Second Year Mathematics Revision

Calculus - Part 2

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26 Apr 2019

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Introduction to vector fields

Definition 1

A **vector field** in 2 or 3-space is a function **F** that assigns a vector to a particular point (x, y) or (x, y, z). The **gradient vector field** is the vector field defined by the function
$\mathbf{F} = \nabla f = \left(\frac{\partial}{\partial x} f, \frac{\partial}{\partial y} f, \frac{\partial}{\partial z} f\right)$.

These vector fields are typically represented as
$\mathbf{F}(x, y) = P(x, y)\hat{i} + Q(x, y)\hat{j}$ or
$\mathbf{F}(x, y + z) = P(x, y, z)\hat{i} + Q(x, y, z)\hat{j} + R(x, y, z)\hat{k}$. Here, $\hat{i}, \hat{j}, \hat{k}$ are the 3 standard basis vectors in the associated space.

Introduction to Vector fields

Definition 2

A **conservative vector field** is a vector field $\mathbf{F}$ which can be expressed as the gradient of a scalar function. That is, there is a function $f : \mathbb{R}^n \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$. The corresponding function f is called the **potential** function.

Properties of Vector Fields

Definition 3

The **del**/**nabla** operator is the vector $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$.

The advantage of using this definition is that we now have the ability to use the vector operations as per what we already know, that is, scalar multiplication, dot product, and cross product. For example, taking ∇ and multiplying it by the scalar function f, we get the **gradient** vector.

Divergence

Here, we let $F : \mathbb{R}^n \to \mathbb{R}^n$ to be a continuous, differentiable function in the appropriate vector space.

Definition 4 (Divergence)

The **divergence** of a vector field describes the rate of change of how much fluid flows out of the neighbourhood of a point. That is, does water have a tendency to flow in or out of the point. The operation is defined as:

$$\mathrm{div}\mathbf{F} = \nabla \cdot \mathbf{F}$$

Here, n could take any value, and the definition will still make sense, but typically we shall only take the case $n = 3$

Curl

Definition 5 (Curl)

The **curl** represents the tendency of water to spin about the point in the neighbourhood of the point. It is computed according to the expression:

$$\text{curl}\mathbf{F} = \nabla \times \mathbf{F}$$

Restrictions for curl

The curl operator is only defined in $\mathbb{R}^3$ because it is fundamentally a cross product.

Examples

Example 1

Consider the vector field defined by $\mathbf{F}(x, y, z) = (x, y, z)$. Evaluate the following:

1. $\nabla \cdot \mathbf{F}$
2. $\nabla \times \mathbf{F}$
3. $\nabla \times (\mathbf{a} \times \mathbf{F})$
4. $\nabla \frac{1}{||\mathbf{F}||^3}$

Parameterising curves

The following are typical parameterisation of the curves:

1. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ corresponds to $x = a\cos(t), y = b\sin(t), 0 \leq t \leq 2\pi$ in the anti-clockwise direction.
2. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ corresponds to $x = a\cos(t), y = -b\sin(t), 0 \leq t \leq 2\pi$ in the clockwise direction
3. A line segment from $\mathbf{x}$ to $\mathbf{y}$ has parameterisation $(1-t)\mathbf{x} + t\mathbf{y}$
4. $y = f(x)$ corresponds to $x = t, y = f(t)$
5. $x = g(y)$ corresponds to $y = t, x = g(t)$

Path Integrals

Definition 6

A **path integral** is given by the formula:

$$\int_C f(x,y)ds = \int_{t=a}^{t=b} f(h(t), g(t)) \left|\left|\mathbf{r}'(t)\right|\right| dt$$

where the curve C has a parameteric vector equation $\mathbf{r}(t)$

Examples

Example 2 [Q151 MATH2011]

1. $\int_C (x+y+z)ds$ where C is the curve parameterised by $(\cos t, \sin t, t)$, $t \in [0, 2\pi]$
2. $\int_C e^{\sqrt{z}} ds$ where C is the curve parameterised by $(1, 2, t^2)$, $t \in [0, 1]$

Surface Integrals

Surfaces are functions in $\mathbb{R}^3$ that are dependent on 2 variables. Thus in order to parameterise surfaces, we need to use 2 variables and express the 3rd variable in terms of the other 2.

Definition 7

A **surface integral** is an integral which measures surface area. We denote such an integral by the formula

$$\iint_S f(x, y, z) ds$$

$f(x, y, z)$ represents some kind of density function, and if this density is equal to 1 over the surface S, then we get the surface area.

Evaluating Surface Integrals I

To evaluate surface integrals, we need to parameterise them in one of the following ways:

1. $\mathbf{s}(x, y) = (x, y, g(x, y))$
2. $\mathbf{s}(u, v) = (x(u, v), y(u, v), z(u, v))$.

In the first scenario, we express the surface as a function of the other 2 variables already existing in the expression [This is common in more elementary expressions for the surface].
In the second scenario, we construct 2 new variables to describe every component in the vector.

Evaluating Surface Integrals II

In either case, we obtain the following definition:

Theorem

$$\iint_S f(x,y,z)ds = \iint_{S'} f(x(u,v),y(u,v),z(u,v))\,||\mathbf{s}_u \times \mathbf{s}_v||\,du\,dv$$

To see how this is applied to the parameterisation $\mathbf{s} = (x, y, g(x,y))$.

$$\begin{aligned}\mathbf{s}_x \times \mathbf{s}_y &= (1,0,g_x) \times (0,1,g_y)\\ &= \det\begin{bmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 0 & g_x\\ 0 & 1 & g_y\end{bmatrix}\\ &= -\hat{i}g_x - \hat{j}g_y + \hat{k}\end{aligned}$$

Surface Integrals Examples

Example 3 [MATH2111 Q11]

Find a parametric representations for the parts of the plane $2x + 3y + z = 4$:

1. $1 \leq x \leq 2, 2 \leq z \leq 4$
2. $0 \leq x + y + z \leq 7, 2 \leq x - y \leq 4$
3. $x \geq 0, x^2 + y^2 \leq 4$

Surface Integrals Examples

Example 4 [MATH2111 Q12]

Let Ω denote the conical region $\sqrt{x^2+y^2} \leq z \leq 2$.

1. Find a parametric representation $\mathbf{x}(u,v)$ for the surface, the boundary of Ω.
2. Use simple geometry to write down the outwards point unit normal vector at each point on S.
3. Verify that these vectors are parallel to the normal vectors obtained from the formula $N(\mathbf{x}) = \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}$
4. Find the surface area of the cone.

Vector Integrals

Vector integrals involve the integration of vector fields across paths and surfaces. However, you can only ever integrate scalar functions. So the actual vector integrals are ever so slightly different in their formulas.

Path integrals of vector fields

Path Integral of a vector field

Consider a vector field $\mathbf{F}(x,y,z) = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}$, and suppose a curve C has a parameterisation $\mathbf{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$. Then we write:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t=a}^{t=b} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \int_C Pdx + Qdy + Rdx$$

In this case, orientation must be preserved, that is, path integrals of vector fields are dependent on what direction the path goes in. Typically, the anti-clockwise direction is the default direction, unless otherwise specified.

Conservative Vector Fields

Theorem 1 (Fundamental Theorem of Line Integrals)

Suppose that C is a differentiable curve given by $\mathbf{r}(t), a \leq t \leq b$, and $\mathbf{F} = \nabla f$ [That is, $\mathbf{F}$ is conservative]. Then the following holds:

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Corollaries

Corollaries

1. $\int_C \nabla f \cdot d\mathbf{r}$ is path independent
2. If $\mathbf{F}$ is a continuous vector field on an open connected region D and if $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent for any path in D, then $\mathbf{F}$ is conservative.
3. $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent, if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C

Examples of vector integrals along paths

Example 5 [MATH2011 Q155, MATH2111 Q2]

Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the vector field $\mathbf{F}(x, y, z) = (x^2 + y^2 + z^2)\hat{i} - z\hat{j} + (y + 1)\hat{k}$ and the curve C from $(0, 0, -1)$ to $(0, 0, 1)$ is

1. The straight line joining the 2 points
2. An arc of the circle $y^2 + z^2 = 1$ in the plane $x = 0$ oriented counterclockwise when viewed from the positive x-axis.

If $\mathbf{F}$ conservative?

Surface Integrals of vector fields

In a rather obvious sense, we can generalise the path integral into a surface integral, and we can arrive at the following conclusion:

Definition

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} dS = \iint_S \mathbf{F} \cdot \mathbf{n} dA$$

where A is image projected by S onto the xy plane.

Examples

Example 6 [MATH2111 Q16]

Evaluate $\int_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = \sinh(x)\hat{i} + \cosh(y)\hat{k}$ and S is given by $z = x + y^2$ for $0 \leq y \leq x, 0 \leq x \leq 1$.

Examples

Example 7 [MATH2011 Q171 a)]

Evaluate $\iint_S \mathbf{F} \cdot \hat{n} dS$ where $\mathbf{F} = y\hat{i} - x\hat{j} + z\hat{k}$ and $z = 4 - x^2 - y^2, z \geq 0$ and $\hat{n}$ pointing upwards.

Integral Theorems

There are 3 major integral theorems:

Theorems

1. Green's Theorem
2. Divergence Theorem
3. Stokes' Theorem

Green's Theorem

Green's Theorem

For a closed region D in $\mathbb{R}^2$ we have the following equivalence:

$$\oint_{\partial D} P(x,y)dx + Q(x,y)dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dx\, dy$$

Conditions

The relevant conditions required for this theorem to hold:

1. D is a closed, non-self intersecting region. [Note: If this is the case, split the region into 2, but this won't be covered]
2. P, Q are $C^1(D)$, meaning they are differentiable functions on D.

Examples

Example 8 [MATH2011 Q161, MATH2111 Q28]

Evaluate $\oint_C (x^2 - 2xy)dx + (x^2 + 3)dy$ around the boundary C taken anti-clockwise around the region contained by $y^2 = 8x$ and $x = 2$ using Green's Theorem.

Frame Title

Example 9 [MATH2111 Q31]

Show that for any planar region Ω that:

$$\text{area}(\Omega) = \frac{1}{2}\oint_{\partial\Omega}(xdy - ydx)$$

Divergence Theorem

Divergence Theorem

For a $C^1(R)$ vector field $\mathbf{F}$ on a closed and bounded solid V. Then the following holds:

$$\iiint_V \nabla \cdot \mathbf{F} dV = \iint_{\partial V} \mathbf{F} \cdot d\mathbf{r}$$

Examples for Divergence Theorem I

Example 10 [MATH2111 Past Paper]

Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector defined by $\mathbf{F} = xy\hat{i} + z^3\hat{j} + y^2\hat{k}$ and W the box defined by $[0,1] \times [0,2] \times [0,4]$ and the orientation defined outward. Use divergence theorem to evaluate:

$$\iint_{\partial W} \mathbf{F} \cdot d\mathbf{S}$$

Examples for Divergence Theorem II

Example 11 [MATH2011 Q180]

Evaluate $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}$ for $\mathbf{F} = (x, -y, z^2 - 1)$ and S is the boundary surface of the solid Ω bounded by the planes
$z = 0, z = 1, x^2 + y^2 = 1$

Stokes' Theorem

Stokes' Theorem

Consider a closed and bounded surface S and a $C^1(S)$ vector field $\mathbf{F}$. Then we have:

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{s} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}$$

Examples for Stokes' Theorem

Example 12 [MATH2011 Q175]

Verify Stokes' Theorem for the vector field $\mathbf{F} = (2x - y, -yx^2, -y^2z)$ and S is the top half-surface of the sphere $x^2 + y^2 + z^2 = 1, z \geq 0$ and C is the bounding curve of S.

Examples for Stokes' Theorem

Example 13 [MATH2011 Q178]

Calculate $\int_C \mathbf{F} \cdot d\mathbf{s}$ where $\mathbf{F} = (4z + x^2, -2x + 3y^5, 2x^2 + 5sin(z)$ and C is the curve of intersection of the surfaces $x^2 + y^2 = 1, z = y + 1$

Orthogonal Curvilinear Co-ordinates

Orthogonal curvilinear co-ordinates are a form of substitution that enables conversion into a more "convenient" form for the co-ordinate system. In effect, it is a parameterisation.

Definition

We say that (ξ_1, ξ_2, ξ_3) are orthogonal co-ordinates if the following hold true:

$$\mathbf{b}_{\xi_i} \cdot \mathbf{b}_{\xi_j} = 0$$

where $\mathbf{b}_{\xi_i} = \frac{d}{d\xi_i}\mathbf{x}$

Example

Example 14 [MATH2111 Q10]

Toroidal coordiantes (w, θ, ψ) are defined by
$x = (a + w\cos(\psi))\cos(\theta), y = (a + w\cos(\psi))\sin(\theta), z = w\sin(\psi)$,
with a being a constant such that $0 < w < a$.

1. Verify that (w, θ, ψ) is a right-handed, orthogonal coordinate system. Find the scale factors and orthonormal basis vectors.
2. Describe the curve C_1 where $\theta = \frac{\pi}{2}, w = b$ if b is a constant such that $0 < b < a$.
3. Describe the curve C_2 where $\theta = \psi, w = b, 0 < b < a$ and b is a constant.
4. Calculate the length of the curves C_1, C_2 for $a = 1, b = \frac{1}{2}$

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Periodic Functions

Fourier Series deals with periodic functions, so we define a few terms.

Definition 1

A function $f : D \to Y$ is L-**periodic** if, for all $x \in D$,

$$f(x + L) = f(x).$$

Further, f has **frequency** or **period** L, and the smallest K such that f is K-periodic is called the **fundamental frequency** of f.

Real Trigonometric Polynomials

To define Fourier series, we first define trigonometric polynomials.

Definition 2

A **real trigonometric polynomial** of degree n and period $2L$ is a function of the form

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \left(a_k \cos \frac{k\pi x}{L} + b_k \sin \frac{k\pi x}{L} \right),$$

where $a_n \neq 0$ or $b_n \neq 0$, and each $a_i, b_i \in \mathbb{R}$.

Dividing a_0 by 2 will simplify things later.

Complex Trigonometric Polynomials (MATH2111)

Since $e^{ik\pi x/L} = \cos\frac{k\pi x}{L} + i\sin\frac{k\pi x}{L}$, we can extend real trigonometric polynomials as follows.

Definition 3

A **complex trigonometric polynomial** of degree n and period $2L$ is a function of the form

$$f(x) = \sum_{k=-n}^{n} c_k e^{ik\pi x/L} = c_0 + \sum_{k=1}^{n}\left(c_k e^{ik\pi x/L} + c_{-k}e^{-ik\pi x/L}\right),$$

where $c_n \neq 0$ or $c_{-n} \neq 0$, and each $c_i \in \mathbb{C}$.

If each $c_i = \overline{c_{-i}}$, then each term is real, and the function is a real trigonometric polynomial.

Fourier Series

Now we define Fourier Series.

Definition 4

Suppose $f : \mathbb{R} \to \mathbb{R}$ is bounded, piecewise continuous, and $2L$-periodic. Then the n^{th} **real Fourier polynomial** of f is

$$(S_n f)(x) = \frac{a_0[f]}{2} + \sum_{k=1}^{n} \left(a_k[f] \cos \frac{k\pi x}{L} + b_k[f] \sin \frac{k\pi x}{L} \right),$$

where

$$a_0[f] = \frac{1}{L} \int_{-L}^{L} f(x) \mathrm{d}x,$$

and for $1 \leq k \leq n$,

$$a_k[f] = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{k\pi x}{L} \mathrm{d}x, \quad b_k[f] = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{k\pi x}{L} \mathrm{d}x.$$

Fourier Series (MATH2111)

Definition 5

Suppose $f : \mathbb{R} \to \mathbb{C}$ is bounded, piecewise continuous, and $2L$-periodic. Then the n^{th} **complex Fourier polynomial** of f is

$$\begin{aligned}(S_n f)(x) &= \sum_{k=-n}^{n} c_k[f]e^{ik\pi x/L} \\ &= c_0[f] + \sum_{k=1}^{n}\left(c_k[f]e^{ik\pi x/L} + c_{-k}[f]e^{-ik\pi x/L}\right),\end{aligned}$$

where

$$c_k[f] = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-ik\pi x/L}\mathrm{d}x$$

for $-n \leq k \leq n$.

Fourier Series

A couple of results help us when dealing with even or odd functions.

Theorem 1

If f is an even function, then

$$a_0[f] = \frac{2}{L}\int_0^L f(x)\mathrm{d}x, \quad a_k[f] = \frac{2}{L}\int_0^L f(x)\cos\frac{k\pi x}{L}\mathrm{d}x,$$

and $b_k[f] = 0$.
If f is an odd function, then

$$b_k[f] = \frac{2}{L}\int_0^L f(x)\sin\frac{k\pi x}{L}\mathrm{d}x,$$

and $a_0[f] = a_k[f] = 0$.

Fourier Series

A **Fourier series** is the limit of the sequence $\{S_n f\}_{n=1}^{\infty}$ denoted Sf.

Example 1

Find the real Fourier series of f where $f(x) = x$ for $-1 < x \leq 1$, and requiring that $f(x+2) = f(x)$ for all $x \in \mathbb{R}$.

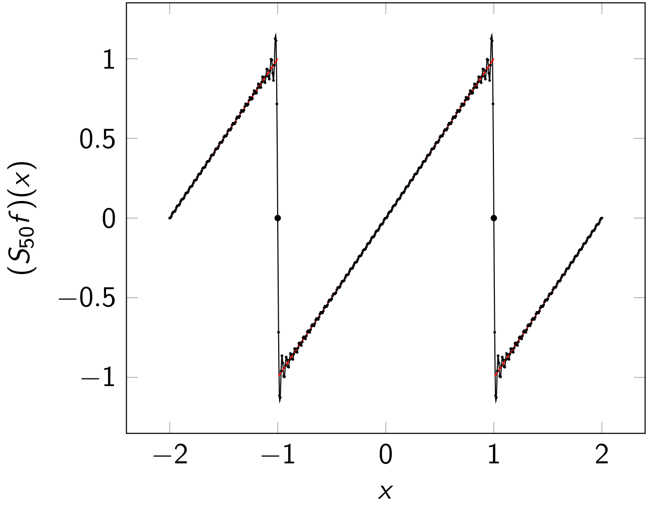
So f is 2-periodic, and odd. Thus $a_0[f] = a_k[f] = 0$. Then

$$b_k[f] = 2\int_0^1 x \sin k\pi x \,\mathrm{d}x = 2\frac{\sin \pi k - \pi k \cos \pi k}{\pi^2 k^2}.$$

Now, $\sin \pi k = 0$ and $\cos \pi k = (-1)^k$ as k is an integer, so

$$(Sf)(x) = \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin k\pi x.$$

Gibbs Phenomenon



Fourier Series

At jump discontinuities, the Fourier series approaches the average of the function value either side of the discontinuity.

Theorem 2

Suppose f has a jump discontinuity at a. That is, the limits

$$f(a^+) = \lim_{x \to a^+} f(x), \quad f(a^-) = \lim_{x \to a^-} f(x)$$

both exist, but $f(a^+) \neq f(a^-)$. Then

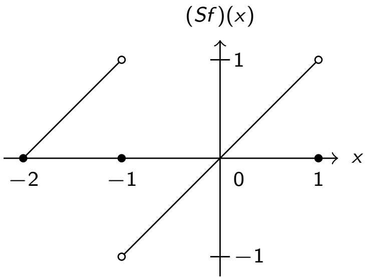
$$\lim_{n \to \infty} (S_n f)(a) = \frac{f(a^+) + f(a^-)}{2}.$$

For those in MATH2111, this is conditioned on the one-sided derivatives existing at a. For the functions you deal with, this will almost always be the case.

Fourier Series

Example 2

For the previous example, draw the graph of $(Sf)(x)$ for $-2 \leq x \leq 1$



Wherever the function is continuous, we can simply draw f. At each jump discontinuity, the Fourier series approaches the average of the function, in this case 0.

Function Convergence

We can say the Fourier series "converges" to the function, but to do so, we need to define what we mean by convergence.

Definition 6

A sequence of functions $\{f_i\}_{i=1}^{\infty}$ **converges pointwise** to f on an interval I if, for every $x \in I$, we have

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Definition 7

A sequence of functions $\{f_i\}_{i=1}^{\infty}$ **converges uniformly** to f on an interval I if, for every $\epsilon > 0$, there exists an $N > 0$ such that

$$|f_n(x) - f(x)| < \epsilon$$

for all $n \geq N$ and $x \in I$.

Function Convergence

Uniform convergence implies pointwise convergence, however the converse is not true.

Example 3

Prove that $f_n(x) = x^n$ converges pointwise to $f(x) = 0$ on $[0, 1)$, but not uniformly.

If $x \in [0, 1)$, then $\lim_{n\to\infty} x^n = 0$, so f_n converges pointwise. Now take $\epsilon = \frac{1}{2}$. Then if the sequence converges uniformly we have

$$|f_n(x) - f(x)| = |x^n| = x^n < \frac{1}{2}$$

for sufficiently large n, say $n \geq N > 0$, and all $x \in [0, 1)$. So,

$$x < \frac{1}{\sqrt[N]{2}} < 1.$$

Take $x \in \left(1/\sqrt[N]{2}, 1\right)$ to derive a contradiction.

Function Convergence

Theorem 3

If a sequence of continuous functions $\{f_i\}_{i=1}^{\infty}$ converges uniformly to f on an interval I, then f is continuous on I.

This means we can disprove uniform convergence by showing that the sequence converges pointwise to a discontinuous limit.

Series Convergence

Just as with sequences, we can define convergence for series of functions similarly.

Definition 8

Let $\{f_i\}_{i=1}^{\infty}$ be a sequence of functions. The infinite series

$$\sum_{k=1}^{\infty} f_k(x)$$

converges pointwise to S on an interval I if the partial sums converge pointwise to S on I. That is, for every $x \in I$,

$$S(x) = \lim_{n\to\infty} \sum_{k=1}^{n} f_k(x).$$

Series Convergence

Definition 9

Let $\{f_i\}_{i=1}^{\infty}$ be a sequence of functions. The infinite series

$$\sum_{k=1}^{\infty} f_k(x)$$

converges uniformly to S on an interval I if the partial sums converge uniformly to S on I. That is, for every $\epsilon > 0$, there exists an $N > 0$ such that

$$|S_n(x) - S(x)| < \epsilon$$

for all $n \geq N$ and $x \in I$, and where S_n is the n^{th} partial sum.

Series Convergence

Theorem 4 (Weierstrass M-test)

If the sequence of functions $\{f_i\}_{i=1}^{\infty}$ satisfies

$$|f_n(x)| \leq M_n$$

for all $x \in I$ and $n \geq 1$, and if

$$\sum_{n=1}^{\infty} M_n$$

converges, then

$$\sum_{n=1}^{\infty} f_n(x)$$

converges uniformly on I.

Series Convergence

Example 4 (Question 10 from MATH2111 2018)

Prove that

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2 + x^2}$$

converges uniformly on $\mathbb{R}$.

$$\left| \frac{\cos nx}{n^2 + x^2} \right| \le n^{-2}.$$

So, since

$$\sum_{n=1}^{\infty} n^{-2} = \frac{\pi^2}{6},$$

by Weierstrass M-test, the series converges uniformly on $\mathbb{R}$.

Convergence of Fourier Series

With these definitions, we can say the following.

Theorem 5

Suppose c_k are the complex, and a_k, b_k the real Fourier coefficients of f. Then if

$$\sum_{k=-\infty}^{\infty} |c_k|, \text{ or } \sum_{k=1}^{\infty} (|a_k| + |b_k|)$$

converge, then the corresponding Fourier series converges uniformly to f. Further, f is continuous.

So, if the function we are trying to represent as a Fourier series is discontinuous, then the series does not converge uniformly.

Convergence of Fourier Series

Example 5

Let f be defined such that $f(x) = x$ for $0 \leq x \leq 1$, f is even, and $f(x+2) = f(x)$ for all $x \in \mathbb{R}$. You are given that

$$(Sf)(x) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k \text{ odd}} \frac{\cos k\pi x}{k^2}.$$

Does Sf converge uniformly on $\mathbb{R}$? What does this say about f?

So

$$\sum_{k=1}^{\infty} \left(\left| \frac{-4}{k^2\pi^2} \right| + |0| \right) = \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2}$$

converges by p-test. Thus, Sf converges uniformly to f. Since the partial sums are continuous, uniform convergence implies that f is continuous.

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Balls

To generalise results about open and closed intervals, we first define a ball in $\mathbb{R}^n$.

Definition 10

A **ball** around $\mathbf{a} \in \mathbb{R}^n$ of radius $\epsilon > 0$ is

$$B(\mathbf{a}, \epsilon) = \{\mathbf{x} \in \mathbb{R}^n : d(\mathbf{a}, \mathbf{x}) < \epsilon\}.$$

Here, $d(\mathbf{a}, \mathbf{x})$ is the Euclidean distance between **a** and **x**.

In $\mathbb{R}^2$ and $\mathbb{R}^3$ this is a disk and sphere, **without** its boundary.

Types of Points

Definition 11

Consider $\Omega \subseteq \mathbb{R}^n$. Then $\mathbf{a} \in \mathbb{R}^n$ is

- An **interior point** of Ω if there is a ball around $\mathbf{a}$ contained entirely in Ω.
- A **boundary point** of Ω if every ball around $\mathbf{a}$ contains both points in Ω and not in Ω.
- A **limit point** or **accumulation point** of Ω if there is a sequence $\{\mathbf{x}_n\}_{n=1}^{\infty} \subseteq \Omega$ with limit $\mathbf{a}$, and $\mathbf{x}_i \neq \mathbf{a}$ for all $\mathbf{x}_i$.

Every interior point is also a limit point, but the converse is not necessarily true.

Types of Points

Example 6

Classify the points $(0,0)$, and $(1,1)$ of the set

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\right\} \cup \{(1,1)\}.$$

There is a ball around $(0,0)$ that is contained entirely in Ω, so it is an interior point. There is a sequence contained in Ω with limit $(0,0)$, so it is a limit point. It is not a boundary point.
No ball around $(1,1)$ is contained entirely in Ω, so it's not an interior point. There is also no sequence in Ω with limit $(1,1)$ that isn't eventually constant, so it's not a limit point. It is a boundary point, as any ball around $(1,1)$ contains an element in Ω (namely $(1,1)$), and not in Ω (namely $(1, 1+\epsilon)$ for sufficiently small ϵ).

Types of Sets

Now, we use these point definitions to define types of sets.

Definition 12

A set $\Omega \subseteq \mathbb{R}^n$ is

- **Open** if every element $\mathbf{a} \in \Omega$ is an interior point.
- **Closed** if its complement $\mathbb{R}^n - \Omega$ is open.

In $\mathbb{R}^n$, $\mathbb{R}^n$ and $\emptyset$ are the only sets which are both open **and** closed.

Theorem 6

The following definitions are equivalent:

- Ω is closed.
- Ω contains all of its boundary points.
- Ω contains all of its limit points.

Types of Sets

Example 7

Prove that $\Omega = (a, b)$ is an open set in $\mathbb{R}$ where $a < b$.

Suppose $x \in \Omega$. Then let $\epsilon = \min\{x - a, b - x\}$. Now, if $c \in (x - \epsilon, x + \epsilon)$, then

$$c > x - \epsilon \geq x - (x - a) = a.$$

Similarly,

$$c < x + \epsilon \leq x + (b - x) = b,$$

so $c \in \Omega$. That is, $B(x, \epsilon) = (x - \epsilon, x + \epsilon) \subseteq (a, b)$. Then every element $x \in \Omega$ has a ball around it contained entirely in Ω, and thus is an interior point. Thus, Ω is open.

Types of Sets

Example 8

Is the set $\Omega = B(\mathbf{0}, 1) \cup \{(x, y) \in \mathbb{R}^2 : x \geq 0\}$ open? Is it closed?

Ω is not open, as $(0, 2) \in \Omega$ but $(-\epsilon, 2) \notin \Omega$ for any $\epsilon > 0$, so any ball around $(0, 2)$ does not lie entirely in Ω. That is, not every point in Ω is an interior point.

Any ball around $(-1, 0)$ contains $(-1, 0) \notin \Omega$ and $(\epsilon - 1, 0) \in \Omega$ for some $\epsilon > 0$. Thus $(-1, 0)$ is a boundary point not contained in Ω, and so Ω is not closed.

Properties of Sets

Definition 13

Suppose $\Omega \subseteq \mathbb{R}^n$. Then

- The **interior** of Ω is the set of all interior points of Ω.
- The **boundary** of Ω is the set of all boundary points of Ω denoted $\partial\Omega$.
- The **closure** of Ω is $\overline{\Omega} = \Omega \cup \partial\Omega$.

The interior is the largest open subset of Ω, and the closure is the smallest closed set containing Ω.

Limits in $\mathbb{R}^n$

A precise definition of limits in $\mathbb{R}^n$ is as follows (similar to $\epsilon - \delta$ in first year).

Definition 14

Suppose $\mathbf{f} : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{a} \in \Omega$. We say

$$\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}$$

if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $\mathbf{x} \in \Omega$ and $0 < d(\mathbf{x}, \mathbf{a}) < \delta$, then $d(\mathbf{f}(\mathbf{x}), \mathbf{b}) < \epsilon$. Alternatively,

$$\mathbf{x} \in \Omega \cap B(\mathbf{a}, \delta) - \{\mathbf{a}\} \implies \mathbf{f}(\mathbf{x}) \in B(\mathbf{b}, \epsilon).$$

Limits in $\mathbb{R}^n$

Example 9 (Question 33ii from MATH2111 2019)

Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2+x^2y^2}{x^2+y^2} = 1$ using the definition of a limit.

Note that $|f(x,y)-1|$ is:

$$\left|\frac{x^2+y^2+x^2y^2}{x^2+y^2}-1\right| = \left|\frac{x^2y^2}{x^2+y^2}\right| \leq \frac{x^4+2x^2y^2+y^4}{x^2+y^2} = x^2+y^2.$$

So for some $\epsilon > 0$ choose $\delta = \sqrt{\epsilon}$. Then if $\sqrt{x^2+y^2} < \delta$, we have $|f(x,y)-1| \leq x^2+y^2 < \delta^2 = \epsilon$, as required.

Limits in $\mathbb{R}^n$

Dealing with limits of vector functions can be difficult, so instead it suffices to consider each component's limit individually.

Theorem 7

Suppose $\mathbf{f} : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and denote the components of $\mathbf{f}$ $f_1, f_2, \cdots, f_m$. Let $\mathbf{a} \in \Omega$. If

$$L_1 = \lim_{\mathbf{x}\to\mathbf{a}} f_1(\mathbf{x}), \quad L_2 = \lim_{\mathbf{x}\to\mathbf{a}} f_2(\mathbf{x}), \quad \cdots, \quad L_m = \lim_{\mathbf{x}\to\mathbf{a}} f_m(\mathbf{x})$$

all exist, then

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = (L_1, L_2, \cdots, L_m)^T.$$

This can be used in conjunction with other methods on scalar functions to evaluate limits of vector functions.

Pinching Principle

Just as single variable limits can be evaluated using the pinching principle, so can multivariable limits.

Theorem 8 (Pinching Principle)

Suppose $f, g, h : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, $\mathbf{a} \in \Omega$, and

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \lim_{\mathbf{x}\to\mathbf{a}} h(\mathbf{x}) = L.$$

If $f(\mathbf{x}) \leq g(\mathbf{x}) \leq h(\mathbf{x})$ for all $\mathbf{x} \in B(\mathbf{a}, \epsilon)$ where $\epsilon > 0$. That is, all $\mathbf{x}$ in a neighbourhood of $\mathbf{a}$. Then

$$\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = L.$$

This can also be called the pinching theorem.

Pinching Principle

Example 10

Find $\lim_{(x,y)\to(0,0)} \left(\frac{x^2+y^2+x^2y^2}{x^2+y^2}, \frac{xy(x+y)}{x^2-xy+y^2} \right)^T$.

So considering each component separately, we have

$$1 \leq \frac{x^2+y^2+x^2y^2}{x^2+y^2} \leq 1 + \frac{x^4+2x^2y^2+y^4}{x^2+y^2} = 1 + x^2 + y^2,$$
$$0 \leq \left| \frac{xy(x+y)}{x^2-xy+y^2} \right| \leq |x+y|.$$

The last inequality follows from expanding $(|x|-|y|)^2 \geq 0$ and noting that $x^2+y^2-xy \geq x^2+y^2-|xy|$.
So, by pinching theorem, the limit is $(1,0)^T$.

Continuity in $\mathbb{R}^n$

Now that we have defined limits, we can define continuity.

Definition 15

$\mathbf{f} : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is **continuous** at $\mathbf{a} \in \Omega$ if

- $\mathbf{a}$ is a limit point of Ω, $\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x})$ exists and equals $\mathbf{f}(\mathbf{a})$, or
- $\mathbf{a}$ is not a limit point of Ω.

Theorem 9

Suppose $\mathbf{f} : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$. Then the following are equivalent.

- $\mathbf{f}$ is continuous at $\mathbf{a} \in \Omega$.
- For all $\epsilon > 0$, there exists some $\delta > 0$ such that if $\mathbf{x} \in \Omega \cap B(\mathbf{a}, \delta) - \{\mathbf{a}\}$ then $\mathbf{f}(\mathbf{x}) \in B(\mathbf{f}(\mathbf{a}), \epsilon)$.
- For every $\{\mathbf{x}_i\}_{i=1}^{\infty} \subseteq \Omega$ with limit $\mathbf{a}$, $\{\mathbf{f}(\mathbf{x}_i)\}_{i=1}^{\infty}$ has limit $\mathbf{f}(\mathbf{a})$.
- $\mathbf{f}(\mathbf{a})$ is in the interior of $\mathbf{f}(\Omega)$ implies $\mathbf{a}$ is in the interior of Ω.

Continuity in $\mathbb{R}^n$

Example 11

Consider $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ where $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ and otherwise

$$\mathbf{f}(x, y) = \left(\frac{x^3}{x^2 + y^2}, \frac{y^3}{x^2 + y^2} \right)^T.$$

Prove $\mathbf{f}$ is continuous on $\mathbb{R}^2$.

Since $\mathbb{R}^2$ is open, every $\mathbf{a} = (a, b) \in \mathbb{R}^2$ is a limit point. When $\mathbf{a} \neq \mathbf{0}$, $\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x})$ exists and is equal to $\mathbf{f}(\mathbf{a})$. When $\mathbf{a} = \mathbf{0}$ we have

$$\lim_{\mathbf{x} \to \mathbf{0}} \frac{x^3}{x^2 + y^2} = \lim_{\mathbf{x} \to \mathbf{0}} \frac{y^3}{x^2 + y^2} = 0.$$

So, $\lim_{\mathbf{x} \to \mathbf{0}} \mathbf{f}(\mathbf{x}) = \mathbf{0} = \mathbf{f}(\mathbf{0})$, and thus $\mathbf{f}$ is continuous on $\mathbb{R}^2$.

Continuity in $\mathbb{R}^n$

Just like limits, we can determine continuity of a vector function by its components.

Theorem 10

Suppose $\mathbf{f} : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$. $\mathbf{f}$ is continuous at $\mathbf{a} \in \Omega$ **if and only if** all components of $\mathbf{f}$ are continuous at $\mathbf{a}$.

Continuous functions preserve openness of sets in their preimage.

Theorem 11 (Preimage Theorems)

Suppose $\mathbf{f} : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$. Then the following are equivalent.

- $\mathbf{f}$ is continuous on Ω.
- $U \subseteq \mathbb{R}^m$ is open implies $\mathbf{f}^{-1}(U)$ is open.
- $U \subseteq \mathbb{R}^m$ is closed implies $\mathbf{f}^{-1}(U)$ is closed.

Here, $\mathbf{f}^{-1}(U) = \{\mathbf{x} \in \Omega : \mathbf{f}(\mathbf{x}) \in U\}$.

More Types of Sets

Now we define a few more types of sets.

Definition 16

A set $\Omega \subseteq \mathbb{R}^n$ is

- **Bounded** if there is some finite $\epsilon > 0$ such that $\Omega \subseteq B(\mathbf{0}, \epsilon)$.
- **Path connected** if, for all $\mathbf{x}, \mathbf{y} \in \Omega$, there is a continuous function $\phi : [0,1] \to \mathbb{R}^n$ such that $\phi(0) = \mathbf{x}$, $\phi(1) = \mathbf{y}$, and $\phi([0,1]) \subseteq \Omega$. That is, there is a continuous path between all $\mathbf{x}$ and $\mathbf{y}$ contained entirely within Ω.
- **Compact** if Ω is both bounded and path connected.
- **Simply connected** if any closed curve in Ω can be continuously deformed to a point without leaving Ω.

More Types of Sets

Example 12

Let $\Omega = B(\mathbf{0},2) - B(\mathbf{0},1)$. Is Ω path connected? Is it compact?

Let $\mathbf{x}_1 = (r_1 \cos\theta_1, r_1 \sin\theta_1)^T, \mathbf{x}_2 = (r_2 \cos\theta_2, r_2 \sin\theta_2)^T \in \Omega$.
Consider

$$\phi(t) = (r(t)\cos\theta(t), r(t)\sin\theta(t))^T,$$

where $r(t) = r_1 + (r_2 - r_1)t$ and $\theta(t) = \theta_1 + (\theta_2 - \theta_1)t$. Then $r(t)$ lies between r_1 and r_2 for $t \in [0,1]$ and thus $\phi([0,1]) \subseteq \Omega$. Further, $\phi(0) = \mathbf{x}_1$ and $\phi(1) = \mathbf{x}_2$. Since the components of ϕ are continuous, ϕ is continuous, and so Ω is path connected. Clearly $\Omega \subseteq B(\mathbf{0},2)$, so Ω is bounded, and thus is compact.

Monotone Convergence Theorem

In $\mathbb{R}$, a sequence that is bounded and monotone has a limit in $\mathbb{R}$. We can say a similar thing for $\mathbb{R}^n$.

Theorem 12 (Bolzano-Weierstrass Theorem)

Suppose $\Omega \subseteq \mathbb{R}^n$. Then the following are equivalent.

- Ω is closed and bounded.
- Every sequence in Ω has a subsequence that converges to an element of Ω.

Since there's no real concept of monotone, we say that there simply is a subsequence that converges in Ω.

Bolzano-Weierstrass Theorem

Example 13

Let $\Omega = \left\{(x, y) \in \mathbb{R}^2 : -1 \leq x, y \leq 1\right\}$. Does the sequence $\left\{\left(\cos \frac{i\pi}{2}, \sin \frac{i\pi}{2}\right)\right\}_{i=1}^{\infty}$ converge in Ω? Does it have a subsequence that converges in Ω?

The sequence does not converge in Ω. It cycles through four points. However, Ω is bounded and closed, so any sequence in Ω has a subsequence that does converge. One such subsequence is the constant sequence $\{(1, 0)\}_{i=1}^{\infty}$.

Image Theorems

Compactness and path connectedness are preserved by continuous functions.

Theorem 13 (Image Theorems)

Suppose $\mathbf{f} : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is a continuous function. Then

- If $K \subseteq \Omega$ is compact, then $\mathbf{f}(K)$ is compact.
- If $B \subseteq \Omega$ is path connected, then $\mathbf{f}(B)$ is path connected.

Image Theorems

Example 14 (Question 65 from MATH2111 2019)

Let

$$S_1 = \left\{(x, y) \in \mathbb{R}^2 : 0 \le x, y \le 1\right\},$$
$$S_2 = \left\{(x, y) \in \mathbb{R}^2 : x, y \ge 0\right\}.$$

Is there a continuous function that maps S_1 onto S_2? S_2 onto S_1?

S_1 is compact, however S_2 is not, so there is no continuous map from S_1 onto S_2.

S_2 is path connected, but not compact, and S_1 is path connected, so there may be a continuous function from S_2 onto S_1.

$$\mathbf{f}(x, y) = (|\sin x|, |\sin y|)^T$$

is such a function.