MATH1131/1141 Algebra Test 2 2014 S1 v1a

May 25, 2017

These solutions were written and typed up by Brendan Trinh. Please be ethical with this resource. It is for the use of MathSoc members, so do not repost it on other forums or groups without asking for permission. If you appreciate this resource, please consider supporting us by coming to our events and buying our T-shirts! Also, happy studying ☺!

We cannot guarantee that our working is correct, or that it would obtain full marks – please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are sometimes multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

1.

$$\begin{aligned}
\operatorname{Im}(z+3iw) &= \operatorname{Im}(1+5i+3i(3-2i)) \\
&= \operatorname{Im}(1+5i+9i+6) \\
&= \operatorname{Im}(7+14i) \\
&= 14.
\end{aligned}$$

$$
\begin{aligned}
\frac{z}{\bar{w}} &= \frac{1+5i}{3+2i} \\
&= \frac{1+5i}{3+2i} \times \frac{3-2i}{3-2i} && \text{(Realising the denominator)} \\
&= \frac{3+10+i\,(15-2)}{9+4} && \text{(Grouping real \& imaginary terms)} \\
&= \frac{13+13i}{13} \\
&= 1+i.
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{Arg}(1-4i-w) &= \operatorname{Arg}(1-4i-3+2i) \\
&= \operatorname{Arg}(-2-2i) \\
&= -\frac{3\pi}{4}.
\end{aligned}
$$

Note: Remember that it is implied that they want the 'principal argument' of $1-4i-w$*. This means that it will take values in* $(-\pi, \pi]$*. So, you shouldn't write something like* $\frac{5\pi}{4}$*.*

2. For the vector $(b_1, b_2, b_3, b_4)^T$ to belong to the span of vectors, there must exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$
\alpha \begin{pmatrix} 1 \\ -2 \\ -2 \\ 6 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ -5 \\ -4 \\ 3 \end{pmatrix} + \gamma \begin{pmatrix} -3 \\ 4 \\ 2 \\ 12 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}.
$$

Writing this as an augmented matrix,

$$
\left(\begin{array}{ccc|c} 1 & 3 & -3 & b_1 \\ -2 & -5 & 4 & b_2 \\ -2 & -4 & 2 & b_3 \\ 6 & 3 & 12 & b_4 \end{array}\right)
$$

We attempt to find a solution for α, β and γ by row-reducing. First, perform $R_2 \leftarrow R_2 + 2R_1$, $R_3 \leftarrow R_3 + 2R_1$ and $R_4 \leftarrow R_4 - 6R_1$,

$$
\left(\begin{array}{ccc|c} 1 & 3 & -3 & b_1 \\ 0 & 1 & -2 & 2b_1 + b_2 \\ 0 & 2 & -4 & 2b_1 + b_3 \\ 0 & -15 & 30 & -6b_1 + b_4 \end{array}\right)
$$

Next, $R_3 \leftarrow R_3 - 2R_2$ and $R_4 \leftarrow R_4 + 15R_2$,

$$\left(\begin{array}{ccc|c} 1 & 3 & -3 & b_1 \\ 0 & 1 & -2 & 2b_1 + b_2 \\ 0 & 0 & 0 & -2b_1 - 2b_2 + b_3 \\ 0 & 0 & 0 & 24b_1 + 15b_2 + b_4 \end{array}\right)$$

Thus, we see that for solutions to exist, we must have the conditions that

$$-2b_1 - 2b_2 + b_3 = 0 \text{ and } 24b_1 + 15b_2 + b_4 = 0.$$

3. Using the fact that $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$, we obtain the following,

$$\begin{aligned} \sin^5\theta &= (\sin\theta)^5 \\ &= \left(\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\right)^5 \\ &= \frac{1}{32i}\left(e^{i\theta} - e^{-i\theta}\right)^5. \end{aligned}$$

Applying the Binomial Theorem,

$$\sin^5\theta = \frac{1}{32i}\left(e^{5i\theta} - 5e^{4i\theta}e^{-i\theta} + 10e^{3i\theta}e^{-2i\theta} - 10e^{2i\theta}e^{-3i\theta} + 5e^{i\theta}e^{-4i\theta} - e^{-5i\theta}\right)$$

Pairing up certain terms together, and using the given identity again, we get

$$\begin{aligned} \sin^5\theta &= \frac{1}{32i}\left[\left(e^{5i\theta} - e^{-5i\theta}\right) - 5\left(e^{3i\theta} - e^{-3i\theta}\right) + 10\left(e^{i\theta} - e^{-i\theta}\right)\right] \\ &= \frac{1}{32i}\left(2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta\right) \\ \therefore \sin^5\theta &= \frac{1}{16}\left(\sin 5\theta - 5\sin 3\theta + 10\sin\theta\right). \end{aligned}$$

UNSW MathSoc

MATH1131 Algebra Test 2 2014 S1 v1b

May 25, 2017

These answers were written and typed up by Brendan Trinh and edited by Aaron Hassan and Henderson Koh. Please be ethical with this resource. It is for the use of MathSoc members, so do not repost it on other forums or groups without asking for permission. If you appreciate this resource, please consider supporting us by coming to our events and buying our T-shirts! Also, happy studying :)

We cannot guarantee that our answers are correct – please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are sometimes multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

---

1. Note that $(1+3i)\,z = (1+3i)\,(-2-3i)$. Hence $\operatorname{Re}\left((1+3i)\,z\right) = -2+9 \Rightarrow \boxed{\operatorname{Re}\left((1+3i)\,z\right) = 7}$.
   Also, $\left|z^2\right| = |z|^2 = 2^2 + 3^2 = \boxed{13}$.
   Lastly, we have

$$\begin{aligned}\frac{z+1}{w} &= \frac{-1-3i}{1-i} && \text{(as } z+1 = -2-3i+1 = -1-3i\text{)}\\ &= \frac{(-1-3i)\,(1+i)}{2} && \text{(multiplying top and bottom by conjugate of denominator)}\\ &= \frac{2-4i}{2}\\ &= \boxed{1-2i}.\end{aligned}$$

2. Suppose that a real solution exists and attempt to solve for it.

In other words, attempt to solve the equation

$$\alpha_1 \begin{pmatrix} 1 \\ 2 \\ 4 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix} + \alpha_3 \begin{pmatrix} -2 \\ 1 \\ -3 \\ -7 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

for the scalars $\alpha_1, \alpha_2, \alpha_3$. Then considering each component as an equation, we can write this in augmented matrix form as:

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & b_1 \\ 2 & 1 & 1 & b_2 \\ 4 & 1 & -3 & b_3 \\ 1 & -1 & -7 & b_4 \end{array}\right).$$

A handy trick is to write the right-hand column in terms of the coefficients of b_1, b_2, b_3 and b_4, so the augmented matrix is:

$$\left(\begin{array}{ccc|cccc} 1 & 0 & -2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 & 0 \\ 4 & 1 & -3 & 0 & 0 & 1 & 0 \\ 1 & -1 & -7 & 0 & 0 & 0 & 1 \end{array}\right).$$

This can now be easier to row-reduce than when the b's are present, as there is no reason to write b's when row-reducing, and there is less chance of algebraic errors. Just keep in mind that the first three columns represent $\alpha_1, \alpha_2, \alpha_3$ respectively, and the right-hand columns represent b_1 to b_4 respectively. If we row-reduce this matrix to row-echelon form, it will be of the form

$$\left(\begin{array}{ccc|cccc} 1 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 2 & 1 & 0 \\ 0 & 0 & 0 & -3 & 1 & 0 & 1 \end{array}\right).$$

Now, real solutions for the α's will exist (i.e. $(b_1, b_2, b_3, b_4)^T$ will be in the span of the given vectors) if and only if the right-hand vector (or matrix in this format) is non-leading. So reading off rows 3 and 4, we need the expression in the right-hand matrix in these rows to be 0, i.e. we need $-2b_1 - b_2 + b_3 = 0$ and $-3b_1 + b_2 + b_4 = 0$. These are the conditions that are necessary and sufficient to ensure that $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$ belongs to the span of the given vectors.

3. By expressing $\cos^5\theta = (\cos\theta)^5$ and applying the identity, the following is obtained:

$$\cos^5\theta = \frac{1}{16}\left(\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right).$$

Proof. For brevity, let $z = e^{i\theta}$. Note that the given identity tells us that $z + z^{-1} = 2\cos\theta$. Since $z^n = e^{i(n\theta)}$, we also have $z^n + z^{-n} = 2\cos n\theta$, for any integer n. Now,

$$\left(z + \frac{1}{z}\right)^5 = 32\cos^5\theta \tag{1}$$

by letting $n = 1$ and taking the fifth power on both sides of the identity $z + \frac{1}{z} = 2\cos\theta$. But by the Binomial Theorem, we have

$$\begin{aligned}\left(z + \frac{1}{z}\right)^5 &= z^5 + 5z^3 + 10z + 10z^{-1} + 5z^{-3} + z^{-5} \\ &= \left(z^5 + z^{-5}\right) + 5\left(z^3 + z^{-3}\right) + 10\left(z + z^{-1}\right) \\ &= 2\cos 5\theta + 5\times 2\cos 3\theta + 10\times 2\cos\theta \\ &= 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta. \qquad (2)\end{aligned}$$

Thus, by equating (1) and (2), we have $32\cos^5\theta = 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$. Rearranging this shows that $\cos^5\theta = \frac{1}{16}\left(\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right)$. ■

UNSW MathSoc

MATH1131 Algebra Test 2 2014 S1 v2a

May 25, 2017

These solutions were written and typed up by Brendan Trinh. Please be ethical with this resource. It is for the use of MathSoc members, so do not repost it on other forums or groups without asking for permission. If you appreciate this resource, please consider supporting us by coming to our events and buying our T-shirts! Also, happy studying :)

We cannot guarantee that our working is correct, or that it would obtain full marks - please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are sometimes multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

---

1. To find the complex square roots of $-24-70i$, we consider $(x+iy)^2 = -24-70i$ where $x, y \in \mathbb{R}$.
   We first expand this out,

$$\begin{aligned}(x+iy)^2 &= -24-70i\\ x^2-y^2+2xyi &= -24-70i.\end{aligned}$$

Equating the real and imaginary terms,

$$x^2 - y^2 = -24 \tag{1}$$
$$2xy = -70. \tag{2}$$

Rearranging (2), we obtain $y = -\frac{35}{x}$.
Substituting this in to (1), we can solve for x,

$$\begin{aligned}x^2 - \frac{35^2}{x^2} &= -24\\ x^4 + 24x^2 - 35^2 &= 0.\end{aligned}$$

To solve this, we let $u = x^2$ and apply the quadratic formula,

$$\begin{aligned}u^2 + 24u - 35^2 &= 0\\ u = x^2 &= \frac{-24 \pm \sqrt{24^2 - 4\left(-35^2\right)}}{2}\\ &= -12 \pm 37\\ &= 25 \text{ or } -49.\end{aligned}$$

However, since $x \in \mathbb{R}$, then $x^2 \geq 0$ and so $u \geq 0$. Thus, $u = 25$ and so $x = \pm 5$. Substituting this in to our rearranged form of (2) gives us the solutions

$$5 - 7i \text{ and } -5 + 7i.$$

Alternatively, the system of equations (1) and (2) can be solved by inspection. Equation (2) shows us that $xy = -35$. Noting the factors of 35, a natural guess is to try x and y to be 5 and 7 (with one having a negative sign as xy needs to be negative). From equation (1), it is evident that trying $x = 5$, $y = -7$ works, since $5^2 - (-7)^2 = 25 - 49 = -24$. So we can have $x = 5$ and $y = -7$. Since the square roots of a complex number come in a $\pm$ pair, the square roots are $\pm(5 - 7i)$, as obtained before.

2. For the two lines to intersect, the "$\mathbf{x}$"s must take the same value for some $\lambda, \mu \in \mathbb{R}$. That is, if we equate the two equations, and there exists solutions for λ and μ, then the two lines intersect.
Suppose there exists $\lambda, \mu \in \mathbb{R}$ such that the two lines intersect, i.e.

$$\begin{pmatrix}-1\\1\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\-5\end{pmatrix} = \begin{pmatrix}0\\-1\\5\end{pmatrix} + \mu\begin{pmatrix}-1\\-1\\6\end{pmatrix}.$$

Rearranging,

$$\lambda\begin{pmatrix}1\\2\\-5\end{pmatrix} + \mu\begin{pmatrix}1\\1\\-6\end{pmatrix} = \begin{pmatrix}1\\-2\\2\end{pmatrix}.$$

Expressing this as an augmented matrix,

$$\left(\begin{array}{cc|c}1 & 1 & 1\\ 2 & 1 & -2\\ -5 & -6 & 2\end{array}\right).$$

Now, we row-reduce. Performing the row operations $R_2 \leftarrow R_2 - 2R_1$ and $R_3 \leftarrow R_3 + 5R_1$,

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -1 & -4 \\ 0 & -1 & 7 \end{array}\right)$$

From the last two rows, we see that $\mu = 4$ as well as -7. Thus, there is no solution for $\lambda, \mu \in \mathbb{R}$ and hence, the two lines do not intersect.

3. (i) To find the complex roots of this, we first rearrange to obtain $z^6 = -64$. Then, we express the RHS in polar form,

$$\begin{aligned} z^6 &= -64 \\ &= 64e^{i\pi + 2k\pi i}. \end{aligned}$$

where $k = 0, 1, 2, \ldots, 5$.

Note: that you can choose to use $k = 1, 2, \ldots, 6$ which will get you the same answer, so long as you choose 6 consecutive numbers. For our solution, we will be using $k = 0, 1, 2, \ldots, 5$.

Now, we raise both sides to the power of $\frac{1}{6}$ and evaluate for each k for each complex root,

$$\begin{aligned} z &= 2e^{\frac{(2k+1)\pi i}{6}} \\ &= 2e^{\frac{\pi}{6}i}, 2e^{\frac{\pi}{2}i}, 2e^{\frac{5\pi}{6}i}, 2e^{\frac{7\pi}{6}i}, 2e^{\frac{3\pi}{2}i}, 2e^{\frac{11\pi}{6}i}. \end{aligned}$$

(ii) Having obtained all the roots of $p(z) = z^6 + 64$, we know all the factors of it.

Furthermore, by the Conjugate Root Theorem, we know that since all the coefficients are real, the complex roots come in conjugate pairs.

Writing the factorised form of $p(z)$ such that the conjugate pairs are next to one another,

$$p(z) = \left[\left(z - 2e^{\frac{\pi}{6}i}\right)\left(z - 2e^{\frac{11\pi}{6}i}\right)\right]\left[\left(z - 2e^{\frac{\pi}{2}i}\right)\left(z - 2e^{\frac{3\pi}{2}i}\right)\right]\left[\left(z - 2e^{\frac{5\pi}{6}i}\right)\left(z - 2e^{\frac{7\pi}{6}i}\right)\right].$$

Note that $(x - a)(x - b) = x^2 - (a + b)x + ab$.

In this case, we have a and b to be a complex conjugate pair.

We know that $z + \bar{z} = 2\cos\theta$ and $z\bar{z} = |z|^2$. Applying this, we can selectively expand the factors to obtain real quadratic factors,

$$p(z) = \left(z^2 - 2\left(2\cos\frac{\pi}{6}\right)z + 4\right)\left(z^2 - 2\left(2\cos\frac{\pi}{2}\right)z + 4\right)\left(z^2 - 2\left(2\cos\frac{5\pi}{6}\right)z + 4\right).$$

Simplifying this, we obtain our answer of

$$p(z) = \left(z^2 - 2\sqrt{3}z + 4\right)\left(z^2 + 4\right)\left(z^2 + 2\sqrt{3}z + 4\right).$$

UNSW Mathζoc

MATH1131/1141 Algebra Test 2 2014 S1 v2b

May 25, 2017

These solutions were written and typed up by Harry Gibbs. Please be ethical with this resource. It is for the use of MathSoc members, so do not repost it on other forums or groups without asking for permission. If you appreciate this resource, please consider supporting us by coming to our events and buying our T-shirts! Also, happy studying :)

We cannot guarantee that our working is correct, or that it would obtain full marks - please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are sometimes multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

---

1. It can be found[1] that the solutions to $z^2 = 16 - 30i$ are $z = \pm(5 - 3i)$

2. To check whether the lines intersect, we construct the equation

$$\begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}.$$

   It can be found[2] that the lines do intersect at $\begin{pmatrix} 5 \\ 6 \\ 12 \end{pmatrix}$, i.e. where $\lambda = 3$ and $\mu = 2$.

[1] Refer to Question 1 in Test 2 2014 S1 v2a

[2] Refer to Question 2 in Test 2 2014 S1 v2a

3. (i) It can be found[3] that the roots of $z^5 - 32 = 0$ are

$$\begin{aligned} z_1 &= 2e^{i\frac{2\pi}{5}} \\ z_2 &= 2e^{i\frac{4\pi}{5}} \\ z_3 &= 2e^{i\frac{6\pi}{5}} \\ z_4 &= 2e^{i\frac{8\pi}{5}} \\ z_5 &= 2. \end{aligned}$$

(ii) It can be found[3] that $p(z)$ can be factorised as

$$p(z) = (z-2)(z^2 - (\sqrt{5}-1)z + 4)(z^2 + (\sqrt{5}+1)z + 4).$$

UNSW Mathζoc

[3] Refer to Question 3 in Test 2 2014 S1 v2a

MATH1131/1141 Algebra Test 2 2014 S1 v3a

May 25, 2017

These solutions were written and typed up by Harry Gibbs and edited by Aaron Hassan. Please be ethical with this resource. It is for the use of MathSoc members, so do not repost it on other forums or groups without asking for permission. If you appreciate this resource, please consider supporting us by coming to our events and buying our T-shirts! Also, happy studying :)!

We cannot guarantee that our working is correct, or that it would obtain full marks – please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are sometimes multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

1.

$$\begin{aligned}(-5-i)\bar{z}+2w &= -(5+i)(-1+i)+2(-11+7i)\\ &= -(-5-i+5i-1)-22+14i\\ &= -16+10i.\end{aligned}$$

To realise the denominator we multiply by the complex conjugate of the denominator, hence

$$\begin{aligned}\frac{w}{1+3i} &= \frac{-11+7i}{1+3i}\times\frac{1-3i}{1-3i}\\ &= \frac{-11+7i+33i+21}{10}\\ &= \frac{10+40i}{10} = 1+4i.\end{aligned}$$

By drawing a picture, we see that z lies in the third quadrant. Notice that multiplying z

by 2 doesn't change the argument. Hence, the argument is

$$\begin{aligned}\operatorname{Arg}(2z) &= \operatorname{Arg}(z)\\ &= -\frac{3\pi}{4}.\end{aligned}$$

2. The polar form of z is given by $|z|e^{i\operatorname{Arg}(z)}$. Note that z lies in the 2nd quadrant, so the principal argument is

$$\begin{aligned}\operatorname{Arg}(z) &= \pi + \tan^{-1}\left(\frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}\right)\\ &= \pi + \tan^{-1}\left(\frac{3}{-\sqrt{3}}\right)\\ &= \pi - \frac{\pi}{3}\\ &= \frac{2\pi}{3}.\end{aligned}$$

The modulus of z is

$$\begin{aligned}|z| &= \sqrt{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2}\\ &= \sqrt{3+9}\\ &= \sqrt{12} = 2\sqrt{3}.\end{aligned}$$

Hence the polar form for z is

$$z = 2\sqrt{3}e^{i\frac{2\pi}{3}}.$$

Calculating z^{19} yields

$$\begin{aligned}z^{19} &= (2\sqrt{3})^{19}e^{i\frac{2\pi}{3}19}\\ &= (2\sqrt{3})^{19}\left(\cos\left(19\times\frac{2\pi}{3}\right) + i\sin\left(19\times\frac{2\pi}{3}\right)\right)\\ &= (2\sqrt{3})^{19}\left(\cos\left(\frac{38\pi}{3}\right) + i\sin\left(\frac{38\pi}{3}\right)\right)\\ &= (2\sqrt{3})^{19}\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) && \text{(Removing revolutions)}\\ &= (2\sqrt{3})^{19}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\\ &= -2^{18}\sqrt{3^{19}} + i2^{18}3^{10}\end{aligned}$$

The principal argument of z^{19} is

$$\operatorname{Arg}\left(z^{19}\right) = \frac{2\pi}{3}.$$

3. The equivalent matrix equation is

$$\begin{pmatrix} 1 & 3 & -2 & 4 \\ -2 & -4 & 5 & -9 \\ -1 & 1 & 4 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix}.$$

So the augmented matrix form is

$$\left(\begin{array}{cccc|c} 1 & 3 & -2 & 4 & 2 \\ -2 & -4 & 5 & -9 & 0 \\ -1 & 1 & 4 & -6 & 6 \end{array}\right).$$

Thus we perform Gaussian elimination:

$$\left(\begin{array}{cccc|c} 1 & 3 & -2 & 4 & 2 \\ -2 & -4 & 5 & -9 & 0 \\ -1 & 1 & 4 & -6 & 6 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 3 & -2 & 4 & 2 \\ 0 & 2 & 1 & -1 & 4 \\ 0 & 4 & 2 & -2 & 8 \end{array}\right), R_2 \leftarrow R_2 + 2R_1, R_3 \leftarrow R_3 + R_1$$

$$\left(\begin{array}{cccc|c} 1 & 3 & -2 & 4 & 2 \\ 0 & 2 & 1 & -1 & 4 \\ 0 & 4 & 2 & -2 & 8 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 3 & -2 & 4 & 2 \\ 0 & 2 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right), R_3 \leftarrow R_3 - 2R_2$$

Hence, to find the solution, we will back substitute. From Row 2 we have

$$2x_2 + x_3 - x_4 = 4.$$

Hence, let $\boxed{x_3 = \lambda}$ and $\boxed{x_4 = \mu}$. Then we have

$$2x_2 + \lambda - \mu = 4 \implies \boxed{x_2 = 2 + \frac{1}{2}\mu - \frac{1}{2}\lambda}.$$

From Row 1, we have

$$\begin{aligned} 2 &= x_1 + 3x_2 + -2x_3 + 4x_4 \\ &= x_1 + 3\left(2 + \frac{1}{2}\mu - \frac{1}{2}\lambda\right) - 2\lambda + 4\mu \\ &= x_1 + 6 - \frac{7}{2}\lambda + \frac{11}{2}\mu \\ &\Rightarrow \boxed{x_1 = -4 - \frac{11}{2}\mu + \frac{7}{2}\lambda}. \end{aligned}$$

Hence using the boxed expressions for the x_i in terms of μ and λ, the solution is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -\frac{11}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \quad \lambda, \mu \in \mathbb{R}$$

UNSW Mathζoc