Semester 1, 2017, MATH1131/1141 Final Exam Revision Session (Algebra)

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Introduction

Today, we will cover

- Chapter 1 – Introduction to Vectors
- Chapter 2 – Vector Geometry
- Chapter 3 – Complex Numbers
- Chapter 4 – Linear Equations & Matrices
- Chapter 5 – Matrices

We will recap key concepts and go through example questions. Most examples in this session will consist of past exam questions where we will show you how to approach such questions for the final exam.

Chapter 1 – Introduction to Vectors

- Dot product: $\boxed{\mathbf{a}\cdot\mathbf{b} = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + \cdots + a_n b_n = \|\mathbf{a}\|\,\|\mathbf{b}\|\cos\theta = \mathbf{a}^T\mathbf{b}}$ for $\mathbf{a},\mathbf{b}\in\mathbb{R}^n$
- Length: $\boxed{\|\mathbf{x}\| = \sqrt{\mathbf{x}^T\mathbf{x}} = \sqrt{\mathbf{x}\cdot\mathbf{x}} = \sqrt{x_1^2+\cdots+x_n^2}}$ $\forall\mathbf{x}\in\mathbb{R}^n$
- Cross product: $\boxed{\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}}$ for $\mathbf{a},\mathbf{b}\in\mathbb{R}^3$
- Cross product (alternative): $\boxed{\mathbf{a}\times\mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\|\sin\theta\,\hat{\mathbf{n}}}$, where $\hat{\mathbf{n}}$ is the (unique) *unit normal vector* (perpendicular to both $\mathbf{a}$ and $\mathbf{b}$) with direction determined by the *right-hand rule*. This formula tells us that the length of the cross product is equal to $\|\mathbf{a}\|\,\|\mathbf{b}\|\sin\theta$.
- Scalar triple product: $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ for $\mathbf{a},\mathbf{b},\mathbf{c}\in\mathbb{R}^3$

Some algebraic properties of vector operations

- $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a} \quad \forall \mathbf{a},\mathbf{b}\in\mathbb{R}^n$
- For any real scalar α, $\alpha\,(\mathbf{a}\cdot\mathbf{b})=(\alpha\mathbf{a}\cdot\mathbf{b})$.
- $(\mathbf{a}+\mathbf{b})\cdot\mathbf{x}=(\mathbf{a}\cdot\mathbf{x})+(\mathbf{b}\cdot\mathbf{x}) \quad \forall \mathbf{a},\mathbf{b},\mathbf{x}\in\mathbb{R}^n$
- Similar properties hold for cross product, **except** that we have $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$ (i.e. switching the variables switches the sign)
- The scalar triple product is invariant under *cyclic permutations* of the input vectors, or under a switch of position of the dot and cross. In symbols, this means $\boxed{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})}$ (shifting the variables along by one space each in the same direction leaves the value unchanged) and $\boxed{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}}$ (switching position of cross and dot leaves value unchanged – remember the cross product operation is done before dot product, else the operation makes no sense)
- Remember: a dot product results in a scalar, a cross product results in a vector, and a scalar triple product results in a scalar (as it is a dot product itself). Cross products are only defined in $\mathbb{R}^3$.

Lines and planes

- Parametric form of line in $\mathbb{R}^n$ through $\mathbf{a}$ and with direction vector $\mathbf{v}$ is $\mathbf{x} = \mathbf{a} + \lambda \mathbf{v}$
- Can use any point on line for $\mathbf{a}$ and any vector $\mathbf{v}$ that points along the line
- Cartesian equation of plane in $\mathbb{R}^3$: $ax + by + cz = d$. The vector $(a, b, c)^T$ is a normal vector for this plane
- Parametric vector form of plane in $\mathbb{R}^n$: $\mathbf{x} = \mathbf{a} + \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2$, where $\mathbf{a}$ is a point on the plane, and $\mathbf{v}_1, \mathbf{v}_2$ are (non-zero, non-parallel) direction vectors that span the plane
- Point-normal form of a plane (only valid in $\mathbb{R}^3$): $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{a} \iff \mathbf{n} \cdot (\mathbf{x} - \mathbf{a}) = 0$, where $\mathbf{n}$ is a normal vector to the plane, $\mathbf{a}$ is a point on the plane
- In $\mathbb{R}^3$, a normal vector for the plane $\mathbf{x} = \mathbf{a} + \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2$ is $\mathbf{n} := \mathbf{v}_1 \times \mathbf{v}_2$ (cross product of the direction vectors)
- In $\mathbb{R}^n$, Cartesian equation of line is $\frac{x_1 - a_1}{v_1} = \frac{x_2 - a_2}{v_2} = \cdots = \frac{x_n - a_n}{v_n}$ if $v_i \neq 0$ for all i. This passes through $(a_1, \ldots, a_n)^T$ and has direction vector $(v_1, \ldots, v_n)^T$.

Converting between types of plane equations in $\mathbb{R}^3$

- Mainly need to be able to convert between parametric vector form and Cartesian form
- Parametric vector form to Cartesian form:
  - find a normal $\mathbf{n}$ to plane by taking cross product of direction vectors ($\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$)
  - then use $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{a}$ (point-normal form) to obtain the Cartesian form. Note we can read off $\mathbf{a}$ and the direction vectors $\mathbf{v}_1, \mathbf{v}_2$ if given Parametric vector form.
- Cartesian form to parametric vector form:
  - Set two variables out of x, y, z to be free parameters like λ, μ
  - then use Cartesian equation to solve for third variable, and write $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in terms of vectors involving λ and μ.

Example I

Consider the triangle ABC with $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AC} = \mathbf{b}$. Let E be the midpoint of AC and D be the midpoint of BC. The lines AD and BE meet at the point F. Assume these points are in $\mathbb{R}^3$ so that the cross product is defined.

(a) Write down the area of the triangle ABC in terms of quantities involving $\mathbf{a}$ and $\mathbf{b}$.

(b) Show that $\overrightarrow{BE} = \dfrac{1}{2}\mathbf{b} - \mathbf{a}$ and find the vector $\overrightarrow{AD}$.

(c) Given that $\overrightarrow{BF} = \dfrac{2}{3}\overrightarrow{BE}$ and $\overrightarrow{AF} = \dfrac{2}{3}\overrightarrow{AD}$, find in terms of quantities involving $\mathbf{a}$ and $\mathbf{b}$ the area of $\triangle ABF$.

Example II – performing vector operations

Let $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix}$. Find:

(a) $\mathbf{a} \cdot \mathbf{b}$

(b) $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})$

(c) the angle θ between $\mathbf{b}$ and $\mathbf{c}$.

Example II – Answers

(a) $\mathbf{a} \cdot \mathbf{b} = 0$

(b) $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) = 10$

(c) $\theta = \cos^{-1}\left(\frac{5}{7\sqrt{2}}\right)$

Example III – Line and plane questions

1. Show that the line $\ell : \mathbf{x} = t \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ lies on the plane $\Pi : x - y + 2z = 0$.
2. Consider the line $\ell : \frac{x-6}{5} = \frac{y-3}{2} = z + 1$ and the plane $\Pi : 2x + y + z = 1$. Find where these intersect.
3. Convert $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ to Cartesian form.

Key points

- The angle between two (non-zero) vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ is the unique θ satisfying $\boxed{\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}}$ and $0 \le \theta \le \pi$.
- The area of the parallelogram spanned by two vectors $\mathbf{a}, \mathbf{b}$ in $\mathbb{R}^3$ is $\boxed{\|\mathbf{a}\times\mathbf{b}\|}$. Area of the *triangle* spanned by these vectors is *half* of this.
- The volume of the parallelipiped spanned by three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbb{R}^3$ is $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$, i.e. $\boxed{\underline{\text{absolute value of}}\ \det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}}$.

Further facts

- Given three points $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R}^2$, the area of the triangle they form is equal to the
$$\boxed{\underline{\text{absolute value of}}\ \frac{1}{2}\det\begin{pmatrix}1 & x_1 & y_1\\ 1 & x_2 & y_2\\ 1 & x_3 & y_3\end{pmatrix}}.$$
- **In 2-D**: Area of the parallelogram spanned by two vectors $\begin{pmatrix}a\\ b\end{pmatrix}$ and $\begin{pmatrix}c\\ d\end{pmatrix}$ in $\mathbb{R}^2$ is the $\boxed{\underline{\text{absolute value of}}\ \det\begin{pmatrix}a & b\\ c & d\end{pmatrix}}$
- The midpoint of points A and B with position vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ is the point with position vector $\frac{1}{2}(\mathbf{a} + \mathbf{b})$.
- The line segment between points $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$ is the set $\ell := \{(1-t)\mathbf{a} + t\mathbf{b} : 0 \le t \le 1\}$.
- The parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$ is the set $\mathfrak{p} := \{\lambda\mathbf{a} + \mu\mathbf{b} : 0 \le \lambda \le 1, 0 \le \mu \le 1\}$.

Geometric properties of vector operations

- The angle between two vectors in $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ is acute iff $\mathbf{a} \cdot \mathbf{b} > 0$, 90° iff $\mathbf{a} \cdot \mathbf{b} = 0$ and obtuse iff $\mathbf{a} \cdot \mathbf{b} < 0$.
- Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$ are parallel iff $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. In particular, we always have $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.
- The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$.
- The points A, B, C, O in $\mathbb{R}^3$ with position vectors $\mathbf{a} = (a_1, a_2, a_3)^T$, $\mathbf{b} = (b_1, b_2, b_3)^T$, $\mathbf{c} = (c_1, c_2, c_3)^T$ are coplanar iff the volume of the parallelipiped spanned by $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is 0, i.e. iff $\det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = 0$.

Summary of projections

- $\boxed{\text{proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b}} = \|\mathbf{a}\|\cos\theta \underbrace{\widehat{\mathbf{b}}}_{=\frac{\mathbf{b}}{\|\mathbf{b}\|}}$, for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n, \mathbf{b} \neq \mathbf{0}$
- Length of the projection is $\|\text{proj}_{\mathbf{b}}\mathbf{a}\| = \frac{|\mathbf{a}\cdot\mathbf{b}|}{\|\mathbf{b}\|} = \|\mathbf{a}\|\cos\theta$ (where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$)
- Distance from a point X to a line ℓ in $\mathbb{R}^n$ through the point A, $n \geq 2$: $d = \left\|\overrightarrow{AX} - \mathbf{p}\right\|$, where $\mathbf{p} = \text{proj}_{\mathbf{v}}\overrightarrow{AX}$. *You should draw a rough diagram to derive this when needed.*
- Distance from point X to a plane in $\mathbb{R}^3$: $\boxed{d = \left\|\text{proj}_{\mathbf{n}}\overrightarrow{AX}\right\|}$, where A is a point on the plane, $\mathbf{n}$ is a normal vector to the plane. *You should draw a rough diagram to derive this when needed.*
- Distance between two skew-lines in $\mathbb{R}^3$: $\boxed{d = \text{proj}_{\mathbf{n}}\overrightarrow{AB}}$, where A and B are points on each line, and $\boxed{\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2}$ is a vector orthogonal to both lines ($\mathbf{v}_1, \mathbf{v}_2$ direction vectors for the two lines).

Example I

Consider two points $\mathbf{u}$, $\mathbf{v}$ in $\mathbb{R}^3$ with position vectors $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ and respectively.

(a) Find the cross product $\mathbf{u} \times \mathbf{v}$.

(b) Hence find the Cartesian equation of the plane parallel to $\mathbf{u}$ and $\mathbf{v}$ and passing through the point $\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$.

(c) Find the distance of $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ to the plane in (b).

Example II

The non-zero point Q has position vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$. The vector $\overrightarrow{OQ}$ makes angles α, β and γ respectively with the X, Y and Z axes.

(a) By considering the vector $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ show that

$$a = \sqrt{a^2 + b^2 + c^2} \cos \alpha.$$

(b) Deduce that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

(c) If the angles α and β are complementary, what can be said about the vector $\overrightarrow{OQ}$?

Chapter 3 – Complex Numbers ($\mathbb{C}$)

- $i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i$. Before or after this, pattern repeats, i.e. $i^n = i^a \quad \forall n \in \mathbb{Z}$, where $a \in \{0, 1, 2, 3\}$ is the remainder when n is divided by 4. E.g. $i^{2017} = i^1 = i$, since 2017 has remainder 1 when divided by 4.
- Cartesian form: $z = x + iy,\ x, y \in \mathbb{R}$; Polar form: $z = re^{i\theta}$, where $r \geq 0, \theta \in \mathbb{R}$
- Conversion from Cartesian to Polar: $\boxed{r = \sqrt{x^2 + y^2}}$, $\boxed{\theta_0 = \operatorname{atan2}(y, x)}$ (Principal argument, i.e. in range $(-\pi, \pi]$; search up $\operatorname{atan2}$ function on Wikipedia for reference).
- Conversion from Polar to Cartesian: $\boxed{x = r\cos\theta, y = r\sin\theta}$.
- $\theta_0 = \operatorname{Arg}(z), r = |z|, x = \operatorname{Re}(z) = \Re(z), y = \operatorname{Im}(z) = \Im(z)$
- A general argument is $\theta = \theta_0 + 2k\pi$, for $k \in \mathbb{Z}$. Note $\operatorname{Arg}(0)$ however is undefined. Arg with a capital A is principal argument.
- To add complex numbers component-wise add real parts and imaginary parts. Multiply complex numbers just like expanding normal algebraic expressions, and remembering $i^2 = -1$. To divide complex numbers, 'realise' the denominator.

Useful identities/facts

There are many complex number identities/facts out there. Here are some.

- Euler's formula: $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$
- $\boxed{\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}}$ and $\boxed{\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}}$
- For $r_1, r_2, \theta_1, \theta_2 \in \mathbb{R}$, we have $r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$.
- De Moivre's Theorem:
  $\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \quad n \in \mathbb{Z}}$
- $z\bar{z} = |z|^2, z + \bar{z} = 2\Re(z), z - \bar{z} = 2i\Im(z), \frac{1}{z} = \frac{\bar{z}}{|z|^2}$ for $z \neq 0$
- $\bar{z} = z^{-1}$ iff $|z| = 1, \quad \overline{zw} = \bar{z}\bar{w} \quad \forall z, w \in \mathbb{C}$,
  $\overline{z+w} = \bar{z} + \overline{w} \quad \forall z, w \in \mathbb{C}, |zw| = |z||w|, |z|^n = |z^n|, \bar{z}^n = \overline{z^n}$, for all $n \in \mathbb{Z}$.
- The n complex n-th roots of unity are: $\boxed{e^{i\frac{2k\pi}{n}}, k = 0, 1, \ldots, n-1}$.
- In general, the n complex n-th roots of the complex number $re^{i\theta}$ (where $r > 0$) are $\boxed{\sqrt[n]{r}\exp\left(\dfrac{\theta}{n} + i\dfrac{2k\pi}{n}\right), k = 0, 1, \ldots, n-1}$.

Example I – Simple complex number operations

Let $z = 5 + 5i$ and $w = 2 + i$. Find:

(a) $2z + 3\overline{w}$

(b) $z(w - 1)$

(c) $\dfrac{z}{w}$.

Answers to Example I

(a) $16 + 7i$

(b) $10i$

(c) $3 + i$.

Example II

Let the set $\mathcal{S}$ in the complex plane defined by

$$\mathcal{S} = \{z \in \mathbb{C} : |z - (1+i)| = 1\}.$$

(a) Sketch the set $\mathcal{S}$ on a labelled Argand diagram.

(b) By considering your sketch, or otherwise, find the maximum value of $|z|$ for $z \in \mathcal{S}$.

Example III

Let $z = \cos\theta + i\sin\theta$, where $\theta \in \mathbb{R}$.

(a) Prove that

$$z^n + \frac{1}{z^n} = 2\cos n\theta \quad \text{and} \quad z^n - \frac{1}{z^n} = 2i\sin n\theta$$

for any positive integer n.

(b) Deduce that

$$(2\cos\theta)^4 (2i\sin\theta)^4 = \left(z^2 - \frac{1}{z^2}\right)^4.$$

(c) Hence show that $\cos^4\theta \sin^4\theta = \dfrac{1}{128}(\cos 8\theta - 4\cos 4\theta + 3)$.

Example IV

Let

$$S = e^{i\theta} + \frac{e^{3i\theta}}{3} + \frac{e^{5i\theta}}{3^2} + \frac{e^{7i\theta}}{3^3} + \cdots .$$

(a) Prove that

$$S = \frac{3\left(3e^{i\theta} - e^{-i\theta}\right)}{10 - 6\cos 2\theta}.$$

(b) Hence, or otherwise, find the sum

$$T := \sin\theta + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{3^2} + \frac{\sin 7\theta}{3^3} + \cdots .$$

Chapter 4 – Linear Equations and Matrices

- System of simultaneous linear equations can always be written in matrix form. Can solve via row-reduction
- To determine nature of solutions to $A\mathbf{x} = \mathbf{b}$, row-reduce the augmented matrix $[A|\mathbf{b}]$ to row-echelon form. Then $\boxed{\text{there exists a solution iff right-hand column is non-leading}}$.
- Nature of solution to linear system has precisely three possibilities: no solution, unique solution, or infinitely many solutions.
- **If** there are solutions (i.e. right-hand column is non-leading in the row-echelon form of augmented matrix), there is a unique solution iff every column is leading (in row echelon form of coefficient matrix), and infinitely many solutions iff there is a non-leading column (in row echelon form of coefficient matrix).
- If there are fewer equations than variables (coefficient matrix has fewer rows than columns, i.e. is a "short, fat matrix"), then there is **never** a unique solution (i.e. there will be either no solutions or infinitely many solutions). This is because in this case, the row-echelon form will always have non-leading columns.

Example I

A system of three equations in three unknowns x, y and z has been reduced to the following form

$$\left[\begin{array}{ccc|c} 2 & 0 & -4 & b_1 \\ 3 & 1 & -2 & b_2 \\ -2 & -1 & 0 & b_3 \end{array}\right].$$

(a) Reduce the matrix to row-echelon form.

(b) For which value(s) of b_1, b_2, b_3 will the system have no solution?

(c) For which value(s) of b_1, b_2, b_3 will the system have infinitely many solutions?

(d) For the value(s) of b_1, b_2, b_3 determined in part (c), find the general solution (in terms of b_1, b_2 and b_3).

Solution 1

So after some work, we get ...

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & \frac{b_1}{2} \\ 0 & 1 & -4 & b_2 - \frac{3b_1}{2} \\ 0 & 0 & 0 & b_3 + b_2 - \frac{b_1}{2} \end{array}\right].$$

Example 2

Consider the following system of linear equations.

$$x + y - z = 2$$
$$2x + 3y + z = 6$$

(a) Using Gaussian Elimination, find the general solution to the system of equations.

(b) Hence or otherwise, find the solution to the system with the property that the sum of the x, y and z coordinates is 0.

Chapter 5 – Matrices

- $m \times n$ matrix: m rows, n columns. E.g. 2×5 matrix has two rows and five columns.
- Matrix product: if A has same number of columns as number of rows in B, then AB is well-defined. Otherwise, AB is not defined.
- If A is $m \times n$ and B is $n \times p$, then AB is $m \times p$ and
  $\boxed{[AB]_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}}$, for any $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, p\}$.
- Transpose: flip the matrix about its main diagonal. i.e. Write its rows as columns, or equivalently, its columns as rows. E.g. if
  $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \end{pmatrix}$ then $A^T = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 0 \end{pmatrix}$.
- $\left(A^T\right)^T = A, (AB)^T = B^T A^T$, $(A+B)^T = A^T + B^T$, $(\alpha A)^T = \alpha A^T$, for any compatible matrices A, B and scalar α.

Inverses

- Only square matrices have an inverse
- Not all square matrices have an inverse. If A is a square matrix with an inverse, it is said to be invertible
- If it exists, inverse of $n \times n$ (square) matrix A is a matrix X such that $AX = I_n$ and $XA = I_n$ ($n \times n$ identity matrix).
- If A has an inverse, it is unique
- To find A^{-1} for a square matrix A, row reduce: $\boxed{[A|I] \xrightarrow{\text{row-reduce}} [I|A^{-1}]}$. Remember, if you end up with a row of 0's in the left-hand matrix partway through this process, it means that A is not invertible. Otherwise, it *is* invertible.
- Formula for inverse of a 2×2 invertible matrix: if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\boxed{A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$ (i.e. divide by determinant, swap main diagonal elements, and negate the others).
- If A, B are invertible and compatible, AB is invertible with $(AB)^{-1} = B^{-1}A^{-1}$

Determinants

- Determinants defined for (and only for) **square** matrices.
- Two main methods for determinant computation: expansion along a row or column, or row-reduction to a triangular matrix (in row-echelon form)
- Remember, can expand across or down any row or column respectively. Must keep in mind the correct sign to associate with each entry. Try and expand along a row or column with lots of 0's in it to reduce calculations.
- If using row-reduction, must keep in mind how the elementary row operations affect determinants: swapping two rows negates the determinant, adding a scalar multiple of a row leaves determinant unchanged, and a scalar multiple from a row can be factored out for the determinant, e.g. $\begin{vmatrix} \alpha a & \alpha b \\ c & d \end{vmatrix} = \alpha \begin{vmatrix} a & b \\ c & d \end{vmatrix}$
- The rules above apply similarly to *columns* as well. This comes down to that fact that $\boxed{\det\left(A^T\right) = \det A}$.

Some key determinant properties

- $\det A^T = \det A$ for all square matrices A
- If A is $n \times n$, $\det(cA) = c^n \det A$ for any scalar c. E.g. if $\det A = 2$ and A is 3×3, then $\det(2A) = \left(2^3\right) \det A = 8 \det A = 16$.
- $\det(AB) = (\det A)(\det B)$ for compatible matrices A, B
- A is invertible iff $\det A \neq 0$. In that case, $\det\left(A^{-1}\right) = \frac{1}{\det A}$.
- If A has a zero row or column, then $\det A = 0$.
- If A has any two rows or columns be scalar multiples of one another, then $\det A = 0$. E.g. $\begin{vmatrix} 1 & 2 \\ 15 & 30 \end{vmatrix} = 0$ (row 2 is 15 times row 1).
- For a system of equations in the form $A\mathbf{x} = \mathbf{b}$, where A is square: a unique solution exists iff $\det A \neq 0$, in which case solution is $\mathbf{x} = A^{-1}\mathbf{b}$. Also $\det A = 0$ iff no solution exists or infinitely many solutions exist.
- Determinant of a triangular matrix = product of diagonal elements
- Determinant of a diagonal matrix = product of diagonal elements

Example I

Consider the matrix $M = \begin{pmatrix} 2 & i \\ 1+i & \alpha \end{pmatrix}$.

(a) Find the conditions on α so that the matrix is invertible.

(b) Hence find M^{-1} for $\alpha = 1$.

A Final Example

Consider the matrix $A = \begin{pmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

- Write down the matrix $A - \lambda I$, where λ is a scalar parameter and I is the identity matrix.
- Find the determinant D of $A - \lambda I$ as a function of λ.
- Find all solutions to $D = 0$.
- For $\lambda = 6$, find the general solution to the linear system $(A - \lambda I)\mathbf{x} = \mathbf{0}$.
- Give a geometric interpretation of your solution to the previous part.

Good Luck!

ANY QUESTIONS?
GOOD LUCK FOR YOUR FINAL EXAMS!
MATHSOC WISHES YOU ALL THE BEST IN YOUR FUTURE STUDY.