UNSW Mathematics Society Presents

MATH2011/2111 Seminar

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Overview I

1. Curves and Surfaces

Curves

Definition

The parameterisation of a curve in $\mathbb{R}^n$ is a vector valued function

$$\mathbf{c} : I \to \mathbb{R}^n$$

where I is an interval on $\mathbb{R}$.

- A multiple point is a point through which the curve passes more than once.
- A curve is closed if $\mathbf{c}(a) = \mathbf{c}(b)$.

Limits and Calculus for Curves

Definition

Let $\mathbf{c} : I \to \mathbb{R}^n$ be a curve with components $c_i, i = 1, 2, \ldots, n$ and $a \in I$.

- If $\lim_{t \to a} c_i(t)$ exists for all i, then $\lim_{t \to a} \mathbf{c}(t)$ exists and

$$\lim_{t \to a} \mathbf{c}(t) = \left(\lim_{t \to a} c_1(t), \ldots, \lim_{t \to a} c_n(t) \right)$$

- If $c_i'(t)$ exists for all i, then

$$\mathbf{c}'(t) = (c_1'(t), \ldots, c_n'(t))$$

$\mathbf{c}'(t)$ can be interpreted as the tangent vector or the velocity at t and $\mathbf{c}''(t)$ is the acceleration.

Surfaces in $\mathbb{R}^3$

- **Graph**: $z = f(x, y)$.
- **Implicitly**: $F(x, y, z) = 0$.
- **Parametrically**: $\mathbf{x}(u, v)$

2. Point Set Topology

Ball

Definition

A ball around $\mathbf{a} \in \mathbb{R}^n$ of radius $\epsilon > 0$ is the set

$$B(\mathbf{a}, \epsilon) = \{\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x} - \mathbf{a}|| < \epsilon\}$$

The punctured ball around $\mathbf{a}$ is the set

$$B^o(\mathbf{a}, \epsilon) = \{\mathbf{x} \in \mathbb{R}^n : 0 < ||\mathbf{x} - \mathbf{a}|| < \epsilon\}$$

Limit of sequences

Definition

For a sequence $\{\mathbf{x}_i\}_{i=1}^{\infty}$ of points in $\mathbb{R}^n$, $\mathbf{x}$ is the limit of the sequence if and only if

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall k \geq N : \mathbf{x}_k \in B(\mathbf{x}, \varepsilon)$$

- A sequence $\{\mathbf{x}_k\}$ converges if and only if the components of $\mathbf{x}_k$ converge to the components of $\mathbf{x}$.
- To show a point is not a limit point, show

$$\exists \varepsilon > 0 \text{ such that } \forall N \in \mathbb{N}\ \exists k \geq N \text{ such that } \mathbf{x}_\mathbf{k} \notin B(\mathbf{x}, \varepsilon)$$

Open and closed sets

Definition

Let $\Omega \subseteq \mathbb{R}^n$.

- $\mathbf{x} \in \Omega$ is an interior point if there exists an $\varepsilon > 0$ such that $B(\mathbf{x}, \varepsilon) \subseteq \Omega$.
- Ω is open if every point is an interior point.
- Ω is closed if its complement is open.
- $\mathbf{x} \in \Omega$ is a boundary point if for every $\varepsilon > 0$, there exists $\mathbf{x}_1, \mathbf{x}_2 \in B(\mathbf{x}, \varepsilon)$ such that $\mathbf{x}_1 \in \Omega$ and $\mathbf{x}_2 \notin \Omega$.

A set is closed if and only if it contains all its boundary points.

Theorem

A finite union/intersection of open sets is open.
A finite union/intersection of closed sets is closed.

Past exam question

2019 Q2(ii)

Use the definition to show that the set $(0, 1] \subseteq \mathbb{R}$ is not open.

Past exam question

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We need to show there exists an $x \in (0, 1]$ such that for all $\varepsilon > 0$, there exists a point $y \in B(x, \varepsilon) : y \notin (0, 1]$.

Past exam question

2019 Q2(ii)

Use the definition to show that the set $(0,1] \subseteq \mathbb{R}$ is not open.

We need to show there exists an $x \in (0,1]$ such that for all $\varepsilon > 0$, there exists a point $y \in B(x,\varepsilon) : y \notin (0,1]$.
Consider $x = 1$. Choose

$$y = 1 + \frac{\varepsilon}{2} \notin (0,1].$$

$y \in B(1,\varepsilon)$ since $\mathrm{d}(y,1) = \frac{\varepsilon}{2} < \varepsilon$. Hence $(0,1]$ is not open.

Interior and boundary

Definition

Suppose $\Omega \subseteq \mathbb{R}^n$.

- The interior of Ω is the set of all interior points of Ω.
- The boundary of Ω is the set of all boundary points of Ω, denoted $\partial\Omega$.
- The closure of Ω is $\overline{\Omega} = \Omega \cup \partial\Omega$.

Lemma

Let $\mathbf{x} \in \mathbb{R}^n$ and $\Omega \subseteq \mathbb{R}^n$.

- $\mathbf{x}$ is an interior point of $\Omega \implies \mathbf{x}$ is a limit point of Ω.
- $\mathbf{x}$ is a limit point of $\Omega \implies \mathbf{x}$ is a boundary point or an interior point of Ω.
- $\mathbf{x}$ is in $\overline{\Omega} \implies$ there is a sequence in Ω with limit $\mathbf{x}$.

Limit of a function at a point

Definition

Let $\mathbf{b} \in \mathbb{R}^m, \Omega \subseteq \mathbb{R}^n, \mathbf{a} \in \overline{\Omega}$ and let $\mathbf{f} : \Omega \to \mathbb{R}^m$ be a function. We say that $\mathbf{f}(\mathbf{x})$ converges to $\mathbf{b}$ as $\mathbf{x} \to \mathbf{a}$ if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } \forall \mathbf{x} \in \Omega : x \in B^o(\mathbf{a}, \delta) \cap \Omega \implies \mathbf{f}(\mathbf{x}) \in B(\mathbf{b}, \varepsilon).$$

If such $\mathbf{b}$ exists, then $\mathbf{b}$ is unique and we write

$$\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}.$$

Limit theorems

Theorems

Let $\mathbf{b} \in \mathbb{R}^m, \Omega \subseteq \mathbb{R}^n, \mathbf{a} \in \overline{\Omega}$ and let $\mathbf{f} : \Omega \to \mathbb{R}^m$ be a function. Then

$$\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b} \iff \lim_{\mathbf{x} \to \mathbf{a}} f_i(\mathbf{x}) = b_i \text{ for all } i = 1, \dots, m$$

$$\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b} \iff \lim_{k \to \infty} \mathbf{f}(\mathbf{x}_k) = \mathbf{b}$$

for every sequence $\{\mathbf{x}_k\}_{k=1}^{\infty} \subseteq \Omega$ with $\lim_{k \to \infty} \mathbf{x}_k = \mathbf{a}$.

Algebra of limits is similar to the single-variable case.

Past exam question

2018 Final Q1(iii)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{x^2+xy+y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$$

Prove that the function f is not continuous at $(0,0)$.

Past exam question

2018 Final Q1(iii)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{x^2+xy+y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$$

Prove that the function f is not continuous at $(0,0)$.

We need to show that there exist a sequence $\{\mathbf{x}_k\}_{k=1}^{\infty} \subseteq \mathbb{R}^2$ such that $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{0}$ but $\lim_{k\to\infty} f(\mathbf{x}_k)) \neq 1$.

Past exam question

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Consider the sequence $\mathbf{x}_k = \left(\frac{1}{k}, \frac{1}{k}\right)$
$\lim_{k\to\infty} \mathbf{x}_k = \mathbf{0}$ but

$$\lim_{k\to\infty} f(\mathbf{x}_k)) = \lim_{k\to\infty} \frac{\frac{1}{k^2} + \frac{1}{k}\cdot\frac{1}{k} + \frac{1}{k^2}}{\frac{1}{k^2} + \frac{1}{k^2}} = \lim_{k\to\infty} \frac{3}{2} = \frac{3}{2} \neq 1$$

Pinching theorem

Theorem

Let $\Omega \subseteq \mathbb{R}^n$, let $\mathbf{a}$ be a limit point of Ω and let $f, g, h : \Omega \to \mathbb{R}$ be functions such that there exists $\delta > 0$ such that $\forall \mathbf{x} \in B(\mathbf{a}, \delta) \cap \Omega$,

$$g(\mathbf{x}) \leq f(\mathbf{x}) \leq h(\mathbf{x}).$$

Then

$$\lim_{\mathbf{x} \to \mathbf{a}} g(\mathbf{x}) = \mathbf{b} = \lim_{\mathbf{x} \to \mathbf{a}} h(\mathbf{x}) \implies \lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = \mathbf{b}$$

Past exam question

2016 Final 1(iv)(c) modified

Given that

$$\left|\frac{xy}{x^2 - xy + y^2}\right| \leq 1, \ \forall (x,y) \neq (0,0)$$

show that

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 - xy + y^2} = 0$$

Past exam question

2016 Final 1(iv)(c) modified

Given that

$$\left|\frac{xy}{x^2 - xy + y^2}\right| \leq 1, \ \forall (x, y) \neq (0, 0)$$

show that

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 - xy + y^2} = 0$$

Since

$$0 \leq \left|\frac{x^2 y}{x^2 - xy + y^2}\right| = |x| \cdot \left|\frac{xy}{x^2 - xy + y^2}\right| \leq |x| \ \forall (x, y) \neq (0, 0)$$

Past exam question

2016 Final 1(iv)(c) modified

Given that

$$\left|\frac{xy}{x^2 - xy + y^2}\right| \leq 1, \ \forall (x, y) \neq (0, 0)$$

show that

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Since

$$0 \leq \left|\frac{x^2 y}{x^2 - xy + y^2}\right| = |x| \cdot \left|\frac{xy}{x^2 - xy + y^2}\right| \leq |x| \ \forall (x, y) \neq (0, 0)$$

and

$$\lim_{(x,y)\to(0,0)} 0 = \lim_{(x,y)\to(0,0)} |x| = 0$$

the result follows by the pinching theorem.

Continuity

Definition

Let $\mathbf{a} \in \Omega \subseteq \mathbb{R}^n$. A function $\mathbf{f} : \Omega \to \mathbb{R}^m$ is continuous at $\mathbf{a}$ if

$$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$$

$\mathbf{f}$ is said to be continuous on Ω if it is continuous at $\mathbf{a}$ for every $\mathbf{a} \in \Omega$.

The sum, product, quotient and composition of functions are continuous as in the one-variable case.

Preimage

Definition

Suppose that $\Omega \subseteq \mathbb{R}^n$ and $\mathbf{f} : \Omega \to \mathbb{R}^m$ is a function. The preimage of a set $U \subseteq \mathbb{R}^m$ is defined by

$$\mathbf{f}^{-1}(U) = \{\mathbf{y} \in \mathbb{R}^n : f(\mathbf{y}) \in U\}.$$

Theorem

$\mathbf{f}$ is continuous on Ω if and only if

$$U \text{ is open in } \mathbb{R}^m \implies \mathbf{f}^{-1}(U) \text{ is open in } \mathbb{R}^n.$$

Theorem

Suppose Ω is open. If $\mathbf{f}(\mathbf{a})$ is an interior point of $\mathbf{f}(\Omega)$ and $B(\mathbf{f}(\mathbf{a}), \varepsilon) \subseteq \mathbf{f}(\Omega)$ then $\mathbf{a}$ is an interior point of $\mathbf{f}^{-1}(B(\mathbf{f}(\mathbf{a}), \varepsilon))$.

Compact sets

Definition

A set $\Omega \subseteq \mathbb{R}^n$ is bounded if there exists an $M \in \mathbb{R}$ such that $d(\mathbf{x}, \mathbf{0}) \leq M$ for all $\mathbf{x} \in \Omega$.

Definition

A set $\Omega \subseteq \mathbb{R}^n$ is compact if it is closed and bounded.

Theorem

Let $\Omega \subseteq \mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}$ be continuous. Then

$$K \subseteq \Omega \text{ and } K \text{ compact} \implies f(K) \text{ compact}.$$

Path connected sets

Definition

A set $\Omega \subseteq \mathbb{R}^n$ is said to be path connected if for any $\mathbf{x}, \mathbf{y} \in \Omega$, there is a continuous function φ such that $\varphi(t) \in \Omega$ for all $t \in [0,1]$ and $\varphi(0) = \mathbf{x}$ and $\varphi(1) = \mathbf{y}$.

Theorem

Let $\Omega \subseteq \mathbb{R}^n$ and $\mathbf{f} : \Omega \to \mathbb{R}^m$ be continuous. Then

$$B \subseteq \Omega \text{ and } B \text{ path connected} \implies \mathbf{f}(B) \text{ path connected}$$

Suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then the function $\varphi : [0,1] \to \mathbb{R}^n$

$$\varphi(t) = (1-t)\mathbf{x} + t\mathbf{y}$$

is the line segment from $\mathbf{x}$ to $\mathbf{y}$.

Example

Example

Prove from the definition that the set

$$S = \{(x, y) \in \mathbb{R}^2 : y + x^2 \geq 0\}$$

is path-connected.

Example

Example

Prove from the definition that the set

$$S = \{(x, y) \in \mathbb{R}^2 : y + x^2 \geq 0\}$$

is path-connected.

Suppose $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^2$. Define

$$\varphi(t) = \begin{cases} (x_1, 3y_1 t) & 0 \leq t \leq \frac{1}{3} \\ (-3x_1(t - \frac{2}{3}) + 3x_2(t - \frac{1}{3}), 0) & \frac{1}{3} \leq t \leq \frac{2}{3} \\ (x_2, 3y_2(t - \frac{2}{3}) & \frac{2}{3} \leq t \leq 1 \end{cases}.$$

Injective map

Definition

A function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is injective if

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{y}) \implies \mathbf{x} = \mathbf{y}.$$

Theorem

If a continuous function $\mathbf{f}$ is injective on Ω, then

$$\partial\left(\mathbf{f}(\Omega)\right) = \mathbf{f}(\partial\Omega)$$

3. Differentiable Functions

Partial derivatives

Definition

Let $\mathbf{a} \in \Omega \subseteq \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ be a function with coordinates x_i and standard basis vectors $\mathbf{e}_i, i \in \{1, \ldots, n\}$. The partial derivative of f in direction i is defined as

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(\mathbf{a} + h\mathbf{e}_i) - f(\mathbf{a})}{h}$$

assuming the limit exists.

Practically, this can be computed by taking derivative with respect to a coordinate assuming the others are constant.

Jacobian matrix

Definition

If all partial derivatives of $\mathbf{f} : \Omega \to \mathbb{R}^m$ exist at $\mathbf{a} \in \Omega \subseteq \mathbb{R}^n$, then the Jacobian matrix of $\mathbf{f}$ at $\mathbf{a}$ is

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \frac{\partial f_1}{\partial x_2}(\mathbf{a}) & \dots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{a}) & \frac{\partial f_2}{\partial x_2}(\mathbf{a}) & \dots & \frac{\partial f_2}{\partial x_n}(\mathbf{a}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{a}) & \frac{\partial f_m}{\partial x_2}(\mathbf{a}) & \dots & \frac{\partial f_m}{\partial x_n}(\mathbf{a}) \end{pmatrix}$$

Theorem

Let $\Omega \subseteq \mathbb{R}^n, \mathbf{a} \in \Omega$ be an interior point and $\mathbf{f} : \Omega \to \mathbb{R}^m$ be a function. If $\mathbf{f}$ is differentiable at $\mathbf{a}$ then all partial derivatives $\frac{\partial f_j}{\partial x_i}$ exist at $\mathbf{a}$ and

$$D\mathbf{f}(\mathbf{a}) = J\mathbf{f}(\mathbf{a})$$

Affine maps

Definition

A function $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^m$ is called affine if there exists a $\mathbf{y}_0 \in \mathbb{R}^m$ and a linear map $\mathbf{L} : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$T(\mathbf{x}) = \mathbf{y_0} + \mathbf{L}(\mathbf{x})$$

Definition

A function $\mathbf{f} : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ has an affine approximation at a point $\mathbf{a} \in \Omega$ if there is a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $\mathbf{b} \in \mathbb{R}^m$ such that

$$\lim_{\mathbf{x} \to \mathbf{a}} \frac{\|\mathbf{f}(\mathbf{x}) - A\mathbf{x} - \mathbf{b}\|}{\|\mathbf{x} - \mathbf{a}\|} = 0$$

The affine approximation is

$$\mathbf{g}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$$

Differentiability

Definition

A function $\mathbf{f}$ is said to be differentiable at $\mathbf{a}$ if $\mathbf{f}$ has an affine approximation at $\mathbf{a}$.

Theorem

A function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\mathbf{a} \in \mathbb{R}^n$ if and only if

$$\lim_{\mathbf{x} \to \mathbf{a}} \frac{\|\mathbf{f}(\mathbf{x}) - J\mathbf{f}(\mathbf{a})(\mathbf{x} - \mathbf{a}) - \mathbf{f}(\mathbf{a})\|}{\|\mathbf{x} - \mathbf{a}\|} = 0.$$

Past exam question

2012 Final Q1(b)

Find the best affine approximation to the function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$\mathbf{f}(x, y) = (x^2 + y^2, \sinh(x^2 - y^2))$$

at the point $(1, 1)$.

Past exam question

2012 Final Q1(b)

Find the best affine approximation to the function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$\mathbf{f}(x, y) = (x^2 + y^2, \sinh(x^2 - y^2))$$

at the point $(1, 1)$.

The best affine approximation at $\mathbf{a} = (1, 1)$ is given by

$$\mathbf{g}(\mathbf{x}) = \mathbf{f}(\mathbf{a}) + J\mathbf{f}(\mathbf{a})(\mathbf{x} - \mathbf{a}).$$

2012 Final Q1(b) Solution

$$\mathbf{f}(x,y) = (x^2+y^2, \sinh(x^2-y^2))$$

The Jacobian matrix is given by

$$\begin{pmatrix} 2x & 2y \\ 2x\cosh(x^2-y^2) & -2y\cosh(x^2-y^2) \end{pmatrix}$$

$$\begin{aligned} \mathbf{g}(\mathbf{x}) &= (2,0) + \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix} (\mathbf{x} - (1,1)) \\ &= (2,0) + (2(x-1)+2(y-1), 2(x-1)-2(y-1)) \\ &= (2x+2y-2, 2x-2y) \end{aligned}$$

Theorems

Theorem

Suppose $\Omega \subseteq \mathbb{R}^n$ is open and $\mathbf{f} : \Omega \to \mathbb{R}^m$. If all partial derivatives exist and are continuous on Ω, then $\mathbf{f}$ is differentiable on Ω.

Theorem

Suppose $\Omega \subseteq \mathbb{R}^n$ is open and $\mathbf{f} : \Omega \to \mathbb{R}^m$ is differentiable on Ω. Then $\mathbf{f}$ is continuous on Ω.

Clariaut's theorem

Theorem

If $f, \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j}, \frac{\partial^2 f}{\partial x_i \partial x_j}, \frac{\partial^2 f}{\partial x_j \partial x_i}$ all exist and are continuous on an open set around $\mathbf{a}$, then

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{a})$$

Past exam question

2016 Final Q2(ii)

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(a) Compute the partial derivatives

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}.$$

(b) Compute the second partial derivatives values at $(0,0)$.

$$\frac{\partial^2 f}{\partial x \partial y} \text{ and } \frac{\partial^2 f}{\partial y \partial x}.$$

(c) Give the statement of Clariaut's Theorem.
(d) Why did Clariaut's Theorem not work for the function f above?

2016 Final Q2(ii) Solution

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

2016 Final Q2(ii) Solution

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(a) From the definition,

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{h\cdot 0(h^2-0)}{h^2+0^2} - 0}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0 \end{aligned}$$

2016 Final Q2(ii) Solution

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(a) From the definition,

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{h\cdot 0(h^2-0)}{h^2+0^2} - 0}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0 \end{aligned}$$

Similarly,

$$\frac{\partial f}{\partial y}(0,0) = 0$$

2016 Final Q2(ii) Solution

$$f(x,y) = \begin{cases} \frac{x^3y - xy^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

For $(x,y) \neq (0,0)$,

2016 Final Q2(ii) Solution

$$f(x,y) = \begin{cases} \frac{x^3y - xy^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

For $(x,y) \neq (0,0)$,

$$\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= \frac{(3x^2y - y^3)(x^2+y^2) - 2x(x^3y - xy^3)}{(x^2+y^2)^2} \\ &= \frac{3x^4y + 3x^2y^3 - x^2y^3 - y^5 - 2x^4y + 2x^2y^3}{(x^2+y^2)^2} \\ &= \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2} \end{aligned}$$

2016 Final Q2(ii) Solution

$$f(x,y) = \begin{cases} \frac{x^3y - xy^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

For $(x,y) \neq (0,0)$,

$$\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= \frac{(3x^2y - y^3)(x^2+y^2) - 2x(x^3y - xy^3)}{(x^2+y^2)^2} \\ &= \frac{3x^4y + 3x^2y^3 - x^2y^3 - y^5 - 2x^4y + 2x^2y^3}{(x^2+y^2)^2} \\ &= \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2} \end{aligned}$$

Similarly,

$$\frac{\partial f}{\partial y}(x,y) = -\frac{xy^4 + 4x^3y^2 - x^5}{(x^2+y^2)^2}$$

2016 Final Q2(ii) Solution

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

2016 Final Q2(ii) Solution

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$$

(b)

$$\begin{aligned} \frac{\partial^2 f}{\partial y \partial x}(0,0) &= \lim_{h\to 0} \frac{f_x(0,h)-f_x(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{0-h^5}{(0^2+h^2)^2}-0}{h} \\ &= \lim_{h\to 0} -1 \\ &= -1 \end{aligned}$$

2016 Final Q2(ii) Solution

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(b)

$$\begin{aligned} \frac{\partial^2 f}{\partial y \partial x}(0,0) &= \lim_{h\to 0} \frac{f_x(0,h) - f_x(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{0-h^5}{(0^2+h^2)^2} - 0}{h} \\ &= \lim_{h\to 0} -1 \\ &= -1 \end{aligned}$$

Similarly,

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = 1$$

2016 Final Q2(ii) Solution

(c) If the first and second order partial derivatives all exist and are continuous around $\mathbf{a}$ then

$$\frac{\partial^2 f}{\partial x \partial y}(\mathbf{a}) = \frac{\partial^2 f}{\partial y \partial x}(\mathbf{a})$$

2016 Final Q2(ii) Solution

(c) If the first and second order partial derivatives all exist and are continuous around **a** then

$$\frac{\partial^2 f}{\partial x \partial y}(\mathbf{a}) = \frac{\partial^2 f}{\partial y \partial x}(\mathbf{a})$$

(d) Second order partial derivatives are not continuous at $(0, 0)$, so the two values in (b) are not equal.

Chain rule

Theorem

Let $\Omega \subseteq \mathbb{R}^n, \Omega' \subseteq \mathbb{R}^m$ and let $\mathbf{a} \in \Omega$. Suppose $\mathbf{f} : \Omega \to \mathbb{R}^m$ and $\mathbf{g} : \Omega' \to \mathbb{R}^k$ are functions such that $\mathbf{f}(\Omega) \subseteq \Omega'$. If $\mathbf{f}$ is differentiable at $\mathbf{a}$ and $\mathbf{g}$ is differentiable at $\mathbf{f}(\mathbf{a})$, then $\mathbf{g} \circ \mathbf{f} : \Omega \to \mathbb{R}^k$ is differentiable at $\mathbf{a}$ and

$$D(\mathbf{g} \circ \mathbf{f})(\mathbf{a}) = D\mathbf{g}(\mathbf{f}(\mathbf{a}))D\mathbf{f}(\mathbf{a}).$$

Past exam question

2019 Final Q1(ii)

Use the chain rule to find $\frac{\partial z}{\partial r}$, $\frac{\partial z}{\partial s}$ and $\frac{\partial^2 z}{\partial s \partial r}$ in terms of $\frac{\partial f}{\partial u}$, $\frac{\partial f}{\partial v}$, $\frac{\partial^2 f}{\partial u^2}$, $\frac{\partial^2 f}{\partial v^2}$ and $\frac{\partial^2 f}{\partial u \partial v}$, if $z = f(u, v)$ and if $u = 2r - s$ and $v = r + s^2$.

Past exam question

2019 Final Q1(ii)

Use the chain rule to find $\frac{\partial z}{\partial r}$, $\frac{\partial z}{\partial s}$ and $\frac{\partial^2 z}{\partial s \partial r}$ in terms of $\frac{\partial f}{\partial u}$, $\frac{\partial f}{\partial v}$, $\frac{\partial^2 f}{\partial u^2}$, $\frac{\partial^2 f}{\partial v^2}$ and $\frac{\partial^2 f}{\partial u \partial v}$, if $z = f(u, v)$ and if $u = 2r - s$ and $v = r + s^2$.

$$\frac{\partial z}{\partial r} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial r} = 2\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$$

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial s} = -\frac{\partial f}{\partial u} + 2s\frac{\partial f}{\partial v}$$

2019 Final Q1(ii) Solution

$$\begin{aligned}
\frac{\partial^2 z}{\partial s \partial r} &= \frac{\partial}{\partial s}\frac{\partial z}{\partial r} \\
&= \frac{\partial}{\partial s}\left(2\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\right) \\
&= 2\frac{\partial}{\partial s}\frac{\partial f}{\partial u} + \frac{\partial}{\partial s}\frac{\partial f}{\partial v} \\
&= \left(2\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial s} + 2\frac{\partial^2 f}{\partial v \partial u}\frac{\partial v}{\partial s}\right) + \left(\frac{\partial^2 f}{\partial u \partial v}\frac{\partial u}{\partial s} + \frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial s}\right) \\
&= -2\frac{\partial^2 f}{\partial u^2} + (4s-1)\frac{\partial^2 f}{\partial u \partial v} + 2s\frac{\partial^2 f}{\partial v^2}
\end{aligned}$$

Gradient

Definition

For $\Omega \subseteq \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$, the Jacobian is referred to as the gradient of f.

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right).$$

$\nabla f(\mathbf{a})$ is the direction of steepest change at $\mathbf{a} \in \Omega$ and is orthogonal to the tangent plane at that point. The tangent plane can be given by the equation

$$\nabla f \cdot (\mathbf{x} - \mathbf{a}) = 0.$$

Directional derivative

Definition

Let $\Omega \subseteq \mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}$ be a function. Let $\mathbf{a}$ be an interior point of Ω and $\mathbf{u}$ be a unit vector in $\mathbb{R}^n$. The directional derivative of f in the direction $\mathbf{u}$ at $\mathbf{a}$ is defined by

$$D_{\mathbf{u}} f(\mathbf{a}) = \frac{\partial f}{\partial \mathbf{u}} = \lim_{h \to 0} \frac{f(\mathbf{a} + h\mathbf{u}) - f(\mathbf{a})}{h}$$

if the limit exists.

Theorem

The directional derivative of f in the direction $\mathbf{u}$ at $\mathbf{a}$ is given by

$$\frac{\partial f}{\partial \mathbf{u}} = \nabla f(\mathbf{a}) \cdot \mathbf{u}.$$

Hessian matrix

Definition

For $\Omega \subseteq \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$, the Hessian of f at $\mathbf{a}$ is the $n \times n$ matrix

$$Hf(\mathbf{a}) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(\mathbf{a}) & \frac{\partial^2 f}{\partial x_2 \partial x_1}(\mathbf{a}) & \dots & \frac{\partial^2 f}{\partial x_n \partial x_1}(\mathbf{a}) \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(\mathbf{a}) & \frac{\partial^2 f}{\partial x_2^2}(\mathbf{a}) & \dots & \frac{\partial^2 f}{\partial x_n \partial x_2}(\mathbf{a}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(\mathbf{a}) & \frac{\partial^2 f}{\partial x_2 \partial x_n}(\mathbf{a}) & \dots & \frac{\partial^2 f}{\partial x_n^2}(\mathbf{a}) \end{pmatrix}$$

Taylor polynomials

Definition

Let $\Omega \subseteq \mathbb{R}^n$ be open and $f : \Omega \to \mathbb{R}$ be a function such that all partial derivatives of at most 2 exist and are continuous. Let $\mathbf{a}, \mathbf{x} \in \Omega$ be such that the line segment joining $\mathbf{a}, \mathbf{a}$ lies in Ω. The polynomial

$$P_{1,\mathbf{a}}(\mathbf{x}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})$$

is the Taylor polynomial of order 1 about $\mathbf{a}$, and

$$P_{2,\mathbf{a}}(\mathbf{x}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + \frac{1}{2}((\mathbf{x} - \mathbf{a}) \cdot Hf(\mathbf{a})(\mathbf{x} - \mathbf{a})$$

is the Taylor polynomial of order 2 about $\mathbf{a}$.

Taylor's theorem for 1st order

Theorem

There exists some point $\mathbf{z}$ on the line segment joining $\mathbf{x}$ and $\mathbf{a}$ such that

$$f(\mathbf{x}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + R_{1,\mathbf{a}}(\mathbf{x}),$$

where

$$R_{1,\mathbf{a}}(\mathbf{x}) = \frac{1}{2}((\mathbf{x} - \mathbf{a}) \cdot Hf(\mathbf{z})(\mathbf{x} - \mathbf{a})).$$

Past exam question

2013 Final Q1(ii)

Let $f(x, y) = e^{x^2 \sin y}$. Find the taylor polynomial of degree 2 for f around the point $(1, 0)$.

Past exam question

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Let $f(x,y) = e^{x^2 \sin y}$. Find the taylor polynomial of degree 2 for f around the point $(1,0)$.

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The gradient and Hessian of f are given by

$$\nabla f(x,y) = e^{x^2 \sin y}(2x \sin y, x^2 \cos y)^T$$

Past exam question

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The gradient and Hessian of f are given by

$$\nabla f(x,y) = e^{x^2 \sin y}(2x \sin y, x^2 \cos y)^T$$

$$Hf(x,y) = e^{x^2 \sin y} \begin{pmatrix} 2\sin y + 4x^2 \sin y & 2x\cos y + 2x^3 \sin y \cos y \\ 2x \cos y + 2x^3 \sin y \cos y & x^4 \sin y \cos y - x^2 \sin y \end{pmatrix}$$

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$$f(1, 0) = 1, \quad \nabla f(1, 0) = (0, e)^T, \quad Hf(1, 0) = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

2013 Final Q1(ii) Solution

$$P_{2,\mathbf{a}}(\mathbf{x}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + \frac{1}{2}((\mathbf{x} - \mathbf{a}) \cdot Hf(\mathbf{a})(\mathbf{x} - \mathbf{a})$$

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$$P_{2,\mathbf{a}}(\mathbf{x}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + \frac{1}{2}((\mathbf{x} - \mathbf{a}) \cdot Hf(\mathbf{a})(\mathbf{x} - \mathbf{a})$$

$$f(1,0) = 1, \quad \nabla f(1,0) = (0,e)^T, \quad Hf(1,0) = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

$$P_2(x,y) = 1 + \begin{pmatrix} 0 \\ e \end{pmatrix} \cdot \begin{pmatrix} x-1 \\ y \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x-1 \\ y \end{pmatrix} \cdot \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x-1 \\ y \end{pmatrix}$$

$$\begin{aligned} P_2(x,y) &= 1 + ey + y(x-1) + y(x-1) \\ &= 2xy + (e-2)y + 1 \end{aligned}$$

Maxima, minima and saddle points

Definition

Let $\mathbf{a} \in \Omega \subseteq \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ be a function. Then $\mathbf{a}$ is a

- stationary point if $\nabla f(\mathbf{a}) = \mathbf{0}$.
- global maximum if $f(\mathbf{a}) \geq f(\mathbf{x})$ for all $\mathbf{x} \in \Omega$.
- global minimum if $f(\mathbf{a}) \leq f(\mathbf{x})$ for all $\mathbf{x} \in \Omega$.
- local maximum if there is an open set around $\mathbf{a}$ such that $f(\mathbf{a}) \geq f(\mathbf{x})$ for all $\mathbf{x} \in \Omega \cap A$.
- local minimum if there is an open set around $\mathbf{a}$ such that $f(\mathbf{a}) \leq f(\mathbf{x})$ for all $\mathbf{x} \in \Omega \cap A$.
- saddle point if it is a stationary point but neither a maximum nor a minimum.

Critical points

Definition

A point $\mathbf{a} \in \Omega \subseteq \mathbb{R}^n$ is a critical point of a function $f : \Omega \to \mathbb{R}$ if $\mathbf{a}$ satisfies one of the following:

- $\mathbf{a}$ is a stationary point.
- $\mathbf{a}$ lies on the boundary of Ω.
- f is not differentiable at $\mathbf{a}$.

Local minima and maxima occur at critical points.

Positive and negative definite

Definition

An $n \times n$ matrix is

- positive definite if all eigenvalues are positive
- positive semidefinite if all eigenvalues are nonnegative
- negative definite if all eigenvalues are negative
- negative semidefinite if all eigenvalues are nonpositive.

Classification of stationary points

Theorem

Let $\Omega \subseteq \mathbb{R}^n$ be open, $\mathbf{a} \in \Omega$ and let $f : \Omega \to \mathbb{R}$ be a function such that all partial derivatives of order at most 2 exist on Ω and $\nabla f(\mathbf{a}) = \mathbf{0}$. Then

- If $Hf(\mathbf{a})$ is positive definite, f has a local minimum at $\mathbf{a}$.
- If $Hf(\mathbf{a})$ is negative definite, f has a local maximum at $\mathbf{a}$.
- If f has a local minimum at $\mathbf{a}$, then $Hf(\mathbf{a})$ is positive semidefinite
- If f has a local maximum at $\mathbf{a}$, then $Hf(\mathbf{a})$ is negative semidefinite.

Sylvester's criterion

Theorem

If Δ_k is the determinant of the upper left $k \times k$ submatrix of H, then H is

- positive definite if and only if $\Delta_k > 0$ for all k.
- positive semidefinite if and only if $\Delta_k \geq 0$ for all k.
- negative definite if and only if $\Delta_k < 0$ for all odd k and $\Delta_k > 0$ for all even k.
- negative semidefinite if and only if $\Delta_k \leq 0$ for all odd k and $\Delta_k \geq 0$ for all even k.

Past exam question

2019 Final Q1(v) modified

Let f be the function

$$f(x, y) = x^3 + 3x^2 + 24xy + 12y^2 + 15x - 2.$$

Find the critical points of f and find out what type they are.

Past exam question

2019 Final Q1(v) modified

Let f be the function

$$f(x,y) = x^3 + 3x^2 + 24xy + 12y^2 + 15x - 2.$$

Find the critical points of f and find out what type they are.

The gradient of f are given by

$$\nabla f(x,y) = (3x^2 + 6x + 24y + 15, 24x + 24y)^T$$

Past exam question

2019 Final Q1(v) modified

Let f be the function

$$f(x,y) = x^3 + 3x^2 + 24xy + 12y^2 + 15x - 2.$$

Find the critical points of f and find out what type they are.

The gradient of f are given by

$$\nabla f(x,y) = (3x^2 + 6x + 24y + 15, 24x + 24y)^T$$

Setting $\nabla f(x,y) = \mathbf{0}$ and solving gives the critical points $(1,-1)$ and $(5,-5)$.

2019 Final Q1(v) solution

The Hessian of f is given by

$$Hf(x,y) = \begin{pmatrix} 6x+6 & 24 \\ 24 & 24 \end{pmatrix}$$

$$Hf(1,-1) = \begin{pmatrix} 12 & 24 \\ 24 & 24 \end{pmatrix}$$

2019 Final Q1(v) solution

The Hessian of f is given by

$$Hf(x,y) = \begin{pmatrix} 6x+6 & 24 \\ 24 & 24 \end{pmatrix}$$

$$Hf(1,-1) = \begin{pmatrix} 12 & 24 \\ 24 & 24 \end{pmatrix}$$

$\det(Hf(1,-1)) = 12 \times 24 - 24 \times 24 = -288 < 0$, which means the eigenvalues have opposite signs. Hence $(1,-1)$ is a saddle point.

2019 Final Q1(v) solution

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$$Hf(5,-5) = \begin{pmatrix} 36 & 24 \\ 24 & 24 \end{pmatrix}$$

2019 Final Q1(v) solution

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$\det(Hf(1,-1)) = 12 \times 24 - 24 \times 24 = -288 < 0$, which means the eigenvalues have opposite signs. Hence $(1,-1)$ is a saddle point.

$$Hf(5,-5) = \begin{pmatrix} 36 & 24 \\ 24 & 24 \end{pmatrix}$$

$\det(Hf(5,-5)) = 36 \times 24 - 24 \times 24 = 288 > 0$ and $12 > 0$ so $(5,-5)$ is a local minimum.

Lagrange multipliers

Theorem

Suppose $f : \mathbb{R} \to \mathbb{R}$ and $\varphi : \mathbb{R}^n \to \mathbb{R}$ are differentiable and

$$S = \{\mathbf{x} \in \mathbb{R}^n : \varphi(\mathbf{x}) = c\}$$

defines a smooth surface in $\mathbb{R}^n$. If f attains a local maximum or minimum at a point $\mathbf{a} \in S$, then $\nabla f(\mathbf{a})$ and $\nabla \varphi(\mathbf{a})$ are parallel. If $\nabla \varphi(\mathbf{a}) \neq \mathbf{0}$, there exist a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$\nabla f(\mathbf{a}) = \lambda \nabla \varphi(\mathbf{a}).$$

Past exam question

2018 Final Q1(vi)

Use the method of Lagrange multipliers to find the maximum and minimum values of the function $f(x, y, z) = 2x - 3y + z$ on the sphere $x^2 + y^2 + z^2 = 14$ (you may assume a maximum and minimum exist).

Past exam question

2018 Final Q1(vi)

Use the method of Lagrange multipliers to find the maximum and minimum values of the function $f(x, y, z) = 2x - 3y + z$ on the sphere $x^2 + y^2 + z^2 = 14$ (you may assume a maximum and minimum exist).

First define $g(x, y, z) = x^2 + y^2 + z^2 - 14$ and find the gradients

$$\nabla f(x, y, z) = (2, -3, 1)^T$$

$$\nabla g(x, y, z) = (2x, 2y, 2z)^T$$

Past exam question

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Use the method of Lagrange multipliers to find the maximum and minimum values of the function $f(x, y, z) = 2x - 3y + z$ on the sphere $x^2 + y^2 + z^2 = 14$ (you may assume a maximum and minimum exist).

First define $g(x, y, z) = x^2 + y^2 + z^2 - 14$ and find the gradients

$$\nabla f(x, y, z) = (2, -3, 1)^T$$

$$\nabla g(x, y, z) = (2x, 2y, 2z)^T$$

Now check that $\nabla g(\mathbf{x}) \neq 0$ for all $\mathbf{x}$ on the constraint. Clearly $\nabla g(\mathbf{x}) = \mathbf{0}$ if and only if $\mathbf{x} = (0, 0, 0)$, which does not lie on the sphere.

2018 Final Q1(vi) Solution

The maximum and minimum on the sphere must satisfy

$$\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x}) \quad \text{and} \quad g(\mathbf{x}) = 0$$

2018 Final Q1(vi) Solution

The maximum and minimum on the sphere must satisfy

$$\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x}) \quad \text{and} \quad g(\mathbf{x}) = 0$$

This gives the following system of equations:

$$2 = 2\lambda x \quad -3 = 2\lambda y \quad 1 = 2\lambda z \quad x^2 + y^2 + z^2 = 14$$

2018 Final Q1(vi) Solution

The maximum and minimum on the sphere must satisfy

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This gives the following system of equations:

$$2 = 2\lambda x \quad -3 = 2\lambda y \quad 1 = 2\lambda z \quad x^2 + y^2 + z^2 = 14$$

$$\text{Since } \lambda \neq 0, \quad x = \frac{1}{\lambda} \quad y = -\frac{3}{2\lambda} \quad z = \frac{1}{2\lambda}.$$

2018 Final Q1(vi) Solution

The maximum and minimum on the sphere must satisfy

$$\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x}) \quad \text{and} \quad g(\mathbf{x}) = 0$$

This gives the following system of equations:

$$2 = 2\lambda x \quad -3 = 2\lambda y \quad 1 = 2\lambda z \quad x^2 + y^2 + z^2 = 14$$

$$\text{Since } \lambda \neq 0, \quad x = \frac{1}{\lambda} \quad y = -\frac{3}{2\lambda} \quad z = \frac{1}{2\lambda}.$$

Substituting into the constraint gives

$$\left(\frac{1}{\lambda}\right)^2 + \left(-\frac{3}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = \frac{7}{2\lambda^2} = 14$$

Solving gives $\lambda = \pm 2$, so the points are $\pm(\frac{1}{2}, -\frac{3}{2}, \frac{1}{4})$, which give function values of $\pm\frac{23}{4}$.

Inverse function theorem

Theorem

Let $\Omega \subseteq \mathbb{R}^n$ be open, $\mathbf{a} \in \Omega$ and $\mathbf{f} : \Omega \to \mathbb{R}^n$ be continuously differentiable.
If $D\mathbf{f}(\mathbf{a})$ is an invertible matrix, then $\mathbf{f}$ is invertible on an open set containing $\mathbf{a}$.

$$\mathbf{f}^{-1} : \mathbf{f}(U) \to U$$

is continuously differentiable and for $\mathbf{x} \in U$,

$$D\mathbf{f}^{-1}(\mathbf{f}(\mathbf{x})) = (D\mathbf{f}(\mathbf{x}))^{-1}.$$

4. Integration

Upper and lower sums

Definition

Consider $f : R \to \mathbb{R}$ where $R = [a, b] \times [c, d]$ is a rectangle in $\mathbb{R}^2$. Let $P_1 = \{a = x_0, x_1, \ldots, x_n = b\}$ be a partition of $[a, b]$ and $P_2 = \{c = y_0, y_1, \ldots, y_m = d\}$ be a partition of $[c, d]$. Define the upper and lower sum by

$$\underline{\mathcal{S}}_{\mathcal{P}_1,\mathcal{P}_2}(f) = \sum_{j,k} \underline{f}_{jk} \Delta x_j \Delta y_k.$$

$$\overline{\mathcal{S}}_{\mathcal{P}_1,\mathcal{P}_2}(f) = \sum_{j,k} \overline{f}_{jk} \Delta x_j \Delta y_k.$$

Riemann integration

Definition

If there exists a unique number $I \in \mathbb{R}$ such that

$$\underline{\mathcal{S}}_{\mathcal{P}_1,\mathcal{P}_2}(f) \leq I \leq \overline{\mathcal{S}}_{\mathcal{P}_1,\mathcal{P}_2}(f)$$

for every pair of partitions $\mathcal{P}_1, \mathcal{P}_2$, then f is Riemann integrable on R and

$$\iint_R f = \iint_R f(x,y)\,\mathrm{d}A = I$$

and I is the Riemann integral of f over R.

Regions

Definition

A region $D \subseteq \mathbb{R}^2$ is y-simple if there exist continuous functions $\varphi_1, \varphi_2 : [a, b] \to \mathbb{R}$ such that $\varphi_1(x) \leq \varphi_2(x)$ for all $\mathbf{x} \in [a, b]$ and

$$D = \{(x, y) : x \in [a, b], \varphi_1(x) \leq y \leq \varphi_2(x)\}.$$

x-simple is similarly defined.
A region D is elementary if it is x-simple or y-simple.

Properties

Properties

If f and g are integrable on D, then

$$\iint_D \alpha f + \beta g = \alpha \iint_D f + \beta \iint_D g$$

$$\left| \iint_D f \right| \leq \iint_D |f|.$$

If $f(\mathbf{x}) \leq g(\mathbf{x})$ for all $\mathbf{x} \in D$, then

$$\iint_D f \leq \iint_D g.$$

If $D = D_1 \cup D_2$ and (interior D)$\cap$(interior D_2) $= \phi$, then

$$\iint_D f = \iint_{D_1} f + \iint_{D_2} f.$$

Fubini's Theorem

Theorem

Let $f : R \to \mathbb{R}$ be continuous on a rectangular domain $R = [a, b] \times [c, d]$. Then

$$\iint_R f = \int_a^b \int_c^d f(x, y)\,\mathrm{d}y\,\mathrm{d}x = \int_c^d \int_a^b f(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

Iterated integrals for elementary regions

Theorem

Suppose D is a y-simple region bounded by $x = a, x = b, y = \varphi_1(x)$ and $y = \varphi_2(x)$ and $f : D \to \mathbb{R}$ is continuous. Then

$$\iint_D f = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) \, \mathrm{d}x \, \mathrm{d}y.$$

A similar result holds for integrals over x-simple regions.

Leibniz' rule

Theorem

Let $a, b \in \mathbb{R}$ be constants, say $a \leq b$ and $\varphi_1, \varphi_2 : [a, b] \to \mathbb{R}$ be continuous functions such that $\varphi_1(x) \leq \varphi_2(x)$ for all $x \in [a, b]$. If f and $\frac{\partial f}{\partial x}$ are continuous on the region

$$D = \{(x, y) : x \in [a, b], \varphi_1(x) \leq y \leq \varphi_2(x)\}$$

then the function

$$g(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) \,\mathrm{d}x \,\mathrm{d}y$$

has derivative

$$g^{'}(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial f}{\partial x}(x, y) \,\mathrm{d}y + f(x, \varphi_2(x))\varphi_2^{'}(x) - f(x, \varphi_1(x))\varphi_1^{'}(x).$$

Past exam question

2014 Final Q2(i)(b)

Calculate

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{\sin(xy)}{y}\,\mathrm{d}y.$$

Past exam question

2014 Final Q2(i)(b)

Calculate
$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{\sin(xy)}{y}\,\mathrm{d}y.$$

Using Leibniz rule,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{\sin(xy)}{y}\,\mathrm{d}y &= \int_0^x \frac{\partial}{\partial x}\frac{\sin(xy)}{y}\,\mathrm{d}y + \frac{\sin(x^2)}{x}\\
&= \int_0^x \cos(xy)\,\mathrm{d}y + \frac{\sin(x^2)}{x}\\
&= \frac{1}{x}\sin(xy)\Big|_{y=0}^{y=x} + \frac{\sin(x^2)}{x}\\
&= \frac{2\sin(x^2)}{x}
\end{aligned}$$

Change of variable

Theorem

Let $\Omega \subseteq \mathbb{R}^n$ and $F : \Omega \to \mathbb{R}^n$ be an injective and continuously differentiable function such that $\det(JF(\mathbf{x})) \neq 0$ for all $\mathbf{x} \in \Omega$. If f is any function that is integrable on $\Omega^{'} = F(\Omega)$ then

$$\iint_{\Omega'} (f \circ F) |\det JF|$$

Polar substitution refers to subbing $x = r\cos\theta$ and $y = r\sin\theta$ where $r > 0, \theta \in [0, 2\pi]$. The Jacobian determinant is r.

Past exam question

2016 Final Q2(iv)

Evaluate

$$\iint_{\Omega'} x^2 \,\mathrm{d}x\,\mathrm{d}y$$

where Ω' is the bounded portion of the first quadrant lying between the hyperbolas

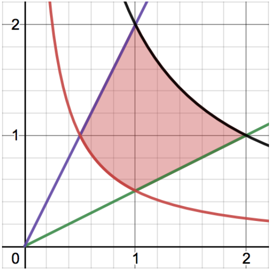
$$xy = \frac{1}{2} \text{ and } xy = 2$$

and the two straight lines

$$y = \frac{x}{2} \text{ and } y = 2x.$$

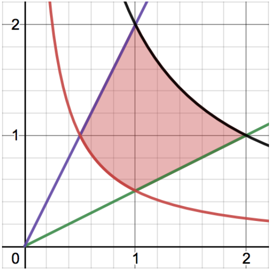
2016 Final Q2(iv) Solution

First we plot $\Omega' = \{(x,y) \in \mathbb{R}^2 : \frac{1}{2} \leq xy \leq 2, \frac{x}{2} \leq y \leq 2x\}$.



2016 Final Q2(iv) Solution

First we plot $\Omega' = \{(x, y) \in \mathbb{R}^2 : \frac{1}{2} \leq xy \leq 2, \frac{x}{2} \leq y \leq 2x\}$.



Now define $u = xy$ and $v = y$. Expressing x and y in terms of u and v gives

$$x = \frac{u}{v} \text{ and } y = v$$

2016 Final Q2(iv) Solution

The Jacobian is given by

$$J = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{1}{v} & -\frac{1}{v^2} \\ 0 & 1 \end{pmatrix}$$

and the determinant is $\det(J) = \frac{1}{v}$.

2016 Final Q2(iv) Solution

The Jacobian is given by

$$J = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{1}{v} & -\frac{1}{v^2} \\ 0 & 1 \end{pmatrix}$$

and the determinant is $\det(J) = \frac{1}{v}$.
The region in the $u - v$ plane can be described by

$$\frac{1}{2} \leq u \leq 2 \quad \text{and} \quad \frac{u}{2v} \leq v \leq \frac{2u}{v}$$

which simplifies to

$$\frac{1}{2} \leq u \leq 2 \quad \text{and} \quad \sqrt{\frac{u}{2}} \leq v \leq \sqrt{2u}$$

2016 Final Q2(iv) Solution

The integral is now

$$\begin{aligned}\int_{\frac{1}{2}}^{2}\int_{\sqrt{\frac{u}{2}}}^{\sqrt{2u}}\left(\frac{u}{v}\right)^2\cdot\frac{1}{v}\,\mathrm{d}v\,\mathrm{d}u &= \int_{\frac{1}{2}}^{2} -\frac{u^2}{v^2}\bigg|_{v=\sqrt{\frac{u}{2}}}^{v=\sqrt{2u}}\,\mathrm{d}u\\ &= \int_{\frac{1}{2}}^{2} -\frac{u^2}{2(2u)}+\frac{u^2}{2(u/2)}\,\mathrm{d}u\\ &= \int_{\frac{1}{2}}^{2}\frac{3}{4}u\,\mathrm{d}u\\ &= \frac{45}{32}\end{aligned}$$

5. Fourier Series

Inner Products I

Definition of an Inner Product

An **inner product** is a binary operation that acts on two vectors in a vector space. It can be anything that has the following properties:

1. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$;
2. $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ iff $\mathbf{u} = \mathbf{0}$;
3. $\langle \lambda \mathbf{u} + \mu \mathbf{v}, \mathbf{w} \rangle = \lambda \langle \mathbf{u}, \mathbf{w} \rangle + \mu \langle \mathbf{v}, \mathbf{w} \rangle$;
4. $\langle \mathbf{v}, \mathbf{u} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$,

for any vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and for any scalars λ, μ.

Inner Products II

Definition of a Norm

A **norm** is a map from a vector space to $\mathbb{R}$. It can be anything that has the following properties:

1. $\|\mathbf{u}\| \geq 0$;
2. $\|\mathbf{u}\| = 0$ iff $\mathbf{u} = \mathbf{0}$;
3. $\|\lambda\mathbf{u}\| = |\lambda|\|\mathbf{u}\|$;
4. $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$,

for any vectors $\mathbf{u}$ and $\mathbf{v}$ and any scalar λ.

Inner Products II

Definition of a Norm

A **norm** is a map from a vector space to $\mathbb{R}$. It can be anything that has the following properties:

1. $\|\mathbf{u}\| \geq 0$;
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3. $\|\lambda \mathbf{u}\| = |\lambda| \|\mathbf{u}\|$;
4. $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$,

for any vectors $\mathbf{u}$ and $\mathbf{v}$ and any scalar λ.

- $\|\mathbf{u}\| = \sqrt{\langle \mathbf{u},\mathbf{u} \rangle}$ is a norm for any inner product $\langle \cdot, \cdot \rangle$.
- (Cauchy-Schwarz inequality) For any vectors $\mathbf{u}$ and $\mathbf{v}$ in a vector space V,

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|.$$

Inner Products III

Inner Products with Functions

- For functions in $C[a,b]$ (the vector space of all functions that are continuous on $[a,b]$), the following inner product is used

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,\mathrm{d}x.$$

- f and g are **orthogonal** if $\langle f, g\rangle = 0$.
    - Note that $1, \dots, \cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)$ are all mutually orthogonal for $n = 1, 2, \dots$ on the interval $[-L, L]$.
- The L^2**-norm** (or 2-norm) is defined as

$$\|f\|_2 = \sqrt{\int_a^b f(x)^2\,\mathrm{d}x}.$$

Introduction to Fourier Series I

Fourier Series

- **Fourier Series** are series used to express periodic functions. They are in the form:

$$S_f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right).$$

- Our task is to find a_0, a_n and b_n as well as determine where the series will converge towards the function.
- Fourier series can replicate functions that are not continuous.

Introduction to Fourier Series II

Fourier Coefficients

- If we have an orthogonal set of vectors $\{\mathbf{u_1}, \dots, \mathbf{u_n}\}$ that spans a vector space V, then any vector $\mathbf{v} \in V$ can be decomposed as

$$\mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{u_i} \text{ where } \alpha_i = \frac{\langle \mathbf{v}, \mathbf{u_i} \rangle}{\|\mathbf{u_i}\|^2}.$$

- We use the same idea to derive the **Fourier coefficients**. Since $\|\frac{1}{2}\|^2 = \|\cos\left(\frac{n\pi x}{L}\right)\|^2 = \|\sin\left(\frac{n\pi x}{L}\right)\|^2 = L$ for $n = 1, 2, \dots,$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x, \qquad n = 0, 1, 2, \dots,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x, \qquad n = 1, 2, \dots.$$

Odd and Even Functions

Odd and Even Functions

- A function f is **odd** if $f(-x) = -f(x)$.
- For any odd function, the Fourier coefficients $\{a_n\}_{n=0,1,\ldots}$ will all be 0 and the Fourier series will be a sum of just sine functions.
- A function f is **even** if $f(-x) = f(x)$.
- For any even function, the Fourier coefficients $\{b_n\}_{n=1,2,\ldots}$ will all be 0 and the Fourier series will be a sum of just cosine functions.

Convergence of Sequences of Functions

Types of Convergences

Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of functions.

- f_k converges to f **pointwisely** on $[a, b]$ to f if $f_k(x) \to f(x)$ as $k \to \infty$ for each $x \in [a, b]$.
- f_k converges to f **uniformly** on $[a, b]$ if $|f_k(x) - f(x)| \to 0$ as $k \to \infty$ for all $x \in [a, b]$.
  - If $f_k : \mathbb{R} \to \mathbb{R}$ is continuous on $[a, b]$ for all k and if f is not continuous on $[a, b]$, then f_k cannot converge uniformly to f on $[a, b]$.
- f_k converges to f **in the mean squared sense** on $[a, b]$ if

$$\lim_{k\to\infty} \int_a^b [f_k(x) - f(x)]^2 \, \mathrm{d}x = 0.$$

Convergence of Fourier Series I

Piecewise Continuity

- A function f is **piecewise continuous** on $[a, b]$ if
  1. For each $x \in [a, b)$, $f(x^+)$ exists;
  2. For each $x \in (a, b]$, $f(x^-)$ exists;
  3. f is continuous on (a, b) except at (at most) a finite number of points.
- A function is piecewise continuous on $\mathbb{R}$ if it is piecewise continuous on any finite interval $[a, b] \subseteq \mathbb{R}$.
- Any continuous function is piecewise continuous.
- Essentially, a piecewise continuous function can be partitioned into a finite set of continuous "pieces".

Convergence of Fourier Series II

- If f is piecewise continuous on a closed and bounded interval $[a, b]$, then $\int_a^b f(x)\,\mathrm{d}x$ exists.
- If f is piecewise continuous, then we can compute its corresponding Fourier coefficients. However, piecewise continuity does not mean the Fourier series will converge to f.
- **Piecewise differentiability** is defined similarly to piecewise continuity. Any function f is piecewise differentiable on $[a, b]$ if
  1. For each $x \in [a, b)$, $D^+(x)$ exists;
  2. For each $x \in (a, b]$, $D^-(x)$ exists;
  3. f is differentiable on (a, b) except at (at most) a finite number of points.
- $D^+f(x)$ is not necessarily the same as $f'(c^+)$.

Convergence of Fourier Series III

Convergence of Fourier Series

Let $c \in \mathbb{R}$ and suppose that a function $f : \mathbb{R} \to \mathbb{R}$ has the following properties:

1. f is $2L$-periodic (i.e. $f(x + 2L) = f(x)$ for all x);
2. f is piecewise continuous on $[-L, L]$;
3. $D^+ f(c)$ and $D^- f(c)$ exist.

Then if f is continuous at c,

$$S_f(c) = f(c).$$

Otherwise

$$S_f(c) = \frac{f(c^+) + f(c^-)}{2}.$$

Convergence of Fourier Series IV

Weierstrass Test

Let $f_k : \mathbb{R} \to \mathbb{R}$ be a sequence of functions defined on $[a, b]$. Suppose that there exists a sequence of numbers c_k such that

$$|f_k(x)| \le c_k \text{ for all } x \in [a, b]$$

and $\displaystyle\sum_{k=1}^{\infty} c_k$ converges. Then $\displaystyle\sum_{k=1}^{\infty} f_k(x)$ converges uniformly to a function f on $[a, b]$.

This is one way to show that a Fourier series converges to the desired function.

Fourier Series Question I

(MATH2111) S2, 2018 – Q4(iii)

Let f be the function defined by

$$f(x) = \begin{cases} -x+\pi, & \text{if } 0 \le x < \pi, \\ x+\pi, & \text{if } -\pi \le x < 0, \end{cases}$$
$$f(x) = f(x+2\pi) \text{ for all } x \in \mathbb{R}.$$

b) Find the coefficients a_0, a_k and b_k $(k \ge 1)$ in the Fourier series of f,

$$F(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(kx) + b_k \sin(kx)].$$

d) Does the Fourier series of f converge uniformly to f on $\mathbb{R}$? Give reasons.

Fourier Series Question I

Fourier Series Question I

b) Since the function is even, we know that $b_k = 0$ for all k and we can focus on a_k.
Using the formula, we obtain the following

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,\mathrm{d}x \\ &= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos(kx)\,\mathrm{d}x \\ &= \frac{2}{\pi}\int_{0}^{\pi} (-x+\pi)\cos(kx)\,\mathrm{d}x \\ &\vdots \quad \text{(apply integration by parts)} \\ &= \frac{2}{\pi}\left(\frac{1}{k^2} - \frac{\cos(\pi k)}{k^2}\right) \quad \text{for } k > 0 \end{aligned}$$

Fourier Series Question I

$$\begin{aligned} a_k &= \frac{4}{\pi k^2} \quad \text{for } k = 1, 3, 5, \ldots \\ &= 0 \quad \text{for } k = 0, 2, 4, \ldots \\ b_k &= 0 \quad \text{for all } k \geq 1 \end{aligned}$$

d) Supposing that F does converge uniformly to f, we apply the Weierstrass Test to confirm. Working from the answer in part b),

$$\begin{aligned} F_k &= \left| \frac{4}{\pi k^2} \cos(kx) \right| \quad \text{for } k = 1, 3, 5, \ldots \quad \text{(0 otherwise)} \\ &\leq \frac{4}{k^2} \quad \text{for all } x \in \mathbb{R}^2 \end{aligned}$$

and $\sum_{k=1}^{\infty} \frac{4}{k^2}$ converges by the p-series test.
So by the Weierstrass test, F converges uniformly to f on $\mathbb{R}$.

Convergence of Fourier Series V

Parseval's Theorem

If a function f is $2L$-periodic and bounded and $\int_{-L}^{L} f(x)^2 \, \mathrm{d}x < \infty$, then the Fourier series of f converges to f in the mean squared sense.

Parseval's Identity

If the above holds, then

$$\int_{-L}^{L} f(x)^2 \, \mathrm{d}x = \|f\|_2^2 = \frac{L}{2} a_0^2 + L \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

where a_0, a_n and b_n are the corresponding Fourier coefficients.

Fourier Series Question II

(MATH2111) T1, 2019 – Q2(ii)

Let f be such that $f(x+2\pi) = f(x)$ and

$$f(x) = \begin{cases} 1, & \text{if } 0 \le |x| \le d, \\ 0, & \text{if } d < |x| \le \pi, \end{cases}$$

where d is some constant such that $0 < d < \pi$.

b) Determine the Fourier series F for the function f.

c) Using Parseval's identity and the Fourier series for f, find

$$\sum_{k=1}^{\infty} \frac{\sin^2(kd)}{k^2}.$$

Fourier Series Question II

b) Since f is even, $b_k = 0$ for all k. Then we need to find a_k:

$$\begin{aligned}
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos(kx)\,\mathrm{d}x \\
&= \frac{2}{\pi}\left(\int_{0}^{d} f(x)\cos(kx)\,\mathrm{d}x + \int_{d}^{\pi} f(x)\cos(kx)\,\mathrm{d}x\right) \\
&= \frac{2}{\pi}\int_{0}^{d} \cos(kx)\,\mathrm{d}x \\
&= \frac{2}{\pi}\left[\frac{1}{k}\sin(kx)\right]_0^d \quad \text{for } k > 0 \\
&= \frac{2}{\pi k}\sin(kd).
\end{aligned}$$

Fourier Series Question II

$$
\begin{aligned}
a_0 &= \frac{2}{\pi}\int_0^d \cos(0)\,\mathrm{d}x \\
&= \frac{2d}{\pi} \\
F(x) &= \frac{d}{\pi} + \sum_{k=1}^{\infty} \frac{2}{\pi k}\sin(kd)\cos(kx)
\end{aligned}
$$

c) It's not difficult to see that the conditions for Parseval's theorem hold, so we can apply Parseval's identity.

$$
\begin{aligned}
\int_{-\pi}^{\pi} f(x)^2\,\mathrm{d}x &= \frac{\pi}{2}\left(\frac{2d}{\pi}\right)^2 + \pi\sum_{k=1}^{\infty}\left(\frac{2}{\pi k}\sin(kd)\right)^2 \\
\int_{-d}^{d} 1^2\,\mathrm{d}x &= \frac{\pi}{2}\cdot\frac{4d^2}{\pi^2} + \pi\cdot\frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{\sin^2(kd)}{k^2}
\end{aligned}
$$

Fourier Series Question II

$$2d = \frac{2d^2}{\pi} + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin^2(kd)}{k^2}$$

$$\sum_{k=1}^{\infty}\frac{\sin^2(kd)}{k^2} = \frac{\pi}{4}\left(2d - \frac{2d^2}{\pi}\right)$$

$$= \frac{\pi d}{2} - \frac{d^2}{2}$$

6. Vector Fields

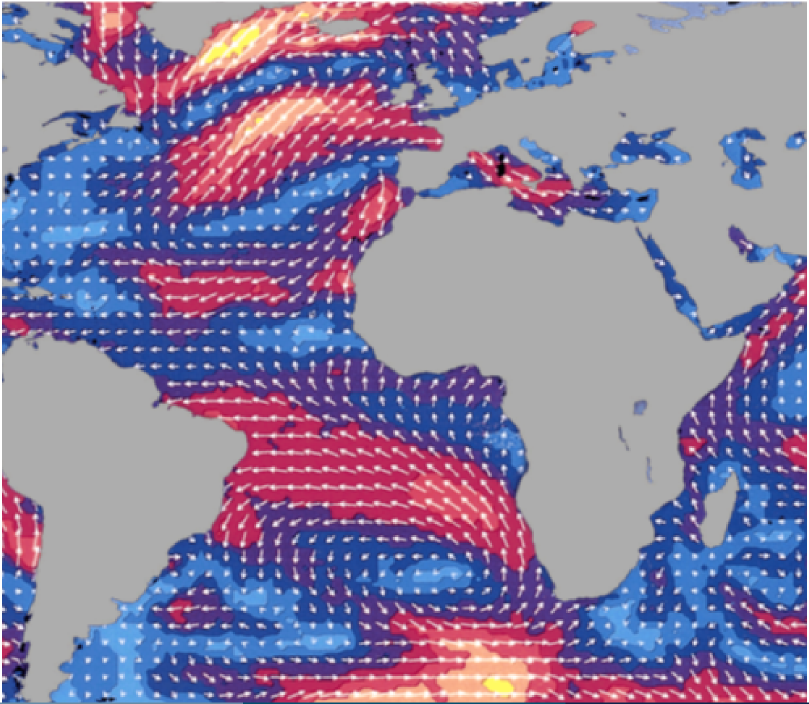
Introduction to Vector Fields I

Vector Fields

- A **vector field** is a map of every point to a vector. A vector field in 3D space can be represented by the vector function:

$$\mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), F_2(\mathbf{x}), F_3(\mathbf{x})).$$

- To sketch a vector field in 2D, draw some arrows from a collection of points on the 2D-plane. The arrow direction represents the direction of the vector while the relative length represents the relative magnitude.



Introduction to Vector Fields II

Definition of Flow Lines

A path $\mathbf{c}(t)$ is a **flow line** for a vector field $\mathbf{F}$ if

$$\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t)).$$

Vector Field Operators I

Gradient

- The "del" operator is a vector-like operator used in vector calculus. It is represented by the ∇ ("nabla") symbol.

$$\nabla f := \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}$$

- The **gradient** of a 3-dimensional scalar function f is

$$\operatorname{grad} f = \nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}.$$

Vector Field Operators II

Divergence

- If $\mathbf{F} = (F_1, F_2, F_3)$, the **divergence** of $\mathbf{F}$ is the scalar function

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

- The divergence at a point P can be interpreted as the total outward flux at P.
- A negative divergence indicates a net inflow while a zero divergence indicates zero net flow (and the flow is **incompressible**).

Vector Fields Question I

(MATH2111) S1, 2016 – Q3(i)

Find div$\mathbf{F}$ if $\mathbf{F} = (x + \cos(yz))\mathbf{i} - \log(x^2 + y^2 + z^2)\mathbf{j} + e^{xy}\mathbf{k}$.

Vector Fields Question I

(MATH2111) S1, 2016 – Q3(i)

Find div$\mathbf{F}$ if $\mathbf{F} = (x + \cos(yz))\mathbf{i} - \log(x^2 + y^2 + z^2)\mathbf{j} + e^{xy}\mathbf{k}$.

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}(x + \cos(yz)) + \frac{\partial}{\partial y}(-\log(x^2 + y^2 + z^2)) + \frac{\partial}{\partial z}e^{xy} \\ &= 1 - \frac{2x}{x^2 + y^2 + z^2}\end{aligned}$$

Vector Field Operators III

Curl

- If $\mathbf{F} = (F_1, F_2, F_3)$, the **curl** of $\mathbf{F}$ is the vector field

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \begin{pmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \\ -\frac{\partial F_3}{\partial x} + \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{pmatrix}$$

- A vector field is **irrotational** if $\operatorname{curl}\mathbf{F} = \mathbf{0}$
- If $\mathbf{F}$ is a 2D vector field, then we calculate $\operatorname{curl} F$ by letting the third component be 0. (i.e. $F_3 = 0$.)

Vector Field Operators IV

Interpretation of the Curl

- The value of $\nabla \times \mathbf{F}$ at a point P can be interpreted as how the vector field swirls around the point P.
- The 3rd component of $\operatorname{curl}\mathbf{F}$ is denoted by $\operatorname{curl}_z\mathbf{F}$. i.e.

$$\operatorname{curl}_z\mathbf{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}.$$

- For 2D vector fields, $\operatorname{curl}_z\mathbf{F}$ at a point (x_0, y_0) gives the anticlockwise rotation around that point while looking down on the x, y-plane.
- $\operatorname{curl}_z\mathbf{F} < 0$ at a point indicates clockwise rotation around that point.

Vector Fields Question II

(MATH2111) S2, 2016 – Q4(ii)

Prove that if φ is a C^2 scalar field, then $\operatorname{curl}(\operatorname{grad}\varphi) = \mathbf{0}$.

Vector Fields Question II

(MATH2111) S2, 2016 – Q4(ii)

Prove that if φ is a C^2 scalar field, then $\operatorname{curl}(\operatorname{grad}\varphi) = \mathbf{0}$.

Assuming that φ is a field on $\mathbb{R}^3$ (for the curl to be defined), then $\operatorname{grad}\varphi$ exists (since φ is C^2) and is

$$\nabla\varphi = \begin{pmatrix} \frac{\partial\varphi}{\partial x} \\ \frac{\partial\varphi}{\partial y} \\ \frac{\partial\varphi}{\partial z} \end{pmatrix}.$$

Similarly, $\operatorname{curl}(\operatorname{grad}\varphi)$ exists and is

$$\nabla \times (\nabla\varphi) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \end{vmatrix}$$

Vector Fields Question II

$$
\begin{aligned}
\nabla \times (\nabla \varphi) &= \left(\frac{\partial^2 \varphi}{\partial y \partial z} - \frac{\partial^2 \varphi}{\partial z \partial y}\right)\mathbf{i} - \left(\frac{\partial^2 \varphi}{\partial x \partial z} - \frac{\partial^2 \varphi}{\partial z \partial x}\right)\mathbf{j} \\
&\quad + \left(\frac{\partial^2 \varphi}{\partial x \partial z} - \frac{\partial^2 \varphi}{\partial z \partial x}\right)\mathbf{k} \\
&= \mathbf{0}^*
\end{aligned}
$$

*Since φ is C^2, mixed partial derivatives commute.

7. Line Integrals

Path Integrals

Path Integrals

- A path $\mathbf{c}(t)$ parametrises a curve if $\mathbf{c}(t) = (x(t), y(t), z(t))$ traces out the curve $\mathcal{C}$ for $a \le t \le b$.
- A **path integral** (or scalar line integral) is defined as the following:
$$\int_{\mathcal{C}} f(x, y, z)\,\mathrm{d}s = \int_a^b f(\mathbf{c}(t))\|\mathbf{c}'(t)\|\,\mathrm{d}t,$$
where $\mathbf{c}(t)$ is a parametrisation for $\mathcal{C}$ for $a \le t \le b$ (assuming f and $\mathbf{c}'$ are continuous).
- $\|\mathbf{c}'(t)\|$ refers to the Euclidean norm (or 2-norm):
$$\|\mathbf{c}'(t)\| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}.$$
- A path integral can be interpreted as a regular integral, but taken with respect to a curve rather than x, y or z.

Path Integrals II

Notes on Path Integrals

- The value of the integral does not depend on the choice of parametrisation.
- If $f(x, y, z) = 1$, then we get the length of $\mathcal{C}$:

$$\text{Length of } \mathcal{C} = \int_a^b \|\mathbf{c}'(t)\| \, \mathrm{d}t.$$

- Just like regular integrals, path integrals have the following properties:

$$\int_{\mathcal{C}} [\lambda f_1 + f_2] \, \mathrm{d}s = \lambda \int_{\mathcal{C}} f_1 \, \mathrm{d}s + \int_{\mathcal{C}} f_2 \, \mathrm{d}s$$

$$\int_{\mathcal{C}_1 + \mathcal{C}_2} f \, \mathrm{d}s = \int_{\mathcal{C}_1} f \, \mathrm{d}s + \int_{\mathcal{C}_2} f \, \mathrm{d}s$$

Line Integrals I

Line Integrals

- A **line integral** is defined as the following:

$$\int_{\mathcal{C}} \mathbf{F} \cdot \,\mathrm{d}\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\,\mathrm{d}t;$$
$$= \int_a^b \left(\mathbf{F}(\mathbf{c}(t)) \cdot \frac{\mathbf{c}'(t)}{\|\mathbf{c}'(t)\|} \right) \|\mathbf{c}'(t)\|\,\mathrm{d}t,$$

  where $\mathbf{c}(t)$ is a continuously differentiable parametrisation of the oriented curve $\mathcal{C}$.
- The line integral is the integral of the tangential component of $\mathbf{F}$.

Line Integrals II

- The work done by a force field $\mathbf{F}$ acting on an object being moved along a curve $\mathcal{C}$ can be calculated by $\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{s}$.
- Line integrals are often written in the form

$$\int_{\mathcal{C}} M \, \mathrm{d}x + N \, \mathrm{d}y + P \, \mathrm{d}z.$$

If we have a parametrisation $\mathbf{c}(t) = (x(t), y(t), z(t))$ for $\mathcal{C}$, then this can be computed using

$$\int_a^b [M(\mathbf{c}(t))x'(t) + N(\mathbf{c}(t))y'(t) + P(\mathbf{c}(t))z'(t)] \, \mathrm{d}t.$$

Line Integrals Question I

(MATH2111) S1, 2018 – Q4(ii)

Let C be the curve defined by $\mathbf{r}(t) = (1, t, e^t)$ from $t = 0$ to $t = 1$. Evaluate the line integral

$$\int_C (2\cos z \,\mathrm{d}x + x \,\mathrm{d}y + 3e^y \,\mathrm{d}z).$$

Line Integrals Question I

(MATH2111) S1, 2018 – Q4(ii)

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$$\int_C (2\cos z \, \mathrm{d}x + x \, \mathrm{d}y + 3e^y \, \mathrm{d}z).$$

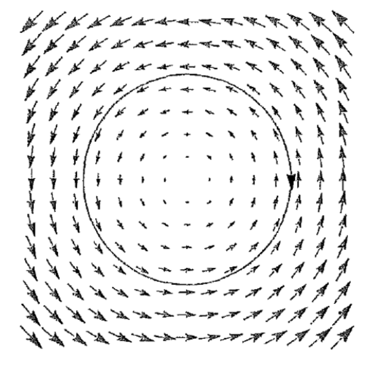
Using the formula from the previous slide, this can be computed as following:

$$\begin{aligned} \int_0^1 \left(2\cos\left(e^t\right)\frac{\mathrm{d}}{\mathrm{dt}}(1) + 1 \cdot \frac{\mathrm{d}}{\mathrm{dt}}(t) + 3e^t \frac{\mathrm{d}}{\mathrm{dt}} e^t\right) \mathrm{d}t &= \int_0^1 (1 + 3e^{2t}) \, \mathrm{d}t \\ &= \left[t + \frac{3}{2}e^{2t}\right]_0^1 \\ &= \frac{3}{2}e^2 - \frac{1}{2}. \end{aligned}$$

Line Integrals Question II

(MATH2111) S1, 2016 – Q3(ii)

Consider the vector field $\mathbf{G}$ and the curve C shown below. Is the path integral $\oint_C \mathbf{G} \cdot d\mathbf{r}$ positive, negative or zero? Why? (Note that C is oriented **clockwise**.)



Line Integrals Question II

While the flux is rotating in the anticlockwise (positive) direction (curl$\mathbf{G} > 0$), C is oriented in the clockwise (negative) direction, so the line integral will be negative.
Alternatively, it can be seen that the path always travels against the flux, meaning that the work done will be negative.

Line Integrals III

Properties of Line Integrals

Again, similarly to regular integrals, line integrals display the following properties.

- Line integrals preserve linearity

$$\int_{\mathcal{C}} (\lambda \mathbf{F} + \mathbf{G}) \cdot \, \mathrm{d}\mathbf{s} = \lambda \int_{\mathcal{C}} \mathbf{F} \, \mathrm{d}\mathbf{s} + \int_{\mathcal{C}} \mathbf{G} \, \mathrm{d}\mathbf{s}$$

- Reversing the orientation reverses the sign

$$\int_{-\mathcal{C}} \mathbf{F} \cdot \, \mathrm{d}\mathbf{s} = - \int_{\mathcal{C}} \mathbf{F} \cdot \, \mathrm{d}\mathbf{s}$$

- If $\mathcal{C}$ is a union of n smooth curves $\mathcal{C}_1 + \cdots + \mathcal{C}_n$, then

$$\int_{\mathcal{C}} \mathbf{F} \cdot \, \mathrm{d}\mathbf{s} = \int_{\mathcal{C}_1} \mathbf{F} \cdot \, \mathrm{d}\mathbf{s} + \cdots + \int_{\mathcal{C}_n} \mathbf{F} \cdot \, \mathrm{d}\mathbf{s}$$

Line Integrals IV

Fundamental Theorem of Calculus for Line Integrals

- If there exists a real-valued function φ such that $\mathbf{F} = \nabla\varphi$, then $\mathbf{F}$ is **conservative** (and called a **gradient field**). φ is the **potential function** for $\mathbf{F}$.
- (Fundamental Theorem of Calculus for Line Integrals) If $\mathbf{F} = \nabla\varphi$ on a domain $\mathcal{D}$, then for every oriented curve $\mathcal{C} \in \mathcal{D}$,

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{s} = \varphi(\mathbf{q}) - \varphi(\mathbf{p})$$

where $\mathcal{C}$ goes from $\mathbf{p}$ to $\mathbf{q}$.

Line Integrals IV

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$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{s} = \varphi(\mathbf{q}) - \varphi(\mathbf{p})$$

  where $\mathcal{C}$ goes from $\mathbf{p}$ to $\mathbf{q}$.
- If $\mathbf{F}$ is conservative and $\mathcal{C}$ is closed (if $\mathbf{p} = \mathbf{q}$), then $\oint_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{s} = 0$.
- If $\mathbf{F}$ is conservative, then the choice of the path $\mathbf{c}$ is irrelevant when computing the line integral.

Line Integrals Question III

(MATH2111) T1, 2019 – Q3(i)

Let $\mathbf{F}(x, y, z) = 2xy\mathbf{i} - z\mathbf{j} + x^2\mathbf{k}$.

a) Evaluate the line integral

$$\int_C \mathbf{F} \cdot \mathrm{d}r,$$

along the straight line segment connecting the point $A = (0, 0, 0)$ to the point $B = (1, 1, 1)$.

b) Would the integral be the same regardless of the path taken between A and B? Explain why or why not.

Line Integrals Question III

a) We can parametrise this line segment by $\mathbf{r}(t) = (t, t, t)^T$ for $0 \leq t \leq 1$.
Then $\mathbf{r'}(t) = (1, 1, 1)^T$ and

$$\begin{aligned}\int_C \mathbf{F} \cdot \, \mathrm{d}r &= \int_0^1 (2t^2 - t + t^2) \, \mathrm{d}t \\ &= \int_0^1 (3t^2 - t) \, \mathrm{d}t \\ &= \left[t^3 - \frac{t^2}{2} \right]_0^1 \\ &= \frac{1}{2}\end{aligned}$$

b) As proven on slide 102, $\nabla \times (\nabla\varphi) = \mathbf{0}$. The contrapositive of this is that if $\nabla \times \mathbf{F} \neq \mathbf{0}$, then we cannot find a φ such that $\mathbf{F} = \nabla\varphi$. We check $\nabla \times \mathbf{F}$:

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{pmatrix} -1 \\ -2x \\ -2x \end{pmatrix} && (1) \\ &\neq \mathbf{0}. && (2) \end{aligned}$$

So, $\mathbf{F}$ is not conservative, meaning that the integral does depend on the path taken.

Green's Theorem

Suppose

- $\mathcal{D}$ is a bounded, simple region in $\mathbb{R}^2$ and $\mathcal{C}$ is its boundary (oriented in the positive (anticlockwise) direction) and
- $M(x,y)$ and $N(x,y)$ are continuously differentiable on $\mathcal{D}$.

Then

$$\oint_{\mathcal{C}} (M\,\mathrm{d}x + N\,\mathrm{d}y) = \iint_{\mathcal{D}} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathrm{d}x\,\mathrm{d}y.$$

Line Integrals Question IV

(MATH2111) Term 1, 2020 – Q3(ii)

Let C be the circle $x^2 + y^2 = 4$ oriented in a counterclockwise direction. Use Green's theorem to evaluate

$$\int_C (2y - e^{x^2})\,\mathrm{d}x + (7x + \sqrt{y^2+1})\,\mathrm{d}y.$$

Line Integrals Question IV

(MATH2111) Term 1, 2020 – Q3(ii)

Let C be the circle $x^2 + y^2 = 4$ oriented in a counterclockwise direction. Use Green's theorem to evaluate

$$\int_C (2y - e^{x^2})\,\mathrm{d}x + (7x + \sqrt{y^2+1})\,\mathrm{d}y.$$

Let $M(x,y) = 2y - e^{x^2}$ and $N(x,y) = 7x + \sqrt{y^2+1}$. Then

$$\begin{aligned} \frac{\partial N}{\partial x} &= 7 \\ \frac{\partial M}{\partial y} &= 2 \\ \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} &= 5 \end{aligned}$$

Line Integrals Question IV

So by Green's theorem,

$$\int_C (2y - e^{x^2})\,\mathrm{d}x + (7x + \sqrt{y^2+1})\,\mathrm{d}y = \iint_D 5\,\mathrm{d}x\,\mathrm{d}y,$$

where D is the region bounded by the circle C. This integral is then just the 5 times the area of D, which is $\pi(2)^2 = 4\pi$. So the solution is

$$20\pi.$$

8. Surface Integrals

Surface Integrals I

Surface Integrals

- $\boldsymbol{\phi} : \mathbb{R}^2 \to \mathbb{R}^3$ parametrises a surface $\mathcal{S}$ if $\boldsymbol{\phi}(u, v)$ maps out $\mathcal{S}$ on some domain $D \in \mathbb{R}^2$.
- A **surface integral** is a double integral of a scalar function or vector field over a surface.

Surface Integrals I

Surface Integrals

- $\boldsymbol{\phi} : \mathbb{R}^2 \to \mathbb{R}^3$ parametrises a surface $\mathcal{S}$ if $\boldsymbol{\phi}(u, v)$ maps out $\mathcal{S}$ on some domain $D \in \mathbb{R}^2$.
- A **surface integral** is a double integral of a scalar function or vector field over a surface.
- Let $\boldsymbol{\phi}(u, v)$ be a parametrisation of a smooth surface $\mathcal{S}$ with parameter domain D. The surface integral of a scalar function f over $\mathcal{S}$ is

  $$\iint_{\mathcal{S}} f(x, y, z)\,\mathrm{d}S = \iint_D f(\boldsymbol{\phi}(u, v))\|\mathbf{T_u} \times \mathbf{T_v}\|\,\mathrm{d}u\,\mathrm{d}v,$$

  where

  $$\mathbf{T_u} = \frac{\partial \boldsymbol{\phi}}{\partial u} \quad \text{and} \quad \mathbf{T_v} = \frac{\partial \boldsymbol{\phi}}{\partial v}.$$

Surface Integrals II

Notes on Surface Integrals

- The surface area of $\mathcal{S}$ is given by

$$\text{Area}(\mathcal{S}) = \iint_D \|\mathbf{T_u} \times \mathbf{T_v}\| \, \mathrm{d}u \, \mathrm{d}v.$$

- Sometimes the following notation is used:

$$\|\mathbf{n}(u,v)\| = \|\mathbf{T_u} \times \mathbf{T_v}\|.$$

- A surface $\mathcal{S}$ is **smooth** if it admits a parametrisation $\boldsymbol{\phi}(u,v) = (x(u,v), y(u,v), z(u,v))$ for $(u,v) \in D$, where
    - D is an elementary region in $\mathbb{R}^2$,
    - $\boldsymbol{\phi}$ is continuously differentiable and one-to-one except possibly on ∂D and
    - $\mathcal{S}$ is regular ($\mathbf{n} \neq \mathbf{0}$) except possibly on ∂D.

Surface Integrals Question I

(MATH2111) T1, 2020 – Q3(iv)

Consider a surface S given by

$$z = xy \quad \text{for} \quad x^2 + y^2 \leq 4.$$

Find the area of the surface S.

Surface Integrals Question I

(MATH2111) T1, 2020 – Q3(iv)

Consider a surface S given by

$$z = xy \quad \text{for} \quad x^2 + y^2 \leq 4.$$

Find the area of the surface S.

Our parametrisation can be written as $\boldsymbol{\phi}(x, y) = (x, y, xy)^T$ for $(x, y) : x^2 + y^2 \leq 4$.
Then

$$\mathbf{T_x} = \begin{pmatrix} 1 \\ 0 \\ y \end{pmatrix} \quad \text{and} \quad \mathbf{T_y} = \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix}.$$

Surface Integrals Question I

$$\mathbf{T_x} \times \mathbf{T_y} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & y \\ 0 & 1 & x \end{vmatrix}$$
$$= \begin{pmatrix} -y \\ -x \\ 1 \end{pmatrix}$$
$$\|\mathbf{T_x} \times \mathbf{T_x}\| = \sqrt{y^2 + x^2 + 1}$$

So we have

$$\text{Area}(S) = \iint_D \sqrt{y^2 + x^2 + 1}\, \mathrm{d}x\, \mathrm{d}y,$$

where $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$.

Surface Integrals Question I

To find this double integral, we convert to polar coordinates $(x = r\cos\theta,\ y = r\sin\theta)$. The bounds become $0 \le r \le 2$ and $-\pi < \theta \le \pi$ and the determinant of the Jacobian is r:

$$\begin{aligned}
\text{Area}(S) &= \int_{-\pi}^{\pi}\int_0^2 r\sqrt{(r\cos\theta)^2 + (r\sin\theta)^2 + 1}\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_{-\pi}^{\pi}\left(\int_0^2 r\sqrt{r^2+1}\,\mathrm{d}r\right)\,\mathrm{d}\theta \\
&= \left[\frac{1}{3}(r^2+1)^{\frac{3}{2}}\right]_0^2 \int_{-\pi}^{\pi}\,\mathrm{d}\theta \\
&= \frac{1}{3}(5^{\frac{3}{2}} - 1)\cdot 2\pi \\
&= \frac{2\pi}{3}(5^{\frac{3}{2}} - 1)
\end{aligned}$$

Surface Integrals on Vector Fields

Where $\boldsymbol{\phi}(u, v)$ is the parametrisation of $\mathcal{S}$ with parameter domain $\mathcal{D}$, the surface integral of a vector field F over an oriented smooth surface $\mathcal{S}$ is

$$\begin{aligned}\iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} &= \iint_{D} \mathbf{F}(\boldsymbol{\phi}(u, v)) \cdot (\mathbf{T_u} \times \mathbf{T_v})\, \mathrm{d}u\, \mathrm{d}v \\ &= \iint_{D} \left(\mathbf{F}(\boldsymbol{\phi}(u, v)) \cdot \left(\frac{\mathbf{T_u} \times \mathbf{T_v}}{\|\mathbf{T_u} \times \mathbf{T_v}\|} \right) \right) \|\mathbf{T_u} \times \mathbf{T_v}\|\, \mathrm{d}u\, \mathrm{d}v\end{aligned}$$

Stokes' Theorem

Stokes' Theorem

Suppose

- $\mathcal{S}$ is a a smooth oriented surface,
- $\partial\mathcal{S}$ is its boundary oriented in the anticlockwise direction when looking down on $\mathcal{S}$ from the positive direction and
- $\mathbf{F}$ is a C^1 vector field on $\mathcal{S}$.

Then

$$\iint_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \oint_{\partial\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{s}.$$

Surface Integrals Question II

(MATH2111) S1, 2016 – Q3(v)(e) (Modified)

Let S denote the surface $x^2 + y^2 = z^2$ for $0 \leq z \leq 1$, oriented so that the z-component of the unit normal is **positive**, and let $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$. Use Stoke's theorem to determine the path integral

$$\oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

By Stoke's theorem,

$$\oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S}.$$

Then,

$$\nabla \times \mathbf{F} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}.$$

Surface Integrals Question II

We can parametrise S by $\boldsymbol{\phi}(r,\theta) = (r\cos\theta, r\sin\theta, r)$, where $0 \le r \le 1$ and $-\pi < \theta \le \pi$. Then,

$$\begin{aligned}
\mathbf{T_r} \times \mathbf{T}_\theta &= \begin{pmatrix} \cos\theta \\ \sin\theta \\ 1 \end{pmatrix} \times \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \\ r \end{pmatrix} \\
(\nabla \times \mathbf{F}) \cdot (\mathbf{T_r} \times \mathbf{T}_\theta) &= (-1)(-r\cos\theta) + (-1)(-r\sin\theta) + (-1)r \\
&= r\cos\theta + r\sin\theta - r
\end{aligned}$$

Surface Integrals Question II

$$
\begin{aligned}
\oint_{\partial S} \mathbf{F} \cdot \, \mathrm{d}\mathbf{r} &= \int_0^1 \int_{-\pi}^{\pi} (r\cos\theta + r\sin\theta - r) \, \mathrm{d}\theta \, \mathrm{d}r \\
&= \int_0^1 \Big[r\sin\theta - r\cos\theta - r\theta \Big]_{-\pi}^{\pi} \mathrm{d}r \\
&= \int_0^1 (-2\pi r) \, \mathrm{d}r \\
&= \left[-\pi r^2 \right]_0^1 \\
&= -\pi
\end{aligned}
$$

Divergence Theorem

Divergence Theorem (or Gauss's Theorem)

Suppose

- the region $W \subseteq \mathbb{R}^3$ is a bounded, solid and simple region,
- $\mathcal{S}$ is its piece-wise smooth, oriented such that the normal vector points outwards and
- $\mathbf{F}$ be a C^1 vector field on W.

Then

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iiint_W \nabla \cdot \mathbf{F} \, \mathrm{d}V.$$

Surface Integrals Question III

(MATH2111) T1, 2019 – 4(i)

The solid V is the cube $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, $-1 \leq z \leq 1$. The closed surface S is the boundary of V. The vector field $\mathbf{F}$ is defined by

$$\mathbf{F}(x, y, z) = yx^2\mathbf{i} + y^3\mathbf{j} + z\mathbf{k}$$

b) Apply the divergence theorem to calculate the flux

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

of the field $\mathbf{F}$ through the surface S.

Surface Integrals Question III

b)

$$
\begin{aligned}
\iint_S \mathbf{F} \cdot \, \mathrm{d}\mathbf{S} &= \iiint_V \nabla \cdot \mathbf{F} \, \mathrm{d}V \\
&= \iiint_V (2yx + 3y^2 + 1) \, \mathrm{d}V \\
&= \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} (2yx + 3y^2 + 1) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\
&= \int_{-1}^{1} \int_{-1}^{1} (6y^2 + 2) \, \mathrm{d}y \, \mathrm{d}z \\
&= \int_{-1}^{1} 8 \, \mathrm{d}z \\
&= 16.
\end{aligned}
$$