



MATH1081 Test 2 2008 S1 v2A

April 7, 2016

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1. (i) The congruence is equivalent to the equation $28x - 66y = 3$. Noting that $\gcd(28, 66) = 2$ and that $2 \nmid 3$, then by the Bezout property, we can conclude that there are no integer solutions.
 - (ii) The congruence is equivalent to the equation $29x - 67y = 3$. Noting that $\gcd(29, 67) = 1$ and that $1 \mid 3$, then by the Bezout property, we know that there are multiple integer solutions, in particular the solutions are congruent modulo 67. To find them, use the extended Euclidean algorithm:

$$67 = 2 \times 29 + 9$$

$$29 = 3 \times 9 + 2$$

$$9 = 4 \times 2 + 1.$$

Reversing this,

$$\begin{aligned}
 1 &= 9 - 4 \times 2 \\
 &= 9 - 4 \times (29 - 3 \times 9) \\
 &= 13 \times 9 - 4 \times 29 \\
 &= 13 \times 67 - 26 \times 29 - 4 \times 29 \\
 &= 13 \times 67 - 30 \times 29.
 \end{aligned}$$

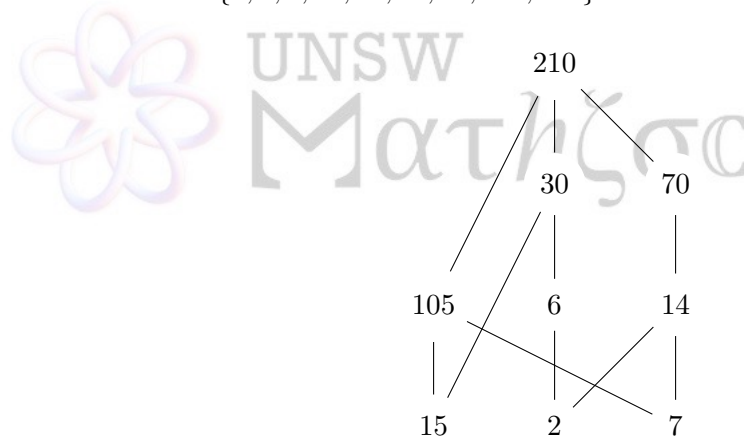
But in order to get the result in a similar form to our original equation, we must multiply both sides by 3, giving

$$-90 \times 29 + 39 \times 67 = 3.$$

Hence the solution is

$$x \equiv -90 \equiv 44 \pmod{67}.$$

2. Divisibility on the set: $S = \{2, 6, 7, 14, 15, 30, 70, 105, 210\}$



- (i) The Hasse diagram is shown above.
- (ii) Recall that maximal elements are those that are related to no element in S except itself (nothing else on top in the Hasse diagram) and minimal elements are those that have no elements except itself that are related to it (nothing below them in the Hasse diagram).

The maximal element of S is 210 and the minimal elements are 2, 7 and 15.

- (iii) Recall that a greatest element of S is one where every element in S is related to it (where all other elements precede it) and the least element of S is one that is related to every element in S (precedes all other elements).

So S has a greatest element 210. But, S does not have a least element as 2, 7, 15 are not comparable (divisible) to each other and do not fit the criterion above.

3. Define a relation \sim on \mathbb{Z}^+ by

$$x \sim y \text{ iff } y = 3^k x \text{ for some integer } k.$$

We must show that \sim is reflexive, symmetric and transitive.

Recall the following definitions for some relation \sim on a set A :

- Reflexive if for all $a \in A$, $a \sim a$,
- Symmetric if for all $a, b \in A$, if $a \sim b$, then $b \sim a$,
- Transitive if for $a, b, c \in A$ where $a \sim b$ and $b \sim c$ implies that $a \sim c$.

We now show that each of these properties are true,

Reflexive: Let $a \in \mathbb{Z}^+$. We need to show that there exists an integer k such that $a = 3^k a$.

But $k = 0$ works here, so we can write $a = 3^0 a$, which implies that $a \sim a$.

Symmetric: Let $a, b \in \mathbb{Z}^+$. Suppose $a \sim b$. Then, there exists an integer k such that $b = 3^k a$. Multiplying both sides by 3^{-k} gives $a = 3^{-k} b$. Noting that $-k$ is also an integer, the result implies that $b \sim a$. Hence, $a \sim b \implies b \sim a$.

Transitive: Let $a, b, c \in \mathbb{Z}^+$. Suppose $a \sim b$ and $b \sim c$. Then there exist integers k and n , such that we can write $b = 3^k a$ and $c = 3^n b$. Then through substitution, we have $c = 3^n(3^k a) = 3^{n+k} a$. Notice that $n + k$ is indeed an integer. So, we can conclude that $a \sim c$. Hence, $a \sim b$ and $b \sim c \implies a \sim c$.

Thus, \sim is an equivalence relation.



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1. (i) Dividing both sides of the congruence $20x \equiv 16 \pmod{92}$ by 4, we have

$$5x \equiv 4 \pmod{23}.$$

This congruence is equivalent to the equation $5x - 23y = 4$. Noting that $\gcd(23, 5) = 1$ and $1 \mid 4$, then by the Bezout property, we can conclude that there are multiple integer solutions, in particular the solutions are congruent modulo 23. Using the extended Euclidean algorithm, we have

$$23 = 4 \times 5 + 3$$

$$5 = 1 \times 3 + 2$$

$$3 = 1 \times 2 + 1.$$

Reversing this,

$$\begin{aligned}
\mathbf{1} &= \mathbf{3} - \mathbf{2} \\
&= \mathbf{3} - (\mathbf{5} - \mathbf{3}) \\
&= 2 \times \mathbf{3} - \mathbf{5} \\
&= 2(\mathbf{23} - 4 \times \mathbf{5}) - \mathbf{5} \\
&= 2 \times \mathbf{23} - 9 \times \mathbf{5}.
\end{aligned}$$

Thus -9 is an inverse of 5 modulo 23 . But, after multiplying both sides by 4 , we have the solution

$$x \equiv -9 \times 4 = -36 \equiv 10 \pmod{23}.$$

Alternatively, we could do this question without the extended Euclidean algorithm by noticing that

$$\begin{aligned}
5x &\equiv 4 \pmod{23} \\
&\equiv 50 \pmod{23}.
\end{aligned}$$

Dividing by 5 gives

$$\begin{aligned}
x &\equiv 10 \left(\pmod{\frac{23}{\gcd(23, 5)}} \right) \\
&= 10 \pmod{23}.
\end{aligned}$$

(ii) $x \equiv 10 \pmod{23}$ implies that $x \equiv 10, 33, 56, 79 \pmod{92}$.

2. Let $a, b, c \in \mathbb{Z}$ such that $a^2 \mid b$ and $b^3 \mid c$. Then we can write $b = ra^2$ and $c = sb^3$ for some integers r and s . We then have:

$$\begin{aligned}
c^3 &= s^3 b^9 \\
&= s^3 b^5 b^4 \\
&= s^3 b^5 (ra^2)^4 \\
&= s^3 r^4 a^8 b^5 \\
&= (s^3 r^4 a^4) a^4 b^5.
\end{aligned}$$

Now $s^3 r^4 a^4 \in \mathbb{Z}$, so we can conclude that $a^4 b^5 \mid c^3$.

3. A relation \sim is defined on \mathbb{R} by $x \sim y$ iff $\sin x = \sin y$.

- (i) We are basically finding the set of values that x can take such that $\sin x = \sin 0 = 0$.
The equivalence class of 0 is then

$$\begin{aligned}\{x \in \mathbb{R} \mid 0 \sim x\} &= \{k\pi, k \in \mathbb{Z}\} \\ &= \{\dots, -2\pi, -\pi, 0, \pi, 2\pi, \dots\}.\end{aligned}$$

- (ii) The equivalence class of a , $a \in \mathbb{R}$ is

$$\{x \in \mathbb{R} \mid a \sim x\}.$$

We are basically finding the set of values that x can take such that $\sin x = \sin a$.
If $\sin a \neq 0$, the equivalence class is

$$\{k\pi + (-1)^k a, k \in \mathbb{Z}\} = \{\dots, a, \pi - a, 2\pi + a, \dots\}.$$

If $\sin a = 0$, see part (i).



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1. (i) Dividing both sides of the congruence $45x \equiv 15 \pmod{78}$ by 3 gives

$$15x \equiv 5 \pmod{26}.$$

This congruence is equivalent to $15x - 26y = 5$. Since $\gcd(15, 26) = 1$ and $1 \mid 5$, there are multiple integer solutions by the Bezout property, in particular the solutions are congruent modulo 26. We can find them using the extended Euclidean algorithm:

$$26 = 1 \times 15 + 11$$

$$15 = 1 \times 11 + 4$$

$$11 = 2 \times 4 + 3$$

$$4 = 1 \times 3 + 1.$$

Reversing this,

$$\begin{aligned}
\mathbf{1} &= \mathbf{4} - \mathbf{3} \\
&= \mathbf{4} - (\mathbf{11} - 2 \times \mathbf{4}) \\
&= 3 \times \mathbf{4} - \mathbf{11} \\
&= 3(\mathbf{15} - \mathbf{11}) - \mathbf{11} \\
&= 3 \times \mathbf{15} - 4 \times \mathbf{11} \\
&= 3 \times \mathbf{15} - 4(\mathbf{26} - \mathbf{15}) \\
&= 7 \times \mathbf{15} - 4 \times \mathbf{26}.
\end{aligned}$$

So 7 is an inverse of 15 modulo 26. Hence, the solution is

$$x \equiv 7 \times 5 \equiv 9 \pmod{26}.$$

Alternatively, we could do this question without the extended Euclidean algorithm by noticing that

$$\begin{aligned}
15x &\equiv 5 \pmod{26} \\
3x &\equiv 1 \left(\pmod{\frac{26}{\gcd(26, 5)}} \right) \\
&= 1 \pmod{26} \\
&\equiv 27 \pmod{26} \\
x &\equiv 9 \left(\pmod{\frac{26}{\gcd(26, 3)}} \right) \\
&\equiv 9 \pmod{26}.
\end{aligned}$$

(ii) $x \equiv 9 \pmod{26}$ implies that $x \equiv 9, 35, 61 \pmod{78}$.

2. Let $a, m \in \mathbb{Z}$ and suppose $a \mid m$ and $(a+1) \mid m$. Since $a \mid m$, $(a+1) \mid m$ and $\gcd(a, a+1) = 1$, we can say that

$$(a+1) \mid \frac{m}{a} \text{ and } a \mid \frac{m}{a+1}.$$

Hence, it follows that $a(a+1) \mid m$. □

Alternatively, Let $a, m \in \mathbb{Z}$ and suppose $a \mid m$ and $(a+1) \mid m$. Then, for some integers r and s , we can write

$$m = ra \quad \text{and} \quad m = s(a+1).$$

First, multiplying both sides of the first equation by $(a + 1)$ gives

$$m = ra \implies m(a + 1) = ra(a + 1).$$

Next, multiplying both sides of the second equation by a gives

$$m = s(a + 1) \implies ma = sa(a + 1).$$

Subtracting the second line from the first, we have

$$m = (r - s)a(a + 1) \text{ where } r - s \in \mathbb{Z}.$$

Hence, $a(a + 1) \mid m$. □

3. Define a relation \preceq on \mathbb{Z}^+ by

$$x \preceq y \quad \text{iff} \quad y = 3^k x \quad \text{for some non-negative integer } k.$$

We must show that \preceq is reflexive, antisymmetric and transitive. Recall the following definitions for some relation \sim on a set A :

- Reflexive if for all $a \in A$, $a \sim a$,
- Antisymmetric if for all $a, b \in A$, if $a \sim b$ and $b \sim a$, then $a = b$,
- Transitive if for $a, b, c \in A$ where $a \sim b$ and $b \sim c$ implies that $a \sim c$.

We now show that each of these properties are true,

Reflexive: Let $a \in \mathbb{Z}^+$. Since $a = 3^0 a$, it follows that $a \preceq a$.

Antisymmetric: Let $a, b \in \mathbb{Z}^+$. Suppose that $a \preceq b$ and $b \preceq a$. Then $b = 3^k a$ and $a = 3^l b$ for some integers k and l . We then have

$$b = 3^k(3^l b) = 3^{k+l} b.$$

Since k and l must be non-negative, the above line implies $k = l = 0$. Hence $a = b$.

Transitive: Let $a, b, c \in \mathbb{Z}^+$. Suppose $a \preceq b$, $b \preceq c$. Then $b = 3^k a$ and $c = 3^l b$ for some integers k and l . So

$$c = 3^l(3^k a) = 3^{l+k} a.$$

As $l + k$ is indeed a non-negative integer, $a \preceq c$.

Since \preceq is reflexive, antisymmetric and transitive, it is a partial order.



MATH1081 Test 2 2009 S2 v2B

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1. (i) The congruence is equivalent to $79x - 98y = 5$. After using the extended Euclidean algorithm¹, you should find that -31 is an inverse of 79 modulo 98 . You should then find that the solution to the congruence is

$$x \equiv -31 \times 5 = -155 \equiv 41 \pmod{98}.$$

- (ii) Note that $\gcd(78, 99) = 3$ and $3 \nmid 5$. So by the Bezout property, there are no integer solutions.

2. Let $x, y, m \in \mathbb{Z}$. Suppose $m \mid (4x + y)$ and $m \mid (7x + 2y)$. Then for some integers a and b ,

$$4x + y = am \quad \text{and} \quad 7x + 2y = bm.$$

Solve simultaneously for x and y . You should find that $x = m(2a - b)$ and $y = m(4b - 7a)$. And so, $m \mid x$ and $m \mid y$.

¹Refer to Question 1. (ii) in Test 2 2008 S1 v2A

3. Let F be the set of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$, and define the relation \preceq on F by

$$f \preceq g \quad \text{iff} \quad f(x) \leq g(x) \quad \text{for all } x \in \mathbb{R}.$$

To prove that \preceq is a partial order, we need to show that it is reflexive, antisymmetric and transitive. To do this, let $a, b, c \in F$ and use these functions appropriately for each property.



MATH1081 Test 1 2010 S1 v1A

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1. (i) The congruence is equivalent to $25x - 109y = 3$. After using the extended Euclidean algorithm¹, you should find that 48 is an inverse of 25 modulo 109. You should find that the solution is

$$x \equiv 48 \times 3 = 144 \equiv 35 \pmod{109}.$$

Alternatively, we can find that

$$\begin{aligned} 25x &\equiv 3 \pmod{109} \\ &\equiv -215 \pmod{109} \\ 5x &\equiv -43 \pmod{109} \\ &\equiv 175 \pmod{109} \\ x &\equiv 35 \pmod{109}. \end{aligned}$$

- (ii) Note that $\gcd(25, 110) = 5$, but $5 \nmid 3$. So by the Bezout property, there are no integer solutions.

¹Refer to Question 1. (ii) in Test 2 2008 S1 v2A

2. (i) $6500 = 2^2 \times 5^3 \times 13$ and $1120 = 2^5 \times 5 \times 7$.

Since $\gcd(a, b) \times \text{lcm}(a, b) = ab$, we have

$$\gcd(6500, 1120) = 2^2 \times 5$$

$$\text{and } \text{lcm}(6500, 1120) = 2^5 \times 5^3 \times 7 \times 13.$$

3. Define a relation \sim on \mathbb{R}^+ by

$$x \sim y \quad \text{iff} \quad x - y \in \mathbb{Z}.$$

We must prove that \sim is reflexive, symmetric and transitive. We let $a, b, c \in \mathbb{Z}$ and use these appropriately to show that each of these properties hold one at a time.