MATH1131/1141 Lab Test 1 Calculus Solutions to Samples

October 7, 2019

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We cannot guarantee that our answers are correct - please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are sometimes multiple methods of solving the same question.

Note that underlined text are answers to 'drop-down' style questions.

Question 1

$$\text{Let } f(x) = \frac{4x^3 + 4x + 2\sin(x) - 3}{-5x^7 - 2x^3 + 4}$$

Find $\lim_{x \to \infty} f(x)$ if it exists.

Answer: Observe that as $x \to \infty$ the $\sin(x)$ term becomes insignificant. Also note that the degree of the denominator is greater than that of the numerator.

Dividing by the highest power

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{\frac{4x^3}{x^7}+\frac{4x}{x^7}+\frac{2}{x^7}-\frac{3}{x^7}}{\frac{-5x^7}{x^7}-\frac{2x^3}{x^7}+\frac{4}{x^7}} \\ &= \frac{0}{-5}\end{aligned}$$

Hence the limit is $f(x) = 0$.

Question 2

The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{x-4}{|x^2-3x+2|} & x \neq 1 \text{ or } 2 \\ 0 & x = 1 \text{ or } 2 \end{cases}$$

has discontinuities at 1 and 2. At the discontinuity at 1 the left and right and limits of f do not exist.

Can the function f can be into a function that is continuous at 1 by redefining the value of $f(1)$? If so what is the value?

Answer: No.

In these types of question, we want to evaluate the left and right hand limits around our point of interest to determine the nature of the discontinuity. We can factorise the denominator

$$f(x) = \begin{cases} \frac{x-4}{|(x-1)(x-2)|} & x \neq 1 \text{ or } 2 \\ 0 & x = 1 \text{ or } 2 \end{cases}$$

Now it becomes clear that

$$\lim_{x\to 1^+} f(x) \to -\infty$$
$$\lim_{x\to 1^-} f(x) \to -\infty$$

Hence, we conclude that at $x = 1$ the left and right hand limits do not exist and there is no single value $f(x)$ can be redefined to that will make the function continuous over the interval. This is also known as an infinite discontinuity.

Question 3

The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{x-4}{x^2-3x-4} & x \neq -1 \text{ or } 4 \\ 0 & x = -1 \text{ or } 4 \end{cases}$$

has discontinuities at -1 and 4. At the discontinuity at 1 the left and right and limits of f exists and are equal.

Can the function f can be into a function that is continuous at 4 by redefining the value of $f(4)$? If so what is the value?

Answer: Yes
By factorising the piecemeal function can be re-written as

$$f(x) = \begin{cases} \frac{1}{x+1} & x \neq -1 \text{ or } 4 \\ 0 & x = -1 \text{ or } 4 \end{cases}$$

Now it becomes clear that around our point of interest

$$\lim_{x \to 4^+} f(x) = \frac{1}{5}$$
$$\lim_{x \to 4^-} f(x) = \frac{1}{5}$$

This means that at $x = 4$ both the left and right limits exist and are equal. Thus we can redefine $f(4)$ to be $\frac{1}{5}$. This resolves the removable discontinuity, by 'filling the hole'.

Question 4

The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{x+4}{|x^2+7x+12|} & x \neq -3 \text{ or } -4 \\ 0 & x = -3 \text{ or } -4 \end{cases}$$

has discontinuities at -4 and -3. At the discontinuity at -4 the left and right and limits of f exists but are not equal.

Can the function f be made into a function that is continuous at -4 by redefining the value of $f(-4)$? If so what is the value?

Answer: By factorising the piecemeal function can be re-written as

$$f(x) = \begin{cases} \frac{x+4}{|(x+4)(x+3)|} & x \neq -3 \text{ or } 4 \\ 0 & x = -3 \text{ or } -4 \end{cases}$$

Now it becomes clear that when approaching from the positive side of our point of interest, the quadratic denominator is positive. Therefore we are allowed to drop the absolute brackets.

$$\begin{aligned} \lim_{x \to -4^+} f(x) &= \lim_{x \to -4^+} \frac{x+4}{(x+4)(x+3)} \\ &= \lim_{x \to -4^+} \frac{1}{x+3} \\ &= -1 \end{aligned}$$

When approaching from the negative side, notice the sign of the quadratic denominator is negative. Hence we multiply the denominator by -1 to ensure it remains positive when breaking the absolute brackets.

$$\begin{aligned} \lim_{x \to -4^-} f(x) &= \lim_{x \to -4^-} \frac{x+4}{-(x+4)(x+3)} \\ &= \lim_{x \to -4^-} \frac{-1}{x+3} \\ &= 1 \end{aligned}$$

Hence, at $x = -4$ the left and right and limits exists but are not equal, therefore we cannot redefine $f(-4)$ to make the interval continuous. This is also known as a jump discontinuity

Question 5

True statements regarding the behaviour of $\cos(x)$ and $\frac{\cos(x)}{x}$ for large values of x.

$$\lim_{x\to\infty} \frac{\cos(x)}{x} \text{ exists and is equal to } 0$$

For large values of x we can find values for which the numerator $\cos(x)$ is equal to 1 and -1. Under such circumstances, the denominator will diverge to infinity whilst the numerator will be bounded between -1 and 1, therefore the limit exists and is equal 0

$$\lim_{x\to\infty} \cos(x) \text{ is undefined}$$

Here the behaviour of the function is uncertain, as it is periodic, so we cannot determine the limit.

$$\lim_{x\to\infty} \left(\frac{\cos(x)}{x} + \cos(x)\right) \text{ is undefined}$$

Note from above that the limit of $\cos(x)$ is undefined and $\frac{\cos(x)}{x}$ is defined. The sum of a defined and undefined limit is undefined, from which the statement follows.

Question 6

Solve the inequality

$$\frac{1}{x+6} \leq -\frac{1}{6}$$

Answer: In these types of questions, the trick is to first multiply both sides of the inequality by the square of the denominator with the x term. This enables further algebraic manipulation as the denominators no longer involve x

$$(x+6) \leq -\frac{(x+6)^2}{6} \quad x \neq -6$$

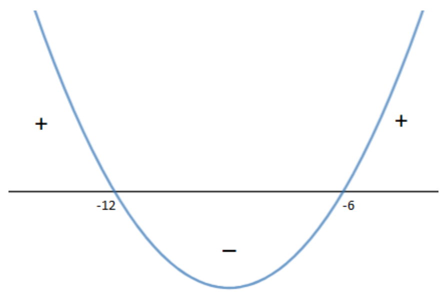
It is important to note $x \neq -6$ as the original expression is undefined here. We specifically multiply by square terms because they are always positive and hence will not alter the sign of inequality

In the next step, we always want to move all the terms to one side to prepare for factorisation.

$$6(x+6)+(x+6)^2 \leq 0 \quad x \neq -6$$

Now factorising gives,

$$(x+6)(x+12) \leq 0 \quad x \neq -6$$



Therefore our solution is the interval $[-12,-6)$

Question 7

Solve the inequality

$$|5x+4| < 6$$

Answer: In these types of questions, the trick is to square both sides to remove the absolute value. We can do this because both sides of the inequality are non-negative, and x^2 is monotonic increasing for $x \geq 0$.

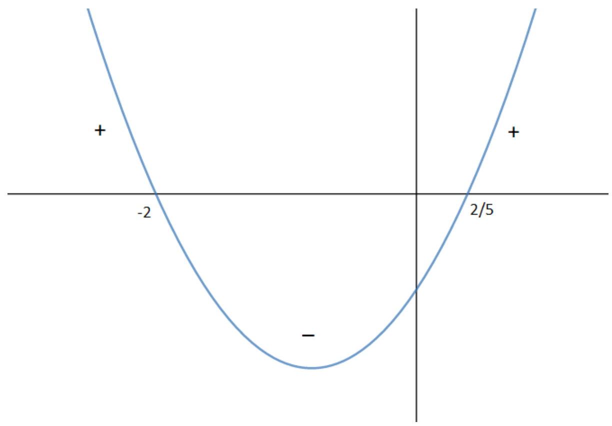
$$(5x+4)^2 < 36$$

Now, we simply follow the standard procedure, factorise the inequality by first moving the terms to one side.

$$(5x+4)^2 - 36 < 0$$

Factorising by difference of two squares,

$$(5x-2)(5x+10) < 0$$



Therefore our solution is the interval $(-2, 2/5)$

Question 8

Find the maximal domain and range for the real function defined by $f(x) = \sqrt{-x^2+5x}$

Answer: The maximal domain of f is the largest set of x values for which the function is defined. Since we can't have the square root of a negative number (in the reals),

$$-x^2 + 5x \geq 0$$

and thus, Domain$(f) = [0, 5]$

Since the square root function is monotonic increasing, we know f is largest when $h(x) = -x^2 + 5x$ is at maximum. And f is at a minimum when h is at minimum.

Now, $h'(x) = -2x + 5$. Stationary points occur when $h'(x) = 0$ i.e.

$$\begin{aligned} -2x + 5 &= 0 \\ x &= \frac{5}{2} \\ h\left(\frac{5}{2}\right) &= \frac{25}{4}. \end{aligned}$$

Since h is a concave down parabola, this is a maximum. Of course, the minimum is then 0, when the parabola cuts the axis. Hence, the respective min and max of $h(x) = -x^2 + 5x$ over Domain(f) are 0 and $\frac{25}{4}$.

Since both f and h are continuous functions, it follows that Range(f) $= [0, \frac{5}{2}]$

Question 9

Find values of a and b such that the piece-wise defined function

$$f(x) = \begin{cases} x^4 + 2 & x \leq 1 \\ ax + b & x > 1 \end{cases}$$

is differentiable.

Answer: For a function to be differentiable it must first be continuous. This means at every point, the left and right hand limits must exist and be equal. But of particular interest is that it must occur at the point $x = 1$, where the function's rule changes.

$$\begin{aligned} \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^-} f(x) \\ \lim_{x \to 1^+} x^4 + 2 &= \lim_{x \to 1^-} ax + b \\ (1)^4 + 2 &= a(1) + b \\ a + b &= 3 \end{aligned}$$

The second condition is that the left and right hand limits of the gradient functions must exist and be equal everywhere. Again, of particular interest is $x = 1$:

$$\begin{aligned}\lim_{x\to 1^+} f'(x) &= \lim_{x\to 1^-} f'(x)\\ \lim_{x\to 1^+} 4x^3 &= \lim_{x\to 1^-} a\\ 4(1)^3 &= a\\ a &= 4\end{aligned}$$

Solving these simultaneous equations, we find that values $a = 4$ and $b = -1$ make the function differentiable.