MATH1081 Test 3 2008 S1 v3A

April 28, 2016

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1. (a) The truth tables are given below for each formula:

$\sim p \to (q \wedge p)$

p	q	$\sim p$	$q \wedge p$	$\sim p \to (q \wedge p)$
T	T	F	T	T
T	F	F	F	T
F	T	T	F	F
F	F	T	F	F

$(p \wedge q) \to q$

p	q	$p \wedge q$	$(p \wedge q) \to q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

(b) First, it will be easier to see if you produce a combined truth table:

p	q	$\sim p \to (q \wedge p)$	$(p \wedge q) \to q$	first $\to$ second	second $\to$ first
T	T	T	T	T	T
T	F	T	T	T	T
F	T	F	T	T	F
F	F	F	T	T	F

Recall that for a formula to logically imply another formula, one only needs to look at each row where the first formula produces a T. If every such row also produces a T in the second formula, then this proves that the first formula logically implies the second.

By looking at the truth table, we can see that every time the first formula produces a T (the first two rows), the second formula also produces a T. Hence, the first formula logically implies the second.

On the other hand, to check whether the second formula logically implies the first, we need to look at the rows where the second formula produces a T (every line). However, consider the case where $p = q = F$ (4th line). While the second formula produces a T, the first formula produces a F. Hence, the second formula does not logically imply the first.

2. **Theorem:** $\log_6 11$ is irrational.

Proof. We prove by contradiction.

Suppose that $\log_6 11$ is rational, that is, there exist integers p and q such that $\log_6 11 = \frac{p}{q}$. Without loss of generality, we can assume $p, q > 0$.

Then,

$$\begin{aligned} \log_6 11 &= \frac{p}{q} \\ 11 &= 6^{\frac{p}{q}} \\ 11^q &= 6^p \end{aligned}$$

which is a contradiction, as the LHS is always odd and RHS is always even.

Hence, our assumption was false, and it follows that $\log_6 11$ is irrational, thus completing the proof. $\square$

3. **Theorem:** $q(n) = 11n^2 + 32n$ is a prime number for two integer values of n, and is composite for all other integer values of n.

Proof. First, note that if a number is prime, then it has exactly two factors, which are 1 and the number itself.

Factorising $q(n)$, we obtain $q(n) = n(11n+32)$. From the definition, for $q(n)$ to be prime, it is necessary that at least one of $n = \pm 1$ or $11n + 32 = \pm 1$ is true.

We now consider the possible cases.

If $n = 1$, then $q(n) = 43$, which is prime.

If $n = -1$, then $q(n) = -21$, which is not prime.

If $11n + 32 = 1$, then $n = \frac{-31}{11} \notin \mathbb{Z}$. This contradicts the constraint that $n \in \mathbb{Z}$ and hence is not possible.

If $11n + 32 = -1$, then $n = -3$, and hence $q(n) = 3$, which is prime.

By exhaustion of all possible cases where $q(n)$ can be prime, we conclude that $q(n)$ is prime in only two cases (namely, $n = 1$ or $n = -3$). For all other values of n, $q(n)$ is divisible by n and $11n + 32$, which are both not equal to 1, implying that it is composite. $\square$

MATH1081 Test 3 2008 S2 v1A

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1. (a) The truth tables for each formula are given below:

$(p \to (\sim q)) \wedge r$

p	q	r	$\sim q$	$p \to (\sim q)$	$(p \to (\sim q)) \wedge r$
T	T	T	F	F	F
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	F
F	T	T	F	T	T
F	T	F	F	T	F
F	F	T	T	T	T
F	F	F	T	T	F

$q \to ((\sim p) \wedge r)$

p	q	r	$\sim p$	$(\sim p) \wedge r$	$q \to ((\sim p) \wedge r)$
T	T	T	F	F	F
T	T	F	F	F	F
T	F	T	F	F	T
T	F	F	F	F	T
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	F	T

(b) First off, note that the second cannot imply the first when $p = q = r = F$.

On the other hand, the first **does** imply the second, which can be seen from this truth table:

p	q	r	$(p \to (\sim q)) \wedge r$	$q \to ((\sim p) \wedge r)$	first $\to$ second
T	T	T	F	F	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	T	F	F	F	T
F	F	T	T	T	T
F	F	F	F	T	T

Hence, the first formula logically implies the second.

2. **Theorem :** If m and n are positive integers, then $m!n! < (m+n)!$

Proof. Let m and n be positive integers.

Now,

$$\begin{aligned} m!n! &= (1 \times 2 \times 3 \times \cdots \times m)(1 \times 2 \times 3 \times \cdots \times n) \\ &< 1 \times 2 \times 3 \times \cdots \times m(m+1)(m+2)(m+3)\dots(m+n) \\ &= (m+n)! \end{aligned}$$

as $1 < m+1$, $2 < m+2$, $\dots$, $n < m+n$.

It follows that $m!n! < (m+n)!$, thus completing the proof. $\square$

Alternative proof. Let m and n be positive integers. Notice that $\binom{m+n}{m} > 1$ since $m, n \in Z^+$ so $m, n > 0$.
This is true since this finds us the total number of ways to select m items from a total of

$m+n$ items which is clearly greater than 1.
Hence,

$$\begin{aligned}\binom{m+n}{m} &> 1\\ \frac{(m+n)!}{m!n!} &> 1\\ m!n! &< (m+n)!\end{aligned}$$

which concludes the proof.

3. **Theorem:** If $a \in \mathbb{R}^+$, then the equation $ax = \cos \pi x$ has exactly one solution x such that $0 \leq x \leq 1$.

Proof. Let a be some element of $\mathbb{R}^+$. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) = ax - \cos \pi x$.

Now, $f(0) = -1$ and $f(1) = a + 1$.

Clearly, f is continuous on the interval $[0, 1]$. Thus, as $-1 < 0 < a+1$, by the intermediate value theorem, there exists at least one number $c \in (0, 1)$ such that $f(c) = 0$.

That is, there is *at least one* solution for $ax = \cos \pi x$ on the specified interval.

Further, on the interval $[0, 1]$,

$$f'(x) = a + \pi \sin \pi x > 0$$

that is, f is monotone increasing, and so it can have at most one root in that interval.

It follows that $ax = \cos \pi x$ has exactly one solution on the interval $[0, 1]$, thus completing the proof. $\square$

More rigorous proof for the uniqueness condition by contradiction. Suppose that the equation $ax = \cos \pi x$ has two distinct solutions for x in the interval $[0, 1]$. Then f has two distinct roots in the interval $[0, 1]$. Let these roots be x_1 and x_2.

Since f is continuous and differentiable over the interval $[0, 1]$, f is also continuous over the closed interval $[x_1, x_2]$ and differentiable over the open interval (x_1, x_2), as $[x_1, x_2]$ and (x_1, x_2) are subsets of $[0, 1]$. By applying Rolle's theorem, there must exist some y in the open interval (x_1, x_2) such that $f'(y) = 0$.

By differentiating the function, we have

$$f'(y) = a + \pi \sin \pi y = 0.$$

However, a and π are positive and $\sin \pi y$ is non-negative for all possible values of y

in $[0,1]$. This implies that f' is non-zero over the interval $[0,1]$, which contradicts our assumption that the equation $ax = \cos \pi x$ has two distinct solutions for x in the interval $[0,1]$. Thus, the equation can only have exactly one solution for x in the interval $[0,1]$ and this completes the proof.

MATH1081 Test 3 2009 S1 v2A

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1.

$$
\begin{aligned}
& p \to (\sim (q \wedge (\sim p))) \\
& \Leftrightarrow (\sim p) \vee (\sim (q \wedge (\sim p))) && (\text{as } v \to u \Leftrightarrow (\sim v) \vee u) \\
& \Leftrightarrow \sim (p \wedge (q \wedge (\sim p))) && (\text{De Morgan's Law}) \\
& \Leftrightarrow \sim (p \wedge ((\sim p) \wedge q)) && (\text{Associative Law}) \\
& \Leftrightarrow \sim ((p \wedge (\sim p)) \wedge q) && (\text{Commutative Law}) \\
& \Leftrightarrow \sim (\mathbf{c} \wedge q) && (\text{Law of Negation}) \\
& \Leftrightarrow \sim (\mathbf{c}) && (\text{Domination Law}) \\
& \Leftrightarrow \mathbf{t} && (\text{Negation of a contradiction})
\end{aligned}
$$

2. **Theorem:** If $n \in \mathbb{Z}^+$, then

$$(1 \times 2) + (2 \times 5) + \cdots + n(3n-1) = n^2(n+1)$$

Proof. We prove by induction.

When $n = 1$, LHS $= 1 \times 2 = 2$, RHS $= 1^2 \times 2 = 2$, and so the theorem is true for $n = 1$.

Assume the theorem is true for $n = k$, where $k \in \mathbb{Z}^+$, that is

$$(1 \times 2) + (2 \times 5) + \cdots + k(3k-1) = k^2(k+1).$$

It is required to prove that the theorem is also true for $n = k + 1$, that is

$$(1 \times 2) + (2 \times 5) + \cdots + (k+1)(3k+2) = (k+1)^2(k+2).$$

We have

$$\begin{aligned}
LHS &= (1 \times 2) + \cdots + k(3k-1) + (k+1)(3k+2) \\
&= k^2(k+1) + (k+1)(3k+2) && \text{(by assumption)} \\
&= (k+1)(k^2 + 3k + 2) \\
&= (k+1)(k+1)(k+2) && \text{(factorising)} \\
&= (k+1)^2(k+2) \\
&= RHS.
\end{aligned}$$

Hence, if the theorem is true for $n = k$ for some $k \in \mathbb{Z}^+$, then it is also true for $n = k + 1$.

Therefore, the theorem is true for all positive integers n, by mathematical induction, and the proof is complete. □

3. **Theorem:** If $x \in \mathbb{R}$ and $2x^2 - 3 = 0$, then x is irrational.

Proof. We prove by contradiction.

Let $x \in \mathbb{R}$ and $2x^2 - 3 = 0$. Solving, we obtain

$$x^2 = \frac{3}{2}.$$

Suppose that x is rational, that is, it can be written as an irreducible ratio of two integers, $x = \frac{p}{q}$ for some $p, q \in \mathbb{Z} \backslash \{0\}$.

Then, $x^2 = \frac{p^2}{q^2} = \frac{3}{2}$ and thus

$$2p^2 = 3q^2. \tag{1}$$

Clearly, the LHS is even, and so q^2 must be even also. This implies that q is even, and hence $q = 2r$ for some $r \in \mathbb{Z}$.

Substituting this result into (1) and simplifying gives

$$p^2 = 2 \times 3 \times r^2.$$

This shows that p^2 is even, which implies that p must be even. Hence, p and q share a common factor of 2. But since we assumed that p and q form an irreducible fraction, this is a contradiction.

Thus, the initial assumption was wrong.

It follows that x is irrational, and the proof is complete. $\square$

MATH1081 Test 3 2009 S2 v1A

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1. (i) Let

$$\begin{aligned} m &= \text{“I earn some money”} \\ h &= \text{“I go for a holiday this summer”} \\ w &= \text{“I work this summer”} \end{aligned}$$

The argument in symbolic form is:

$$\begin{array}{c} m \to h \\ h \vee w \\ \hline \therefore (\sim h) \to (\sim m) \wedge w \end{array}$$

(ii) Using a truth table, we consider critical rows where the hypotheses are true:

m	h	w	$m \to h$	$h \vee w$	$(\sim m) \wedge w$	$(\sim h) \to (\sim m) \wedge w$
T	T	T	T	T	F	T
T	T	F	T	T	F	T
T	F	T	F	*	*	*
T	F	F	F	*	*	*
F	T	T	T	T	T	T
F	T	F	T	T	F	T
F	F	T	T	T	T	T
F	F	F	T	F	*	*

Note that the rows we disregard are the ones where at least one of the statements $m \to h$ and $h \vee w$ are not true.

The conclusion is always true whenever the hypotheses are true, and therefore the above argument is logically valid.

2. **Theorem :** Between any two different rational numbers there is another rational number.

Proof. Let x and y be two distinct rational numbers. Suppose without loss of generality that $x < y$.

We claim that $\frac{x+y}{2}$ is a rational number between x and y.

First, we try to prove that $\frac{x+y}{2}$ is indeed rational. Since x and y are rational, then we can write $x = \frac{a}{b}$ and $y = \frac{c}{d}$, where a, b, c and d are integers and $b, d \neq 0$. So, we have

$$\frac{x+y}{2} = \frac{\frac{a}{b} + \frac{c}{d}}{2} = \frac{ad+bc}{2bd},$$

where $ad + bc$ and $2bd$ are integers. As $\frac{x+y}{2}$ can be expressed as a fraction of two integers where $2bd \neq 0$, then it is indeed rational.

Now, we try to prove that $\frac{x+y}{2}$ is between x and y, that is, $x < \frac{x+y}{2}$ and $\frac{x+y}{2} < y$.

Recall that $x < y$. This is equivalent to saying that $\frac{y-x}{2} > 0$. Furthermore, this is also equivalent to $\frac{x+y-2x}{2} > 0$, which can simplified to $\frac{x+y}{2} > x$.

Alternatively, we can express $\frac{y-x}{2} > 0$ as $\frac{2y-x-y}{2} > 0$, which simplifies to $\frac{x+y}{2} < y$.

(Note that these steps were found by working backwards, i.e. start from $x < \frac{x+y}{2}$ and then end up with $\frac{y-x}{2} > 0$. The proof is then written with the steps reversed.)

This shows that $\frac{x+y}{2}$ is indeed a rational number between x and y. $\square$

3. **Theorem:** Prove that if n is a positive integer then $4^{2n} + 10n - 1$ is a multiple of 25.

Proof. We prove by induction.

For $n = 1$, $4^{2n} + 10n - 1$ is equal to 25, which is indeed a multiple of 25.

Now, assume that the theorem holds for some positive integer $n = k$, that is $4^{2k} + 10k - 1$ is a multiple of 25, or $4^{2k} = 25m - 10k + 1$ for some integer m.

We will try to prove that the theorem holds when $n = k+1$, that is $4^{2(k+1)} + 10(k+1) - 1$ is a multiple of 25.

We have

$$\begin{aligned}
4^{2(k+1)} + 10(k+1) - 1 &= 4^{2k+2} + 10k + 9 \\
&= 16 \times 4^{2k} + 10k + 9 \\
&= 16 \times (25m - 10k + 1) + 10k + 9 \qquad \text{(induction hypothesis)} \\
&= 16 \times 25m - 150k + 25 \\
&= 25(16m - 6k + 1),
\end{aligned}$$

which is divisible by 25, as $16m - 6k + 1$ is an integer.

Hence, we have proved that the theorem holds when $n = k+1$, whenever it is true for $n = k$ for some integer k. And so, the theorem holds for all positive integers, by mathematical induction. $\square$

MATH1081 Test 3 2010 S1 v2B

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1.

$$
\begin{aligned}
&(p \to q) \wedge (q \to (\sim p \vee r)) \\
&\Leftrightarrow ((\sim p) \vee q) \wedge ((\sim q) \vee (\sim p \vee r)) && \text{(as } v \to u \Leftrightarrow (\sim v) \vee u\text{)} \\
&\Leftrightarrow ((\sim p) \vee q) \wedge ((\sim p) \vee (\sim q \vee r)) && \text{(Associative, commutative laws)} \\
&\Leftrightarrow (\sim p) \vee (q \wedge (\sim q \vee r)) && \text{(Distributive law)} \\
&\Leftrightarrow (\sim p) \vee ((q \wedge \sim q) \vee (q \wedge r)) && \text{(Distributive law)} \\
&\Leftrightarrow (\sim p) \vee (\mathbf{F} \vee (q \wedge r)) && \text{(Law of negation)} \\
&\Leftrightarrow (\sim p) \vee (q \wedge r) && \text{(Identity law)} \\
&\Leftrightarrow p \to (q \wedge r) && \text{(as } v \to u \Leftrightarrow (\sim v) \vee u\text{)}
\end{aligned}
$$

as required.

2. **Theorem:** If n is a positive integer then $(n{+}1)(n{+}2)\dots(2n) = 2^n{\times}1{\times}3{\times}5{\times}\dots{\times}(2n{-}1)$.

   Proof. We prove this by induction. When $n = 1$, $LHS = 2$ and $RHS = 2$. So the theorem holds for $n = 1$.

   Assume now that the theorem holds for some particular integer k, that is

$$(k+1)(k+2)\dots(2k) = 2^k \times 1 \times 3 \times 5 \times \dots \times (2k-1).$$

   We will try to prove that the theorem also holds for $n = k + 1$, that is

$$(k+2)(k+3)\dots(2(k+1)) = 2^{k+1} \times 1 \times 3 \times 5 \times \dots \times (2k+1).$$

   We have

$$\begin{aligned}
LHS &= (k+2)(k+3)\dots(2k)(2k+1)(2k+2)\\
&= \frac{(2k+2)(2k+1)}{k+1} \times (k+1)(k+2)\dots(2k)\\
&= \frac{(2k+2)(2k+1)}{k+1} \times 2^k \times 1 \times 3 \times 5 \times \dots \times (2k-1) \qquad \text{(induction hypothesis)}\\
&= \frac{2(k+1)(2k+1)}{k+1} \times 2^k \times 1 \times 3 \times 5 \times \dots \times (2k-1)\\
&= 2^{k+1} \times 1 \times 3 \times 5 \times \dots \times (2k-1) \times (2k+1)\\
&= RHS.
\end{aligned}$$

   So, we have just proven that the theorem holds for $n = k + 1$, whenever it is true for $n = k$. Hence, the theorem holds for all positive integers, by mathematical induction. □

3. **Theorem:** Prove that if n is any positive integer then $\sqrt{4n-2}$ is irrational.

   Proof. We will prove by contradiction.

   Let n be a positive integer. Suppose $\sqrt{4n-2}$ is rational.

   Then there exist coprime integers p and $q \neq 0$, such that $\sqrt{4n-2} = \frac{p}{q}$. Squaring both sides yields $4n - 2 = \frac{p^2}{q^2}$. Upon multiplying both sides by q^2, we then get

$$2(2n-1)q^2 = p^2. \tag{1}$$

   Since, the left hand side has a factor of 2, then it is even. This implies that p^2 must also be even. One can then deduce that p is also even, that is there exists some integer r such that $p = 2r$.

Substituting this back into (1) yields

$$2(2n-1)q^2 = 4r^2 \Leftrightarrow (2n-1)q^2 = 2r^2.$$

Now, with a similar argument as before, the right hand side is even, which implies that the left hand side must also be even. However, as $2n-1$ is odd, the only way that the left hand side can be even, is if q^2 is even, which implies that q is even.

However, we assumed that p and q are coprime, which is a contradiction. So our original assumption was incorrect. And so, $\sqrt{4n-2}$ is indeed irrational. $\square$

Use a proof by contradiction by first letting $\sqrt{4n-2} = \frac{p}{q}$ where $p, q \in \mathbb{Z}^+$ and are coprime.