\theta/2)$, followed by a rotation by $\pi/2$, which means that $\text{Arg}(z-1) = \text{Arg}(z+1) + \frac{\pi}{2}$.

If $\tan(\theta/2) < 0$, then $\tan(-\theta/2) > 0$ and we can write

$$z-1 = -e^{i\pi/2}\tan\left(-\frac{\theta}{2}\right)(z+1).$$

By noting that $-e^{i\pi/2} = -i = e^{-i\pi/2}$, we have

$$z-1 = e^{-i\pi/2}\tan\left(-\frac{\theta}{2}\right)(z+1),$$

which we can interpret in a similar way as before. We see that $z-1$ is a scaling of $z+1$ by $\tan(-\theta/2)$, followed by a rotation by $-\pi/2$, which means that $\text{Arg}(z-1) = \text{Arg}(z+1) - \frac{\pi}{2}$.

Putting these two cases together allows us to deduce that

$$\operatorname{Arg}(z-1) - \operatorname{Arg}(z+1) = \pm\frac{\pi}{2}$$

**(iii)** Angles inscribed in a semicircle are always right angles.

# MATH1251 ALGEBRA S2 2010 TEST 2 VERSION 1A

*Sample Solutions*

September 13, 2017

These answers were written and typed up by Aaron Hassan. Please be ethical with this resource. It is for the use of MathSoc members, so do not repost it on other forums or groups without asking for permission. If you appreciate this resource, please consider supporting us by coming to our events and buying our T-shirts! Also, happy studying !

We cannot guarantee that our answers are error-free or would obtain full marks – please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are often multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

Please note quiz papers that are NOT in your course pack will not necessarily reflect the style or difficulty of questions in your quiz.

---

## Question 1

**(i)** Place $p_1, p_2, p_3, q$ (as vectors) into an augmented matrix and row-reduce. We have

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & -1 & 3 \\ 3 & 5 & -4 & 7 \\ -1 & 3 & -1 & -7 \end{array}\right) \xrightarrow[R_2 \rightsquigarrow R_2 - 3R_1]{R_3 \rightsquigarrow R_3 + R_1} \left(\begin{array}{ccc|c} \boxed{1} & 1 & -1 & 3 \\ 0 & \boxed{2} & -1 & -2 \\ 0 & 4 & -2 & -4 \end{array}\right)$$

$$\xrightarrow{R_3 \rightsquigarrow R_3 - 2R_2} \left(\begin{array}{ccc|c} \boxed{1} & 1 & -1 & 3 \\ 0 & \boxed{2} & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This is now in row-echelon form, and the right-hand column is non-leading. Hence, $q$ **is** an element of span $(p_1, p_2, p_3)$.

**(ii)** Since the left-hand part of the row-echelon form in the matrix above contains a zero row, the set $\{p_1, p_2, p_3\}$ is **not** spanning.

**Remark.** Make sure you know why the claims made above are true! For (i), the polynomial $q$

is in span $(p_1, p_2, p_3)$ if and only if there exist scalars $\alpha_1, \alpha_2, \alpha_3$ such that

$$\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 = q \iff \alpha_1 \left(1 + 3x - x^2\right) + \alpha_2 \left(1 + 5x + 3x^2\right) + \alpha_3 \left(-1 - 4x - x^2\right) \equiv 3 + 7x - 7x^2.$$

If we then equate coefficients of like powers of $x$ above, we will end up with a system of linear equations that is exactly described by the augmented matrix formed by placing the polynomials $p_1, p_2, p_3, q$ (as vectors) into an augmented matrix. This is the reason why we consider such a matrix. Also, such $\alpha_i$ will exist if and only if this linear system has a solution, and we know from first semester that the linear system will have a solution if and only if the right-hand column is non-leading in the row-echelon form. This explains what we did, and is the general explanation for why we do what we do for these types of questions.

For (ii), to check if the polynomials span $P_2$, it is equivalent to the fact that their coordinate vectors span $\mathbb{R}^3$ or $\mathbb{C}^3$ (why?), and this is the case if and only if the matrix formed by placing them as columns has no zero rows in the row echelon form. This is because a set of vectors is spanning if and only if any vector in the ambient space can be written as a linear combination of them, i.e. for any right-hand vector in the augmented matrix, there should exist a solution to the corresponding linear system $A\mathbf{x} = \mathbf{b}$. There exists a solution to this for every $\mathbf{b}$ if and only if the row-echelon form of $A$ has no zero rows, as we know from first semester.

## Question 2

**(i)** The domain and codomain of $T$ are vector spaces. We must show that $T$ preserves addition and preserves scalar multiplication. Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$ and $\alpha \in \mathbb{R}$. Then $\alpha\mathbf{x} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \\ \alpha x_4 \end{pmatrix} \in \mathbb{R}^4$, so

$$\begin{aligned} T(\alpha\mathbf{x}) &= \begin{pmatrix} \alpha x_1 + 2(\alpha x_2) - (\alpha x_3) \\ 3(\alpha x_3) + (\alpha x_4) \end{pmatrix} \\ &= \begin{pmatrix} \alpha(x_1 + 2x_2 - x_3) \\ \alpha(3x_3 + x_4) \end{pmatrix} \\ &= \alpha \begin{pmatrix} x_1 + 2x_2 - x_3 \\ 3x_3 + x_4 \end{pmatrix} \\ &= \alpha T(\mathbf{x}). \end{aligned}$$

Thus $T$ preserves scalar multiplication. Now, let $\mathbf{x} \in \mathbb{R}^4$ as before and let $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \in \mathbb{R}^4$.

Then $\mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 \end{pmatrix} \in \mathbb{R}^4$, so

$$
\begin{aligned}
T(\mathbf{x} + \mathbf{y}) &= \begin{pmatrix} (x_1 + y_1) + 2(x_2 + y_2) - (x_3 + y_3) \\ 3(x_3 + y_3) + (x_4 + y_4) \end{pmatrix} \\
&= \begin{pmatrix} (x_1 + 2x_2 - x_3) + (y_1 + 2y_2 - y_3) \\ (3x_3 + x_4) + (3y_3 + y_4) \end{pmatrix} \\
&= \begin{pmatrix} x_1 + 2x_2 - x_3 \\ 3x_3 + x_4 \end{pmatrix} + \begin{pmatrix} y_1 + 2y_2 - y_3 \\ 3y_3 + y_4 \end{pmatrix} \\
&= T(\mathbf{x}) + T(\mathbf{y}).
\end{aligned}
$$

Thus $T$ preserves addition. As $T$ preserves scalar multiplication and addition, it is a linear map.

**(ii)** By inspection of the formula for $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, the $A$ is $A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix}$.

## Question 3

**(i)** Anticipating we would need to augment $A$ with $\mathbf{b}$ in question (ii), we shall do this now, so that we would not need to do the whole row-reduction again. We have

$$
\begin{aligned}
[A \mid \mathbf{b}] = \left(\begin{array}{cccc|c} \boxed{-1} & 2 & 1 & 0 & 4 \\ 1 & -1 & 0 & -1 & 1 \\ 3 & -2 & 1 & -4 & 8 \end{array}\right) &\xrightarrow[R_3 \rightsquigarrow R_3 + 3R_1]{R_2 \rightsquigarrow R_2 + R_1} \left(\begin{array}{cccc|c} \boxed{-1} & 2 & 1 & 0 & 4 \\ 0 & \boxed{1} & 1 & -1 & 5 \\ 0 & 4 & 4 & -4 & 20 \end{array}\right) \\
&\xrightarrow{R_3 \rightsquigarrow R_3 - 4R_2} \left(\begin{array}{cccc|c} \boxed{-1} & 2 & 1 & 0 & 4 \\ 0 & \boxed{1} & 1 & -1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).
\end{aligned}
$$

The leading columns in the row echelon form of $A$ are thus columns 1 and 2. Therefore, the columns 1 and 2 *of the original matrix* $A$ form a basis for the image of $A$. So a basis for $\operatorname{im}(A)$

is

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \right\}.$$

**(ii)** The right-hand column is non-leading in the row-echelon form we found in part (i). Therefore, $\mathbf{b}$ *is* in the image of $A$.

# MATH1251 ALGEBRA S2 2010 TEST 2 VERSION 1B

*Sample Solutions*

September 13, 2017

These answers were written and typed up by Aaron Hassan. Please be ethical with this resource. It is for the use of MathSoc members, so do not repost it on other forums or groups without asking for permission. If you appreciate this resource, please consider supporting us by coming to our events and buying our T-shirts! Also, happy studying!

We cannot guarantee that our answers are error-free or would obtain full marks – please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are often multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

Please note quiz papers that are NOT in your course pack will not necessarily reflect the style or difficulty of questions in your quiz.

---

## Question 1

Let $S = \{A_1, A_2, A_3\}$. This set is linearly independent if and only if the set $\widetilde{S} := \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, where for each $j \in \{1, 2, 3\}$, $\mathbf{v}_j$ is the coordinate vector of $A_j$ with respect to the standard basis of the space of $2 \times 2$ matrices (in other words, $\mathbf{v}_j$ is $A_j$ written as a vector). Therefore, we place the vectors $\mathbf{v}_j$ into a matrix and row-reduce. We get

$$(\mathbf{v}_1\, \mathbf{v}_2\, \mathbf{v}_3) = \begin{pmatrix} \boxed{1} & 3 & -1 \\ -1 & -2 & 3 \\ 3 & 9 & 5 \\ -1 & 1 & 9 \end{pmatrix} \xrightarrow[R_3 \rightsquigarrow R_3 - 3R_1,\, R_4 \rightsquigarrow R_4 + R_1]{R_2 \rightsquigarrow R_2 + R_1} \begin{pmatrix} \boxed{1} & 3 & -1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 8 \\ 0 & 4 & 8 \end{pmatrix}$$

$$\xrightarrow{R_4 \rightsquigarrow R_4 - 2R_2} \begin{pmatrix} \boxed{1} & 3 & -1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & \boxed{8} \\ 0 & 0 & 0 \end{pmatrix}.$$

Every column in the row-echelon form is leading. Therefore, $\widetilde{S}$ is a linearly independent set. Thus $S$ is a linearly independent set, so the answer to the question is "yes".

## Question 2

By inspection, we see that

$$\begin{pmatrix} 9 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} + 4 \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \text{i.e.} \quad \mathbf{b} = \mathbf{v}_1 + 4\mathbf{v}_2.$$

This means that the coordinate vector of $\mathbf{b}$ with respect to the basis $\mathcal{B} := \{\mathbf{v}_1, \mathbf{v}_2\}$ is $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

**Remark.** If you couldn't spot the required linear combination by inspection like was done above, you could always just let the desired coordinate vector be $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, which gives us (by definition of coordinate vector)

$$\alpha \mathbf{v}_1 + \beta \mathbf{v}_2 = \mathbf{b}, \quad \text{i.e.} \quad \alpha \begin{pmatrix} 1 \\ -3 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \end{pmatrix} \iff \begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \end{pmatrix}.$$

You could then solve this linear system (e.g. using Gaussian Elimination, inverse matrix, high school methods, or whatever method you like) to obtain the required values $\alpha$ and $\beta$.

## Question 3

Note that $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{R}^2$, $2 \in \mathbb{R}$, and $2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \in \mathbb{R}^2$. Therefore,

$$\begin{aligned} T\left(2\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = T\begin{pmatrix} 2 \\ 0 \end{pmatrix} &= \begin{pmatrix} 2^2 \\ 0^2 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \end{aligned}$$

but

$$\begin{aligned} 2T\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= 2\begin{pmatrix} 1^2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 0 \end{pmatrix} \\ &\neq T\left(2\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ 0 \end{pmatrix}. \end{aligned}$$

Hence $T$ does not preserve scalar multiplication and so cannot be linear.

*Remember, to show that a map is* not *linear, it suffices to come up with a single numerical example demonstrating that one of the properties of linear maps (preservation of addition or scalar multiplication) is not obeyed.*

## Question 4

**(i)** We row-reduce $A$, and augment $\mathbf{b}$ as this will be useful for part (ii). We have

$$[A \mid \mathbf{b}] = \left(\begin{array}{ccccc|c} \boxed{-1} & 2 & 1 & 0 & 1 & 4 \\ 1 & -1 & 0 & -1 & 0 & 1 \\ 3 & -2 & 1 & -4 & 1 & -3 \end{array}\right) \xrightarrow[R_3 \rightsquigarrow R_3+3R_1]{R_2 \rightsquigarrow R_2+R_1} \left(\begin{array}{ccccc|c} \boxed{-1} & 2 & 1 & 0 & 1 & 4 \\ 0 & \boxed{1} & 1 & -1 & 1 & 5 \\ 0 & 4 & 4 & -4 & 4 & 9 \end{array}\right)$$

$$\xrightarrow{R_3 \rightsquigarrow R_3-4R_2} \left(\begin{array}{ccccc|c} \boxed{-1} & 2 & 1 & 0 & 1 & 4 \\ 0 & \boxed{1} & 1 & -1 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & -11 \end{array}\right).$$

As we can see, there are precisely two leading columns in the row-echelon form of $A$. Therefore, we have $\operatorname{rank}(A) = 2$. The number of non-leading columns in the row-echelon form of $A$ is 3, so $\operatorname{nullity}(A) = 3$.

**(ii)** No, because the right-hand column in the row-echelon form in part (i) is leading.

# MATH1251 ALGEBRA S2 2010 TEST 2 VERSION 2A

*Sample Solutions*

September 13, 2017

These answers were written and typed up by Aaron Hassan. Please be ethical with this resource. It is for the use of MathSoc members, so do not repost it on other forums or groups without asking for permission. If you appreciate this resource, please consider supporting us by coming to our events and buying our T-shirts! Also, happy studying !

We cannot guarantee that our answers are error-free or would obtain full marks – please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are often multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

Please note quiz papers that are NOT in your course pack will not necessarily reflect the style or difficulty of questions in your quiz.

---

## Question 1

**(i)** We place (the coordinate vectors of) the matrices $A_1, A_2, A_3, A_4$ and $B$ into an augmented matrices (as columns), and row-reduce. We get

$$\left(\begin{array}{cccc|c} \boxed{1} & 1 & 3 & 1 & 2 \\ 2 & 1 & 7 & 0 & 1 \\ 1 & 0 & 4 & -1 & -1 \\ -1 & 1 & -5 & 3 & 4 \end{array}\right) \xrightarrow[R_3 \rightsquigarrow R_3 - R_1\ R_4 \rightsquigarrow R_4 + R_1]{R_2 \rightsquigarrow R_2 - 2R_1} \left(\begin{array}{cccc|c} \boxed{1} & 1 & 3 & 1 & 2 \\ 0 & \boxed{-1} & 1 & -2 & -3 \\ 0 & -1 & 1 & -2 & -3 \\ 0 & 2 & -2 & 4 & 6 \end{array}\right)$$

$$\xrightarrow[R_4 \rightsquigarrow R_4 + 2R_2]{R_3 \rightsquigarrow R_3 - R_2} \left(\begin{array}{cccc|c} \boxed{1} & 1 & 3 & 1 & 2 \\ 0 & \boxed{-1} & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

As the right-hand column in the row-echelon form is non-leading, $B$ **is** an element of span $\{A_1, A_2, A_3, A_4\}$.

**(ii)** No, because not every row is leading in the row-echelon form of the left-hand part of the augmented matrix above (i.e. there exists a zero row in the left-hand part).

## Question 2

### Method 1 – Inspection

By inspection, we see that

$$\begin{pmatrix} 7 \\ 8 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 4 \end{pmatrix} - 2 \begin{pmatrix} -1 \\ 6 \end{pmatrix}, \quad \text{i.e.} \quad \mathbf{b} = 5\mathbf{v}_1 - 2\mathbf{v}_2.$$

This means that the coordinate vector of $\mathbf{v}$ with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ is $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$.

### Method 2 – Usual procedure

Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ and let the desired coordinate vector be $[\mathbf{b}]_{\mathcal{B}} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. Then by definition, we have

$$\alpha \mathbf{v}_1 + \beta \mathbf{v}_2 = \mathbf{b}, \quad \text{i.e.} \quad \alpha \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}.$$

Defining $B$ to the matrix with columns $\mathbf{v}_1$ and $\mathbf{v}_2$, i.e. $B = (\mathbf{v}_1\, \mathbf{v}_2) = \begin{pmatrix} 1 & -1 \\ 4 & 6 \end{pmatrix}$, the above equation can be equivalently expressed in matrix form as

$$B \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \mathbf{b} \iff \begin{pmatrix} 1 & -1 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$$

This is a linear system that we can solve for $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ using a number of methods (e.g. Gaussian Elimination, high school methods, inverse matrix, etc.). For example, using the inverse matrix method, we obtain

$$\begin{aligned} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} &= \begin{pmatrix} 1 & -1 \\ 4 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 7 \\ 8 \end{pmatrix} \\ &= \frac{1}{6 \times 1 - 1 \times (-4)} \begin{pmatrix} 6 & 1 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \end{pmatrix} \\ &= \frac{1}{10} \begin{pmatrix} 50 \\ -20 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ -2 \end{pmatrix}. \end{aligned}$$

Thus the answer is $[\mathbf{b}]_{\mathcal{B}} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$.

*Remember, if we have a* $2 \times 2$ *real or complex matrix* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, *then its inverse is given by* $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, *provided that* $\det(A) \neq 0$. *In other words, switch the diagonal entries, negate the others, and remember to divide by the determinant.*

### Question 3

Since

$$T(\mathbf{0}) = 0 + 2 \times 0 + 3 \times 0 + 4 = 4 \neq 0,$$

$T$ is not a linear transformation.

**Tip.** Remember, a linear map must map the zero vector of its domain to the zero vector of its codomain. So if you can show that a map does not map the zero vector of its domain to the zero vector of its codomain, it cannot be linear. Of course, if a map *does* map the zero vector of its domain to the zero vector of its codomain, we cannot conclude just from this whether or not the map is linear.

## Question 4



To find the kernel, we solve $A\mathbf{x} = \mathbf{0}$. To do this, we just row-reduce $A$, augmented with $\mathbf{0}$ (but we don't need to write the augmented part all the time since the zero vector does not change when applying elementary row operations to it. So we will only write it at the last step.). We have (keeping in mind that column $j$ corresponds to $x_j$, for $j = 1, 2, 3, 4, 5$)

$$A = \begin{pmatrix} \boxed{1} & -1 & 3 & 2 & 0 \\ 2 & -1 & 5 & 4 & -1 \\ 3 & -2 & 8 & 7 & 2 \end{pmatrix} \xrightarrow[R_3 \rightsquigarrow R_3 - 3R_1]{R_2 \rightsquigarrow R_2 - 2R_1} \begin{pmatrix} \boxed{1} & -1 & 3 & 2 & 0 \\ 0 & \boxed{1} & -1 & 0 & -1 \\ 0 & 1 & -1 & 1 & 2 \end{pmatrix}$$

$$\xrightarrow{R_3 \rightsquigarrow R_3 - R_2} \left(\begin{array}{ccccc|c} \boxed{1} & -1 & 3 & 2 & 0 & 0 \\ 0 & \boxed{1} & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} & 3 & 0 \end{array}\right).$$

We set $\boxed{x_3 = \alpha}$ and $\boxed{x_5 = \beta}$ (free parameters) as columns 3 and 5 in the row-echelon form are non-leading. Then from row 3, we have

$$x_4 = -3x_5 \Rightarrow \boxed{x_4 = -3\beta}.$$

Now, from row 2, we have

$$x_2 = x_3 + x_5 \Rightarrow \boxed{x_2 = \alpha + \beta}.$$

Now, from row 1, we have

$$x_1 = x_2 - 3x_3 - 2x_4 = (\alpha + \beta) - 3\alpha - 2\,(-3\beta) \Rightarrow x_1 = \boxed{-2\alpha + 7\beta}.$$

Therefore, we have $A\mathbf{x} = \mathbf{0}$ if and only if

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2\alpha + 7\beta \\ \alpha + \beta \\ \alpha \\ -3\beta \\ \beta \end{pmatrix} \quad \text{for some scalars } \alpha, \beta$$

$$= \alpha \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 7 \\ 1 \\ 0 \\ -3 \\ 1 \end{pmatrix}.$$

Therefore, a basis for $\ker(A)$ is

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ 1 \\ 0 \\ -3 \\ 1 \end{pmatrix} \right\}.$$



**Tip.** If you have time, you should check your answer by checking that $A$ multiplied by each of the vectors you found for the basis of the kernel results in the zero vector, i.e. $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. If this does not happen, you must have made a mistake somewhere in your working, and should go back and find it.