MATH1151 Calculus Test 2 2008 S1 v2a

January 28, 2015

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1. First, we let $L = \lim_{x\to\infty}\left(\sqrt{4x^2+7x-2}-(2x+1)\right)$.
   Now, in order to help us investigate the behaviour of this limit, multiply the numerator and denominator by the conjugate, i.e. by $\sqrt{4x^2+7x-2}+(2x+1)$.
   Simplifying,

$$\begin{aligned}
L &= \lim_{x\to\infty}\frac{4x^2+7x-2-(2x+1)^2}{\sqrt{4x^2+7x-2}+2x+1}\\
&= \lim_{x\to\infty}\frac{4x^2+7x-2-\left(4x^2+4x+1\right)}{\sqrt{4x^2+7x-2}+2x+1}\\
&= \lim_{x\to\infty}\frac{3x-3}{\sqrt{4x^2+7x-2}+2x+1}\\
&= \lim_{x\to\infty}\frac{3x-3}{x\sqrt{4+\frac{7}{x}-\frac{2}{x^2}}+2x+1}
\end{aligned}$$

Dividing the numerator and the denominator by x will help us see the limiting behaviour as $x \to \infty$,

$$= \lim_{x\to\infty} \frac{3 - \frac{3}{x}}{\sqrt{4 + \frac{7}{x} - \frac{2}{x^2}} + 2 + \frac{1}{x}}$$

By the Algebra of Limits, we can change this to,

$$\begin{aligned} &= \frac{\lim_{x\to\infty}\left(3 - \frac{3}{x}\right)}{\lim_{x\to\infty}\left(\sqrt{4 + \frac{7}{x} - \frac{2}{x^2}} + 2 + \frac{1}{x}\right)} \\ &= \frac{3}{\sqrt{4} + 2} \\ \therefore L &= \frac{3}{4}. \end{aligned}$$

2. To determine the number of real roots of a cubic polynomial, we can consider the product of the y-values of the stationary points. If this value is negative, that means the y-values of the stationary points are opposite in sign, implying that there must be 3 real roots (try draw sketches of this scenario if you're unsure about this).
First, we let $f(x) = x^3 - 6x^2 + 1$.
Differentiating, $f'(x) = 3x^1 - 12x$.
Now, for stationary points, let $f'(x) = 0$.
So,

$$\begin{aligned} 3x^2 - 12x &= 0 \\ x^2 - 4x &= 0 \\ x(x-4) &= 0 \\ \\ x &= 0 \text{ or } 4. \end{aligned}$$

Consider the product of the y-values of the two stationary points,

$$\begin{aligned} f(0)\cdot f(4) &= \left(0^3 - 6(0)^2 + 1\right)\cdot\left(4^3 - 6(4)^2 + 1\right) \\ &= 1 \times (64 - 96 + 1) \\ &= -31 < 0. \end{aligned}$$

Thus, both stationary points lie on opposite sides of the x-axis and hence there must be 3 real solutions (roots) to the function.
Alternatively, note that this is a cubic with a positive leading coefficient. We can choose x such that we have the change of signs $- \to + \to - \to +$ which would allow us to use the Intermediate Value Theorem and the Fundamental Theorem of Algebra to argue that

there are exactly three real roots. An example of possible x values where we would see these sign changes are $x = -1, 0, 1, 10$.

3.

$$f(x) = \begin{cases} ax + b, & \text{for } x \leq 1 \\ \tan \frac{\pi x}{4}, & \text{for } 1 < x < 2. \end{cases}$$

For the function to be differentiable at $x = 1$, it must first be continuous, and the derivative must exist.
First, ensuring continuity at $x = 1$, the two sided limits must agree with the function evaluated at this point, i.e.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1).$$

So, $a + b = 1 = a + b$ implying that $a + b = 1$.
Now, ensuring differentiability, we use the definition of the derivative,

$$\begin{aligned} \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{1} &= \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} \\ \lim_{h \to 0^-} \frac{a(1+h) + b - (a+b)}{h} &= \lim_{h \to 0^+} \frac{\tan\left(\frac{\pi(1+h)}{4}\right) - 1}{h} \end{aligned}$$

Note that the RHS takes the form $\left(\frac{0}{0}\right)$, so

$$\begin{aligned} \lim_{h \to 0^-} \frac{ah}{h} &= \lim_{h \to 0^+} \frac{\frac{\pi}{4} \sec^2\left(\frac{\pi(1+h)}{4}\right)}{1} \qquad \text{(L'Hopital's rule)} \\ &= a = \frac{\pi}{4} \sec^2(\pi) 4 \\ &= \frac{\pi}{4} \times 2 \\ \therefore a &= \frac{\pi}{2}. \end{aligned}$$

From the first condition, we have $b = 1 - \frac{\pi}{2}$.
Hence,

$$a = \frac{\pi}{2}, b = 1 - \frac{\pi}{2}.$$

4. Mean Value Theorem: If a function f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a constant $c \in (a, b)$ such that

$$f'(c) = \frac{f(a) - f(b)}{a - b}.$$

Now, let $f : [0, x] \to \mathbb{R},\ f(x) = \tan^{-1} x$ where $x > 0$.
We choose this domain for f by first inspecting the required answer.
Note that f is continuous on $[0, x]$ and differentiable on $(0, x)$. Also, $f'(x) = \frac{1}{1+x^2}$.

Thus, by the Mean Value Theorem, there exists $c \in (0, x)$ such that

$$\begin{aligned} f'(c) &= \frac{f(0) - f(x)}{0 - x} \\ \frac{1}{1 + c^2} &= \frac{0 - f(x)}{0 - x} \\ &= \frac{f(x)}{x} \\ &= \frac{\tan^{-1} x}{x}. \end{aligned}$$

Since $c > 0$, this implies that $\frac{1}{1+c^2} < 1$.
Hence, $\frac{\tan^{-1} x}{x} < 1$ and so $\tan^{-1} x < x$ for $x > 0$.
At $x = 0$, $\tan^{-1} 0 = 0$.
Therefore,

$$\tan^{-1} x \leq x \text{ for } x \geq 0.$$

MATH1151 Calculus Test 2 2009 S1 v1b

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1.
$$f(x) = \begin{cases} ax+b, & \text{for } x \leq 1 \\ \ln x, & \text{for } x > 1. \end{cases}$$

For the function to be differentiable at $x = 1$, it must be both continuous and differentiable at this point.

First, ensuring continuity, we want the two sided limits to agree with the function evaluated at this point, i.e.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$
$$a + b = \ln 1 = a + b$$

This implies that $a + b = 0$.

Now, ensuring differentiability, we use the definition of the derivative,

$$\begin{aligned}
\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}\\
\lim_{h\to 0^-}\frac{a(1+h)+b-(a+b)}{h} &= \lim_{h\to 0^+}\frac{\ln(1+h)-\ln(1)}{h}\\
\lim_{h\to 0^-}\frac{ah}{h} &= \lim_{h\to 0^+}\frac{\ln(1+h)}{h}
\end{aligned}$$

Since the RHS takes the indeterminate form $\left(\frac{0}{0}\right)$,

$$\begin{aligned}
\lim_{h\to 0^-} a &= \lim_{h\to 0^+}\frac{1}{1+h} && \text{(L'Hopital's rule)}\\
&= 1
\end{aligned}$$

Hence, we have $a=1$, and from the first equation, $b=-1$.

2. Mean Value Theorem: If f is continuous on the interval $[a,b]$ and differentiable on (a,b), then there exists a constant $c\in(a,b)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

Since $f(x)=(x-1)^3$ is continuous and differentiable on $\mathbb{R}$ (as it is a polynomial), it must be continuous on $[1,4]$ and differentiable on $(1,4)$.
Also, note that $f'(x)=3(x-1)^2$. By the Mean Value Theorem, there exists a constant $c\in(1,4)$ such that

$$3(c-1)^2=\frac{f(4)-f(1)}{4-1}=\frac{3^3-0}{3-0}=3^2=9.$$

Solving for c, we find that $c=1\pm\sqrt{3}$. But $c\in(1,4)$, so $c=1+\sqrt{3}$.

3. Let $L=\lim_{x\to 0}\frac{e^{2x}+e^{-2x}-2}{x^2}$.
Note that this takes the indeterminate form $\left(\frac{0}{0}\right)$, so we can apply L'Hopital's rule,

$$L=\lim_{x\to 0}\frac{2e^{2x}-2e^{-2x}}{2x} \qquad \text{(L'Hopital's rule)}$$

This also takes the indeterminate form $\left(\frac{0}{0}\right)$,

$$\begin{aligned}
&= \lim_{x\to 0}\frac{4e^{2x}+4e^{-2x}}{2} && \text{(L'Hopital's rule)}\\
&= \frac{4+4}{2}\\
&= 4.
\end{aligned}$$

4. Factorising the denominator,

$$f\left(x\right)=\frac{\left|x-2\right|}{x^{2}-6x+8}=\frac{\left|x-2\right|}{\left(x-4\right)\left(x-2\right)}.$$

<u>Case 1</u>: $x\geq 2$,
When $x\geq 2$, we know that $\left|x-2\right|=x-2$. Rewriting $f\left(x\right)$,

$$f\left(x\right)=\frac{x-2}{\left(x-4\right)\left(x-2\right)}=\frac{1}{x-4}.$$

So, as $x\to 2^{+}$, $f\left(x\right)\to\frac{1}{2-4}=-\frac{1}{2}$.
<u>Case 2</u>: $x<0$,
When $x<2$, we know that $\left|x-2\right|=-\left(x-2\right)$. Rewriting $f\left(x\right)$,

$$f\left(x\right)=\frac{-\left(x-2\right)}{\left(x-4\right)\left(x-2\right)}=-\frac{1}{x-4}.$$

So, as $x\to 2^{-}$, $f\left(x\right)\to-\frac{1}{2-4}=\frac{1}{2}$.
Thus, since the 2-sided limits do not agree, $\lim_{x\to a}f\left(x\right)$ does not exist.

5. Let $f\left(x\right)=x^{2}+1-2\cos x$.
We choose two simple x-values that would result in opposite signs of $f(x)$ and would allow to use the Intermediate Value Theorem.
Choosing $x=0$ and $x=\frac{\pi}{2}$, and evaluating the function at these points,

$$f\left(0\right)=-2<0,\quad f\left(\frac{\pi}{2}\right)=\frac{\pi^{2}}{4}+1>0.$$

These points were chosen to obtain simple function values.
Since f is continuous on $\left[0,\frac{\pi}{2}\right]$, then by the Intermediate Value Theorem, there exists $c\in\left(0,\frac{\pi}{2}\right)$ such that $f\left(0\right)\leq f\left(c\right)\leq f\left(\frac{\pi}{2}\right)$.
Thus, we can choose $c\in\left[0,\frac{\pi}{2}\right]$ such that $f(c)=0$. This implies that there exists at least one positive root.

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1. Recall the definition of the derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

2.

$$c = \frac{3}{2}.$$

3. L'Hopital's rule may or may not help. Limit is 1.

4. Multiply the numerator and denominator by the conjugate and do something similar as in Test 2 2008 S1 v2a. Limit is -2.

5. Consider cases when $x > 2$ and when $x < 2$. The two sided limits do not agree, i.e. $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$ implying there is a jump discontinuity (an essential discontinuity). This means that it cannot be removed by simply defining a function value for

$x = 2.$