MATH1251 2018 S2 Algebra Quiz Solutions

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We cannot guarantee that our answers are correct - please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are sometimes multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

Test 1 Version 1A

1. Let $z = x + iy$:

$$\begin{aligned}
|z+i| &= 2|z| \\
|x+i(y+1)| &= 2|x+iy| \\
\sqrt{x^2+(y+1)^2} &= 2\sqrt{x^2+y^2} \\
x^2+y^2+2y+1 &= 4x^2+4y^2 \\
1 &= 3x^2+3y^2-2y \\
\frac{1}{3} &= x^2+y^2-\frac{2y}{3} \\
\frac{1}{3}+\frac{1}{9} &= x^2+\left(y-\frac{1}{3}\right)^2
\end{aligned}$$

This is the equation of a circle with center $\left(0, \frac{1}{3}\right)$ and radius $\frac{2}{3}$.

2. We know that $\vec{0}X = \vec{0}$ and $X\vec{0}^T = X\vec{0} = \vec{0}$. So $\vec{0} \in S$. Let $A, B \in S$. Then:

$$AX = XA^T$$
$$BX = XB^T$$

Adding:

$$\begin{aligned} AX + BX &= XA^T + XBT \\ (A+B)X &= X(A^T + B^T) && \text{distributive laws} \\ (A+B)X &= X(A+B)^T && \text{transpose of sum equals sum of transpose} \end{aligned}$$

$$\therefore A + B \in S$$
$$S \text{ is closed under addition}$$

Let $A \in S$ and $\lambda \in \mathbb{R}$. Then:

$$\begin{aligned} AX &= XA^T \\ \therefore \lambda(AX) &= \lambda(XA^T) \\ \Rightarrow A(\lambda X) &= X(\lambda A^T) \\ \Rightarrow A(\lambda X) &= \lambda(\lambda A)^T \end{aligned}$$

$$\therefore \lambda A \in S$$
$$S \text{ is closed under multiplication}$$

Hence S is a subspace of $\mathbb{M}_{2,2}\mathbb{R}$ by the Subspace Theorem.

3.

$$
\begin{aligned}
(-\sqrt{3}+i)^{28} &= (2e^{-\frac{5\pi i}{6}})^{28} \\
&= 2^{28}2e^{-\frac{28\cdot 5\pi i}{6}} \\
&= 2^{28}e^{-\frac{70\pi i}{3}} \\
&= 2^{28}\left(\cos\left(\frac{70\pi}{3}\right) - i\sin\left(\frac{70\pi}{3}\right)\right) \\
&= 2^{28}\left(\cos\left(22\pi + \frac{4\pi}{3}\right) - i\sin\left(22\pi + \frac{4\pi}{3}\right)\right) \\
&= 2^{28}\left(\cos\left(\frac{4\pi}{3}\right) - i\sin\left(\frac{4\pi}{3}\right)\right) \\
&= 2^{28}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2i}\right) \\
&= -2^{27} - 2^{27}\sqrt{3}i
\end{aligned}
$$

4.

$$
\begin{aligned}
\cos(5\theta) + i\sin(5\theta) &= (\cos(\theta) + i\sin(\theta))^5 \\
&= \cos^5(\theta) + 5i\cos^4(\theta)\sin(\theta) - 10\cos^3(\theta)\sin^2(\theta) + i\sin^5(\theta)
\end{aligned}
$$

Equating imaginary parts, we obtain:

$$
\begin{aligned}
\sin(5\theta) &= 5\cos^4(\theta)\sin(\theta) - 10\cos^2(\theta)\sin^3(\theta) + \sin^3(\theta) \\
&= 5\left(1 - 2\sin^2(\theta) + \sin^4(\theta)\right)\sin(\theta) - 10\left(1 - \sin^2(\theta) + \sin^4(\theta)\right)\sin(\theta) + \sin^3(\theta) \\
&= 5\sin(\theta) - 20\sin^3(\theta) + 16\sin^5(\theta)
\end{aligned}
$$

Test 1 Version 2A

1. Take $\vec{x} = (1,1)$ and $\vec{y} = (-1,1)$. Then $\vec{x} \in S$, since $|1| \le 2\,|1|$ and $\vec{y} \in S$, since $|1| \le 2\,|-1|$. But $\vec{x}+\vec{y} = (0,2) \notin S$, since $|2| \not\le 2\,|0|$. Therefore, S is not closed under addition and hence, is not a subspace of $\mathbb{R}^2$.

2. (a)

$$
\begin{aligned}
z^6 + 27 &= 0 \\
z^6 &= -27 \\
&= 3^3 e^{i\pi} \\
&= 3^3 e^{2k\pi i + i\pi} \qquad k \in \mathbb{Z} \\
\therefore z &= 3^{\frac{1}{2}} e^{\frac{k\pi i}{3} + \frac{\pi i}{6}} \\
&= 3^{\frac{1}{2}} e^{-\frac{5\pi i}{6}}, 3^{\frac{1}{2}} e^{-\frac{\pi i}{2}}, 3^{\frac{1}{2}} e^{-\frac{\pi i}{6}}, 3^{\frac{1}{2}} e^{\frac{\pi i}{6}}, 3^{\frac{1}{2}} e^{\frac{\pi i}{2}}, 3^{\frac{1}{2}} e^{\frac{5\pi i}{6}}
\end{aligned}
$$

$$\text{taking } k = -3, -2, -1, 0, 1, 2.$$

(b) Hence, we may express $p(z)$ as

$$p(z) = (z - 3^{\frac{1}{2}} e^{\frac{\pi i}{6}})(z - 3^{\frac{1}{2}} e^{-\frac{\pi i}{6}})(z - 3^{\frac{1}{2}} e^{\frac{\pi i}{2}})(z - 3^{\frac{1}{2}} e^{-\frac{\pi i}{2}})(z - 3^{\frac{1}{2}} e^{\frac{5\pi i}{6}})(z - 3^{\frac{1}{2}} e^{-\frac{5\pi i}{6}})$$

But since $\overline{e^{i\theta}} = e^{-i\theta}$ and $(z-\omega)(z-\overline{\omega}) = z^2 - 2\operatorname{Re}(\omega) + |\omega|^2$, we have

$$
\begin{aligned}
p(z) &= (z^2 - 2 \cdot 3^{\frac{1}{2}} \cos(\frac{\pi}{6}) \cdot z) + (3^{\frac{1}{2}})^2)(z^2 - 2 \cdot 3^{\frac{1}{2}} \cos(\frac{\pi}{2}) \cdot z) + (3^{\frac{1}{2}})^2)(z^2 - 2 \cdot 3^{\frac{1}{2}} \cos(\frac{5\pi}{6}) \cdot z) + (3^{\frac{1}{2}})^2) \\
&= (z^2 - 3z + 3)(z^2 + 3)(z^2 + 3z + 3).
\end{aligned}
$$

3. (a) Solving the characteristic equation, we have

$$\begin{aligned} 2\lambda^2 - \lambda + 1 &= 0 \\ \lambda &= \frac{1 \pm \sqrt{1-8}}{4} \\ &= \frac{1}{4} \pm \frac{\sqrt{7}i}{4} \end{aligned}$$

$$\therefore \text{Re}(\lambda) = \tfrac{1}{4} > 0$$

Hence, the system is unstable.

(b) Solving the characteristic equation, we have

$$\begin{aligned} 3\lambda^2 - 2\lambda + 1 &= 0 \\ \lambda &= \frac{2 \pm \sqrt{4-12}}{6} \\ &= \frac{2 \pm \sqrt{8}i}{6} \\ &= \frac{1}{3} \pm \frac{\sqrt{2}}{3}i \end{aligned}$$

$$\begin{aligned} \therefore |\lambda| &= \frac{1}{9} + \frac{4}{9} \\ &= \frac{5}{9} \\ &< 1 \end{aligned}$$

Hence, the system is stable.

4. Since $\vec{z} \in W$, write $\vec{z} = \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

We know that $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \in W$ and $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \in W$.

$\therefore \vec{x} - \vec{z} \perp \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\vec{x} - \vec{z} \perp \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

$$\therefore \begin{pmatrix} 1 - \lambda - 2\mu \\ 2 - 2\lambda - \mu \\ 3 - \lambda + \mu \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 0 \text{ and } \begin{pmatrix} 1 - \lambda - 2\mu \\ 2 - 2\lambda - \mu \\ 3 - \lambda + \mu \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 0$$

$$\implies \begin{cases} (1 - \lambda - 2\mu) + 2(2 - 2\lambda - \mu) + (3 - \lambda + \mu) = 0 \\ 2(1 - \lambda - 2\mu) + (2 - 2\lambda - \mu) - (3 - \lambda + \mu) = 0 \end{cases}$$

$$\implies \begin{cases} 6\lambda + 3\mu = 8 \\ 3\lambda + 6\mu = 1 \end{cases}$$

$$\left(\begin{array}{cc|c} 3 & 6 & 1 \\ 6 & 3 & 8 \end{array}\right) \xrightarrow{r_2 = r_2 - r_1} \left(\begin{array}{cc|c} 3 & 6 & 1 \\ 0 & -9 & 6 \end{array}\right)$$

$$\begin{aligned} \therefore -9\mu &= 6 \\ \mu &= -\frac{2}{3} \\ 3\lambda - 6(-\frac{2}{3}) &= 1 \\ 3\lambda - 4 &= 1 \\ \lambda &= \frac{5}{3} \end{aligned}$$

$\therefore \vec{z} = \frac{5}{3} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - \frac{2}{3} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

Test 1 Version 2B

Q1. We prove that S is not a subspace, by disproving closure under addition. Take

$$\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \vec{y} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Then $\vec{x} \in S$ as $1^2 = 1^2 + 0^2$ and $\vec{y} \in S$ as $1^2 = 0^2 + 1^2$. However,

$$\vec{x} + \vec{y} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \notin S$$

since $2^2 \neq 1^2 + 1^2$. Thus, S is not closed under addition, and hence not a subspace of $\mathbb{R}^2$.

Q2. Since $\vec{z} \in W$, we write

$$\vec{z} = \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

We know that $\vec{x} - \vec{z} \perp \vec{w}$ for all $\vec{w} \in W$. Therefore,

$$\vec{x} - \vec{z} \perp \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \qquad \text{and} \qquad \vec{x} - \vec{z} \perp \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

That is,

$$\begin{pmatrix} 1 - 2\lambda - \mu \\ 3 - \lambda - \mu \\ 2 - \lambda + 2\mu \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 0, \qquad \text{and} \qquad \begin{pmatrix} 1 - 2\lambda - \mu \\ 3 - \lambda - \mu \\ 2 - \lambda + 2\mu \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = 0.$$

Expanding these both gives us

$$\begin{cases} 2(1-2\lambda-\mu)+(3-\lambda-\mu)+(2-\lambda+2\mu) &= 0 \\ (1-2\lambda-\mu)+(3-\lambda-\mu)-2(2-\lambda+2\mu) &= 0 \end{cases},$$

$$\begin{cases} 6\lambda+\mu &= 7 \qquad \cdots \text{①} \\ \lambda+6\mu &= 0 \qquad \cdots \text{②} \end{cases}.$$

Substituting ② into ① we have

$$\begin{aligned} -36\mu+\mu &= 7 \\ -35\mu &= 7 \\ \mu &= -\frac{1}{5}. \end{aligned}$$

Then substituting this into ① we get

$$\begin{aligned} 6\lambda &= 7+\frac{1}{5} \\ &= \frac{36}{5} \\ \lambda &= \frac{6}{5}. \end{aligned}$$

Thus,

$$\vec{z} = \frac{6}{5}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

Q3. a) The characteristic equation is $3\lambda^2+4\lambda+2=0$. Solving this for λ we find

$$\begin{aligned} \lambda &= \frac{-4\pm\sqrt{16-24}}{6} \\ &= \frac{-4\pm\sqrt{8}i}{6} \\ &= \frac{-2\pm\sqrt{2}i}{3}. \end{aligned}$$

Then,

$$\begin{aligned}|\lambda| &= \frac{4}{9} + \frac{2}{9} \\ &= \frac{6}{9} \\ &< 1.\end{aligned}$$

Thus, the time system is stable.

b) As with the last part, we find the solution to $3\lambda^2 + 5\lambda - 1 = 0$:

$$\begin{aligned}\lambda &= \frac{-5 \pm \sqrt{25+12}}{6} \\ &= \frac{-5 \pm \sqrt{37}}{6}.\end{aligned}$$

Then, we note that

$$\frac{-5-\sqrt{37}}{6} < 0, \qquad \frac{-5+\sqrt{37}}{6} > \frac{-5+\sqrt{25}}{6} = 0,$$

So one of the solutions has $\mathrm{Re}(\lambda) > 0$ and thus the time system is unstable.

Q4. a)

$$\begin{aligned}z^5 - 32 &= 0 \\ z^5 &= 32 \\ z^5 &= 32e^{2k\pi i} && \text{for } k \in \mathbb{Z} \\ \therefore z &= 2e^{\frac{2k\pi i}{5}} \\ &= 2e^{\frac{-4\pi i}{5}}, 2e^{\frac{-2\pi i}{5}}, 2, 2e^{\frac{2\pi i}{5}}, 2e^{\frac{4\pi i}{5}},\end{aligned}$$

taking $k = \pm 2, \pm 1, 0$.

b) So, using the roots from the previous part, we have

$$\begin{aligned}
p(z) &= (z-2)\left(z-2e^{\frac{2\pi i}{5}}\right)\left(z-2e^{\frac{-2\pi i}{5}}\right)\left(z-2e^{\frac{4\pi i}{5}}\right)\left(z-2e^{\frac{-4\pi i}{5}}\right) \\
&= (z-2)\left(z^2-2z\cdot 2\cos\frac{2\pi}{5}+2^2\right)\left(z^2-2z\cdot 2\cos\frac{4\pi}{5}+2^2\right) \\
&= (z-2)\left(z^2-4z\cos\frac{2\pi}{5}+4\right)\left(z^2-4z\cos\frac{4\pi}{5}+4\right),
\end{aligned}$$

where we have used the identities

$$\begin{aligned}
(z-w)(z-\overline{w}) &= z^2-2\mathrm{Re}(w)z+|w|^2, \\
\overline{e^{i\theta}} &= e^{-i\theta}.
\end{aligned}$$