MATH1231/1241 Lab Test 2
Algebra Sample Solutions

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These answers were written and typed by Abdellah Islam, and edited by Supriya Segal. Please be ethical with this resource. If you appreciate this resource, please consider supporting us by coming to our events! Also, happy studying :)

We cannot guarantee that our answers are correct - please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are sometimes multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

NOTE: Any text presented `like this` is presented as required in Maple syntax.

Question 1

Let $S = \{v_1, v_2, v_3, v_4, v_5\}$ be a set of five vectors in $\mathbb{R}^3$. Let $W = \text{span}\{S\}$. When these vectors are placed as columns into a matrix A as $A = (v_1 \mid v_2 \mid v_3 \mid v_4 \mid v_5)$ and A is row-reduced to echelon form U, we have

$$U = \begin{pmatrix} 1 & -4 & -3 & -3 & -2 \\ 0 & 1 & 3 & -2 & 1 \\ 0 & 0 & 1 & -3 & 3 \end{pmatrix}.$$

1. *State the dimension of W.*
2. *State a basis B for W, using vectors $\mathbf{v}_i$ with i as small as possible.*
3. *Express $\mathbf{v}_5$ as a linear combination of the basis vectors in B.*

$W = \text{span}\{S\} = A\mathbf{x}$, $\mathbf{x} \in \mathbb{R}^5$. Since A can be row-reduced to echelon form U and U has 3 leading columns, then W has dimension 3.

Since $\dim(\text{span}\{S\}) = 3$ and the first three columns of U are leading columns, we can take the first three vectors of S to form a basis of W: B =`{v1, v2, v3}`.

If we take $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)^T$ then $A\mathbf{x} = \mathbf{0}$ can be row reduced to $U\mathbf{x} = \mathbf{0}$:

$$(A \mid \mathbf{0}) \sim \left(\begin{array}{ccccc|c} 1 & -4 & -3 & -3 & -2 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 3 & 0 \end{array}\right).$$

Let $x_4 = 0$ and $x_5 = 1$. Hence we have $x_3 = -3$, $x_2 = 8$ and $x_1 = 25$, i.e. $25\mathbf{v}_1 + 8\mathbf{v}_2 - 3\mathbf{v}_3 + \mathbf{v}_5 = \mathbf{0}$. So, $\mathbf{v}_5$ =`-25*v1-8*v2+3*v3`.

Question 7

1. *Find the nullity of the matrix*

$$\begin{pmatrix} -16 & 100 & -36 & -72 & -188 \\ 160 & 40 & 112 & 164 & 224 \\ -126 & -175 & -194 & -116 & 205 \\ -46 & 55 & -184 & 122 & 511 \end{pmatrix}.$$

2. *Select all statements below which are true.*

- $(-1, -2, 1, -2, 1)^T$ *is in* $\ker A$.
- $(0, 0, 0, 0)^T$ *is in* $\ker A$.
- $(54, -20, -56, 39)^T$ *is in* $\ker A$.
- $\ker A$ *is a subspace of* $\mathbb{R}^5$.
- $(-1, 1, 2, -2, 1)^T$ *is in* $\ker A$.
- $(46, 60, -119, 16)^T$ *is in* $\ker A$.
- $\ker A$ *is a subspace of* $\mathbb{R}^4$.
- $(0, 0, 0, 0, 0)^T$ *is in* $\ker A$.

By defining A in Maple as given, we can find a basis for the kernel of A using the command

`NullSpace(A)`. This gives us the basis

$$\left\{\begin{pmatrix}-1\\1\\2\\-2\\1\end{pmatrix}\right\}.$$

So $\operatorname{nullity}(A) = 1$.

If $\mathbf{x} \in \ker(A)$ then $\mathbf{x} \in \mathbb{R}^5$. Hence $\ker(A)$ is a subspace of $\mathbb{R}^5$, and $(0,0,0,0,0)^T$ is in $\ker(A)$. From our basis of $\ker(A)$, we know that $(-1,1,2,-2,1)^T$ is in $\ker(A)$. $(-1,-2,1,-2,1)^T$ is not a multiple of $(-1,1,2,-2,1)^T$ so it is not in $\ker(A)$. All vectors in $\mathbb{R}^4$ cannot be in $\ker(A)$.

Question 9

1. *Find the rank of the matrix*

$$\begin{pmatrix}-44 & -128 & -14 & -92 & -72\\ 122 & -65 & -170 & 44 & 615\\ 158 & 77 & 190 & -192 & -683\\ -58 & 87 & 106 & 10 & -337\end{pmatrix}.$$

2. *Select all statements below which are true.*

   - $(0,0,0,0,0)^T$ *is in* $\operatorname{im} A$.
   - $\operatorname{im} A$ *is a subspace of* $\mathbb{R}^4$.
   - $(-2,2,4,-4,2)^T$ *is in* $\operatorname{im} A$.
   - $(0,0,0,0)^T$ *is in* $\operatorname{im} A$.
   - $(-70,-99,89,34)^T$ *is in* $\operatorname{im} A$.
   - $\operatorname{im} A$ *is a subspace of* $\mathbb{R}^5$.
   - $(2,4,-2,4,-2)^T$ *is in* $\operatorname{im} A$.
   - $(-53,-63,-1,58)^T$ *is in* $\operatorname{im} A$.

Define A in Maple as the given matrix. Then by using the Maple command `Rank(A)`, we can find that the rank of A is 3.

For some $\mathbf{x} \in \mathbb{R}^5$, $A\mathbf{x} \in \mathbb{R}^4$. Hence $\mathrm{im}(A)$ is a subspace of $\mathbb{R}^4$, and $(0,0,0,0)$ is in $\mathrm{im}(A)$. If we enter the line `LinearSolve(A,<-70,-99,89,34>);` into Maple, then we receive the error "Inconsistent System" which means no solution. So $(-70,-99,89,34)^T$ is not in $\mathrm{im}(A)$. Entering `LinearSolve(A,<-53,-63,-1,58>);` into Maple, we find a solution and so $(-53,-63,-1,58)^T$ is in $\mathrm{im}(A)$.

Question 11

Let A be a 3×4 matrix with column vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4$ so

$$A = (\boldsymbol{a}_1 \mid \boldsymbol{a}_2 \mid \boldsymbol{a}_3 \mid \boldsymbol{a}_4).$$

A has row-reduced echelon form

$$U = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

1. *State the vales of* $\mathrm{rank}\, A$ *and* $\mathrm{nullity}\, A$.
2. *Find a basis for the column space of A,* $\mathrm{col}\, A$.

Since A can be row-reduced to echelon form U, which has two leading columns, then $\mathrm{rank}(A) = 2$. By the Rank-Nullity Theorem, $\mathrm{nullity}(A) = 2$.

The column space of A is given by the columns of A which correspond to the leading columns with non-zero coefficients, which in this case are the first two column vectors. Hence a basis for $\mathrm{col}(A) =$`{a1,a2}`.

Question 18

Let A be a 3×5 matrix with column vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5$ so

$$A = (\boldsymbol{a}_1 \mid \boldsymbol{a}_2 \mid \boldsymbol{a}_3 \mid \boldsymbol{a}_4 \mid \boldsymbol{a}_5).$$

A has row-reduced echelon form

$$U = \begin{pmatrix} 1 & 0 & -1 & 0 & -4 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -3 \end{pmatrix}.$$

1. *State the values of* rank A *and* nullity A.
2. *Find a basis for the kernel or nullspace of* A, ker A.

A can be row-reduced to echelon form U, where U has 3 leading columns. Then $\text{rank}(A) = 3$ and $\text{nullity}(A) = 2$.

To find a basis for the kernel of A, we should consider the row echelon form of $A\mathbf{x} = \mathbf{0}$. Since we have the echelon form of A, we can row-reduce $A\mathbf{x} = \mathbf{0}$ to

$$(A \mid \mathbf{0}) \sim \left(\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & -4 & 0 \\ 0 & 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 \end{array}\right).$$

Taking $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)^T$, we have

$$\mathbf{x} = x_3 \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 4 \\ -2 \\ 0 \\ 3 \\ 1 \end{pmatrix} = \text{span} \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \\ 0 \\ 3 \\ 1 \end{pmatrix} \right\}.$$

Hence a basis for $\ker(A) =$ `{<1,-1,1,0,0>,<4,-2,0,3,1>}`.

Question 25

The 2×2 matrix

$$A = \begin{pmatrix} -5 & 3 \\ 8 & 0 \end{pmatrix}$$

has two distinct real eigenvalues.

1. *Give the characteristic polynomial for A.*
2. *Find the set of eigenvalues for A.*
3. *Find one eigenvector for each eigenvalue.*

Characteristic polynomial $p(t) = \det(A - tI) =$`t^2+5*t-24`.

The eigenvalues are the solutions to the characteristic polynomial, so the set of eigenvalues for $A =$`{-8, 3}`.

If $\mathbf{v}_1$ and $\mathbf{v}_2$ are the eigenvectors for -8 and 3 respectively, then we must consider the matrix equation $(A - tI)\mathbf{v} = \mathbf{0}$. For $t = -8$, we have

$$\begin{pmatrix} 3 & 3 \\ 8 & 8 \end{pmatrix} \mathbf{v}_1 = \mathbf{0}.$$

Hence $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. For $t = 3$ we have

$$\begin{pmatrix} -8 & 3 \\ 8 & -3 \end{pmatrix} \mathbf{v}_2 = \mathbf{0}.$$

Hence $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 8 \end{pmatrix}$.

Question 26

The 2×2 matrix $A = \begin{pmatrix} 0 & 3 \\ -4 & 7 \end{pmatrix}$ *has eigenvalue/eigenvector pairs:*

$$3, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \textit{and} \qquad 4, \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

1. *Write down an expression in Maple notation for the general solution to the vector-values function of a real variable t,*

$$\boldsymbol{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

to the first order system using two arbitrary constants c1 and c2:

$$\frac{dy_1}{dt} = 3y_2$$
$$\frac{dy_2}{dt} = -4y_1 + 7y_2.$$

2. *Use the initial condition condition* $\boldsymbol{y}(0) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ *to determine the constants c1 and c2 and find the solution* $\boldsymbol{y}(t)$.

Since we know the eigenvalues and eigenvectors, we can plug them into the general solution: $\mathbf{y}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Given our initial condition, we can find the coefficients by solving the matrix equation

$$\left(\begin{array}{cc|c} 1 & 3 & -1 \\ 1 & 4 & -1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 3 & -1 \\ 0 & 1 & 0 \end{array}\right).$$

This gives us $c_1 = -1$ and $c_2 = 0$. So $\mathbf{y}(t) = -e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Question 27

Let A be a 2×2 matrix with eigenvalue, eigenvector pairs:

$$-5, \begin{pmatrix} -4 \\ 3 \end{pmatrix} \quad \textit{and} \quad 2, \begin{pmatrix} 4 \\ -2 \end{pmatrix}.$$

1. *Find an invertible matrix M and a diagonal matrix D such that* $A = MDM^{-1}$.
2. *For any integer n, find the matrix* A^n *as a single matrix.*

When diagonalising a matrix, $M = (\mathbf{v}_1 | \mathbf{v}_2)$ and $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Hence

$$M = \begin{pmatrix} -4 & 4 \\ 3 & -2 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} -5 & 0 \\ 0 & 2 \end{pmatrix}.$$

Since $A = MDM^{-1}$ then $A^n = MD^nM^{-1}$. Hence by defining M and $F = D$ in Maple (since D is a restricted variable), we can use the Maple command `A:=M.MatrixPower(F,n).M^(-1)` to find that

$$A^n = \begin{pmatrix} -2(-5)^n + 3.2^n & -4(-5)^n + 4.2^n \\ \frac{-3}{2}(-5)^n - 3.2^{n-1} & 3(-5)^n - 2^{n+1} \end{pmatrix}.$$

So A^n =`<<-2*(-5)^n+3*2^n,(-3/2)*(-5)^n-3*2^(n-1)>|<-4*(-5)^n+4*2^n,3*(-5)^n-2^(n+1)>>`.

Question 28

1. *Construct a "for loop" in order to evaluate the sum*

$$\sum_{n=12}^{23} \sin\left(\frac{k}{n}\right)$$

   for k from 2 to 50.

   Note: *This question is one in which we must select the correct options from drop down menus. I will present the correct sequence of lines below. After, I will present some logic.*

2. *Consider the sequence* $\{a_n\}$ *generated by the recurrence relation*

$$a_{n+1} = a_n - 5a_{n-1} + a_{n-2} \quad \textit{for} \quad n\textit{=3,4,5,...}$$

   given that $a_1 = 4$, $a_2 = -1$, *and* $a_3 = 1$. *Write a for loop to find the value of* a_{60}.

In order to get the desired outcome in Maple, enter:
`for k from 2 to 50 do evalf(add(sin(k/n), n=12..23)) end do;`.
We need `for k from 2 to 50` because the question specifies that the **sequence** is with respect to k. Then, we evaluate `add(sin(k/n), n=12..23))` because the **sum** is with respect to n.

The recurrence relation for loop can be written as:

```
f:= proc(n)
local a,i;
a[1]:= 4;
a[2]:= -1;
a[3]:= 1;
for i from 4 to n do
a[i]:= a[i-1]-5*a[i-2]+a[i-3];
end do;
return a[n];
end proc;
```

Then entering `f(60)` will give us a_{60} =-108431555483984760558.

The first line declares a function f of n. Then we define variables a and i, as well as initial conditions $a[1] = 4$, $a[2] = -1$, and $a[3] = 1$. The lines 6 to 9 finds the value of $a[i]$, and returns the numerical value.

Question 29

A simple iteration procedure with $a_0 = 0$ and

$$a_{n+1} = \sin\left((1 + \frac{1}{6}a_n)^2\right), \quad n \geq 0,$$

is being used to find an approximate solution to the equation $x = \sin\left(\left(1 + \frac{1}{6}x\right)^2\right)$. Write a procedure which takes a positive integer m and uses a for loop to calculate a_m. The procedure should return a_m if $|a_m - a_{m-1}| < 10^{-7}$, and -1 otherwise. All calculations are done using 30 significant figures.
***Note**: This question is one in which we must select the correct options from drop down menus. I will present the correct sequence of lines below. After, I will present some logic.*

The correct lines of code are:

```
Digits:= 30;
f:= proc(m)
local a,i;
a[0]:= 0;
for i from 1 to m do
a[i]:= evalf(sin((1+a[i-1]/6)^2));
end do;
if abs(a[m]-a[m-1]) < 10^(-7) then
a[m]
else
-1
end if
end proc;
```

Typing `f(4)` into Maple gives us `-1`, and `f(11)` gives us `0.976125175777253100707794674276`.

The first line ensures that our results are given in 30 significant figures. The next line declares a function f of m. Then we define variables a and i, and the initial condition $a[0] = 0$.

From line 5 to 7, we have Maple evaluate the desired expression for i from 1 to m, whatever the entered m may be. After we end do, we check if $|a[m] - a[m-1]|$ is less than 10^{-7}. Finally the function returns the appropriate number; -1 or $a[m]$.