MATH1081 S1 2008 Test 4 v3b

Sample Solutions

September 27, 2017

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1. 8 letter words

   (i) No letter used twice

   1. Pick the first letter of the word: 26 ways
   2. Pick a different letter for the second letter/place: 25 ways.
   3. Pick a different letter for the third letter/place. 24 ways:

      . . . etc . . .

   8. Pick a different letter for the 8th letter/place: 19 ways.

   Answer[1]*:* $26 \times 25 \times 24 \ldots \times 19$ or $P(26, 8)$.

[1] In these solutions, $P(n, r)$ is the number of permutations of r objects chosen from a set of n objects.

(ii) One letter used twice and six letters used once at most

1. Pick the letter to be used twice: 26 ways.
2. Choose two places from 8 for this double letter: $\binom{8}{2}$ ways.
3. Choose 6 different letters for the remaining spots, and note that order is important: $P(25, 6)$ ways.

Answer: $26 \times \binom{8}{2} \times P(25, 6)$.

2. An ordinary die is rolled ten times. There are 6^{10} possible outcomes in total.

There are $\binom{6}{5}$ ways to choose which five numbers of the die will be the ones that appear twice each.
Now, the number of sequences of 10 die-rolls that have five different faces come up twice each is

$$\binom{6}{5} \times \frac{10!}{2! \times 2! \times 2! \times 2! \times 2!} = \frac{\binom{6}{5} \times 10!}{(2!)^5}.$$

This is because there were 6 ways to choose which five numbers of the die will be the ones that appear twice each, and once these are chosen, there are $\frac{10!}{(2!)^5}$ ways to get these (i.e. just the number of ways to arrange five pairs of same numbers in a line, e.g. number of ways to arrange $1, 1, 2, 2, 3, 3, 4, 4, 5, 5$). Therefore, the required probability is "number of favourable outcomes" divided by "total number of possibilities", i.e. the answer is

$$\frac{\frac{\binom{6}{5} \times 10!}{(2!)^5}}{6^{10}} = \frac{\binom{6}{5} \times 10!}{(2!)^5 \times 6^{10}}.$$

3. Particular solution of $a_n - 8a_{n-1} + 15a_{n-2} = 21 \times 2^n$.

The characteristic equation is $r^2 - 8r + 15 = 0$, and 2 is not a root of this equation, since $2^2 - 8 \times 2 + 15 = 4 - 16 + 15 = 19 - 16 \neq 0$. Hence we can try a particular solution $p_n = c \times 2^n$, for some constant c. Substitute this into the recurrence to get

$$(c \times 2^n) - 8(c \times 2^{n-1}) + 15(c \times 2^{n-2}) = 21 \times 2^n.$$

Divide both sides by 2^{n-2} (noting that $\frac{2^n}{2^{n-2}} = 2^2 = 4$ and $\frac{2^{n-1}}{2^{n-2}} = 2$) to get

$$\begin{aligned} 4c - 8(2)c + 15c &= 21(4) \\ 4c - 16c + 15c &= 84 \\ 3c &= 84 \\ \implies c &= 28. \end{aligned}$$

So $p_n = 28 \times 2^n$ is a particular solution.

3. Let

$$\begin{aligned} A_1 &= \{\text{hands with exactly 5 hearts}\} \\ A_2 &= \{\text{hands with exactly 5 diamonds}\} \\ A_3 &= \{\text{hands with exactly 5 spades}\} \\ A_4 &= \{\text{hands with exactly 5 clubs}\}. \end{aligned}$$

We want to find $|A_1 \cup A_2 \cup A_3 \cup A_4|$.
By using the Inclusion-Exclusion principle, we see that

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = \sum_{i=1}^{4} |A_i| - \sum_{i \neq j} |A_i \cap A_j|.$$

This is because the terms with three-way or four-way intersections have size 0, because there is no way to have a 13-card hand with three suits or four suits having exactly five cards (since this would require at least 15 cards in the hand, but we only have 13). We find $\sum_{i=1}^{4} |A_i|$, i.e., we want hands of 13 cards with exactly 5 cards in some suit.

1. Choose the suit to have 5 cards: 4 ways.
2. Choose 5 from that suit: $\binom{13}{5}$ ways.
3. Choose a further 8 from the remaining cards in the deck: $\binom{39}{8}$ ways.

Now we must find $\sum_{i \neq j} |A_i \cap A_j|$, i.e. we must account for hands where the remaining 8 (in step 3. above) feature 5 cards of another suit.

1. Choose the 2 suits to have 5 cards each in the 13 card hand: $\binom{4}{2}$ ways.
2. Choose 5 cards from the first of these suits: $\binom{13}{5}$ ways.
3. Repeat for the other suit: $\binom{13}{5}$ ways.

4. Choose 3 cards from the remaining 26 cards in the deck: $\binom{26}{3}$ ways.

Therefore,

$$\begin{aligned}|A_1 \cup A_2 \cup A_3 \cup A_4| &= \sum_{i=1}^{4} |A_i| - \sum_{i \neq j} |A_i \cap A_j| \\ &= 4\binom{13}{5}\binom{39}{8} - \binom{4}{2}\binom{13}{5}^2\binom{26}{3}.\end{aligned}$$

Note: Be careful with questions like this, since the big trap is accidentally overcounting/undercounting in situations like these. To overcome this issue, we must use the Inclusion-Exclusion Principle.

MATH1081 S2 2008 Test 4 v1b

Sample Solutions

September 27, 2017

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1. Note that the English alphabet contains five vowels and 21 consonants. We want seven-letter words with at least six consonants. We break the question down in to the following two cases to help us find the answer:

   Case 1: 6 consonants

   1. Pick a vowel. 5 ways.
   2. Choose the position in the word for the vowel to be in. 7 ways.
   3. Pick a consonant for the first available position in the word. 21 ways.
   4. Repeat for the second available position. 21 ways.

      . . . etc . . .

   8. Pick a consonant for the last available position. 21 ways.

   Total: $5 \times 7 \times 21^6$ ways.

Case 2: 7 consonants

1. Pick a consonant for the first position. 21 ways.

   ...etc ...

7. Pick a consonant for the last position. 21 ways.

Total: 21^7 ways.

Final answer: $5 \times 7 \times 21^6 + 21^7$ ways.

2. Total possibilities: $\binom{52}{8}$.

   1. Exclude the kings from the 52 and choose 5 cards from the remaining deck. $\binom{48}{5}$ ways.

   2. Choose the three kings (out of four kings in the deck) that will be included. $\binom{4}{3} = 4$ ways.

   Answer:

$$\frac{4 \times \binom{48}{5}}{\binom{52}{8}}.$$

3. (i) We want the number of solutions to $x_1+x_2+x_3+x_4+x_5 = 16$, where $x_1, \ldots, x_5 \geq 0$. We can imagine a solution as a series of 16 identical dots and 4 identical lines arranged in a row. For example,

$$\circ \quad \circ \quad \circ \quad \circ \quad | \quad \circ \quad \circ \quad \circ \quad \circ \quad | \quad \circ \quad \circ \quad \circ \quad \circ \quad | \quad \circ \quad \circ \quad | \quad \circ \quad \circ$$

   represents the solution $x_1 = 4, x_2 = 4, x_3 = 4, x_4 = 2$ and $x_5 = 2$.

   We count the number of ways we can arrange these identical dots and identical lines in a row to obtain the number of possibilities. But, since they're all identical, We just need to count how many ways we can choose 5 spots in the 20 available positions for the lines. The total number of arrangements is $\binom{20}{4}$. So the answer is $\binom{20}{4}$.

   (ii) We want the number of solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 16$, where $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$, $x_4 \geq 2$ and $x_5 \geq 1$.
   Let

$$\begin{aligned} x_1 &= y_1 + 1 \\ x_2 &= y_2 + 2 \\ x_3 &= y_3 + 3 \\ x_4 &= y_4 + 2 \\ x_5 &= y_5 + 1 \end{aligned}$$

Adding the equations above and substituting into the original equation yields $y_1 + y_2 + y_3 + y_4 + y_5 = 7$, and require that $y_i \geq 0$. Note that this new equation is equivalent to the original equation.
Similar to part (i), we use the dots and lines analogy: 7 dots and 4 lines. We arrange them in a row, and choose the positions for the lines. This can be done $\binom{11}{4}$ ways.
Answer:

$$\binom{11}{4}.$$

4. We seek the general solution to $a_n - 16a_{n-2} = 9 \times 2^n$. To do this, we first solve the homogeneous equation. The characteristic equation is

$$r^2 - 16 = 0 \implies r = \pm 4.$$

So the homogeneous solution is $h_n = A \times 4^n + B \times (-4)^n$, where A and B are arbitrary constants.

For a particular solution, try $p_n = c \times 2^n$. Substituting into the recurrence, we get

$$(c \times 2^n) - 16(c \times 2^{n-2}) = 9 \times 2^n.$$

Divide through by 2^{n-2} (noting that $\frac{2^n}{2^{n-2}} = 2^{n-(n-2)} = 2^2 = 4$):

$$4c - 16c = 9 \times 4 \implies -12c = 36 \implies c = -3.$$

Thus a particular solution is $p_n = -3 \times 2^n$. Therefore the general solution is $a_n = h_n + p_n$, that is,

$$a_n = A \times 4^n + B \times (-4)^n - 3 \times 2^n,$$

where A and B are arbitrary constants.

MATH1081 S1 2009 Test 4 v1a

Sample Solutions

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1. We seek the solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 66$.

   (i) $x_1, \cdots, x_5$ are non-negative.

   Use the dots and lines analogy[1]. We have 66 identical dots and 4 identical lines in a row. We arrange these 70 items in a row, which is done in $\frac{70!}{66! \times 4!} = \binom{70}{4}$ ways.

   Answer: $\binom{70}{4}$.

   Note: For brevity, the explanation of the "dots and line" analogy has been omitted here. However, you should write your explanation in full in the real test, similar to what was written in Question 3 on the previous two pages. Note what it says on the past tests: 'To obtain full marks for this test, your solutions must be fully and clearly explained!'

   (ii) Here, $x_1, x_2, \cdots x_5$ are even numbers (and we interpret the question as requiring the x_k to be non-negative; if negative even numbers are also allowed, there are infinitely many solutions). Use the dots and lines analogy.

[1] Refer to Question 3 in Test 4 2008 S2 v1b

Here, we pair the 66 dots into 33 pairs, and arrange these instead. This ensures that all of the x_k terms are even (and conversely, any solution where the x_k are all even has a unique corresponding dots and line picture with “paired up” dots). So we have 33 pairs of dots and 4 lines. Arrange in a row and choose positions for the lines.

Answer: $\binom{37}{4}$.

Another way to think of it is that the x_k are non-negative even numbers iff for each k, we have $x_k = 2y_k$ for some non-negative integer y_k. Substituting this into the equation, we have

$$2y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5 = 66,$$

which is equivalent to

$$y_1 + y_2 + y_3 + y_4 + y_5 = 33.$$

Therefore, we are just finding the number of solutions to the above equation (for non-negative integers y_k), which is now a standard “dots and lines” problem, and will give an answer of $\binom{37}{4}$.

2. Number of ways to fill out form:

   1. Choose 1st favourite. 20 ways.
   2. Choose 2nd favourite. 19 ways.
   3. Choose 3rd favourite. 18 ways.
   4. For each of the 17 remaining programs, tick or do not tick. 2^{17} ways.

   Answer: $20 \times 19 \times 18 \times 2^{17}$.

3. The characteristic equation is

$$r^2 - 10r + 21 = 0 \implies (r-7)(r-3) = 0 \implies r = 3 \text{ or } r = 7.$$

Thus the general solution to the recurrence relation is $a_n = A \cdot 3^n + B \cdot 7^n$.

We must now work out the values of the constants A and B. To do this, we use the initial conditions:

$$a_0 = -1 = A \cdot 3^0 + B \cdot 7^0 \implies A + B = -1$$

$$a_1 = 1 = A \cdot 3^1 + B \cdot 7^1 \implies 3A + 7B = 1.$$

We must now solve this system of linear equations for A and B. We have

$$A + B = -1 \implies 3A + 3B = -3, \text{ and}$$

$$3A + 7B = 1.$$

Subtract:

$$
\begin{aligned}
(3A+3B)-(3A+7B) &= -3-1 \\
-4B &= -4 \\
B &= 1.
\end{aligned}
$$

Since $A+B=1$, we have $A=-2$. Therefore, the solution is $a_n = (-2)\cdot 3^n + 7^n$.

4. This is a pigeonhole principle problem. Let the students be pigeons to be placed in the pigeonholes

$$
\begin{gathered}
(\,1,\ 6,\ 11, \ldots\ 46) \\
(\,2,\ 7,\ 12, \ldots\ 47) \\
(\,3,\ 8,\ 13, \ldots\ 48) \\
(\,4,\ 9,\ 14, \ldots\ 49) \\
(\,5,\ 10,\ 15, \ldots\ 50)
\end{gathered}
$$

where each number denotes the house number on one side of the street.

There are five pigeonholes and 26 pigeons. As $5\times 5 = 25 < 26$, it follows that at least one pigeonhole has at least six pigeons in it.

Now, each pigeonhole has 10 houses. Without loss of generality, assume the pigeonhole $(\,1,\ 6,\ 11, \ldots\ 46)$ is the pigeonhole with at least six pigeons.

We split the 10 houses into pairs:

$$(\,1,\ 6),\ (\,11,\ 16),\ (\,21,\ 26),\ (\,31,\ 36),\ (\,41,\ 46).$$

There are at least six students and only five pairs of houses. By the pigeonhole principle again, it follows that since $6 > 5$, at least one of these pairs of houses will have two pigeons in it. That is, two students live exactly five houses away from each other.

MATH1081 S2 2009 Test 4 v2b

Sample Solutions

September 27, 2017

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1. One method is to consider cases of 8 consonants, then 9, then 10, and add them all together[1].
   The answer may take the form of $\binom{21}{8} \times \binom{5}{2} \times 10! + \binom{21}{9} \times \binom{5}{1} \times 10! + \binom{21}{10} \times 10!$.
   Although your answer may look different, ensure that they take the same value.

2. It can be found[2] that there are

$$\binom{2}{1}\binom{13}{4}\binom{39}{9} - \binom{13}{4}^2\binom{26}{5} \text{ hands.}$$

   Note: Because it is possible that the hand could have exactly four spades AND exactly four diamonds, we need to use the Inclusion-Exclusion Principle.
   Here is a sample solution:
   Let S be the set of all thirteen-card hands that contain exactly four spades and D the set of all such hands that contain exactly four diamonds. We are asked to find $|S \cup D|$. We

[1] Refer to Question 1 in Test 4 2008 S2 v1B

[2] Refer to Question 3 in Test 4 2008 S1 v3b

know that

$$|S \cup D| = |S| + |D| - |S \cap D|,$$

by the Inclusion-Exclusion Principle. Note that by symmetry, $|S| = |D|$, and so

$$|S \cup D| = 2|S| - |S \cap D|.$$

Now, we have

$$|S| = \binom{13}{4} \times \binom{39}{9}.$$

This is because to get a hand with exactly four spades, we pick 4 spades from 13 possible spades (in $\binom{13}{4}$ ways), and then pick the remaining 9 cards for the thirteen-card hand from the 39 non-spade cards (in $\binom{39}{9}$ ways).

Furthermore, we have

$$|S \cap D| = \binom{13}{4} \times \binom{13}{4} \times \binom{26}{5}.$$

This is because $S \cap D$ refers to the set of all hands with exactly four spades **and** exactly four diamonds. To pick such a hand, we pick 4 spades from 13 possible spades, then pick 4 diamonds from 13 possible diamonds, and finally pick the remaining 5 cards for the thirteen-card hand from the 26 non-spade and non-diamond cards in the deck.

Thus the final answer is

$$|S \cap D| = 2|S| - |S \cap D| = 2\binom{13}{4} \times \binom{39}{9} - \binom{13}{4} \times \binom{13}{4} \times \binom{26}{5}.$$

3. It can be found[3] that the homogeneous solution is

$$h_n = A \cdot (-5)^n + B \cdot 2^n,$$

for some constants A and B.

It can also be found that a particular solution is

$$p_n = \frac{2}{7} n \cdot 2^n.$$

Note: Since 2^n is a solution to the homogeneous equation, our 'guess' needs to be adjusted to $cn \cdot 2^n$, for some constant c.

The general solution should be

$$a_n = h_n + p_n = A(-5)^n + B \cdot 2^n + \frac{2}{7} n \cdot 2^n.$$

Full working:

Solve $a_n + 3a_{n-1} - 10a_{n-2} = 2^n$.

[3] Refer to Question 4 in Test 4 2008 S2 v1b

The characteristic equation is

$$\begin{aligned} r^2 + 3r - 10 &= 0 \\ \implies (r-2)(r+5) &= 0 \\ \implies r = 2 \quad &\text{or} \quad r = -5. \end{aligned}$$

Hence the homogeneous solution is

$$h_n = A(-5)^n + B \cdot 2^n.$$

For a particular solution, we know from lectures that $p_n = cn \cdot 2^n$ will be a particular solution for some constant c (since 2 is a simple root of the characteristic polynomial and the right-hand side of the recurrence is 2^n). Then $p_{n-1} = c(n-1) \cdot 2^{n-1}$ and $p_{n-2} = c(n-2) \cdot 2^{n-2}$. Substitute these into the recurrence:

$$\begin{aligned} p_n + 3p_{n-1} - 10p_{n-2} &= 2^n \\ \implies cn \cdot 2^n + 3c(n-1) \cdot 2^{n-1} - 10c(n-2) \cdot 2^{n-2} &= 2^n. \end{aligned}$$

This needs to hold for every n, so we can substitute $n = 2$ to find c:

$$\begin{aligned} 2c \times 2^2 + 3c \cdot (2-1) \times 2^{2-1} - 0 &= 2^2 \\ \implies 8c + 6c &= 4 \\ \implies c &= \frac{2}{7}. \end{aligned}$$

Thus $p_n = \frac{2}{7}n \cdot 2^n$ and the general solution to the recurrence is

$$a_n = h_n + p_n = A(-5)^n + B \cdot 2^n + \frac{2}{7}n \cdot 2^n.$$

4. This is a pigeonhole principle problem. Let the pigeonholes be the different combinations of choosing three electives. There should be $\binom{7}{3}$ pigeonholes. Let the students be the pigeons.
So there are 35 pigeonholes and 200 pigeons. As $5 \times 35 = 175 < 200$, it follows from the pigeonhole principle that at least one pigeonhole has at least 6 pigeons in it. In other words, at least six of the students must have completed the same electives as each other.

MATH1081 S1 2010 Test 4 v2a

Hints and Sample Solutions

September 27, 2017

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1. **Hints**

   (i) Note that the 4 possible time slots each day are each 1 hour long. With 4 possible timeslots in each of the five days, it can be found that there are $\binom{20}{6}$ ways.

   (ii) By first choosing which day of the week contains the two timeslots, it can be found that there are $5 \times \binom{4}{2} \times 4^4$ ways.

   Sample Solutions

   (i) We assume each "timeslot" is one class hour. Since there are four possible timeslots each day, and there are five days, there are a total of $4 \times 5 = 20$ available timeslots. So if we can choose any six of these, the total number of possible choices is $\binom{20}{6}$.

   (ii)
   - Choose which day contains the two hours of class (5 ways to do this since there are fives days to choose from).
   - Given this day, choose two hours out of the four hours of that day to have classes ($\binom{4}{2}$ ways to do this).

- For the remaining four days, choose one hour of each day (4^4 ways to do this, since 4 choices for each day, and 4 days).

Hence the answer is $5 \times \binom{4}{2} \times 4^4$.

2. **Hints**

(i) Using the dots and lines analogy[1], it can be found that there are $\binom{59}{4}$ solutions.

(ii) Let $x_k = 2y_k + 1$. This reduces the equality[1] of x to $\sum_{k=1}^{5} y_k = 25$. Note that number of solutions for the y equation is equivalent to the number of solutions to the x equation. Using the dots and lines method again, it can be found that there are $\binom{29}{4}$ ways.

Sample Solutions

(i) As usual, we can represent non-negative integer solutions to the equation equivalently as arrangements of 55 identical dots and 4 identical lines (since there are 5 variables in the equation). The number of ways to arrange these is $\frac{59!}{55!4!} = \binom{59}{4}$. Therefore, the answer is $\binom{59}{4}$.

(ii) Note that a non-negative integer x_k is odd iff $x_k = 2y_k + 1$ for some non-negative integer y_k. Making this substitution in the given equation gives us

$$\begin{aligned} \sum_{k=1}^{5}(2y_k + 1) &= 55 \\ \iff 2\sum_{k=1}^{5} y_k + 5 &= 55 \\ \iff \sum_{k=1}^{5} y_k &= 25. \end{aligned}$$

So we just need to find the number of non-negative integer solutions to

$$y_1 + y_2 + y_3 + y_4 + y_5 = 25.$$

This corresponds to finding the number of ways to arrange 25 identical dots and 4 identical lines in a row, which is $\frac{29!}{25!4!} = \binom{29}{4}$. Hence the answer is $\binom{29}{4}$.

3. **Hints**

It can be found[2] that $a_n = -3 \cdot 5^n + 8 \cdot 2^n$.

Sample Solutions

The characteristic equation is

$$r^2 - 7r + 10 = 0 \iff (r-2)(r-5) = 0.$$

[1] Refer to Question 3 in Test 4 2008 S2 v1b

[2] Refer to Question 3 in Test 4 2009 S1 v1a

So the characteristic roots are 2 and 5, so the solution is

$$a_n = A \times 2^n + B \times 5^n,$$

for some constants A and B. To find these constants, we use the initial conditions:

$$a_0 = 5 \implies A \times 2^0 + B \times 5^0 = 5 \implies A + B = 5.$$

Also,

$$a_1 = 1 \implies A \times 2^1 + B \times 5^1 = 1 \implies 2A + 5B = 1.$$

So we solve simultaneously. From $A + B = 5$, we have $A = 5 - B$. Substituting this into $2A + 5B = 1$ gives us

$$2(5 - B) + 5B = 1 \implies 10 + 3B = 1 \implies B = -3.$$

As $A = 5 - B$, we have $A = 5 - (-3) = 8$. Hence the solution is

$$a_n = 8 \times 2^n - 3 \times 5^n.$$

4. **Hints**

Let 'FRED' be considered one 'letter'. So in a word that we want, there is the string of letters 'FRED'; and 7 other letters. In order to find the answer, ensure to account for the possibility of having two 'FRED's in the word. The answer should be

$$8 \cdot 26^7 - \binom{5}{2} \cdot 26^3.$$

Sample Solutions

We must place 11 letters

$$* * * * * * * * * * *.$$

We need to have a string of 'FRED'. Note that our word of 11 letters will contain either exactly one string of FRED or exactly two strings of FRED. Consider words of the form

$$FRED * * * * * * *, *FRED * * * * * *, * * FRED * * * * *, \ldots, * * * * * * * FRED,$$

where the $*$'s are allowed to be any letter. If we naïvely count these, there are 8 places to place the F (after which we write RED), and for each of these placements of the F, we can choose the remaining 7 letters in 26^7 ways. This gives a naïve count of 8×26^7 words. This naïve counting has single-counted all words with FRED appearing exactly once, and *double-counted* all words with FRED appearing exactly twice. This is because for each placement of the F in the above word formations, there exists at least one other place we can place the F to get the word FRED appearing twice; for this other place, we are already counting ALL words with FRED appearing here (*including with FRED occurring in the originally mentioned position*), so we are double-counting the occurrences of two FREDS.

Therefore, we must subtract k from our naïve count to get the answer, where k is the number of ways to get exactly two strings of FRED. This is

$$k = 10 \times 26^3.$$

This is because if we are to have two 'FRED's appear in the string of 11 letters, there are 10 ways to choose the placements of the two 'FRED's, and once we have chosen the positions of the 'FRED's, we pick the remaining 3 letters in 26^3 ways. To see that there are 10 ways to choose the placements of the two 'FRED's, note that in

$$***********,$$

the first F can go in the first position (so the first four letters are FRED, in which case the F for the next FRED can go in either position $5, 6, 7$ or 8, giving 4 possibilities), OR the first F can go in the second position (so the F of the second FRED can go in either position $6, 7$ or 8, giving a further 3 possibilities), OR the first F can go in the third position (so the F of the second FRED can go in either position 7 or 8, giving a further 2 possibilities), OR the first F can go in the fourth position (so the F of the second FRED can go in only position 8, giving a further 1 possibility, so total $4 + 3 + 2 + 1 = 10$). Note that the second 'FRED's F can never go in a position later than 8, because then we would not be able to fit the word 'FRED' in.
Therefore, the answer is our naïve count minus k, i.e.

$$8 \cdot 26^7 - \binom{5}{2} \cdot 26^3.$$