MATH1231/1241 Lab Test 2 Algebra Sample Solutions

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We cannot guarantee that our answers are correct - please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are sometimes multiple methods of solving the same question.

NOTE: Any text presented `like this` is presented as required in Maple syntax.

Question 1

Let $S = \{v_1, v_2, v_3, v_4, v_5\}$ be a set of five vectors in $\mathbb{R}^3$. Let $W = \text{span}\{S\}$. When these vectors are placed as columns into a matrix A as $A = (v_1 \mid v_2 \mid v_3 \mid v_4 \mid v_5)$ and A is row-reduced to echelon form U, we have

$$U = \begin{pmatrix} 1 & -4 & -3 & -3 & -2 \\ 0 & 1 & 3 & -2 & 1 \\ 0 & 0 & 1 & -3 & 3 \end{pmatrix}.$$

1. *State the dimension of W.*
2. *State a basis B for W, using vectors $\mathbf{v}_i$ with i as small as possible.*
3. *Express $\mathbf{v}_5$ as a linear combination of the basis vectors in B.*

$W = \text{span}\{S\} = A\mathbf{x}$, $\mathbf{x} \in \mathbb{R}^5$. Since A can be row-reduced to echelon form U and U has 3 leading columns, then W has dimension 3.

Since $\dim(\text{span}\{S\}) = 3$ and the first three columns of U are leading columns, we can take the first three vectors of S to form a basis of W: $B =$`{v1, v2, v3}`.

If we take $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)^T$ then $A\mathbf{x} = \mathbf{0}$ can be row reduced to $U\mathbf{x} = \mathbf{0}$:

$$(A \mid \mathbf{0}) \sim \left(\begin{array}{ccccc|c} 1 & -4 & -3 & -3 & -2 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 3 & 0 \end{array}\right).$$

Let $x_4 = 0$ and $x_5 = 1$. Hence we have $x_3 = -3$, $x_2 = 8$ and $x_1 = 25$, i.e. $25\mathbf{v}_1 + 8\mathbf{v}_2 - 3\mathbf{v}_3 + \mathbf{v}_5 = \mathbf{0}$. So, $\mathbf{v}_5 =$`-25*v1-8*v2+3*v3`.

Question 7

1. *Find the nullity of the matrix*

$$\begin{pmatrix} -16 & 100 & -36 & -72 & -188 \\ 160 & 40 & 112 & 164 & 224 \\ -126 & -175 & -194 & -116 & 205 \\ -46 & 55 & -184 & 122 & 511 \end{pmatrix}.$$

2. *Select all statements below which are true.*
   - $(-1, -2, 1, -2, 1)^T$ *is in* $\ker A$.
   - $(0, 0, 0, 0)^T$ *is in* $\ker A$.
   - $(54, -20, -56, 39)^T$ *is in* $\ker A$.
   - $\ker A$ *is a subspace of* $\mathbb{R}^5$.
   - $(-1, 1, 2, -2, 1)^T$ *is in* $\ker A$.
   - $(46, 60, -119, 16)^T$ *is in* $\ker A$.
   - $\ker A$ *is a subspace of* $\mathbb{R}^4$.
   - $(0, 0, 0, 0, 0)^T$ *is in* $\ker A$.

By defining A in Maple as given, we can find a basis for the kernel of A using the command

`NullSpace(A)`. This gives us the basis

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 2 \\ -2 \\ 1 \end{pmatrix} \right\}.$$

So $\text{nullity}(A) = 1$.

If $\mathbf{x} \in \ker(A)$ then $\mathbf{x} \in \mathbb{R}^5$. Hence $\ker(A)$ is a subspace of $\mathbb{R}^5$, and $(0,0,0,0,0)^T$ is in $\ker(A)$. From our basis of $\ker(A)$, we know that $(-1,1,2,-2,1)^T$ is in $\ker(A)$. $(-1,-2,1,-2,1)^T$ is not a multiple of $(-1,1,2,-2,1)^T$ so it is not in $\ker(A)$. All vectors in $\mathbb{R}^4$ cannot be in $\ker(A)$.

Question 9

1. *Find the rank of the matrix*

$$\begin{pmatrix} -44 & -128 & -14 & -92 & -72 \\ 122 & -65 & -170 & 44 & 615 \\ 158 & 77 & 190 & -192 & -683 \\ -58 & 87 & 106 & 10 & -337 \end{pmatrix}.$$

2. *Select all statements below which are true.*
   - $(0,0,0,0,0)^T$ *is in* $\operatorname{im} A$.
   - $\operatorname{im} A$ *is a subspace of* $\mathbb{R}^4$.
   - $(-2,2,4,-4,2)^T$ *is in* $\operatorname{im} A$.
   - $(0,0,0,0)^T$ *is in* $\operatorname{im} A$.
   - $(-70,-99,89,34)^T$ *is in* $\operatorname{im} A$.
   - $\operatorname{im} A$ *is a subspace of* $\mathbb{R}^5$.
   - $(2,4,-2,4,-2)^T$ *is in* $\operatorname{im} A$.
   - $(-53,-63,-1,58)^T$ *is in* $\operatorname{im} A$.

Define A in Maple as the given matrix. Then by using the Maple command `Rank(A)`, we can find that the rank of A is 3.

For some $\mathbf{x} \in \mathbb{R}^5$, $A\mathbf{x} \in \mathbb{R}^4$. Hence $\text{im}(A)$ is a subspace of $\mathbb{R}^4$, and $(0,0,0,0)$ is in $\text{im}(A)$. If we enter the line `LinearSolve(A,<-70,-99,89,34>);` into Maple, then we receive the error "Inconsistent System" which means no solution. So $(-70,-99,89,34)^T$ is not in $\text{im}(A)$. Entering `LinearSolve(A,<-53,-63,-1,58>);` into Maple, we find a solution and so $(-53,-63,-1,58)^T$ is in $\text{im}(A)$.

Question 11

Let A be a 3×4 matrix with column vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4$ so

$$A = (\boldsymbol{a}_1 \mid \boldsymbol{a}_2 \mid \boldsymbol{a}_3 \mid \boldsymbol{a}_4).$$

A has row-reduced echelon form

$$U = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

1. *State the vales of* $\operatorname{rank} A$ *and* $\operatorname{nullity} A$.
2. *Find a basis for the column space of A,* $\operatorname{col} A$.

Since A can be row-reduced to echelon form U, which has two leading columns, then $\text{rank}(A) = 2$. By the Rank-Nullity Theorem, $\text{nullity}(A) = 2$.

The column space of A is given by the columns of A which correspond to the leading columns with non-zero coefficients, which in this case are the first two column vectors. Hence a basis for $\text{col}(A) =$`{a1,a2}`.

Question 18

Let A be a 3×5 matrix with column vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5$ so

$$A = (\boldsymbol{a}_1 \mid \boldsymbol{a}_2 \mid \boldsymbol{a}_3 \mid \boldsymbol{a}_4 \mid \boldsymbol{a}_5).$$

A has row-reduced echelon form

$$U = \begin{pmatrix} 1 & 0 & -1 & 0 & -4 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -3 \end{pmatrix}.$$

1. *State the values of* $\operatorname{rank} A$ *and* $\operatorname{nullity} A$.
2. *Find a basis for the kernel or nullspace of* A, $\ker A$.

A can be row-reduced to echelon form U, where U has 3 leading columns. Then $\operatorname{rank}(A) = 3$ and $\operatorname{nullity}(A) = 2$.

To find a basis for the kernel of A, we should consider the row echelon form of $A\mathbf{x} = \mathbf{0}$. Since we have the echelon form of A, we can row-reduce $A\mathbf{x} = \mathbf{0}$ to

$$(A \mid \mathbf{0}) \sim \left(\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & -4 & 0 \\ 0 & 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 \end{array}\right).$$

Taking $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)^T$, we have

$$\mathbf{x} = x_3 \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 4 \\ -2 \\ 0 \\ 3 \\ 1 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \\ 0 \\ 3 \\ 1 \end{pmatrix} \right\}.$$

Hence a basis for $\ker(A) =$ {<1,-1,1,0,0>,<4,-2,0,3,1>}.

Question 25

The 2×2 matrix

$$A = \begin{pmatrix} -5 & 3 \\ 8 & 0 \end{pmatrix}$$

has two distinct real eigenvalues.

1. *Give the characteristic polynomial for* A.
2. *Find the set of eigenvalues for* A.
3. *Find one eigenvector for each eigenvalue.*

Characteristic polynomial $p(t) = \det(A - tI) =$`t^2+5*t-24`.

The eigenvalues are the solutions to the characteristic polynomial, so the set of eigenvalues for $A =$`{-8, 3}`.

If $\mathbf{v}_1$ and $\mathbf{v}_2$ are the eigenvectors for -8 and 3 respectively, then we must consider the matrix equation $(A - tI)\mathbf{v} = \mathbf{0}$. For $t = -8$, we have

$$\begin{pmatrix} 3 & 3 \\ 8 & 8 \end{pmatrix} \mathbf{v}_1 = \mathbf{0}.$$

Hence $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. For $t = 3$ we have

$$\begin{pmatrix} -8 & 3 \\ 8 & -3 \end{pmatrix} \mathbf{v}_2 = \mathbf{0}.$$

Hence $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 8 \end{pmatrix}$.

Question 26

The 2×2 matrix $A = \begin{pmatrix} 0 & 3 \\ -4 & 7 \end{pmatrix}$ *has eigenvalue/eigenvector pairs:*

$$3, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \textit{and} \qquad 4, \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

1. *Write down an expression in Maple notation for the general solution to the vector-values function of a real variable t,*

$$\boldsymbol{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

to the first order system using two arbitrary constants c1 and c2:

$$\frac{dy_1}{dt} = 3y_2$$
$$\frac{dy_2}{dt} = -4y_1 + 7y_2.$$

2. *Use the initial condition condition* $\boldsymbol{y}(0) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ *to determine the constants c1 and c2 and find the solution* $\boldsymbol{y}(t)$.

Since we know the eigenvalues and eigenvectors, we can plug them into the general solution: $\mathbf{y}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Given our initial condition, we can find the coefficients by solving the matrix equation

$$\left(\begin{array}{cc|c} 1 & 3 & -1 \\ 1 & 4 & -1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 3 & -1 \\ 0 & 1 & 0 \end{array}\right).$$

This gives us $c_1 = -1$ and $c_2 = 0$. So $\mathbf{y}(t) = -e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Question 27

Let A be a 2×2 matrix with eigenvalue, eigenvector pairs:

$$-5, \begin{pmatrix} -4 \\ 3 \end{pmatrix} \quad \textit{and} \quad 2, \begin{pmatrix} 4 \\ -2 \end{pmatrix}.$$

1. *Find an invertible matrix M and a diagonal matrix D such that* $A = MDM^{-1}$.
2. *For any integer n, find the matrix* A^n *as a single matrix.*

When diagonalising a matrix, $M = (\mathbf{v}_1|\mathbf{v}_2)$ and $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Hence

$$M = \begin{pmatrix} -4 & 4 \\ 3 & -2 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} -5 & 0 \\ 0 & 2 \end{pmatrix}.$$

Since $A = MDM^{-1}$ then $A^n = MD^nM^{-1}$. Hence by defining M and $F = D$ in Maple (since D is a restricted variable), we can use the Maple command `A:=M.MatrixPower(F,n).M^(-1)` to find that

$$A^n = \begin{pmatrix} -2(-5)^n + 3.2^n & -4(-5)^n + 4.2^n \\ \frac{-3}{2}(-5)^n - 3.2^{n-1} & 3(-5)^n - 2^{n+1} \end{pmatrix}.$$

So $A^n =$`<<-2*(-5)^n+3*2^n,(-3/2)*(-5)^n-3*2^(n-1)>|<-4*(-5)^n+4*2^n,3*(-5)^n-2^(n+1)>>`.

Question 28

1. *Construct a "for loop" in order to evaluate the sum*

$$\sum_{n=12}^{23} \sin\left(\frac{k}{n}\right)$$

for k from 2 to 50.
Note: *This question is one in which we must select the correct options from drop down menus. I will present the correct sequence of lines below. After, I will present some logic.*

2. *Consider the sequence $\{a_n\}$ generated by the recurrence relation*

$$a_{n+1} = a_n - 5a_{n-1} + a_{n-2} \quad \textit{for} \quad n\text{=}3,4,5,\ldots$$

given that $a_1 = 4$, $a_2 = -1$, and $a_3 = 1$. Write a for loop to find the value of a_{60}.

In order to get the desired outcome in Maple, enter:
`for k from 2 to 50 do evalf(add(sin(k/n), n=12..23)) end do;`.
We need `for k from 2 to 50` because the question specifies that the **sequence** is with respect to k. Then, we evaluate `add(sin(k/n), n=12..23))` because the **sum** is with respect to n.

The recurrence relation for loop can be written as:

```
f:= proc(n)
local a,i;
a[1]:= 4;
a[2]:= -1;
a[3]:= 1;
for i from 4 to n do
a[i]:= a[i-1]-5*a[i-2]+a[i-3];
end do;
return a[n];
end proc;
```

Then entering `f(60)` will give us a_{60} =-108431555483984760558.

The first line declares a function f of n. Then we define variables a and i, as well as initial conditions $a[1] = 4$, $a[2] = -1$, and $a[3] = 1$. The lines 6 to 9 finds the value of $a[i]$, and returns the numerical value.

Question 29

A simple iteration procedure with $a_0 = 0$ and

$$a_{n+1} = \sin\left((1 + \frac{1}{6}a_n)^2\right), \quad n \geq 0,$$

is being used to find an approximate solution to the equation $x = \sin\left(\left(1 + \frac{1}{6}x\right)^2\right)$. Write a procedure which takes a positive integer m and uses a for loop to calculate a_m. The procedure should return a_m if $|a_m - a_{m-1}| < 10^{-7}$, and -1 otherwise. All calculations are done using 30 significant figures.
***Note**: This question is one in which we must select the correct options from drop down menus. I will present the correct sequence of lines below. After, I will present some logic.*

The correct lines of code are:

```
Digits:= 30;
f:= proc(m)
local a,i;
a[0]:= 0;
for i from 1 to m do
a[i]:= evalf(sin((1+a[i-1]/6)^2));
end do;
if abs(a[m]-a[m-1]) < 10^(-7) then
a[m]
else
-1
end if
end proc;
```

Typing `f(4)` into Maple gives us `-1`, and `f(11)` gives us `0.976125175777253100707794674276`.

The first line ensures that our results are given in 30 significant figures. The next line declares a function f of m. Then we define variables a and i, and the initial condition $a[0] = 0$.

From line 5 to 7, we have Maple evaluate the desired expression for i from 1 to m, whatever the entered m may be. After we end do, we check if $|a[m] - a[m-1]|$ is less than 10^{-7}. Finally the function returns the appropriate number; -1 or $a[m]$.