UNSW Mathematics Society presents

MATH2501/2601 Revision Seminar

(Higher) Linear Algebra

Seminar I / II

T2, 2020

Presented by Henry Lam and Alex Zhu

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Group Theory (MATH2601 Only)

Groups

From first-year linear algebra, you have gone through some core concepts of vector spaces. Before we dive back into that, we'll look at essentially a simpler variation as to build up some intuition towards later topics.

Groups

Definition 1: Group

A **group** G is an non-empty set with an operation $(*)$ defined onto it. They satisfy the four conditions:

- **Closure**: Suppose $a, b \in G$ then $a * b \in G$
- **Associativity**: $(a * b) * c = a * (b * c)$ for any $a, b, c \in G$.
- **Existence of identity**: There exists some $e \in G$ such that $a * e = e * a$ for all $a \in G$.
- **Existence of inverses**: For any $a \in G$, there exists $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$.

If the following condition is also met, we call it an **Abelian** group.

- **Commutativity**: $a * b = b * a$ for any $a, b \in G$.

This is typically denoted as $(G, *)$.

Group Theory Example

Ex 1: Properties of groups

Suppose that G is a finite group and $a \in G$. Prove that there exists some positive integer, n, such that $a^n = e$, where e is the identity element of G.

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Suppose that $|G| = m$. Consider some $a \in G$ and $k > m$. From the **closure property**, we have that: $a, a^2, \ldots, a^k \in G$.

As G only has m unique elements, then there must exist some integers $i > j$ such that: $a^i = a^j$.

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As G only has m unique elements, then there must exist some integers $i > j$ such that: $a^i = a^j$.
Now as every element in a group has an **inverse**, we can see that:

$$a^i \times a^{-j} = a^j \times a^{-j}$$

$$a^{i-j} = e$$

Hence, there exists some positive integer $n = i - j$ such that $a^n = e$.

Properties of Groups

From the base definition of groups, we can prove some basic properties that we have become accustomed to assuming.

- Uniqueness of identity and inverses.
- $(a^{-1})^{-1} = a$.
- $(a * b)^{-1} = b^{-1} * a^{-1}$.
- If $a * b = a * c$, then $b = c$, where $a, b, c \in (G, *)$.

Fields

Extending the definition of a group by including an additional operation and a few more conditions gives us a **field** $\mathbb{F}$.

Definition 2: Field

A **field** $(\mathbb{F}, +, \times)$ is the set $\mathbb{F}$ with two operations defined on it, such that:

- $(\mathbb{F}, +)$ is **abelian**;
- $(\mathbb{F} \setminus \{0\}, \times)$ is **abelian**;
- **Multiplicative Distributivity**: $a \times (b + c) = a \times b + a \times c$, for any $a, b, c \in \mathbb{F}$.

Looks familiar?!

These rules are very reminiscent of the **10** axioms of vector spaces, although they aren't exactly the same.

Subgroups and Subfields

From first year, you have dealt with the idea of **subspaces**. The idea here is fairly similar to it, as we only have to prove a portion of the properties are satisfied.

Theorem 1: Subgroup Theorem

Consider a (non-empty) set $A \subset G$, where $(G, *)$ is a group. Then $(A, *)$ is a **subgroup** of $(G, *)$ iff all elements of A satisfy:

- **Closure under operation**: Suppose $a, b \in A$ then $a * b \in A$.
- **Existence of inverse**: For any $a \in A$, $a^{-1} \in A$.

Subgroup Example

Ex 2: Subgroup Theorem

Consider $b \in G$, where G is a group under the operation $\circ$.
Prove that $H_b := \{b \circ a \circ b^{-1} : a \in G\}$ is a group under $\circ$.

Subgroup Example

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Consider $b \in G$, where G is a group under the operation $\circ$.
Prove that $H_b := \{b \circ a \circ b^{-1} : a \in G\}$ is a group under $\circ$.

- We must firstly check that H_b is non-empty, which can be verified as $e \in H_b$ (consider $a = e$).

Subgroup Example

Ex 2: Subgroup Theorem

Consider $b \in G$, where G is a group under the operation $\circ$.
Prove that $H_b := \{b \circ a \circ b^{-1} : a \in G\}$ is a group under $\circ$.

Closure: Consider $x, y \in H_b$, such that $x = b \circ a \circ b^{-1}$ and $y = b \circ c \circ b^{-1}$, for some $a, c \in G$.

We can see that:

$$\begin{aligned} x \circ y &= (b \circ a \circ b^{-1}) \circ (b \circ c \circ b^{-1}) \\ &= (b \circ a) \circ (b^{-1} \circ b) \circ (c \circ b^{-1}) && \text{(Associativity)} \\ &= (b \circ a) \circ (c \circ b^{-1}) && \text{(Identity)} \\ &= b \circ (a \circ c) \circ b^{-1} && \text{(Associativity)} \end{aligned}$$

As $a, c \in G$ and G is a group under $\circ$ then $a \circ c \in G$, i.e. $x \circ y \in H_b$

Subgroup Example

Ex 2: Subgroup Theorem

Consider $b \in G$, where G is a group under the operation $\circ$.
Prove that $H_b := \{b \circ a \circ b^{-1} : a \in G\}$ is a group under $\circ$.

Inverse: Consider $z = b \circ a^{-1} \circ b^{-1}$ and x from before. $z \in H_b$ as $a^{-1} \in G$.

Following the working out before, we just replace c with a^{-1}.

$$\begin{aligned} x \circ z &= \dots \\ &= b \circ (a \circ a^{-1}) \circ b^{-1} \\ &= b \circ b^{-1} && \text{(Identity)} \\ &= e. \end{aligned}$$

A similar argument applies for $z \circ x = e$, and so we have that every $x \in H_b$ has an inverse.

Subgroup Example

Ex 2: Subgroup Theorem

Consider $b \in G$, where G is a group under the operation $\circ$.
Prove that $H_b := \{b \circ a \circ b^{-1} : a \in G\}$ is a group under $\circ$.

Hence, by the Subgroup theorem, H_b is also a group under $\circ$.
Notation-wise, we say that $H_b \leq G$ under the operation, $\circ$.

Morphisms

Just like how we can define a mapping between any two sets, like from $[0, 1)$ to $\mathbb{R}$, we can also define something similar between two groups.

Definition 2: Morphism

Consider two groups $(G, *)$ and $(H, \circ)$. The mapping $f : G \to H$, is defined as a **homomorphism** from G to H if it satisfies the following:

$$f(a * b) = f(a) \circ f(b)$$

for any $a, b \in G$.

If this mapping is a bijection, we call say that both groups are **isomorphic** to each other.

Properties of Homomorphisms

From the definition in the previous slide, some neat properties that hold are:

- **Inverse maps to inverse**: $f(a^{-1}) = f(a)^{-1}$.
- **Identity maps to identity**: $f(e) = e'$, where e, e' are the identity elements of G and H, respectively.

Homomorphism Example

MATH2601 Final 2008 Q3c)(ii)

Suppose that $G = \left\{ \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}$ is a group under matrix multiplication. Is $(G, \times)$ isomorphic to $(\mathbb{R}, +)$?

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Consider $f : (\mathbb{R}, +) \to (G, \times)$ where $f(t)$ gives the matrix above, and $s, t \in \mathbb{R}$. We'll firstly show that this is a homomorphism.

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Consider $f : (\mathbb{R}, +) \to (G, \times)$ where $f(t)$ gives the matrix above, and $s, t \in \mathbb{R}$. We'll firstly show that this is a homomorphism.

$$\begin{aligned} f(s) \times f(t) &= \begin{pmatrix} 1 & s & \frac{s^2}{2} \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & s+t & \frac{t^2}{2} + st + \frac{s^2}{2} \\ 0 & 1 & s+t \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

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$$f(s) \times f(t) = \begin{pmatrix} 1 & s & \frac{s^2}{2} \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & s+t & \frac{t^2}{2} + st + \frac{s^2}{2} \\ 0 & 1 & s+t \\ 0 & 0 & 1 \end{pmatrix} = f(s+t)$$

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Suppose that $G = \left\{ \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}$ is a group under matrix multiplication. Is $(G, \times)$ isomorphic to $(\mathbb{R}, +)$?

From this, $(G, \times)$ is homomorphic to $(\mathbb{R}, +)$ and as f is a bijection as well. Hence these two groups are also **isomorphic** to each other.

Kernel and Image

Consider a homomorphism $f : G \to H$. We define two special types of sets that come from this mapping, namely the **kernel** and **image**.

Definition 3: Kernel

The **kernel** is the 'roots' of f:

$$ker(f) := \{a \in G : f(a) = e'\}$$

where e' is the identity element of H.

Definition 4: Image

The **image** of f is the 'projections' of f:

$$im(f) := \{y \in H : f(x) = y, \quad \text{for some } x \in G\}.$$

Vector Spaces

Vector Spaces

The formal definition is the following:

Definition 5: Vector Spaces

Those 10 axioms that you kinda remember but not really.

Vector Subspaces

Theorem 2: Subspace Theorem

Suppose U is a non-empty subset of the vector space, $(V, \mathbb{F})$. Then $(U, \mathbb{F})$ is a **subspace** of $(V, \mathbb{F})$ if the following is condition is met:

$$\lambda \mathbf{x} + \mathbf{y} \in U$$

for any $\lambda \in \mathbb{F}$ and $\mathbf{x}, \mathbf{y} \in U$.

Why does this work?

If this condition is met, then we can show that:

- **Closure under scalar multiplication**: Set $\mathbf{y} = \mathbf{0}$.
- **Closure under vector addition**: Set $\lambda = 1$.
- **Existence of vector inverse**: Set $\lambda = -1$ and $\mathbf{y} = \mathbf{x}$.

Subspace Example

Ex 4: Subspace Theorem

Consider the set:

$$W_1 = \left\{ \begin{pmatrix} x_1 & -x_2 \\ x_1 + x_2 & 3x_1 - x_2 \end{pmatrix} : x_1, x_2 \in \mathbb{R} \right\}.$$

Prove that this is a subspace of $M_{2,2}(\mathbb{R})$.

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Prove that this is a subspace of $M_{2,2}(\mathbb{R})$.

Clearly W_1 is a (non-empty) subset of $M_{2,2}(\mathbb{R})$. Consider $X, Y \in G$ and $\lambda \in \mathbb{R}$.

$$X + \lambda Y = \begin{pmatrix} x_1 & -x_2 \\ x_1 + x_2 & 3x_1 - x_2 \end{pmatrix} + \lambda \begin{pmatrix} y_1 & -y_2 \\ y_1 + y_2 & 3y_1 - y_2 \end{pmatrix}$$

Subspace Example

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$$\begin{aligned} X + \lambda Y &= \begin{pmatrix} x_1 & -x_2 \\ x_1 + x_2 & 3x_1 - x_2 \end{pmatrix} + \lambda \begin{pmatrix} y_1 & -y_2 \\ y_1 + y_2 & 3y_1 - y_2 \end{pmatrix} \\ &= \begin{pmatrix} x_1 + \lambda y_1 & -(x_2 + \lambda y_2) \\ x_1 + \lambda y_1 + x_2 + \lambda y_2 & 3(x_1 + \lambda y_1) - (x_2 + \lambda y_2) \end{pmatrix} \end{aligned}$$

Subspace Example

Ex 4: Subspace Theorem

Consider the set:

$$W_1 = \left\{ \begin{pmatrix} x_1 & -x_2 \\ x_1 + x_2 & 3x_1 - x_2 \end{pmatrix} : x_1, x_2 \in \mathbb{R} \right\}.$$

Prove that this is a subspace of $M_{2,2}(\mathbb{R})$.

Clearly each term matches the required form of W_1 and so $X + \lambda Y \in W_1$. Thus, by the Subspace Theorem, we have shown that $W_1 \leq M_{2,2}(\mathbb{R})$.

Span

Suppose we have a set of vectors $S := \{v_1, v_2, \cdots, v_n\} \subset V$. We define the span of W as the set of all possible linear combinations of the elements of W, i.e.

$$\text{span}(S) = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_n \mathbf{v}_n$$

where $\lambda_i \in \mathbb{F}$.
It is quite easy to see that $\text{span}(S) \leq V$, and in the case when $\text{span}(S) = V$, we call S a **spanning set**.

Linear Independent and Dependent

We call a set **linearly independent** if only the trivial linear combination maps to the zero vector, i.e. if we have

$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_n \mathbf{v}_n = \mathbf{0}$$

then $\lambda_i = 0$ for all $i = 1, 2, \cdots, n$.
We call the set **linearly dependent** otherwise.

Linearly 'independent'

Linear independence means that we are unable to express any of the vectors in the set as a linear combination of all the other vectors.

Linear Independence/Dependence

Ex 5: Linearly independent and dependent sets

Consider the following set of vectors:
$S = \left\{(1, 2, 3)^T, (3, 2, 9)^T, (5, 2, -1)^T\right\}$. Is this linearly independent or linearly dependent?

Linear Independence/Dependence

Ex 5: Linearly independent and dependent sets

Consider the following set of vectors:
$S = \left\{(1,2,3)^T, (3,2,9)^T, (5,2,-1)^T\right\}$. Is this linearly independent or linearly dependent?

Setting up the relevant matrix yields:

$$\left(\begin{array}{ccc|c} 1 & 3 & 5 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 9 & -1 & 0 \end{array}\right) \xrightarrow{R_1 - \frac{1}{2}R_2} \xrightarrow{3R_1 - R_3} \left(\begin{array}{ccc|c} 1 & 3 & 5 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 16 & 0 \end{array}\right)$$

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RREF has all columns being leading columns, then solutions to this set of equations are **unique**. In other words, S must be linearly independent.

Linear Independence/Dependence

Ex 5: Linearly independent and dependent sets

Is the following set linearly independent?
$\mathcal{A} = \{\cos x, \sin x, \sin(x + \pi/4)\}$

Linear Independence/Dependence

Ex 5: Linearly independent and dependent sets

Is the following set linearly independent?
$\mathcal{A} = \{\cos x, \sin x, \sin(x + \pi/4)\}$

No, as we can observe the following property:

$$\sin(x + \pi/4) = \frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x$$

Rearrange and we have non-trivial coefficients that map to 0, i.e. **linearly dependent.**

Basis

Naturally we'll be interested in ways we can **minimally** represent a vector space, such as through **spanning sets**. The most basic sets we can choose are called **basis sets**, which obtain the following two properties:

- $\text{span}(S) = V$ i.e. spanning set of V
- S is linearly independent

Why these two properties only?

The first one is simple as we want to describe the whole set. By obtaining l.i., we are saying that each vector is 'pulling their own weight'.

Notable qualities of Basis sets

- **Uniqueness of representation**: Each $\mathbf{x} \in V$ is uniquely represented as a linear combination of the basis vectors.
- **Dependent on type of field**: Valid choices of basis also depend on the field accompanying the vector space, e.g. $(\mathbb{C}, +, \times, \mathbb{R})$ vs $(\mathbb{C}, +, \times, \mathbb{C})$.

Standard Basis

The most simplistic basis we generally call 'standard' basis, some of which you should be very familiar with:

- $\mathbb{R}^n : \{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$
- $M_{n,m}(\mathbb{R}) : \{E_{1,1}, E_{1,2}, \cdots, E_{n,m}\}$

Dimensions

Not only are there multiple basis for each vector space, but they all have the **same number of elements**. As a result, we define the following useful concept of the **dimension** of a vector space.

Definition 6: Dimension

Consider a vector space, V, which has a **finite spanning basis**, S. Then, we define the size of S to be the dimension of V, $\dim(V) = |S|$.

Theorem:

The following three statements are equivalent (where $\dim(V) = n$):

- S is a basis of V
- S is linearly independent and $|S| = n$
- S is a spanning set and $|S| = n$

Basis Sets

Ex 7: Basis Vectors

Consider the degree-2 polynomial vector space, $\mathcal{P}_2(\mathbb{R})$. Show that the following set is a basis of this vector space,
$S = \{2 + 3x, 4x - x^2, 1 + x^2\}$.

Basis Sets

Ex 7: Basis Vectors

Consider the degree-2 polynomial vector space, $\mathcal{P}_2(\mathbb{R})$. Show that the following set is a basis of this vector space, $S = \{2 + 3x, 4x - x^2, 1 + x^2\}$.

We know that $\dim(\mathcal{P}_2(\mathbb{R})) = 3$ and so we only need to show that S is linearly independent. To do this, we'll consider three values: $x = -1, 0, 1$.

$$\left(\begin{array}{ccc|c} -1 & -5 & 2 & 0 \\ 2 & 0 & 1 & 0 \\ 5 & 3 & 2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 6 & 8 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 0 & 10 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 \end{array}\right)$$

Basis Sets

Ex 7: Basis Vectors

Consider the degree-2 polynomial vector space, $\mathcal{P}_2(\mathbb{R})$. Show that the following set is a basis of this vector space, $S = \{2 + 3x, 4x - x^2, 1 + x^2\}$.

We can clearly see that the only solution is the trivial solution. Hence, the only way to make sure that the linear sum of these functions maps to zero, is to make all coefficients zero, i.e. S is linearly independent.

Thus, by the previous dimension theorem, we can say that S is a basis of $\mathcal{P}_2(\mathbb{R})$.

Coordinates

When looking back at the Cartesian plane, we described all of the points **uniquely** as 'coordinates'. This was possible as the x-y directional vectors were **basis vectors of** $\mathbb{R}^2$. We'll be extending this idea to apply to more general basis'.

Definition 7: Coordinate Vector

Suppose we have a basis $\mathcal{B} = \{\mathbf{v}_i\}_{i=1}^n$ and $\mathbf{x} \in V$ such that:

$$\mathbf{x} = \sum_{i=1}^n \alpha_i \mathbf{v}_i.$$

Then, we define the vector $[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$ to be the **coordinate vector** of $\mathbf{x}$ w.r.t $\mathcal{B}$.

Properties of Coordinate Vectors

By noting that how these coordinate vectors are **unique** to each pair of $(\mathbf{x}, \mathcal{B})$, it is also possible to work within this representation, rather than directly in the space of V. This is notable as V can be very abstract and we are (generally) more comfortable working with field vectors.

How to find Coordinate Vectors?

The most basic approach for finding these coordinate vectors is solving the matrix equation: $V\boldsymbol{\alpha} = \mathbf{x}$, where the columns of V are the basis vectors.

Other approaches exist, but are mostly circumstantial, e.g. for polynomials we can sub in n different values and find the coefficients.

Coordinate Vector Example

Ex 8: Coordinate Vectors

Let V be the vector space of all 2×2 real symmetric matrices, and $A = \begin{pmatrix} 9 & 5 \\ 5 & -4 \end{pmatrix}$. Find the corresponding coordinate vector w.r.t the following basis:

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 4 & -1 \\ -1 & -5 \end{pmatrix} \right\}.$$

We want to find a vector α such that:

$$A = \sum_{i=1}^{3} \alpha_i B_i.$$

Coordinate Vector Example

Rearranging this, so that we focus on each individual component of the matrices yields:

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 9 \\ -2 & 1 & -1 & 5 \\ 1 & 3 & -5 & -4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 4 & 9 \\ 0 & 5 & 7 & 23 \\ 0 & -1 & 9 & 13 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 4 & 9 \\ 0 & 5 & 7 & 23 \\ 0 & 0 & 52 & 88 \end{array}\right)$$

Coordinate Vector Example

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Solving this augmented matrix leads to:

$$\alpha_3 = \frac{22}{13},\ \alpha_2 = \frac{29}{13},\ \alpha_1 = -\frac{29}{13}$$

Hence, the coordinate vector of A w.r.t. $\mathcal{B}$ is:

$$[A]_{\mathcal{B}} = \frac{1}{13}(-29, 29, 22)^T$$

Sums of Vector Spaces (MATH2601)

Suppose that we have two subspaces, $U, W \subseteq V$. We define their **sum** as:

Definition 8

$U + W = \{\mathbf{y} \in V : \mathbf{y} = \mathbf{u} + \mathbf{w}, \text{for some } \mathbf{u} \in U, \mathbf{w} \in W\}$.

In the case that these intersection of these two subspaces only contains the $\mathbf{0}$ vector, then we call this a **direct sum** (denoted by $U \oplus W$).

Sums of Vector Spaces (MATH2601)

The concept of summing vector spaces actually leads to a very familiar result about sets, i.e. the cardinality relationship.

Theorem

Suppose that U, W are finite subspaces of V, then we have

$$\dim(U) + \dim(W) = \dim(U + W) + \dim(U \cap W).$$

This can be used to help determine whether the sum is direct or not.

Sum of Subspaces Example (MATH2601)

2010 Q1.b)

Let W be the vector space over $\mathbb{R}$ defined by

$$W = \{(z_1, z_2, x) : z_1, z_2 \in \mathbb{C}, x \in \mathbb{R}\}$$

with the usual addition and scalar multiplication (with scalar field being $\mathbb{R}$).

a) What is $\dim(W)$?

Sum of Subspaces Example (MATH2601)

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with the usual addition and scalar multiplication (with scalar field being $\mathbb{R}$).

a) What is $\dim(W)$?

It can be easily shown that the following set is a basis of W:

$$\mathcal{B}_W = \{(1,0,0), (i,0,0), (0,1,0), (0,i,0), (0,0,1)\}.$$

Hence, $\dim(W) = 5$.

Sum of Subspaces Example (MATH2601)

2010 Q1.b)

b) Consider the following subspaces of W:

$$U = \{(z_1, z_2, x) \in W : iz_1 + x = 0\}$$
$$V = \{(z_1, z_2, x) \in W : (1+i)z_2 - x = 0\}.$$

What is the dimension of U? V?

Sum of Subspaces Example (MATH2601)

2010 Q1.b)

b) Consider the following subspaces of W:

$$U = \{(z_1, z_2, x) \in W : iz_1 + x = 0\}$$
$$V = \{(z_1, z_2, x) \in W : (1+i)z_2 - x = 0\}.$$

What is the dimension of U? V?

For $\mathbf{u} \in U$, we can see the following:

$$\mathbf{u} = \begin{pmatrix} -ix \\ z_2 \\ x \end{pmatrix} = x \begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix} + \mathrm{Re}(z_2) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \mathrm{Im}(z_2) \begin{pmatrix} 0 \\ i \\ 0 \end{pmatrix}.$$

This is clearly a **linearly independent** set of vectors and they also **span** U, hence this is a basis for U. Thus $\dim(U) = 3$.

Sum of Subspaces Example (MATH2601)

2010 Q1.b)

b) Consider the following subspaces of W:

$$U = \{(z_1, z_2, x) \in W : iz_1 + x = 0\}$$
$$V = \{(z_1, z_2, x) \in W : (1+i)z_2 - x = 0\}.$$

What is the dimension of U? V?

For $\mathbf{v} \in V$, we can see the following:

$$\mathbf{v} = \begin{pmatrix} z_1 \\ \frac{x}{1+i} \\ x \end{pmatrix} = \mathrm{Re}(z_1) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mathrm{Im}(z_1) \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix} + x \begin{pmatrix} 0 \\ \frac{1-i}{2} \\ 1 \end{pmatrix}.$$

A similar argument applies here as before, and so we have $\dim(V) = 3$.

Sum of Subspaces Example (MATH2601)

2010 Q1.b)

c) Find a basis for $U \cap V$. What is $\dim(U + V)$?

Sum of Subspaces Example (MATH2601)

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We can see that all elements of $U \cap V$ have these properties:

$$z_1 = \frac{x}{-i} = ix$$

$$z_2 = \frac{x}{1+i} = \frac{1-i}{2}x$$

Sum of Subspaces Example (MATH2601)

2010 Q1.b)

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$$z_2 = \frac{x}{1+i} = \frac{1-i}{2}x$$

Doing the same thing as in **b)**, we have:

$$\begin{pmatrix} z_1 \\ z_2 \\ x \end{pmatrix} = x \begin{pmatrix} i \\ \frac{i-1}{2} \\ 1 \end{pmatrix}.$$

From this we can see that $\dim(U \cap V) = 1$.

Sum of Subspaces Example (MATH2601)

2010 Q1.b)

c) Find a basis for $U \cap V$. What is $\dim(U + V)$?

Using the previous theorem, we can find $\dim(U + V)$,

$$\begin{aligned}\dim(U + V) &= \dim(U) + \dim(V) - \dim(U \cap V) \\ &= 3 + 3 - 1 \\ &= 5.\end{aligned}$$

Linear Transformations

Linear Transformations

Definition

The mapping $T : V \to W$ (over the same field) is called a linear transformation if:

- $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$
- $T(\lambda\mathbf{x}) = \lambda T(\mathbf{x})$

for any $\mathbf{x}, \mathbf{y} \in V, \lambda \in \mathbb{F}$.

Linear Transformations

The way to prove that something is a linear map is slightly easier than first-year, as we'll combine them both into one proof.

Lemma

Consider a mapping $T : V \to W$, in which V, W are both vector spaces. Then T is a linear map iff

$$T(\mathbf{x} + \lambda \mathbf{y}) = T(\mathbf{x}) + \lambda T(\mathbf{y})$$

for any $\mathbf{x}, \mathbf{y} \in V, \lambda \in \mathbf{F}$.

Linear Transform Example

Linear Transform Example

The function $T : \mathbb{P}_2(\mathbb{R}) \to \mathbb{R}^3$ is defined by:

$$T(p) = (p(1), p(3), p'(2)).$$

Prove that this is linear.

Linear Transform Example

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Consider $p, q \in \mathbb{P}_2(\mathbb{R})$ and $\lambda \in \mathbb{R}$.

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Prove that this is linear.

Consider $p, q \in \mathbb{P}_2(\mathbb{R})$ and $\lambda \in \mathbb{R}$.

$$T(p + \lambda q) = ((p + \lambda q)(1), (p + \lambda q)(3), (p + \lambda q)'(2))$$

As the sum of functions and derivatives are linear, we have:

Linear Transform Example

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$$T(p + \lambda q) = ((p + \lambda q)(1), (p + \lambda q)(3), (p + \lambda q)'(2))$$

As the sum of functions and derivatives are linear, we have:

$$\begin{aligned} &= (p(1) + \lambda q(1), p(3) + \lambda q(3), p'(2) + \lambda q'(2)) \\ &= (p(1), p(3), p'(2)) + \lambda(q(1), q(3), q'(2)) \\ &= T(p) + \lambda T(q) \end{aligned}$$

Thus, T is a linear transform.

Kernel and Image of T

Similarly with groups, we also have kernel and image for linear transforms, $T : V \to W$.

Kernel

The **kernel** is the set of 'roots' of T, i.e.

$$\ker(T) := \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}$$

Image

The **image** is the set of 'projections' of T, i.e.

$$\operatorname{im}(T) := \{\mathbf{y} \in W : T(\mathbf{x}) = \mathbf{y}, \text{for some } \mathbf{x} \in V\}$$

These are also both **vector spaces** under the field, $\mathbb{F}$, as well. (How can we show this?)

Rank and Nullity

The dimensions of the **kernel** and **rank** are given special names: **nullity** and **rank**, respectively.

Rank-Nullity Theorem

Consider a linear transform, $T : V \to W$ in which V is finite-dimensional vector space. We have:

$$\text{rank}(T) + \text{nullity}(T) = \dim(V).$$

Suppose that $\dim(V) = \dim(W) = n$, then we have the following **equivalent statements**:

Properties from Rank & Nullity

- $\text{nullity}(T) = 0$.
- $\text{rank}(T) = \dim(V) = n$.
- T is an invertible mapping.

Rank and Nullity Example

Matrix Time

Consider the following matrix:

$$A = \begin{pmatrix} 1 & 1 & 0 & 2 & 1 \\ 0 & 0 & -1 & -2 & 2 \\ -1 & -1 & 1 & 4 & -1 \\ 1 & 1 & 0 & 4 & 2 \end{pmatrix}.$$

Find its rank and nullity of A. What is a basis for image(A)?

Rank and Nullity Example

For matrices, we just do regular row-reductions until we reach the row-reduced echelon form.

$$\begin{pmatrix} 1 & 1 & 0 & 2 & 1 \\ 0 & 0 & -1 & -2 & 2 \\ -1 & -1 & 1 & 4 & -1 \\ 1 & 1 & 0 & 4 & 2 \end{pmatrix} \xrightarrow{R_1+R_3} \xrightarrow{R_3+R_4} \begin{pmatrix} 1 & 1 & 0 & 2 & 1 \\ 0 & 0 & -1 & -2 & 2 \\ 0 & 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & 2 & 1 \\ 0 & 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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$$\begin{pmatrix} 1 & 1 & 0 & 2 & 1 \\ 0 & 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

As this has three leading columns, then that means $\text{rank}(A) = 3$ ($\text{nullity}(A) = 2$, no. of non-leading columns).

Rank and Nullity Example

To find a basis for the image(A) we simply select the corresponding leading columns i.e.

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 4 \\ 4 \end{pmatrix} \right\}.$$

Matrix Representation of Linear Maps

Rather than working directly with **linear transforms**, we can work with their equivalent **matrix representations** (while noting what the chosen basis for V and W are).

Theorem: Matrix Representation of Linear Maps

Consider the following linear transform, $T : V \to W$, with basis $\mathcal{B}$ and $\mathcal{C}$ respectively. We can find an equivalent unique matrix representation, A such that:

$$[T(\mathbf{v})]_{\mathcal{C}} = A[\mathbf{v}]_{\mathcal{B}}$$

for any $\mathbf{v} \in V$.

From here on, we'll denote this matrix representation as A.

Procedure to find A

Suppose $\dim(V) = p$ and $T : V \to W$ is a linear map.

1. Choose a basis set for the domain and co-domain, $\mathcal{B}$ and $\mathcal{C}$.
2. Find $T(\mathbf{v}_1)$ and determine the corresponding coordinate vector w.r.t $\mathcal{C}$.
3. The vector from 2 is the first column of A.
4. Repeat steps 2 and 3 for all other $p - 1$ basis vectors, with each one becoming the next column of A.

Procedure to find A

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3. The vector from 2 is the first column of A.
4. Repeat steps 2 and 3 for all other $p - 1$ basis vectors, with each one becoming the next column of A.

The above process is simple enough, but the real issue is to do with **our choice of basis** for the domain and co-domain. Generally speaking it'll be quite **cumbersome to directly apply this approach**.

Indirect Approach to find $[T]^{\mathcal{B}}_{\mathcal{C}}$

Firstly we'll utilise the following results:

Theorem: Composition of Linear Mappings

Suppose $S : U \to V$ and $T : V \to W$ are both linear maps. Then $T \circ S$ is also a linear map.

Theorem: Matrix Representation of Linear Composition

Consider basis $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ for vector spaces U, V and W. Then the matrix representation of $T \circ S$ is

$$[T \circ S]^{\mathcal{A}}_{\mathcal{C}} = [T]^{\mathcal{B}}_{\mathcal{C}}[S]^{\mathcal{A}}_{\mathcal{B}}.$$

Using this, we can see that $T = \mathrm{id}_W \circ (T \circ \mathrm{id}_V)$ is another way to think of any linear map, T.

Matrix Representation Example

2016 FE

Consider the linear transform from earlier: $T : \mathbb{P}_2(\mathbb{R}) \to \mathbb{R}^3$, $T(p) = (p(1), p(2), p'(2))$ and the following two respective bases: $\mathcal{B} = \{1 + t, 2 - t^2, 3t - t^2\}, \mathcal{C} = \{(1, 0, 0), (0, 1, 0), (-1, 2, 1)\}$. Find the representation matrix of T w.r.t to these two bases.

Matrix Representation Example

2016 FE

$T : \mathbb{P}_2(\mathbb{R}) \to \mathbb{R}^3,\ T(p) = (p(1), p(2), p'(2))$
$\mathcal{B} = \{1 + t, 2 - t^2, 3t - t^2\},\ \mathcal{C} = \{(1, 0, 0), (0, 1, 0), (-1, 2, 1)\}$

Directly finding this matrix will be somewhat cumbersome, so we'll go with the indirect approach. Namely,

$$A = C^{-1} \times S \times B$$

where

1. B: changes the basis from $\mathcal{B}$ to the standard basis of $\mathbb{P}_2(\mathbb{R})$
2. S: matrix representation with standard basis
3. C^{-1}: changes the basis from the standard basis of $\mathbb{R}^3$ back to $\mathcal{C}$.

Matrix Representation Example

2016 FE

$T : \mathbb{P}_2(\mathbb{R}) \to \mathbb{R}^3,\ T(p) = (p(1), p(2), p'(2))$
$\mathcal{B} = \{1 + t, 2 - t^2, 3t - t^2\},\ \mathcal{C} = \{(1, 0, 0), (0, 1, 0), (-1, 2, 1)\}$

1. Change basis from $\mathcal{B}$ to the standard basis of degree-2 polynomials:

$$\mathcal{B} = \{1 + t, 2 - t^2, 3t - t^2\} \xrightarrow{\text{id}} \{1, t, t^2\}.$$

$$B = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 3 \\ 0 & -1 & -1 \end{pmatrix}.$$

Matrix Representation Example

2016 FE

$T : \mathbb{P}_2(\mathbb{R}) \to \mathbb{R}^3,\ T(p) = (p(1), p(2), p'(2))$
$\mathcal{B} = \{1 + t, 2 - t^2, 3t - t^2\},\ \mathcal{C} = \{(1, 0, 0), (0, 1, 0), (-1, 2, 1)\}$

2. Matrix representation with standard basis:

$$\{1, t, t^2\} \xrightarrow{T} \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

$$\begin{aligned}[T(1)]_{\mathcal{S}_2} &= [(1, 1, 0)]_{\mathcal{S}_2} \\ &= (1, 1, 0)^T\end{aligned}$$

Matrix Representation Example

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$$\begin{aligned} [T(1)]_{\mathcal{S}_2} &= [(1, 1, 0)]_{\mathcal{S}_2} \\ &= (1, 1, 0)^T \\ [T(t)]_{\mathcal{S}_2} &= [(1, 2, 1)]_{\mathcal{S}_2} \\ &= (1, 2, 1)^T \end{aligned}$$

Matrix Representation Example

2016 FE

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$$\begin{aligned}
[T(1)]_{\mathcal{S}_2} &= [(1, 1, 0)]_{\mathcal{S}_2} \\
&= (1, 1, 0)^T \\
[T(t)]_{\mathcal{S}_2} &= [(1, 2, 1)]_{\mathcal{S}_2} \\
&= (1, 2, 1)^T \\
[T(t^2)]_{\mathcal{S}_2} &= [(1, 4, 4)]_{\mathcal{S}_2} \\
&= (1, 4, 4)^T.
\end{aligned}$$

Matrix Representation Example

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2. Matrix representation with standard basis:

$$\{1, t, t^2\} \xrightarrow{T} \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

So the matrix representation for standard basis is:

$$S = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 0 & 1 & 4 \end{pmatrix}.$$

Matrix Representation Example

2016 FE

$T : \mathbb{P}_2(\mathbb{R}) \to \mathbb{R}^3,\ T(p) = (p(1), p(2), p'(2))$
$\mathcal{B} = \{1 + t, 2 - t^2, 3t - t^2\},\ \mathcal{C} = \{(1, 0, 0), (0, 1, 0), (-1, 2, 1)\}$

3. Change basis from standard basis back to $\mathcal{C}$:

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \xrightarrow{\text{id}} \mathcal{C}.$$

$$C^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

Matrix Representation Example

2016 FE

$T : \mathbb{P}_2(\mathbb{R}) \to \mathbb{R}^3,\ T(p) = (p(1), p(2), p'(2))$
$\mathcal{B} = \{1 + t, 2 - t^2, 3t - t^2\},\ \mathcal{C} = \{(1, 0, 0), (0, 1, 0), (-1, 2, 1)\}$

From this, we have the full representation matrix:

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 3 \\ 0 & -1 & -1 \end{pmatrix}$$

with respect to basis $\mathcal{B}$ and $\mathcal{C}$.

Isomorphism (MATH2601)

Definition: Isomorphic

Consider any two vector spaces V, W. If there exists an **invertible linear map** T between them, then they are **isomorphic to each other**.

This statement also applies to their matrix representations, regardless of the basis chosen.

Isomorphism Example (MATH2601)

Isomorphic

Consider the following vector space pairs:

- $\mathbb{C}^2$ and $\mathbb{R}^2$ (under the field $\mathbb{R}$)
- $\mathcal{P}_2(\mathbb{R})$ and $\mathbb{R}^3$

Isomorphism Example (MATH2601)

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Consider the following vector space pairs:

- $\mathbb{C}^2$ and $\mathbb{R}^2$ (under the field $\mathbb{R}$)
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Not isomorphic, as the dimensions of the two vector spaces are different from each other ($\dim(\mathbb{C}^2) = 4$ under $\mathbb{R}$).

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Consider the following vector space pairs:

- $\mathbb{C}^2$ and $\mathbb{R}^2$ (under the field $\mathbb{R}$)
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Dimensions are the same, so we have to go further with our checks. By noting that matrix representations are essentially the same thing as a linear transform, we can check the **invertibility** of the earlier found representation.

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 1 & 4 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = 8 - 4 - (4 - 1) = 1 \neq 0$$

Invariant Transformations (MATH2601)

Definition: Invariant Subspace

Consider a linear mapping $T : V \to V$, with $X \leq V$. If $T(X)$ is a subspace of X then we call X **an invariant subspace under** T.

In other words, X maps back to a subset of itself. If we can break up our initial vector space into two **disjoint spanning groups** then we have:

Invariant Transforms Example (MATH2601)

2018 FE Q3b)

Let V be a vector space, and S, T be linear transforms from V to V. Define $W = \ker(S - T)$. Show that if $ST = TS$ then W is invariant under T.

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2018 FE Q3b)

Let V be a vector space, and S, T be linear transforms from V to V. Define $W = \ker(S - T)$. Show that if $ST = TS$ then W is invariant under T.

Want to show: $T(W) \subseteq W$.
Consider any $\mathbf{w} \in W$, and we'll consider its transform $T(\mathbf{w})$.

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Consider any $\mathbf{w} \in W$, and we'll consider its transform $T(\mathbf{w})$.

$$
\begin{aligned}
(T - S) \circ (T(\mathbf{w})) &= T^2(\mathbf{w}) - ST(\mathbf{w}) \\
&= T^2(\mathbf{w}) - TS(\mathbf{w}) \qquad \text{(assumption of } TS = ST\text{)} \\
&= T \circ [(T - S) \circ (\mathbf{w})]
\end{aligned}
$$

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As $\mathbf{w} \in W$, then we have:

$$= T \circ [\mathbf{0}] = \mathbf{0}.$$

As this argument was applied to any $\mathbf{w}$, then we have shown that W is invariant under T.

Normal Form

The idea here is that we can break down any linear transformation based on their **image** and **kernel**. Simply find the basis for the **kernel** and then **extend it to be a basis for the domain**, and do the same for the **image projections**. This leads to the following matrix representation:

$$N_{p;q;r} = \begin{pmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = I_r \oplus \mathbf{0}.$$

This matrix has p columns and q rows.
Note: $\operatorname{rank}(T) = r, \dim(V) = p, \dim(W) = q$.

Similarity

Definition: Similarity

Suppose we have two matrices $A, B \in M_{p,p}(\mathbb{F})$ and an invertible $P \in \mathrm{GL}(\mathbb{F}, p)$ s.t.

$$B = P^{-1}AP.$$

We say that A is **similar to** B.

The idea here is that these matrices essentially **represent the same linear transformation** (just with different basis!). As such, a few properties remain intact:

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The idea here is that these matrices essentially **represent the same linear transformation** (just with different basis!). As such, a few properties remain intact:

- $rank(A) = rank(B)$;
- $nullity(A) = nullity(B)$;
- $tr(A) = tr(B)$;
- $|A| = |B|$

Similarity Example

FE 2008 2.b)

Show that

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -2 & 5 \\ 3 & 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -4 & -2 & 0 \\ 2 & 3 & 0 \\ 1 & -5 & 1 \end{pmatrix}$$

are not similar.

Similarity Example

FE 2008 2.b)

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are not similar.

Looking at the similarity invariant properties, we try to find a mismatch:

- **Trace:** $tr(A) = 1 + -2 + 1 = 0$ and $tr(B) = -4 + 3 + 1 = 0$

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- **Trace:** $tr(A) = 1 + -2 + 1 = 0$ and $tr(B) = -4 + 3 + 1 = 0$
- **Determinant:** $|A| = 1(-2(1) - 5(1)) - 0 + 0 = -7$ and $|B| = 1(-4(3) - -2(2)) = -8$.

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FE 2008 2.b)

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- **Determinant:** $|A| = 1(-2(1) - 5(1)) - 0 + 0 = -7$ and $|B| = 1(-4(3) - -2(2)) = -8$.

We stop here as the determinants aren't the same, and so they **can't be similar to each other**.

Inner Product Spaces

Dot Product

Definition: Dot Product

Consider any two $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. The dot product is defined as:

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i.$$

Complex Dot Product

Consider any two $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$. The complex dot product is defined as:

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} \bar{x}_i y_i.$$

Dot Product

Definition: Dot Product

Consider any two $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. The dot product is defined as:

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i.$$

For non-zero $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we define the **angle** between these vectors as θ, where:

$$\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \|\mathbf{y}\|}.$$

If the dot product is 0, we say that $\mathbf{x}$ and $\mathbf{y}$ are **orthogonal** to each other.

Inner Product

Definition: Inner Product

Consider a vector space, V with its respective field, $\mathbb{F}$. An inner product is a function between any two vectors in V that maps to $\mathbb{F}$ with the following properties:

- $\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$
- $\langle \mathbf{x}, \lambda \mathbf{y} \rangle = \lambda \langle \mathbf{x}, \mathbf{y} \rangle$
- $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$
- $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ with equality iff $\mathbf{x} = \mathbf{0}$.

Generalises Dot Products

Just like in everything pretty much everything before in this course, the inner product is a generalisation of the **dot product**.

Orthogonality

Definition: Orthogonal Vectors

Any two vectors $\mathbf{x}, \mathbf{y} \in V$ with $\langle \mathbf{x}, \mathbf{y} \rangle = 0$, are said to be **orthogonal to each other**. This can be denoted as $\mathbf{x} \perp \mathbf{y}$.

Definition: Orthonormal Vectors

Suppose that you have a set of vectors, $S = \{\mathbf{v}_i\}_{i=1}^{n}$. This set is said to be **orthonormal** if for $i, j = 1, 2, \ldots, n$:

$$\langle \mathbf{x}_i, \mathbf{x}_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Projection

Definition: Projection

Suppose that $\mathbf{y}, \mathbf{x} \in V$. We define the **projection** of $\mathbf{x}$ onto $\mathbf{y}$ as:

$$\text{proj}_{\mathbf{y}}(\mathbf{x}) = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\langle \mathbf{y}, \mathbf{y} \rangle}\mathbf{y}.$$

It turns out that we also have: $\text{proj}_{\mathbf{y}}(\mathbf{x}) \perp \mathbf{y} - \mathbf{x}$.

Interpretation of Projection

The projection is essentially the 'portion' of $\mathbf{x}$ explained by $\mathbf{y}$. This can be observed by looking at the above result.

Projection Example

FE 2017 Q1.c)

For the degree-2 polynomial vector space, we consider

$$f(t) = 3 + t^2 \quad g_1(t) = 1 - t \quad g_2(t) = 2 + t - t^2.$$

Find the projection of f onto $W = \text{span}\{g_1, g_2\}$, under the following inner product:

$$\langle p, q \rangle = p(0)q(0) + p(1)q(1) + p(2)q(2).$$

Projection Example

FE 2017 Q1.c)

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$$\langle p, q \rangle = p(0)q(0) + p(1)q(1) + p(2)q(2).$$

We know that $f(t) = \alpha_1 g_1(t) + \alpha_2 g_2(t) + h(t)$ where $h \in W^\perp$ (Why?). We want the projection of f onto W, and so we want to find α_1, α_2.

Projection Example

FE 2017 Q1.c)

For the degree-2 polynomial vector space, we consider

$$f(t) = 3 + t^2 \quad g_1(t) = 1 - t \quad g_2(t) = 2 + t - t^2.$$

Find the projection of f onto $W = \text{span}\{g_1, g_2\}$, under the following inner product:

$$\langle p, q \rangle = p(0)q(0) + p(1)q(1) + p(2)q(2).$$

Inner producting w.r.t g_1 and g_2 on both sides will yield:

$$\langle f, g_1 \rangle = \alpha_1 \langle g_1, g_1 \rangle + \alpha_2 \langle g_2, g_1 \rangle \quad \langle f, g_2 \rangle = \alpha_1 \langle g_1, g_2 \rangle + \alpha_2 \langle g_2, g_2 \rangle.$$

Projection Example

Finding the intermediate calculations as follows:

$$\begin{aligned}
\langle f, g_1 \rangle &= 3(1) + 4(0) + 7(-1) \\
&= -4, \\
\langle g_1, g_1 \rangle &= 1(1) + 0(0) + -1(-1) \\
&= 2, \\
\langle g_2, g_1 \rangle &= 2(1) + 2(0) + 0(-1) \\
&= 2.
\end{aligned}$$

Projection Example

Finding the intermediate calculations as follows:

$$\begin{aligned}\langle f, g_1\rangle &= 3(1) + 4(0) + 7(-1)\\ &= -4,\\ \langle g_1, g_1\rangle &= 1(1) + 0(0) + -1(-1)\\ &= 2,\\ \langle g_2, g_1\rangle &= 2(1) + 2(0) + 0(-1)\\ &= 2.\end{aligned}$$

$$\begin{aligned}\langle f, g_2\rangle &= 3(2) + 4(2) + 7(0)\\ &= 14,\\ \langle g_1, g_2\rangle &= \overline{\langle g_2, g_1\rangle}\\ &= \bar{2} = 2,\\ \langle g_2, g_2\rangle &= 2(2) + 2(2) + 0(0)\\ &= 8.\end{aligned}$$

Projection Example

FE 2017 Q1.c)

For the degree-2 polynomial vector space, we consider

$$f(t) = 3 + t^2 \quad g_1(t) = 1 - t \quad g_2(t) = 2 + t - t^2.$$

Find the projection of f onto $W = \text{span}\{g_1, g_2\}$, under the following inner product:

$$\langle p, q \rangle = p(0)q(0) + p(1)q(1) + p(2)q(2).$$

Subbing all of this in leads to:

$$2\alpha_1 + 2\alpha_2 = -4 \qquad 2\alpha_1 + 8\alpha_2 = 14.$$

We see that $\alpha_1 = -5, \alpha_2 = 3$. Hence, the projection of f onto W is,

$$\text{proj}_W(f) = -5g_1 + 3g_2 = 1 + 8t - 3t^2.$$

Gram-Schmidt Process

Theorem: Gram-Schmidt Process

Any finite-dimensional inner product space has an orthonormal basis.

This pretty much comes from the fact that we can 'swap' out vectors in our basis and replace them with orthogonal vectors, one at a time.

Procedure:

$$\begin{aligned}
\mathbf{w}_1 &= \mathbf{v}_1 \\
\mathbf{w}_2 &= \mathbf{v}_2 - \mathrm{proj}_{\mathbf{w}_1}(\mathbf{v}_2) \\
\mathbf{w}_3 &= \mathbf{v}_3 - \mathrm{proj}_{\mathbf{w}_1}(\mathbf{v}_3) - \mathrm{proj}_{\mathbf{w}_2}(\mathbf{v}_3) \\
&\vdots \\
\mathbf{w}_n &= \mathbf{v}_n - \mathrm{proj}_{\mathbf{w}_1}(\mathbf{v}_n) - \ldots - \mathrm{proj}_{\mathbf{w}_{n-1}}(\mathbf{v}_n)
\end{aligned}$$

After this, just normalise each vector.

Gram-Schmidt Example

FE 2011 Q3

Let W be a subspace of $\mathbb{R}^4$ that is spanned by the following vectors: $\mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ -2 \\ 0 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} -3 \\ -1 \\ 4 \\ 1 \end{pmatrix}$. Find an orthonormal basis for W (under the regular dot product).

Gram-Schmidt Example

FE 2011 Q3

Let W be a subspace of $\mathbb{R}^4$ that is spanned by the following vectors: $\mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ -2 \\ 0 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} -3 \\ -1 \\ 4 \\ 1 \end{pmatrix}$. Find an orthonormal basis for W (under the regular dot product).

Using the Gram-Schmidt approach, we can find an orthonormal basis for W.

$$
\begin{aligned}
\mathbf{w}_1 &= \mathbf{v}_1 = (2, -1, 0, 2)^T \\
\mathbf{w}_2 &= \mathbf{v}_2 - \mathrm{proj}_{\mathbf{w}_1}(\mathbf{v}_2) \\
&= (3, 0, -2, 0)^T - \frac{(2, -1, 0, 2)^T \cdot (3, 0, -2, 0)^T}{\|(2, -1, 0, 2)^T\|^2}(2, -1, 0, 2)^T
\end{aligned}
$$

Gram-Schmidt Example

FE 2011 Q3

Let W be a subspace of $\mathbb{R}^4$ that is spanned by the following vectors: $\mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ -2 \\ 0 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} -3 \\ -1 \\ 4 \\ 1 \end{pmatrix}$. Find an orthonormal basis for W (under the regular dot product).

$$\begin{aligned}
\mathbf{w}_2 &= (3,0,-2,0)^T - \frac{6+0+0+0}{4+1+0+4}(2,-1,0,2)^T \\
&= (3,0,-2,0)^T - \frac{1}{3}(4,-2,0,4)^T \\
&= \frac{1}{3}(5,2,-6,-4)^T.
\end{aligned}$$

Gram-Schmidt Example

FE 2011 Q3

Let W be a subspace of $\mathbb{R}^4$ that is spanned by the following vectors: $\mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ -2 \\ 0 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} -3 \\ -1 \\ 4 \\ 1 \end{pmatrix}$. Find an orthonormal basis for W (under the regular dot product).

Intermediate calculations:

$$\begin{aligned} \mathbf{w}_1 \cdot \mathbf{v}_3 &= (2, -1, 0, 2)^T \cdot (-3, -1, 4, 1)^T \\ &= -6 + 1 + 0 + 2 = -3 \end{aligned}$$

Gram-Schmidt Example

FE 2011 Q3

Let W be a subspace of $\mathbb{R}^4$ that is spanned by the following vectors: $\mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ -2 \\ 0 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} -3 \\ -1 \\ 4 \\ 1 \end{pmatrix}$. Find an orthonormal basis for W (under the regular dot product).

Intermediate calculations:

$$\begin{aligned}
\mathbf{w}_1 \cdot \mathbf{v}_3 &= (2, -1, 0, 2)^T \cdot (-3, -1, 4, 1)^T \\
&= -6 + 1 + 0 + 2 = -3 \\
\mathbf{w}_2 \cdot \mathbf{v}_3 &= \frac{1}{3}(5, 2, -6, -4)^T \cdot (-3, -1, 4, 1)^T \\
&= \frac{1}{3}(-15 - 2 - 24 - 4) = -15
\end{aligned}$$

Gram-Schmidt Example

$$
\begin{aligned}
\mathbf{w}_3 &= \mathbf{v}_3 - \operatorname{proj}_{\mathbf{w}_1}(\mathbf{v}_3) - \operatorname{proj}_{\mathbf{w}_2}(\mathbf{v}_3) \\
&= (-3,-1,4,1)^T - \frac{-3}{9}(2,-1,0,2)^T - \frac{-15}{9}(5,2,-6,-4)^T \\
&= \frac{1}{3}\left[(-9,-3,12,3)^T + (2,-1,0,2)^T + (25,10,-30,-20)^T\right] \\
&= (6,2,-6,-5)^T.
\end{aligned}
$$

Gram-Schmidt Example

So an orthogonal basis of W is:

$$\mathcal{B}_2 = \left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \\ 2 \end{pmatrix}, \frac{1}{3}\begin{pmatrix} 5 \\ 2 \\ -6 \\ -4 \end{pmatrix}, \begin{pmatrix} 6 \\ 2 \\ -6 \\ -5 \end{pmatrix} \right\}.$$

Gram-Schmidt Example

Orthonormal basis is:

$$\mathcal{B}_3 = \left\{ \frac{1}{3}\begin{pmatrix} 2 \\ -1 \\ 0 \\ 2 \end{pmatrix}, \frac{1}{9}\begin{pmatrix} 5 \\ 2 \\ -6 \\ -4 \end{pmatrix}, \frac{1}{\sqrt{101}}\begin{pmatrix} 6 \\ 2 \\ -6 \\ -5 \end{pmatrix} \right\}.$$

QR Factorisation

Theorem: QR Factorisation

Suppose that we have **full-rank** $p \times q$ matrix, A, we can represent it as $A = QR$. Q is a $p \times q$ orthogonal matrix and R is an invertible $q \times q$ upper-right triangular matrix.

Method to find the QR Factorisation

This result simply follows from the Gram-Schmidt Process, as we can think of the column space of A as a finite-dimensional vector space (for which we can find an orthonormal basis).

QR Factorisation

$$A = \left(\mathbf{q}_1 \;\middle|\; \mathbf{q}_2 \;\middle|\; \dots \;\middle|\; \mathbf{q}_q\right) \begin{pmatrix} \|\mathbf{w}_1\| & \langle \mathbf{q}_1, \mathbf{a}_2 \rangle & \dots & \langle \mathbf{q}_1, \mathbf{a}_q \rangle \\ & \|\mathbf{w}_2\| & \dots & \langle \mathbf{q}_2, \mathbf{a}_q \rangle \\ & & \ddots & \vdots \\ & & & \|\mathbf{w}_q\| \end{pmatrix}$$

where

- $\mathbf{w}_k$ are the vectors found directly by using the Gram-Schmidt process on the columns of A
- $\mathbf{q}_k$ is the corresponding normalised vector
- $\mathbf{a}_k$ are the column vectors of A.

QR Factorisation

Theorem: QR Factorisation

Suppose that we have **full-rank** $p \times q$ matrix, A, we can represent it as $A = QR$. Q is a $p \times q$ orthogonal matrix and R is an invertible $q \times q$ upper-right triangular matrix.

Make Q square

If A is non-square, then so will our Q. If we want to **make Q a square matrix** simply extend the Gram-Schmidt process to $\mathbb{F}^p$ and make the constructed vectors the column vectors of Q. For R, simply add $p - q$ extra rows full of zeroes.

QR Factorisation Example

2501 2017 FE

Find a QR factorisation of $B = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$.

QR Factorisation Example

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Find a QR factorisation of $B = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$.

As the goal is to find a QR factorisation, for now we'll just use the Gram-Schmidt approach on the columns.

$$\mathbf{w}_1 = (1, 2, 0, 2)^T$$

QR Factorisation Example

2501 2017 FE

Find a QR factorisation of $B = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$.

As the goal is to find a QR factorisation, for now we'll just use the Gram-Schmidt approach on the columns.

$$\begin{aligned} \mathbf{w}_1 &= (1,2,0,2)^T \\ \mathbf{w}_2 &= (2,1,1,1)^T - \text{proj}_{\mathbf{w}_1}((2,1,1,1)^T) \\ &= (2,1,1,1)^T - \frac{[2+2+0+2]}{3^2}\left((1,2,0,2)^T\right) \\ &= \frac{1}{3}(4,-1,3,-1)^T \end{aligned}$$

QR Factorisation Example

$$\mathbf{w}_3 = (1,0,1,1)^T - \frac{\langle \mathbf{w}_1, (1,0,1,1)^T \rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{w}_2, (1,0,1,1)^T \rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2$$

QR Factorisation Example

$$\begin{aligned}
\mathbf{w}_3 &= (1,0,1,1)^T - \frac{\langle \mathbf{w}_1, (1,0,1,1)^T\rangle}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\langle \mathbf{w}_2, (1,0,1,1)^T\rangle}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 \\
&= (1,0,1,1)^T - \frac{3}{3^2}(1,2,0,2)^T - \frac{2}{\sqrt{3}^2} \times \frac{1}{3}(4,-1,3,-1)^T \\
&= \frac{1}{9}(-2,-4,3,5)^T
\end{aligned}$$

QR Factorisation Example

Now we have to **normalise** each vector, leading to:

$$\hat{\mathbf{q}}_1 = \frac{1}{3}(1, 2, 0, 2)^T$$

$$\hat{\mathbf{q}}_2 = \frac{1}{3\sqrt{3}}(4, -1, 3, -1)^T$$

$$\hat{\mathbf{q}}_3 = \frac{1}{3\sqrt{6}}(-2, -4, 3, 5)^T$$

QR Factorisation Example

Now we have to **normalise** each vector, leading to:

$$\hat{\mathbf{q}}_1 = \frac{1}{3\sqrt{6}}(\sqrt{6}, 2\sqrt{6}, 0, 2\sqrt{6})^T$$
$$\hat{\mathbf{q}}_2 = \frac{1}{3\sqrt{6}}(4\sqrt{2}, -\sqrt{2}, 3\sqrt{2}, -\sqrt{2})^T$$
$$\hat{\mathbf{q}}_3 = \frac{1}{3\sqrt{6}}(-2, -4, 3, 5)^T$$

These vectors will form the columns of Q:

$$Q = \frac{1}{3\sqrt{6}}\begin{pmatrix} \sqrt{6} & 4\sqrt{2} & -2 \\ 2\sqrt{6} & -\sqrt{2} & -4 \\ 0 & 3\sqrt{2} & 3 \\ 2\sqrt{6} & -\sqrt{2} & 5 \end{pmatrix}$$

QR Factorisation Example

For the R matrix part, we simply need to find the relevant inner products:

$$\begin{aligned}\|\mathbf{w}_1\| &= \sqrt{1^2+2^2+0^2+2^2} = 3\\ \langle \hat{\mathbf{q}}_1, \mathbf{b}_2\rangle &= \frac{1}{3}(1,2,0,2)\cdot(2,1,1,1)\\ &= 2\end{aligned}$$

and so on...

QR Factorisation Example

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and so on...

This leads us to

$$R = \begin{pmatrix} 3 & 2 & 1 \\ & \sqrt{3} & \frac{2}{\sqrt{3}} \\ & & \frac{2}{3} \end{pmatrix}.$$

QR Factorisation Example

Hence, a QR factorisation of B is

$$B = \frac{1}{3\sqrt{6}} \begin{pmatrix} \sqrt{6} & 4\sqrt{2} & -2 \\ 2\sqrt{6} & -\sqrt{2} & -4 \\ 0 & 3\sqrt{2} & 3 \\ 2\sqrt{6} & -\sqrt{2} & 5 \end{pmatrix} \times \begin{pmatrix} 3 & 2 & 1 \\ & \sqrt{3} & \frac{2}{\sqrt{3}} \\ & & \frac{2}{3} \end{pmatrix}.$$

Adjoint (MATH2601)

Theorem: Adjoint Linear Maps

Consider the linear mapping, $T : V \to W$. There exists a unique linear mapping, $T^* : W \to V$ such that:

$$\langle T(\mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{v}, T^*(\mathbf{w}) \rangle$$

for all $\mathbf{v} \in V, \mathbf{w} \in W$. We call any such T^* as the **adjoint of** T.

Some properties of adjoints include:

- $(S + T)^* = S^* + T^*$
- $(\alpha T)^* = \bar{\alpha} T^*$ for any scalar α.
- $(T^*)^* = T$
- $U : W \to V$ is a linear map, then we have: $(U \circ T)^* = T^* \circ U^*$

Adjoint Example (MATH2601)

2016 2601 FE

Consider the following linear mapping, $T : \mathbb{R}^2 \to \mathbb{P}_1(\mathbb{R})$, defined by:

$$T(x_1, x_2) = (x_1 + x_2) - (2x_2)t.$$

Here, we'll consider the **standard inner product for** $\mathbb{R}^2$ and for $\mathbb{P}_2(\mathbb{R})$ we have:

$$\langle p, q \rangle = p(0)q(0) + p(1)q(1).$$

Find the adjoint of T w.r.t to these inner products.

Adjoint Example (MATH2601)

2016 2601 FE

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Here, we'll consider the **standard inner product for** $\mathbb{R}^2$ and for $\mathbb{P}_2(\mathbb{R})$ we have:

$$\langle p, q \rangle = p(0)q(0) + p(1)q(1).$$

Find the adjoint of T w.r.t to these inner products.

Consider $\mathbf{x} = (x_1, x_2)$ and $p(t) = p_0 + p_1 t$. The first thing we should do is find the inner product of $T(\mathbf{x})$ and p:

$$\langle T(\mathbf{x}), p \rangle = (x_1 + x_2)p_0 + (x_1 - x_2)(p_0 + p_1)$$

Adjoint Example (MATH2601)

$$\langle T(\mathbf{x}), p \rangle = (x_1 + x_2)p_0 + (x_1 - x_2)(p_0 + p_1)$$

Adjoint Example (MATH2601)

$$\begin{aligned}
\langle T(\mathbf{x}), p\rangle &= (x_1 + x_2)p_0 + (x_1 - x_2)(p_0 + p_1) \\
&= x_1(p_0 + p_0 + p_1) + x_2(p_0 - p_0 - p_1) \\
&= x_1(2p_0 + p_1) + x_2(-p_1) \\
&= \langle \mathbf{x}, T^*(p)\rangle
\end{aligned}$$

Adjoint Example (MATH2601)

$$\begin{aligned}
\langle T(\mathbf{x}), p\rangle &= (x_1 + x_2)p_0 + (x_1 - x_2)(p_0 + p_1) \\
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&= x_1(2p_0 + p_1) + x_2(-p_1) \\
&= \langle \mathbf{x}, T^*(p)\rangle
\end{aligned}$$

Hence, we can see that:

$$T^*(p) = (2p_0 + p_1, -p_1)$$

where $p(t) = p_0 + p_1 t$.

Types of Maps (MATH2601)

Consider the linear transform $T : V \to V$. We have some special names for T if it possesses some properties, such as:

- **Unitary** if $T^* = T^{-1}$
- **Isometry** if $\|T(\mathbf{v})\| = \|\mathbf{v}\|$ for all $\mathbf{v} \in V$
- **Hermitian** if $T^* = T$

Equivalent Properties of Linear Maps (MATH2601)

Equivalent Properties of Linear Maps

The following properties are equivalent:

- T is an isometry
- $\langle T(\mathbf{v}), T(\mathbf{w})\rangle = \langle \mathbf{v}, \mathbf{w}\rangle$ for all $\mathbf{v}, \mathbf{w} \in V$
- T is unitary
- T^* is an isometry
- $\{\mathbf{a}_i\}_{i=1}^n$ is an orthonormal basis of V, then so is $\{T(\mathbf{a}_i)\}_{i=1}^n$.

Type of Linear Map (MATH2601)

2601 2007 FE

Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by $T(w, z) = (-z, w)$. Is T Unitary? Isometric? Hermitian?

Type of Linear Map (MATH2601)

2601 2007 FE

Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by $T(w, z) = (-z, w)$. Is T Unitary? Isometric? Hermitian?

Before we can discuss any of these properties, we first need to find the adjoint, T^*. Consider $\mathbf{x}, \mathbf{y} \in \mathbb{C}^2$.

$$\begin{aligned}\langle T(\mathbf{x}), \mathbf{y}\rangle &= \langle(-x_2, x_1), (y_1, y_2)\rangle \\ &= -\bar{x}_2 y_1 + \bar{x}_1 y_2 \\ &= \bar{x}_1 y_2 + \bar{x}_2(-y_1) \\ &= \langle \mathbf{x}, T^*(\mathbf{y})\rangle.\end{aligned}$$

Thus, $T^*(y_1, y_2) = (y_2, -y_1)$.

Type of Linear Map (MATH2601)

2601 2007 FE

Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by $T(w,z) = (-z,w)$. Is T Unitary? Isometric? Hermitian?

Unitary?: We can see that $T^{-1}(z,w) = (w,-z)$ i.e. $T^* = T^{-1}$, and so T is **unitary**.

Type of Linear Map (MATH2601)

2601 2007 FE

Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by $T(w, z) = (-z, w)$. Is T Unitary? Isometric? Hermitian?

Unitary?: We can see that $T^{-1}(z, w) = (w, -z)$ i.e. $T^* = T^{-1}$, and so T is **unitary**.

Isometric?: From the previous theorem, we know that unitary is equivalent to isometry, and so T **is also isometric**.

Type of Linear Map (MATH2601)

2601 2007 FE

Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by $T(w, z) = (-z, w)$. Is T Unitary? Isometric? Hermitian?

Unitary?: We can see that $T^{-1}(z, w) = (w, -z)$ i.e. $T^* = T^{-1}$, and so T is **unitary**.

Isometric?: From the previous theorem, we know that unitary is equivalent to isometry, and so T **is also isometric**.

Hermitian?: The last thing to check is whether it's Hermitian, i.e. $T = T^*$. Consider $\mathbf{y} = (1, 0)$.

$$T(1, 0) = (0, 1) \neq (0, -1) = T^*(1, 0)$$

i.e. **not Hermitian**.

Method of Least Squares

Context: We want to solve a system of linear equations $A\mathbf{x} = \mathbf{b}$, but we have fewer unknowns than relations (i.e. less columns than rows). In a general setting, there **won't be unique** solution. So we try to find the **best solution (in the least squares sense)**.

Theorem: Method of Least Squares

Least squares solution to $A\mathbf{x} = \mathbf{b}$ is a solution to the **normal equations**:

$$A^*A\mathbf{x} = A^*\mathbf{b}.$$

For MATH2501, it's simply:

$$A^TA\mathbf{x} = A^T\mathbf{b}.$$

Method of Least Squares Example

FE 2017 Q1.a)

Find the line $y = a + bx$ which best fits in the least squares sense to the points:

$$(-1, 6), \quad (1, 2), \quad (2, -1), \quad (2, 7).$$

Method of Least Squares Example

FE 2017 Q1.a)

Find the line $y = a + bx$ which best fits in the least squares sense to the points:

$$(-1, 6), \quad (1, 2), \quad (2, -1), \quad (2, 7).$$

Setting up the relevant matrices/vectors:

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 6 \\ 2 \\ -1 \\ 7 \end{pmatrix}, \quad A^* = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 2 & 2 \end{pmatrix}$$

Method of Least Squares Example

FE 2017 Q1.a)

Find the line $y = a + bx$ which best fits in the least squares sense to the points:

$$(-1, 6), \quad (1, 2), \quad (2, -1), \quad (2, 7).$$

Now we find the required matrix and vector components that we need to solve.

$$A^*A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 4 \\ 4 & 10 \end{pmatrix}$$

Method of Least Squares Example

FE 2017 Q1.a)

Find the line $y = a + bx$ which best fits in the least squares sense to the points:

$$(-1, 6), \quad (1, 2), \quad (2, -1), \quad (2, 7).$$

Now we find the required matrix and vector components that we need to solve.

$$A^*A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 10 \end{pmatrix}$$

$$A^*\mathbf{y} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ 2 \\ -1 \\ 7 \end{pmatrix} = \begin{pmatrix} 14 \\ 8 \end{pmatrix}$$

Method of Least Squares Example

FE 2017 Q1.a)

Find the line $y = a + bx$ which best fits in the least squares sense to the points:

$$(-1, 6), \quad (1, 2), \quad (2, -1), \quad (2, 7).$$

Solving the system of equations yields:

$$\left(\begin{array}{cc|c} 4 & 4 & 14 \\ 4 & 10 & 8 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 4 & 4 & 14 \\ 0 & 6 & -6 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 4 & 0 & 10 \\ 0 & 1 & -1 \end{array}\right)$$

Thus, the least squares solution is:

$$y = \frac{5}{2} - x.$$