Semester 1, 2018, MATH1131/1141 Final Exam Revision Session (Calculus)

UNSW Mathematics Society

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Introduction

What we will cover today –

- Part 1 – Limits
- Part 2 – Continuous Functions
- Part 3 – Mean Value Theorem and its Applications
- Part 4 – Inverse Functions
- Part 5 – Integration

Most examples in this session will consist of past exam questions where we will show you how to approach such questions for the final exam.

$\varepsilon - \delta$ Definition of a Limit

Limit at a point

Let $a \in \mathbb{R}$ and let f be a real-valued function defined in some open interval around a. Let $L \in \mathbb{R}$. We say $\lim_{x \to a} f(x) = L$ if for every positive number ε, there is a $\delta > 0$ such that whenever $0 < |x - a| < \delta$, we have $|f(x) - L| < \varepsilon$.

Limit to ∞

Let f be a real-valued function. Let $L \in \mathbb{R}$. We say $\lim_{x \to \infty} f(x) = L$ if for every positive number ε, there is an M such that whenever $x > M$, we have $|f(x) - L| < \varepsilon$.

Example

Example

Show from the definition of the limit to infinity that

$$\lim_{x \to \infty} e^{-2x} \sin x = 0$$

(MATH1141 2009 Q4)

L'Hôpital's rule

L'Hôpital's rule

Let f and g be differentiable and suppose as $x \to a$ either $f(x) \to 0$ and $g(x) \to 0$ or $f(x) \to \infty$ and $g(x) \to \infty$.
If $\frac{f'(x)}{g'(x)} \to \omega$ as $x \to a$ then $\frac{f(x)}{g(x)} \to \omega$ as $x \to a$.

Note

- ω can be any real number or $\pm\infty$.
- Can replace $x \to a$ by $x \to \pm\infty$ or $x \to a^{\pm}$.
- If $\frac{f'(x)}{g'(x)}$ doesn't converge to any limit or $\pm\infty$ (e.g. if it just oscillates forever), then we cannot apply L'Hôpital's rule. An example of this is in the next slide.

Requirement that limit exist for L'Hôpital's rule

Suppose $f(x) = x - \sin x$, $g(x) = x$ and $a = \infty$. Then

$$\frac{f'(x)}{g'(x)} = \frac{1 - \cos x}{1},$$

which has *no limit* as $x \to \infty$ since the cosine oscillates indefinitely between -1 and $+1$.
Note however that

$$\begin{aligned}\frac{f(x)}{g(x)} &= \frac{x - \sin x}{x} \\ &= 1 - \frac{\sin x}{x} \\ &\to 1 \text{ as } x \to \infty.\end{aligned}$$

Examples

Example

Use L'Hôpital's rule to find the following limit:

$$\lim_{x\to\infty} \frac{\ln x}{x}.$$

Example

Find the following limit or explain why it doesn't exist:

$$\lim_{x\to 0^+} \frac{(1-\cos x)^{\frac{3}{2}}}{x-\sin x}.$$

Definition of Continuity

Definition

Suppose that f is defined on some open interval containing the point $a \in \mathbb{R}$ (and hence is also defined at a).

If $\lim_{x \to a} f(x) = f(a)$, then we say that f is continuous at a.

If f is defined at a but $\lim_{x \to a} f(x) \neq f(a)$, then we say that f is discontinuous (or 'not continuous') at a.

If $\mathcal{I}$ is a subset of $\mathbb{R}$, we say that f is continuous on $\mathcal{I}$ if it is continuous at *every* point $a \in \mathcal{I}$.

Understanding the definition

Requirement 1: f is **defined** at the point a.

Requirement 2: $\lim_{x \to a} f(x)$ exists, i.e. $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$.

Requirement 3: $\lim_{x \to a} f(x)$ **is equal to** the function's value at that point, i.e. $\lim_{x \to a} f(x) = f(a)$.

Example

Example

Show that the function $f(x) = |x - 2|$ is continuous on $\mathbb{R}$.

Max-Min Theorem (aka Extreme Value Theorem)

Max-Min Theorem

If f is continuous on the closed interval $[a, b]$ then f attains both a global maximum and global minimum on $[a, b]$. That is, there exist $c, d \in [a, b]$ such that

$$f(c) \leq f(x) \leq f(d) \quad \text{for all } x \in [a, b] .$$

Intermediate Value Theorem (IVT)

Intermediate Value Theorem

Suppose f is continuous on the closed interval $[a, b]$. If z lies between $f(a)$ and $f(b)$ then there is at least one $c \in (a, b)$ such that $f(c) = z$.

Useful corollary

Let f be continuous on the closed interval $[a, b]$ with $f(a)$ and $f(b)$ having opposite signs (i.e. one is strictly positive and the other is strictly negative). Then there exists a c such that $a < c < b$ with $f(c) = 0$ (i.e. f has a zero between a and b).

Examples

Example

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with $\lim_{x\to\pm\infty} f(x) = 0$.
Show that if there is a number $\eta \in \mathbb{R}$ with $f(\eta) > 0$, then f attains a maximum value on $\mathbb{R}$.
(Note that the Max-Min Theorem applies to *finite* closed intervals $[a, b]$ only.)

Example

Show that the function $f(x) = x^3 - x^2 + 1$ has a zero between $x = -1$ and $x = 0$.

Definition of Differentiability

Definition

Suppose that f is defined on some open interval containing the point $a \in \mathbb{R}$. We say that f is differentiable at a if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. If the limit exists, we denote it by $f'(a)$. We call $f'(a)$ the derivative of f at a.

In other words, f is said to be differentiable at a point a if its derivative there exists.

An important theorem

Theorem (differentiable $\Rightarrow$ continuous)

If f is differentiable at a, then f is continuous at a.
Equivalently, **if** f is not continuous at a point a, then it cannot be differentiable there.

So differentiable $\Rightarrow$ continuous. But not the other way round in general! (E.g. consider $f(x) = |x|$ at the origin.)

Example

Example

Show that the function $f(x) = x^2$ is differentiable everywhere.

Implicit Differentiation

Example

For the curve $y^4 + x^3 - x^2 \mathrm{e}^{3y} = 4$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

Using implicit differentiation on the curve's equation, we have

$$\begin{aligned} 4y^3 y' + 3x^2 - 2x\mathrm{e}^{3y} - 3x^2\mathrm{e}^{3y}y' &= 0 \\ \Rightarrow y'\left(4y^3 - 3x^2\mathrm{e}^{3y}\right) &= 2x\mathrm{e}^{3y} - 3x^2 \\ \Rightarrow y' \equiv \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2x\mathrm{e}^{3y} - 3x^2}{4y^3 - 3x^2\mathrm{e}^{3y}}. \end{aligned}$$

Mean Value Theorem (MVT)

Suppose that f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is at least one real number c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Tip

Remember in the exam when using the MVT, state that the function you're using is continuous on the closed interval in question and differentiable on the corresponding open interval (you may lose marks if you don't state these, since the MVT applies only to functions like this).

You should similarly try and state the hypotheses for other theorems you're using and that they hold for the question at hand. E.g. if using the IVT, you should state that the interval in question is closed and the function in question is continuous over this interval.

Special Case: Rolle's Theorem

Rolle's Theorem

Suppose that g is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Suppose also that $g(a) = g(b)$. Then there is a real number c in (a, b) such that $g'(c) = 0$.

Mean Value Theorem Example

Example

Use the Mean Value Theorem to show that

$$\frac{x-a}{x} < \ln\frac{x}{a} < \frac{x-a}{a}$$

if $0 < a < x$.

Inverse Functions

Suppose that f is a one-to-one function. Then the *inverse function* of f is a function g such that

- $g(f(x)) = x \qquad \forall x \in \mathrm{Dom}(f)$
- $f(g(x)) = x \qquad \forall x \in \mathrm{Range}(f)$

  The following are properties of the inverse function g if it exists for a given f:
- $\mathrm{Dom}(g) = \mathrm{Range}(f)$
- $\mathrm{Range}(g) = \mathrm{Dom}(f)$
- g is one-to-one with inverse f
- A function f can only have one inverse (i.e. g is unique for given f). The inverse function of f is commonly denoted f^{-1}.

Example 1

Evaluate

(a) $\sin^{-1}(\sin \frac{5\pi}{4})$

(b) $\sin(\cos^{-1} \frac{3}{5})$.

Hints

(a) Remember that $\sin^{-1}$ of anything must be between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$. So the answer will be that angle between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$ whose sine is the same as that of $\frac{5\pi}{4}$. In general, for $t \in [-1, 1]$, think of $\sin^{-1} t$ as meaning "that angle between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$ whose sine is t".

(b) Consider drawing a particular right-angled triangle (sides $3, 4, 5$).

Inverse function theorem

Let f be one-to-one and continuously differentiable on an open interval $\mathcal{I}$ containing the point a. Suppose that $f'(a) \neq 0$. Then

1) f is invertible on some open interval containing a, i.e. it is locally invertible around a
2) The aforementioned local inverse f^{-1} is itself continuously differentiable and

$$\boxed{\left(f^{-1}\right)'(f(a)) = \frac{1}{f'(a)}.}$$

Corollary to previous theorem

Corollary

If f is continuously differentiable on some interval $\mathcal{I} = \mathrm{Dom}(f)$ and $f'(x)$ is never 0 for $x \in \mathcal{I}$ (which implies f is one-to-one and hence invertible on $\mathcal{I}$), then the inverse $g \equiv f^{-1}$ exists and has domain $\mathrm{Range}(f)$ and range $\mathrm{Dom}(f)$, is continuously differentiable on $\mathrm{Range}(f)$, and $g'(f(x)) = \frac{1}{f'(x)}$ for all $x \in \mathrm{Dom}(f)$.

Example

Find the slope of the inverse function of $f(x) = x^3 + x - 1$ at $x = 1$.

Definition – Oblique asymptotes

The line $y = ax + b$ is an oblique asymptote of $f(x)$ if:

$$\lim_{x \to +\infty} (f(x) - (ax + b)) = 0 \text{ or } \lim_{x \to -\infty} (f(x) - (ax + b)) = 0.$$

Example

Show that the function $f(x) = x + 2 + \frac{1}{x}$ has $y = x + 2$ as an oblique asymptote.

Curve sketching

Example

Sketch the curve defined parametrically by

$$x = \frac{t}{2}, y = 2 + t^3, \quad t \in \mathbb{R}.$$

Example

Sketch the graph of the polar curve

$$r = 1 + \cos\theta.$$

Integration

Key things to know include

- Riemann sums
- Basic properties of integrals
- The Fundamental Theorems of Calculus
- Indefinite integration, integration by parts, substitution
- Improper integrals and comparison tests

The course pack covers this theory comprehensively, so use it! We will mainly focus on doing sample questions now.

Integration

Example

Suppose $a \in (0, 1)$.

a) Show that

$$\int_0^a \ln(1-x)\ \mathrm{d}x = (a-1)\ln(1-a) - a.$$

b) By using a) and an appropriate Riemann sum, show that

$$\lim_{n\to\infty} \left\{ \left(1 - \frac{a}{n}\right)\left(1 - \frac{2a}{n}\right)\cdots\left(1 - \frac{na}{n}\right) \right\}^{\frac{a}{n}} = e^{-a}(1-a)^{a-1}.$$

c) Hence find

$$\lim_{a\to 1^-} \left\{ \lim_{n\to\infty} \left\{ \left(1 - \frac{a}{n}\right)\left(1 - \frac{2a}{n}\right)\cdots\left(1 - \frac{na}{n}\right) \right\}^{\frac{a}{n}} \right\}.$$

Integration

Example

Find $\int_1^b x^{-\frac{1}{2}}\,\mathrm{d}x$ by Riemann sums, using a uniform partition and:

$$x_i^* = \left(\frac{\sqrt{x_{i-1}} + \sqrt{x_i}}{2}\right)^2.$$

Fundamental Theorems of Calculus

The First Fundamental Theorem of Calculus

If f is a continuous function defined on $[a,b]$, then the function $F : [a,b] \to \mathbb{R}$, defined by

$$F(x) = \int_a^x f(t)\,\mathrm{d}t,$$

is continuous on $[a,b]$, continuously differentiable on (a,b), and has derivative F' given by $F'(x) = f(x)$ for all x in (a,b).

The Second Fundamental Theorem of Calculus

Suppose that f is a continuous function on $[a,b]$. If F is an antiderivative of f on $[a,b]$, then

$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a).$$

Tests for convergence/divergence

We have the following tests for convergence/divergence of improper integrals:

- p-**test**: The improper integral $\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x$ is convergent if $p > 1$ and divergent if $p \le 1$.
- **comparison test (inequality form)**: Suppose that f and g are integrable functions and that $0 \le f(x) \le g(x)$ whenever $x > a$. If $\int_a^\infty g(x)\,\mathrm{d}x$ converges then $\int_a^\infty f(x)\,\mathrm{d}x$ converges. If $\int_a^\infty f(x)\,\mathrm{d}x$ diverges then $\int_a^\infty g(x)\,\mathrm{d}x$ diverges. (You can visualise these graphically to remember them.)
- **comparison test (limit form)**: Suppose that f and g are non-negative and continuous on $[a, \infty)$. If $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = L$ and $0 < L < \infty$, then $\int_a^\infty f(x)\,\mathrm{d}x$ and $\int_a^\infty g(x)\,\mathrm{d}x$ either both converge or both diverge. I.e. f and g have similar behaviour at infinity. In particular, their improper integrals from a to ∞ have the same convergence status.

Example

Example

Does the integral:

$$\int_1^\infty \frac{\mathrm{d}x}{\sqrt{x + x^3}}$$

converge? Prove your answer.
(MATH1151 2015 Q4)

Example

Does the integral:

$$\int_{69}^\infty \frac{1}{x \ln x} \, \mathrm{d}x$$

converge? Give reasons for your answer.

Further examples

Example

Does the integral:

$$\int_2^\infty \frac{x^2 \ln x}{x^5 - 3}\, \mathrm{d}x$$

converge? Give reasons for your answer.
(MATH1151 2011 Q1)

Example

Does the integral:

$$\int_1^\infty \frac{1}{\sqrt{1 + x + x^2}}\, \mathrm{d}x$$

converge? Give reasons for your answer.

Hyperbolic Functions

Basic facts

- $\cosh x = \frac{1}{2}(e^x + e^{-x})$. Domain: $\mathbb{R}$. Range: $[1, \infty)$. Even function.
- $\sinh x = \frac{1}{2}(e^x - e^{-x})$. Domain: $\mathbb{R}$. Range: $\mathbb{R}$. Odd function.
- $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$. Domain: $\mathbb{R}$. Range: $(-1, 1)$. Odd function.
- $\frac{\mathrm{d}}{\mathrm{d}x}(\cosh x) = \sinh x$
- $\frac{\mathrm{d}}{\mathrm{d}x}(\sinh x) = \cosh x$
- $\cosh^2 x - \sinh^2 x = 1$

Hyperbolic identities are often analogous to their trigonometric counterparts, i.e. replace $\sin x$ by $\sinh x$ and $\cos x$ by $\cosh x$, but some signs may change.

Identities

Some hyperbolic identities

- $\cosh^2 x - \sinh^2 x = 1$ (must-know)
- $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
- $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
- $\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$
- $\cosh x + \sinh x = e^x$ and $\cosh x - \sinh x = e^{-x}$
- $(\cosh x + \sinh x)^\alpha = \cosh(\alpha x) + \sinh(\alpha x)$ $(= e^{\alpha x})$ for all real x, α
- $\sinh(2x) = 2 \sinh x \cosh x$
- $\cosh(2x) = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x = \cosh^2 x + \sinh^2 x$

Note in red some sign changes compared to the analogous trigonometric identities. The last identity can be rearranged to establish half-angle identities (i.e. get $\sinh^2 x$ and $\cosh^2 x$ in terms of $\cosh(2x)$).

Hyperbolic Functions

Example

Prove that $\cosh^2 x - \sinh^2 x = 1$.
(MATH1141, 2013 Q1)

s

Example

a) State the domain and range of $\tanh$ and $\tanh^{-1}$ and sketch their graphs.
b) How many real numbers x satisfy the equation

$$\tanh^{-1} x = x + 1?$$

Prove your answer explaining which theorems you have used.
(MATH1151, 2015 Q3)

SAMPLE QUESTIONS

The following slides contain further sample questions.

Limits

Question

Prove using the ε-δ definition of limits that $\lim_{x \to 2} x^3 = 8$

Question

Evaluate the following limit or explain why it doesn't exist:

$$\lim_{x \to 2^-} \frac{|x^2 - 4|}{x - 2}$$

(MATH1151 2015 Q2)

Limits

Question

Calculate the limit

$$L = \lim_{x\to\infty} \frac{1}{x} \int_{x}^{4x} \cos\left(\frac{1}{u}\right) \mathrm{d}u.$$

Inverse functions

Question

The area $A(t)$ of an arbitrary depends convex quadrilateral $\mathcal{Q}$ with given side lengths a, b, c, d depends on the sum $t = \alpha + \beta$ of either pair of opposite angles, and is given by Bretschneider's formula:

$$A(t) = \sqrt{(s-a)(s-b)(s-c)(s-d) - \frac{1}{2}abcd\,(1+\cos t)},$$

where $s = \frac{a+b+c+d}{2}$ is the *semi-perimeter* of $\mathcal{Q}$.

a) Explain why the area A of a convex quadrilateral with fixed sides a, b, c, d is maximal if the sum of either pair of opposite angles is π.

b) Show that the area function $A : [0, \pi] \to \mathbb{R}$ as defined above is invertible, and that the inverse function B is differentiable on $(A(0), A(\pi))$.

Inverse functions

Cont.

c) Show that

$$B'\left(A_0\right) = \frac{4A_0}{abcd},$$

where

$$A_0 = \sqrt{\left(s-a\right)\left(s-b\right)\left(s-c\right)\left(s-d\right) - \frac{1}{2}abcd}.$$

Mean Value Theorem

Question

a) State carefully the Mean Value Theorem.

b) Use the Mean Value Theorem to prove that if $a < b$, then

$$0 < \tan^{-1} b - \tan^{-1} a \leq b - a.$$

c) Using (b) or otherwise, prove that the improper integral

$$I = \int_1^{\infty} \left[\tan^{-1}\left(t + \frac{1}{t^2}\right) - \tan^{-1} t\right] \mathrm{d}t$$

converges.

Integration

Question

1. Determine whether the following improper integrals converge
   (a) $\int_0^\infty e^{-\sqrt{x}}\,\mathrm{d}x$
   (b) $\int_3^\infty \frac{x}{\sqrt{x^6-1}}\,\mathrm{d}x$.
2. [Absolute integrability implies integrability] Suppose that f is a continuous real-valued function on $[a,\infty)$, where $a \in \mathbb{R}$ is a constant. Show that if

$$\int_a^\infty |f(x)|\,\mathrm{d}x \quad \text{converges,}$$

then

$$\int_a^\infty f(x)\,\mathrm{d}x \quad \text{converges.}$$

(Bonus exercise: Show by finding a counterexample that the converse is not true.)

Good Luck!

ANY QUESTIONS?
GOOD LUCK FOR YOUR FINAL EXAMS!
MATHSOC WISHES YOU ALL THE BEST IN YOUR FUTURE STUDY.