UNSW Mathematics Society

(Higher) Mathematics 1A

Algebra

Presented by: Donald Tang, Shayekh Rouf

Term 1, 2020

Table of contents

Introduction to Vectors

OH YEAH

Introduction to Vectors - Addition and Subtraction

Addition of Vectors

If we want to add vectors, say **x** and **y**, in any dimension $\mathbb{R}^n$,then it's as easy as simply adding together each component of each vector. The same concept applies with subtraction. The result is always a new vector. We can do this as follows:

Example 1

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix} \text{ or } \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} - \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \\ \vdots \\ x_n - y_n \end{pmatrix}$$

Note:

Vector addition is **commutative** and **associative**.

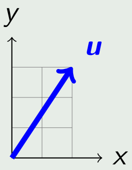
Geometric Interpretation - Addition

Visualising Vectors in Space

Vectors can be interpreted as the **net movement** of a certain object. For example, consider the vector $\mathbf{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ in $\mathbb{R}^2$.

Example 2

We can interpret this as 2 units to the right in the x-axis and 3 units up in the y-axis. This would look something like below:



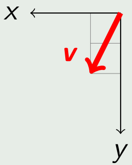
This thought process applies to $\mathbb{R}^3$ and higher dimensions, though this may be harder to mentally visualise.

Visualising Negative Vectors in Space

Negative values mean that we need to consider movement in the opposite direction. For example, consider the vector $\mathbf{v} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$.

Example 3

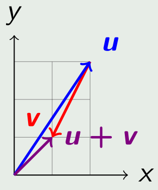
We can interpret this as 1 unit to the left in the x-axis and 2 units down in the y-axis. This would look something like below:



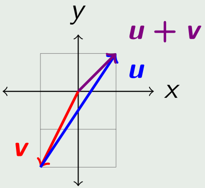
Visualising Vector Addition in Space

We add vectors geometrically from tip to tail, so $\mathbf{v} + \mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ can be interpreted as moving $\mathbf{v}$ so its tail meets the tip of $\mathbf{u}$ (or vice versa) and drawing a vector from the origin to the resultant point.

Example 4



or



Note:

Vector subtraction is the same as vector addition, but we flip the direction of the vector that we are subtracting, then add it.

Introduction to Vectors - Multiplication

Scalar multiplication of Vectors

If we want to multiply our vector $\mathbf{x}$ in any $\mathbb{R}^n$ by some real number λ, then we just multiply each component of the vector by λ. This looks like:

Example 5

$$\lambda \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{pmatrix} \text{ or } \lambda \left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \right) = \begin{pmatrix} \lambda x_1 + \lambda y_1 \\ \lambda x_2 + \lambda y_2 \\ \vdots \\ \lambda x_n + \lambda y_n \end{pmatrix}$$

Note:

Scalar multiplication is **commutative** and **associative**.

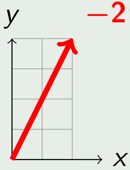
Geometric Interpretation - Multiplication

Visualising Vectors in Space

Multiplication with some λ can be interpreted as multiplying the length of the vector by λ. This can preserve the vector's direction ($\lambda > 0$), change its direction to the opposite ($\lambda < 0$) or produce the zero vector ($\lambda = 0$). Consider the vector $-2\mathbf{v} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

Example 6

We can interpret this as doubling the length of $\mathbf{v}$ and changing its direction to the opposite. This would look something like below:



Vector Length

Length

The length of a vector can be determined through many iterations of Pythagoras' theorem through applications in each dimension. Or, we can just use the following formula (which is derived from many iterations of Pythagoras' theorem in each dimension):

$$|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$$

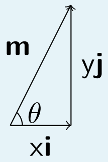
Also note that:

$$|\lambda\mathbf{x}| = |\lambda||\mathbf{x}| = |\lambda|\sqrt{x_1^2 + x_2^2 + ... + x_n^2}$$

Vector Components in $\mathbb{R}^2$

Angles and Magnitudes

Given a vector $\mathbf{m}$ of magnitude $|\mathbf{m}|$ with an angle of θ to the x-axis, we can draw it up as a sum of its x- and y-components:



Using some epic trigonometry, it is clear that:

$$x = |\mathbf{m}| \cos\theta \text{ and } y = |\mathbf{m}| \sin\theta$$

Which means that $\mathbf{m}$ as a sum of its components is just:

$$\mathbf{m} = |\mathbf{m}| cos\theta \mathbf{i} + |\mathbf{m}| sin\theta \mathbf{j}$$

Lines! Cartesian and Parametric

Parametric Lines

A straight line L is just a really long vector, so we can represent any line we want in terms of some real number parameter (e.g. λ). We just need a point **a** on the line and a vector **v** which is parallel to the line.

$$L = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a} + \lambda \mathbf{v}\}$$

Alternatively, we can represent it in some weird Cartesian form:

$$L = \left\{\mathbf{x} \in \mathbb{R}^n : \frac{x_1 - a_1}{b_1} = \frac{x_2 - a_1}{b_2} = \ldots = \frac{x_n - a_n}{b_n}\right\}$$

Where $(a_1, \ldots, a_n)^T$ is a point on the line and $(b_1, \ldots, b_n)^T$ is the direction of the line.

Lines! Cartesian and Parametric

Note:

Parametric vectors can have multiple values depending on what your choice of $\mathbf{a}$, $\mathbf{v}$ or λ is, so you can have several valid answers.

Lines! Converting Between Cartesian and Parametric

Cartesian to Parametric Lines

From the Cartesian equation, we can see that each term is equal, so we can actually set them all as some parameter.

Let $\mu = \frac{x_k - a_k}{b_k}$ for some integer $1 \leq k \leq n$.

Making x_k the subject, $x_k = a_k + \mu b_k$.

Thus, a parametric form of our line can be given by:

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \mu \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Lines! Converting Between Cartesian and Parametric

Parametric to Cartesian Lines

We can also go the other way, from parametric to Cartesian.

Example 7 Let's consider a specific example this time:

$\mathbf{v} = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 3 \\ 5 \end{pmatrix}$. We can equate the components $x_1,\ x_2$ and x_3 of $\mathbf{v}$ such that: $x_1 = 2 + 5\lambda,\ x_2 = 4 + 3\lambda,\ x_3 = 3 + 5\lambda$.

If we rearrange each component to be in terms of λ, then equate them all, we can find a Cartesian equation for our line:

$$\frac{x_1 - 2}{5} = \frac{x_2 - 4}{3} = \frac{x_3 - 3}{5}.$$

Lines! An Interval Between Two Points

Intervals

We can use our parametric representation to represent an interval $\mathbf{I}$ between two points, let's say $\mathbf{a}$ and $\mathbf{b}$. This can be done as follows:

$$\mathbf{I} = \lambda\mathbf{a} + (1 - \lambda)\mathbf{b}$$

For $0 \leq \lambda \leq 1$ - A fact we can use for many geometric applications.

Note:

When $\lambda = 0$, all the 'weight' of the vector is at $\mathbf{b}$, and when $\lambda = 1$, all the 'weight' of the vector is at $\mathbf{a}$.

Planes

Properties of Planes

When you really think about it, a plane is just the **span** of all points you get from adding two non-parallel, non-zero vectors.

In other words, it's a possible **linear combination** of the vectors.

This means we can represent planes in a bunch of ways involving vectors. Planes also have some cool properties.

Planes can intersect with other planes. This intersection can be another plane if they're both parallel (it's actually the exact same plane) or the intersection can occur in the form of a line.

Span

The span of two vectors can be formally denoted by:

$$\text{span}(\mathbf{v}_1,\mathbf{v}_2) = \{x : x = \lambda v_1 + \mu v_2 \text{ where } \lambda, \mu \in \mathbb{R}\}$$

Planes

Cartesian Hyperplanes

Hyperplanes in $\mathbb{R}^n$ can be expressed in Cartesian form:

$$a_1x_1 + a_2x_2 + ... + a_nx_n = d$$

Hyperplanes in $\mathbb{R}^3$ are regular 2D planes, whereas hyperplanes in $\mathbb{R}^4$ are three-dimensional!

We'll most commonly be looking at hyperplanes in three dimensions, so it is likely that you'll see either:

$$ax + by + cz = d$$

or

$$a_1x_1 + a_2x_2 + a_3x_3 = b$$

or a similar equation for the 2D planes we are familiar with.

Planes

Parametric Planes

A parametric way to express a 2D plane $\prod$ in $\mathbb{R}^n$ is:

$$\prod : \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} + \lambda \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \mu \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \quad \textbf{or} \quad \prod : \mathbf{c} + \lambda \mathbf{x} + \mu \mathbf{y}$$

For some real numbers λ and μ.

Note:

$\mathbf{x}$ and $\mathbf{y}$ are non-parallel, non-zero vectors that are parallel to the plane, and $\mathbf{c}$ is the position vector for some point on the plane. If the plane passes through the origin, it's easiest to just let $\mathbf{c} = 0$.

Converting from Parametric to Cartesian form

Example 8

We are given the parametric form of a plane:

$$\Pi = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ -4 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 0 \\ 7 \end{pmatrix}$$

So,

$$x_1 = 3 + \lambda - 2\mu \ (1), \ \ x_2 = 4\lambda \ (2) \text{ and } \ x_3 = 9 - 4\lambda + 7\mu \ (3).$$

We can do $7 \times (1) + 2 \times (3)$ to get: $7x_1 + 2x_3 = 39 - \lambda$
Substituting (2) into this, we can see that: $7x_1 + 2x_3 = 39 - \frac{1}{4}x_2$
So a Cartesian equation of this plane is given by:

$$28x_1 + x_2 + 8x_3 = 156$$

Converting from Cartesian to Parametric form

Example 9

We are given the Cartesian form of a plane:

$$\prod : 4x + 3y - 9z = 3$$

To convert this to some parametric form, we parametrise two chosen variables. In this case, we will let $x = \lambda_1$ and $y = \lambda_2$, which means that:

$$z = \frac{4}{9}\lambda_1 + \frac{1}{3}\lambda_2 - \frac{1}{3}$$

So we can see that:

$$\prod = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \frac{4}{9}\lambda_1 + \frac{1}{3}\lambda_2 - \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{3} \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 0 \\ \frac{4}{9} \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ \frac{1}{3} \end{pmatrix}$$

Vector Geometry

Dot Product

The Dot Product

The dot product of two vectors **x** and **y** can be written as

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$$

This is also known as the scalar product, since the result is scalar.

The dot product is also equivalent to the expression:

$$\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}||\mathbf{y}| \cos\theta$$

Where θ is the angle between two vectors. This means we can use the dot product to the find the angle between two vectors, and if vectors are orthogonal! Cool.

Dot Product

Properties of the Dot Product

The dot product also has some nice properties which we can use:

- **Commutative Law** - $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- **Distributive Law** - $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
- $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
- $(\lambda \mathbf{a}) \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b})$

We can also derive some important results using the dot product, notably:

- $-|\mathbf{a}||\mathbf{b}| \leq \mathbf{a} \cdot \mathbf{b} \leq |\mathbf{a}||\mathbf{b}|$ (Cauchy-Schwarz Inequality)
- $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$ (Triangle Inequality)

Orthogonality

How do I know I'm RIGHT? (poor joke my bad)

If the dot product is equal to 0, then that must mean $\cos\theta = 0$, implying $\theta = \frac{\pi}{2}$. Using this fact, we know two vectors are orthogonal when their dot product equals zero.

An important definition is that of the **orthonormal** set of vectors. These describe sets of vectors in $\mathbb{R}^n$ that are of unit length and mutually orthogonal to each other. e.g. in $\mathbb{R}^3$ the standard basis vectors are a set of orthonormal vectors:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Dot Product

Projections

Projections

Projections are given by the simple equation:

$$\text{proj}_{\mathbf{v}}(\mathbf{u}) = \frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v}$$

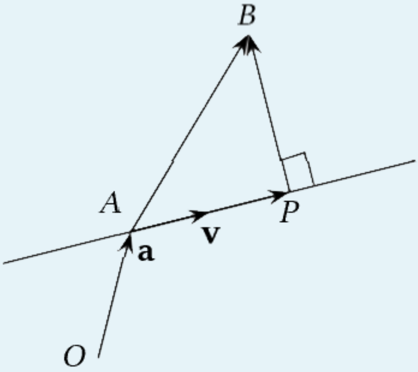
Where we are projecting **u** onto **v**.

The projection is pretty much what we would obtain if we were sliding the vector **u** along a line perpendicular to **v** and touching the tip of **u** and forcing **u** to face the same direction as **v**. This lets us find a whole bunch of features, like shortest distances.

Shortest Distance

Distance between a point and line in $\mathbb{R}^2$ and $\mathbb{R}^3$

We have a point B and we want to find the shortest distance between B and $\mathbf{x} = \mathbf{a} + \lambda\mathbf{v}$. Here is a diagram of the situation:



Shortest Distance

Distance between a point and line in $\mathbb{R}^2$ and $\mathbb{R}^3$

Our shortest distance is just the magnitude of $\vec{PB}$. We can also note that $\vec{AP}$ is the projection of $\vec{AB}$ onto $\mathbf{v}$.
In other words,

$$\vec{AP} = \text{proj}_{\mathbf{v}}(\mathbf{b} - \mathbf{a})$$

We can find the distance by finding an expression for $\vec{PB}$ in terms of our known quantities.
Using some vector magic, we note that:

$$\vec{PB} = \mathbf{b} - \mathbf{a} - \text{proj}_{\mathbf{v}}(\mathbf{b} - \mathbf{a})$$

So our shortest distance is simply the magnitude of this new vector.

Shortest Distance

Example 10 - MATH1141 June 2015, Q4i

Find the shortest distance between the point B(1,2,0) to the line with parametric vector equation:

$$\begin{pmatrix}1\\0\\1\end{pmatrix} + \lambda \begin{pmatrix}0\\1\\1\end{pmatrix}, \lambda \in \mathbb{R}$$

We'll start by making our vector $\mathbf{b} - \mathbf{a}$:

$$\mathbf{b} - \mathbf{a} = \begin{pmatrix}1\\2\\0\end{pmatrix} - \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}0\\2\\-1\end{pmatrix}$$

Shortest Distance

Example 10 - MATH1141 June 2015, Q4i (ct'd)

Next, we'll find the projection of $\mathbf{b} - \mathbf{a}$ onto $\mathbf{v}$:

$$\text{proj}_{\mathbf{v}}(\mathbf{b} - \mathbf{a}) = \frac{\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

So our vector $\vec{PB}$ is:

$$\vec{PB} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{3}{2} \\ -\frac{3}{2} \end{pmatrix}$$

And $|\vec{PB}| = \sqrt{(0)^2 + (\frac{3}{2})^2 + (-\frac{3}{2}^2)} = \frac{3\sqrt{2}}{2}$ units. This is the shortest distance between the point and our line!

The Cross Product

Definition

The cross product is a useful operation that allows us to find a vector that is orthogonal to two vectors to which we are applying it to. It is given by:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - b_2 a_3 \\ b_1 a_3 - a_1 b_3 \\ a_1 b_2 - b_1 a_2 \end{pmatrix}$$

Note:

The cross product is only applicable if both vectors are in $\mathbb{R}^3$ and has no definition outside of it. The direction of the resultant vector can be ascertained using the right hand rule.

The Cross Product

Properties of the Cross Product

- $\mathbf{a} \times \mathbf{a} = 0$
- $\mathbf{a} \times \mathbf{b} = -\,\mathbf{b} \times \mathbf{a}$
- $(\mathbf{a}\lambda) \times \mathbf{b} = \mathbf{a} \times (\mathbf{b}\lambda) = \lambda(\mathbf{a} \times \mathbf{b})$
- **Distributive Law** - $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
- $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3,\ \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1,\ \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2$, where $\mathbf{e}_k$ is one of the standard basis vectors of $\mathbb{R}^3$
- $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

Note:

The cross product is **NOT** associative. This is demonstrated by the fact that:

$(\mathbf{e}_1 \times \mathbf{e}_2) \times \mathbf{e}_2 = -\mathbf{e}_1$, but

$\mathbf{e}_1 \times (\mathbf{e}_2 \times \mathbf{e}_2) = 0$

The Cross Product

Normal to a Plane

Recalling that the parametric form of a plane in $\mathbb{R}^3$ can be given by:

$$\prod : \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \lambda_1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \lambda_2 \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \textbf{ or } \prod : \mathbf{c} + \lambda_1 \mathbf{x} + \mu_1 \mathbf{y}$$

We can deduce that a normal to the plane itself needs to be perpendicular to both vectors $\mathbf{x}$ and $\mathbf{y}$ that are parallel to the plane. In this case, this means that the normal $\mathbf{n}$ can be given by:

$$\mathbf{n} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \textbf{ or } \mathbf{n} = \mathbf{x} \times \mathbf{y}$$

Areas

Area of a Parallelogram

The area of a parallelogram is simply the magnitude of the cross product of its sides.

Areas

Example 11 - MATH1131 June 2012 Q2iv

The points C and D have position vectors:

$$\mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \ \mathbf{d} = \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}$$

Find the area of the parallelogram with adjacent sides OC and OD, where O is the origin.

Clearly, $A = |\mathbf{c} \times \mathbf{d}| = \left| \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix} \right|$

$= \left| \begin{pmatrix} 7 \\ 7 \\ -7 \end{pmatrix} \right|$

$= \sqrt{7^2 + 7^2 + (-7)^2} = \sqrt{147} = 7\sqrt{3}$

Volumes

The Scalar Triple Product

This cool-sounding product is defined as:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

And has the following properties:

- $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
- $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b})$
- $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \times \mathbf{a}) \cdot \mathbf{b} = 0$

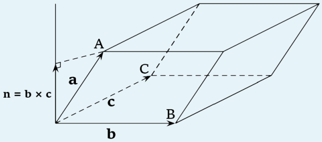
Note:

The cross product must ALWAYS be performed first because the dot product is a scalar, and the cross product of a vector and scalar has no meaning.

Volumes

Parallelepipeds

A parallelepiped is a 3 dimensional object. All its faces are parallelograms (note: rectangles are parallelograms!). Its volume V is given by the area r of its base times its perpendicular height h.



The length of the projection of **a** onto **n** is the perpendicular height, and the area r of the base is given by $|\mathbf{b} \times \mathbf{c}|$.

Thus, $h = \frac{|\mathbf{a} \cdot \mathbf{n}|}{|\mathbf{n}|} = \frac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{b} \times \mathbf{c}|} = \frac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{r}$.

Rearranging,

$$h \times r = V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|.$$

Planes

Point-Normal Form

We've learnt about Cartesian and parametric forms of a plane, but what's a point-normal form? A point normal form is just:

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \cdot \left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \right) = 0$$

Where the vector **n** is a normal to the plane and the vector **a** is a point on the plane itself.

This is a somewhat intuitive formula, because we recall that the dot product of a vector with its normal is equal to 0.

Expanding this form, we find a Cartesian equation for the plane:

$$n_1 x_1 + n_2 x_2 + n_3 x_3 = n_1 a_1 + n_2 a_2 + n_3 a_3$$

Playing with Planes

Converting between the three forms

- **Point-Normal to Cartesian** - Simply expand.
- **Cartesian to Point-Normal** - With some careful inspection we can see that the coefficients of $x_1,\ x_2$ and x_3 are the coordinates for a normal to the plane. To find a point on the plane itself, just let 2 variables equal 0. You know the rest.
- **Parametric to Point-Normal** - If a plane is defined by $\prod_1 = \mathbf{a} + \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2$, then to find a normal we simply find $\mathbf{v}_1 \times \mathbf{v}_2$. By definition, $\mathbf{a}$ is a point on the plane, so we can now put the ingredients into point-normal form.

Playing with Planes

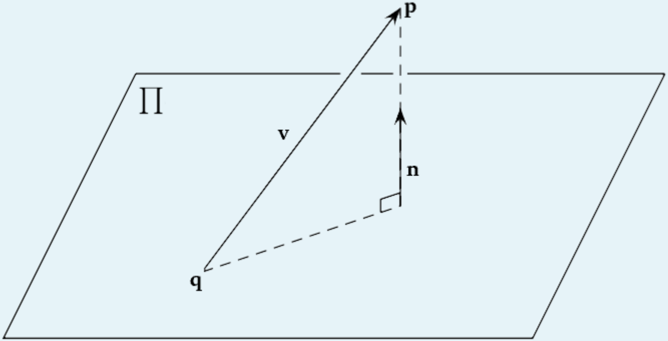
Converting between the three forms (ct'd)

- **Parametric to Cartesian** - The method with least algebra is to just find the point-normal form, then expand to find the Cartesian form.
- **Point-Normal to Parametric** - This involves expanding the point-normal form into Cartesian form, and then into parametric form.

Shortest Distance

Shortest Distance Between a Plane and a Line

The process is similar to finding the shortest distance between a point and a line. Find the normal to a plane through the cross product of its parallel vectors or otherwise, then perform vector magic as before. Here's a diagram illustrating the situation:



Shortest Distance

Example 12 - MATH1131 June 2015 Q3iiic

Find the shortest distance d between the point $P(4,2,2)$ and the plane $\prod : x - 3y + 2z = 1$.
We can easily find a normal here by recalling that the variable coefficients represent such a normal, so we know that:

$$\mathbf{n} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$$

A point $\mathbf{q}$ on the plane can be found if we let $y = z = 0$, so:

$$\mathbf{q} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

So a vector $\mathbf{v}$ from $\prod$ to P is just:

$$\mathbf{v} = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$$

Shortest Distance

Example 12 - MATH1131 June 2015 Q3iiic (ct'd)

We have everything we need, so we can just plug our values into our formula.

$$d = |proj_{\mathbf{n}}(\mathbf{v})| = \left|\frac{\mathbf{v}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n}\right| = \frac{|\mathbf{v}\cdot\mathbf{n}|}{|\mathbf{n}|^2}|\mathbf{n}| = \frac{|\mathbf{v}\cdot\mathbf{n}|}{|\mathbf{n}|}$$

$$= \frac{\left|\begin{pmatrix}3\\2\\2\end{pmatrix}\cdot\begin{pmatrix}1\\-3\\2\end{pmatrix}\right|}{\left|\begin{pmatrix}1\\-3\\2\end{pmatrix}\right|} = \frac{1}{\sqrt{14}}$$

Shortest Distance

Shortest Distance Between 2 Lines in $\mathbb{R}^3$ (MATH1141 ONLY)

- If the lines are parallel, then just find a point on one line, then do what we did for the distance between a line and point in $\mathbb{R}^2$.
- If the lines are not parallel, find their cross product (to find a vector perpendicular to both), then find the magnitude of the projection of any vector with one end on one line, and the other end on the other line.

Complex Numbers

Complex Numbers

Complex Arithmetic

Addition and Subtraction

Ironically, adding and subtracting complex numbers is anything but complex. Like adding together vector components, we add together the real components and imaginary components separately.

Consider two complex numbers, z and ω, such that:

$$z = a + bi \text{ and } \omega = c + di, \text{ where } a, b, c, d \in \mathbb{R}$$

- $z + \omega = a + c + (b + d)i$
- $z - \omega = a - c + (b - d)i$

Complex Arithmetic

Multiplication and Division

Multiplying and dividing complex numbers is similar to multiplying real numbers, but keeping in mind that $i^2 = -1$.

- $z\omega = (a+bi)(c+di) = ac+adi+bci-bd = ac-bd+(ad+bc)i$
- $\frac{z}{\omega} = \frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$

Note:

We multiply the denominator by its *complex conjugate* so we can *realise* the denominator, which makes it easier to distinguish certain properties such as finding $\mathrm{Re}(\frac{z}{\omega})$ or $\mathrm{Im}(\frac{z}{\omega})$ if we wish to do so.

Complex Notation

The Polar Form

Polar forms consider a complex number as a vector, with a magnitude and a direction. We can represent a complex number z in the form:

$$z = r(\cos\theta + i\sin\theta)$$

Or, with less writing, in the form:

$$z = re^{i\theta}$$

Where r is the length (modulus) of our complex vector, and θ is the angle (argument) it makes with the first quadrant of the x-axis.

$$x = r\cos\theta \text{ and } y = r\sin\theta. \text{ Thus, } r = \sqrt{x^2 + y^2}.$$

Complex Notation

Note:

Since the value of θ is not unique, we can choose a value of

- θ such that $-\pi < \theta \leq \pi$. This is known as our *principal argument*.
- Also keep in mind, arg(z) is asking for any argument of z, whereas Arg(z) is asking specifically for the principal argument.
- cisθ notation from the good ol' 4U days is no longer used, so be mindful of this!

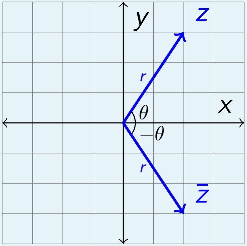
Complex Conjugates

The Conjugate

If $z = re^{i\theta}$, then its reflection along the x-axis is simply

$$\bar{z} = re^{-i\theta} \text{ or } \bar{z} = r(\cos(-\theta) + i\sin(-\theta)) = r(\cos\theta - i\sin\theta)$$

$\bar{z}$ is also known as the *complex conjugate* of z. On an Argand diagram, it looks like this:



Complex Properties

De Moivre's Theorem

This simple yet highly important theorem is perhaps one of 'De' most versatile and useful concepts related to complex numbers. It states that:

$$(r(\cos\theta + i\sin\theta))^n = r^n(\cos(n\theta) + i\sin(n\theta)) \text{ for } n \in \mathbb{R}.$$

This can be proved by induction, give it a go if you want!

Note:

This result also applies to negative and fractional values of n too.

Complex Properties

Simple Complex Properties

Complex numbers have some pretty cool properties:

- $|z||\omega| = |z\omega|$
- $\frac{|z|}{|\omega|} = \left|\frac{z}{\omega}\right|$
- $|z^n| = |z|^n$
- $z\bar{z} = |z|^2$
- $\bar{z} = \frac{1}{z}$ (ONLY if $|z| = 1$)
- $\arg(z\omega) = \arg(z) + \arg(\omega)$
- $\arg(\frac{z}{\omega}) = \arg(z) - \arg(\omega)$
- $\arg(z^n) = n \times \arg(z)$ (a consequence of De Moivre's theorem)

Complex Roots and Roots of Polynomials

Applications of De Moivre's Theorem

Using De Moivre's theorem and the properties of complex numbers, we can now attempt to find the *n*th roots of numbers as well as solving various polynomial roots.

Polynomials

Polynomial theorems

Some important polynomial theorems:

- **Remainder Theorem**: The remainder when the polynomial is divided by $z - \alpha$ is given by simply evaluating $p(\alpha)$.
- **Factor Theorem**: A number α is a root of $p(z)$ only if $z - \alpha$ is a factor of $p(z)$.
- **The Fundamental Theorem of Algebra**: A polynomial of degree n has precisely n roots over the complex field.
- Complex roots always occur in conjugate pairs in a polynomial with real number coefficients. If complex roots exist, there is always an even quantity (including 0) of them in a real polynomial.

Polynomials

More polynomial theorems

Some more important polynomial theorems:

- Every polynomial can be factorised in the form:

$$p(z) = a(z - \alpha_1)(z - \alpha_2)...(z - \alpha_n)$$

Where a is the coefficient of z^n and α_i are roots of $p(z)$.

Note:

As a side note, we can convert linear factors with complex coefficients into a quadratic with real coefficients (provided that the initial polynomial has real coefficients).

$$(z - \alpha)(z - \bar{\alpha}) = z^2 - 2\mathrm{Re}(\alpha)z + |\alpha|^2$$

Complex Roots

Example 13

Finding all ninth roots of 1:
We begin by stating our required equation:

$$z^9 = 1$$

$$\text{i.e. } z^9 = e^{0+2ki\pi} \text{ for k} = 0, 1, 2, .., 8$$

$$z = e^{(2ki\pi)/9}$$

$$z = 1, e^{2i\pi/9}, e^{4i\pi/9}, e^{6i\pi/9}, e^{8i\pi/9}, e^{10i\pi/9}, e^{12i\pi/9}, e^{14i\pi/9}, e^{16i\pi/9}$$

$$z = 1, e^{2i\pi/9}, e^{4i\pi/9}, e^{2i\pi/3}, e^{8i\pi/9}, e^{-8i\pi/9}, e^{-2i\pi/3}, e^{-4i\pi/9}, e^{-2i\pi/9}$$

Roots of Unity and Polynomials Shenanigans

Example 14 - MATH1141 June 2014 Q4iv

iv) You may assume that $z^9 - 1 = (z^3 - 1)(z^6 + z^3 + 1)$

a) Explain why the roots of $z^6 + z^3 + 1 = 0$ are $e^{\pm 2i\pi/9}, e^{\pm 4i\pi/9}, e^{\pm 8i\pi/9}$.

b) Divide $z^6 + z^3 + 1$ by z^3 and let $x = z + \frac{1}{z}$. Find a cubic equation satisfied by x.

c) Deduce that $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9} = 0$

Roots of Unity and Polynomials Shenanigans

Example 14 - MATH1141 June 2014 Q4iva

For this part, we can actually cheat a little since we already know the roots of $z^9 - 1 = 0$ from example 13.

If we factorise $z^9 - 1 = 0$ to $(z^3 - 1)(z^6 + z^3 + 1) = 0$, then we can deduce that either the cubic or the sextic factor must equal 0 to satisfy the equation.

Roots of Unity and Polynomials Shenanigans

Example 14 - MATH1141 June 2014 Q4iva

We also know that the polynomial can be expressed as a combination of 9 linear factors, some with complex constants.

After appropriately expanding these linear factors with complex constants, we end up with our real cubic and sextic polynomials, which means that when one real polynomial equals 0, the other cannot equal 0 since they have distinct factors.

Roots of Unity and Polynomials Shenanigans

Example 14 - MATH1141 June 2014 Q4iva

Finally, we note the roots of $z^3 - 1 = 0$ are $z = 1$ and $e^{\pm 2i\pi/3}$, so the roots of $z^6 + z^3 + 1 = 0$ must be the remaining roots of $z^9 - 1$, i.e. $z = e^{\pm 2i\pi/9}, e^{\pm 4i\pi/9}, e^{\pm 8i\pi/9}$.

Roots of Unity and Polynomials Shenanigans

Example 14 - MATH1141 June 2014 Q4ivb

Upon dividing, we get:

$$\frac{z^6 + z^3 + 1}{z^3} = z^3 + \frac{1}{z^3} + 1$$

$$= \left(z + \frac{1}{z}\right)^3 - 3\left(z + \frac{1}{z}\right) + 1$$

$$= x^3 - 3x + 1$$

As required.

Roots of Unity and Polynomials Shenanigans

Example 14 - MATH1141 June 2014 Q4ivc

The new roots of this equation are $z + \frac{1}{z} = z + \bar{z}$, which means that the roots are simply $e^{\frac{2i\pi}{9}} + e^{\frac{-2i\pi}{9}}, e^{\frac{4i\pi}{9}} + e^{\frac{-4i\pi}{9}}, e^{\frac{8i\pi}{9}} + e^{\frac{-8i\pi}{9}}$. Which equal:

$$2\cos\frac{2\pi}{9}, 2\cos\frac{4\pi}{9}, 2\cos\frac{8\pi}{9}, \text{ respectively.}$$

Taking the sum of roots of this polynomial, we find:

$$2\cos\frac{2\pi}{9} + 2\cos\frac{4\pi}{9} + 2\cos\frac{8\pi}{9} = -\frac{0}{1}$$

Which gives us our desired result of:

$$\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9} = 0$$

Polynomials

Example 15 - MATH1131 June 2015 Q3ii

ii) Consider the complex polynomial

$$p(z) = z^4 - 3z^3 + 6z^2 - 12z + 8.$$

a) Given that $p(2i) = 0$, factorise $p(z)$ into linear and quadratic factors with real coefficients.

b) Find all roots of p.

Polynomials

Example 15 - MATH1131 June 2015 Q3ii

Thanks to the factor theorem, we know if $z-2i$ is a factor, then so is $z+2i$ since all coefficients are real. Thus, we know that:

$$(z-2i)(z+2i) = z^2+4$$

Is a factor of $p(z)$.

Applying some nifty polynomial division using this quadratic factor, we discover that:

$$p(z) = (z^2+4)(z^2-3z+2) = (z^2+4)(z-2)(z-1)$$

Which means the solutions to $p(z)$ are $z = \pm 2i, 1, 2$

De Moivre's Theorem and Powers of Trig Functions

Example 16 - MATH1141 June 2012 Q2iv

iv) Use De Moivre's theorem to prove that

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$$

First, we note that $(\cos\theta + i\sin\theta)^4$
$= \cos 4\theta + i\sin 4\theta$.

But also,
$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$.
Equating the real parts of both equations,

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$$

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$$

Complex Geometry

Common Loci

- $|z-\omega| = a$ - A circle with centre ω and radius a.
- $|z-a| = |z-b|$ - The perpendicular bisector of points a and b.
- $\arg(z-\omega) = \theta$ - A ray from the point w at an angle of θ from the positive x-axis. The point $z = w$ is excluded (because $\arg(0)$ is undefined).
- $\mathrm{Re}(z) = a$ - The line $x = a$.
- $\mathrm{Im}(z) = a$ - The line $y = a$.

Complex Geometry

Common Regions

- $|z-\omega| \geq / \leq a$ - Area outside/inside circle.
- $|z-a| \geq / \leq |z-b|$ - Region on the side of b/a.
- $\alpha \leq \arg(z-\omega) \leq \beta$ - The region in between $\arg(z-\omega)=\alpha$ and $\arg(z-\omega)=\beta$.
- $\text{Re}(z) \geq / \leq a$ - The region to the right/left of $x=a$.
- $\text{Im}(z) \geq / \leq a$ - The region above/below $y=a$.

Note:

Replacing $\geq / \leq$ with $> / <$ means that the line used as the border of each region must be a dotted line.

Complex Geometry

Example 17 - MATH1131 June 2015 - Q2iii

Sketch the following region on the Argand diagram:

$$S = \{z \in \mathbb{C} : |z - i - 1| \leq 1 \text{ or } |\text{Im}(z)| \geq 1\}.$$

Linear Equations

What is a Linear Equation?

Definition

A linear equation is defined as

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n + b = 0$$

where $x_1 \ldots x_n$ are variables, whereas $a_1 \ldots a_n$ and b are the coefficients.

Linear Equations

Systems of linear equations are very frequently used in the world of mathematics, engineering, physics, computer science, etc. A reason for this is because systems that are non-linear are often approximated well with linear equations (which makes computation far easier).

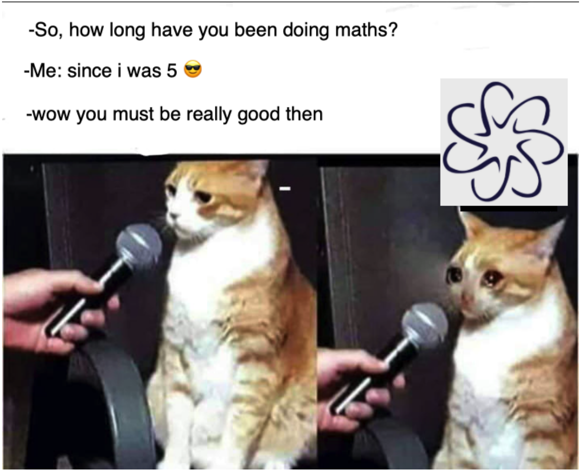
Solving systems of linear equations

Hopefully by now, we are all able to solve systems of equations such as

$$\begin{aligned} 2x + y &= 0 \\ 3x - y &= 7. \end{aligned}$$

We can recognise that this will have a unique intersection as it is the intersection of two lines.

Cute cat


-So, how long have you been doing maths?
-Me: since i was 5 😎
-wow you must be really good then

A less subtle example

How about this pair of simultaneous equations?

$$x + 2y + z = 1 \tag{1}$$

$$2x + 3y - 2z = -2. \tag{2}$$

From a quick glance, we can see that since there are 3 unknowns (x, y, z) and only 2 equations, there will be infinitely many solutions. Now how do we solve this system of equations?

A less subtle example

There are many different approaches to this question, but a quick method is to set any of the variables equal to a parameter. In this case, let $z = \lambda, \lambda \in \mathbb{R}$. This gives us

$$x + 2y + \lambda = 1 \tag{1}$$

$$2x + 3y - 2\lambda = -2. \tag{2}$$

Now doing $2(1) - (2)$ gives us

$$\begin{aligned} y + 4\lambda &= 4 \\ y &= 4 - 4\lambda \end{aligned}$$

Subbing this into (1) gives us $x = -7 + 7\lambda$.

A less subtle example

Hence, our solution is $x = -7 + 7\lambda$, $y = 4 - 4\lambda$, $z = \lambda$, which we can see is a line. A prettier form may be:

$$\ell : \begin{pmatrix} -7 \\ 4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ -4 \\ 1 \end{pmatrix}.$$

Note: this is the case because the geometrical interpretation for this system is finding the intersection of two planes (which gives us the line of intersection).

...

What if we had a lot more equations and variables? For example...

$$\begin{aligned} 2a + 3b - c + d &= 6 \\ x - 4b + 5c &= 2 \\ 1 - 6c &= 5 \\ a + 2b - d &= 7. \end{aligned}$$

This is where matrices come in to make the process a lot more easier on the eyes.

The augmented matrix

Definition

The **augmented matrix** is defined as

$$\left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} & b_m \end{array}\right)$$

Where m, n are the columns and rows. Each row is just a simple way of expressing $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$.

Elementary Row Operations

The three elementary row operations

1. Interchange of equations (swapping rows)
2. Adding a multiple of one equation to another
3. Multiplication by a scalar

Any combination of the three can be applied in one step as well.
Each of these operations will give us a new matrix that is equivalent to the old one : their solutions will remain the same.

Row-echelon Form

A combination of all these techniques is called **Gaussian Elimination**. We use Gaussian elimination to try and obtain a row-echelon form of our matrix.

Definition

A matrix is considered in **row-echelon form** if

1. All zero rows are at the bottom of the matrix.
2. In every leading row, the leading entry is further to the right than the leading entry in any row higher up in the matrix (it forms a diagonal downwards from left to right as best as it can).

Row-echelon form can help us determine if a matrix has no solutions, a unique solution or infinitely many solutions.

An example

An example of a matrix in row-echelon form is

$$\left(\begin{array}{ccc|c} a & x & x & d \\ 0 & b & x & e \\ 0 & 0 & c & f \end{array}\right)$$

Reduced Row-Echelon form

Reduced row-echelon

A matrix is said to be in reduced row echelon form if it is in row-echelon form and

1. Every leading entry is 1
2. Every leading entry is the only non-zero entry in its column.

An example

An example of a matrix in reduced row-echelon form is

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & d \\ 0 & 1 & 0 & e \\ 0 & 0 & 1 & f \end{array}\right)$$

A question

Now let's try applying these techniques.

MATH1131 June 2011 q3 (ii)

A pet shop has x hamsters, y rabbits and z guinea pigs.
Each hamster eats $50g$ of dry food and $40g$ of fresh vegetables, and needs $1m^2$ of space.
Each rabbit eats $300g$ dry food and $320g$ of fresh vegetables, and needs $5m^2$ of space.
Each guinea pig eats $100g$ of dry food and $200g$ of fresh vegetables, and needs $3m^2$ of space.
Altogether they eat $2900g$ of dry food and $3920g$ of fresh vegetables, and need $63m^2$ of space.
(a) Explain why $5x + 30y + 10z = 290$.

Solution

(a)

We examine the equation regarding how much dry fruit has been eaten. In this case, we have

$$50x + 300y + 100z = 2900.$$

Dividing by 10 will give us the solution to part (i), or $5x + 30y + 10z = 290$.

A question

MATH1131 June 2011 q3 (ii)

A pet shop has x hamsters, y rabbits and z guinea pigs.
Each hamster eats $50g$ of dry food and $40g$ of fresh vegetables, and needs $1m^2$ of space.
Each rabbit eats $300g$ dry food and $320g$ of fresh vegetables, and needs $5m^2$ of space.
Each guinea pig eats $100g$ of dry food and $200g$ of fresh vegetables, and needs $3m^2$ of space.
Altogether they eat $2900g$ of dry food and $3920g$ of fresh vegetables, and need $63m^2$ of space.
(b) Write down a system of linear equations that determine x, y and z.

Solution

(b)

We do the same thing we did for part (a) with fresh vegetables and space. This will give us the system of equations:

$$\begin{aligned} 5x + 30 + 10 &= 290 \\ 4x + 32y + 20z &= 392 \\ x + 5y + 3z &= 63. \end{aligned}$$

A question

MATH1131 June 2011 q3 (ii)

A pet shop has x hamsters, y rabbits and z guinea pigs.
Each hamster eats $50g$ of dry food and $40g$ of fresh vegetables, and needs $1m^2$ of space.
Each rabbit eats $300g$ dry food and $320g$ of fresh vegetables, and needs $5m^2$ of space.
Each guinea pig eats $100g$ of dry food and $200g$ of fresh vegetables, and needs $3m^2$ of space.
Altogether they eat $2900g$ of dry food and $3920g$ of fresh vegetables, and need $63m^2$ of space.
(c) Reduce your system to echelon form and solve to find the number of hamsters, rabbits and guinea pigs.

Solution

(c)

We can form an augmented matrix using part (b) like so:

$$\left(\begin{array}{ccc|c} 5 & 30 & 10 & 290 \\ 4 & 32 & 20 & 392 \\ 1 & 5 & 3 & 63 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 5 & 3 & 63 \\ 4 & 32 & 20 & 392 \\ 5 & 30 & 10 & 290 \end{array}\right) \xrightarrow{R_3 = 5R_1 - R_3}$$

$$\left(\begin{array}{ccc|c} 1 & 5 & 3 & 63 \\ 4 & 32 & 20 & 392 \\ 0 & -5 & 5 & 25 \end{array}\right) \xrightarrow{R_2 = 4R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 5 & 3 & 63 \\ 0 & -12 & -8 & -140 \\ 0 & -5 & 5 & 25 \end{array}\right)$$

Solution

(c) continued

$$\left(\begin{array}{ccc|c} 1 & 5 & 3 & 63 \\ 0 & -12 & -8 & -140 \\ 0 & -5 & 5 & 25 \end{array}\right) \underset{R_3=\frac{1}{5}R_3}{\overset{R_2=-\frac{1}{4}R_2}{\Longrightarrow}} \left(\begin{array}{ccc|c} 1 & 5 & 3 & 63 \\ 0 & 3 & 2 & 35 \\ 0 & -1 & 1 & 5 \end{array}\right)$$

$$\overset{R_3=R_2+3R_3}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 5 & 3 & 63 \\ 0 & 3 & 2 & 35 \\ 0 & 0 & 5 & 50 \end{array}\right)$$

and now the matrix is in row-echelon form. But! We haven't finished yet. We still need to solve for x, y, z.

Back Substitution

Back Substitution

This is the final step in solving a matrix in row-echelon form. First, if there are any non-leading variables, assign an arbitrary parameter to them.
Then, from the bottom row, we can deduce the solutions for each variable and substitute them for the equations above.

Solution

From our previous question, our final matrix is like so:

$$\left(\begin{array}{ccc|c} 1 & 5 & 3 & 63 \\ 0 & 3 & 2 & 35 \\ 0 & 0 & 5 & 50 \end{array}\right)$$

From the matrix above, we can clearly see that $5z = 50$ and thus $z = 10$. In the second equation, we have $3y + 2z = 35$. Substituting z in will give us $3y + 20 = 35$ and thus $y = 5$. Finally, we do the same for the last (or first, depending how you look at it) equation, $x + 5y + 3z = 63$. This gives us $x + 25 + 30 = 63$ and thus $x = 8$.

Deducing the number of solutions from row-echelon form

Deducing solubility from row-echelon form

By examining the leading-terms of a matrix in row-echelon form, we can determine if it has a unique, infinite or no solution.

1. If the right-most column is a leading column, there are no solutions (there is a row of zeros equalling a number).
2. There is a unique solution if every variable is a leading variable (every column has a leading variable).
3. Infinite solutions if and only if there is at least one non-leading variable on the left hand side.

A question

1131 June 2012 q3 (iii)

A system of three equations in three unknowns x, y and z has been reduced to the following echelon form: $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & \alpha^2 - 9 & \alpha - 3 \end{array}\right)$

(a) For which value of α will the system have no solution?

Solution

(a)

A matrix has no solutions if, in row-echelon form, the right-most column is a leading column. Therefore, if we let $\alpha = -3$, the entire bottom row will be 0s and the right-most column will be -6. This will give us a matrix with no solutions as any linear combination of 0s will never give us -6.

A question

MATH1131 June 2012 q3 (iii)

A system of three equations in three unknowns x, y and z has been reduced to the following echelon form: $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & \alpha^2 - 9 & \alpha - 3 \end{array}\right)$

(b) For which value of α will the system have infinitely many solutions?

Solution

(b)

A matrix has infinitely many solutions if, in row-echelon form, there is at least one non-leading variable on the left-hand side. Therefore, if we let $\alpha = 3$, this will result in the third column being non-leading (as well as the fourth).

A question

1131 June 2012 q3 (iii)

A system of three equations in three unknowns x, y and z has been reduced to the following echelon form: $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & \alpha^2 - 9 & \alpha - 3 \end{array}\right)$

(c) For the value of α determined in part (b), find the general solution.

Solution

(c)

From part (b), we let $\alpha = 3$. Therefore, this gives us the matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Since the z variable is non leading, we assign it an arbitrary parameter, such as λ. The second row will thus give us the equation $2y + 4\lambda = 8$. Rearranging, we get $y = 4 - 2\lambda$. Back-substituting this into the first row will give us $x + 2(4 - 2\lambda) + 3\lambda = 5$. Therefore, $x = -3 + \lambda$.

Solution

$$\begin{aligned}\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} -3+\lambda \\ 4-2\lambda \\ \lambda \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ 4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix},\end{aligned}$$

where $\lambda \in \mathbb{R}$.

Geometric Applications - Problems involving lines and planes

Question

Is the line $\frac{x-1}{2} = \frac{y}{3} = \frac{z+1}{-1}$ parallel to the plane

$$\mathbf{x} = \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \text{ where } \lambda_1, \lambda_2 \in \mathbb{R}?$$

Solution

We first convert the line to parametric form by setting everything equal to a parameter. So we have $\frac{x-1}{2} = \frac{y}{3} = \frac{z+1}{-1} = \lambda$. Therefore,

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 1+2\lambda \\ 3\lambda \\ -1-\lambda \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}. \end{aligned}$$

We then have to take the direction vector of this line and see if it can be expressed as a linear combination (or alternatively, in the span) of the direction vectors of the plane.

Solution

This gives us the equation (LHS is direction vector of line, RHS is plane):

$$\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

We can then put everything into an augmented matrix like so:

$$\left(\begin{array}{cc|c} 2 & 3 & 1 \\ 1 & 1 & 3 \\ 0 & -1 & -1 \end{array}\right)$$

We can immediately perform back substitution to give $\lambda_2 = 1$. Substituting this into the second row gives us $\lambda_1 = 2$. Substituting these 2 values into the first row confirms these values as true, which means they are parallel.

Matrices

Matrices

Important facts about matrices

A $m \times n$ matrix is one with m rows and n columns (in that order).

1. A matrix is square if the number of rows $=$ number of columns (it looks like a square).
2. Addition can only be performed when the two matrices have the same number of rows AND columns. We just add the corresponding elements together into a new matrix (and same for subtraction).
3. Multiplying a matrix by a scalar means multiplying every entry by that scalar.
4. A zero matrix is one where every entry is 0.

Matrices are non-commutative!

Careful!

Most matrices are NOT commutative, that is, $AB \neq BA$, where A and B are matrices. Be careful when ordering them.

Matrix Multiplication

More facts about matrices

1. We can multiply two matrices together IF the number of columns in the first matrix equals the number of rows in the second matrix.
2. The resulting matrix will have the number of **rows** from the **first** matrix, and the number of **columns** from the **second** matrix.
3. To multiply them together, go through the rows of the first matrix and multiply them by the columns of the second.

Question

Question

Find AB, where

$$A = \begin{pmatrix} -2 & 1 \\ 3 & 4 \\ -1 & 5 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 0 & 2 \\ 1 & -2 & -3. \end{pmatrix}$$

Solution

Since A has 2 columns and B has 2 rows, we can multiply them together. We can also tell that the end result will be a 3×3 matrix.

$$\begin{pmatrix} -2+1 & 0-2 & -4-3 \\ 3+4 & 0-8 & -2-15 \\ -1+5 & 0-10 & -2-15 \end{pmatrix} = \begin{pmatrix} -1 & -2 & -7 \\ 7 & -8 & -17 \\ 4 & -10 & -17 \end{pmatrix}.$$

The identity matrix

Definition

The **identity** matrix is a very important matrix that consists of 1's along the diagonal and 0's elsewhere. An example of a 3×3 identity matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We can view this as the value of 1 in terms of matrix multiplication, that is, $IA = AI = A$.

Transpose of a matrix

Definition

The **transpose** (denoted as A^T) of an $m \times n$ matrix is the $n \times m$ matrix with entries

$$[A^T]_{ij} = [A]_{ji}.$$

In other words, we swap the columns and the rows.

Properties of the transpose

Properties of the transpose

1. $(A^T)^T = A$
2. $(\lambda A + \mu B)^T = \lambda A^T + \mu B^T$
3. $(AB)^T = B^T A^T$
4. If $A^T = A$, the matrix is **symmetric**
5. If $A^T = -A$, the matrix **skew symmetric**

More on the transpose

Examples of transpose

1. $$\begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 3 \\ 0 & -1 & -1 \end{pmatrix}^T = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 1 & -1 \\ 1 & 3 & -1 \end{pmatrix}$$

2. A symmetric matrix:

$$\begin{pmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \\ 4 & 5 & -1 \end{pmatrix}$$

The determinant of a matrix

The determinant

The concept of a matrix determinant is a very important topic in mathematics - and is something you will revisit (many times) again in 1B and later year math courses. An example of something cool about determinants is that an $n \times n$ matrix has a unique solution if and only if its determinant is not equal to 0.

The first thing to note is: the determinant is **only** defined for **square** matrices.

The determinant of a matrix

Definition

The determinant of a matrix is written as $det(A)$ or $|A|$. For a 1×1 matrix, it is its sole entry. For a 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the determinant is given as $det(A) = ad - bc$.

The determinant of a matrix

The determinant of a 3×3 matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

is given as

$$det(A) = a(ei - hf) - b(di - gf) + c(dh - ge).$$

The determinant of a matrix

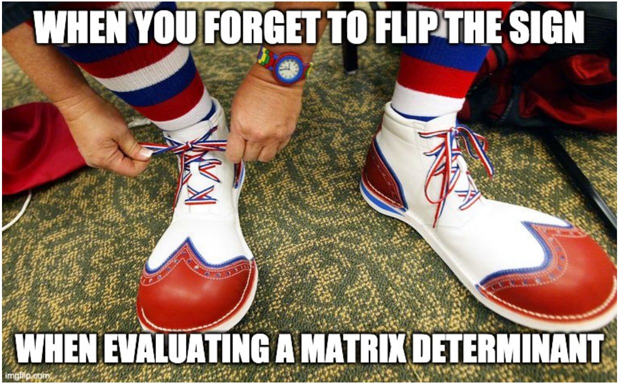
Sign changes

Be very careful when calculating the determinant of a 3×3 matrix! Note the equation:

$$det(A) = a(ei - hf) - b(di - gf) + c(dh - ge)$$

has a negative sign in front of the middle coefficient. Be careful about these sign changes! The signs are alternating, starting with positive.

:(


WHEN YOU FORGET TO FLIP THE SIGN
WHEN EVALUATING A MATRIX DETERMINANT
imgflip.com

The determinant of a matrix

Definition

The exam will (hopefully) not examine you on the determinants of matrices greater than 3×3, but... the determinant of an $n \times n$ matrix is defined as

$$det(A) = \sum_{k=1}^{n} (-1)^{1+k} a_{1k} |A_{1k}|$$

The determinant of a matrix

Example

The calculation of the determinant of a 3×3 matrix is:

$$\begin{aligned}
\begin{vmatrix} 5 & 1 & 7 \\ -2 & 3 & -4 \\ 6 & -1 & 2 \end{vmatrix} &= 5\begin{vmatrix} 3 & -4 \\ -1 & 2 \end{vmatrix} - 1\begin{vmatrix} -2 & -4 \\ 6 & 2 \end{vmatrix} + 7\begin{vmatrix} -2 & -4 \\ 6 & 2 \end{vmatrix} \\
&= 5(6-4) - (-4+24) + 7(2-18) \\
&= -122.
\end{aligned}$$

The determinant of a matrix

Properties of the determinant

1. If any two rows of a determinant swap, the sign of the determinant is reversed.
2. If a row of a matrix is multiplied by a scalar, then the entire determinant will be multiplied by that scalar.
3. The determinant is not changed by adding two rows.
4. If a matrix contains a zero row/column, then the determinant is 0.
5. If a row/column of a matrix is a scalar multiple of another row/column in the same matrix, the determinant is 0.

The determinant of a matrix

More properties

1. $det(A^T) = det(A)$
2. If AB exists, then $det(AB) = det(A)det(B)$
3. $det(A + B) \neq det(A) + det(B)$
4. $det(mA) = m^n det(A)$

The determinant of a matrix

An efficient way to calculate the determinant

If A is a square matrix in row-echelon form, then $det(A)$ is equivalent to the diagonals multiplied together!

Question

2016 MATH1141 q2 vii

Let

$$A = \begin{pmatrix} 1 & 1 & 4 \\ 0 & 3 & 8 \\ 0 & 2 & 6 \end{pmatrix}$$

(a) Calculate the determinant of A.

Solution

$$det(A) = \begin{vmatrix} 1 & 1 & 4 \\ 0 & 3 & 8 \\ 0 & 2 & 6 \end{vmatrix}$$

$$3 \times 2 \times det(A) = \begin{vmatrix} 1 & 1 & 4 \\ 0 & 6 & 16 \\ 0 & 6 & 18 \end{vmatrix}$$

$$6det(A) = \begin{vmatrix} 1 & 1 & 4 \\ 0 & 6 & 16 \\ 0 & 0 & 2 \end{vmatrix}$$

$$det(A) = \frac{1}{6} \times 12 = 2$$

Question

June 2015 MATH1141 q3 iv

Consider the matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$. Suppose that $det(A) = 7$.

Find the value of the determinant of the following matrices:

$$B = \begin{pmatrix} d & e & f \\ a & b & c \\ d - 3a & e - 3b & f - 3c \end{pmatrix} \quad C = \begin{pmatrix} g - 3a & 2a & a - d \\ h - 3b & 2b & b - e \\ i - 3c & 2c & c - f \end{pmatrix}$$

Solution

In matrix B, we can see that the third row is just a linear combination of the first two rows. Therefore, the determinant will be 0.

In matrix C, we have to recall that $det(A^T) = det(A)$. The transpose will thus be:

$$\begin{pmatrix} g-3a & h-3b & i-3c \\ 2a & 2b & 2c \\ a-d & b-e & c-f \end{pmatrix}.$$

We can then perform two swaps to obtain the matrix:

$$\begin{pmatrix} 2a & 2b & 2c \\ -(d-a) & -(e-b) & -(f-c-) \\ g-3a & h-3b & i-3c \end{pmatrix}.$$

Solution continued

$$\begin{pmatrix} 2a & 2b & 2c \\ -(d-a) & -(e-b) & -(f-c-) \\ g-3a & h-3b & i-3c \end{pmatrix}.$$

We can observe that we have matrix A, but the 1st row is scaled by 2, the second row was multiplied by -1, the second and third being just addition of rows (which doesn't affect the det). Thus, $2 \times -1 \times (-1)^2 \times det(A) = -14 = det(C)$.

The inverse of a matrix

Definition

The inverse of a matrix A, denoted A^{-1}, is a matrix which follows the rules:

1. $AA^{-1} = I$ or $A^{-1}A = I$
2. All invertible matrices are square
3. $(A^{-1})^{-1} = A$
4. $det(A^{-1}) = (det(A))^{-1}$ or $\frac{1}{det(A)}$

Your exam will always have a couple of inverse matrix questions, so it's good to be prepared for them!

The inverse of a matrix

Condition

A matrix has an inverse if and only if

$$det(A) \neq 0.$$

The inverse of a matrix

Definition

The inverse of a 2×2 matrix is defined as

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The inverse of a matrix

Definition

To calculate the general inverse of any matrix, we can use row reduction.

1. We write the matrix in the form $(A|I)$, where I is the identity matrix.
2. Row-reduce the matrix to obtain I on the left hand side.
3. We will thus have the form $(I|A^{-1})$, the RHS being the inverse matrix.

The inverse of a matrix

Question

Find the inverse of the matrix

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$$

Solution

In order to apply the formula on $A = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$, we first evaluate its determinant: $det(A) = 2 \times 4 - 3 \times 3 = -1$. Hence, its determinant is

$$\frac{1}{det(A)} \begin{pmatrix} 4 & -3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix}$$

The inverse of a matrix

MATH1131 2017 q4 ii

Calculate the inverse of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Solutions

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Therefore the inverse matrix is the RHS, or $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

Questions

MATH1131 June 2013 q3 ii

Let $M = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & 1 \\ 0 & 2 & \alpha \end{pmatrix}$.

(a) Evaluate the determinant of M.

Solutions

(a)

$$\begin{aligned} det(M) &= 1\begin{vmatrix} 5 & 1 \\ 2 & \alpha \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 0 & \alpha \end{vmatrix} + 0\begin{vmatrix} 2 & 5 \\ 0 & 2 \end{vmatrix} \\ &= 1(5\alpha - 2) - 2(2\alpha - 0) + 0 \\ &= \alpha - 2. \end{aligned}$$

Questions

MATH1131 June 2013 q3 ii

Let $M = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & 1 \\ 0 & 2 & \alpha \end{pmatrix}$.

(b) Determine the value(s) of α for which M does not have an inverse.

Solutions

(b)

Since the inverse of a matrix does not exist if and only if the determinant is 0, then we have to solve the equation
$det(M) = \alpha - 2 = 0$.
Clearly, $\alpha = 2$ for which M does not have an inverse.

Questions

MATH1131 June 2013 q3 ii

Let $M = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & 1 \\ 0 & 2 & \alpha \end{pmatrix}$.

(c) Find the inverse of M when $\alpha = 1$.

Solutions

(c)

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 5 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -4 & 2 & -1 \end{array}\right) = \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & -4 & 2 & -1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & -2 \\ 0 & 1 & 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & -4 & 2 & -1 \end{array}\right)$$

Therefore, the inverse matrix is the RHS, or $\begin{pmatrix} -3 & 2 & -2 \\ -2 & -1 & 1 \\ -4 & 2 & -1 \end{pmatrix}$.

Very important learning

Questions

MATH1141 2019 2iv

Let A be an $n \times n$ real matrix with the property $A^T = -A$. Such a matrix is called skew-symmetric.

(a) Show that A is not invertible if n is odd.

Solutions

(a)

A matrix is not invertible if the determinant is 0. Since the only thing we really have to go by is $A^T = -A$, we apply the equate the determinants on both sides.

$$\begin{aligned} det(A^T) &= det(-A) \\ det(A) &= (-1)^n det(A) \end{aligned}$$

If n is an odd number, then $(-1)^n$ will be -1. Therefore, $det(A) = -det(A)$ and thus $det(A)$ can only be 0. This means that for odd n's, there will be no inverse.

Questions

MATH1141 2019 2iv

Let A be an $n \times n$ real matrix with the property $A^T = -A$. Such a matrix is called skew-symmetric.

(b) Let $\mathbf{x}$ be a solution to the linear system

$$A\mathbf{x} = \mathbf{x}.$$

Show that $\mathbf{x}^T\mathbf{x} = 0$.

Solutions

(b)

$$
\begin{aligned}
\mathbf{x} &= A\mathbf{x} \\
\mathbf{x}^T &= (A\mathbf{x})^T \\
\mathbf{x}^T &= \mathbf{x}^T A^T \\
\mathbf{x}^T &= -\mathbf{x}^T A \\
\mathbf{x}^T\mathbf{x} &= -\mathbf{x}^T A\mathbf{x} \\
\mathbf{x}^T\mathbf{x} &= -\mathbf{x}^T\mathbf{x}.
\end{aligned}
$$

Therefore, $\mathbf{x}^T\mathbf{x} = 0$ as $\mathbf{x}^T\mathbf{x} = -\mathbf{x}^T\mathbf{x}$ (similar to the previous question).

Questions

MATH1141 2019 2iv

Let A be an $n \times n$ real matrix with the property $A^T = -A$. Such a matrix is called skew-symmetric.

(c) Show that the matrix $I_n - A$ is invertible.

Solutions

(b) Begin by recalling that a non-zero matrix M is invertible if and only if $M\mathbf{x} = \mathbf{0}$ has only the zero solution. This is because we can multiply both sides of the equation $M\mathbf{x} = \mathbf{0}$ by M^{-1}. So we resume with the aim to prove that $(I_n - A)\,\mathbf{x} = \mathbf{0}$ has only the zero solution.

(Proof by contradiction) Suppose that there is some $\mathbf{x} \neq \mathbf{0}$ such that $(I_n - A)\,\mathbf{x} = \mathbf{0}$. Then

$$\begin{aligned}(I - A)\mathbf{x} &= \mathbf{0}\\ A\mathbf{x} &= \mathbf{x}.\end{aligned}$$

However from part (b), this means that $\mathbf{x}^T\mathbf{x} = |\mathbf{x}|^2 = 0$ which implies that $\mathbf{x} = 0$. This is a contradiction and therefore $I_n - A$ is invertible.

Farewell

Sadly, this is the end of our seminar! We hope you all learn a lot and good luck for your exams :)