UNSW Mathematics Society

MATH1131/1141 final exam workshop

Handout

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Contents

Chapter 1

Algebra

1.1 Complex Numbers

Question 1 (1131)

For $z = -1 + i$ and $3 + 4i$, find the following in $a + ib$ form:

a) $z + \overline{w}$.

b) $\dfrac{z}{w}$.

Question 2 (1131)

Let the set S in the complex plane be defined by

$$S = \left\{ z \in \mathbb{C} : -\frac{\pi}{4} \leq \operatorname{Arg}(z) \leq \frac{\pi}{4} \quad \text{and} \quad \operatorname{Re}(z) \leq 3 \right\}.$$

a) Sketch the set S on a labelled Argand diagram.

b) Let w be the complex number in S with the greatest imaginary part. By considering your sketch or otherwise, find w in $a + ib$ form.

Question 3 (1131)

Suppose that $z = 1 + i$ and $w = \sqrt{3} + i$.

a) Find zw in Cartesian form.

b) Show that $\operatorname{Arg}(zw) = \dfrac{5\pi}{12}$.

c) Hence show that

$$\cos\left(\frac{5\pi}{12}\right) = \frac{-1+\sqrt{3}}{2\sqrt{2}}.$$

Question 4 (1131)

Let $z = \sqrt{2} - \sqrt{2}i$.

a) Find $|z|$.

b) Find $\operatorname{Arg}(z)$.

c) Use the polar form of z to evaluate z^6 in Cartesian form.

Question 5 (1131)

Let $p(z) = z^7 + 4z^5 - z^2 - 4$.

a) Show that $2i$ is a root of $p(z)$.

b) Explain why it follows from (a) that $z^2 + 4$ is a factor of $p(z)$.

c) Divide $p(z)$ by $z^2 + 4$ and hence find all the roots of $p(z)$ in polar form.

Question 6 (1131)

Suppose that w and z are non-zero complex numbers such that

$$|w - z| = |w + z|.$$

Prove that $\dfrac{w}{z}$ is purely imaginary.

Question 7 (1141)

Use De Moivre's theorem to express $\sin(5\theta)$ as a polynomial in terms of $\sin\theta$.

1.2 Vector Geometry

Question 1 (1131)

Find a vector parametric form for the plane passing through the three points with position vectors

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} -2 \\ 1 \\ -5 \end{pmatrix}.$$

Question 2 (1131)

Consider the line in $\mathbb{R}^3$,

$$x - 4 = -y = z - 5.$$

a) Write this line in parametric vector form.

b) Find the point on the line closest to the origin.

Question 3 (1141)

a) Define what it means for a set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ to be an orthonormal set in $\mathbb{R}^n$.

b) Let M be the matrix whose columns consist of the n orthonormal vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ in $\mathbb{R}^n$. By considering $M^T M$ or otherwise, find, with reasons, all possible values for $\det(M)$.

Question 4 (1131)

A plane Π has parametric vector equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}; \qquad \lambda, \mu \in \mathbb{R}.$$

Find the Cartesian equation of this plane.

Question 5 (1131)

Let ℓ_1 and ℓ_2 be the lines

$$\ell_1 : \qquad \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}; \qquad \lambda \in \mathbb{R},$$

$$\ell_2 : \qquad \mathbf{x} = \begin{pmatrix} -2 \\ 6 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}; \qquad \mu \in \mathbb{R}.$$

a) Show that the point B with coordinates $(-1, 4, 3)$ lies on the line ℓ_1.
b) Find the point A at which the lines ℓ_1 and ℓ_2 intersect.
c) Find the projection of the vector $\overrightarrow{AB}$ onto the line ℓ_2.

Question 6 (1131)

Suppose that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are distinct non-zero vectors with the property that

$$\text{proj}_{\mathbf{w}}(\mathbf{u}) = \text{proj}_{\mathbf{w}}(\mathbf{v}).$$

Prove that $\mathbf{u} - \mathbf{v}$ is perpendicular to $\mathbf{w}$.

Question 7 (1141)

Suppose that $\mathbf{u}$ and $\mathbf{v}$ are non-zero, non-parallel vectors of the same magnitude. Prove that $\mathbf{u} - \mathbf{v}$ is perpendicular to $\mathbf{u} + \mathbf{v}$.

1.3 Matrices

Question 1 (1131)

Let $P = \begin{pmatrix} 1 & 4 \\ 3 & 5 \\ 0 & 7 \end{pmatrix}$ and $Q = \begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}$.

a) Evaluate PQ^T.

b) What is the size of PQP^T.

Question 2 (1131)

Let A and B be 2×2 matrices.

a) Use a counterexample to show that $\det(A + B)$ does not equal $\det(A) + \det(B)$ in general.

b) Use the fact that $\det(AB) = \det(A)\det(B)$ to prove that if A is an invertible matrix then $\det\left(A^{-1}\right) = \det(A)^{-1}$.

Question 3 (1151)

Prove that if an $n \times n$ matrix A is invertible and both A and A^{-1} have only integer entries, then $\det(A) = \pm 1$.

Question 4 (1141)

A square matrix Q is said to be unitary if it has the property that $\overline{Q}^T Q = I$, where $\overline{Q}$ is the matrix obtained from Q by taking complex conjugates of each entry of Q.

a) Give an example of a 2×2 unitary matrix with non-real entries.

b) Show that the determinant of a unitary matrix has the form $e^{i\theta}$ for some real number θ.

Question 5 (1141)

A matrix $Q \in M_{nn}(\mathbb{R})$ is said to be nilpotent (of degree 2) if $Q^2 = \mathbf{0}$, the zero matrix.

a) Give an example of a non-zero 2×2 nilpotent matrix.
b) Explain why a nilpotent matrix cannot be invertible.

Suppose now that $S, Q \in M_{nn}(\mathbb{R})$ commute, that S is invertible and that Q is nilpotent (of degree 2).

c) Prove that $S^{-1}Q = QS^{-1}$.
d) Show that $S + Q$ is invertible by finding an integer k such that

$$(S+Q)(S^{-1} - S^{-k}Q) = I.$$

Question 6 (1131)

Matrices P and Q are said to be orthogonal if $Q^T Q = P^T P = I$. Given that P and Q are orthogonal, simplify

$$\left(P^{-1}Q\right)^T \left(Q^T P\right)^{-1}.$$

Question 7 (1131)

Given that the invertible matrix $n \times n$ matrix A satisfies

$$A^2 = 2A + I,$$

express the inverse of A in terms of A and I.

Chapter 2

Calculus

2.1 Limits

Question 1 (1131)

Evaluate the limits

a)

$$\lim_{x\to\infty} \frac{6x^2+\sin(x)}{4x^2+\cos(x)};$$

b)

$$\lim_{x\to 0} \frac{e^{2x}-2x-1}{4x^2}.$$

Question 2 (1131)

Find the limit, if it exists:

$$\lim_{x\to\infty} \left(\sqrt{x^2+4x}-x\right).$$

Question 3 (1141)

Let $a \in \mathbb{R}$. Find the limit, if it exists:

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x.$$

Question 4 (1131)

Use the ϵ-M definition of the limit to prove that

$$\lim_{x\to\infty}\frac{x^2-2}{x^2+3}=1.$$

Question 5 (1141)

Use the ϵ-M definition of the limit to prove that

$$\lim_{x\to\infty}\frac{e^x}{\cosh x}=2.$$

Question 6 (1131)

Use the Pinching theorem to evaluate

$$\lim_{x\to\infty} e^{-x}\sin(x).$$

Question 7 (1141)

Let f be a differentiable function on (a, b), and take $c \in (a, b)$. Define

$$q(x) = \frac{f(x) - f(c) - f'(c)(x - c)}{(x - c)^2},$$

where $a < x < b$ and $x \neq c$.
Show that if $f''(c)$ exists then

$$\lim_{x \to c} q(x) = \frac{f''(c)}{2}.$$

2.2 Differentiation

Question 1 (1131)

Let

$$f(x) = \begin{cases} x^2 & x \geq 0 \\ 0 & x < 0. \end{cases}$$

a) Show that $f(x)$ is differentiable at $x = 0$ and find $f'(0)$.

b) Determine $f'(x)$ for all x.

Question 2 (1131)

Find a and b such that the function

$$f(x) = \begin{cases} x^2 + ax + b, & x < 0 \\ \cos 2x, & x \geq 0 \end{cases}$$

is differentiable.

Question 3 (1131)

Use logarithmic differentiation to find the derivative of

$$y = (\cosh x)^{2x}.$$

Question 4 (1131)

Let $f(x) = x^3 + 5x - \cos x$.

a) Use the Intermediate Value Theorem to show that $f(x)$ has at least one positive root.

b) Show that $f(x)$ has exactly one root.

Question 5 (1141)

Use the Mean Value Theorem to prove that if $a < b$ then

$$0 < \tan^{-1} b - \tan^{-1} a \leq b - a.$$

Question 6 (1131)

Use the Mean Value Theorem to prove that, for $x > 0$,

$$\ln(1 + x) > \frac{x}{1 + x}.$$

Question 7 (1131)

Consider the function

$$f(x) = x - \frac{1}{x}$$

defined on the interval $(1, \infty)$.

a) Show that f is an increasing function.

b) Let g be the inverse function of f. What is the domain of g?

c) Find $g(\frac{3}{2})$ and $g'(\frac{3}{2})$.

Question 8 (1141)

Consider the function $f : (0, 2\sqrt{\pi}] \to \mathbb{R}$ defined by

$$f(x) = x^2 + \cos(x^2).$$

a) Find all critical points of f and determine their nature.

b) Explain why f is invertible, state the domain of f^{-1} and find $f^{-1}\left(\frac{5\pi}{2}\right)$.

c) Where is f^{-1} differentiable?

2.3 Integration

Question 1 (1131)

a)

$$\int \frac{dx}{x\,(1+(\ln x)^2)}.$$

b)

$$\int x \sinh(2x)\,dx.$$

c)

$$\int x^2\sqrt{3+x^3}\,dx.$$

d)

$$\int \sqrt{1+x^2}\,dx.$$

Question 2 (1131)

Use the fundamental theorem of calculus to find

$$\frac{d}{dx}\int_x^{x^2} \cosh\left(\sqrt{t}\right)dt.$$

Question 3 (1141)

Suppose that f is a function whose derivative is continuous and hence bounded on $[a, b]$, with $|f'(x)| \leq L$ for all $x \in [a, b]$.

a) Show that for any $n > 0$,

$$\int_a^b f(x) \sin nx \, dx = \frac{K(n)}{n} + \frac{1}{n} \int_a^b f'(x) \cos nx \, dx,$$

where $K(n) = f(a) \cos(na) - f(b) \cos(nb)$.

b) Explain why

$$\left| \int_a^b f'(x) \cos nx \, dx \right| \leq (b - a) L.$$

c) Find, with reasons,

$$\lim_{n \to \infty} \int_a^b f(x) \sin nx \, dx.$$

Question 4 (1131)

Determine, with reasons, whether the following improper integrals converge or diverge:

a) $\displaystyle\int_0^\infty \frac{dx}{x^2 + e^x}$ b) $\displaystyle\int_e^\infty \frac{dx}{x + \ln x}$ c) $\displaystyle\int_0^\infty \frac{dx}{e^{2x} + \cos^2 x}$

d) $\displaystyle\int_0^\infty x e^{-x^2} \, dx$ e) $\displaystyle\int_1^\infty \frac{1}{\sqrt{1 + x^6}} \, dx$

Question 5 (1141)

Consider the function $f(x) = \dfrac{1}{1+x}$ defined on $[0,1]$ and let P be the partition $\left\{0, \frac{1}{n}, \frac{2}{n}, ..., 1\right\}$.

a) Show that the lower Riemann sum $L_p(f)$ is given by

$$L_p(f) = \sum_{k=1}^{n} \frac{1}{n+k}.$$

b) Assuming that the limits of the upper and lower Riemann sums are equal, evaluate

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k}.$$