MATH2111 - Analysis of $\mathbb{R}^n$

UNSW Mathematics Society - Abdellah Islam

1 Metrics

Metric functions are how we classify distance between elements of sets in analysis. To begin, we look at norms which measure the 'size' of an element of a vector space.

Definition 1.1. *A **norm** is a function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ defined by $\mathbf{x} \mapsto \|\mathbf{x}\|$ which satisfies*

1. $\|\mathbf{x} - \mathbf{y}\| \geq 0 \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, *equality iff* $\mathbf{x} = \mathbf{y}$,
2. $\|\lambda \mathbf{x}\| = |\lambda| \|\mathbf{x}\| \quad \forall\ \mathbf{x} \in \mathbb{R}^n,\ \lambda \in \mathbb{R}$,
3. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\| \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

The standard p-norm is given by

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} x_i^p \right)^{1/p}, \quad \forall p \in \mathbb{Z}^+.$$

Note that $\|\mathbf{x}\|$ denotes the standard 2-norm, unless stated otherwise.
We now define a metric, which measures the 'distance' between two elements of a set. Since we are looking at $\mathbb{R}^n$ we restrict ourselves to the set $\mathbb{R}^n$.

Definition 1.2. *A **metric** is a function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ which satisfies*

1. $d(\mathbf{x}, \mathbf{y}) \geq 0 \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, *equality iff* $\mathbf{x} = \mathbf{y}$,
2. $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x}) \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,
3. $d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) \quad \forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$.

Since the idea of a metric and a norm look similar, we are encouraged to define a metric by a norm. We can define the standard p-metric by

$$d_p(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_p \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

Note that d denotes the metric given by the standard 2-norm (i.e. the Euclidean distance), unless stated otherwise.
We also have an equivalence relation on metrics.

Definition 1.3. *Two metrics d and δ are **equivalent** if there exists constants $0 < C_1 < C_2 < \infty$ such that*

$$C_1 \delta(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{y}) \leq C_2 \delta(\mathbf{x}, \mathbf{y}) \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

2 Sequences in $\mathbb{R}^n$

To look at the limit of a multi-variable function, we will need some tools first. We can take the concept of a metric and define a ball in $\mathbb{R}^n$.

Definition 2.1. *A **ball** around a point $\mathbf{a} \in \mathbb{R}^n$ of radius $\epsilon > 0$ is defined as*

$$B(\mathbf{a}, \epsilon) = \{\mathbf{x} \in \mathbb{R}^n : d(\mathbf{a}, \mathbf{x}) < \epsilon\}.$$

Note that a ball does not contain its boundary.
The idea of a 'ball' can be thought of as the collection of points 'close' to the point $\mathbf{a}$. This idea allows us to define the limit point of a sequence as the point which the sequence eventually is always 'close' to, for all ϵ.

Definition 2.2. *Let $\{\mathbf{x}_i\}$ be a **sequence** in $\mathbb{R}^n$. We say $\mathbf{x}$ is the **limit** of the sequence $\{\mathbf{x}_i\}$ iff*

$$\forall \epsilon > 0 \quad \exists N : \quad n \geq N \Rightarrow \mathbf{x}_n \in B(\mathbf{x}, \epsilon).$$

Example 2.1. *Suppose we have a sequence $\{\mathbf{x}_n\}$ given by*

$$\mathbf{x}_n = \left(3 + \frac{2}{n}, 5 - \frac{1}{2^n}\right) \quad \forall n \in \mathbb{N}.$$

Prove that the limit point of this sequence is $(3, 5)$.

Proof. We claim that the limit point is $(3, 5)$, and observe the ball centred at $(3, 5)$ with arbitrary radius ϵ:

$$B((3, 5), \epsilon) = \left\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x} - (3, 5)\| < \epsilon\right\}.$$

If $\mathbf{x}_n \in B((3, 5), \epsilon)$ then

$$\left\|\left(\frac{2}{n}, -\frac{1}{2^n}\right)\right\| < \epsilon,$$

which is what we are aiming for. Suppose $n \geq N$ for some $N \in \mathbb{N}$. Then $n^2 \geq N^2$ and $2^{2n} \geq N^2$, hence

$$\frac{4}{n^2} + \frac{1}{2^{2n}} \leq \frac{4}{N^2} + \frac{1}{N^2} = \frac{5}{N^2}.$$

We want to turn the RHS of this inequality into ϵ^2, so for each $\epsilon > 0$ choose $N = \dfrac{3}{\epsilon}$. Then

$$\begin{aligned} & \frac{4}{n^2} + \frac{1}{2^{2n}} \leq \frac{5}{9}\epsilon^2 \\ \Rightarrow \quad & \frac{4}{n^2} + \frac{1}{2^{2n}} < \epsilon^2 \\ \Rightarrow \quad & \left\|\left(\frac{2}{n}, -\frac{1}{2^{2n}}\right)\right\| < \epsilon. \end{aligned}$$

We now can state that for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ (given by $N = 3/\epsilon$) such that if $n \geq N$ then $\mathbf{x}_n \in B((3, 5), \epsilon)$. Hence $(3, 5)$ is the limit of the sequence. □

When we want to simply compute a limit rather than prove it, we can use the following theorem.

Theorem 2.1. *A* ***sequence*** $\{\mathbf{x}_n\}$ ***converges*** *to limit* $\mathbf{x}$ ***iff***

- *The* ***components*** *of* $\mathbf{x}_n$ ***converge*** *to the* ***components*** *of* $\mathbf{x}$, *or*
- $d(\mathbf{x}_n, \mathbf{x}) \to 0$.

Example 2.2. *Suppose we have a sequence* $\{\mathbf{x}_n\}$ *defined by*

$$\mathbf{x}_n = \left(2 - \frac{1}{n+1}, \frac{n+1}{n^2}\right).$$

Find its limit point.

Proof. We will be using part 1 of Theorem 2.1 to find the limit point, and check part 2 using the result. Since

$$\lim_{n\to\infty}\left(2 - \frac{1}{n+1}\right) = 2 \quad \text{and} \quad \lim_{n\to\infty}\frac{n+1}{n^2} = 0$$

then the components of $\mathbf{x}_n$ converge to the components of $(2,0)$. By Theorem 2.1 the sequence $\{\mathbf{x}_n\}$ converges to $(2,0)$. Now the distance between $\mathbf{x}_n$ and $(2,0)$ is given by

$$d(\mathbf{x}_n, (2,0)) = \left\|\left(-\frac{1}{n+1}, \frac{n+1}{n^2}\right)\right\| = \sqrt{\frac{1}{(n+1)^2} + \frac{(n+1)^2}{n^4}}.$$

Clearly this approaches zero, hence $d(\mathbf{x}_n, ((2,0)) \to 0$. □

We also define a special type of sequence, called a **Cauchy Sequence**.

Definition 2.3. *A sequence* $\{\mathbf{x}_n\}$ *is a* ***Cauchy Sequence*** *if*

$$\forall \epsilon > 0 \quad \exists K : \quad k, l > K \Rightarrow d(\mathbf{x}_k, \mathbf{x}_l) < \epsilon.$$

This idea is useful because it allows us to know whether or not a sequence converges without finding its limit point.

Theorem 2.2. *A sequence* $\{\mathbf{x}_n\}$ *converges in* $\mathbb{R}^n$ ***iff*** $\{\mathbf{x}_n\}$ *is Cauchy.*

Example 2.3. *Decide, using Theorem 2.2, if the sequence* $\left\{\frac{1}{n}\right\}$ *converges.*

Proof. We claim the sequence converges, and we will show it by proving that the sequence is Cauchy. Begin by choosing two integers k, l such that $d(\mathbf{x}_k, \mathbf{x}_l) < \epsilon$. Then

$$\left\|\frac{1}{k} - \frac{1}{l}\right\| < \epsilon,$$

which is true if $\frac{1}{k}, \frac{1}{l} < \frac{\epsilon}{2}$. So we choose $K = \frac{2}{\epsilon}$, and we can see that the sequence is Cauchy. Hence by Theorem 2.2 the sequence converges. □

3 Sets

We now look at sets in $\mathbb{R}^n$ and attempt to classify sets as open or closed, as well as classify the elements of a set. We use the notion of balls to make this classification.

Definition 3.1. *Consider a set $\Omega \subset \mathbb{R}^n$.*

- *We say that a point $\mathbf{x}_0 \in \Omega$ is an **interior point** of Ω if*

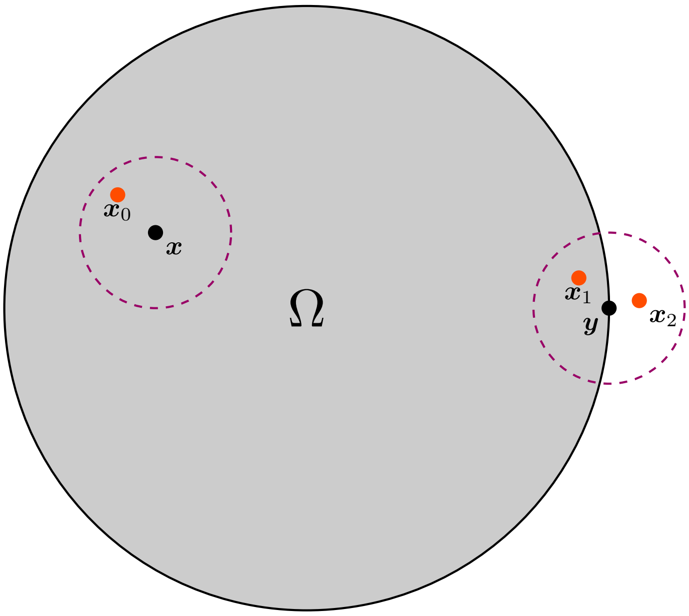
$$\exists \epsilon > 0 : \quad B(\mathbf{x}_0, \epsilon) \subseteq \Omega.$$

- *We say that Ω is **open** if every point in the set is an interior point. If the complement set Ω^c is open then we say that Ω is **closed**.*
- *A point $\mathbf{x}_0 \in \Omega$ is a **boundary point** of Ω if*

$$\forall \epsilon > 0 \quad \exists \mathbf{x}_1 \in \Omega, \mathbf{x}_2 \notin \Omega : \quad \mathbf{x}_1, \mathbf{x}_2 \in B(\mathbf{x}_0, \epsilon).$$

- *A point $\mathbf{x}_0 \in \mathbb{R}^n$ is called a limit point of a set Ω if there exists sequence $\{\mathbf{x}_i\}$ in Ω such that $\mathbf{x}_i \neq \mathbf{x}_0$ and $\mathbf{x}_0$ is the limit of the sequence $\{\mathbf{x}_i\}$.*

An intuitive way to think of interior points and boundary points is outlined below.



The ball centred at $\mathbf{x}$ contains only points which are elements of Ω (like $\mathbf{x}_0$), hence $\mathbf{x}$ is an interior point of Ω. The ball centred at $\mathbf{y}$ contains some points in Ω (like $\mathbf{x}_1$) and some points not in Ω (like $\mathbf{x}_2$), hence $\mathbf{y}$ is a boundary point of Ω. The process of deciding whether a set is closed requires us to look at the complement set. The following theorem gives an easier method for proving this.
Let us attempt a problem involving these concepts.

Example 3.1. *Is the set*

$$S = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 < y < x\right\}$$

open or not open?

Proof. Fix a point $(x, y) \in S$, and consider a point $\mathbf{x}_0 = (x + \delta_1, y + \delta_2) \in B((x, y), \epsilon)$ such that $|\delta_1|, |\delta_2| < 1$. If $\mathbf{x}_0 \in S$ then

$$\begin{aligned} & y + \delta_2 < x + \delta_1 \\ \Leftrightarrow \quad & |x - y| > |\delta_2 - \delta_1| . \end{aligned}$$

Also

$$\begin{aligned} & (x + \delta_1)^2 < y + \delta_2 \\ \Leftrightarrow \quad & x^2 + 2x\delta_1 + \delta_1^2 < y + \delta_2 \\ \Leftrightarrow \quad & |(2x + \delta_1)\delta_1 - \delta_2| < \left|y - x^2\right| . \end{aligned}$$

However if $\mathbf{x}_0 \in B((x, y), \epsilon)$ then $\|(\delta_1, \delta_2)\| < \epsilon$. Hence

$$\begin{aligned} |\delta_2 - \delta_1| &\leq |\delta_1| + |\delta_2| \\ &< 2\epsilon. \end{aligned}$$

Setting $\epsilon < \frac{1}{2}|x - y|$, then $|x - y| > |\delta_2 - \delta_1|$ whenever $\mathbf{x}_0 \in B((x, y), \epsilon)$.
Also,

$$\begin{aligned} |(2x + \delta_1)\delta_1 - \delta_2| &\leq (2|x| + 1)|\delta_1| + |\delta_2| \\ &< (2x + 1)\epsilon + \epsilon \\ &= (2|x| + 2)\epsilon \end{aligned}$$

so setting $\epsilon < \dfrac{\left|y - x^2\right|}{2|x| + 2}$ will ensure that $|(2x + \delta_1)\delta_1 - \delta_2| < \left|y - x^2\right|$ whenever $\mathbf{x} \in B((x, y), \epsilon)$.
Now, set $\epsilon < \min\left\{\dfrac{1}{2}|x - y|, \dfrac{\left|y - x^2\right|}{2|x| + 2}\right\}$. If $\mathbf{x}_0 \in B((x, y), \epsilon)$ then

$$|x - y| > |\delta_2 - \delta_1| \quad \text{and} \quad |(2x + \delta_1)\delta_1 - \delta_2| < \left|y - x^2\right|$$

and so $\mathbf{x}_0 \in S$. Hence by definition, S is open. $\square$

So we can prove that a set is open. However, how does one prove a set is closed? The following theorems are nice ways to prove or disprove whether a set is closed.

Theorem 3.1. *A set $\Omega \subset \mathbb{R}^n$ is closed* ***iff*** *Ω contains* ***all*** *its boundary points.*

We can essentially deduce that a set is open iff it does not contain any boundary points, and a set is closed iff it contains its boundary.
This theorem is useful because counterexamples are easy to produce in the case that the set is not closed, and we can study boundary points easily as well (more on that later). The next theorem is in the same vein.

Theorem 3.2. *A set $\Omega \subset \mathbb{R}^n$ is closed* ***iff*** *Ω contains* ***all*** *its limit points.*

Before we try some examples, we will define the set of boundary points and the set of interior points.

Definition 3.2. *Let Ω be a set.*

- *We call the set of all interior points* ***the interior of*** *Ω, which is also the largest open subset of Ω.*
- *We call the set of all boundary points* ***the boundary of*** *Ω, i.e. $\partial\Omega$.*
- *The smallest closed set containing Ω is called* ***the closure of*** *Ω, and is given by $\overline{\Omega} = \Omega \cup \partial\Omega$.*

Example 3.2. *Is the set*

$$\Omega = \left\{(x, y) \in \mathbb{R}^2 \mid y^2 \leq x \leq 1\right\}$$

closed?

Proof. We note that the boundary points are where the inequalities become equality, namely at

$$y^2 = x \quad \text{and} \quad x = 1.$$

Hence the boundary of Ω is

$$\partial\Omega = \left\{(x, y) \in \mathbb{R}^2 \mid y^2 = x\right\} \cup \left\{(x, y) \in \mathbb{R}^2 \mid x = 1\right\}.$$

Since $\partial\Omega \subseteq \Omega$ then Ω contains all its boundary points, and by Theorem 3.1 Ω is closed. $\square$

Example 3.3. *Is the set*

$$\Omega = \left\{(x, y) \in \mathbb{R}^2 \mid y = x, \quad x \in [0, 3)\right\}$$

closed? If not, what is the smallest closed set containing Ω?

Proof. Consider the point $(x, y) = (3, 3) \notin \Omega$. We can create a ball of arbitrary radius ϵ around this point, $B\left((3, 3), \epsilon\right)$. Choose a point

$$\left(3 - \frac{\epsilon}{2\sqrt{2}}, 3 - \frac{\epsilon}{2\sqrt{2}}\right) \in B\left((3, 3), \epsilon\right).$$

Clearly $\left(3 - \frac{\epsilon}{2\sqrt{2}}, 3 - \frac{\epsilon}{2\sqrt{2}}\right) \in \Omega$. Now choose a point

$$\left(3 + \frac{\epsilon}{2\sqrt{2}}, 3 + \frac{\epsilon}{2\sqrt{2}}\right) \in B\left((3, 3), \epsilon\right).$$

This point is not in Ω, hence any ball we create of radius ϵ around the point $(3, 3)$ will contain a point in Ω and a point not in Ω. This means that $(3, 3)$ must be a boundary point and so Ω is not closed since it does not contain a boundary point.

By definition 3.2 we know that the closure of Ω is the smallest closed set containing Ω. In fact, the only boundary point not in Ω is $(3, 3)$. So the closure is given by

$$\overline{\Omega} = \left\{(x, y) \in \mathbb{R}^2 \mid y = x, \quad x \in [0, 3]\right\}.$$

$\square$

4 Limits in $\mathbb{R}^n$

To define the limit of a multivariable function, we again use the notion of a ball in $\mathbb{R}^n$.

Definition 4.1. *For a function* $\boldsymbol{f} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, *we say*

$$\lim_{\mathbf{x}\to\mathbf{a}} \boldsymbol{f}(\mathbf{x}) = \mathbf{b}$$

if

$$\forall \epsilon > 0 \quad \exists \delta > 0 : \quad \mathbf{x} \in \Omega \cap B(\mathbf{a}, \delta) \setminus \{\mathbf{a}\} \implies \boldsymbol{f}(\mathbf{x}) \in B(\mathbf{b}, \epsilon).$$

Note that for $n = m = 1$ this definition reduces to the single variable limit. The reason why we use balls to define the limit in several variables is so that we can be sure that the limit approaches $\mathbf{b}$ along **any** path. This idea is highlighted in the following example.

Example 4.1. *does the limit*

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^3 + y^3}$$

exist? If it does, find its value.

Proof. Consider the path on the x-axis, $(x, y) = (t, 0)$ as $t \to 0$. Then

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^3 + y^3} = \lim_{t\to 0} \frac{t \cdot (0)^2}{t^3 + 0^3} = 0.$$

Now consider the path along the line $y = x$ given by $(x, y) = (t, t)$ as $t \to 0$. Then

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^3 + y^3} = \lim_{t\to 0} \frac{t \cdot t^2}{t^3 + t^3} = \lim_{t\to 0} \frac{t^3}{2t^3} = \frac{1}{2}.$$

Different paths give us different values, so the limit does not exist. $\square$

While it is easy to prove that the limit does not exist, it is generally harder to prove a limit exists.

Example 4.2. *Prove, using the definition of the limit, that*

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2 + y^2} = 0.$$

Proof. Let $(x, y) \in B((0, 0), \delta)$. Then

$$\|(x, y)\| < \delta \Rightarrow |x| < \delta.$$

Hence

$$\begin{aligned} & \left|\frac{x^3}{x^2 + y^2}\right| \leq \left|\frac{x^3}{x^2 + 0}\right| \\ \Rightarrow \quad & \left|\frac{x^3}{x^2 + y^2}\right| \leq |x| \\ \Rightarrow \quad & \left|\frac{x^3}{x^2 + y^2}\right| < \delta. \end{aligned}$$

For each $\epsilon > 0$, set $\delta = \epsilon$. Then

$$\|(x,y)\| < \delta \Rightarrow \left| \frac{x^3}{x^2+y^2} \right| < \epsilon,$$

which satisfies the definition of the limit. Hence

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0.$$

□

When dealing with functions which output vectors, the process of proving whether a limit exists can be difficult. Instead we can use the following theorem.

Theorem 4.1. *Let $\boldsymbol{f} : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a function with* ***components*** *$f_1, f_2, ..., f_m$. Then for some point $\mathbf{a} \in \Omega$, if*

$$\lim_{\mathbf{x}\to\mathbf{a}} f_1(\mathbf{x}) = L_1, \quad \lim_{\mathbf{x}\to\mathbf{a}} f_2(\mathbf{x}) = L_2, \quad ..., \quad \lim_{\mathbf{x}\to\mathbf{a}} f_m(\mathbf{x}) = L_m$$

then

$$\lim_{\mathbf{x}\to\mathbf{a}} \boldsymbol{f}(\mathbf{x}) = (L_1, L_2, ..., L_m).$$

This tells us that when we are evaluating the limit of a vector-valued function, it suffices to consider each component of the function individually.

When we simply want to compute a limit rather than prove it, the following theorem may be helpful. Observe that all the usual limit rules can be extended to the multi-variable case.

Theorem 4.2. *Let*

$$f : \mathbb{R}^n \to \mathbb{R} \quad \text{and} \quad g : \mathbb{R}^n \to \mathbb{R}$$

be two functions such that

$$\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = a \quad \text{and} \quad \lim_{\mathbf{x}\to\mathbf{c}} g(\mathbf{x}) = b.$$

Then

$$\lim_{\mathbf{x}\to\mathbf{c}} (f+g)(\mathbf{x}) = a+b, \quad \lim_{\mathbf{x}\to\mathbf{c}} (fg)(\mathbf{x}) = ab, \quad \text{and} \quad b \neq 0 \Rightarrow \lim_{\mathbf{x}\to\mathbf{c}} (f/g)(\mathbf{x}) = a/b.$$

5 Continuity

Now that we have a definition for the limit of a multi-variable function, we can define continuity.

Definition 5.1. *Consider a function $\boldsymbol{f} : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$. We say that $\boldsymbol{f}$ is* ***continuous at a point*** $\mathbf{x}_0 \in \Omega$ *if either*

- $\mathbf{x}_0$ *is a* ***limit point*** *of Ω and* $\lim_{\mathbf{x}\to\mathbf{x}_0} \boldsymbol{f}(\mathbf{x}) = \boldsymbol{f}(\mathbf{x}_0)$, *or*
- $\mathbf{x}_0$ *is not a* ***limit point*** *of Ω.*

The definition is similar to that of the single variable continuity definition, except when the point is not a limit point. We can write the definition of continuity in the sense of a limit as

$$\forall \epsilon > 0 \quad \exists \delta > 0 : \quad \mathbf{x} \in \Omega \cap B(\mathbf{x}_0, \delta)\backslash\{\mathbf{x}_0\} \Rightarrow \boldsymbol{f}(\mathbf{x}) \in B(\boldsymbol{f}\left(\mathbf{x}_0\right), \epsilon).$$

We can also define continuity in terms of limit points or interior points with the following theorem.

Theorem 5.1. *Let $\boldsymbol{f} : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be continuous at a point $\mathbf{x}_0$. This is* ***equivalent*** *to saying*

- $\forall \{\mathbf{x}_i\} \subseteq \Omega,\ \{\mathbf{x}_i\} \to \mathbf{a} \Rightarrow \{\boldsymbol{f}(\mathbf{x}_i)\} \to \boldsymbol{f}(\mathbf{a})$, *or*
- $\boldsymbol{f}(\mathbf{x}_0)$ *is an interior point of $\boldsymbol{f}(\Omega) \Rightarrow \mathbf{x}_0$ is an interior point of Ω.*

Example 5.1. *Show that the following function is continuous on $\mathbb{R}^2$:*

$$f(x, y) = \begin{cases} \dfrac{x^4 + y^4}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First suppose that $f(\mathbf{x})$ is an interior point of $f\left(\mathbb{R}^2\backslash\{0\}\right)$. Then $\mathbf{x} \in \mathbb{R}^2\backslash\{0\}$, which is an open set. Hence $\mathbf{x}$ is an interior point of $\mathbb{R}^2\backslash\{0\}$ and so by Theorem 5.1 f is continuous on $\mathbb{R}^2\backslash\{0\}$. Now we consider the point $(x, y) = (0, 0)$. Suppose $\|(x, y)\| < \delta$. Then

$$\begin{aligned} & \left|\frac{x^4 + y^4}{x^2 + y^2}\right| \leq \left|\frac{x^4 + 2x^2y^2 + y^4}{x^2 + y^2}\right| \\ \Rightarrow \quad & \left|\frac{x^4 + y^4}{x^2 + y^2}\right| \leq \left|x^2 + y^2\right| \\ \Rightarrow \quad & \qquad\qquad\quad < \delta^2. \end{aligned}$$

Setting $\delta = \sqrt{\epsilon}$ then for each choice of epsilon we have a choice of delta such that

$$\|(x, y)\| < \delta \Rightarrow \left|\frac{x^4 + y^4}{x^2 + y^2}\right| < \epsilon.$$

Hence by the definition of the limit,

$$\lim_{(x,y)\to(0,0)} f(x, y) = 0 = f(0),$$

and so f is continuous at $(0, 0)$. So f is continuous on $\mathbb{R}^2$. □

Much like the limit, we can also determine whether a function is continuous component-wise.

Theorem 5.2. *Suppose* $\boldsymbol{f} : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$, *with components* $f_1, f_2, ..., f_m$ *all continuous at* $\mathbf{x}_0$. *Then* $\boldsymbol{f}$ *is continuous at* $\mathbf{x}_0$.

This makes it much easier to determine if a vector-valued function is continuous. There is another way to classify continuity for multi-variable functions, using the pre-image.

Definition 5.2. *Suppose* $\boldsymbol{f} : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$. *Then we define the* ***pre-image*** *of* $U \subseteq \mathbb{R}^m$ *under* $\boldsymbol{f}$ *as*

$$\boldsymbol{f}^{-1}(U) = \{\mathbf{x} \in \Omega \mid \boldsymbol{f}(\mathbf{x}) \in U\}.$$

In other words, the pre-image of U is the set of all inputs that get mapped to U. We now use this definition to give the following theorem.

Theorem 5.3. *(Pre-Image Theorem).*
Suppose $\boldsymbol{f} : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$. *Then* $\boldsymbol{f}$ *is continuous* ***iff***

- $\forall U \subseteq \mathbb{R}^m$, U *is open* $\Rightarrow \boldsymbol{f}^{-1}(U)$ *is open, or*
- $\forall U \subseteq \mathbb{R}^m$, U *is closed* $\Rightarrow \boldsymbol{f}^{-1}(U)$ *is closed.*

This is called the **Pre-image Theorem** and is a very useful theorem for proving continuity, as well as proving a set to be open or closed.

Example 5.2. *Find whether the following set is open or closed or neither:*

$$S = \left\{(x, y) \in \mathbb{R}^2 \mid 0 < x \leq \sqrt{y}\right\}.$$

Proof. To find whether this set is open or closed or neither, we will define a continuous function $\boldsymbol{f} : \mathbb{R} \to \mathbb{R}^2$. Note that we choose to have domain $\mathbb{R}$ so it is simple to deduce whether $\boldsymbol{f}^{-1}(S)$ is open or closed or neither. If we define the value of $\boldsymbol{f}(x)$ to have second component constant, we can easily deduce the pre-image. So let $\boldsymbol{f} : \mathbb{R} \to \mathbb{R}^2$ be defined by

$$\boldsymbol{f}(x) = (x, 1).$$

Since the components of $\boldsymbol{f}$ are continuous, then $\boldsymbol{f}$ is continuous by Theorem 5.2. If $\boldsymbol{f}(x) \in S$ then $x \in (0, 1]$. Hence $\boldsymbol{f}^{-1}(S) = (0, 1]$ which is clearly neither open nor closed. Hence by Theorem 5.3 S is neither open nor closed. □

Note that an example of a subset of $\mathbb{R}^3$ would involve the same method, except we would define a function $\boldsymbol{f} : \mathbb{R} \to \mathbb{R}^3$ with two constant components.

6 Compact & Path-Connected Sets

In this final section we define two key classes of sets, and theorems based around them.

Definition 6.1. *Consider a set $\Omega \subseteq \mathbb{R}^n$. Then*

- *Ω is **bounded** if there exists $M > 0$ such that $\Omega \subseteq B(\mathbf{0}, M)$,*
- *Ω is **compact** if it is closed and bounded.*

Boundedness can be thought of as the idea of containing a set inside of a finite ball. The idea of compactness allows us to study the next theorem, known as the **Bolzano-Weierstrass Theorem**.

Theorem 6.1. *(Bolzano-Weierstrass Theorem).*
*Suppose $\Omega \subseteq \mathbb{R}^n$. Then the following statements are **equivalent**.*

- *Ω is compact.*
- *Each sequence in Ω has a convergent subsequence in Ω.*

Example 6.1. *Consider the set*

$$\Omega = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 2\right\}.$$

Define a sequence $\{\mathbf{x}_k\}$ by $\mathbf{x}_k = \left(i^k, (-1)^k\right)$. Does $\{\mathbf{x}_k\}$ converge in Ω? Does it has a subsequence which converges in Ω?

Proof. The sequence $\{\mathbf{x}_k\}$ clearly does not converge; it oscillates between the points $(1, 1), (i, -1), (-1, 1)$ and $(-i, -1)$. If we want to find a subsequence which converges in Ω we need to make sure the components are real, hence $k = 2n$. This gives us the subsequence $\{((-1)^n, 1)\}$ which still oscillates, but only between two points this time. So we can see that $k = 4n$ will give us a subsequence which converges in Ω, $\{(1, 1)\}$. $\square$

Now we define what it means for a set to be path-connected and simply-connected.

Definition 6.2. *Consider a set $\Omega \subseteq \mathbb{R}^n$. Then*

- *We say that Ω is **Path-Connected** if for each $\mathbf{x}, \mathbf{y} \in \Omega$ there exists a continuous function $\phi : [0, 1] \to \Omega$ such that $\phi(0) = \mathbf{x}$ and $\phi(1) = \mathbf{y}$.*
- *We say that Ω is **Simply-Connected** if any simple closed curve in Ω can be continuously shrunk down to a point while remaining in Ω.*

In $\mathbb{R}^2$, path-connected means that all points can be connected by some line and simply-connected means that the set has no 'holes'. We can see this more closely in the following example.

Example 6.2. *Is the set*

$$S = \left\{(x, y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2\right\}$$

path-connected? Is it simply-connected?

Proof. Choose two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ such that

$$\mathbf{x} = (r_1 \cos\theta_1, r_1 \sin\theta_1) \quad \text{and} \quad \mathbf{y} = (r_2 \cos\theta_2, r_2 \sin\theta_2)\,.$$

Then we can define $\phi : [0,1] \to \mathbb{R}^2$ by

$$\phi(t) = \left((tr_2 + (1-t)r_1)\cos\left(t\theta_2 + (1-t)\theta_1\right), (tr_2 + (1-t)r_1)\sin\left(t\theta_2 + (1-t)\theta_1\right)\right).$$

Then $\phi(0) = \mathbf{x}$ and $\phi(1) = \mathbf{y}$. Also, if we set $r = \min\{r_1, r_2\}$ and $R = \max\{r_1, r_2\}$ then

$$\begin{aligned}
& & \|\phi(t)\| = tr_2 + (1-t)r_1 \\
\Rightarrow & & tr + (1-t)r \leq \|\phi(t)\| \leq tR + (1-t)R \\
\Rightarrow & & r \leq \|\phi(t)\| \leq R \\
\Rightarrow & & 1 < \|\phi(t)\| < 2.
\end{aligned}$$

So $\phi(t) \in S$ for all $t \in [0,1]$, hence S is path-connected.
Any simple closed curve in S contains the set $B(\mathbf{0}, 1) \not\subset S$, so S is not simply connected. □

Compactness and Path-connectedness are special in the sense that these properties are preserved under a continuous map. This is outlined in the following theorem.

Theorem 6.2. *Let $\boldsymbol{f} : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be* ***continuous****. Then for some subset $K \subseteq \Omega$,*
- *K is compact $\Rightarrow \boldsymbol{f}(K)$ is compact.*
- *K is path-connected $\Rightarrow \boldsymbol{f}(K)$ is path-connected.*

This theorem can be useful when finding a continuous map between two sets.

Example 6.3. *Consider the sets*

$$\begin{aligned}
S_1 &= \left\{(x,y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 2\right\}, \\
S_2 &= \left\{(x,y) \in \mathbb{R}^2 \mid 1 \leq xy \leq 2\right\}.
\end{aligned}$$

Is there a continuous map from S_1 to S_2? Is there a continuous map from S_2 to S_1?

Proof. First note that S_1 is compact and path-connected, and S_2 is compact but not path-connected.
If we have a map $\boldsymbol{f}$ from S_1 to S_2 then $\boldsymbol{f}(S_1) = S_2$. However by Theorem 6.2, since S_1 is path-connected then S_2 must be path-connected - a contradiction. Hence there does not exist a continuous map from S_1 to S_2.
If we have a map $\boldsymbol{g}$ from S_2 to S_1 then $\boldsymbol{g}(S_2) = S_1$. By Theorem 6.2, since S_2 is compact then S_1 must be compact, which is true. Hence there may be a continuous function from S_2 to S_1. □