MATH1131/1141 Calculus Test 2 2008 S1 v8a

October 11, 2017

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1. Note that f is a polynomial function, so it is indeed **continuous** (note that you should mention this or you could lose a mark) on the intervals $[0,1]$ and $[2,3]$.
   Note that $f(0) = 3$ and $f(1) = -1$.
   Since $f(1) < 0 < f(0)$, then from the Intermediate Value Theorem, we know that there exists a real number $c \in (0,1)$ such that $f(c) = 0$. Hence, f has a zero on the interval $[0,1]$.
   Similarly, note that $f(2) = -3$ and $f(3) = 3$. Since $f(2) < 0 < f(3)$, then from the Intermediate Value Theorem, we know that there exists a real number $d \in (2,3)$ such that $f(d) = 0$. Hence, f also has a zero on the interval $[2,3]$.

2. If $f(x) = -2x^2 + x$, then from the definition of the derivative, we have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\left(-2(x+h)^2 + (x+h)\right) - \left(-2x^2 + x\right)}{h} \\
&= \lim_{h\to 0} \frac{-2\left(x^2 + 2xh + h^2\right) + x + h + 2x^2 - x}{h} \\
&= \lim_{h\to 0} \frac{-2x^2 - 4xh - 2h^2 + x + h + 2x^2 - x}{h} \\
&= \lim_{h\to 0} \frac{h(-4x - 2h + 1)}{h} \\
&= \lim_{h\to 0} (-4x - 2h + 1) \\
&= -4x + 1.
\end{aligned}$$

3. The area of a rectangle is given by

$$A = LW.$$

Noting that both W and L depend on time, we implicitly differentiate both sides with respect to t (using the product rule),

$$\begin{aligned}
\frac{dA}{dt} &= L\frac{dW}{dt} + W\frac{dL}{dt} \\
&= 4L - 2W,
\end{aligned}$$

by substituting in the information given, noting that $\frac{dL}{dt}$ is negative since L is decreasing. Thus, when $L = 13$ and $W = 10$,

$$\frac{dA}{dt} = 4 \times 13 - 2 \times 10 = 32\text{cm}^2/\text{s}.$$

Note: Whenever the questions asks for 'rate of change', then you're most likely going to be differentiating with respect to t, i.e. time.

4. The Mean Value Theorem states that if f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

For $f(x) = x^3 - 2x^2 + 5$, we have $f'(x) = 3x^2 - 4x$. Also, note that f is continuous on the interval $[0, 2]$ and differentiable on $(0, 2)$ since it is a polynomial function.
Thus, from the Mean Value Theorem, we know that there exists a number $c \in (0, 2)$ such that

$$3c^2 - 4c = \frac{f(2) - f(0)}{2 - 0}.$$

Simplifying this and solving for c, we get

$$\begin{aligned}
3c^2 - 4c &= \frac{8 - 8 + 5 - 5}{2} \\
&= 0 \\
3c^2 - 4c &= 0 \\
c(3c - 4) &= 0 \\
\therefore c &= 0 \text{ or } \frac{4}{3}.
\end{aligned}$$

But $c \in (0, 2)$, so $c = \frac{4}{3}$ is the point which satisfies the conclusions of the Mean Value Theorem.

5. Currently, this limit takes the indeterminate form of $\left(\frac{0}{0}\right)$. So, we can try L'Hôpital's rule,

$$\begin{aligned}
\lim_{x \to 0} \frac{x \sin x}{1 - \cos x} &= \lim_{x \to 0} \frac{x \cos x + \sin x}{\sin x} && \text{(L'Hôpital's rule)} \\
&= \lim_{x \to 0} \frac{x}{\sin x} \times \cos x + 1 \\
&= \lim_{x \to 0} \frac{x}{\sin x} \times \lim_{x \to 0} \cos x + 1 && \text{(by the Algebra of Limits)} \\
&= 1 \times 1 + 1 && \text{(L'Hôpital's rule)} \\
&= 2.
\end{aligned}$$

MATH1131/1141 Calculus Test 2 2008 S2 v4b

October 11, 2017

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1. The Mean Value Theorem states that if f is continuous on $[a,b]$, and differentiable on (a,b), then there exists a real number $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

Note that the function $f(x) = \sqrt{x-1}$ is continuous on $[1,3]$ and differentiable on $(1,3)$. Then, from the mean value theorem, we know that there exists a real number $c \in (1,3)$ such that

$$f'(c) = \frac{f(3) - f(1)}{3-1}.$$

Since $f'(x) = \frac{1}{2\sqrt{x-1}}$, we substitute this into the equation above to get

$$\begin{aligned} \frac{1}{2\sqrt{c-1}} &= \frac{\sqrt{2}-0}{2-0} \\ &= \frac{1}{\sqrt{2}}. \end{aligned}$$

Squaring both sides, we have

$$\begin{aligned} \frac{1}{4(c-1)} &= \frac{1}{2} \\ c-1 &= \frac{1}{2} \\ \therefore\ c &= \frac{3}{2}. \end{aligned}$$

Note that $\frac{3}{2} \in (1,3)$. Thus, $c = \frac{3}{2}$ is the point that satisfies the conclusions of the mean value theorem.

2. Note that the limit takes the indeterminate form $\left(\frac{0}{0}\right)$, and so we can try L'Hôpital's rule,

$$\lim_{x\to 0} \frac{1-\cos 3x}{x^2} = \lim_{x\to 0} \frac{3\sin 3x}{2x} \qquad \text{(L'Hôpital's rule.)}$$

This also takes the indeterminate form $\left(\frac{0}{0}\right)$, so we can try L'Hôpital's rule again

$$\begin{aligned} &= \lim_{x\to 0} \frac{9\cos 3x}{2} \qquad \text{(L'Hôpital's rule)} \\ &= \frac{9}{2}. \end{aligned}$$

3. Note that f is continuous on the intervals $[-2,-1]$ and $[1,2]$ (indeed it is actually continuous everywhere, since f is a polynomial function).
Note that $f(-2) = -11$ and $f(-1) = 3$.
Since $f(-2) < 0 < f(-1)$, then from the Intermediate Value Theorem, we know that there exists a number $c \in (-2,-1)$ such that $f(c) = 0$. Hence, f has a zero on the interval $[-2,-1]$.
Similarly, note that $f(1) = 1$ and $f(2) = -3$. Since $f(2) < 0 < f(1)$, then from the Intermediate Value Theorem, we know that there exists a number $d \in (1,2)$ such that $f(d) = 0$. Hence, f also has a zero on the interval $[1,2]$.

Note: Always remember to check or at least mention that the function is continuous, before you apply the intermediate value theorem. In general, whenever you are using a theorem, like the min-max theorem for example, always check the conditions before you continue.
Note 2: It's important that you distinguish c and d in this question, as they are different

numbers.

4. If $f(x) = -x^3$, then from the definition of the derivative, we know that

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{-(x+h)^3 - \left(-x^3\right)}{h} \\
&= \lim_{h \to 0} \frac{-\left(x^3 + 3x^2h + 3xh^2 + h^3\right) + x^3}{h} \\
&= \lim_{h \to 0} \frac{-3x^2h - 3xh^2 - h^3}{h} \\
&= \lim_{h \to 0} \frac{h(-3x^2 - 3xh - h^2)}{h} \\
&= \lim_{h \to 0} (-3x^2 - 3xh - h^2) \\
&= -3x^2.
\end{aligned}
$$

5. We want the line tangent to $x^2 + y^3 - x^2y = 1$ at $(1,1)$. By differentiating implicitly with respect to x (applying the product rule to x^2y), we obtain

$$2x + 3y^2\frac{dy}{dx} - \left(2xy + x^2\frac{dy}{dx}\right) = 0.$$

Rearranging, we see that

$$\frac{dy}{dx} = \frac{2x - 2xy}{x^2 - 3y^2}.$$

So, at $(1,1)$,

$$\frac{dy}{dx} = \frac{2(1) - 2(1)(1)}{1^2 - 3(1)^2} = 0.$$

Using the point-gradient formula at the point $(1,1)$, we get

$$
\begin{aligned}
y - y_1 &= m\,(x - x_1) \\
y - 1 &= 0\,(x - 1) \\
y &= 1.
\end{aligned}
$$

Therefore, the tangent line at $(1,1)$ is $y = 1$.

MATH1131/1141 Calculus Test 2 2009 S1 v8b

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---

1. The Mean Value Theorem states that if f is continuous on $[a,b]$, and differentiable on (a,b), then there exists a number $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

Note that f is continuous on $[0,1]$ and differentiable on $(0,1)$, as it is a polynomial function. Then, from the Mean Value Theorem, we know that there exists a number $c \in (0,1)$ such that

$$f'(c) = \frac{f(1) - f(0)}{1-0}.$$

Note that $f'(x) = 3x^2 - 2x$. So we have

$$\begin{aligned}
3c^2 - 2c &= \frac{3-3}{1} \\
3c^2 - 2c &= 0 \\
c\,(3c-2) &= 0 \\
\therefore c &= 0 \text{ or } \frac{2}{3}.
\end{aligned}$$

But $c \in (0,1)$, and so $c = \frac{2}{3}$ is a point which satisfies the conclusions of the Mean Value Theorem.

2. The limit currently takes the indeterminate form $\left(\frac{0}{0}\right)$. So we try L'Hôpital's rule

$$\lim_{x\to 1} \frac{3x^3 - 5x^2 + x + 1}{x^2 - 2x + 1} = \lim_{x\to 1} \frac{9x^2 - 10x + 1}{2x - 2}. \qquad \text{(L'Hopital's rule)}$$

This limit also takes the indeterminate form $\left(\frac{0}{0}\right)$, so we try L'Hôpital's rule again and find that

$$\begin{aligned}
&= \lim_{x\to 1} \frac{18x - 10}{2} \qquad \text{(L'Hôpital's rule)} \\
&= \frac{18 - 10}{2} \\
&= 4.
\end{aligned}$$

3. Clearly, the function is continuous at every point except $x = 1$ since it is composed of polynomials. We need to be more careful about whether it is continuous at $x = 1$.
For f to be continuous at $x = 1$, $\lim_{x\to 1} f(x)$ must exist and equal to $f(1)$.
So first, we will check whether the limit actually exists. Since the function splits up at $x = 1$, we need to take the two-sided limit at $x = 1$.
We have

$$\begin{aligned}
\lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} x^3 \\
&= 1
\end{aligned}$$

and

$$\begin{aligned}
\lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} (x-1)^3 + 2 \\
&= 0^3 + 2 \\
&= 2.
\end{aligned}$$

Since the two sided limits do not agree, then $\lim_{x\to 1} f(x)$ doesn't actually exist. So, we can safely conclude that f is not continuous at $x = 1$.

Thus, f is continuous for $x \neq 1$.

4. If $f(x) = -x^3$, then from the definition of the derivative we know that

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{-(x+h)^3 - \left(-x^3\right)}{h} \\ &= \lim_{h \to 0} \frac{-\left(x^3 + 3x^2h + 3xh^2 + h^3\right) + x^3}{h} \\ &= \lim_{h \to 0} \frac{-3x^2h - 3xh^2 - h^3}{h} \\ &= \lim_{h \to 0} -3x^2 - 3xh - h^2 \\ &= -3x^2. \end{aligned}$$

MATH1131/1141 Calculus Test 2 2009 S1 v1a

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1. Recall the definition that a function f is continuous at $x = 2$ if

$$\lim_{x\to 2} f(x) = f(2). \tag{1}$$

At the moment, f is not continuous, since $f(2)$ isn't defined. So to find a value that will make f continuous at 2, we simply need to just assign $f(2)$ to be whatever $\lim\limits_{x\to 2} f(x)$ is. We can see that

$$\begin{aligned}\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x-2}{x^2-3x+2} &= \lim_{x\to 2} \frac{x-2}{(x-2)(x-1)} \\ &= \lim_{x\to 2} \frac{1}{x-1} && \text{(since } x \neq 2\text{)} \\ &= 1.\end{aligned}$$

So if we assign $f(2)$ to be equal to 1, then we satisfy the continuity condition (1). So the answer is $\boxed{1}$.

2.

$$f(x) = \begin{cases} ax + b, & \text{for } x \le 1 \\ \tan \frac{\pi x}{4}, & \text{for } 1 < x < 2. \end{cases}$$

For the function to be differentiable at $x = 1$, it must be continuous there, and the derivative must exist there.
First, to ensure continuity at $x = 1$, the two sided limits must agree with the function value at this point, i.e.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1).$$

But this is simply $a + b = 1 = a + b$. So f is continuous at $x = 1$ if $a + b = 1$.
Next, to ensure that the derivative exists at $x = 1$, we use the definition of the derivative,

$$\begin{aligned} f'(1) = \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} &= \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} \\ \Leftrightarrow \lim_{h \to 0^-} \frac{a(1+h) + b - (a+b)}{h} &= \lim_{h \to 0^+} \frac{\tan\left(\frac{\pi(1+h)}{4}\right) - (a+b)}{h} \end{aligned}$$

Note that from the continuity condition, we know that $a + b = 1$. And so, the LHS simplifies to a.

The RHS takes the indeterminate form $\left(\frac{0}{0}\right)$, so trying L'Hôpital's rule

$$\begin{aligned} \lim_{h \to 0^-} \frac{ah}{h} = a &= \lim_{h \to 0^+} \frac{\frac{\pi}{4} \sec^2\left(\frac{\pi(1+h)}{4}\right)}{1} && \text{(L'Hôpital's rule)} \\ &= \frac{\pi}{4} \sec^2 \frac{\pi}{4} \\ &= \frac{\pi}{4} \times 2 \\ \therefore a &= \frac{\pi}{2}. \end{aligned}$$

From the continuity condition, we have $b = 1 - \frac{\pi}{2}$.
Hence, we require

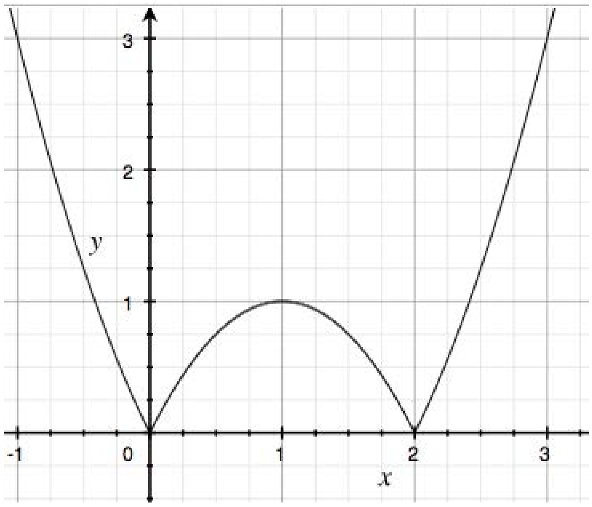
$$a = \frac{\pi}{2}, \; b = 1 - \frac{\pi}{2}$$

for f to be differentiable at $x = 1$.

Note 1: In these types of questions, there are always two things we must ensure: that f is continuous at the point and differentiable at the point.
Note 2: Because the function is piecewise and splits up at $x = 1$, when we are ensuring continuity at $x = 1$, we need to consider the two-sided limits $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$,

not just $\lim_{x\to 1} f(x)$.

3. (i) This is most easily done by first sketching the graph of $y = f(x)$. This is done by first sketching $y = x^2 - 2x$ and then reflecting anything below the x-axis above.



From this, note that in this case, the critical points are the endpoints of the interval $[0, 5]$, the stationary points and the points where the derivative does not exist (the cusps).
The endpoints are clearly $(0, 0)$ and $(5, 15)$.
The stationary points can be found by noting that the x-coordinates of the stationary points of $y = x^2 - 2x$ and $y = f(x)$ are actually the same. So we first find the x-coordinate of the stationary point of the graph $y = x^2 - 2x$. Letting $y' = 0$, we find that

$$y' = 2x - 2 = 0$$

which gives $x = 1$. Substituting $x = 1$ back into $y = f(x)$ gives $y = f(1) = |-1| = 1$. Therefore, the stationary point is $(1, 1)$.
The points where the derivative does not exist are the points where the graph cuts sharply. From the graph, it is easy to see that those points are $(0, 0)$ and $(2, 0)$.
Hence, the critical points are $(0, 0)\,, (2, 0)\,, (1, 1)\,,$ and $(5, 15)$.

Note: Be careful of the internet when it comes to critical points. A lot of places on the internet, including Wikipedia define critical points as just the stationary points. However, in your course notes/pack, critical points are actually defined as the end-points of the interval that the domain takes (as long as it's not the whole of $\mathbb{R}$*), the points where* f *isn't differentiable and the stationary points.*

(ii) From looking at the graph, we can easily see that the absolute minimum values of $f(x)$ are attained when $x = 0$ and at $x = 2$. So the absolute minimum value of $f(x)$

is 0.

Since the domain of f is restricted to $[0,5]$, then we simply look for the greatest value that f attains on this interval. It is clear that the absolute maximum value of $f(x)$ is attained at $x=5$. So the absolute maximum value of $f(x)$ is 15.

4. The limit at the moment takes the indeterminate form $\frac{0}{0}$. So, we can try L'Hôpital's rule. Have a go at it yourself! The steps are relatively simple[1]. With a couple applications of L'Hôpital's rule, you should find that the limit equals to 3.

   Alternate method.

   Here is a method that does not use L'Hôpital's rule. We notice by inspection, polynomial long division, or synthetic division (aka Horner's Method) that $\frac{2x^4-3x^3+x}{x-1} \equiv 2x^3-x^2-x$. Let $f(x)=2x^3-x^2-x$. Note that $f(1)=0$. Now, denoting the limit by L, we have

$$\begin{aligned}
L \equiv \lim_{x\to 1} \frac{2x^4-3x^3+x}{(x-1)^2} &= \lim_{x\to 1} \frac{\frac{2x^4-3x^3+x}{x-1}}{x-1} \\
&= \lim_{x\to 1} \frac{2x^3-x^2-x}{x-1} \\
&= \lim_{x\to 1} \frac{f(x)-f(1)}{x-1} \quad \text{(as we noted } f(1)=0\text{)} \\
&\stackrel{\text{def}}{\equiv} f'(1). \quad \text{(recalling one of the definitions of the derivative).}
\end{aligned}$$

   But $f'(1)$ exists as $f(x)=2x^3-x^2-x$ is a polynomial and is thus differentiable everywhere. So $L=f'(1)=6\times 1^2-2\times 1-1 \Rightarrow \boxed{L=3}$.

[1]Refer to Question 2 in Test 2 2008 S2 v4b

MATH1131/1141 Calculus Test 2 2009 S2 v3a

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---

1. To find the line tangent to the curve at $(1,1)$, we first need to find $\frac{\mathrm{d}y}{\mathrm{d}x}$, in order to obtain the gradient of the tangent at $(1,1)$.
   It can be easily found through implicit differentiation[1] that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y(x+1)}{x(2+y)}$. So the gradient of the tangent at $(1,1)$ is $\frac{2}{3}$. From here, it can be found[1] that the equation of the line tangent is
   $$y = \frac{2}{3}x + \frac{1}{3}.$$

2. We first need to interpret the problem in a mathematical setting. Drawing a diagram will make this much easier. Let b denote the (perpendicular) distance from the wall to the base of the ladder in metres. Also, let h denote the distance between the top of the ladder and the floor in metres and t denote time in seconds. The question is asking us to find $\frac{\mathrm{d}h}{\mathrm{d}t}$.
   The variables b and h are related by the equation $b^2 + h^2 = 9$ (Pythagoras' Theorem).

[1] Refer to Question 5 in Test 2 2008 S2 v4b

Implicitly differentiating both sides with respect to t, gives

$$2b\frac{\mathrm{d}b}{\mathrm{d}t} + 2h\frac{\mathrm{d}h}{\mathrm{d}t} = 0. \tag{1}$$

But we know that $\frac{\mathrm{d}b}{\mathrm{d}t} = 0.5$ and $b = 1$. Substituting this into equation (1) gives

$$1 + 2h\frac{\mathrm{d}h}{\mathrm{d}t} = 0.$$

There's one more thing to do before we can get $\frac{\mathrm{d}h}{\mathrm{d}t}$: we need to find what h is. We use Pythagoras' Theorem to see that $h = \sqrt{9-b^2} = \sqrt{9-1^2} = 2\sqrt{2}$. And so, we can conclude that

$$\frac{dh}{dt} = -\frac{\sqrt{2}}{8}\text{m/s}.$$

That is, the top of the ladder is dropping at a rate of $\frac{\sqrt{2}}{8}$ m/s. *Note: that the answer was negative because the top of the ladder was "dropping" down the wall.*

3. It can be found[2] that the point $c = \frac{3}{2}$ satisfies the conclusions of the Mean Value Theorem.

4. (i) Critical points are end points, points where the function is not differentiable and where the derivative is zero. It can be found[3] that the critical points on the given interval are $(0,1)$, $(2,1)$ and $(1,0)$.

   (ii) As f is continuous on $[0,2]$, the max-min theorem guarantees that f attains a global maximum and global minimum on $[a,b]$, on a critical point. By looking at the critical points we found in the previous part, we can easily see that the absolute minimum value of $f(x)$ is 0 and the absolute maximum value of $f(x)$ is 1.

5. Since the limit as it is now seems to be in the indeterminate form $\frac{0}{0}$, we can try L'Hôpital's Rule and it can be found[4] that the limit is equal to 1.

[2]Refer to Question 1 in Test 2 2008 S2 v4b
[3]Refer to Question 3 in Test 2 2009 S1 v1a
[4]Refer to Question 5 in Test 2 2008 S1 v8a

MATH1131/1141 Calculus Test 2 2010 S1 v2a

October 11, 2017

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1. For f to be differentiable at $x = 0$,

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

must exist and it must be continuous at $x = 0$.

For $\lim_{h\to 0} \frac{f(h)-f(0)}{h}$ to exist, it must be true that

$$\begin{aligned}
\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} &= \lim_{h\to 0^-} \frac{f(h)-f(0)}{h} \\
\lim_{h\to 0^+} \frac{\sin h - 0}{h} &= \lim_{h\to 0^-} \frac{ae^h + b - a - b}{h} \\
\lim_{h\to 0^+} \frac{\sin h}{h} &= \lim_{h\to 0^-} \frac{ae^h - a}{h}
\end{aligned}$$

LHS is a standard limit, while RHS is in the indeterminate form "$\frac{0}{0}$", so we try L'Hôpital's

Rule and see that

$$\begin{aligned} 1 &= a \lim_{h\to 0^-} \frac{e^h}{1} \\ \therefore a &= 1. \end{aligned}$$

For the function to be continous at $x = 0$, we require that

$$\begin{aligned} f(0) &= \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) \\ 0 &= a + b \\ \therefore b &= -1. \end{aligned}$$

2. $\lim_{x\to 0} \frac{\tan x}{e^{3x}-1}$ is in the indeterminate form $\left(\frac{0}{0}\right)$, so we try L'Hopital's rule,

$$\begin{aligned} \lim_{x\to 0} \frac{\tan x}{e^{3x}-1} &= \lim_{x\to 0} \frac{\sec^2 x}{3e^{3x}} \\ &= \frac{1}{3}. \end{aligned}$$

3. Differentiating implicitly:

$$\begin{aligned} 3x^2 + 3y^2\frac{dy}{dx} - 1 - 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx}(3y^2 - 2y) &= 1 - 3x^2 \\ \frac{dy}{dx} &= \frac{1-3x^2}{3y^2-2y} \end{aligned}$$

At (1,1),

$$\frac{dy}{dx} = \frac{-2}{1} = -2$$

Using the point-gradient formula,

$$\begin{aligned} y - 1 &= -2(x-1) \\ y &= -2x + 3 \end{aligned}$$

4. Let $p(x) = x^3 - 6x^2 + 1$.

$$p'(x) = 3x^2 - 12x$$

Stationary points are at $(0, 1)$ and $(4, -31)$. Note that

$$\begin{aligned} p(-1) &= -6 \\ p(0) &= 1 \\ p(4) &= -31 \\ p(10) &= 401 \end{aligned}$$

Since polynomials are continuous on $\mathbb{R}$, then by the Intermediate Value Theorem, there is at least one root in each of the intervals $[-1, 0]$, $[0, 4]$ and $[4, 10]$ since the polynomial changes signs at the endpoints of each of these intervals.
As it is a cubic polynomial, it only has at most three roots. Therefore $p(x)$ has exactly 3 real roots.

5. From the chain rule, we know that

$$\begin{aligned} \frac{d}{dx}(\tan^{-1}(4x+1)) &= \frac{1}{1+(4x+1)^2} \times 4 \\ &= \frac{4}{16x^2+8x+2} \\ &= \frac{2}{8x^2+4x+1}. \end{aligned}$$