MATH1231/1241 Algebra S2 2009 Test 2 v3B

August 22, 2017

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1. **Hints**

   Simply show that T preserves addition and scalar multiplication.

   Worked Sample Solutions

   Let $p, q \in \mathbb{P}_3$. Then $p+q$ is a polynomial in $\mathbb{P}_3$ and

$$\begin{aligned} T((p+q)(x)) &\stackrel{\text{def}}{=} \begin{pmatrix} (p+q)(1) \\ (p+q)(3) \end{pmatrix} \\ &= \begin{pmatrix} p(1)+q(1) \\ p(3)+q(3) \end{pmatrix} \\ &= \begin{pmatrix} p(1) \\ p(3) \end{pmatrix} + \begin{pmatrix} q(1) \\ q(3) \end{pmatrix} \end{aligned}$$

$$= T(p(x)) + T(q(x)).$$

And so, T satisfies the addition property.

Next, let $p \in \mathbb{P}_3$ and $\alpha \in \mathbb{R}$. Then αp is a polynomial in $\mathbb{P}_3$ with

$$\begin{aligned} T(\alpha p(x)) &\stackrel{\text{def}}{=} \begin{pmatrix} \alpha p(1) \\ \alpha p(3) \end{pmatrix} \\ &= \alpha \begin{pmatrix} p(1) \\ p(3) \end{pmatrix} \\ &= \alpha T(p(x)). \end{aligned}$$

Thus, T also satisfies the scalar multiplication property. Therefore, T is linear.

2. (i) **<u>Hints</u>**

As usual, row-reduce the matrix A, and then the rank will be the number of leading columns in the row-echelon form of A, whilst the nullity is the number of *non*-leading columns in the row-echelon form. You should find that $\operatorname{rank}(A) = 2$ and $\operatorname{nullity}(A) = 3$. Here is a sample solution.

<u>Worked Sample Solution</u>

We row-reduce A. But anticipating part (ii) of this question, we do so with $\mathbf{b}$ augmented in the right-hand part (the augmented part of the matrix will be irrelevant to the answer for (i), but will be handy for (ii) and putting it in now will save us from having to redo row-reductions for part (ii)). The appropriate augmented matrix is given by

$$\left(\begin{array}{ccccc|c} \boxed{-1} & 2 & 1 & 0 & 1 & 4 \\ 1 & -1 & 0 & -1 & 0 & 1 \\ 3 & -2 & 1 & -4 & 1 & 8 \end{array}\right) \xrightarrow[R_3 \rightsquigarrow R_3 + 3R_1]{R_2 \rightsquigarrow R_2 + R_1} \left(\begin{array}{ccccc|c} \boxed{-1} & 2 & 1 & 0 & 1 & 4 \\ 0 & \boxed{1} & 1 & -1 & 1 & 5 \\ 0 & 4 & 4 & -4 & 4 & 20 \end{array}\right)$$

$$\xrightarrow{R_3 \rightsquigarrow R_3 - 4R_2} \left(\begin{array}{ccccc|c} \boxed{-1} & 2 & 1 & 0 & 1 & 4 \\ 0 & \boxed{1} & 1 & -1 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Only the left-hand part of the matrix is relevant to part (i). As we can see, there are two leading columns in the row-echelon form, with three non-leading columns (remember, only looking to the left of the vertical line). This means that $\operatorname{rank}(A) = 2$ and $\operatorname{nullity}(A) = 3$.

(ii) **<u>Hints</u>**

Look at the row-echelon form of the augmented matrix from our working in part (i). If there is a solution to the matrix equation (i.e. if the right-hand column is non-leading), then $\mathbf{b}$ is indeed in the image of A. Otherwise, it is not.

Here is a worked sample solution.

Worked Sample Solution

From part (i), we see that there exists a solution to the matrix equation $A\mathbf{x} = \mathbf{b}$ since the right-hand column is non-leading in the row-echelon form. Therefore, $\mathbf{b}$ is in the image of A.

3. **Hints**

Recall that the general solution to the matrix differential equation $\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = A\mathbf{y}$ for some given constant 2×2 matrix A is[1] $\mathbf{y} = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$, where λ_k and $\mathbf{v}_k$ are eigenvalue-eigenvector pairs of A.

Worked Sample Solution

The equivalent matrix form of the system of differential equations is

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \begin{pmatrix} 11 & -3 \\ 8 & 1 \end{pmatrix}\mathbf{y}, \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Using the given eigenvalues and eigenvectors of the matrix A, we thus see that the general solution to the system of differential equations is

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} 3 \\ 4 \end{pmatrix} e^{7t}, \quad c_1, c_2 \text{ arbitrary constants.}$$

Writing it out component-wise, we equivalently have that the general solution to the given system of differential equations is

$$\begin{aligned} y_1 &= c_1 e^{5t} + 3c_2 e^{7t} \\ y_2 &= 2c_1 e^{5t} + 4c_2 e^{7t}. \end{aligned}$$

4. (i) **Hints**

Make use of the fact that $p_k = ck^2$ is a *probability distribution.*

Worked Sample Solution

Since p_k is a (discrete) probability distribution function, then we know that

$$\sum_{k=1}^{5} p_k = \sum_{k=1}^{5} ck^2 = 1.$$

Solving for c, we find that

$$\begin{aligned} c(1 + 2^2 + 3^2 + 4^2 + 5^2) &= 1 \\ 55c &= 1, \end{aligned}$$

[1] This is the general solution **provided** that the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent (i.e. the matrix A is not what is called *defective*), which will *probably* always be the case for the purposes of MATH1231/41.

and hence

$$c = \frac{1}{55}.$$

(ii) **Hints**

Since, p_k is a (discrete) probabilty (mass) function, to find the probability that the random variable X takes a certain value, we simply substitute this into p_k. This should be easy now that we have c from part (i).

Worked Sample Solution

We have

$$\mathbb{P}(X = 3) = \frac{1}{55} \times 3^2 = \frac{9}{55}.$$

(iii) **Hints**

Since X is a discrete random variable, we can simply add up the probabilities of the outcomes that satisfy the event $\{X \leq 4\}$. Alternatively, we can note that this event is the complement of the event $\{X = 5\}$ and use our knowledge of probability of complementary events to answer the question.

Worked Sample Solution

Using a similar method to part (ii) for each case, we know that

$$\begin{aligned}
\mathbb{P}(X \leq 4) &= \mathbb{P}(X = 4) + \mathbb{P}(X = 3) + \mathbb{P}(X = 2) + \mathbb{P}(X = 1) \\
&= c \times 4^2 + c \times 3^2 + c \times 2^2 + c \times 1^2 \\
&= \frac{30}{55} \\
&= \frac{6}{11}.
\end{aligned}$$

Simpler calculation. Note that since X only takes integer values from 1 to 5, the event $\{X \leq 4\}$ is the complement of the event $\{X = 5\}$. Therefore,

$$\begin{aligned}
\mathbb{P}(X \leq 4) &= 1 - \mathbb{P}(X = 5) && (1) \\
&= 1 - \frac{1}{55} \times 5^2 = \frac{55}{55} - \frac{25}{55} && (2) \\
&= \frac{30}{55} && (3) \\
&= \frac{6}{11}. && (4)
\end{aligned}$$

MATH1231/1241 Algebra S2 2009 Test 2 v7A

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1. (i) **Hints**

   As usual, row-reduce the matrix A, and then the rank will be the number of leading columns in the row-echelon form of A, whilst the nullity is the number of *non*-leading columns in the row-echelon form. You should find that $\text{rank}\,(A) = 3$ and $\text{nullity}\,(A) = 1$. Here is a sample solution.

   Worked Sample Solution

   We row-reduce A:

$$A = \begin{pmatrix} \boxed{1} & -3 & -1 & 1 \\ 2 & -5 & 0 & 1 \\ -3 & 5 & -6 & 3 \end{pmatrix} \xrightarrow[R_3 \rightsquigarrow R_3 + 3R_1]{R_2 \rightsquigarrow R_2 - 2R_1} \begin{pmatrix} \boxed{1} & -3 & -1 & 1 \\ 0 & \boxed{1} & 2 & -1 \\ 0 & -4 & -9 & 6 \end{pmatrix}$$

$$\xrightarrow{R_3 \rightsquigarrow R_3+4R_2} \begin{pmatrix} \boxed{1} & -3 & -1 & 1 \\ 0 & \boxed{1} & 2 & -1 \\ 0 & 0 & \boxed{-1} & 2 \end{pmatrix}.$$

So there are three leading columns in the row-echelon form, with one non-leading column. This means that $\operatorname{rank}(A) = 3$ and $\operatorname{nullity}(A) = 1$.

(ii) **Hints**

Construct the appropriate augmented matrix to represent the corresponding matrix equation $A\mathbf{x} = \mathbf{0}$. Row-reduce this matrix to help find the kernel of A. However, instead of performing the calculations over again, notice that having the vector $\mathbf{0}$ on the right hand side of the matrix doesn't change any calculations/row operations, nor does it change the vector $\mathbf{0}$ itself. So, we can simply use the results from part (i). You should find that a basis for the kernel of A is

$$\left\{ \begin{pmatrix} -8 \\ -3 \\ 2 \\ 1 \end{pmatrix} \right\}.$$

Worked Sample Solution

When we set up the augmented matrix and row-reduced, we achieve the exact same matrix as in part (i) (as the column furthest to the right is all "0"s, which stays unchanged when performing elementary row operations).

So when we write $A\mathbf{x} = \mathbf{0}$ $\left(\text{where } \mathbf{x} = (x_1, x_2, x_3, x_4)^T\right)$, as an augmented matrix and row-reduce, we end up with

$$\left(\begin{array}{cccc|c} \boxed{1} & -3 & -1 & 1 & 0 \\ 0 & \boxed{1} & 2 & -1 & 0 \\ 0 & 0 & \boxed{-1} & 2 & 0 \end{array}\right).$$

Let $\boxed{x_4 = \alpha}$ where $\alpha \in \mathbb{R}$.

Then from row 3, we find that $x_3 = 2x_4 \implies \boxed{x_3 = 2\alpha}$.

From row 2, we have $x_2 = x_4 - 2x_3 \implies x_2 = \alpha - 4\alpha \implies \boxed{x_2 = -3\alpha}$.

Now, from row 1, we have $x_1 = 3x_2 + x_3 - x_4 = -9\alpha + 2\alpha - \alpha \implies \boxed{x_1 = -8\alpha.}$

Hence, our solution is given by

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \alpha \begin{pmatrix} -8 \\ -3 \\ 2 \\ 1 \end{pmatrix}.$$

Therefore, a basis for the kernel of A is

$$\left\{\begin{pmatrix}-8\\-3\\2\\1\end{pmatrix}\right\}.$$

2. To find the eigenvalues, we know that they are the solutions to the equation $\det(A-\lambda I) = 0$.

   So for eigenvalues λ,

$$\begin{aligned}\det(A-\lambda I) &= \begin{vmatrix}-7-\lambda & 5\\ 2 & -4-\lambda\end{vmatrix}\\ &= (7+\lambda)(4+\lambda)-10\\ &= \lambda^2+11\lambda+28-10\\ &= \lambda^2+11\lambda+18\\ &= (\lambda+9)(\lambda+2)=0.\end{aligned}$$

   Therefore, the eigenvalues are $\lambda_1=-9$ and $\lambda_2=-2$. For the eigenvectors, they are the basis vectors of the eigenspaces $E_\lambda = \ker(A-\lambda I)$.

   For $\lambda_1=-9$,

$$\begin{aligned}E_{\lambda_1} = E_{-9} = \ker(A+9I) &= \ker\begin{pmatrix}2 & 5\\ 2 & 5\end{pmatrix}\\ &= \text{span}\left\{\begin{pmatrix}-5\\2\end{pmatrix}\right\}. \quad \text{(by inspection)}\end{aligned}$$

   For $\lambda_2=-2$,

$$\begin{aligned}E_{\lambda_2} = E_{-2} = \ker(A+2I) &= \ker\begin{pmatrix}-5 & 5\\ 2 & -2\end{pmatrix}\\ &= \text{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}. \quad \text{(by inspection)}\end{aligned}$$

   Hence the eigenvalues are $\lambda_1 = -9$ with eigenvectors $\alpha\begin{pmatrix}-5\\2\end{pmatrix}$ for $\alpha\in\mathbb{R}, \alpha\neq 0$ and $\lambda_2=-2$ with eigenvectors $\beta\begin{pmatrix}1\\1\end{pmatrix}$ for $\beta\in\mathbb{R},\beta\neq 0$ (remember, $\mathbf{0}$ is not an eigenvector, by definition).

3. <u>**Hints**</u>

(i) Conditional probability: $\mathbb{P}\left(\text{Positive} \cap \text{Diseased}\right) = \mathbb{P}\left(\text{Positive}|\text{Diseased}\right) \times \mathbb{P}\left(\text{Diseased}\right) = 0.054$.

(ii) To find $\mathbb{P}\left(\text{Positive}\right)$, consider all the different cases (condition it on being diseased and not diseased).

$$\mathbb{P}(\text{Positive}) = 0.101.$$

(iii) Use Bayes' Rule, or note that $\mathbb{P}(A|B)\mathbb{P}(B) = \mathbb{P}(A \cap B) = \mathbb{P}(B|A)\mathbb{P}(A)$ and rearrange with appropriate events A and B.

$$\mathbb{P}\left(\text{Diseased}|\text{Positive}\right) = \frac{54}{101}.$$

Generally for probability questions, recall the following probability properties/rules

- $\mathbb{P}\left(A \cap B\right) = \mathbb{P}\left(A \mid B\right)\mathbb{P}\left(B\right) = \mathbb{P}\left(B \mid A\right)\mathbb{P}\left(A\right)$
- $\mathbb{P}\left(A \mid B\right) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}$
- $\mathbb{P}\left(A\right) = \mathbb{P}\left(A \cap B\right) + \mathbb{P}\left(A \cap B^c\right) = \mathbb{P}\left(A \mid B\right)\mathbb{P}\left(B\right) + \mathbb{P}\left(A \mid B^c\right)\mathbb{P}\left(B^c\right).$

Worked Sample Solution

(i) We have

$$\begin{aligned}\mathbb{P}\left(\text{diseased and positive}\right) = \mathbb{P}\left(\text{diseased} \cap \text{positive}\right) &= \mathbb{P}\left(\text{positive} \mid \text{diseased}\right)\mathbb{P}\left(\text{diseased}\right)\\ &= 0.9 \times 0.06\\ &= 0.054.\end{aligned}$$

Note: a common error in this question is to write

$$\mathbb{P}\left(\textit{diseased and positive}\right) = \mathbb{P}\left(\textit{diseased}\right) \times \mathbb{P}\left(\textit{positive}\right).$$

In general, it is ***not true*** *that $\mathbb{P}(A \cap B) = \mathbb{P}(A) \times \mathbb{P}(B)$, as this is true if and only if A and B are independent. And in this question, the events are not independent, since the probability of testing positive depends on whether the person is diseased.*

(ii) Using the Law of Total Probability, we have

$$\begin{aligned}\mathbb{P}\left(\text{positive}\right) &= \underbrace{\mathbb{P}\left(\text{positive} \mid \text{diseased}\right)\mathbb{P}\left(\text{diseased}\right)}_{=0.054 \text{ from part (i)}} + \mathbb{P}\left(\text{positive} \mid \text{not diseased}\right)\mathbb{P}\left(\text{not diseased}\right)\\ &= 0.054 + 0.05 \times (1 - 0.06)\\ &= 0.101.\end{aligned}$$

(iii) We have

$$\begin{aligned}
\mathbb{P}\left(\text{diseased} \mid \text{positive}\right) &= \frac{\mathbb{P}\left(\text{diseased and positive}\right)}{\mathbb{P}\left(\text{positive}\right)} \\
&\qquad \text{(by definition of conditional probability)} \\
&= \frac{0.054}{0.101} \\
&= \frac{54}{101}.
\end{aligned}$$

Alternatively, you can use Bayes' Rule to get the same result.

MATH1231/1241 Algebra S2 2011 Test 2 v1A

Full Solutions

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1. To prove that $T : \mathbb{P}_2 \to \mathbb{R}^2$ is a linear transformation, we need to show that it satisfies the *addition and scalar multiplication conditions*. Note that from basic calculus, we know that differentiation is a *linear operator*, i.e. that $(p+q)' = p' + q'$ (in words, the derivative of a sum is the sum of the derivatives) and $(\lambda p)' = \lambda p'$ (in words, constants can come outside the differentiation operator), for any $p, q \in \mathbb{P}_2$ and scalar $\lambda \in \mathbb{R}$. We will use these facts in our proof.

   - Let $p_1, p_2 \in \mathbb{P}_2$, so note

$$T\left(p_1(x)\right) = \begin{pmatrix} p_1(1) \\ p_1'(2) \end{pmatrix} \quad \text{and} \quad T\left(p_2(x)\right) = \begin{pmatrix} p_2(1) \\ p_2'(2) \end{pmatrix}.$$

Then, $p_1 + p_2$ is a polynomial in $\mathbb{P}_2$ and we have,

$$\begin{aligned}
T\left(\left(p_1+p_2\right)(x)\right) &\stackrel{\text{def}}{=} \begin{pmatrix} \left(p_1+p_2\right)(1) \\ \left(p_1+p_2\right)'(2) \end{pmatrix} \\
&= \begin{pmatrix} p_1(1)+p_2(1) \\ p_1'(2)+p_2'(2) \end{pmatrix} \quad \text{(since derivative of a sum is sum of derivatives)} \\
&= \begin{pmatrix} p_1(1) \\ p_1'(2) \end{pmatrix} + \begin{pmatrix} p_2(1) \\ p_2'(2) \end{pmatrix} \qquad \text{(definition of addition of vectors in } \mathbb{R}^2\text{)} \\
&\stackrel{\text{def}}{=} T\left(p_1(x)\right) + T\left(p_2(x)\right).
\end{aligned}$$

This implies that T satisfies the addition condition.

- Secondly, let $\lambda \in \mathbb{R}$ and $p \in \mathbb{P}_2$. Then λp is a polynomial in $\mathbb{P}_2$ that satisfies

$$\begin{aligned}
T\left(\lambda p\left(x\right)\right) &\stackrel{\text{def}}{=} \begin{pmatrix} (\lambda p)(1) \\ (\lambda p)'(2) \end{pmatrix} \\
&= \begin{pmatrix} \lambda p(1) \\ \lambda p'(2) \end{pmatrix} \quad \text{(using the facts mentioned at the start about differentiation)} \\
&\equiv \begin{pmatrix} \lambda p(1) \\ \lambda p'(2) \end{pmatrix} \\
&= \lambda \begin{pmatrix} p(1) \\ p'(2) \end{pmatrix} \qquad \text{(definition of scalar multiplication of vectors in } \mathbb{R}^2\text{)} \\
&\stackrel{\text{def}}{=} \lambda T\left(p\left(x\right)\right).
\end{aligned}$$

Thus, T satisfies the scalar multiplication condition.

Since T satisfies the addition and scalar multiplication conditions, it is a linear transformation.

2. (i) To find rank (A), we row-reduce A and find how many leading columns there are in the row-echelon form. Recall that

$$A = \begin{pmatrix} \boxed{1} & 2 & 1 & -1 & 3 \\ -2 & -5 & -1 & 3 & -8 \\ 1 & 5 & -2 & -3 & 5 \end{pmatrix}.$$

Performing the row operations $R_2 \rightsquigarrow R_2 + 2R_1$ and $R_3 \rightsquigarrow R_3 - R_1$, we see

$$A \sim \begin{pmatrix} \boxed{1} & 2 & 1 & -1 & 3 \\ 0 & \boxed{-1} & 1 & 1 & -2 \\ 0 & 3 & -3 & -2 & 2 \end{pmatrix}.$$

(Aside: The "$\sim$" in this context means "is row-equivalent to", which basically just means "can be row-reduced to (using elementary row operations)". For those of you who have done MATH1081 Discrete Mathematics, it can be shown that $\sim$ is an equivalence relation on the set of $m \times n$ matrices for any given positive integers m, n. You can find out more about row-equivalence at its Wikipedia page: `https://en.wikipedia.org/wiki/Row_equivalence`.)

Finally, performing $R_3 \rightsquigarrow R_3 + 3R_2$, we have

$$A \sim \begin{pmatrix} \boxed{1} & 2 & 1 & -1 & 3 \\ 0 & \boxed{-1} & 1 & 1 & -2 \\ 0 & 0 & 0 & \boxed{1} & -4 \end{pmatrix}.$$

As there are 3 leading columns in the row-echelon form, there are 3 linearly independent vectors of the 5 column vectors, i.e. $\operatorname{rank}(A) = 3$.

By the Rank-Nullity Theorem, we know that $\operatorname{rank}(A) + \operatorname{nullity}(A) = \#$ of columns of A.

Thus, $\operatorname{nullity}(A) = 5 - 3 = 2$.

(Alternative (but equivalent) reasoning: the nullity is the number of non-leading columns in the row-echelon form, i.e. 2.)

(ii) The kernel of A, written as $\ker(A)$, is the set of all solutions to $A\mathbf{x} = \mathbf{0}$, i.e. $\left\{\mathbf{x} \in \mathbb{R}^5 : A\mathbf{x} = \mathbf{0}\right\}$. This means that to find the kernel, we are essentially just solving $A\mathbf{x} = \mathbf{0}$. When we set up the augmented matrix and row-reduced, we achieve the exact same matrix as before (as the column furthest to the right is all "0"s, which stays unchanged when performing elementary row operations).

So when we write $A\mathbf{x} = \mathbf{0}$ $\left(\text{where } \mathbf{x} = (x_1, x_2, x_3, x_4, x_5)^T\right)$, as an augmented matrix and row-reduce, we end up with

$$\left(\begin{array}{ccccc|c} \boxed{1} & 2 & 1 & -1 & 3 & 0 \\ 0 & \boxed{-1} & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & \boxed{1} & -4 & 0 \end{array}\right).$$

As usual, set the non-leading columns' variables (in this case x_3 and x_5) to free parameters. So let $\boxed{x_3 = \beta}$ and $\boxed{x_5 = \alpha}$, where $\alpha, \beta \in \mathbb{R}$ are free parameters. Then from row 3, we obtain $x_4 = 4x_5 \implies \boxed{x_4 = 4\alpha}$.

From row 2, we have $x_2 = x_3 + x_4 - 2x_5 \implies x_2 = \beta + 4\alpha - 2\alpha \implies \boxed{x_2 = \beta + 2\alpha}$.

Now, from row 1, we have $x_1 = -2x_2 - x_3 + x_4 - 3x_5 = -2(\beta + 2\alpha) - \beta + 4\alpha - 3\alpha \implies \boxed{x_1 = -3\beta - 3\alpha}$.

Hence our solution is given by

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -3\beta - 3\alpha \\ \beta + 2\alpha \\ \beta \\ 4\alpha \\ \alpha \end{pmatrix}$$

$$= \alpha \begin{pmatrix} -3 \\ 2 \\ 0 \\ 4 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \text{where } \alpha, \beta \in \mathbb{R}.$$

Therefore, we see that $\ker(A) = \operatorname{span}\left\{ \begin{pmatrix} -3 \\ 2 \\ 0 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$, and so a basis for $\ker(A)$ is:

$$\left\{ \begin{pmatrix} -3 \\ 2 \\ 0 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

3. **Hints.** This whole question revolves around conditional probability, i.e.

$$\mathbb{P}(A \cap B) = \mathbb{P}(A \mid B)\,\mathbb{P}(B) = \mathbb{P}(B \mid A)\,\mathbb{P}(A).$$

Also note that *by definition* $\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$ (if $\mathbb{P}(B) \neq 0$). Remember also that $\cap$ (set intersection) generally refers to "and" for probability (whilst $\cup$, or set unions, generally refers to "or"). So if you see an "and" being used for probability, it generally refers to the intersection of sets/events. We also use the *Law of Total Probability*. Basically, it states that a probability can be broken down into the sum of all its cases. Google it and have a read of it as it can be quite tedious to *carefully* explain mathematically, despite actually being relatively intuitive! You can check out an example (and read about the Law) on its Wikipedia page here: `https://en.wikipedia.org/wiki/Law_of_total_probability#Example`. Sample solutions are below.

Sample solutions.

(i) We have

$$\begin{aligned} \mathbb{P}(\text{red die selected and 1 was thrown}) &= \mathbb{P}(\text{1 was thrown} \mid \text{red die selected})\,\mathbb{P}(\text{red die selected}) \\ &= \frac{4}{6} \times \frac{4}{7} \\ &= \frac{8}{21}. \end{aligned}$$

Note: a common error in this question is to write

$$\mathbb{P}\left(\textit{red die selected and 1 was thrown}\right) = \mathbb{P}\left(\textit{red die selected}\right) \times \mathbb{P}\left(\textit{1 was thrown}\right).$$

In general, it is ***not true*** *that* $\mathbb{P}(A \cap B) = \mathbb{P}(A) \times \mathbb{P}(B)$, *as this is true if and only if* A *and* B *are* *independent* *events. And in this question, the events are not independent, since the probability of a 1 being thrown heavily depends on the colour of the die.*

(ii) Using the Law of Total Probability, we have

$$\begin{aligned}
\mathbb{P}\left(1 \text{ was thrown}\right) &= \underbrace{\mathbb{P}\left(1 \text{ was thrown} \mid \text{red die selected}\right)\mathbb{P}\left(\text{red die selected}\right)}_{=\frac{8}{21} \text{ from part (i)}} \\
&\quad + \mathbb{P}\left(1 \text{ was thrown} \mid \text{blue die selected}\right)\mathbb{P}\left(\text{blue die selected}\right) \\
&= \frac{8}{21} + \frac{1}{6} \times \frac{3}{7} \\
&= \frac{16}{42} + \frac{3}{42} \\
&= \frac{19}{42}.
\end{aligned}$$

(iii) We have

$$\begin{aligned}
\mathbb{P}\left(\text{red die was selected} \mid 1 \text{ was thrown}\right) &= \frac{\mathbb{P}\left(\text{red die selected and 1 was thrown}\right)}{\mathbb{P}\left(1 \text{ was thrown}\right)} \\
&\qquad \text{(definition of conditional probability)} \\
&= \frac{8/21}{19/42} \times \frac{42}{42} = \frac{8 \times 2}{19} \\
&\qquad \text{(using our previous results)} \\
&= \frac{16}{19}.
\end{aligned}$$

MATH1231/1241 Algebra S2 2011 Test 2 v1B

Full Solutions

August 22, 2017

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---

1. (i) The rank of a matrix is basically the number of vectors required to span the column space of the matrix. In order to find this, we need to find how many linearly independent columns there are. Equivalently, the rank of A is the number of leading columns in the row-echelon form (in fact, this also always equals the number of leading *rows* in the row-echelon form).
So we row-reduce the matrix A. We have

$$A = \begin{pmatrix} \boxed{1} & -1 & 2 & 0 \\ 0 & 1 & 3 & 1 \\ 2 & -1 & 7 & 5 \\ -1 & 1 & -2 & 0 \end{pmatrix}.$$

Firstly, we perform $R_3 \rightsquigarrow R_3 - 2R_1$ and $R_4 \rightsquigarrow R_4 + R_1$,

$$A \sim \begin{pmatrix} \boxed{1} & -1 & 2 & 0 \\ 0 & \boxed{1} & 3 & 1 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Next, performing $R_3 \rightsquigarrow R_3 - R_2$,

$$A \sim \begin{pmatrix} \boxed{1} & -1 & 2 & 0 \\ 0 & \boxed{1} & 3 & 1 \\ 0 & 0 & 0 & \boxed{4} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(Aside: The "$\sim$" in this context means "is row-equivalent to", which basically just means "can be row-reduced to (using elementary row operations)". For those of you who have done MATH1081 Discrete Mathematics, it can be shown that $\sim$ is an equivalence relation on the set of $m \times n$ matrices for given positive integers m, n. You can find out more about row-equivalence at its Wikipedia page: `https://en.wikipedia.org/wiki/Row_equivalence`.)

Hence, we see that the first, second and fourth columns are linearly independent (as they are the leading columns of this row-reduced matrix). This means that these three columns span the column space of A.

Thus, $\text{rank}\,(A) = 3$. By the Rank-Nullity Theorem, which states that $\text{rank}\,(\text{A}) + \text{nullity}\,(\text{A}) = \#$ of columns of A, we know that $\text{nullity}\,(\text{A}) = 4 - 3 = 1$.

Alternative reasoning for rank and nullity part. Since A has three leading columns in the row-echelon form, $\text{rank}\,(A) = 3$. Since A has one non-leading column in the row-echelon form, $\text{nullity}\,(A) = 1$.

(ii) Note that the image of A refers to the vector space $\left\{\mathbf{y} \in \mathbb{R}^4 : A\mathbf{x} = \mathbf{y} \text{ for } \mathbf{x} \in \mathbb{R}^4\right\}$. So to see if $\mathbf{b} = (1, 1, 6, -7)^T$ is in the image of A, we need to see if there exists a solution for $\mathbf{x} \in \mathbb{R}^4$ such that $A\mathbf{x} = \mathbf{b}$.

Constructing an augmented matrix,

$$\left(\begin{array}{cccc|c} 1 & -1 & 2 & 0 & 1 \\ 0 & 1 & 3 & 1 & 1 \\ 2 & -1 & 7 & 5 & 6 \\ -1 & 1 & -2 & 0 & -7 \end{array}\right).$$

Recalling from above, we can perform one row operation to make the last row basically all "0"s, we give it a go to see if the last row becomes invalid (in the hope that it does to say that $\mathbf{b}$ is not in the image of A).

So performing the row operations $R_4 \rightsquigarrow R_4 + R_1$, we obtain

$$\left(\begin{array}{cccc|c} 1 & -1 & 2 & 0 & 1 \\ 0 & 1 & 3 & 1 & 1 \\ 2 & -1 & 7 & 5 & 6 \\ 0 & 0 & 0 & 0 & -6 \end{array}\right).$$

Now, the last row is a contradictory equation of $0 = -6$, so $\mathbf{b} \notin \operatorname{im}(A)$ as there does not exist $\mathbf{x} \in \mathbb{R}^4$ such that this is true.

Tip. Here we really got lucky that we could quickly get to the answer from the original matrix. In general though this would not be the case and you may need to end up doing a lot of row-reduction again that you already did in part (i). To overcome this, you should actually augment $\mathbf{b}$ when you do the row-reductions initially in question (i), so that you'll have reduced with $\mathbf{b}$ present, and will need to do no further row-reductions in part (ii) and can just read off the answer from your hard work row-reducing in part (i). (E.g. refer to our Worked Sample Solution to the Question # 2 on page 2 of this document.)

A lot of students jump into row-reducing A when they look at question (i) and then are forced to do basically the same row-reduction again for part (ii). So make sure you read all parts of the question before starting writing, and if you see a part asking if a vector is in the image of A or something like this, make sure to augment $\mathbf{b}$ straightaway before beginning row-reducing.

2. (i) The eigenvalues λ are the solutions to the equation $\det(A - \lambda I) = 0$.
So for eigenvalues λ,

$$\begin{aligned}\det(A - \lambda I) &= \begin{vmatrix} 7-\lambda & -1 \\ 6 & 2-\lambda \end{vmatrix} \\ &= (7-\lambda)(2-\lambda) + 6 \\ &= \lambda^2 - 9\lambda + 14 + 6 = \lambda^2 - 9\lambda + 20 \\ &= (\lambda - 4)(\lambda - 5) = 0.\end{aligned}$$

Thus, the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = 5$. For the eigenvectors, they are the basis vectors of the eigenspaces $E_\lambda = \ker(A - \lambda I)$.
For $\lambda_1 = 4$,

$$\begin{aligned}E_{\lambda_1} = E_4 = \ker(A - 4I) &= \ker\begin{pmatrix} 3 & -1 \\ 6 & -2 \end{pmatrix} \\ &= \operatorname{span}\left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right\} \quad \text{(by inspection, explained below).}\end{aligned}$$

$\Big($Tip – Explanation of inspection (included for your understanding, you don't need to provide this in your working): We can quickly find the eigenvector by thinking about what vector we have to multiply the matrix by to get 0. If you have done things right, we first consider the top row (or whichever row looks easier). Then, we think about what vector it multiplies with to obtain 0. This is then your eigenvector! You can double check it by multiplying it with the bottom row too if you wish. Since we are dealing with distinct eigenvalues here, the eigenspaces must all be one-dimensional, so we only need to find one eigenvector to get the basis for the eigenspace (for eigenvalues of algebraic multiplicity (multiplicity in the characteristic polynomial) more than 1, this will not necessarily be the case though).$\Big)$

For $\lambda_2 = 5$,

$$\begin{aligned} E_{\lambda_2} = E_5 = \ker(A - 5I) &= \ker \begin{pmatrix} 2 & -1 \\ 6 & -3 \end{pmatrix} \\ &= \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}. \end{aligned}$$

Hence, the eigenvalues are $\lambda_1 = 4$ with corresponding eigenvector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\lambda_2 = 5$ with corresponding eigenvector $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(ii) Yes, A is diagonalisable as it can be written in the form $A = PDP^{-1}$ where $P = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$ is invertible and $D = \begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}$ is diagonal.

Alternate Solution. Yes, A is diagonalisable since its eigenvalues are distinct.

Further tips

When finding the eigenvectors for an eigenvalue λ, when you row-reduce the matrix $A - \lambda I$, this matrix *must* have at least one zero row in the row echelon form. This is because λ being an eigenvalue means that $A - \lambda I$ (a square matrix) has determinant 0, which is equivalent to saying it has a zero row in the row echelon form. Therefore, if you end up row-reducing and seeing that there are no zero rows, you must have made a mistake somewhere, and should go back and find it!

In the 2×2 matrix case, it is even easier to check your work here. In the 2×2 case, the matrix $A - \lambda I$ must be such that one of its rows is a constant multiple of the other (this also applies to the columns), if (and only if) λ is an eigenvalue. (This is because a 2×2 matrix has a zero row in the row echelon form if and only if one of its rows is a multiple of the other.) In other words, if when you write the matrix $A - \lambda I$ when finding eigenvectors for an eigenvalue λ, and you see that neither rows is a multiple of the other, you must have made a mistake somewhere (because the matrix won't end up with a zero row in the row-echelon form), and should find it!

Finally, in the Alternate Solution for part (ii) above, it is important to keep in mind that while any matrix A with distinct eigenvalues is diagonalisable, *the converse is not true.* In other words, there exist diagonalisable matrices with repeated eigenvalues. So *if* a matrix A has all distinct eigenvalues, *then* A is diagonalisable, but not necessarily the other way round.

3. To show this is a (discrete) probability distribution, we need to show that $p_k \geq 0$ for $k = 2, 3, 4, \ldots$, and $\sum\limits_{k=2}^{\infty} p_k = 1$.

   It is clear that $p_k \geq 0$ for all $k = 2, 3, 4, \ldots$, because 5^k is non-negative for all such k, and hence so is $\frac{20}{5^k}$. Now, consider the sum $S \equiv \sum\limits_{k=2}^{\infty} p_k = \sum\limits_{k=2}^{\infty} \frac{20}{5^k}$. Note that this is a geometric series with starting term $a = \frac{20}{5^2} = \frac{20}{25} = \frac{4}{5}$ and common ratio $r = \frac{1}{5}$. As the geometric series has common ratio smaller than 1, this series converges and

$$\begin{aligned} S &= \frac{a}{1-r} \\ &= \frac{4/5}{1-\frac{1}{5}} \\ &= \frac{4/5}{4/5} \\ &= 1. \end{aligned}$$

   Therefore, the given sequence is a probability distribution.

MATH1231/1241 Algebra S2 2011 Test 2 v2A

Full Solutions

August 22, 2017

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1. (i) Anticipating we would need to augment A with $\mathbf{b}$ in question (ii), we shall do this now, so that we would not need to do the whole row-reduction again. Writing the components of $\mathbf{b}$ as separate columns in the right-hand part of the augmented matrix, we row-reduce:

$$\left(\begin{array}{ccccc|ccc} \boxed{1} & 3 & 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 5 & 0 & 3 & 1 & 0 & 1 & 0 \\ -2 & 4 & -7 & 2 & 4 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 \rightsquigarrow R_3+2R_1]{R_2 \rightsquigarrow R_2-R_1} \left(\begin{array}{ccccc|ccc} \boxed{1} & 3 & 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & \boxed{2} & -1 & 1 & 1 & -1 & 1 & 0 \\ 0 & 10 & -5 & 6 & 4 & 2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \rightsquigarrow R_3 - 5R_2} \left(\begin{array}{ccccc|ccc} \boxed{1} & 3 & 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & \boxed{2} & -1 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} & -1 & 7 & -5 & 1 \end{array}\right).$$

So the row-echelon form of A (the left-hand matrix above) has three leading columns and two non-leading columns. This means that $\operatorname{rank}(A) = 3$ and $\operatorname{nullity}(A) = 2$.

(ii) Since the row-echelon form of the matrix A (found above) has no zero rows, there will be a solution of $\mathbf{x} \in \mathbb{R}^5$ to the linear system $A\mathbf{x} = \mathbf{b}$ for any given $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \mathbb{R}^3$.

In other words, $\operatorname{im}(A)$ is the whole of $\mathbb{R}^3$, so there are no further conditions needed on b_1, b_2, b_3.

2. We need to find the eigenvalues and eigenvectors of A.
The eigenvalues are the solutions λ to the equation $\det(A - \lambda I) = 0$. But

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & -5 \\ -3 & 4-\lambda \end{vmatrix} \\ &= (2-\lambda)(4-\lambda) - 15 \\ &= \lambda^2 - 6\lambda + 8 - 15 \\ &= \lambda^2 - 6\lambda - 7 \\ &= (\lambda - 7)(\lambda + 1). \end{aligned}$$

Therefore, the eigenvalues are $\lambda_1 = 7$ and $\lambda_2 = -1$.
Now, to find the eigenvectors, we compute the eigenspaces $E_\lambda = \ker(A - \lambda I)$ for each eigenvalue λ. Note that since the eigenvalues are distinct (no repeated ones), the eigenspaces are guaranteed to be one-dimensional, so we just need one independent eigenvector for each eigenvalue. This makes it relatively easy to find these by "inspection" from the matrix $A - \lambda I$.
For $\lambda_1 = 7$,

$$\begin{aligned} E_{\lambda_1} = E_7 = \ker(A - 7I) &= \ker \begin{pmatrix} -5 & -5 \\ -3 & -3 \end{pmatrix} \\ &= \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}. \qquad \text{(by inspection)} \end{aligned}$$

For $\lambda_2 = -1$,

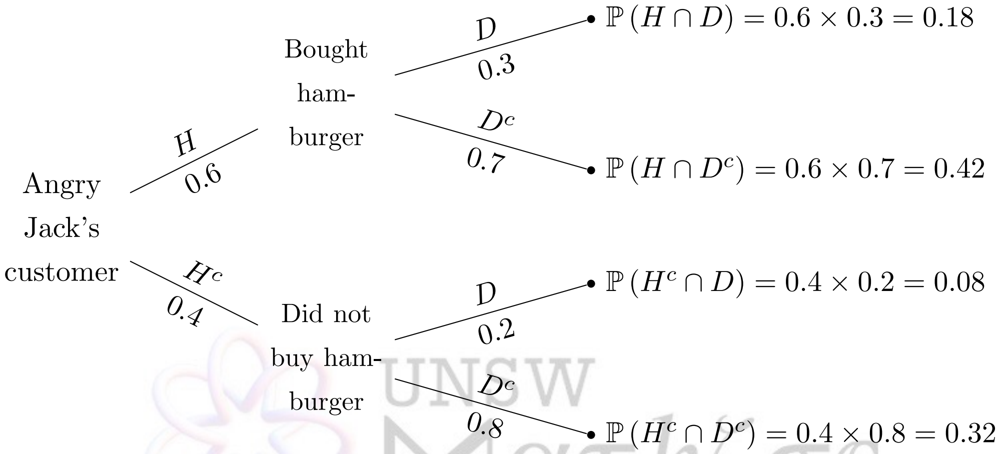
$$E_{\lambda_2} = E_{-1} = \ker(A + I) = \ker \begin{pmatrix} 3 & -5 \\ -3 & 5 \end{pmatrix}$$

$$= \operatorname{span}\left\{\begin{pmatrix}5\\3\end{pmatrix}\right\}. \qquad \text{(by inspection)}$$

That is, the eigenvalues and corresponding eigenvectors are $\lambda_1 = 7$ with eigenvector $\mathbf{v}_1 = \begin{pmatrix}1\\-1\end{pmatrix}$ and $\lambda_2 = -1$ with eigenvector $\mathbf{v}_2 = \begin{pmatrix}5\\3\end{pmatrix}$.

Thus, letting $D = \begin{pmatrix}7 & 0\\0 & -1\end{pmatrix}$ and $M = \begin{pmatrix}1 & 5\\-1 & 3\end{pmatrix}$, we have $A = MDM^{-1}$.

3. Drawing a *tree diagram* usually helps organise our thoughts immensely. Let H be the event that the customer bought a hamburger and similarly D for the event they bought a drink. We then have the following tree diagram based on the given data.



(i) Recall that "and" generally refers to set intersection in probability. In terms of our notation, we are asked to find $\mathbb{P}(H \cap D)$. We have

$$\begin{aligned}
\mathbb{P}(H \cap D) &= \mathbb{P}(D \mid H)\,\mathbb{P}(H) \\
&\qquad \text{(this corresponds to the top-most node in the tree diagram)} \\
&= 0.3 \times 0.6 \\
&= 0.18.
\end{aligned}$$

(ii) We have

$$\begin{aligned}
\mathbb{P}(D) &= \mathbb{P}(D \mid H)\,\mathbb{P}(H) \\
&\quad + \mathbb{P}(D \mid H^c)\,\mathbb{P}(H^c) && \text{(Law of Total Probability)} \\
&= 0.18 + 0.2 \times 0.4 && \text{(using part (i)'s answer for the first summand)} \\
&= 0.26.
\end{aligned}$$

(Or using our tree diagram, we can see the answer is $0.18 + 0.08$, i.e. the sum of the nodes ending in D, which are the ones where a drink was bought.)

(iii) We have

$$\begin{aligned}
\mathbb{P}(H \mid D) &= \frac{\mathbb{P}(H \cap D)}{\mathbb{P}(D)} && \text{(definition of conditional probability)} \\
&= \frac{0.18}{0.26} = \frac{18}{26} && \text{(from earlier parts)} \\
&= \frac{9}{13}. && \text{(simplifying)}
\end{aligned}$$

Remark. While tree diagrams are very helpful for getting a feel for questions, it may be good, even if you draw a tree diagram, to still try and provide reasoning for your answers in terms of probability rules if you can (like Law of Total Probability etc.), in case the marker does not consider just tree diagrams to be rigorous enough working. Also, if you can see how to do these questions without drawing a tree diagram (i.e. just with the reasoning given in our sample answers), by all means do so.

MATH1231/1241 Algebra S2 2012 Test 2 v1A

Answers/Hints & Worked Sample Solutions

August 22, 2017

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1. **Hints**

   Show that it satisfies the addition and scalar multiplication conditions.

   Worked Sample Solution

   Let $p, q \in \mathbb{P}_2$. Then $p+q$ is a polynomial in $\mathbb{P}_2$ and

$$T\left((p+q)(x)\right) \stackrel{\text{def}}{=} \begin{pmatrix} (p+q)(0) \\ (p+q)(1) \\ (p+q)(2) \end{pmatrix}$$

$$
\begin{aligned}
&= \begin{pmatrix} p(0)+q(0) \\ p(1)+q(1) \\ p(2)+q(2) \end{pmatrix} && \text{(definition of } p+q\text{)} \\
&= \begin{pmatrix} p(0) \\ p(1) \\ p(2) \end{pmatrix} + \begin{pmatrix} q(0) \\ q(1) \\ q(2) \end{pmatrix} && \text{(definition of vector addition in } \mathbb{R}^3\text{)} \\
&\stackrel{\text{def}}{=} T(p(x)) + T(q(x)).
\end{aligned}
$$

Thus T satisfies the addition condition.
Now, let $p \in \mathbb{P}_2$ and $\alpha \in \mathbb{R}$. Then αp is a polynomial in $\mathbb{P}_2$ with

$$
\begin{aligned}
T(\alpha p(x)) &\stackrel{\text{def}}{=} \begin{pmatrix} \alpha p(0) \\ \alpha p(1) \\ \alpha p(2) \end{pmatrix} \\
&= \alpha \begin{pmatrix} p(0) \\ p(1) \\ p(2) \end{pmatrix} && \text{(definition of scalar multiplication in } \mathbb{R}^3\text{)} \\
&\stackrel{\text{def}}{=} \alpha T(p(x)).
\end{aligned}
$$

Thus T also satisfies the scalar multiplication condition. Therefore, T is linear. Q.E.D.

2. (i) **<u>Hints</u>**

As usual, row-reduce the matrix A, and then the rank will be the number of leading columns in the row-echelon form of A, whilst the nullity is the number of *non*-leading columns in the row-echelon form. You should find that $\operatorname{rank}(A) = 3$ and $\operatorname{nullity}(A) = 1$. Here is a sample solution. Try to do both parts (i) and (ii) at the same time for efficiency!

Worked Sample Solution

We row-reduce A:

$$A = \begin{pmatrix} \boxed{1} & 2 & 1 & 1 \\ 3 & 5 & 0 & 4 \\ 2 & 6 & 8 & 1 \end{pmatrix} \xrightarrow[R_3 \rightsquigarrow R_3 - 2R_1]{R_2 \rightsquigarrow R_2 - 3R_1} \begin{pmatrix} \boxed{1} & 2 & 1 & 1 \\ 0 & \boxed{-1} & -3 & 1 \\ 0 & 2 & 6 & -1 \end{pmatrix}$$

$$\xrightarrow{R_3 \rightsquigarrow R_3 + 2R_2} \begin{pmatrix} \boxed{1} & 2 & 1 & 1 \\ 0 & \boxed{-1} & -3 & 1 \\ 0 & 0 & 0 & \boxed{1} \end{pmatrix}.$$

So there are three leading columns in the row-echelon form, with one non-leading column. This means that $\operatorname{rank}(A) = 3$ and $\operatorname{nullity}(A) = 1$.

(ii) **Hints**

Recall that the kernel of a matrix is simply the set of vectors that multiply the matrix to yield the zero vector. Mathematically, $\ker(A) = \left\{\mathbf{x} \in \mathbb{R}^4 : A\mathbf{x} = \mathbf{0}\right\}$, where $\mathbf{0}$ is the zero vector of $\mathbb{R}^3$. In other words, we are solving for $\mathbf{x}$ in the equation $A\mathbf{x} = \mathbf{0}$. So as usual, to find the basis for the kernel, we just row-reduce A (conveniently done in part (i)) and find the solutions from that (using back-substitution etc.). You may think that normally to solve $A\mathbf{x} = \mathbf{b}$, we would augment A with the vector $\mathbf{b}$ too. The reason we don't need to do this here is that when finding the kernel, the vector $\mathbf{b}$ is just $\mathbf{0}$, and no matter what elementary row operations we perform, this vector would stay as $\mathbf{0}$, so there is no need to actually augment it. You can just imagine it is there in your mind.

It can be found that a basis is:

$$\left\{ \begin{pmatrix} 5 \\ -3 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Here is a worked sample solution.

Worked Sample Solution

Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ be a vector in $\ker(A)$, i.e. let $A\mathbf{x} = \mathbf{0}$. Then from our row-echelon form of A in part (i), we see column three is non-leading, so we set $\boxed{x_3 = \alpha}$, where $\alpha \in \mathbb{R}$ is a free parameter. From row 4, we see that $\boxed{x_4 = 0}$.

Now from row 3, we have

$$x_2 = -3x_3 + x_4$$

$$\implies x_2 = -3\alpha + 0,$$

so $\boxed{x_2 = -3\alpha}$.
Now, from row 1, we have

$$\begin{aligned} x_1 &= -2x_2 - x_3 - x_4 \\ &= -2\left(-3\alpha\right) - \alpha - 0 \\ &= 6\alpha - \alpha \\ &= 5\alpha, \end{aligned}$$

so $\boxed{x_1 = 5\alpha}$. Thus $\mathbf{x}$ is in $\ker\left(A\right)$ if and only if

$$\begin{aligned} \mathbf{x} \equiv \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} &= \begin{pmatrix} 5\alpha \\ -3\alpha \\ \alpha \\ 0 \end{pmatrix} \\ &= \alpha \begin{pmatrix} 5 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \end{aligned}$$

for some $\alpha \in \mathbb{R}$. Thus a basis for the kernel of A is

$$\left\{ \begin{pmatrix} 5 \\ -3 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

3. **General Hints**
These questions test your understanding of the rules of basic probability. So make sure to keep in mind these rules, such as the definition of conditional probability, the *multiplication rule* $\mathbb{P}\left(A \cap B\right) = \mathbb{P}\left(A\right)\mathbb{P}\left(B \mid A\right)$, the rule $\mathbb{P}\left(A\right) = \mathbb{P}\left(A \cap B\right) + \mathbb{P}\left(A \cap B^c\right)$, etc. The aforementioned ideas are the main ones needed to do this particular problem, as the worked sample solutions will show. In general though for probability questions, other important rules you may find useful to be familiar with include *Bayes' Rule*, the *Law of Total Probability*, and for harder questions, the *Inclusion/Exclusion Principle*. You can read up on these various rules online or from your course pack if you are unsure of them

before tackling this question.

Numerical Answers

(i) $\mathbb{P}\left(\text{car was red and caught speeding}\right) = 0.12$.

(ii) $\mathbb{P}\left(\text{car caught speeding}\right) = 0.15$.

(iii) $\mathbb{P}\left(\text{car was red} \mid \text{caught speeding}\right) = 0.8$.

Worked Sample Solutions

We will use the following notation. Let R be the event that the car was red and S the event that the car was caught speeding. Then we are given the following data: $\mathbb{P}\left(R\right) = 0.4$, $\mathbb{P}\left(S \mid R\right) = 0.3$, $\mathbb{P}\left(S \mid R^c\right) = 0.05$. Note that $\mathbb{P}\left(R^c\right) = 1 - \mathbb{P}(R) = 1 - 0.4 = 0.6$.

(i) We are asked to find $\mathbb{P}\left(R \cap S\right)$. We have

$$\begin{aligned}\mathbb{P}\left(R \cap S\right) &= \mathbb{P}\left(R\right)\mathbb{P}\left(S \mid R\right) \\ &= 0.4 \times 0.3 && \text{(given data)} \\ &= 0.12.\end{aligned}$$

(ii) We are asked to find $\mathbb{P}\left(S\right)$. Using the probability rule $\mathbb{P}\left(A\right) = \mathbb{P}\left(A \cap B\right) + \mathbb{P}\left(A \cap B^c\right)$ (which follows from the facts that $A = (A \cap B) \cup (A \cap B^c)$, $A \cap B$ and $A \cap B^c$ are disjoint and the axiom of probability that states that the probability of a union is the sum of probability for *pairwise disjoint events*; it is easily seen from a Venn diagram too), we have

$$\mathbb{P}\left(S\right) = \mathbb{P}\left(S \cap R\right) + \mathbb{P}\left(S \cap R^c\right). \qquad (*)$$

Now, $S \cap R$ and $R \cap S$ are the same event, so we know from part (i) that $\mathbb{P}\left(S \cap R\right) = 0.12$. To find $\mathbb{P}\left(S \cap R^c\right)$, we have

$$\begin{aligned}\mathbb{P}\left(S \cap R^c\right) &= \mathbb{P}\left(R^c\right)\mathbb{P}\left(S \mid R^c\right) \\ &= 0.6 \times 0.05 && \text{(using given data)} \\ &= 0.03.\end{aligned}$$

Putting these findings into Equation $(*)$, we have

$$\begin{aligned}\mathbb{P}\left(S\right) &= 0.12 + 0.03 \\ &= 0.15.\end{aligned}$$

(iii) We are asked to find $\mathbb{P}\left(R \mid S\right)$. We have

$$\mathbb{P}\left(R \mid S\right) = \frac{\mathbb{P}\left(R \cap S\right)}{\mathbb{P}\left(S\right)} \qquad \text{(definition of conditional probability)}$$

$$
\begin{aligned}
&= \frac{0.12}{0.15} && \text{(from parts (i) and (ii))} \\
&= \frac{12}{15} \div \frac{3}{3} \\
&= \frac{4}{5} \\
&= 0.8.
\end{aligned}
$$

MATH1231/1241 Algebra S2 2012 Test 2 v1B

Hints & Worked Sample Solutions

August 22, 2017

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1. **Hints**

   (i) It can be found[1] that $\text{rank}(A) = 3$ and $\text{nullity}(A) = 0$, by row-reducing A and looking at the number of leading and non-leading columns in the row-echelon form.

   (ii) It can also be found[1] that $\mathbf{b}$ is not in the image of A.

   Worked Sample Solutions

   (i) Anticipating we would need to augment A with $\mathbf{b}$ in question (ii), we shall do this now, so that we would not need to do the whole row-reduction again. Writing the

[1]Refer to Question 2 in Test 2 S2 2009 v3b

components of $\mathbf{b}$ as separate columns in the right-hand part of the augmented matrix, we row-reduce:

$$\left(\begin{array}{ccc|c} \boxed{1} & 2 & -1 & 3 \\ -3 & -3 & 2 & -8 \\ 1 & 2 & 1 & 4 \\ 2 & 7 & -3 & 0 \end{array}\right) \xrightarrow[R_3 \rightsquigarrow R_3 - R_1, R_4 \rightsquigarrow R_4 - 2R_1]{R_2 \rightsquigarrow R_2 + 3R_1} \left(\begin{array}{ccc|c} \boxed{1} & 2 & -1 & 3 \\ 0 & \boxed{3} & -1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 3 & -1 & -6 \end{array}\right)$$

$$\xrightarrow{R_4 \rightsquigarrow R_4 - R_2} \left(\begin{array}{ccc|c} \boxed{1} & 2 & -1 & 3 \\ 0 & \boxed{3} & -1 & 1 \\ 0 & 0 & \boxed{2} & 1 \\ 0 & 0 & 0 & -7 \end{array}\right)$$

So the row-echelon form of A (the left-hand matrix above) has three leading columns and no non-leading columns. This means that $\operatorname{rank}(A) = 3$ and $\operatorname{nullity}(A) = 0$.

(ii) From the row-echelon form above, we see that the right-hand column is leading. This means that $\mathbf{b}$ is **not** in the image of A.

2. **Hints**

(i) It can be found[2] that the eigenvalues are $\lambda = 3$ and 5 with corresponding eigenvectors being (up to constant non-zero multiples) $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ respectively.

(ii) Also, this implies[2] that the matrix A is diagonalisable, with $A = PDP^{-1}$, where $D = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & 1 \\ 4 & 2 \end{pmatrix}$.

Worked Sample Solutions

(i) We have

$$A - \lambda I = \begin{pmatrix} 7-\lambda & -1 \\ 8 & 1-\lambda \end{pmatrix}.$$

For the eigenvalues, we set the determinant of this equal to 0 and solve for λ. We have

$$\begin{aligned} \det(A - \lambda I) &= (7-\lambda)(1-\lambda) + 8 \\ &= \lambda^2 - 8\lambda + 15 \\ &= (\lambda - 3)(\lambda - 5). \end{aligned}$$

Hence the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 5$. To find the corresponding eigenvectors, we find bases for the eigenspaces $E_\lambda = \ker(A - \lambda I)$ for each eigenvalue λ. We have

$$E_{\lambda_1} = E_3 = \ker(A - 3I)$$

[2]Refer to Question 2 in Test 2 S2 2011 v1b

$$
\begin{aligned}
&= \ker \begin{pmatrix} 7-3 & -1 \\ 8 & 1-3 \end{pmatrix} \\
&= \ker \begin{pmatrix} 4 & -1 \\ 8 & -2 \end{pmatrix} \\
&= \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right\}, \qquad \text{(by inspection)}
\end{aligned}
$$

and so a corresponding eigenvector for eigenvalue $\lambda_1 = 3$ is $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$. Also,

$$
\begin{aligned}
E_{\lambda_2} = E_5 &= \ker (A - 5I) \\
&= \ker \begin{pmatrix} 7-5 & -1 \\ 8 & 1-5 \end{pmatrix} \\
&= \ker \begin{pmatrix} 2 & -1 \\ 8 & -4 \end{pmatrix} \\
&= \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}, \qquad \text{(by inspection)}
\end{aligned}
$$

and so a corresponding eigenvector for eigenvalue $\lambda_2 = 5$ is $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. In summary, the eigenvalues of A are $\lambda_1 = 3$ with corresponding eigenvectors being (non-zero multiples of) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $\lambda_2 = 5$ with corresponding eigenvectors being (non-zero multiples of) $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(ii) Yes, A is diagonalisable, because it is 2×2 and has two linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2$.
Alternative answer. Yes, A is diagonalisable. This is implied by the fact that its eigenvalues are distinct.

3. **Hints**

This is a probability distribution if $p_k \geq 0$ for $k = 2, 3, \ldots$ and if $\sum_{k=2}^{\infty} p_k = 1$. It can be shown[3] that these results indeed hold. And so, the sequence is a probability distribution.

Worked Sample Solution

Clearly, $p_k \geq 0$ for every $k = 2, 3, 4, \ldots$, because for all such k, $3^k > 0$ and also $6 > 0$. Furthermore, we have

$$\sum_{k=2}^{\infty} p_k = \sum_{k=2}^{\infty} \frac{6}{3^k}.$$

[3] Refer to Question 3 in Test 2 S2 2011 v1b

This is an infinite geometric series with common ratio $r = \frac{1}{3}$ and starting term $a = \frac{6}{3^2} = \frac{2}{3}$. Since the common ratio satisfies $|r| < 1$, the series converges and is equal to

$$\frac{a}{1-r} = \frac{\frac{2}{3}}{1-\frac{1}{3}} = 1.$$

Hence the sum of all the p_k's is 1, and the p_k's are all non-negative. Therefore, the given sequence defines a probability distribution.

MATH1231/1241 Algebra S2 2012 Test 2 v2A

Hints/Answers & Worked Sample Solutions

August 22, 2017

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1. **Hints/Answers**

   (i) It can be found[1] that $\text{rank}(A) = 3$ and $\text{nullity}(A) = 2$, by row-reducing A looking at the number of leading and non-leading columns.

   (ii) $\mathbf{b}$ always belongs to $\text{im}(A)$, since $\text{rank}(A) = 3$ and $\dim\left(\mathbb{R}^3\right) = 3$.

[1] Refer to Question 1 in Test 2 S2 2009 v3b

Worked Sample Solutions

(i) As usual, we row-reduce A. We have

$$A = \begin{pmatrix} \boxed{-1} & 1 & 3 & 2 & -1 \\ 2 & 0 & -5 & -3 & -1 \\ 1 & 1 & -2 & -1 & 3 \end{pmatrix} \xrightarrow[R_3 \rightsquigarrow R_3 + R_1]{R_2 \rightsquigarrow R_2 + 2R_1} \begin{pmatrix} \boxed{-1} & 1 & 3 & 2 & -1 \\ 0 & \boxed{2} & 1 & 1 & -3 \\ 0 & 2 & 1 & 1 & 2 \end{pmatrix}$$

$$\xrightarrow{R_3 \rightsquigarrow R_3 - R_2} \begin{pmatrix} \boxed{-1} & 1 & 3 & 2 & -1 \\ 0 & \boxed{2} & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 & \boxed{5} \end{pmatrix}.$$

As we can see, there are three leading columns in the row-echelon form, so $\text{rank}(A) = 3$. Also, there are two non-leading columns in the row-echelon form, so $\text{nullity}(A) = 2$.

(ii) No extra conditions are required on the components of the vector $\mathbf{b} \in \mathbb{R}^3$. This is because A has rank 3 (and its image is a subspace of $\mathbb{R}^3$ as it has three rows), so its columns span the entire $\mathbb{R}^3$, i.e. every vector in $\mathbb{R}^3$ belongs to the image of A.

2. **Hints/Answers**

It can be found[2] that the eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = -4$, with corresponding eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$ respectively. And so, $D = \begin{pmatrix} 5 & 0 \\ 0 & -4 \end{pmatrix}$ and $M = \begin{pmatrix} 1 & -4 \\ 2 & 1 \end{pmatrix}$.

Note that your columns of matrix M may differ by non-zero multiples. For example, M could also be $M = \begin{pmatrix} 3 & 4 \\ 6 & -1 \end{pmatrix}$ where the first column is multiplied by 3, and the second column is multiplied by -1. But either answer would be correct. You could also have written your diagonal matrix in the opposite order. This is fine too, but you would also need to write the M matrix's columns in the opposite order to what we have done then. (Remember, the order of the columns of M going from left to right must match the order of the eigenvalues placed into the diagonal entries in D.)

Worked Sample Solution

We are being asked to diagonalise A. We have

$$A - \lambda I = \begin{pmatrix} -3 - \lambda & 4 \\ 2 & 4 - \lambda \end{pmatrix}.$$

Hence

$$\begin{aligned} \det(A - \lambda I) &= (-3 - \lambda)(4 - \lambda) - 8 \\ &= \lambda^2 - \lambda - 20 \end{aligned}$$

[2] Refer to Question 2 in Test 2 S2 2009 v3b

$$= (\lambda - 5)(\lambda + 4).$$

So the eigenvalues of A are $\lambda_1 = 5, \lambda_2 = -4$. Now we find corresponding eigenvectors. We have

$$\begin{aligned}
\ker(A - \lambda_1 I) = \ker(A - 5I) &= \ker\begin{pmatrix} -3-5 & 4 \\ 2 & 4-5 \end{pmatrix} \\
&= \ker\begin{pmatrix} -8 & 4 \\ 2 & -1 \end{pmatrix} \\
&= \operatorname{span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}. && \text{(by inspection)}
\end{aligned}$$

So we may take $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ as the corresponding eigenvector for eigenvalue $\lambda_1 = 5$. Also,

$$\begin{aligned}
\ker(A - \lambda_2 I) = \ker(A + 4I) &= \ker\begin{pmatrix} -3+4 & 4 \\ 2 & 4+4 \end{pmatrix} \\
&= \ker\begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix} \\
&= \operatorname{span}\left\{\begin{pmatrix} 4 \\ -1 \end{pmatrix}\right\}. && \text{(by inspection)}
\end{aligned}$$

So we may take $\mathbf{v}_2 = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ as the corresponding eigenvector for eigenvalue $\lambda_2 = -4$. So putting the eigenvalues into a diagonal matrix D and the corresponding eigenvectors as columns of the matrix M (in the order they appear from top to bottom in the diagonal of D), we can write $A = MDM^{-1}$, where

$$M = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 5 & 0 \\ 0 & -4 \end{pmatrix}.$$

If instead you used $D = \begin{pmatrix} -4 & 0 \\ 0 & 5 \end{pmatrix}$*, your* M *would need to have its columns in the opposite order to what we wrote, i.e. it would be (up to constant non-zero multiples for the columns)* $M = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}$.

3. **Hints/Answers**

   Using probability theory concepts like the definition of conditional probability, law of total probability and Bayes' Rule, it can be found[3] that

[3] Refer to Question 3 in Test 2 S2 2009 v7a

(i) $\mathbb{P}\left(\text{hamburger and drink}\right) = 0.18$.

(ii) $\mathbb{P}\left(\text{drink}\right) = 0.26$.

(iii) $\mathbb{P}\left(\text{hamburger} \mid \text{drink}\right) = \frac{9}{13}$.

Worked Sample Solutions

We introduce some notation. Let H be the even that the customer bought a hamburger and D the event they bought a drink. Translating the given data into our notation, we are thus given that

$$\mathbb{P}\left(H\right) = 0.6, \mathbb{P}\left(D \mid H\right) = 0.3, \mathbb{P}D \mid H^c = 0.2.$$

(i) We are asked to find $\mathbb{P}\left(H \cap D\right)$ (remember, $\cap$ in probability essentially corresponds to "and", and formally refers to the intersection of sets or events). We have

$$\begin{aligned} \mathbb{P}\left(H \cap D\right) &= \mathbb{P}H\mathbb{P}D \mid H && \text{(multiplication rule)} \\ &= 0.6 \times 0.3 && \text{(given data)} \\ &= 0.18. \end{aligned}$$

(ii) We are asked to find $\mathbb{P}\left(D\right)$. We know from basic probability theory that

$$\mathbb{P}\left(D\right) = \mathbb{P}\left(D \cap H\right) + \mathbb{P}D \cap H^c. \qquad (\star)$$

We found in (i) already that $\mathbb{P}\left(D \cap H\right) = 0.18$ (remember, $D \cap H$ and $H \cap D$ are the same event). Also, we have

$$\begin{aligned} \mathbb{P}\left(D \cap H^c\right) &= \mathbb{P}\left(H^c\right)\mathbb{P}\left(D \mid H^c\right) && \text{(multiplication rule)} \\ &= (1 - 0.6) \times 0.2 && \text{(given data, noting } \mathbb{P}\left(H^c\right) = 1 - \mathbb{P}\left(H\right)) \\ &= 0.08. \end{aligned}$$

So from $(\star)$, we have

$$\begin{aligned} \mathbb{P}\left(D\right) &= 0.18 + 0.08 \\ &= 0.26. \end{aligned}$$

(iii) We are asked to find $\mathbb{P}\left(H \mid D\right)$. We have

$$\begin{aligned} \mathbb{P}\left(H \mid D\right) &= \frac{\mathbb{P}\left(H \cap D\right)}{\mathbb{P}\left(D\right)} && \text{(definition of conditional probability)} \\ &= \frac{0.18}{0.26} = \frac{18}{26} && \text{(using answers to (i) and (ii))} \\ &= \frac{9}{13}. \end{aligned}$$