MATH1131/1141 Revision Sheet

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We would like to preface this document by saying that this resource is first and foremost meant to be used as a reference and should NOT be used as a replacement for the course notes or lecture recordings. The course notes provided on moodle are wonderfully written and contain an abundance of fully worked solutions and in depth explanations. Studying for this course using *only* this revision sheet would not be sufficient.

In addition, although the authors have tried their best to include everything essential taught in the course, it was ultimately up to their discretion on whether or not to include results/theorems/definitions etc. Anything that is missing is most definitely a conscious choice made by the authors.

Finally, any and all errors found within this document are most certainly our own. If you have found an error, please contact us via our Facebook page, or give us an email.

Sets, Inequalities & Functions

Essential Sets

- The set $\mathbb{N}$ of *natural numbers* is given by

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}.$$

  Note well that 0 is by convention included within $\mathbb{N}$ at UNSW. Other institutions and websites online might *not* include 0 within $\mathbb{N}$.

- The set $\mathbb{Z}$ of *integers* is given by

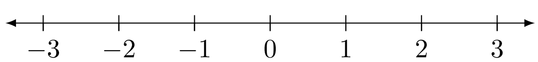
$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}.$$

  Think of $\mathbb{Z}$ as the set of all negative and positive whole numbers with zero included.

- The set $\mathbb{Q}$ of *rational numbers* is the collection of all numbers of the form p/q, where p and q are integers and $q \neq 0$.

  That is, $\mathbb{Q}$ is the set of all numbers that can be represented as fractions where the numerator and denominator are integers.

- The set $\mathbb{R}$ of *real numbers* may be represented as the collection of all points lying on the number line.



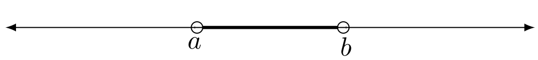
  The set $\mathbb{R}$ can be thought of as the metric completion of $\mathbb{Q}$. That is, it extends $\mathbb{Q}$ in such a way as to allow for the notion of distances to work and subsequently, that of limits and continuity.

Suppose that A and B are two sets. We say that A is a *subset* of B if $x \in A$ implies that $x \in B$. We write this as $A \subseteq B$.

Interval Notation

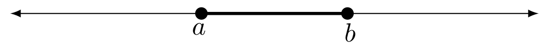
Whenever there is an interval, a round bracket means *excluding* and a square bracket means *including*. This is demonstrated below.

- $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ is represented on the number line as:



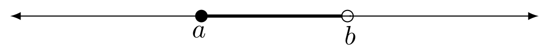
  This set contains all the real numbers between a and b *excluding both* a and b.

- $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ is represented on the number line as:



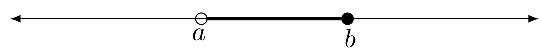
  This set contains all the real numbers between a and b *including both* a and b.

- $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ is represented on the number line as:



  This set contains all the real numbers between a and b *including* a but *excluding* b.

- $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$ is represnted on the number line as:



  This set contains all the real numbers between a and b *excluding* a but *including* b.

Note well that if there is an interval containing infinity, then the bracket adjacent to the ∞ should always be round because there is no $x \in \mathbb{R}$ such that $x = \infty$.

Some terminology regarding these intervals are as follows:

- a and b are called the *endpoints*.
- An interval $[a, b]$ that includes *both* its endpoints is called *closed*.
- An interval (a, b) that excludes *both* its endpoints is called *open*.

Absolute Values

The absolute value of a real number x is denoted by $|x|$ and defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Some useful properties of $|x|$ when x is real are:

- $|-x| = |x|$,
- $|xy| = |x||y|$,
- $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$,
- $|x + y| \leq |x| + |y|$. This property is known as the *triangle inequality* and is *extremely* useful to know.

- $|x| = \sqrt{x^2}$ and $|x|^2 = x^2$.

- $|x-a|$ can be thought of as the distance between x and a on the real number line.

- The statement $|y| < a$ is equivalent to

$$-a < y < a.$$

Similarly, the statement $|y| > a$ is equivalent to

$$y < -a \text{ or } y > a.$$

Functions

A function is a rule, often denoted by f. It takes an element of one set, say A, and maps it to *exactly one element* belonging to the set B. This is commonly written as $f : A \to B$.

- The set A is called the *domain*. It is the set of all allowed inputs of the function f and can be written as $\text{Dom}(f) = A$.

- The set B is called the *co-domain*. This is the set that contains all possible outputs of f and can be written as $\text{Codom}(f) = B$.

- It is important to note that the *co-domain* is different from the *range*, which is the set of all actual outputs of f.

- The expression $f(x)$ is read as 'f of x'.

Function 'Operations'

Suppose that $f : A \to B$ and $g : A \to B$ are real valued functions. Then:

- $(f+g)(x) = f(x) + g(x)$ for all $x \in A$.

- $(f-g)(x) = f(x) - g(x)$ for all $x \in A$.

- $(f \cdot g)(x) = f(x)g(x)$ for all $x \in A$.

- $(f/g)(x) = \dfrac{f(x)}{g(x)}$ for all $x \in A$ provided that $g(x) \neq 0$.

- Suppose that $f : C \to D$ and $g : A \to B$ are functions such that $\text{Range}(g)$ is a subset of C. Then the *composition* $f \circ g : A \to D$ is defined by the rule

$$(f \circ g)(x) = f(g(x)) \text{ for all } x \in A.$$

Polynomials & Rational Functions

A function $f : \mathbb{R} \to \mathbb{R}$ is called a *polynomial* if it can be expressed in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a natural number, the constants $a_0, a_1, \ldots, a_n$ are real numbers and $a_n \neq 0$.

- We call n the *degree* of f.

- The constants $a_0, a_1, \ldots, a_n$ are called the *coefficients* of f.

- The constant a_n is called the *leading coefficient* of f.

Suppose p and q are polynomials. A function f is called a *rational function* if

$$\text{Dom}(f) = \{x \in \mathbb{R} : q(x) \neq 0\}$$

and

$$f(x) = \frac{p(x)}{q(x)}, \text{ for all } x \in \text{Dom}(f).$$

Trigonometric Functions & Identities

Defining Relations

- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

- $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$.

- $\sec\theta = \dfrac{1}{\cos\theta}$.

- $\csc\theta = \dfrac{1}{\sin\theta}$.

Pythagorean Identities

- $\cos^2\theta + \sin^2\theta = 1$.

- $1 + \tan^2\theta = \sec^2\theta$.

- $\cot^2\theta + 1 = \csc^2\theta$.

Sum & Difference Identities

- $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$.

- $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$.

- $\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$.

Complementary Angles

- $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$.
- $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$.
- $\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$.
- $\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$.
- $\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$.
- $\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$.

Double Angle Formulas

- $\sin 2x = 2\sin x \cos x$.
- $$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x \\ &= 2\cos^2 x - 1 \\ &= 1 - 2\sin^2 x.\end{aligned}$$
- $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$.

Periodicity of Trig Functions

- $\sin(\theta + 2\pi) = \sin\theta$.
- $\cos(\theta + 2\pi) = \cos\theta$.
- $\tan(\theta + \pi) = \tan\theta$.

Identities for Negative Angles

- $\sin(-\theta) = \sin\theta$.
- $\cos(-\theta) = \cos\theta$.
- $\tan(-\theta) = \cos\theta$.

Half Angle Formulas

- $\sin\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1 - \cos\theta}{2}}$.
- $\cos\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1 + \cos\theta}{2}}$.
- $\tan\dfrac{\theta}{2} = \dfrac{\sin\theta}{1 + \cos\theta} = \dfrac{1 - \cos\theta}{\sin\theta}$.

Limits

Limits of Functions "at" Infinity

The following results are relevant when one considers the behaviour of $f(x)$ as $x \to \infty$.

Technical Definition

Suppose that L is a real number and f is a real-valued function defined on some interval (b, ∞). We say that $\lim_{x\to\infty} f(x) = L$ if:

For every positive real number ε, there exists a real number M, such that if $x > M$ then $\lvert f(x) - L\rvert < \varepsilon$.

Properties

Suppose that $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ exist and are finite real numbers. Then the following hold:

(i) $\lim_{x\to\infty} [f(x) + g(x)] = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x)$.

(ii) $\lim_{x\to\infty} [f(x) - g(x)] = \lim_{x\to\infty} f(x) - \lim_{x\to\infty} g(x)$.

(iii) $\lim_{x\to\infty} [f(x)g(x)] = [\lim_{x\to\infty} f(x)][\lim_{x\to\infty} g(x)]$.

(iv) If $\lim_{x\to\infty} g(x) \neq 0$ then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)}.$$

The Pinching Theorem

Suppose that f, g and h are all defined on the interval (b, ∞), where $b \in \mathbb{R}$. If

$$f(x) \leq g(x) \leq h(x) \quad \forall x \in (b, \infty)$$

and

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} h(x) = L$$

then

$$\lim_{x\to\infty} g(x) = L.$$

Limits of Functions at a Point

The following results are relevant when one considers the behaviour of $f(x)$ as $x \to a$.

Definition

If the left-hand limit $\lim_{x\to a^-} f(x)$ and the right-hand limit $\lim_{x\to a^+} f(x)$ both exist and equal the same real number L, then we say that the limit of $f(x)$ as $x \to a$ exists and is equal to L, and we write

$$\lim_{x\to a} f(x) = L.$$

If any one of the above conditions fail, then we say that $\lim_{x\to a} f(x)$ does not exist.

Limits and Continuous Functions

- Suppose that f is defined on some interval containing the point a. If $\lim_{x \to a} f(x) = f(a)$, then we say f is *continuous* at a; otherwise, we say that f is *discontinuous* at a.

- If it is the case that $f : \mathbb{R} \to \mathbb{R}$ is continuous at every point a in $\mathbb{R}$ then we say that f is continuous everywhere.

- If $\lim_{x \to a} f(x) = L$ and g is continuous at L then

  $$\lim_{x \to a} g(f(x)) = g\left(\lim_{x \to a} f(x)\right).$$

  If the function f and g are continuous everywhere then the above result implies that

  $$\lim_{x \to a} g(f(x)) = g\left(\lim_{x \to a} f(x)\right)$$

  for any point a in $\mathbb{R}$.

Continuity Continued

Definition for Open Intervals

If f is defined on an open interval (a, b), then f is continuous on (a, b) if f is continuous at every point in (a, b).

Definition for Closed Intervals

Suppose f is defined on the closed interval $[a, b]$.
If

- f is continuous at the endpoint a;
- f is continuous at the endpoint b;
- f is continuous on the open interval (a, b);

then f is continuous on $[a, b]$.

Intermediate Value Theorem

Suppose that f is continuous on the closed interval $[a, b]$. If z lies between $f(a)$ and $f(b)$ then there is at least one real number c in $[a, b]$ such that $f(c) = z$.

The Maximum Minimum Theorem

If f is continuous on a closed interval $[a, b]$ then f attains it's minimum and maximum on $[a, b]$. That is there exists points c and d in $[a, b]$ such that

$$f(c) \leq f(x) \leq f(d)$$

for all x in $[a, b]$.

Definition of a Bounded Function

A function f is said to be *bounded* on an interval I if there is some positive number M such that $|f(x)| \leq M$ for all x in I.

Differentiable Functions

Definition of a Derivative

Suppose that f is defined on some open interval containing the point x. We say that f is differentiable at x if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. If the limit exists we denote it by $f'(x)$.

Standard Derivatives of Common Functions

$f(x)$	$f'(x)$
C, where C is a constant	0
x^n where n is a positive integer.	nx^{n-1}
$\sin(x)$	$\cos(x)$
e^x	e^x

Definition of Stationary Point

If a function f is differentiable at a point c and $f'(c) = 0$ then c is called the *stationary point* of f.

The Mean Value Theorem

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). Then there is at least one real number c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

The Sign of a Derivative

Suppose that f is defined on an interval I. We say that

(a) f *is increasing* on I if for x_1 and x_2 in I,

$$x_1 < x_2 \text{ implies that } f(x_1) < f(x_2);$$

(b) f *is decreasing* on I if for x_1 and x_2 in I,

$$x_1 < x_2 \text{ implies that } f(x_1) > f(x_2).$$

Second Derivative Test

Suppose that a function f is twice differentiable on (a, b) and that $c \in (a, b)$.

(i) If $f'(c) = 0$ and $f''(c) > 0$ then c is a local minimum point of f;

(ii) If $f'(c) = 0$ and $f''(c) < 0$ then c is a local maximum point of f.

Classification of Critical Points

Suppose that f is defined on $[a, b]$. We say that a point c in $[a, b]$ is a *critical point* for f on $[a, b]$ if c satisfies one of the following properties:

(a) c is an endpoint a or b of the interval $[a, b]$,

(b) f is not differentiable at c, or

(c) f is differentiable at c and $f'(c) = 0$.

N.B that the maximum and minimum values of a continuous function on a closed interval are critical values of said function.

L'Hôpital's Rule

Suppose that f and g are both differentiable functions and a is a real number. Suppose that either one of the following two conditions hold:

- $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$;
- $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$.

If

$$\lim_{x \to a} \frac{f'(x)}{g'(x)}$$

exists then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

This theorem holds when a is infinity and for one-sided limits as well.

Inverse Functions

Definition of One-to-One Functions

A function f is one-to-one (or *injective*) if

$$f(x_1) = f(x_2) \text{ implies that } x_1 = x_2$$

whenever $x_1, x_2 \in \mathrm{Dom}(f)$.

One can check for injectivity by using the *horizontal line test*. If every horizontal line crosses the graph $y = f(x)$ at most once, the function is injective.

Definition of Onto Functions

A function f is onto (or *surjective*) if for every y in $\mathrm{Codom}(f)$, there exists an x in $\mathrm{Dom}(f)$ such that $f(x) = y$ (i.e. $\mathrm{Codom}(f) = \mathrm{Range}(f)$).

One can similarly check for surjectivity by seeing if every horizontal line (within $\mathrm{Codom}(f)$) crosses the graph $y = f(x)$ at least once.

Definition of Inverse Function

Suppose that f is one-to-one and onto (or *bijective*). Then there exists a unique function f^{-1} satisfying

$$f^{-1}(f(x)) = x \quad \text{for all } x \in \mathrm{Dom}(f),$$

and

$$f(f^{-1}(x)) = x \quad \text{for all } x \in \mathrm{Range}(f).$$

Moreover,

$$\mathrm{Dom}(f^{-1}) = \mathrm{Range}(f), \quad \mathrm{Range}(f^{-1}) = \mathrm{Dom}(f)$$

and f^{-1} is also bijective.

Inverse Function Theorem

Suppose that I is an open interval, $f : I \to \mathbb{R}$ is differentiable and $f'(x) \neq 0$ for all x in I. Then,

(i) f is one-to-one and has an inverse function
$f^{-1} : \text{Range}(f) \to \text{Dom}(f)$,

(ii) f^{-1} is differentiable at all points in Range(f), and

(iii) the derivative of f^{-1} is given by the formula

$$[f^{-1}]'(x) = \frac{1}{f'(f^{-1}(x))}$$

for all x in Range(f).

Applications to Trig Functions

Function	Domain	Range	Derivative
$\sin x$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	$[-1, 1]$	$\cos x$
$\sin^{-1} x$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos x$	$[0, \pi]$	$[-1, 1]$	$-\sin x$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\tan x$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	$(-\infty, \infty)$	$\sec^2 x$
$\tan^{-1} x$	$(-\infty, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	$\dfrac{1}{1+x^2}$

Curve Sketching

Even and Odd Functions

- An *even* function is one which satisfies the identity, $f(x) = f(-x)$ for all x in Dom(f). Even functions are symmetric over the y-axis.

- An *odd* function is one which satisfies the identity, $f(-x) = -f(x)$ for all x in Dom(f). Odd functions have rotational symmetry around the origin.

Definition of Oblique Asymptotes

Suppose that a and b are real numbers and that $a \neq 0$. We say that a straight line, given by the equation

$$y = ax + b,$$

is an oblique asymptote for a function f if

$$\lim_{x\to\infty} (f(x) - (ax + b)) = 0$$

Summary of Cartesian and Parametric Forms of Conic Sections

Conic	Cartesian Form	Parametric Form
Parabola	$4ay = x^2$	$x(t) = 2at$ $y(t) = at^2$
Circle	$x^2 + y^2 = a^2$	$x(t) = a\cos t$ $y(t) = a\sin t$
Ellipse	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$	$x(t) = a\cos t$ $y(t) = b\sin t$
Hyperbola	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$	$x(t) = a\sec t$ $y(t) = b\tan t$

Derivative of a Function's Parametrisation

Suppose that x and y are both differentiable functions of t and that y is a function of x. If $x'(t) \neq 0$ then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y'(t)}{x'(t)}.$$

Polar Coordinates

A polar coordinate (r, θ) of a point P is related to the Cartesian coordinate (x, y) of P by the formulae:

- $x = r\cos\theta$.

- $y = r\sin\theta$.

- $r = \sqrt{x^2 + y^2}$.

- $\tan\theta = \dfrac{y}{x}$, provided that $x \neq 0$.

Integration

Definition of partition

A finite set $\mathcal{P}$ of points in $\mathbb{R}$ is said to be a partition of $[a, b]$ if

$$\mathcal{P} = \{a_0, a_1, a_2, \ldots, a_n\}$$

and

$$a = a_0 < a_1 < a_2 < \cdots < a_n = b.$$

Definition of Riemann Integral

Suppose that a function f is bounded on $[a, b]$ and that $f(x) \geq 0$ for all x in $[a, b]$. If there exists a unique real number I such that

$$\underline{S}_{\mathcal{P}}(f) \leq I \leq \overline{S}_{\mathcal{P}}(f) \quad \text{for every partition } \mathcal{P} \text{ of } [a, b],$$

then we say that f is Riemann Integrable on the interval $[a, b]$.

I is found by the definite integral of f from a to b, written as

$$I = \int_a^b f(x)\,\mathrm{d}x.$$

Properties of the Riemann Integral

Suppose that f and g are integrable functions over $[a, b]$.

(i) For all real numbers α, we have

$$\int_a^b (\alpha f + \beta g)(x)\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x + \beta \int_a^b g(x)\,\mathrm{d}x.$$

(ii) If $a < c < b$ then

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$$

(iii) If $f(x) \geq 0$ for all $x \in [a, b]$ then

$$\int_a^b f(x)\,\mathrm{d}x \geq 0.$$

(iv) If $f(x) \leq g(x)$ for all $x \in [a, b]$ then

$$\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x.$$

(v) If $m \leq f(x) \leq M$ for all $x \in [a, b]$ then

$$m(b - a) \leq \int_a^b f(x)\,\mathrm{d}x \leq M(b - a).$$

(vi) If $|f|$ is integrable on $[a, b]$, then

$$\left| \int_a^b f(x)\,\mathrm{d}x \right| \leq \int_a^b |f(x)|\,\mathrm{d}x.$$

(vii) For all $x \in [a, b]$ we have

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x.$$

(viii) For all $x \in [a, b]$ we have

$$\int_a^a f(x)\,\mathrm{d}x = 0.$$

The First Fundamental Theorem of Calculus

If f is a continuous function defined on $[a, b]$ then the function $F : [a, b] \to \mathbb{R}$, defined by

$$F(x) = \int_a^x f(t)\,\mathrm{d}t,$$

is continuous on $[a, b]$, differentiable on (a, b) and has a derivative F' given by

$$F'(x) = f(x)$$

for all $x \in (a, b)$.

The Second Fundamental Theorem of Calculus

Suppose that f is a continuous function on $[a, b]$. If F is an anti-derivative of f on $[a, b]$, then

$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a).$$

Integration by Parts

The indefinite integral version is

$$\int u \frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

while for definite integrals it is

$$\int_a^b u \frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = \big[uv\big]_a^b - \int_a^b v \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x.$$

The Comparison Test

Suppose that f and g are integrable functions and that $0 \leq f(x) \leq g(x)$ whenever $x > a$.

(i) If $\int_a^\infty g(x)\,\mathrm{d}x$ converges then $\int_a^\infty f(x)\,\mathrm{d}x$ also converges.

(ii) If $\int_a^\infty f(x)\,\mathrm{d}x$ diverges then $\int_a^\infty g(x)\,\mathrm{d}x$ also diverges.

The Limit Form of the Comparison Test

Suppose that f and g are non-negative and bounded on $[a, \infty)$. If

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L$$

and $0 < L < \infty$ then either

$$\int_a^\infty f(x)\,\mathrm{d}x \quad \text{and} \quad \int_a^\infty g(x)\,\mathrm{d}x$$

both converge or

$$\int_a^\infty f(x)\,\mathrm{d}x \quad \text{and} \quad \int_a^\infty g(x)\,\mathrm{d}x$$

both diverge.

The Logarithm and Exponential Functions

Definition of the Logarithm

Suppose that b is a positive real number not equal to one, c is a rational number and $a = b^c$. Then $\log_b a$ is defined by

$$\log_b a = c.$$

Properties of the Logarithm

In all the below, $x > 0, y > 0$ and $b > 0$ with $b \neq 1$.

- $\log_b 1 = 0$.
- $\log_b(xy) = \log_b x + \log_b y$.
- $\log_b(b^r) = r$.
- $\log_b(b) = 1$.
- $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$.
- $\log_b(x^r) = r\log_b(x)$.

Definition of the Natural Logarithm

The function $\ln : (0,\infty) \to \mathbb{R}$ is rigorously defined by the formula

$$\ln(x) = \int_1^x \frac{1}{x}\,\mathrm{d}t.$$

Properties of the Natural Logarithm

- $\ln$ is differentiable on $(0,\infty)$ and $\dfrac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \dfrac{1}{x}$.
- $\ln x > 0$ if $x > 1$, $\ln 1 = 0$ and $\ln x < 0$ if $x < 1$.
- $\ln x \to -\infty$ as $x \to 0^+$ and $\ln x \to \infty$ as $x \to \infty$.
- $\ln(xy) = \ln x + \ln y$ for all positive real numbers x and y.
- $\ln\left(\dfrac{x}{y}\right) = \ln x - \ln y$ for all positive real numbers x and y.
- $\ln(x^r) = r\ln(x)$ whenever r is a rational number and x is a positive real number.

Definition of the Number e

The real number e is defined to be the unique number x that satisfies the equation

$$\int_1^x \frac{1}{t}\,\mathrm{d}t = 1.$$

The Definition(s) of the Exponential Function

In the course notes the following definition is provided:

- The function $\exp : \mathbb{R} \to (0,\infty)$ is defined as the inverse of the function $\ln : (0,\infty) \to \mathbb{R}$.

This is necessary and sufficient to know for all students taking MATH1131/1141. However, if you're planning on taking more maths courses, the following definitions are equivalent to the one above, and are used just as regularly (but they are not necessary knowledge for this course).

- The function $\exp : \mathbb{R} \to (0,\infty)$ can be defined as the following power series,

  $$\exp(x) = \sum_{k=0}^\infty \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots.$$

  An explanation of this result can be found in MATH1B.

- The function $\exp : \mathbb{R} \to (0,\infty)$ can be defined as

  $$\exp(x) = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n.$$

  The keen eye can tell that this looks awfully similar to the compound interest formula.

Properties of the Exponential Function

- $\exp(\ln x) = x$ for all x in $(0,\infty)$ and $\ln(\exp x) = x$ for all x in $\mathbb{R}$.
- $\exp(1) = e$ and $\exp(0) = 1$.
- $\exp x \to \infty$ as $x \to \infty$ and $\exp x \to 0$ as $x \to -\infty$.
- $\exp$ is differentiable on $\mathbb{R}$ and $\dfrac{\mathrm{d}}{\mathrm{d}x}\exp x = \exp x$ for all x in $\mathbb{R}$. (This property can also be used as a definition for the exponential!)
- $\exp(x + y) = \exp(x)\exp(y)$ for all x and y in $\mathbb{R}$.
- $\exp(rx) = (\exp x)^r$ for every real number x and every rational number r.
- $\exp x = e^x$.

Change of Base Formula

Suppose that b is a positive real number, $b \neq 1$. Then,

$$\log_b(x) = \frac{\ln x}{\ln b}$$

for all x in $(0, \infty)$.

Hyperbolic Trigonometry

Definition of sinh and cosh

The function $\cosh : \mathbb{R} \to \mathbb{R}$ is defined as

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x}).$$

The function $\sinh : \mathbb{R} \to \mathbb{R}$ is defined as

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x}).$$

The Hyperbolic Identity

It is very easy to verify that

$$\cosh^2 x - \sinh^2 x = 1.$$

Other Hyperbolic Trig Functions

- $\tanh x = \dfrac{\sinh x}{\cosh x}$.
- $\coth x = \dfrac{\cosh x}{\sinh x}$.
- $\operatorname{sech} x = \dfrac{1}{\cosh x} x$.
- $\operatorname{cosech} x = \dfrac{1}{\sinh x}$.

Hyperbolic Identities

Difference of squares identities:

- $1 - \tanh^2 x = \operatorname{sech}^2 x$.
- $\coth^2 x - 1 = \operatorname{cosech}^2 x$.

Sum and difference formula:

- $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$.
- $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$.
- $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$.

Double-angle formula:

- $\sinh(2x) = 2 \sinh x \cosh x$.
- $\cosh(2x) = \cosh^2 x + \sinh^2 x$.
- $\tanh(2x) = \dfrac{2 \tanh x}{1 + \tanh x}$.

Hyperbolic Derivatives

If you know the definitions of these functions then the following results can be recovered very quickly.

- $\dfrac{\mathrm{d}}{\mathrm{d}x} \sinh(x) = \cosh(x)$.
- $\dfrac{\mathrm{d}}{\mathrm{d}x} \cosh(x) = \sinh(x)$.
- $\dfrac{\mathrm{d}}{\mathrm{d}x} \tanh(x) = \operatorname{sech}^2 x$.
- $\dfrac{\mathrm{d}}{\mathrm{d}x} \coth(x) = -\operatorname{cosech}^2(x)$.
- $\dfrac{\mathrm{d}}{\mathrm{d}x} \operatorname{sech}(x) = -\operatorname{sech} x \tanh x$.
- $\dfrac{\mathrm{d}}{\mathrm{d}x} \operatorname{cosech} x = -\operatorname{cosech} x \coth x$.

By the first fundamental theorem of calculus, one can immediately construct the integrals of the hyperbolic trig functions.

Inverse Hyperbolic Trig

- $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$ for all $x \in \mathbb{R}$.
- $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$ for all $x \in [1, \infty)$.
- $\tanh^{-1} x = \dfrac{1}{2} \ln \left(\dfrac{1 + x}{1 - x} \right)$ for all $x \in (-1, 1)$.

Derivatives of Inverse Hyperbolic Trig

- $\dfrac{\mathrm{d}}{\mathrm{d}x} \sinh^{-1}(x) = \dfrac{1}{\sqrt{x^2 + 1}}$.
- $\dfrac{\mathrm{d}}{\mathrm{d}x} \cosh^{-1}(x) = \dfrac{1}{\sqrt{x^2 - 1}}$.
- $\dfrac{\mathrm{d}}{\mathrm{d}x} \tanh^{-1}(x) = \dfrac{1}{1 - x^2}$.

Introduction to Vectors

- A **scalar** quantity can be represented by a single number while a **vector** quantity must have both a magnitude and a direction.
- Vector quantities are printed in bold ($\mathbf{u}$) and handwritten with either a tilde underneath ($\underset{\sim}{u}$) or an arrow above ($\vec{u}$).
- Vector and scalars have the following properties (where $\mathbf{u}$ and $\mathbf{v}$ are vectors and λ and μ are scalars):
  1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
  2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
  3. $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$
  4. $0\mathbf{v} = \mathbf{0}$
  5. $\lambda(\mu\mathbf{v}) = (\lambda\mu)\mathbf{v}$
  6. $(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$
  7. $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$

Vector quantities and $\mathbb{R}^n$

- Vectors are often expressed in coordinate form:
$$\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.$$
- Vector addition and scalar multiplication can be performed element-wise:
$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix}$$
and
$$\lambda\mathbf{a} = \lambda \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \lambda a_1 \\ \lambda a_2 \\ \vdots \\ \lambda a_n \end{pmatrix}.$$
- $\mathbb{R}^n$ (where n is a positive integer) is the set of vectors with n real components.
- The zero vector in $\mathbb{R}^n$, denoted by $\mathbf{0}$, is the vector with all n components 0.

Parallel

- Two vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if only $\mathbf{a} = \lambda\mathbf{b}$ for some non-zero real number λ.
- If $\lambda > 0$ the vectors are in the same direction and if $\lambda < 0$, the vectors are in opposite directions.

Magnitude

- The magnitude of a vector $\mathbf{a} \in \mathbb{R}^n$ is defined by
$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$
$|\mathbf{a}| \geq 0$ with equality if and only if $\mathbf{a} = \mathbf{0}$.
- $|\lambda\mathbf{a}| = |\lambda||\mathbf{a}|$ for all $\lambda \in \mathbb{R}$
- A vector $\mathbf{a}$ is a unit vector if $|\mathbf{a}| = 1$.
- For geometric vectors, the magnitude can be interpreted as the length of the vector.
- The distance between points with coordinate vectors $\mathbf{a}$ and $\mathbf{b}$ is given by $|\mathbf{a} - \mathbf{b}|$.

Lines

A line in $\mathbb{R}^n$ is a set of vectors in the form
$$S = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{a} + \lambda\mathbf{v}, \lambda \in \mathbb{R}^n\}$$
where $\mathbf{a}$ and $\mathbf{v} \neq 0$ are fixed vectors in $\mathbb{R}^n$.

- $\mathbf{x} = \mathbf{a} + \lambda\mathbf{v}, \lambda \in \mathbb{R}$ is known as the parametric vector form.
- Restricting λ can give rays and line segments.

Linear Combinations

A linear combination of vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ is a vector of the form
$$\lambda_1\mathbf{v_1} + \lambda_2\mathbf{v_2} + \cdots + \lambda_n\mathbf{v_n}$$
where $\lambda_i \in \mathbb{R}$ for $i = 1, 2, \ldots n$

- The vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ are linearly independent if the only solution to the equation
$$\lambda_1\mathbf{v_1} + \lambda_2\mathbf{v_2} + \cdots + \lambda_n\mathbf{v_n} = \mathbf{0}$$
is $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$. Otherwise the vectors are linearly dependent, and each vector can be written as a linearly combination of the other vectors.
- The span of a set S, denoted by $\text{span}(S)$, is the set of all linear combinations of vectors in S. That is,
$$\text{span}(S) = \{\mathbf{x} \in \mathbb{R} : \mathbf{x} = \lambda_1\mathbf{v_1} + \cdots + \lambda_n\mathbf{v_n}, \\ \text{for all } \lambda_1, \ldots, \lambda_n \in \mathbb{R}\}.$$

Vector Geometry

Dot Product

The dot product of two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \ldots a_n b_n = \sum_{k=1}^{n} a_k b_k$$

Properties of the Dot Product

For any two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$,

- $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
- $\mathbf{a} \cdot \mathbf{b} \in \mathbb{R}$
- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- $\mathbf{a} \cdot (\lambda \mathbf{b}) = \lambda(\mathbf{a} \cdot \mathbf{b})$
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
- The angle θ between $\mathbf{a}$ and $\mathbf{b}$ satisfies

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}, \text{ where } \theta \in [0, \pi]$$

- (Cauchy-Schwarz Inequality)

$$-|\mathbf{a}||\mathbf{b}| \leq \mathbf{a} \cdot \mathbf{b} \leq |\mathbf{a}||\mathbf{b}|$$

- (Triangle Inequality)

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$$

Orthogonality and Projections

- Two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ are orthogonal if $\mathbf{a} \cdot \mathbf{b} = 0$.
- An orthonormal set of vectors in $\mathbb{R}^n$ is a set of mutually orthogonal unit vectors.
- For any two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ and $\mathbf{b} \neq 0$, the projection of $\mathbf{a}$ on $\mathbf{b}$ is the unique vector $\lambda \mathbf{b}$ that satisfies

$$(\mathbf{a} - \lambda \mathbf{b}) \cdot \mathbf{b} = 0.$$

- The formula is given by

$$\text{proj}_{\mathbf{b}} \mathbf{a} = \left(\frac{\mathbf{a} \cdot \mathbf{a}}{|\mathbf{b}|^2} \right) \mathbf{b} = \lambda \mathbf{b}.$$

Cross Product

The cross product of two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ is defined by

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}.$$

- $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$ and $\mathbf{b}$.
- The cross product is only defined in $\mathbb{R}^3$.
- $\mathbf{a} \times \mathbf{a} = \mathbf{0}$
- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
- $\mathbf{a} \times (\lambda \mathbf{b}) = \lambda(\mathbf{a} \times \mathbf{b})$ and $(\lambda \mathbf{a}) \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b})$ for all $\lambda \in \mathbb{R}$
- $\mathbf{a} \times (\lambda \mathbf{a}) = \mathbf{0}$
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ and $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$ for all $\mathbf{c} \in \mathbb{R}^3$
- The standard basis in $\mathbb{R}^3$ satisfies

$$\mathbf{e_1} \times \mathbf{e_2} = \mathbf{e_3}, \mathbf{e_1} \times \mathbf{e_3} = \mathbf{e_2}, \mathbf{e_2} \times \mathbf{e_3} = \mathbf{e_1}$$

- $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$
- $|\mathbf{a} \times \mathbf{b}|$ is the area of a parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$

Scalar Triple Product

For $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$, the scalar triple product is

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

- $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c})$
- $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b})$
- $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \times \mathbf{a}) \cdot \mathbf{b} = 0$
- $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$ is the volume of a parallelepiped spanned by $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$

Planes

Let $\mathbf{a}, \mathbf{v_1}$ and $\mathbf{v_2}$ be fixed vectors in $\mathbb{R}^n$ and suppose $\mathbf{v_1}$ and $\mathbf{v_2}$ are not parallel. Then the set

$$S = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{a} + \lambda_1 \mathbf{v_1} + \lambda_2 \mathbf{v_2}, \text{ for all } \lambda_1, \lambda_2 \in \mathbb{R}\}$$

is the plane through the position vector $\mathbf{a}$, parallel to $\mathbf{v1}$ and $\mathbf{v2}$.

Equations of planes in $\mathbb{R}^3$

- Parametric vector form: $\mathbf{x} = \mathbf{a} + \lambda_1 \mathbf{v_1} + \lambda_2 \mathbf{v_2}$
- Cartesian form: $a_1x_1 + a_2x_2 + a_3x_3 = b$
- Point-normal form: $\mathbf{n} \cdot (\mathbf{x} - \mathbf{c}) = 0$

Projections onto Planes

For $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ with $\mathbf{b}$ and $\mathbf{c}$ not parallel, the projection of $\mathbf{a}$ onto $\pi = \text{span}(\mathbf{b}, \mathbf{c})$ is

$$\text{proj}_\pi \mathbf{a} = \lambda \mathbf{b} + \mu \mathbf{c}$$

where $(\mathbf{b}\cdot\mathbf{b})\lambda + (\mathbf{b}\cdot\mathbf{c})\mu = \mathbf{b}\cdot\mathbf{a}$ and $(\mathbf{c}\cdot\mathbf{b})\lambda + (\mathbf{c}\cdot\mathbf{c})\mu = \mathbf{c}\cdot\mathbf{a}$.

Complex Numbers

Introduction to Complex Numbers

- Polynomial equations like $x^2 + 1 = 0$ have no solution over the real numbers. It would be useful to extend the number system to account for these roots. This can be solved by defining $i = \sqrt{-1}$.
- A complex number is a number of the form $z = a + ib$ where $a, b \in \mathbb{R}$. The set of complex numbers is denoted by $\mathbb{C}$.
- The real part of $z = a + ib$, where $a, b \in \mathbb{R}$ is given by $\text{Re}(z) = a$ and the imaginary part is $\text{Im}(z) = b$.
- Suppose $z = a + ib$ and $w = c + id$ where $a, b, c, d \in \mathbb{R}$. Then:
  1. $z + w = (a + c) + i(b + d)$
  2. $z - w = (a - c) + i(b - d)$
  3. $zw = (ac - bd) + i(ad + bc)$
  4. $\frac{z}{w} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$

Complex Conjugate

If $z = a + ib$ where $a, b \in \mathbb{R}$ then the complex conjugate of z is given by

$$\bar{z} = a - bi$$

- $\bar{\bar{z}} = z$
- $\text{Re}(z) = \frac{1}{2}(z + \bar{z})$ and $\text{Im}(z) = \frac{1}{2i}(z - \bar{z})$
- $z\bar{z}$ is a nonnegative real number
- $\overline{z + w} = \bar{z} + \overline{w}$ and $\overline{z - w} = \bar{z} - \overline{w}$
- $\overline{zw} = \bar{z}\bar{w}$ and $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\overline{w}}$

The Argand Diagram

A complex number $z = a + ib$ can be represented on the x-y plane with the coordinates (a, b). The x-axis is the real axis and y-axis the imaginary axis.

Polar Form and Euler's Formula

- The modulus of a complex number $z = x + iy$ is given by
$$|z| = \sqrt{x^2 + y^2}.$$
This can be interpreted as the distance from the origin on the Argand diagram.
- The principal argument of $z = x + iy, z \neq 0$, written $\text{Arg}(z)$ is the angle θ such that
$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}},\ \sin\theta = \frac{y}{\sqrt{x^2+y^2}} \text{ and } -\pi < \theta \leq \pi.$$
Geometrically, this is the angle z makes with the positive real axis.
- (Euler's Formula) For real θ,
$$e^{i\theta} = \cos\theta + i\sin\theta.$$
- The polar form of a complex number z is $re^{i\theta} = r(\cos\theta + i\sin\theta)$ where $r = |z|$ and θ is the argument of z.
- $r_1e^{i\theta_1} \times r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$
- (De Moivre's Theorem)
$$(re^{i\theta})^n = r^n e^{in\theta}$$

Complex polynomials

- A complex polynomial is a function $p : \mathbb{C} \to \mathbb{C}$
$$p(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$$
where $n \in \mathbb{N}$ and $a_0, \ldots, a_n \in \mathbb{C}$.
- The degree of the polynomial $p(z) = \sum_{k=0}^n a_kz^k$, written $\deg(p)$ is the largest integer k such that $a_k \neq 0$.
- A number α is a root (or zero) of a polynomial p if $p(\alpha) = 0$.
- If there exists polynomials p_1 and p_2 such that $p(z) = p_1(z)p_2(z)$ for all $z \in \mathbb{C}$ then p_1 and p_2 are factors of p.
- (Fundamental Theorem of Algebra) Every non-constant polynomial has at least one root.

- (Factorisation Theorem) Every non-constant polynomial has a factorisation into n linear factors.

$$p(z) = a(z - \alpha_1)(z - \alpha_2)\dots(z - \alpha_n)$$

where $\alpha_1, \dots, \alpha_n$ are roots of p and a is the leading coefficient.

- If α is a root of a polynomial p with real coefficients, then $\bar{\alpha}$ is also a root.

- A polynomial with real coefficients can be factorised into linear and quadratic factors with real coefficients. This uses the fact that

$$(z - \alpha)(z - \bar{\alpha}) = z^2 - (\alpha + \bar{\alpha})z + \alpha\bar{\alpha}$$

where $\alpha + \bar{\alpha} = 2\mathrm{Re}(\alpha)$ and $|\alpha|^2$ are real.

Matrices

Introduction to matrices

An $m \times n$ matrix A is a rectangular array with m rows and n columns. The set of all $m \times n$ matrices with real elements is denoted by $M_{m,n}(\mathbb{R})$.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

- a_{ij} denotes the element in the ith row and jth column.
- The size of A is $m \times n$.
- If $m = n$, then A is a square matrix.
- The diagonal of a square matrix refers to the entries from the top left to bottom right.
- A and B are equal if and only if they have the same size and $a_{ij} = b_{ij}$ for all i, j.
- A column vector is an $n \times 1$ matrix and a row vector is a $1 \times m$ matrix.

Matrix Addition

Suppose $A, B, C \in M_{m,n}$

- The sum $M = A + B$ is the $m \times n$ matrix such that
$$m_{ij} = a_{ij} + b_{ij}$$
for all i, j.

- Matrix addition is commutative
$$A + B = B + A$$

- Matrix addition is associative
$$(A + B) + C = A + (B + C)$$

- A zero matrix $\mathbf{0}$ is a matrix with all entries being zero.
$$A + \mathbf{0} = A$$
Note that here it is assumed that the zero matrix is of size $m \times n$.

Scalar Multiplication

Suppose $A, B, C \in M_{m,n}(\mathbb{R})$ and λ, μ are scalars.

- The scalar multiple $M = \lambda A$ is the $m \times n$ matrix such that
$$m_{ij} = \lambda a_{ij}$$
for all i, j.

- Scalar multiplication is associative
$$\lambda(\mu A) = (\lambda\mu)A$$

- Scalar multiplication is distributive over scalar addition
$$(\lambda + \mu)A = \lambda A + \mu A$$

- Scalar multiplication is distributive over matrix addition
$$\lambda(A + B) = \lambda A + \lambda B$$

Matrix multiplication

- If $A \in M_{m,n}(\mathbb{R})$ and $B \in M_{n,p}(\mathbb{R})$, then the product $M = AB$ is the $m \times p$ matrix with entries given by
$$m_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

- In general, $AB \neq BA$.
- If AB exists, then $A(\lambda B) = \lambda(AB) = (\lambda A)B$.
- Matrix multiplication is associative. If AB and BC exist, then
$$(AB)C = A(BC)$$

- Distributive laws $(A + B)C = AC + BC$ and $A(B + C) = AB + AC$ hold.
- An identity matrix I is a square matrix with 1's on the diagonal and 0's everywhere else.
- $AI = A$ and $IA = A$ where the I's represent identity matrices of appropriate sizes.

Transpose

Suppose $A, B, C \in M_{m,n}$ and λ, μ are scalars.

- The transpose of A, denoted by A^T, is the $n \times m$ matrix with entries given by
$$(a^T)_{ij} = a_{ji}.$$
- For two vectors $\mathbf{a}, \mathbf{b}$, the scalar product can also be written as $\mathbf{a}^T\mathbf{b}$.
- $(A^T)^T = A$.
- If $A, B \in M_{m,n}$ and λ, μ are scalars, then
$$(\lambda A + \mu B)^T = \lambda A^T + \mu B^T.$$
- If AB exists, then
$$(AB)^T = B^T A^T.$$
- A matrix A is said to be symmetric if $A = A^T$.

Inverse Matrix

- A matrix X is a left inverse of A if $XA = I$. A matrix Y is a right inverse of A if $AY = I$.
- If A has both a left inverse X and a right inverse Y, then $X = Y$.
- A matrix A is invertible if there exists a matrix X such that $XA = I$ and $AX = I$. X is the inverse of A, denoted by A^{-1}.
- An invertible matrix is called a non-singular matrix.
- Only square matrices are invertible.
- If A is invertible, then A^{-1} is also invertible and
$$(A^{-1})^{-1} = A.$$
- If A and B are invertible and AB exists, then
$$(AB)^{-1} = B^{-1}A^{-1}.$$
- A matrix A is invertible if and only if it can be reduced by elementary row operations to an identity matrix I. If $(A|I)$ can be reduced to $(I|B)$, then $B = A^{-1}$.
- If A is a square matrix, then A is invertible if and only if $A\mathbf{x} = 0$ implies $\mathbf{x} = 0$.
- Let A be an $n \times n$ matrix. Then A is invertible if and only if the matrix equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for all vectors $\mathbf{b} \in \mathbb{R}^n$. The unique solution is $\mathbf{x} = A^{-1}\mathbf{b}$.

Elementary Row Operations

The three elementary row operations are:

- Multiplying row i by a scalar λ.
- Interchanging rows i and j.
- Adding λ times of row i to row j.

These operations can be effected by left multiplication by an invertible matrix.

Determinants

- The determinant of a 2×2 matrix
$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ is } \det(A) = |A| = a_{11}a_{22} - a_{12}a_{21}$$
- For a matrix A, the (row i, column j) minor, denoted by $|A_{ij}|$, is the determinant of the matrix obtained from A by deleting row i and column j.
- The determinant of an $n \times n$ matrix A is
$$|A| = \sum_{k=1}^{n} (-1)^{k+1} a_{1k} |A_{1k}|$$
- $\det(A^T) = \det(A)$
- Interchanging two rows (or two columns) reverses the sign of the determinant.
- If a matrix contains a zero row or column, then the determinant is zero.
- Multiplying a row (or column) by a scalar λ multiplies the determinant by λ. Hence
$$\det(\lambda A) = \lambda^n \det(A).$$
- Adding a multiple of a row (or column) to another row (or column) does not change the determinant.
- If A and B are square matrices such that AB exists, then
$$\det(AB) = \det(A)\det(B).$$
- The determinant of a row-echelon form matrix is the product of the diagonal entries.
- A square matrix A is invertible if and only if $\det(A) \neq 0$.
- If A is invertible, then
$$\det(A^{-1}) = \frac{1}{\det(A)}.$$