MATH1031 Mastery Lab Test 1 Solutions to Samples

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We cannot guarantee that our answers are correct - please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are sometimes multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

Question 2

Let

$$A = \begin{pmatrix} -7 & -3 & -7 \\ -1 & 2 & -1 \\ 4 & 13 & 5 \end{pmatrix}, \qquad B = \begin{pmatrix} -9 & -8 & 17 \\ 14 & -3 & 6 \end{pmatrix},$$

$$C = \begin{pmatrix} -1 & -3 & -1 \\ 6 & 5 & -3 \end{pmatrix}, \qquad E = \begin{pmatrix} -7 & -8 \\ 2 & 6 \\ -8 & 3 \end{pmatrix},$$

and I is the 3×3 identity matrix. Some of the matrix expressions below are undefined. Select all the undefined expressions.

[Choices: $EC + 3I$, $B + 2C$, BC^T, $AE - 2I$, BC, BE, AB, $B - E$].

Solution: The answers are $AE - 2I$, BC, AB, $B - E$.
Recall that matrix addition (and subtraction) is only defined for matrices of the same size.

Whereas for the product of two matrices F and G of size $m_1 \times n_1$ and $m_2 \times n_2$ respectively, matrix multiplication FG is only defined when $m_2 = n_1$, and furthermore FG has size $m_1 \times n_2$. Using these rules, we can check the size of the resulting matrix in each option to find the answer.

- EC: E has 2 columns and C has 2 rows, so EC is well defined, and is in particular a 3×3 matrix. This certainly can be added with a 3×3 matrix.
- Clearly B and $2C$ are both 2×3.
- Note that C^T is a 3×2 matrix, so BC multiplies a 2×3 by a 3×2 which is valid.
- A has 3 columns and E has 3 rows so AE is well defined. However AE will be a 3×2 matrix, which cannot be added to a 3×3 matrix.
- B has 3 columns but C has 2 rows, so BC is not defined.
- B has 3 columns and E has 3 rows, so BE is defined.
- A has 3 columns but B has 2 rows, so AB is not defined.
- Clearly B is 3×2 whilst E is 2×3, so their difference is not defined.

Question 3

Given $A = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} -2 & 1 \\ -5 & 3 \end{pmatrix}$, *compute* $3A - B$.

Solution: The answer is $\begin{pmatrix} -1 & -4 \\ 11 & 0 \end{pmatrix}$.

Steps:

$$\begin{aligned} 3A - B &= 3\begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} -2 & 1 \\ -5 & 3 \end{pmatrix} \\ &= \begin{pmatrix} -3 & -3 \\ 6 & 3 \end{pmatrix} - \begin{pmatrix} -2 & 1 \\ -5 & 3 \end{pmatrix} \\ &= \begin{pmatrix} -1 & -4 \\ 11 & 0 \end{pmatrix}. \end{aligned}$$

Question 5

Given $A = \begin{pmatrix} -4 & -5 & -5 \\ 3 & -1 & 0 \\ -3 & 5 & 0 \end{pmatrix}$ *and* $B = \begin{pmatrix} 4 & -1 & 2 \\ -2 & -2 & -4 \\ 0 & -5 & -5 \end{pmatrix}$, *and* $AB = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$, *find the values of* c *and* e.

Solution: The answer is $c = 37, e = -1$.
In general, to find AB_{ij} (i.e. the entry in the matrix AB that's in the ith row and jth column), we multiply the ith row of A by the jth column of B.
To find c, we multiply the first row of A by the last column of B:

$$-4 \times 2 + -5 \times -4 + -5 \times -5 = 37.$$

To find e we multiply the second row of A by the second column of B:

$$3 \times -1 + -1 \times -2 + 0 \times -5 = -1.$$

Question 6

Given $A = \begin{pmatrix} -3 & 2 \\ 2 & 0 \end{pmatrix}$ *and* $B = \begin{pmatrix} -5 & -1 \\ -3 & 0 \end{pmatrix}$, *compute* $A + B^T$.

Solution: The answer is $\begin{pmatrix} -8 & -1 \\ 1 & 0 \end{pmatrix}$.
To find B^T, swap the rows and the columns of B. So,

$$\begin{aligned} A + B^T &= \begin{pmatrix} -3 & 2 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} -5 & -3 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -8 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Question 7

Given $A = \begin{pmatrix} 0 & -1 \\ 3 & 2 \end{pmatrix}$ *and* $B = \begin{pmatrix} 9 \\ 3 \end{pmatrix}$, *solve for* X *in the equation* $AX = B$. *You are given that* $A^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -1 & 0 \end{pmatrix}$.

Solution: The answer is $X = \begin{pmatrix} 7 \\ -9 \end{pmatrix}$.

Multiply both sides of $AX = B$ on the left by A^{-1} to obtain $A^{-1}AX = A^{-1}B$. Now, the inverse of A multiplied by A is just the identity matrix, so $A^{-1}A = I$. Then, we have $IX = X = A^{-1}B$. We now have a formula for finding X. By substituting A^{-1} and B into this,

$$\begin{aligned} X &= A^{-1}B \\ &= \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 9 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 7 \\ -9 \end{pmatrix}. \end{aligned}$$

(Note: if the question asked to solve for X in the equation $XA = B$, we would multiply both sides of this equation on the right by A^{-1} to obtain $X = BA^{-1}$. Remember, BA^{-1} is different to $A^{-1}B$ – in general, you can't swap the order of matrix multiplication)

Question 9

Suppose that $8 \sin x - 11 \cos x = R \sin(x - a)$. *What is the value of* R *and* a?

Solution: The answer is $R = \sqrt{185}$ and $a = \tan^{-1}\left(\frac{11}{8}\right)$.

To find R, note that $R = \sqrt{8^2 + (-11)^2} = \sqrt{185}$. To find $\tan a$, for $R \sin(x - a)$ disregarding any negative signs, take the coefficient of $\cos x$ and divide by the coefficient of $\sin x$ to get $\frac{11}{8}$. We can then take the inverse tan of both sides to get the answer.

Question 11

Let $f(x)$ be the sinusoidal function $-7\sin(9\pi x)$. What is the amplitude and period of $f(x)$?

Solution: The answer is: amplitude $= 7$, period $= \frac{2}{9}$.
The amplitude is simply the coefficient of $\sin(9\pi x)$ disregarding any negative signs. The period is given by $\frac{2\pi}{9\pi} = \frac{2}{9}$.

Question 12

Let the function f be defined by

$$f(x) = \begin{cases} 3x^2 - 2, & \text{for } 0 \leq x < 3 \\ 37 - 4x, & \text{for } 3 \leq x < 5 \\ f(x+5), & \text{for all } x \end{cases}$$

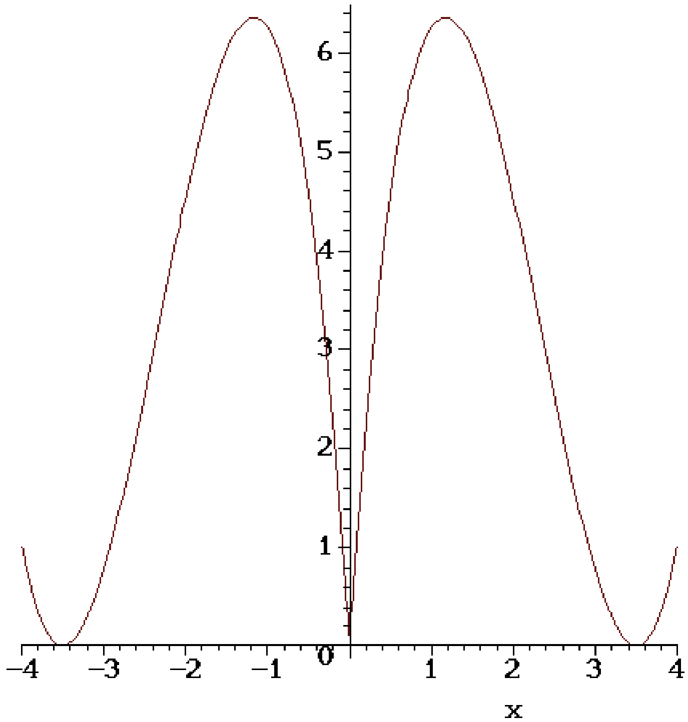
Find $f(0), f(2), f(4), f(32)$ and the period of the function.

Solution: The answers are: period $= 5$, $f(0) = -2, f(2) = 10, f(4) = 21, f(32) = 10$.
To find $f(0)$, we notice that it lies in the first case, as 0 certainly satisfies $0 \leq x < 3$. So we substitute 0 into $3x^2 - 2$ to get -2.
To find $f(2)$, we notice that it also lies in the first case, so substituting 2 into $3x^2 - 2$ gives 10.
To find $f(4)$, we notice that it lies in the second case, so substituting 4 into $37 - 4x$ gives 21.
To find $f(32)$, we use the last case which is $f(x) = f(x+5)$. This means the function repeats every 5 units, so $f(32) = f(27) = f(22) = ... = f(2)$ which now lies in the first case, and equals 10.

Question 13

The function $f(x)$, defined on $[-4, 4]$ except 0, has the following property $f(x) = x(x-3.5)^2$ when $0 < x \leq 4$, and the function is even. What is the graph of the function?

Solution: The answer is:



Since $f(x)$ is an even function defined on $0 < x \leq 4$, we can plot the function on the domain $0 < x \leq 4$, then reflect it along the y axis. To plot the graph, we could either use a table of values or differentiate the function to find the stationary points.

Question 15

Calculate the limit

$$\lim_{x \to -2} \frac{x^2 + 4x + 4}{x + 2}.$$

Solution: The answer is: 0.
Factorise the numerator to get $\dfrac{(x+2)^2}{x+2}$. Now the $(x+2)$ cancels, so we are left with $\lim_{x \to -2} x + 2$, which is 0.

Question 16

Calculate the limit

$$\lim_{x\to\infty} \frac{4x^3 - x + 8}{\sqrt{4x^6 - 6x^3 - 11}}.$$

Solution: The answer is: 2.

When we are taking the limit as x goes to infinity, we look at the largest terms to see what the function "behaves like".

In this case, the function behaves like $\frac{4x^3}{\sqrt{4x^6}}$, or $\frac{4x^3}{2x^3}$, for large x. In this case, we can simply take the leading coefficient of the numerator and divide by the leading coefficient of the denominator to find the limit. Therefore the limit is $\frac{4}{2} = 2$.

Question 18

Calculate the limit

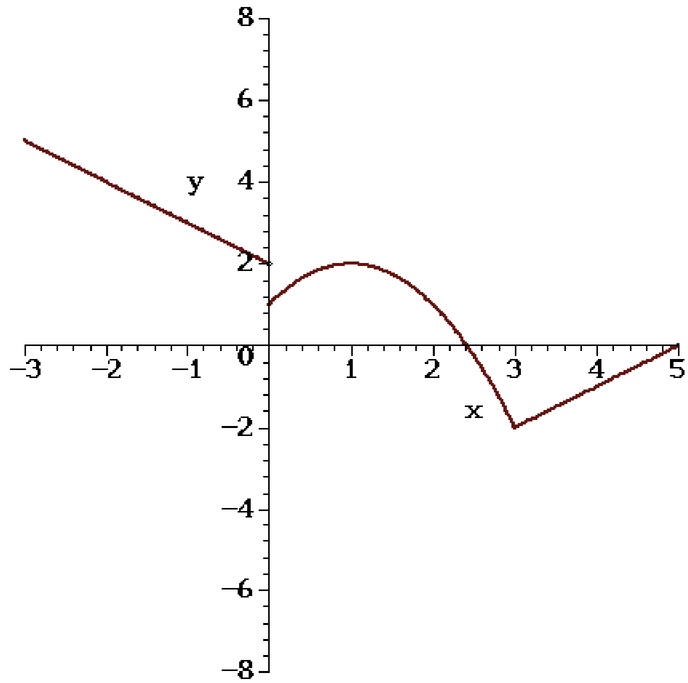
$$\lim_{x\to 4} \frac{x - 4}{x^2 - 7x + 12}.$$

Solution: The answer is: 1.

The denominator can be factorised to $(x-3)(x-4)$. Hence, the fraction can be simplified to $\frac{1}{x-3}$, and substituting 4 into this gives 1.

Question 20

Given the graph of $y = f(x)$ below, state all the values of x where $f(x)$ is not continuous, and all the values where $f(x)$ is not differentiable.



Solution: The answer is: not continuous at $x = 0$, not differentiable at $x = 0, 3$.
There is a jump discontinuity at $x = 0$, so it is not continuous or differentiable here. It is also not differentiable at $x = 3$ since there is a sharp point.

Question 21

Find $\dfrac{dy}{dx}$ *if* $y = \dfrac{2x^7 + 2}{x}$.

Solution: The answer is: $12x^5 - \dfrac{2}{x^2}$.
We can split up the fraction and differentiate:

$$\begin{aligned} y &= 2x^6 + \frac{2}{x} \\ \frac{dy}{dx} &= 12x^5 - \frac{2}{x^2}. \end{aligned}$$

Alternatively, we could use the quotient rule which gives:

$$\begin{aligned}\frac{dy}{dx} &= \frac{14x^6(x) - (2x^7 + 2)}{x^2} \\ &= \frac{14x^7 - 2x^7 - 2}{x^2} \\ &= \frac{12x^7 - 2}{x^2} \\ &= 12x^5 - \frac{2}{x^2}.\end{aligned}$$

Question 22

Find the exact value of the gradient of $y = x^3 - 2x^2 - 6 + \dfrac{2}{x^2}$ *at* $x = -1$.

Solution: The answer is: 11.
Differentiate y to get $y' = 3x^2 - 4x - 4x^{-3}$ then substitute in $x = -1$ to find that the gradient is 11.

Question 24

Find $\dfrac{dy}{dx}$ *if* $y = (3x - 1)(\sin x)$.

Solution: The answer is: $3\sin x + (3x - 1)\cos x$
The product rule gives:

$$y' = 3\sin x + (3x - 1)\cos x.$$

Question 25

Find $\dfrac{dy}{dx}$ *if* $y = (\ln x)^4$.

Solution: The answer is: $4(\ln x)^3(\frac{1}{x})$.
We can find the answer by using the chain rule, and the fact that the derivative of $\ln x$ is $\dfrac{1}{x}$.

Question 26

Find $\frac{dy}{dx}$ *if* $y = \sin(4x^3 + 1)$

Solution: The answer is: $12x^2 \cos(4x^3 + 1)$.
We apply chain rule, by changing the sin to cos and then multiplying by the derivative of $4x^3 + 1$ to get our answer.