MATH1031 Mastery Lab Test 1 Solutions to Samples

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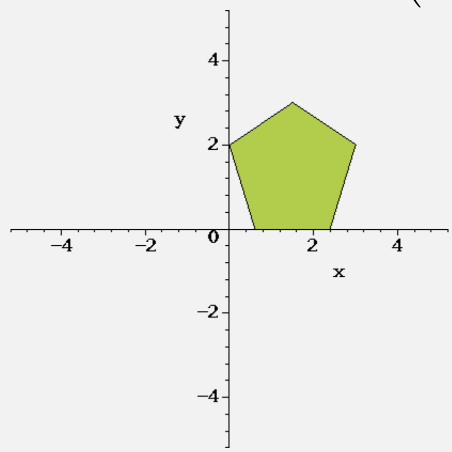
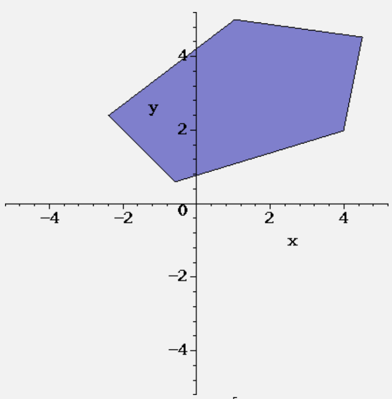
Question 2

A point represented by $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ on the xy-plane is transformed by being premultiplied by the matrix $\begin{pmatrix} 1 & -2 \\ -1 & -1 \end{pmatrix}$ to the point represented by $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$.

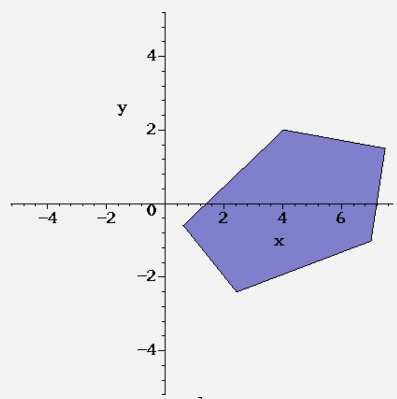


Under this transformation, the pentagon above is transformed to:

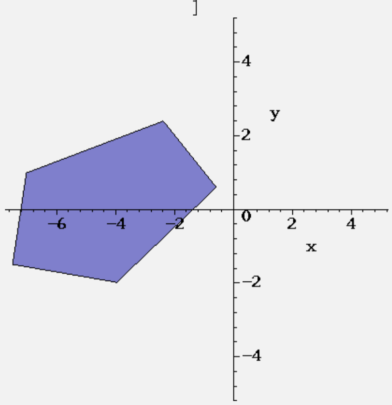
A)



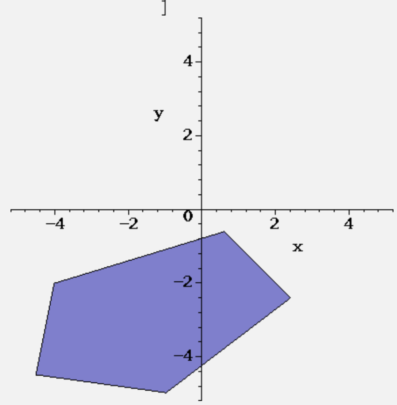
C)



B)



D)



Since $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is on the boundary of the pentagon, we expect it to be mapped to the boundary of the resulting shape. So,

$$\begin{pmatrix} 1 & -2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \times 1 + 0 \times (-2) \\ 1 \times (-1) + 0 \times (-1) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

which we can see is hit by C and D. So, we try another point, $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$. So,

$$\begin{pmatrix} 1 & -2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \times 1 + 2 \times (-2) \\ 0 \times (-1) + 2 \times (-1) \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}.$$

This point is hit by B and D, so since we require the graph to hit both points, the answer is D.

Question 3

A point represented by $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ on the xy-plane is transformed by being premultiplied by the matrix $\begin{pmatrix} -1 & -3 \\ 3 & 3 \end{pmatrix}$ to the point represented by $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} -1 & -3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$.

(a) $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ is transformed to the point A. Enter the column matrix representing A in the space below.

(b) $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ is transformed to the point B. Enter the column matrix representing B in the space below.

We can simply matrix multiply:

$$A = \begin{pmatrix} -1 & -3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \times (-1) + 5 \times (-3) \\ 3 \times 3 + 5 \times 3 \end{pmatrix} = \begin{pmatrix} -18 \\ 24 \end{pmatrix},$$
$$B = \begin{pmatrix} -1 & -3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -5 \end{pmatrix} = \begin{pmatrix} 3 \times (-1) + (-5) \times (-3) \\ 3 \times 3 + (-5) \times 3 \end{pmatrix} = \begin{pmatrix} 12 \\ -6 \end{pmatrix}.$$

Question 4

To encode a message by the 3×3 scrambler matrix $A = \begin{pmatrix} -1 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$ we first convert it to a sequence of integers using the following table:

A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	2	3	4	5	6	7	8	9	10	11	12	13	14

O	P	Q	R	S	T	U	V	W	X	Y	Z	
15	16	17	18	19	20	21	22	23	24	25	26	0

We then put the integers in order into a matrix M of 3 rows, column by column. We calculate the product $C = AM$. Finally, we extract the entries column by column from C to get the coded message.

For example, if we want to encode the message `HAPPY`, we first convert it into a sequence of integers

`[8, 1, 16, 16, 25, 0]`

We then put the integers in order into a matrix and form $M = \begin{pmatrix} 8 & 16 \\ 1 & 25 \\ 16 & 0 \end{pmatrix}$.

From the matrix $C = AM = \begin{pmatrix} -38 & 34 \\ 1 & 25 \\ -15 & 25 \end{pmatrix}$ we extract the coded message:

`[-38, 1, -15, 34, 25, 25]`.

For the recipient, who knows the matrix A, to decode the message, the coded message will be put column by column to form the matrix C of 3 rows, then the matrix M can be calculated by

$$M = A^{-1}C, \text{ where } A^{-1} = \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

Then the message can be recovered using the above table.

Now we received a code message `[-10, 15, -1, -35, 6, -15, -12, 0, 0]`. Enter the first three letters of the decoded message into the boxes below.

We first form the matrix C:

$$C = \begin{pmatrix} -10 & -35 & -12 \\ 15 & 6 & 0 \\ -1 & -15 & 0 \end{pmatrix}.$$

Now, we can calulate M:

$$M = A^{-1}C = \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} -10 & -35 & -12 \\ 15 & 6 & 0 \\ -1 & -15 & 0 \end{pmatrix} = \begin{pmatrix} 8 & 5 & 2 \\ 15 & 6 & 0 \\ 16 & 21 & 0 \end{pmatrix}.$$

Reading column-by-column, we have the message `[8, 15, 16, 5, 6, 21, 2, 0, 0]`. The first three characters are thus `HOP`, from 8, 15, and 16 respectively. To reduce time required to answer this question, you can just calculate the first column of M, as those three numbers are all you need.

Question 5

To encode a message by the 3×3 scrambler matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ -2 & -2 & 1 \\ 3 & 4 & -1 \end{pmatrix}$ we first convert it to a sequence of integers using the following table:

A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	2	3	4	5	6	7	8	9	10	11	12	13	14

O	P	Q	R	S	T	U	V	W	X	Y	Z	
15	16	17	18	19	20	21	22	23	24	25	26	0

We then put the integers in order into a matrix M of 3 rows, column by column. We calculate the product $C = AM$. Finally, we extract the entries column by column from C to get the coded message.

For example, if we want to encode the message `HAPPY`, we first convert it into a sequence of integers

`[8, 1, 16, 16, 25, 0]`

We then put the integers in order into a matrix and form $M = \begin{pmatrix} 8 & 16 \\ 1 & 25 \\ 16 & 0 \end{pmatrix}$.

From the matrix $C = AM = \begin{pmatrix} 1 & 25 \\ -2 & -82 \\ 12 & 148 \end{pmatrix}$ we extract the coded message:

`[1, -2, 12, 25, -82, 148]`.

For the recipient, who knows the matrix A, to decode the message, the coded message will be put column by column to form the matrix C of 3 rows, then the matrix M can be calculated by

$$M = A^{-1}C, \text{ where } A^{-1} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & 0 & 0 \\ -2 & 3 & 2 \end{pmatrix}.$$

Then the message can be recovered using the above table.

Now we are going to encode the message `DELIGHTED`.

Suppose that the sequence of the nine-integer coded message is $a, b, c, \cdots$

Enter the first three integers of the coded message into the boxes below.

The first step is to convert the message `DELIGHTED` into a list of numbers:

`[4, 5, 12, 9, 7, 8, 20, 5, 4].`

Then, we turn this into a matrix, filling column-by-column:

$$M = \begin{pmatrix} 4 & 9 & 20 \\ 5 & 7 & 5 \\ 12 & 8 & 4 \end{pmatrix}.$$

Now, we can calculate the coded message, by multiplying A and M:

$$C = AM = \begin{pmatrix} 0 & 1 & 0 \\ -2 & -2 & 1 \\ 3 & 4 & -1 \end{pmatrix} \begin{pmatrix} 4 & 9 & 20 \\ 5 & 7 & 5 \\ 12 & 8 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 7 & 5 \\ -10 & -24 & -46 \\ 20 & 47 & 76 \end{pmatrix}.$$

Reading column-by-column, we get out coded message:

`[5, -6, 20, 7, -24, 47, 5, -46, 76].`

From this, we find that

$$a = 5, \qquad b = -6, \qquad c = 20.$$

Just as with the previous question, we don't have to calculate all of C. The first column is enough to read off the three numbers we need.

Question 6

The transition matrix of a Markov Process is given by

$$T = \begin{pmatrix} .4 & .2 & z \\ .3 & y & .3 \\ x & .1 & .1 \end{pmatrix}.$$

(a) Enter the values of the unknowns in the boxes below.

(b) The initial distribution vector is $v = \begin{pmatrix} .4 \\ .2 \\ .4 \end{pmatrix}$.

Enter the distribution vector after one time unit in the box below.

Transitions matrices for Markov Processes must have columns adding to 1, so we have

$$\begin{cases} 0.4 + 0.3 + x &= 1, \\ 0.2 + y + 0.1 &= 1, \\ z + 0.3 + 0.1 &= 1. \end{cases}$$

The solutions to these equations is

$$x = 0.3, \qquad y = 0.7, \qquad z = 0.6.$$

Now, the distribution after one time unit is given by

$$Tv = \begin{pmatrix} 0.4 & 0.2 & 0.6 \\ 0.3 & 0.7 & 0.3 \\ 0.3 & 0.1 & 0.1 \end{pmatrix} \begin{pmatrix} 0.4 \\ 0.2 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 0.44 \\ 0.38 \\ 0.18 \end{pmatrix}.$$

As a sanity check, we can see that our answer has $0.44 + 0.38 + 0.18 = 1$, which is what we expect (distribution vectors should have components adding to one). This doesn't mean we are correct, but it's a good way to check for simple mistakes in working out.

Question 7

The transition matrix of a Markov Process is given by

$$T = \begin{pmatrix} \frac{7}{10} & \frac{3}{5} \\ \frac{3}{10} & \frac{2}{5} \end{pmatrix}.$$

The steady state probability distribution vector for this Markov Process is denoted by $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. Hence $v_1 + v_2 =$ __.
Making use of the above condition and solving a matrix equation, find the values of v_1 and v_2. Enter their exact values in the boxes below.

Any probability distribution vector must have components that add up to 1. Thus,

$$v_1 + v_2 = 1.$$

Now, if we have a steady state distribution for a Markov process, we have $T\mathbf{v} = \mathbf{v}$. That is,

$$T\mathbf{v} = \begin{pmatrix} \frac{7}{10} & \frac{3}{5} \\ \frac{3}{10} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \frac{7}{10}v_1 + \frac{3}{5}v_2 \\ \frac{3}{10}v_1 + \frac{2}{5}v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{v}.$$

Now, solving this system of equations will result in infinitely many conditions (both equations from this system are actually the same, just rearranged differently), so we instead solve

$$\begin{aligned} &\begin{cases} \frac{7}{10}v_1 + \frac{3}{5}v_2 &= v_1, \\ v_1 + v_2 &= 1 \end{cases} \\ =&\begin{cases} -15v_1 + 30v_2 &= 0, \\ 30v_1 + 30v_2 &= 30 \end{cases} \\ =&\begin{cases} -15v_1 + 30v_2 &= 0, \\ 45v_1 &= 30. \end{cases} \end{aligned}$$

So,

$$v_1 = \frac{30}{45} = \frac{2}{3}, \qquad v_2 = 1 - v_1 = \frac{1}{3}.$$

Question 8

The transition matrix of a Markov Process is given by

$$T = \begin{pmatrix} .7 & .9 \\ .3 & .1 \end{pmatrix}.$$

As a certain time t, the distribution vector is $v = \begin{pmatrix} .86 \\ .14 \end{pmatrix}$.

You are given that $T^{-1} = \begin{pmatrix} -.5 & 4.5 \\ 1.5 & -3.5 \end{pmatrix}$.

(a) Enter the distribution vector one time unit after t in the box below.

(b) Enter the distribution vector one time unit before t in the box below.

The distribution at $t+1$ (one time unit after t), is given by

$$Tv = \begin{pmatrix} 0.7 & 0.9 \\ 0.3 & 0.1 \end{pmatrix} \begin{pmatrix} 0.86 \\ 0.14 \end{pmatrix} = \begin{pmatrix} 0.728 \\ 0.272 \end{pmatrix}.$$

To find the distribution at $t-1$ (one time unit before t), we have to instead solve $Tx = v$. This can be rearranged as $x = T^{-1}v$, so

$$T^{-1}v = \begin{pmatrix} -0.5 & 4.5 \\ 1.5 & -3.5 \end{pmatrix} \begin{pmatrix} 0.86 \\ 0.14 \end{pmatrix} = \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix}.$$

In both cases, the resulting vectors have components that add to 1, as we expect.

Question 9

For the difference equation,

$$3x_{n+1} + x_n = 0, \quad n \geq 0,$$

select all the correct statements from the list below:

A) If $x_1 = -1$ then $x_7 = -6561$.

B) The general solution: $x_n = A(-3)^n$.

C) The general solution: $x_n = A\left(-\frac{1}{3}\right)^n$.

D) If $x_1 = -1$ then $x_7 = 6561$.

E) The general solution: $x_n = A(3)^n$.

F) If $x_1 = -1$, then $x_7 = -\frac{1}{729}$

G) The general solution: $x_n = A\left(\frac{1}{3}\right)^n$.

H) If $x_1 = -1$ then $x_7 = \frac{1}{729}$.

The auxiliary equation is

$$3m + 1 = 0$$

and when we solve it, we obtain

$$m = -\frac{1}{3}.$$

Hence, the general solution we require is

$$x_n = A\left(-\frac{1}{3}\right)^n.$$

Therefore option C in the above list must be selected.

Now if $x_1 = -1$, we have:

$$\begin{aligned} -1 &= A\left(-\frac{1}{3}\right)^1 \\ -1 &= -\frac{A}{3} \\ A &= 3 \end{aligned}$$

So our difference equation now becomes

$$x_n = 3\left(-\frac{1}{3}\right)^n.$$

Plugging $n = 7$ in, from our calculator we obtain

$$x_7 = 3\left(-\frac{1}{3}\right)^7 = -\frac{1}{729}.$$

So option F must also be selected.

Question 10

For the difference equation,

$$2x_{n+2} = -13x_{n+1} - 21x_n, \quad n \geq 0$$

select the auxiliary equation and the general solution from the list below:

A) $x_n = A3^n + B\left(-\frac{7}{2}\right)^n$
B) $2m^2 + 13m + 21 = 0$
C) $2m^2 = 13m - 21 = 0$
D) $x_n = A(-3)^n + B\left(\frac{7}{2}\right)^n$
E) $21m^2 + 13m + 2 = 0$
F) $x_n = A(-3)^n + B\left(-\frac{7}{2}\right)^n$
G) $21m^2 = 13m - 2 = 0$
H) $x_n = A3^n + B\left(\frac{7}{2}\right)^n$

Note that the auxiliary equation is deduced only after everything is moved to the same side of the equation. So here, we move everything into the left hand side first.

$$2x_{n+2} + 13x_{n+1} + 21x_n = 0.$$

Hence the auxiliary equation is

$$2m^2 + 13m + 21 = 0.$$

Therefore option B must be selected in the list above. Now we solve the auxiliary equation to deduce an expression for the general form. You may alternatively use completing the square or the quadratic formula, but factorisation is certainly possible here.

$$\begin{aligned}(m+3)(2m+7) &= 0 \\ m &= -3, -\frac{7}{2}\end{aligned}$$

So our general solution is

$$x_n = A(-3)^n + B\left(-\frac{7}{2}\right)^n.$$

Thus option F in the list must also be selected.

Question 11

> Find the gradient of the tangent to $x^2 - xy - 3y^2 = -29$ at the point $(1, 3)$.
>
> Enter the exact value of the answer in the box.

We proceed by implicit differentiation. Note that to differentiate xy, we need to use the product rule, so you may find this extra working out helpful. Note that the derivative of y is just $\frac{dy}{dx}$.

$$u = x \qquad\qquad v = y$$

$$u' = 1 \qquad\qquad v' = \frac{dy}{dx}$$

So upon differentiating with respect to x, we obtain:

$$2x - \left(x\frac{dy}{dx} + y\right) - 6y\frac{dy}{dx} = 0.$$

We isolate all the terms with $\frac{dy}{dx}$ on the LHS so that we can make it the subject.

$$\begin{aligned}
2x - x\frac{dy}{dx} - y - 6y\frac{dy}{dx} &= 0 \\
-x\frac{dy}{dx} - 6y\frac{dy}{dx} &= -2x + y \\
-(x+6y)\frac{dy}{dx} &= -2x + y \\
(x+6y)\frac{dy}{dx} &= 2x - y \\
\frac{dy}{dx} &= \frac{2x-y}{x+6y}
\end{aligned}$$

Subbing $x = 1$ and $y = 3$ in gives

$$\frac{dy}{dx} = \frac{2-3}{1+18} = -\frac{1}{19}.$$

This can be submitted as `-1/19`.

Question 12

If $y = 2t^2 - 2t$ and $x = \sin(2t)$, find $\frac{dy}{dx}$.

Enter the expression, in Maple syntax, for $\frac{dy}{dx}$ in terms of t in the box.

Start by computing the derivatives with respect to t, for both x and y.

$$\begin{aligned}
y &= 2t^2 - 2t \\
\frac{dy}{dt} &= 4t - 2
\end{aligned}$$

$$\begin{aligned}
x &= \sin(2t) \\
\frac{dx}{dt} &= 2\cos(2t)
\end{aligned}$$

The useful result to note here is that

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

From this result, all that's left over is to substitute everything in.

$$\frac{dy}{dx} = \frac{4t-2}{2\cos(2t)}.$$

This can be submitted as `(4*t-2)/(2*cos(2*t))`.

Question 13

The level of pollution P in a city is related to the number of people N (in millions) by

$$7P^3 - N^2 = 55; \quad N, P \geq 0.$$

(a) Find the value of P when $N = 1$ in the box.

(b) Find the relationship between $\frac{dP}{dt}$ and $\frac{dN}{dt}$ when $N = 1$, and write it in the form of

$$\frac{dN}{dt} = c\frac{dP}{dt}.$$

Enter the value of c in the box.

(c) At what rate will the level of pollution change if the population of a city is 1 million and is increasing at a rate of $\frac{1}{2}$ million per year? Enter the answer in the box.

For part a), upon substituting in $N = 1$:

$$\begin{aligned} 7P^3 - 1^2 &= 55 \\ 7P^3 &= 56 \\ P^3 &= 8 \\ P &= 8^{1/2} \\ P &= 2 \end{aligned}$$

This can be submitted as `2`.

For part b), we treat *both* P and N as functions of t. We then use implicit differentiation to differentiate both with respect to t. This gives

$$\begin{aligned} 7 \times 3P^2 \frac{dP}{dt} - 2N\frac{dN}{dt} &= 0 \\ 21P^2 \frac{dP}{dt} - 2N\frac{dN}{dt} &= 0 \end{aligned}$$

We only require the relationship between the derivatives when $N = 1$. From part a), we also know that $P = 2$ when $N = 1$, so we simply substitute in both values. Then, we solve for $\frac{dN}{dt}$.

$$\begin{aligned} 84\frac{dP}{dt} - 2\frac{dN}{dt} &= 0 \\ 42\frac{dP}{dt} - \frac{dN}{dt} &= 0 \\ 42\frac{dP}{dt} &= \frac{dN}{dt} \end{aligned}$$

We see that we require $c = 42$. This can be submitted as `42`.

In part c), translating the English to mathematics, we are told that $\frac{dN}{dt} = \frac{1}{2}$. What we require is $\frac{dP}{dt}$, so we just solve the resulting equatio.

$$\begin{aligned} 42\frac{dP}{dt} &= \frac{1}{2} \\ \frac{dP}{dt} &= \frac{1}{84} \end{aligned}$$

This can be submitted as `1/84`.

Question 15

Find

$$\int x^3 + 7 + \frac{8}{x^2}\,dx$$

and enter your answer in the box using Maple syntax. Do not include the constant of integration in your answer, that is, do not include the usual "+C".

Note: For this question, we use $\int \frac{1}{x}\,dx = \ln(x)$. Don't use $\int \frac{1}{x}\,dx = \ln(|x|)$.

We essentially integrate each term one at a time, using the rule $\int x^n\,dx = \frac{x^{n+1}}{n+1}$, for $n \neq 1$. Note that we may rewrite $\frac{1}{x^2} = x^{-2}$. (Note also that there is a hidden x^0 multiplied to the 7, since $x^0 = 1$.)

$$\begin{aligned}\int x^3 + 7 + \frac{8}{x^2}\,dx &= \int x^3 + 7x^0 + 8x^{-2}\,dx \\ &= \frac{x^4}{4} + 7x^1 + \frac{8x^{-1}}{-1} \\ &= \frac{x^4}{4} + 7x - \frac{8}{x}.\end{aligned}$$

This can be submitted as `(x^4)/4 + 7*x - 8/x`.

Question 16

Use integration by parts to find

$$\int 4xe^{2x}\,dx$$

and enter your answer in the box using Maple syntax. Do not include the constant of integration in your answer, that is, do not include the usual "+C".

For this problem, we have a polynomial function $4x$ multiplied to an exponential function e^{2x}. Recall the rule of LIATE of picking the function to differentiate for IBP:

- L - Logarithm
- I - Inverse trigonometric
- A - Algebraic (polynomials)
- T - Trigonometric
- E - Exponential

This suggests that we should differentiate the algebraic function $4x$. This will work, but personally I prefer to avoid fractions. So I'll instead consider x and $4e^{2x}$ as my functions for IBP. We have:

$$\begin{aligned} u &= x & v' &= 4e^{2x} \\ u' &= 1 & v &= 2e^{2x} \end{aligned}$$

Recall our integration by parts formula:

$$\int uv' = uv - \int vu'.$$

So here, upon substituting in and evaluating,

$$\begin{aligned} \int 4xe^{2x}\,dx &= \int x \times 4e^{2x}\,dx \\ &= x \times 2e^{2x} - \int 1 \times 2e^{2x}\,dx \\ &= 2xe^{2x} - \int 2e^{2x}\,dx \\ &= 2xe^{2x} - e^{2x} \end{aligned}$$

Note: $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b}$.

This can be submitted as `2*x*exp(2*x) - exp(2*x)`.

Question 17

Use integration by substitution to find

$$\int xe^{x^2+6}\,dx$$

and enter your answer in the box using Maple syntax. Do not include the constant of integration in your answer, that is, do not include the usual "+C".

When picking the substitution, try to look for the pattern of 'some function' and 'something similar to its derivative'. Here, we set that first function to be x^2+6, noting that its derivative $2x$ is just a single number multiplied to x, and that x also appears in the expression.

$$\begin{aligned} u &= x^2+6 \\ du &= 2x\,dx \\ \frac{1}{2}\,du &= x\,dx \end{aligned}$$

We now proceed to substitute. Ensure that you're careful substituting away the dx as well!

$$\begin{aligned} \int xe^{x^2+6}\,dx - \int \frac{1}{2}e^u\,du \\ = \frac{1}{2}e^u \\ = \frac{1}{2}e^{x^2+6} \end{aligned}$$

Note that it is a common mistake to forget to substitute back for x at the end. Ensure that you do not fall into this trap!

This can be submitted as `(1/2) * exp(x^2+6)`

Question 18

Calculate the **exact** average value of the following function

$$f(x) = 6x^2 + \frac{3}{x^2}$$

over the interval $[1, 3]$.

Enter the exact value of the average, in Maple syntax, in the box

Recall that the average value of a function $f(x)$ on an interval $[a, b]$ is defined as

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

So we just compute the relevant integral, without forgetting the fraction in front!

$$\begin{aligned}
\text{Average value} &= \frac{1}{3-1}\int_1^3 6x^2 + \frac{3}{x^2}\,dx \\
&= \frac{1}{2}\int_1^3 6x^2 + 3x^{-2}\,dx \\
&= \frac{1}{2}\left[\frac{6x^3}{3} + \frac{3x^{-1}}{-1}\right]_1^3 \\
&= \frac{1}{2}\left[2x^3 - \frac{3}{x}\right]_1^3 \\
&= \frac{1}{2}\left[\left(2(3)^3 - \frac{3}{3}\right) - \left(2(1)^3 - \frac{3}{1}\right)\right] \\
&= 27
\end{aligned}$$

This can be submitted as `27`.

Question 19

Calculate the **exact** mass of a thin rod given that the density (in unit weight per unit length) is

$$\rho(x) = 3x^2 + \frac{3}{x^2}$$

over the interval $[1, 3]$.

Enter the exact mass, in Maple syntax, in the box; unit is not required.

The only thing to be aware of is that the mass is computed by simply evaluating the definite integral of the density (over the given interval). So here we simply provide the answer, and leave the verification as your exercise.

$$M = \int_1^3 3x^2 + \frac{3}{x^2}\, dx = 28.$$

This can be submitted as `28`.

Question 20

When an unexpected major event occurs, the percentage P of the population that is aware of the event is initially zero and then climbs as media and gossip begin to take effect. Suppose the rate of change $\frac{dP}{dt}$ of P (t measured in hours) is governed by the differential equation

$$\frac{dP}{dt} = 60 - \frac{3}{5}P.$$

Using Maple, we obtained the solution to the differential equation

$$P = 100 - 100e^{-\frac{3}{5}t}.$$

(a) What is the rate of change of P when 50% of the population are aware of the event? Enter your answer in the box (in % per hour).

(b) What is the percentage of population aware of the event 5 hours after the event occurred? Enter your answer corrected to the near whole number.

(c) After how many hours will 50% of the population be aware of the the event? Enter the exact value in Maple syntax in the box (in hours).

Note: Maple treat 0.5 as an approximation. To enter half in exact form, you need to enter 1/2.

Note that P is already a percentage, so for part a) we can simply sub in $P = 50$ to the differential equation. Note that we were originally given $\frac{dP}{dt}$ in terms of P here, which is really handy!

$$\frac{dP}{dt} = 60 - \frac{3}{5} \times 50 = 30.$$

This can be submitted as `30`.

Part (b) now requires us to simply sub $t = 5$ into the solution of the differential equation.

$$P = 100 - 100e^{-\frac{3}{5}\times 5} = 95.02129316$$

Rounded to the nearest whole number, this value is 95. This is submitted as `95`.

In part c), we again have $P = 50$, but we now need to solve for t. This requires knowledge of how to solve exponetial equations using logarithms.

$$\begin{aligned} 50 &= 100 - 100e^{-\frac{3}{5}t} \\ -50 &= -100e^{-\frac{3}{5}t} \\ \frac{1}{2} &= e^{-\frac{3}{5}t} \\ \ln\left(\frac{1}{2}\right) &= -\frac{3}{5}t \\ -\frac{5}{3}\ln\left(\frac{1}{2}\right) &= t \end{aligned}$$

Note that this can be re-expressed as $\frac{5}{3}\ln(2)$, using the law $-\log(x) = \log\frac{1}{x}$. This is not a mandatory step; it just makes typing a bit easier to do on Maple.

This simplified expression can be submitted as `(5/3) * ln(2)`.

Question 21

Find the solution of the differential equation

$$\frac{dy}{dx} - y = 2\sin(2x)e^{x}; \quad y(0) = 1.$$

Enter the answer in the box as only the function of x which defines the solution.

Recognise that this is a (first-order) *linear* differential equation, so we approach by the method of the integrating factor. Here, the function multiplied to y is just -1, so we consider

$$\int -1\,dx = -x,$$

and hence our integrating factor is $e^{-}x$. The differential equation thus be expressed as

$$\begin{aligned} \frac{d}{dx}\left(e^{-x}y\right) &= 2\sin(2x)e^{x} \times e^{-x} \\ \frac{d}{dx}\left(e^{-x}y\right) &= 2\sin(2x) \end{aligned}$$

We then integrate and make y the subject:

$$\begin{aligned} e^{-x}y &= -\cos(2x) + c \\ y &= -e^{x}\cos(2x) + ce^{x} \end{aligned}$$

Our initial condition is that when $x = 0$, $y = 1$. Substituting this in, noting that $e^0 = 1$ and $\cos 0 = 1$, we obtain

$$\begin{aligned} 1 &= -e^0 \cos 0 + ce^0 \\ 1 &= -1 + c \\ 2 &= c \end{aligned}$$

Hence our solution is

$$y = -e^x \cos(2x) + 2e^x.$$

This can be submitted as `-exp(-x)*cos(2*x) + 2*exp(-x)`.

Question 22

Find the solution of the differential equation

$$\frac{dy}{dx} = \frac{sin(x)}{y}; \quad y(0) = 2.$$

Enter the answer in the box as only the function of x which defines the solution.

Recognise that this is a *separable* differential equation, as we can multiply over the dx and divide down the $\sin(y)$ to separate all expressions involving x and y.

$$y\,dy = \sin(x)\,dx$$

Now proceed to integrate and solve for y.

$$\begin{aligned}
\int y\,dy &= \int \sin(x)\,dx \\
\frac{y^2}{2} &= -\cos(x) + \frac{c}{2} \\
y^2 &= -2\cos(x) + c
\end{aligned}$$

Note: You may be uncomfortable with the use of $\frac{c}{2}$. If you do this problem with just c as your constant of integration, it should still work out. This is just a common trick to reduce how many coefficients we have to deal with later on.

Here, we now find the value of c before solving for y. When $x = 0$, $y = 2$ so we obtain:

$$\begin{aligned}
2^2 &= -2\cos 0 + c \\
4 &= -2 + c \\
6 &= c
\end{aligned}$$

So our solution is

$$\begin{aligned}
y^2 &= -2\cos(x) + 6 \\
y &= \pm\sqrt{-2\cos(x) + 6}
\end{aligned}$$

To choose between the positive and negative square root, observe again that $y(0) = 2$. Had we chose the negative square root, we would've instead obtainend $y(0) = -2$. Hence the positive square root gives the required answer.

$$y = \sqrt{-2\cos(x) + 6}$$

This can be entered as `sqrt(-2*cos(x) + 6)`