MATH1081 Lab Test 1

Sample Solutions

October 6, 2019

These answers were written and typed by Laeeque Jamdar and Ethan Brown. Please be ethical with this resource. If you appreciate this resource, please consider supporting us by coming to our events! Also, happy studying :)

We cannot guarantee that our answers are correct - please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are sometimes multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

Note: Any text presented `like this` is presented as required in Numbas syntax.

Question 1

In a class of 37 students:

- 21 study French,
- 17 study Physics,
- 10 study both English and French,
- 8 study both English and Physics,
- 9 study both French and Physics,
- 3 study all three subjects, and
- 4 study none of these subjects

How many people study English?

Firstly, of the 37 students, 4 study none of the three subjects, so there are 33 students who study at least one subject. Let the sets F, E, P represent the students studying in French, English and Physics respectively. By the inclusion-exclusion principle:

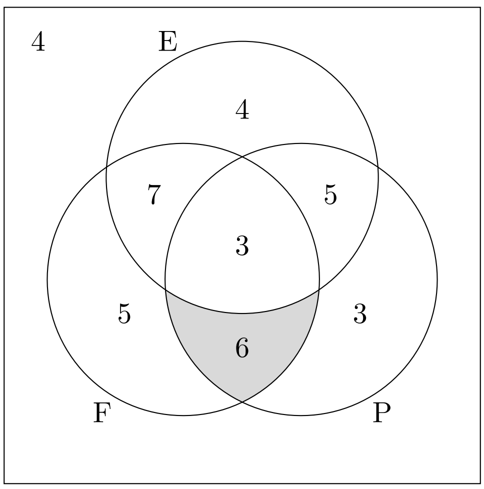
$$|F \cup E \cup P| = |F| + |E| + |P| - (|F \cap E| + |E \cap P| + |P \cap F|) + |F \cap E \cap P|$$
$$33 = 21 + |E| + 17 - (10 + 8 + 9) + 3$$
$$19 = |E|.$$

Writing E, F, P for the sets of students studying English, French and Physics respectively, evaluate $|E^c \cap (F^c \cup P^c)^c|$

Solution 1:

$$|E^c \cap (F^c \cup P^c)^c| = |E^c \cap F \cap P| \quad \text{(De Morgan's laws)}$$

Then we manually fill out the Venn Diagram:



Solution 2:

In this question we will use the fact that $|A - B| = |A| - |A \cap B|$. For justification, we have provided the proof below, but you will not need to prove this during the lab test.

$$\begin{aligned}
|A| &= |A \cap U| && \text{(Identity laws)} \\
&= |A \cap (B \cup B^c)| && \text{(Union with complement)} \\
&= |(A \cap B) \cup (A \cap B^c)| && \text{(Distributive laws)} \\
&= |A \cap B| + |A \cap B^c| - |A \cap B \cap A \cap B^c| && \text{(Inclusion-Exclusion)} \\
&= |A \cap B| + |A \cap B^c| - |\emptyset| && \text{(Intersection with complement)} \\
|A| &= |A \cap B| + |A - B| && \text{(Difference law)} \\
|A - B| &= |A| - |A \cap B|
\end{aligned}$$

We will call this Lemma 1.1. Now we can quickly solve the problem:

$$\begin{aligned}
|E^c \cap (F^c \cup P^c)^c| &= |E^c \cap (F \cap P)| && \text{(De Morgan's laws)} \\
&= |(F \cap P) \cap E^c| && \text{(Commutative law)} \\
&= |(F \cap P) - E|, && \text{(Difference law)} \\
&= |F \cap P| - |F \cap P \cap E| && \text{(Lemma 1.1)} \\
&= 9 - 3 \\
&= 6.
\end{aligned}$$

Question 2

For any integer k, let S_k be the set defined by:

$$S_k = \left\{ n \in \mathbb{Z} \mid k + 8 \le n \le \frac{1}{2}k + 16 \right\}.$$

Recall that the Numbas syntax for the set $\{a, b, c\}$ is `set(a,b,c)`.

For use in the following questions we've listed out the relevant sets:

$$\begin{aligned} S_1 &= \{9, 10, 11, 12, 13, 14, 15, 16\} \\ S_5 &= \{13, 14, 15, 16, 17, 18\} \\ S_1 \cap S_5 &= \{13, 14, 15, 16\} \\ S_1 \cap (S_5)^c &= \{9, 10, 11, 12\} \end{aligned}$$

Note that the last one is determined by considering the elements in S_1, that are *not* in S_5.

What is $S_1 - S_5$?

Relatively straightforward, but make sure you answer as a set.

$$\begin{aligned} S_1 - S_5 &= S_1 \cap (S_5)^c && \text{(Difference law)} \\ &= \texttt{set(9,10,11,12)}. \end{aligned}$$

Find $|P(S_1) \times P(S_5)|$.

The set product rule is that $|A \times B| = |A| \cdot |B|$, so:

$$|P(S_1) \times P(S_5)| = |P(S_1)| \cdot |P(S_5)|.$$

The power set's cardinality is $|P(S)| = 2^{|S|}$, so:

$$\begin{aligned} |P(S_1)| \cdot |P(S_5)| &= 2^{|S_1|} \cdot 2^{|S_5|} \\ &= 2^{|\{9,10,11,12,13,14,15,16\}|} \cdot 2^{|\{13,14,15,16,17,18\}|} \\ &= 2^8 \cdot 2^6 \\ &= 16384. \end{aligned}$$

Find $|P(S_1 \times S_5)|$.

Using the same rules in the opposite order:

$$\begin{aligned}|P(S_1 \times S_5)| &= 2^{|S_1 \times S_5|} \\ &= 2^{|S_1|\cdot|S_5|} \\ &= 2^{|\{9,10,11,12,13,14,15,16\}|\cdot|\{13,14,15,16,17,18\}|} \\ &= 2^{8\cdot 6} \\ &= 2^{48}.\end{aligned}$$

Find $|P(S_1) \cap P(S_5)|$

In this question we will use the fact that $P(A) \cap P(B) = P(A \cap B)$. For justification, we have provided the proof below, but you will not need to prove this during the lab test.

Note that $x \in P(A) \iff x \subseteq A$, by definition of a power set.

$$\begin{aligned}x \in (P(A) \cap P(B)) &\iff x \in P(A) \text{ and } x \in P(B) \\ &\iff x \subseteq A \text{ and } x \subseteq B \\ &\iff x \subseteq (A \cap B) \\ &\iff x \in P(A \cap B)\end{aligned}$$

So all the elements of $P(A) \cap P(B)$ are in $P(A \cap B)$ and vice versa. For the sake of the following questions we will call this Lemma 2.1.

$$\begin{aligned}|P(S_1) \cap P(S_5)| &= |P(S_1 \cap S_5)| \quad \text{(Lemma 2.1)} \\ &= 2^{|S_1 \cap S_5|} \\ &= 2^{|\{13,14,15,16\}|} \\ &= 2^4 \\ &= 16.\end{aligned}$$

Find $|P(S_1) \cup P(S_5)|$

The method for this question is similar to the previous one, but we first need to use the inclusion-exclusion principle to break up the union. We will use Lemma 2.1 from the question above:

$$P(A) \cap P(B) = P(A \cap B).$$

$$\begin{aligned}
|P(S_1) \cup P(S_5)| &= |P(S_1)| + |P(S_5)| - |P(S_1) \cap P(S_5)| && \text{(Inclusion-Exclusion)} \\
&= |P(S_1)| + |P(S_5)| - |P(S_1 \cap S_5)| && \text{(Lemma 2.1)} \\
&= 2^{|S_1|} + 2^{|S_5|} - 2^{|S_1 \cap S_5|} \\
&= 2^{|\{9,10,11,12,13,14,15,16\}|} + 2^{|\{13,14,15,16,17,18\}|} - 2^{|\{13,14,15,16\}|} \\
&= 2^8 + 2^6 - 2^4 \\
&= 304.
\end{aligned}$$

Find $|P(S_1) - P(S_5)|$

This question will combine our knowledge of sets. We will use Lemma 1.1 from Question 1:

$$|A - B| = |A| - |A \cap B|,$$

and Lemma 2.1 from the question above:

$$P(A) \cap P(B) = P(A \cap B).$$

$$\begin{aligned}
|P(S_1) - P(S_5)| &= |P(S_1)| - |(P(S_1) \cap P(S_5))| && \text{(Lemma 1.1)} \\
&= |P(S_1)| - |P(S_1 \cap S_5)| && \text{(Lemma 2.1)} \\
&= 2^{|S_1|} - 2^{|S_1 \cap S_5|} \\
&= 2^{|\{9,10,11,12,13,14,15,16\}|} - 2^{|\{13,14,15,16\}|} \\
&= 2^8 - 2^4 \\
&= 240.
\end{aligned}$$

Question 3

Suppose $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and that the function $f : S \to S$ is given by:

$$f(x) = 5x^2 + 10x + 7 \pmod{11}.$$

Let $T = \{2, 6\}$.
Recall that the Numbas syntax for the set $\{a, b, c\}$ is `set(a,b,c)`.

Constructing a reference table:

x	0	1	2	3	4	5	6	7	8	9	10
$f(x)$	7	0	3	5	6	6	5	3	0	7	2

What is $f(T)$?

From the table, $f(2) = 3$ and $f(6) = 5$. Therefore $f(\{2, 6\}) = \{3, 5\}$, giving us

`set(3, 5)`.

What is $f^{-1}(T)$?

$f^{-1}(T)$ is the set of elements that map to an element in T. From the table, $f^{-1}(2) = \{10\}$ and $f^{-1}(6) = \{4, 5\}$. Therefore $f^{-1}(\{2, 6\}) = \{4, 5, 10\}$, giving us

`set(4, 5, 10)`.

Question 4

Complete the following:

$$\frac{3}{k-2} - \frac{5}{k} + \frac{2}{k+1} = \frac{Ak+B}{(k-2)(k)(k+1)},$$

where $A =$ __ and $B =$ __.

$$\begin{aligned}\frac{3}{k-2} - \frac{5}{k} + \frac{2}{k+1} &= \frac{3(k)(k+1) - 5(k-2)(k+1) + 2(k-2)(k)}{(k-2)(k)(k+1)} \\ &= \frac{3k^2 + 3k - 5k^2 + 5k + 10 + 2k^2 - 4k}{(k-2)(k)(k+1)} \\ &= \frac{4k+10}{(k-2)(k)(k+1)},\end{aligned}$$

so $A = 4$ and $B = 10$.

Hence find a closed form for $\displaystyle\sum_{k=3}^{n} \frac{2k+5}{(k-2)(k)(k+1)}$.

In this question we change the bounds of summation frequently. Note that:

$$\sum_{k=1}^{n-1} \frac{1}{k+1} = \sum_{k=2}^{n} \frac{1}{k} = \sum_{k=3}^{n+1} \frac{1}{k-1}$$

and that

$$\sum_{k=2}^{n} \frac{1}{k} = \frac{1}{2} + \frac{1}{3} + \left(\sum_{k=4}^{n-2} \frac{1}{k}\right) + \frac{1}{n-1} + \frac{1}{n}$$

$$\begin{aligned}
&\sum_{k=3}^{n} \frac{2k+5}{(k-2)(k)(k+1)} \\
&= \frac{1}{2}\left[\sum_{k=3}^{n}\left(\frac{3}{k-2} - \frac{5}{k} + \frac{2}{k+1}\right)\right] \\
&= \frac{1}{2}\left[\sum_{k=3}^{n}\frac{3}{k-2} - \sum_{k=3}^{n}\frac{5}{k} + \sum_{k=3}^{n}\frac{2}{k+1}\right] \\
&= \frac{1}{2}\left[\sum_{k=1}^{n-2}\frac{3}{k} - \sum_{k=3}^{n}\frac{5}{k} + \sum_{k=4}^{n+1}\frac{2}{k}\right] \\
&= \frac{1}{2}\left[\left(\frac{3}{1} + \frac{3}{2} + \frac{3}{3} + \sum_{k=4}^{n-2}\frac{3}{k}\right) - \left(\frac{5}{3} + \sum_{k=4}^{n-2}\frac{5}{k} + \frac{5}{n-1} + \frac{5}{n}\right) + \left(\sum_{k=4}^{n-2}\frac{2}{k} + \frac{2}{n-1} + \frac{2}{n} + \frac{2}{n+1}\right)\right] \\
&= \frac{1}{2}\left[\frac{3}{1} + \frac{3}{2} + \frac{3}{3} + \frac{-5}{3} + \frac{-5}{n-1} + \frac{-5}{n} + \frac{2}{n-1} + \frac{2}{n} + \frac{2}{n+1} + \sum_{k=4}^{n-2}\frac{3-5+2}{k}\right] \\
&= \frac{1}{2}\left[\frac{23}{6} + \frac{-3}{n-1} + \frac{-3}{n} + \frac{2}{n+1}\right]
\end{aligned}$$

Question 5

Answer the following, given that

$$3780 = 2^2 \cdot 3^3 \cdot 5 \cdot 7, \qquad 40020750 = 2 \cdot 3^3 \cdot 5^3 \cdot 7^2 \cdot 11^2, \qquad 45 = 3^2 \cdot 5.$$

What is $\operatorname{lcm}(3780, 40020750)$?

Using their prime factorisations, we take the highest of each exponent:

$$\begin{aligned}
\operatorname{lcm}(3780, 40020750) &= 2^{\max(2,1)} \cdot 3^{\max(3,3)} \cdot 5^{\max(1,3)} \cdot 7^{\max(1,2)} \cdot 11^{\max(0,2)} \\
&= 2^2 \cdot 3^3 \cdot 5^3 \cdot 7^2 \cdot 11^2 \\
&= 80041500.
\end{aligned}$$

What is $\gcd(40020750, 45)$?

Similary, we take the lowest of each exponent:

$$\begin{aligned}
\gcd(40020750, 45) &= 2^{\min(1,0)} \cdot 3^{\min(3,2)} \cdot 5^{\min(1,1)} \cdot 7^{\min(2,0)} \cdot 11^{\min(2,0)} \\
&= 3^2 \cdot 5^1 \\
&= 45
\end{aligned}$$

Question 6

When evaluating modulo m, give your answer in its lowest non-negative form - that is, as an element of $\{0, 1, 2, ...m-1\}$

Evaluate $5^{108} \pmod{11}$.

Since 11 is prime, we can use Fermat's little theorem to reduce the required working:

$$5^{10} \equiv 1 \pmod{11}$$

So substituting it in, we get

$$\begin{aligned}
5^{108} &\equiv (5^{10})^{10} \cdot 5^8 \pmod{11} \\
&\equiv 5^8 \pmod{11} \\
&\equiv 25^4 \pmod{11} \\
&\equiv (22+3)^4 \pmod{11} \\
&\equiv 3^4 \pmod{11} \\
&\equiv 81 \pmod{11} \\
&\equiv 4 \pmod{11}.
\end{aligned}$$

Evaluate $2^{178} \pmod{18}$

Since $2^4 = 16 \equiv -2 \pmod{18}$, we can quickly reduce 2^{178} to

$$\begin{aligned}
2^{178} &\equiv (2^4)^{44} \cdot 2^2 \pmod{18} \\
&\equiv (-2)^{44} \cdot 2^2 \pmod{18} \\
&\equiv 2^{44} \cdot 2^2 \pmod{18} \\
&\equiv (2^4)^{11} \cdot 2^2 \pmod{18} \\
&\equiv -2^{11} \cdot 2^2 \pmod{18} \\
&\equiv -(2^4)^3 \cdot 2 \pmod{18} \\
&\equiv -(-2)^3 \cdot 2 \pmod{18} \\
&\equiv 16 \pmod{18}.
\end{aligned}$$

Question 7

Solve each of the following modular arithmetic equations, giving your answer as a set of all possible solutions in the given modulus.

- If there are no solutions, enter `set()`.
- If there is one solution, say 1, enter `set(1)`.
- If there are multiple solutions, say 1 and 2, enter `set(1, 2)`.

When evaluating in modulo m, give each answer in its lowest non-negative form - that is, as an element of $\{0, 1, 2, ...m-1\}$.

Solve $123x \equiv 3 \pmod{217}$

This is equivalent to solving $123x + 217y = 3$.
We are going to first solve $123x + 217y = 1$, and then multiply our solution by 3.
Using the Euclidean Algorithm:

$$\begin{aligned}
\underline{217} &= 1 \cdot \underline{123} \ + \underline{94} \\
\underline{123} &= 1 \cdot \underline{94} \ + \underline{29} \\
\underline{94} &= 3 \cdot \underline{29} \ + \underline{7} \\
\underline{29} &= 4 \cdot \underline{7} \ + \underline{1} \\
\underline{7} &= 7 \cdot \underline{1} \ + \underline{0}
\end{aligned}$$

We rearrange the second last equation and work our way up again:

$$\begin{aligned}
1 &= \underline{29} - 4 \cdot \underline{7} \\
&= \underline{29} - 4(\underline{94} - 3 \cdot \underline{29}) \\
&= 13 \cdot \underline{29} - 4 \cdot \underline{94} \\
&= 13(\underline{123} - \underline{94}) - 4 \cdot \underline{94} \\
&= 13 \cdot \underline{123} - 17 \cdot \underline{94} \\
&= 13 \cdot \underline{123} - 17(\underline{217} - \underline{123}) \\
&= 30 \cdot \underline{123} - 17 \cdot \underline{217}
\end{aligned}$$

So thus far we have the solution $(x = 30, y = -17)$ to the equation $123x + 217y = 1$.
To solve for $123x + 217y = 3$, we need to multiply our solution by 3

$$\begin{aligned}
x &\equiv 30 \cdot 3 \pmod{217} \\
x &\equiv 90 \pmod{217}
\end{aligned}$$

Our final solution is:

```
set(90)
```

Solve $208x \equiv 5 \pmod{663}$

This is equivalent to solving $208x + 663y = 5$.
The GCD of 208 and 663 is 13, but that is not a factor of 5, so there are no solutions for x that can work
Our final solution is:

```
set()
```

Solve $484x \equiv 20 \pmod{1340}$.

This is equivalent to solving $484x + 1340y = 20$.
Since all three numbers are multiples of 4, we can divide them all by 4 and keep in mind that we'll end up with 4 solutions in the end.
So now we are solving $121x \equiv 5 \pmod{335}$, or $121x + 335y = 5$.
We are going to first solve $121x + 335y = 1$, and then multiply our solution by 5.
Using the Euclidean Algorithm:

$$\begin{aligned}
\underline{335} &= 2 \cdot \underline{121} \; + \underline{93} \\
\underline{121} &= 1 \cdot \underline{93} \; + \underline{28} \\
\underline{93} &= 3 \cdot \underline{28} \; + \underline{9} \\
\underline{28} &= 3 \cdot \underline{9} \; + \underline{1} \\
\underline{9} &= 9 \cdot \underline{1} \; + \underline{0}.
\end{aligned}$$

We rearrange the second last equation and work our way up again:

$$\begin{aligned}
1 &= \underline{28} - 3 \cdot \underline{9} \\
&= \underline{28} - 3(\underline{93} - 3 \cdot \underline{28}) \\
&= 10 \cdot \underline{28} - 3 \cdot \underline{93} \\
&= 10(\underline{121} - \underline{93}) - 3 \cdot \underline{93} \\
&= 10 \cdot \underline{121} - 13 \cdot \underline{93} \\
&= 10 \cdot \underline{121} - 13(\underline{335} - 2 \cdot \underline{121}) \\
&= 36 \cdot \underline{121} - 13 \cdot \underline{335}.
\end{aligned}$$

So thus far we have the solution $(x = 36, y = -13)$ to the equation $121x + 335y = 1$.

To solve for $121x + 335y = 5$, we need to multiply our solution by 5

$$x \equiv 36 \cdot 5 \pmod{335}$$
$$x \equiv 180 \pmod{335}$$

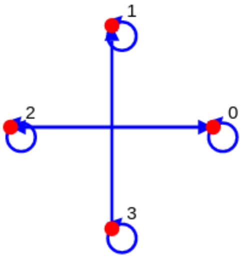
In truth, there are infinite solutions to $121x + 335y = 5$, of the form $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + k \begin{pmatrix} 335 \\ -121 \end{pmatrix}$ for all integers k, and where $\begin{pmatrix} x \\ y \end{pmatrix}$ is one of the solutions. Since we divided everything by 4 early on, x has 4 solutions in the range $[0, 1340)$ (the original modulus) each separated by 335 (the divided modulus).

Our final solution is:

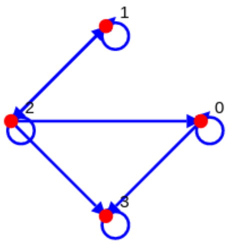
```
set(180, 515, 850, 1185)
```

Question 8

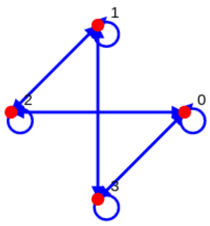
For each of the adjacency graphs below, indicate whether whether they are reflexive, symmetric and/or transitive relations.
Marks will be deducted for incorrect selection, but the minimum possible total mark for this question is 0.



All nodes have self-loops, so the relation is **reflexive**
not all edges are bi-directional, so the relation is **not symmetric**
Every pair of nodes satisfies the transitive property, so the relation is **transitive**



All nodes have self-loops, so the relation is **reflexive**
Not all edges are bi-directional, so the relation is **not symmetric**
$1 \to 2$ and $2 \to 0$ but $1 \not\to 0$ so the relation is **not transitive**



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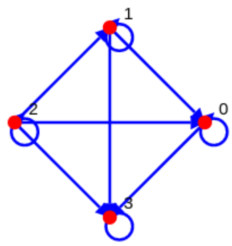

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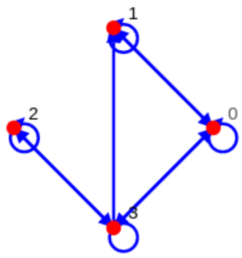

All nodes have self-loops, so the relation is **reflexive**
Not all edges are bi-directional, so the relation is **not symmetric**
$2 \to 3$ and $3 \to 1$ but $2 \not\to 1$ so the relation is **not transitive**