MATH1231/1241 Algebra S2 2007 Test 1 v1B

Full Solutions

September 4, 2018

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1. (i) Rewrite the set as $S = \left\{\mathbf{x} \in \mathbb{R}^4 | x_1 - 5x_3 - 2x_4 = 0\right\}$.

- Clearly, $\mathbf{0} \in S$, since $0 - 5\,(0) - 2\,(0) = 0$.
- Now, suppose that $\mathbf{x}, \mathbf{y} \in S$, where $\mathbf{x} = (x_1, x_2, x_3, x_4)^T$ and $\overrightarrow{y} = (y_1, y_2, y_3, y_4)^T$. Then we have the two conditions,

$$x_1 - 5x_3 - 2x_4 = 0 \text{ and } y_1 - 5y_3 - 2y_4 = 0. \qquad (\star)$$

Now, for $\mathbf{x}+\mathbf{y}$ to lie in set S, the following condition must be true:

$$(x_1+y_1)-5(x_3+y_3)-2(x_4+y_4)=0.$$

Considering the LHS,

$$\begin{aligned}(x_1+y_1)-5(x_3+y_3)-2(x_4+y_4) &= (x_1-5x_3-2x_4)+(y_1-5y_3-2y_4)\\ &= 0+0 \quad \text{(using the conditions } (\star))\\ &= 0.\end{aligned}$$

Hence, $\mathbf{x}+\mathbf{y}\in S$, and so set S is closed under addition.

- Suppose that $\lambda \in \mathbb{R}$. Now, for $\lambda\mathbf{x}$ to lie in set S, the following condition must be true,

$$(\lambda x_1)-5(\lambda x_3)-2(\lambda x_4)=0. \qquad (\dagger)$$

Considering the LHS,

$$\begin{aligned}(\lambda x_1)-5(\lambda x_3)-2(\lambda x_4) &= \lambda(x_1-5x_3-2x_4)\\ &= \lambda(0)\,(\dagger) \qquad \text{(using the condition)}\\ &= 0.\end{aligned}$$

Hence, $\lambda\mathbf{x}\in S$, and so set S is closed under scalar multiplication.

Thus, by the Subspace Theorem, S is a subspace of $\mathbb{R}^4$.

(THOUGHT PROCESS: *To prove that S is a subspace of a vector space V, we need to show three things: non-empty (or equivalently, whether* $\mathbf{0}$ *is an element), closure under addition, closure under scalar multiplication. Review the Subspace Theorem on your course notes if you are not familiar with it. As always, make sure you name the theorem you use in a test or an exam to get full marks.*
For more information on this content, please refer to course notes pp.13-16.
For more practice on this type of question, go to course notes p.66 Problems 6.3 (Q12b, 14, 16, 18, 22, 24a in particular).)

(ii) There are an infinite possible number of choices. For example, we can pick a simple one by choosing $x_2=0, x_3=0, x_4=1$ such that $x_1=2$. Thus, one non-zero element in S is $\begin{pmatrix}2\\0\\0\\1\end{pmatrix}$.

2. For these polynomials to span $\mathbb{P}_2$, every polynomial in $\mathbb{P}_2$ in the form $b_0+b_1t+b_2t^2$ where $b_0,b_1,b_2\in\mathbb{R}$ must be expressible as a linear combination of $p_1(t), p_2(t)$ and $p_3(t)$. Let

$\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\alpha p_1(t) + \beta p_2(t) + \gamma p_3(t) + \delta p_4(t) = b_0 + b_1 t + b_2 t^2.$$

Substituting in the polynomials and grouping the terms, we obtain

$$(\alpha + 2\beta - \gamma) + (-\alpha - \beta + 5\gamma + \delta)\, t + (2\alpha + 3\beta - 5\gamma - 2\delta)\, t^2 = b_0 + b_1 t + b_2 t^2.$$

By equating the coefficients, we can obtain 4 simultaneous equations which can be written in an augmented matrix,

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & b_0 \\ -1 & -1 & 5 & 1 & b_1 \\ 2 & 3 & -5 & -2 & b_2 \end{array}\right).$$

Perform the row operations $R_2 \leftarrow R_2 + R_1$ and $R_3 \leftarrow R_3 - 2R_1$,

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & b_0 \\ 0 & 1 & 4 & 1 & b_0 + b_1 \\ 0 & -1 & -3 & -2 & -2b_0 + b_2 \end{array}\right).$$

Next, $R_3 \leftarrow R_3 + R_2$,

$$\left(\begin{array}{cccc|c} \boxed{1} & 2 & -1 & 0 & b_0 \\ 0 & \boxed{1} & 4 & 1 & b_0 + b_1 \\ 0 & 0 & \boxed{1} & -1 & -b_0 + b_1 + b_2 \end{array}\right).$$

If we let $\delta = c$ where c can take any value from $\mathbb{R}$, back-substitution can be performed to find expressions for α, β and γ in terms of c, b_0, b_1, b_2. We observe that a solution always exists for the system, and hence the polynomials form a spanning set for $\mathbb{P}_2$.

Alternatively, for these polynomials to span $\mathbb{P}_2$, exactly three of these polynomials must be linearly independent. Use this fact to construct a matrix to show that three of them are linearly independent.

3. Let $\alpha, \beta, \gamma \in \mathbb{R}$. For these vectors to be linearly independent, the following must only be true for $\alpha = \beta = \gamma = 0$,

$$\alpha \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 5 \\ 5 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Rewriting this in matrix form,

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 4 & 5 & 3 & 0 \\ -2 & 5 & 1 & 0 \end{array}\right).$$

Row-reducing, we first perform the row operations $R_2 \leftarrow R_2 - 4R_1$ and $R_3 \leftarrow R_3 + 2R_1$,

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 9 & 3 & 0 \end{array}\right).$$

Next, $R_3 \leftarrow R_3 + 3R_2$,

$$\left(\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 0 \\ 0 & \boxed{-3} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

We observe that there are infinitely many solutions for α, β and γ as not all columns are leading, and hence the vectors are <u>linearly dependent</u>.

(THOUGHT PROCESS: *The essential point of the definition of linear independence is that the only way* $\lambda_1 \overrightarrow{v_1} + \lambda_2 \overrightarrow{v_2} + \cdots + \lambda_n \overrightarrow{v_n} = \overrightarrow{0}$ *is satisfied is that all the scalars are zero. For more information and some detailed, worked examples on this content, please refer to course notes pp.25-27.*
For more practice on this type of question, go to course notes pp.69-70 Problems 6.5.)

MATH1231/1241 Algebra Test 1 2008 S0 v1A

Full Solutions

August 23, 2018

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1. To show that it is a subspace, we need to show 3 things,

   - Clearly, the 2×2 zero matrix lies in S, since $0 + 0 = 3\,(0)$.
   - Suppose $A, B \in S$ where $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ such that

$$a_{11} + a_{12} = 3a_{22}, \qquad (1)$$

$$b_{11} + b_{12} = 3b_{22}. \qquad (2)$$

Adding together equations (1) and (2), we obtain

$$(a_{11}+b_{11})+(a_{12}+b_{12})=3\,(a_{22}+b_{22})$$

which implies that $A+B=\begin{pmatrix}a_{11}+b_{11} & a_{12}+b_{12}\\ a_{21}+b_{21} & a_{22}+b_{22}\end{pmatrix}\in S$.
Thus, S is closed under addition.

- Suppose $\lambda\in\mathbb{R}$. Now consider $\lambda\times(1)$,

$$\lambda\,(a_{11}+a_{12})=\lambda\,(3a_{22})\,.$$

This is the same as

$$(\lambda a_{11})+(\lambda a_{12})=3\,(\lambda a_{22})\,.$$

This implies that $\lambda A=\begin{pmatrix}\lambda a_{11} & \lambda a_{12}\\ \lambda a_{21} & \lambda a_{22}\end{pmatrix}\in S$.
Thus, S is closed under scalar multiplication.

Hence, by the Subspace Theorem, S is a subspace of $M_{2,2}(\mathbb{R})$.

(THOUGHT PROCESS: *To prove that S is a subspace of a vector space V, we need to show three things: non-empty (or equivalently, whether* **0** *is an element), closure under addition, closure under scalar multiplication. Review the Subspace Theorem on your course notes if you are not familiar with it. As always, make sure you name the theorem you use in a test or an exam to get full marks.*
For more information on this content, please refer to course notes pp.13-16.
For more practice on this type of question, go to course notes p.66 Problems 6.3 (Q12b, 14, 16, 18, 22, 24a in particular).)

2. (a) The vectors $\overrightarrow{v_1},\overrightarrow{v_2},\ldots,\overrightarrow{v_n}$ are linearly dependent in a vector space V if and only if we can find scalars $\lambda_1,\lambda_2,\ldots,\lambda_n$ <u>not all zero</u> such that

$$\lambda_1\overrightarrow{v_1}+\lambda_2\overrightarrow{v_2}+\cdots+\lambda_n\overrightarrow{v_n}=\overrightarrow{0}.$$

(b) Let $\alpha,\beta,\gamma\in\mathbb{R}$. We consider

$$\alpha A_1+\beta A_2+\gamma A_3=0.$$

Substituting in the matrices, we find

$$\begin{pmatrix}\alpha+2\gamma & 2\alpha+\beta+\gamma\\ 2\alpha+\beta+2\gamma & \alpha+\gamma\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}.$$

Equating the elements, we obtain 4 linear equations which can be expressed in matrix form,

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right).$$

Performing the row operations $R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 - 2R_1$ and $R_4 \leftarrow R_4 - R_1$,

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right).$$

Next, $R_3 \leftarrow R_3 - R_2$,

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right).$$

Next, $R_4 \leftarrow R_4 + R_3$,

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The solution is $\alpha = \beta = \gamma = 0$, and hence the matrices are linearly independent.

(THOUGHT PROCESS: *The essential point of the definition of linear independence is that the only way* $\lambda_1 \overrightarrow{v_1} + \lambda_2 \overrightarrow{v_2} + \cdots + \lambda_n \overrightarrow{v_n} = \overrightarrow{0}$ *is satisfied is that all the scalars are zero. For more information and some detailed, worked examples on this content, please refer to course notes pp.25-27.*
For more practice on this type of question, go to course notes pp.69-70 Problems 6.5.)

3. (i) A basis for W consists of a set of linearly independent vectors that span to form W. To determine this, we construct a matrix with each vector as a column,

$$\begin{pmatrix} 1 & 2 & -1 & 1 \\ 3 & 7 & 0 & 1 \\ 1 & 1 & -4 & 3 \end{pmatrix}.$$

Performing the row-operations $R_2 \leftarrow R_2 - 3R_1$ and $R_3 \leftarrow R_3 - R_1$, we obtain

$$\begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & -1 & -3 & 2 \end{pmatrix}.$$

Next, $R_3 \leftarrow R_3 + R_2$,

$$\begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We observe that removing $\overrightarrow{v_3}$ and $\overrightarrow{v_4}$ (the columns that are non-leading) leaves us with the linearly independent vectors $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$. Hence, $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ form a basis for W.

(ii) The vector in question is $\overrightarrow{v_4}$. We are simply trying to find $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\overrightarrow{v_1} + \beta\overrightarrow{v_2} = \overrightarrow{v_4}.$$

Constructing an augmented matrix,

$$\left(\overrightarrow{v_1} \quad \overrightarrow{v_2} \,\middle|\, \overrightarrow{v_4}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 7 & 1 \\ 1 & 1 & 3 \end{array}\right).$$

Now, row-reducing, we perform the row operations $R_2 \leftarrow R_2 - 3R_1$ and $R_3 \leftarrow R_3 - R_1$,

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{array}\right).$$

Next, $R_3 \leftarrow R_3 + R_2$,

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right).$$

Hence, we have $\beta = -2$ and $\alpha = 5$. That is,

$$5\overrightarrow{v_1} - 2\overrightarrow{v_2} = \overrightarrow{v_4}.$$

(THOUGHT PROCESS: *Definition of bases for a vector space is on p.35 of course notes. For more information and worked examples on this content, please refer to course notes pp.36-38.*
For more practice on this type of question, go to course notes pp.71-73 Problems 6.6.)

MATH1231/1241 Algebra Test 1 2008 S2 v3b

Answers/Hints

August 23, 2018

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1. Very similar to other questions - you must prove that the set is closed under addition, closed under scalar multiplication, and contains a zero vector. This set satisfies these conditions and thus is a subspace by the Subspace Theorem.
(THOUGHT PROCESS: *To prove that S is* **NOT** *a subspace of a vector space V, you show a counterexample. To prove that S is a subspace, you show three things: non-empty (or equivalently, whether* $\mathbf{0}$ *is an element), closure under addition, closure under scalar multiplication. Review the Subspace Theorem on your course notes if you are not familiar with it. As always, make sure you name the theorem you use to get full marks.*
For more information on this content, please refer to course notes pp.13-16.
For more practice on this type of question, go to course notes p.66 Problems 6.3 (Q12b, 14, 16, 18, 22, 24a in particular).)

2. (i) Suppose that $\vec{w}$ is an element of $\mathbb{R}^3$. By considering the augmented matrix with $\vec{w}$ as the subject and using Gaussian elimination, we observe that the system of linear equations do not hold for all values of $\vec{w}$ (due to the bottom row of zeros equaling $12a_1 - 5a_2 + a_3$). Thus the set does not span $\mathbb{R}^3$. Vector $\vec{w}$ only spans the set of vectors iff the condition $12a_1 - 5a_2 + a_3 = 0$ holds.

   (ii) Vector $\vec{u}$ lies in the span of vectors if the above condition holds. Substituting u into the equation, we obtain $LHS = 7 \neq 0$ and thus $\vec{u}$ is not in the span of vectors.

   (*For more information on this content, please refer to course notes pp.20-24.*
   For more practice on this type of question, go to course notes pp.68-69 Problems 6.4.)

3. Consider the linear combination of the vectors and construct an augmented matrix of these vectors with a column of zeros. If the solutions to these system of equations are when the coefficients of the vectors are zero, then the vectors are linearly independent from each other. In this scenario, we can observe that this matrix has infinitely many solutions for the system, and thus the vectors are linearly dependent.
   (THOUGHT PROCESS: *The essential point of the definition of linear independence is that the only way* $\lambda_1\overrightarrow{v_1} + \lambda_2\overrightarrow{v_2} + \cdots + \lambda_n\overrightarrow{v_n} = \overrightarrow{0}$ *is satisfied is that all the scalars are zero.*
   For more information and some detailed, worked examples on this content, please refer to course notes pp.25-27.
   For more practice on this type of question, go to course notes pp.69-70 Problems 6.5.)

MATH1231/1241 Algebra S2 2009 Test 1 v2B

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1. Note that S does not contain the zero vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (because $3 \times 0 - 2 \times 0 = 0 \neq 1$). Therefore, S cannot be a subspace of $\mathbb{R}^2$.

   THOUGHT PROCESS:
   It takes just a single violation of the vector space properties. See §6.3 in the course notes for many similar examples.

2. We first need to establish whether the set containing these polynomials is linearly independent or not. Noticing that we are working with quadratic polynomials, if it is linearly independent, there can only be at most 3 polynomials in the set. But since we are working with 4 polynomials, the set containing these polynomials is thus linearly dependent .
   Next, to find which one of these polynomials can be written as a linear combination of

the others, we let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\alpha p_1(x) + \beta p_2(x) + \gamma p_3(x) + \delta p_4(x) = 0.$$

Substituting in the polynomials, we obtain

$$\alpha\left(1 - 3x + 2x^2\right) + \beta\left(3 - 8x + 5x^2\right) + \gamma\left(1 + x - 2x^2\right) + \delta\left(-x + 8x^2\right) = 0.$$

Grouping together like terms,

$$(\alpha + 3\beta + \gamma) + (-3\alpha - 8\beta + \gamma - \delta)\,x + (2\alpha + 5\beta - 2\gamma + 8\delta)\,x^2 = 0.$$

By equating the coefficients, we see that

$$\alpha + 3\beta + \gamma = 0$$

$$-3\alpha - 8\beta + \gamma - \delta = 0$$

$$2\alpha + 5\beta - 2\gamma + 8\delta = 0.$$

We set up an augmented matrix,

$$\left(\begin{array}{cccc|c} 1 & 3 & 1 & 0 & 0 \\ -3 & -8 & 1 & -1 & 0 \\ 2 & 5 & -2 & 8 & 0 \end{array}\right). \tag{$\star$}$$

Row-reducing, we perform the operations $R_2 \leftarrow R_2 + 3R_1$ and $R_3 \leftarrow R_3 - 2R_1$,

$$\left(\begin{array}{cccc|c} 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & -1 & 0 \\ 0 & -1 & -4 & 8 & 0 \end{array}\right).$$

Next, $R_3 \leftarrow R_3 + R_2$,

$$\left(\begin{array}{cccc|c} \boxed{1} & 3 & 1 & 0 & 0 \\ 0 & \boxed{1} & 4 & -1 & 0 \\ 0 & 0 & 0 & \boxed{7} & 0 \end{array}\right).$$

As the first, second and fourth columns are leading, p_1, p_2, p_4 are linearly independent polynomials. However, the third column is non-leading. This implies that we can express p_3 as a linear combination of p_1, p_2 and p_4.

To find this linear combination, set γ to be a free parameter (as it belongs to column 3, which is non-leading). From row 3, we see $\delta = 0$. Now, row 2 implies that $\beta + 4\gamma - \delta = 0 \Rightarrow \beta = -4\gamma$ (as $\delta = 0$. Now row 1 gives us $\alpha = -3\beta - \gamma = 12\gamma - \gamma = 11\gamma$, since $\beta = -4\gamma$. Hence substituting these into the original equation $\alpha p_1(x) + \beta p_2(x) + \gamma p_3(x) + \delta p_4(x) = 0$, we have

$$\begin{aligned}11\gamma p_1(x) - 4\gamma p_2(x) + \gamma p_3(x) + 0p_4(x) = 0, \quad \forall \gamma \in \mathbb{R}\\ \Rightarrow \gamma p_3(x) = -11\gamma p_1(x) + 4\gamma p_2(x)\\ \Rightarrow p_3(x) = -11p_1(x) + 4p_2(x) \quad \text{(setting } \gamma = 1\text{)}.\end{aligned}$$

Remarks/Tips. In your working out, you can just go straight to the augmented matrix in ($\star$). We have just provided detailed explanations before that step for your own benefit (so that you may understand why we set up that augmented matrix). Also, near the end, we set $\gamma = 1$. You could also just set $\gamma = 1$ at the start of the back-substitution (rather than using an arbitrary free parameter γ).

3. As we know, the vectors will form a basis for $\mathbb{R}^3$ if and only if the matrix whose columns are $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ has every row and column leading in the row echelon form. Set this up as a matrix:

$$\begin{pmatrix} \boxed{1} & 3 & 4 \\ 1 & 5 & 3 \\ -2 & 4 & -7 \end{pmatrix}.$$

Row-reducing by performing the row operations $R_2 \leftarrow R_2 - R_1$ and $R_3 \leftarrow R_3 + 2R_1$,

$$\begin{pmatrix} \boxed{1} & 3 & 4 \\ 0 & \boxed{2} & -1 \\ 0 & 10 & 1 \end{pmatrix}.$$

Next, $R_3 \leftarrow R_3 - 5R_2$,

$$\begin{pmatrix} \boxed{1} & 3 & 4 \\ 0 & \boxed{2} & -1 \\ 0 & 0 & \boxed{6} \end{pmatrix}.$$

As we can see, every row and column in the row echelon form is leading. Hence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form a basis for $\mathbb{R}^3$.

THOUGHT PROCESS:
For the vectors to form a basis, they must (i) span the space, and (ii) be linearly independent. We show both here by placing $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ into a matrix and row-reducing. See the examples in §6.6.1 of the course notes.

4. (i) False. If they are linearly dependent (e.g. because one of them is the zero vector), then it cannot be a basis.

   (ii) True. A basis contains the smallest number of linearly independent vectors to span the set, that is, there can only be 5 vectors in a basis for $\mathbb{R}^5$.

   (iii) False. For example, the set $\{\mathbf{e}_1, 2\mathbf{e}_1, 3\mathbf{e}_1, \ldots, 7\mathbf{e}_1\}$ (where $\mathbf{e}_1$ is the first standard basis vector of $\mathbb{R}^5$) will not span $\mathbb{R}^5$ (its span is just span $\{\mathbf{e}_1\}$, so for example $\mathbf{e}_2$ is not in this span, so the set is not spanning).

THOUGHT PROCESS:

See §6.5 and §6.6 of the course notes, on spans, linear independence and bases.

MATH1231/1241 Algebra Test 1 2011 S2 v1A

Full Solutions

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1. To show that S is a subspace of $\mathbb{R}^3$, we must show three things: that S is non-empty, closed under addition and closed under scalar multiplication.

   - Clearly $\mathbf{0} = (0,0,0)^T \in S$ as $7(0) + 3(0) = 0$ and $(0) - 4(0) = 0$. So S is non-empty.
   - To show that it is closed under addition, let $\boldsymbol{x}, \boldsymbol{y} \in S$ where $\boldsymbol{x} = (x_1, x_2, x_3)^T$ and

$\boldsymbol{y} = (y_1, y_2, y_3)^T$. That is,

$$\begin{aligned} 7x_1 + 3x_2 &= 0 \quad (1) \\ x_2 - 4x_3 &= 0 \quad (2) \\ 7y_1 + 3y_2 &= 0 \quad (3) \\ y_2 - 4y_3 &= 0. \quad (4) \end{aligned}$$

Adding (1) and (2) together, and (3) and (4) together, we obtain

$$\begin{aligned} (1) + (3): \quad & 7x_1 + 3x_2 + 7y_1 + 3y_2 = 0 \\ \implies & 7(x_1 + y_1) + 3(x_2 + y_2) = 0. \\ (1) + (4): \quad & x_2 - 4x_3 + y_2 - 4y_3 = 0 \\ & (x_2 + y_2) - 4(x_3 + y_3) = 0. \end{aligned}$$

Hence, $\boldsymbol{x} + \boldsymbol{y} \in S$ and S is closed under addition.

- Next, to show that it is closed under scalar multiplication, let $\lambda \in \mathbb{R}$ and consider the following,

$$\begin{aligned} \lambda \times (1): \quad & \lambda(7x_1 + 3x_2) = \lambda \times 0 \\ \implies & 7(\lambda x_1) + 3(\lambda x_2) = 0 \\ \lambda \times (2): \quad & \lambda(x_2 - 4x_3) = \lambda \times 0 \\ \implies & (\lambda x_2) - 4(\lambda x_3) = 0. \end{aligned}$$

Hence, $\lambda\boldsymbol{x} \in S$ and S is closed under scalar multiplication.

As S is non-empty, is closed under addition and scalar multiplication, by the Subspace Theorem, S is a subspace of $\mathbb{R}^3$.

THOUGHT PROCESS: *The solution is fairly straight forward and contains the necessary explanation. The first property requires you to show that* $\mathbf{0}$ *satisfies the equation. Remember to revise the conditions, as they may provide more complicated equations.*
For more practice on this type of question, please refer to page 14 of the course pack.

2. For $\boldsymbol{b}$ to be an element of span $(\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3})$, then there must exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\alpha\boldsymbol{v_1} + \beta\boldsymbol{v_2} + \gamma\boldsymbol{v_3} = \boldsymbol{b}.$$

Substituting in the vectors writing this as an augmented matrix,

$$\left(\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 1 & 3 & 1 & b_2 \\ 2 & 4 & 1 & b_3 \\ -1 & 1 & 6 & b_4 \end{array}\right).$$

For $\boldsymbol{b}$ to be in the span, we must be able to row-reduce to find solutions for α, β and γ which will be in terms of b_1, b_2, b_3 and b_4.
Row-reducing, $R_2 \leftarrow R_2 - R_1$, $R_3 \leftarrow R_3 - 2R_1$ and $R_4 \leftarrow R_4 + R_1$,

$$\left(\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 0 & 1 & 3 & -b_1 + b_2 \\ 0 & 0 & 5 & -2b_1 + b_3 \\ 0 & 3 & 4 & b_1 + b_4 \end{array}\right).$$

Next, $R_4 \leftarrow R_4 - 3R_2$,

$$\left(\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 0 & 1 & 3 & -b_1 + b_2 \\ 0 & 0 & 5 & -2b_1 + b_3 \\ 0 & 0 & -5 & 4b_1 - 3b_2 + b_4 \end{array}\right).$$

Finally, $R_4 \leftarrow R_4 + R_3$,

$$\left(\begin{array}{ccc|c} 1 & 2 & -2 & b_1 \\ 0 & 1 & 3 & -b_1 + b_2 \\ 0 & 0 & 5 & -2b_1 + b_3 \\ 0 & 0 & 0 & 2b_1 - 3b_2 + b_3 + b_4 \end{array}\right).$$

Hence, from R_4, a condition for $\boldsymbol{b}$ to be an element of $\text{span}(\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4})$ is that

$$2b_1 - 3b_2 + b_3 + b_4 = 0.$$

This is true since once this is true, we can perform back substitution for the rest of the values of α, β and γ without any problems (thus we can find a solution).

THOUGHT PROCESS: *The first step follows from the fact that a vector can only be an element of a span if it can be written as a linear combination of the other elements. Hence there must be solutions for the three parameters.*
For more practice on this type of question, please refer to page 22 of the course pack.

3. (i) False. A basis for P_4 requires 5 linearly independent polynomials in P_4. Suppose the set of 5 polynomials were linearly dependent. Then it would be untrue.

   (ii) True. We require 5 linearly independent polynomials in P_4 for a basis in P_4.

THOUGHT PROCESS: *The solution is fairly straight forward and contains the necessary explanation.*

For more practice on this type of question, please refer to page 41 of the course pack.

4. To show that T is not a linear transformation, we simply need to show that it does not satisfy a property of them. Knowing that linear transformations satisfy the scalar multiplication condition, i.e.

$$T(\lambda x) = \lambda T(x),$$

(and noticing that the expression is not going to satisfy this), we try apply this to the given transformation.

Let $\lambda \in \mathbb{R}$ and consider $T(\lambda x)$,

$$\begin{aligned} T(\lambda x) &= (\lambda x)\cos(\lambda x) \\ &= \lambda x \cos(\lambda x) \neq \lambda x \cos(x) \qquad \text{(as } \cos(\lambda x) \neq \cos(x) \text{ for all } \lambda \in \mathbb{R}.) \end{aligned}$$

Thus, T is not a linear transformation as it does not satisfy the multiplicative condition.

THOUGHT PROCESS: *The solution is fairly straight forward and contains the necessary explanation. Save time in the exam by choosing the property which the proposed linear transformation clearly does not satisfy.*

For more practice on this type of question, please refer to page 82 of the course pack.

MATH1231/1241 Algebra Test 1 2011 S2 v1B

Answers/Hints

September 4, 2018

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1. (i) Many, many choices for your answer. Choose a value for x_1 or x_2 and solve for the other. An example answer is $\mathbf{x} = \binom{2}{-1}$.

   (ii) You can do this generally by using $\mathbf{x} = \binom{x_1}{x_2}$ where $\mathbf{x} \in S$ or by using the $\mathbf{x}$ possibility you found in part (i). Let $\lambda \in \mathbb{R}$ and consider $\lambda\mathbf{x}$ and go from here to show that it is not closed under scalar multiplication.

   THOUGHT PROCESS: *Show that if* $\mathbf{x}$ *satisfies the equation, then* $\lambda\mathbf{x}$ *does not for all real values of* λ.
   For more practice on this type of question, please refer to page 14 of the course pack.

2. (i) For $q \in \operatorname{span}(p_1, p_2, p_3)$, there exists $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\alpha p_1 + \beta p_2 + \gamma p_3 = q$. Form an augmented matrix and row-reduce to eventually find that q does lie in the span!

(ii) Recall to have a spanning set for P_2, we require there to be 3 linearly independent polynomials in P_2. You need to try and show this. In the process, you will find that only 2 are linearly independent, and thus the set $\{p_1, p_2, p_3\}$ does not span P_2.
THOUGHT PROCESS: *You can show this using a matrix and row reducing. You will find that there is a non leading column, hence one of the vectors is not linearly independent.*
For more practice on this type of question, please refer to page 23 of the course pack.

3. (a) To evaluate this, recall the property of linear transformations that for constants $\alpha, \beta \in \mathbb{R}$, we have

$$T\begin{pmatrix}1\\0\end{pmatrix} = T\left(\alpha\begin{pmatrix}2\\3\end{pmatrix} + \beta\begin{pmatrix}1\\2\end{pmatrix}\right) = \alpha T\begin{pmatrix}2\\3\end{pmatrix} + \beta T\begin{pmatrix}1\\2\end{pmatrix}.$$

That is, you need to find α and β such that $\alpha\binom{2}{3} + \beta\binom{1}{2} = \binom{1}{0}$. You will find that $\alpha = 2$ and $\beta = -3$, and eventually that $T\binom{1}{0} = \binom{3}{-2}$.
THOUGHT PROCESS: *The solution is fairly straight forward and contains the necessary explanation.*
For more practice on this type of question, please refer to page 84 of the course pack.

(b) We are basically trying to find a matrix A that will multiply with the vector $\mathbf{x}$ you plug in to get the result of the linear transformation. To do this, we can plug in the standard basis vectors of the domain $\mathbb{R}^2$ to obtain the columns of the matrix A one-by-one.
To prove this fact for you, notice that

$$\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\\c\end{pmatrix}.$$

You will find that $A = \begin{pmatrix}3 & 1\\-2 & 4\end{pmatrix}$.

THOUGHT PROCESS: *The previous question required you to find $T\binom{1}{0}$, which is equal to $\binom{a}{c}$. Use a similar method with $T\binom{0}{1}$ to find $\binom{b}{d}$.*
For more practice on this type of question, please refer to page 87 of the course pack.

MATH1231/1241 Algebra S2 2011 Test 1 v2A

Answers/Hints & Worked Sample Solutions

September 4, 2018

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1. **Hints**

   To show that S is a subspace of $\mathbb{R}^2$, we need to show that S is non-empty, is closed under addition and closed under scalar multiplication. From there, we can use the Subspace Theorem.

   Recall matrix properties such as $A\mathbf{x} + A\mathbf{y} = A(\mathbf{x} + \mathbf{y})$ to help answer the question. Here is a sample answer.

   Worked Sample Solutions

   Note that S is a sub*set* of $\mathbb{R}^2$, which is a vector space. Now, clearly S is nonempty, as the zero vector of $\mathbb{R}^2$, $\mathbf{0}_{\mathbb{R}^2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, indeed satisfies $A\mathbf{0}_{\mathbb{R}^2} = \mathbf{0}_{\mathbb{R}^3}$, where $\mathbf{0}_{\mathbb{R}^3}$ is the zero vector of $\mathbb{R}^3$, i.e. $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ (since any matrix times the relevant zero vector results in a zero vector).

Now, suppose $\mathbf{x}, \mathbf{y}$ are vectors in S. Then $\mathbf{x} + \mathbf{y} \in \mathbb{R}^2$ and

$$\begin{aligned} A(\mathbf{x} + \mathbf{y}) &= A\mathbf{x} + A\mathbf{y} \quad \text{(Distributive Law of matrix multiplication)} \\ &= \mathbf{0}_{\mathbb{R}^3} + \mathbf{0}_{\mathbb{R}^3} \quad (\text{as } \mathbf{x}, \mathbf{y} \in S) \\ &= \mathbf{0}_{\mathbb{R}^3} \\ &\Rightarrow \mathbf{x} + \mathbf{y} \in S. \end{aligned}$$

Hence S is closed under addition.
Finally, suppose $\mathbf{x}$ is a vector in S, and let α be a scalar. Then $\alpha\mathbf{x}$ is a vector in $\mathbb{R}^2$ satisfying

$$\begin{aligned} A(\alpha\mathbf{x}) &= \alpha(A\mathbf{x}) \\ &= \alpha\mathbf{0}_{\mathbb{R}^3} \quad (\text{as } \mathbf{x} \in S) \\ &= \mathbf{0}_{\mathbb{R}^3} \\ &\Rightarrow \alpha\mathbf{x} \in S. \end{aligned}$$

Hence S is closed under scalar multiplication. It follows from the Subspace Theorem that S is a subspace of $\mathbb{R}^2$.

2. **Hints**

The column space of A is the span of the columns, i.e. if $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 4 \\ -3 \\ 2 \\ 7 \end{pmatrix}$, then we want to see if $\mathbf{b} \in \operatorname{span}(\mathbf{v_1}, \mathbf{v_2})$.
To do this, we need to check if there exists $\alpha, \beta \in \mathbb{R}$ such that $\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 = \mathbf{b}$.
You will find that no solutions exist for α and β such that $\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 = \mathbf{b}$, and so $\mathbf{b}$ is not in the column space of A. Here is a sample answer.

Worked Sample Solutions

Note that $\mathbf{b}$ is in the column space of A if and only if there exist scalars α, β such that $\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 = \mathbf{b}$, where $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 4 \\ -3 \\ 2 \\ 7 \end{pmatrix}$.

So we try and solve $\alpha \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 4 \\ -3 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 5 \\ -6 \end{pmatrix}$.

Comparing the third entry of each vector ($0\alpha + 2\beta = 5$), it is evident that β must be 2.5. So comparing fourth entries now, we have

$$\begin{aligned} 2\alpha + 7\beta &= -6 \\ 2\alpha + 7 \times 2.5 &= -6 \\ \Rightarrow 2\alpha &= -23.5 \\ \Rightarrow \alpha &= -11.75. \end{aligned}$$

But now the first entries tell us that

$$\begin{aligned} \alpha + 4\beta &= -1 \\ \Rightarrow -11.75 + 4 \times 2.5 &= -1 \\ \Rightarrow -11.75 + 10 &= -1 \\ \Rightarrow -1.75 &= -1, \end{aligned}$$

which is a contradiction. So there is no solution for α, β, so $\mathbf{b}$ is *not* in the column space of A.

Alternatively, you can create the corresponding augmented matrix:

$$\left(\begin{array}{cc|c} 1 & 4 & -1 \\ -1 & -3 & 5 \\ 0 & 2 & 5 \\ 2 & 7 & -6 \end{array}\right)$$

and after row-reducing, realise that no solutions exist for α and β.

3. (i) **Sample Answer**

If S is a basis for V, then it is a set of linearly independent vectors, and so $m \leq n$. But also, $\text{span}\,(S) = V$, and so $n \leq m$. This implies that $m = n$.

Alternative answer: Recall that the definition of the dimension of a vector space is the number of vectors in any basis for it (provided this is finite). Hence by definition, m and n are equal.

(ii) **Sample Answer**

If S is linearly dependent, no relationship can be inferred between m and n. S can be linearly dependent no matter what value m takes, so long as another one of the vectors in S can be written as a linear combination of the others.

Alternative answer: Note that for *any* positive integer $m \geq 2$, the following set is linearly dependent: $\{\mathbf{0}, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$, where $\mathbf{v}_2, \ldots, \mathbf{v}_m$ are any vectors from V. (Recall that any set with the zero vector is linearly dependent.)
For the case when S only has one vector, S can still be linearly dependent if its only vector is the zero vector, $\mathbf{0}$.
This shows that there is no relation between m and n, since for *any* m, there exist linearly dependent sets.

4. **Hints**
Very similar to MATH1231/1241 Algebra 2011 S2 Test 1 v1B Q3.
To check if T is a linear transformation, it would be easiest to check if the following property holds: that $T(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha T(\mathbf{x}) + \beta T(\mathbf{y})$ where $\alpha, \beta \in \mathbb{R}$. A sample answer is provided below.
Worked Sample Solutions
We will answer this question with a proof by contradiction.
Suppose that T is linear. Let $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$. Note that $\mathbf{w} = 5\mathbf{e}_1 - \mathbf{e}_2$. So we have

$$\begin{aligned} T(\mathbf{w}) &= T(5\mathbf{e}_1 - \mathbf{e}_2) \\ &= 5T(\mathbf{e}_1) - T(\mathbf{e}_2) \quad \text{(linearity)} \\ &= 5\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \quad \text{(using given values)} \\ &= \begin{pmatrix} 3 \\ 2 \\ -7 \end{pmatrix}. \end{aligned}$$

But we were told that $T(\mathbf{w}) = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$, which is not equal to $\begin{pmatrix} 3 \\ 2 \\ -7 \end{pmatrix}$. So we have a contradiction. Hence T is not linear.

MATH1231/1241 Algebra S2 2012 Test 1 v1A

Answers/Hints & Worked Sample Solutions

September 4, 2018

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1. **Hints**

   To show that S is a subspace of $\mathbb{R}^3$, you need to show that it is non-empty, is closed under addition and under scalar multiplication. A core step you'll be required to make is $\mathbf{x} = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ such that $x_1 - 2x_2 = 0$ and $5x_2 + x_3 = 0$.

   Worked Sample Solution

   Note that clearly S is a sub*set* of $\mathbb{R}^3$, which is a vector space. Since $0 - 2 \times 0 = 0$ and $5 \times 0 + 0 = 0$, the zero vector from $\mathbb{R}^3$, namely $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, is in S, so S is non-empty.

Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ be in S. Then $\mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix}$ is an element of $\mathbb{R}^3$ since

$$\begin{aligned}
(x_1 + y_1) - 2\,(x_2 + y_2) &= (x_1 - 2x_2) + (y_1 - 2y_2) \quad \text{(rearranging terms)} \\
&= 0 + 0 \quad (\text{since } \mathbf{x}, \mathbf{y} \in S) \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
5\,(x_2 + y_2) + (x_3 + y_3) &= (5x_2 + x_3) + (5y_2 + y_3) \quad \text{(rearranging terms)} \\
&= 0 + 0 \quad (\text{since } \mathbf{x}, \mathbf{y} \in S) \\
&= 0.
\end{aligned}$$

Thus S is closed under addition.

Now, suppose $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in S$ and $\alpha \in \mathbb{R}$, then $\alpha\mathbf{x} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{pmatrix}$ is a vector in $\mathbb{R}^3$. Moreover,

$$\begin{aligned}
\alpha x_1 - 2\,(\alpha x_2) &= \alpha\,(x_1 - 2x_2) \\
&= \alpha \times 0 \quad (\text{as } \mathbf{x} \in S) \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
5\,(\alpha x_2) + \alpha x_3 &= \alpha\,(5x_2 + x_3) \\
&= \alpha \times 0 \quad (\text{as } \mathbf{x} \in S) \\
&= 0.
\end{aligned}$$

Hence S is closed under scalar multiplication.

It follows by the Subspace Theorem that S is a subspace of $\mathbb{R}^3$.

2. **Hints**

   Set up an augmented matrix and row-reduce. Your conditions on $\mathbf{b}$ to lie in the span$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ should look like

$$15b_1 + 6b_2 - b_3 + b_4 = 0.$$

Worked Sample Solution

As usual we set up the relevant augmented matrix, namely $\left(\begin{array}{ccc|c} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{b} \end{array}\right)$, and row-reduce. Note that this matrix corresponds to the equation $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{b}$. This is because (by definition of span) $\mathbf{b} \in \text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ if and only if this equation has a solution for α_1, α_2 and α_3.

We can represent $\mathbf{b}$ in the augmented matrix either as $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$ or by creating a column each for b_1, b_2, b_3, b_4. Here, we do the latter. And so, we have

$$
\left[\begin{array}{ccc|cccc} \boxed{1} & 3 & -2 & 1 & 0 & 0 & 0 \\ -2 & -5 & 5 & 0 & 1 & 0 & 0 \\ 0 & 7 & 1 & 0 & 0 & 1 & 0 \\ -3 & -8 & 1 & 0 & 0 & 0 & 1 \end{array}\right]
\xrightarrow[R_4 \rightsquigarrow R_4 + 3R_1]{R_2 \rightsquigarrow R_2 + 2R_1}
\left[\begin{array}{ccc|cccc} \boxed{1} & 3 & -2 & 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & 1 & 2 & 1 & 0 & 0 \\ 0 & 7 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -5 & 3 & 0 & 0 & 1 \end{array}\right]
$$

$$
\xrightarrow[R_4 \rightsquigarrow R_4 - R_2]{R_3 \rightsquigarrow R_3 - 7R_2}
\left[\begin{array}{ccc|cccc} \boxed{1} & 3 & -2 & 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & \boxed{-6} & -14 & -7 & 1 & 0 \\ 0 & 0 & -6 & 1 & -1 & 0 & 1 \end{array}\right]
$$

$$
\xrightarrow{R_4 \rightsquigarrow R_4 - R_3}
\left[\begin{array}{ccc|cccc} \boxed{1} & 3 & -2 & 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & \boxed{-6} & -14 & -7 & 1 & 0 \\ 0 & 0 & 0 & 15 & 6 & -1 & 1 \end{array}\right].
$$

Recall that $\mathbf{b}$ will be in the span if and only if the columns on the right-hand side of the augmented matrix in row-echelon form are all non-leading. We see that from row 4 (the only row with a zero row in the left-hand part), the necessary and sufficient condition on b_1, b_2, b_3, b_4 for $\mathbf{b}$ to be in the span of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is $15b_1 + 6b_2 - b_3 + b_4 = 0$.

3. **<u>Hints</u>**

Remember that the dimension of P_5 is 6 (not 5). (In general, the dimension of P_n is $n+1$, for any $n \in \mathbb{N}$.) Also make sure to recall the relationships between dimension, basis, and the number of elements in a set.

<u>Worked Sample Solution</u>

(i) False. A counterexample will be a set of six polynomials that are all constant polynomials. Then, this set will not span $\mathbb{P}_5$, and so will not be a basis.

(ii) True, as the dimension of $\mathbb{P}_5$ is 6 and so six linearly independent polynomials in $\mathbb{P}_5$ will form a basis for $\mathbb{P}_5$. For example, the standard basis for $\mathbb{P}_5$ $\big($the set $\{1, t, t^2, t^3, t^4, t^5\}\big)$ is a set of six polynomials in $\mathbb{P}_5$ that form a basis for $\mathbb{P}_5$.

3. **<u>Hints</u>**

We need to check which properties of linear transformations do not hold. Remember, to show that a map T is *not* linear, we just need to produce a single example that demonstrates that T does not satisfy a condition of linearity.

Worked Sample Solution

Note that $T\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)^2 \sin\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)^2$. Also, $T\left(2\left(\frac{\pi}{2}\right)\right) = T(\pi) = \pi^2 \sin \pi = 0$. Hence $T\left(2\left(\frac{\pi}{2}\right)\right) \neq 2T\left(\frac{\pi}{2}\right)$, and so T does not satisfy the scalar multiplication condition, whence T is not linear.

MATH1231/1241 Algebra S2 2012 Test 1 v1B

Answers/Hints & Worked Sample Solutions

September 4, 2018

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1. (i) There are many, many different choices. But one easy example you can try is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, since $1 - 5 \times 0 < 2$.

   (ii) **Hints**

   To show that a set is not closed under scalar multiplication, we need to find a vector in the set, such that when we multiply this vector by a scalar, the resulting vector will not be in S.

   Worked Sample Solution

   Using the vector in part (i) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we can multiply it by 2, so that the resulting vector is $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$. Clearly, $2 - 5(0) = 2$. So the resulting vector is not in S. Hence, S is not closed under scalar multiplication.

2. (i) **Hints**

To check if a particular polynomial is an element of the span of the given 3 vectors, we just need to see if we can express this particular polynomial as a linear combination of these 3 vectors, i.e. if there exist α_1, α_2 and α_3 such that $\alpha_1 p_1(x) + \alpha_2 p_2(x) + \alpha_3 p_3(x) = q(x)$.

Worked Sample Solution

We construct the corresponding augmented matrix to represent this equation.

$$\left(\begin{array}{ccc|c} \boxed{1} & 3 & -1 & 1 \\ -1 & -1 & 0 & 2 \\ 2 & 0 & 1 & -7 \end{array}\right).$$

Performing the row operations $R_2 \rightsquigarrow R_2 + R_1$ and $R_3 \rightsquigarrow R_3 - 2R_1$ gives the augmented matrix

$$\left(\begin{array}{ccc|c} \boxed{1} & 3 & -1 & 1 \\ 0 & \boxed{2} & -1 & 3 \\ 0 & -6 & 3 & -9 \end{array}\right).$$

Performing $R_3 \rightsquigarrow R_3 + 3R_2$ gives us the row-echelon form

$$\left(\begin{array}{ccc|c} \boxed{1} & 3 & -1 & 1 \\ 0 & \boxed{2} & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The right-hand column is non-leading, so q **is** an element of $\text{span}(p_1, p_2, p_3)$.

(ii) **Hints**

Look at the row-echelon form of the matrix whose columns contain p_1, p_2, p_3 (as vectors), i.e. the left-hand part of the matrix in (i). Remember what the row-echelon form has to look like in order for the set to be spanning.

Worked Sample Solution

The given set would span P_2 if and only if the left-hand part of the augmented matrix in part (i) (the matrix whose columns contain the coordinate vectors of p_1, p_2, p_3 (with respect to the standard basis of P_2)) had no zero rows in its row-echelon form. But looking at its row-echelon form that we found, we see that this is not the case (since the third row is a zero row). Therefore, the given set does **not** span P_2.

3. (i) **Hints**

To evaluate this, recall the property of linear transformations that for scalars $\alpha, \beta \in$

$\mathbb{R}$, we have

$$T\begin{pmatrix}1\\0\end{pmatrix} = T\left(\alpha\begin{pmatrix}5\\3\end{pmatrix} + \beta\begin{pmatrix}2\\1\end{pmatrix}\right) = \alpha T\begin{pmatrix}5\\3\end{pmatrix} + \beta T\begin{pmatrix}2\\1\end{pmatrix} \quad (*).$$

We're already given what $T\begin{pmatrix}5\\3\end{pmatrix}$ and $T\begin{pmatrix}2\\1\end{pmatrix}$ are. So, all we really need to do is find α and β such that $\alpha\begin{pmatrix}5\\3\end{pmatrix} + \beta\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$. We can find using Gaussian Elimination (or inspection) that the solution is $\alpha = -1$ and $\beta = 3$. From this, we find that $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\-2\end{pmatrix}$, using Equation $(*)$. Here is a worked sample solution.

Worked Sample Solution

We solve for α and β in the equation

$$\alpha\begin{pmatrix}5\\3\end{pmatrix} + \beta\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}.$$

After row-reducing the augmented matrix

$$\left(\begin{array}{cc|c}5 & 2 & 1\\3 & 1 & 0\end{array}\right) \xrightarrow{R_2 \rightsquigarrow R_2 - \frac{3}{5}R_1} \left(\begin{array}{cc|c}5 & 2 & 1\\0 & -\frac{1}{5} & -\frac{3}{5}\end{array}\right)$$

we know that $-\frac{1}{5}\beta = -\frac{3}{5}$ and $5\alpha + 2\beta = 1$. Solving these simultaneously, we find that $\alpha = -1$ and $\beta = 3$.

Thus

$$\begin{aligned}
T\begin{pmatrix}1\\0\end{pmatrix} &= T\left(-\begin{pmatrix}5\\3\end{pmatrix} + 3\begin{pmatrix}2\\1\end{pmatrix}\right)\\
&= -T\begin{pmatrix}5\\3\end{pmatrix} + 3T\begin{pmatrix}2\\1\end{pmatrix} && \text{(linearity of } T\text{)}\\
&= -\begin{pmatrix}8\\-1\end{pmatrix} + 3\begin{pmatrix}3\\-1\end{pmatrix} && \text{(given data)}\\
&= \begin{pmatrix}-8+9\\1-3\end{pmatrix}\\
&= \begin{pmatrix}1\\-2\end{pmatrix}.
\end{aligned}$$

(ii) **Hints**

We want to find the matrix of T (with respect to standard bases). The standard method to do this is to apply T to the standard basis vectors and place them in as columns of the matrix. In other words, column 1 of A should be $T\begin{pmatrix}1\\0\end{pmatrix}$ and column 2 should be $T\begin{pmatrix}0\\1\end{pmatrix}$. We have already calculated $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\-2\end{pmatrix}$ earlier. We just need to find the second column, which is simply $T\begin{pmatrix}0\\1\end{pmatrix}$. To calculate $T\begin{pmatrix}0\\1\end{pmatrix}$, do a similar process to the first part of the question. A sample solution is below.

Worked Sample Solution

We have already calculated $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\-2\end{pmatrix}$ earlier, so this is column 1 of A, so A is of the form $\begin{pmatrix}1 & ?\\-2 & ?\end{pmatrix}$.

Observe that $\begin{pmatrix}0\\1\end{pmatrix} = 2\begin{pmatrix}2\\3\end{pmatrix} - 5\begin{pmatrix}1\\2\end{pmatrix}$ (find this either by inspection or Gaussian Elimination similar to the previous part). So

$$\begin{aligned}
T\begin{pmatrix}0\\1\end{pmatrix} &= T\left(2\begin{pmatrix}5\\3\end{pmatrix} - 5\begin{pmatrix}2\\1\end{pmatrix}\right)\\
&= 2T\begin{pmatrix}5\\3\end{pmatrix} - 5T\begin{pmatrix}2\\1\end{pmatrix} && \text{(linearity)}\\
&= 2\begin{pmatrix}8\\-1\end{pmatrix} - 5\begin{pmatrix}3\\-1\end{pmatrix} && \text{(given data)}\\
&= \begin{pmatrix}1\\3\end{pmatrix}.
\end{aligned}$$

So this is the second column, and thus the required matrix is $A = \begin{pmatrix}1 & 1\\-2 & 3\end{pmatrix}$.