UNSW Mathematics Society Presents
MATH1131/1141 Workshop

Presented by Lachlan Tobin, Noah Rubin, and Michael Pogrebinsky

Overview I

1. Algebra
   - Introduction to Vectors
   - Vector Geometry
   - Matrices
   - Complex numbers

2. Calculus
   - Sets, Inequalities and Functions
   - Limits
   - Properties of Continuous Functions
   - Differentiable Functions
   - Mean Value Theorem

1. Algebra

Planes

Definition

In $\mathbb{R}^3$, the **parametric vector form** of a plane is

$$\mathbf{x} = \mathbf{a} + \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2, \text{ for } \lambda_1, \lambda_2 \in \mathbb{R}.$$

The **Cartesian equation** of a plane is

$$ax_1 + bx_2 + cx_3 = d, \text{ where } a, b, c \text{ and } d \text{ are fixed reals.}$$

Question 1

Find the parametric form of $\mathrm{x}_1 + 2x_2 + 5x_3 = -3$.

Solution. We set $x_2 = \lambda_1$ and $x_3 = \lambda_2$. Then solving for x_1 gives $x_1 = -3 - 2\lambda_1 - 5\lambda_2$. Therefore, the parametric vector form of the equation is

$$\begin{aligned}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} -3 - 2\lambda_1 - 5\lambda_2 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}, \text{ where } \lambda_1, \lambda_2 \in \mathbb{R}.\end{aligned}$$

Planes

Question 2

Consider the three points A, B, C in $\mathbb{R}^3$ with position vectors

$$\begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} -5 \\ 1 \\ 2 \end{pmatrix} \text{ respectively.}$$

Find a parametric vector form for the plane Π that passes through points A, B, and C.

Planes

Solution. Let **OA** be the position vector of the plane Π, now find **AB** and **AC** which are direction vectors of the plane Π, this gives,

$$\mathbf{AB} = \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \\ 5 \end{pmatrix},$$

$$\mathbf{AC} = \begin{pmatrix} -5 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ 3 \end{pmatrix}.$$

Therefore, by definition, the parametric form of the plane Π is

$$\mathbf{x} = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 7 \\ -3 \\ 5 \end{pmatrix} + \lambda_2 \begin{pmatrix} -3 \\ -2 \\ 3 \end{pmatrix}, \text{ for } \lambda_1, \lambda_2 \in \mathbb{R}.$$

Lines and Planes

Definition

In $\mathbb{R}^n$, the parametric vector form of a line is

$$\mathbf{x} = \mathbf{a} + \lambda \mathbf{v}, \text{ for } \lambda \in \mathbb{R}.$$

Lines and Planes

Question 3 (from 2010Nov MATH1131 Paper)

Determine the coordinates of the point of intersection of the line

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \text{ for } t \in \mathbb{R},$$

and the plane

$$x_1 - 3x_2 + x_3 = 15.$$

Lines and Planes

Solution. From the parametric vector form of the line, we have

$$x_1 = 1 + t, x_2 = 2 - t \text{ and } x_3 = 5 + t.$$

Substituting these into the Cartesian equation of the plane, we get

$$\begin{aligned}(1 + t) - 3(2 - t) + (5 + t) &= 15 \\ 5t &= 15 \\ t &= 3.\end{aligned}$$

At $t = 3$, we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 8 \end{pmatrix}.$$

Hence, the point of intersection is (4, -2, 8).

Cross Product

Definition

The **cross product** of two vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ in $\mathbb{R}^3$ is

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

Application of the Cross Product

Question 4

Consider the three points A, B, C in $\mathbb{R}^3$ with position vectors

$$\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -4 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} \text{respectively.}$$

(i) Find $|\mathbf{CB}|$.
(ii) Find the area of the triangle with vertices A, B and C.
(iii) Hence, or otherwise, find the shortest distance from the point A to the line passing through B and C.

Question 4 Solution

Solution.

(i) Subtracting the position vector of C from the position vector of B, we get

$$\mathbf{CB} = \begin{pmatrix} 0 \\ -4 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ -9 \\ 4 \end{pmatrix}.$$

Thus, we have

$$|\mathbf{CB}| = \sqrt{(-3)^2 + (-9)^2 + 4^2} = \sqrt{106}.$$

(ii) Recall that the area of a parallelogram spanned by vectors $\mathbf{a}$ and $\mathbf{b}$ is $|\mathbf{a} \times \mathbf{b}|$ units^2.The area of triangle ABC is simply half the area of the parallelogram that is spanned by $\mathbf{AB}$ and $\mathbf{AC}$, so

$$Area(\triangle ABC) = \frac{1}{2}\left|\mathbf{AB} \times \mathbf{AC}\right|.$$

Question 4 Continued

(ii) Continued.
We have

$$\mathbf{AB} = \begin{pmatrix} 0 \\ -4 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix},$$

and

$$\mathbf{AC} = \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ -3 \end{pmatrix}.$$

Using the formula for the cross product, we get

$$\mathbf{AB} \times \mathbf{AC} = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 7 \\ -3 \end{pmatrix} = \begin{pmatrix} (-2) \cdot (-3) - 1 \cdot 7 \\ 1 \cdot 1 - (-2) \cdot (-3) \\ (-2) \cdot 7 - (-2) \cdot 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \\ -12 \end{pmatrix}.$$

(ii) Cont.
Thus, we can conclude that

$$\begin{aligned} Area(\triangle ABC) &= \frac{1}{2}\left|\begin{pmatrix} -1 \\ -5 \\ -12 \end{pmatrix}\right|, \\ &= \frac{1}{2}\sqrt{(-1)^2 + (-5)^2 + (-12)^2}, \\ &= \frac{\sqrt{170}}{2}\text{units}^2. \end{aligned}$$

(iii) Let d be the shortest distance from A to the line passing through B and C. Note that d is the perpendicular height of triangle ABC, so

$$Area(\triangle ABC) = \frac{1}{2}d\,|\mathbf{CB}|$$

$$\frac{\sqrt{170}}{2} = \frac{\sqrt{106}}{2}d$$

Question 4 Continued

(iii) Cont.

$$d = \sqrt{\frac{85}{53}}.$$

Therefore, the shortest distance from A to the line passing through B and C is $\sqrt{\frac{85}{53}}$ units.

Projections

Definition

The **projection** of vector $\mathbf{a} \in \mathbb{R}^n$ onto vector $\mathbf{b} \in \mathbb{R}^n$ is

$$\text{proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\mathbf{b}.$$

The **length** of the projection is given by

$$|\text{proj}_{\mathbf{b}}\mathbf{a}| = \frac{|\mathbf{a} \cdot \mathbf{b}|}{|\mathbf{b}|}.$$

Distance Between a Point and a Plane

Question 5 (from 2013June MATH1141 Paper)

Find the shortest distance between the point (3, 2, 3) and the plane

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}, \text{ for } \lambda_1, \lambda_2 \in \mathbb{R}.$$

Question 5 Solution

Solution. The cross product of a plane's direction vectors is normal to the plane. Since the plane has direction vectors $\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$, a vector normal to the plane is

$$\mathbf{n} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \cdot 3 - (-1) \cdot 1 \\ (-1) \cdot (-1) - 0 \cdot 3 \\ 0 \cdot 1 - 2 \cdot (-1) \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix}.$$

The vector from the point $(1, 2, 4)$ on the plane to the point $(3, 2, 3)$ is

$$\mathbf{p} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}.$$

Question 5 Continued

The length of the projection of this vector **p** onto **n** is the shortest distance from the point (3, 2, 3) to the plane,

$$|\text{proj}_{\mathbf{n}}\mathbf{p}| = \frac{\left|\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix}\right|}{\left|\begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix}\right|} = \frac{|2 \cdot 7 + 0 \cdot 1 + (-1) \cdot 2|}{\sqrt{7^2 + 1^2 + 2^2}} = \frac{12}{\sqrt{54}} = \frac{4}{\sqrt{6}}.$$

Thus, the shortest distance from the point (3, 2, 3) to the plane is $\frac{4}{\sqrt{6}}$ units.

Dot Product

Definition

The **dot product** of two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$ is

$$\mathbf{a} \cdot \mathbf{b} = a_1 \cdot b_1 + a_2 \cdot b_2 + ... + a_n \cdot b_n.$$

Application of the Dot Product

Question 6

(i) For any vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, show that

$$|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2\,|\mathbf{a}|^2 + 2\,|\mathbf{b}|^2 \, .$$

(ii) Explain the significance of this result geometrically.

Solution.

(i) Recall that $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$, so

$$\begin{aligned} LHS &= |\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2 \\ &= (\mathbf{a}+\mathbf{b}) \cdot (\mathbf{a}+\mathbf{b}) + (\mathbf{a}-\mathbf{b}) \cdot (\mathbf{a}-\mathbf{b}) \\ &= (\mathbf{a} \cdot \mathbf{a} + 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b}) + (\mathbf{a} \cdot \mathbf{a} - 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b}) \\ &= 2\mathbf{a} \cdot \mathbf{a} + 2\mathbf{b} \cdot \mathbf{b} \\ &= 2\,|\mathbf{a}|^2 + 2\,|\mathbf{b}|^2 \\ &= RHS, \text{ as required.} \end{aligned}$$

(ii) Consider the parallelogram spanned by vectors $\mathbf{a}$ and $\mathbf{b}$. It is then clear that vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$ represent the diagonals of this parallelogram. Thus, the result suggests that the sum of the squares of a parallelogram's diagonals equals the sum of the squares of the parallelogram's sides.

MATH1131 June 2012

Let $P = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & 4 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 5 & 0 \end{pmatrix}$

1. Evaluate PQ^T
2. What is the size of P^TQ
3. Does the matrix product PQ exist? Explain your answer.

Written Solution

Solution

1. Observe that since:
$$Q^T = \begin{pmatrix} 1 & 2 \\ -1 & 5 \\ 1 & 0 \end{pmatrix}, PQ^T = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 5 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 12 \\ 8 & 1 \end{pmatrix}$$
2. P^T is a (3×2) matrix and Q is of dimension (2×3), hence $P^T Q$ is a (3×3) matrix
3. PQ does not exist since we are multiplying a (2×3) matrix by a (2×3) matrix. The last number in the first bracket (3) needed to have matched the first number in the last bracket (2).

Solubility from row-echelon form

After transforming the augmented matrix for a system of linear equations into row-echelon form $(U|\mathbf{y})$,

1. The system has **no solution** if and only if the right hand column is a leading column.
2. The system has a **unique solution** if and only if every column on the left is a leading column.
3. The system has **infinite** solutions otherwise.

1141 2017 S1 2(iv)

Question

For some values of the real parameters a, b, c and d, the curve $ax^2 + by^2 + cx + dy = 1$ passes through the points $A(1, 1)$, $B(2, 3)$ and $C(0, 1)$.

1. Explain why the following equations can be used to determine the values of a, b, c and d for which the curve passes through the points.

$$\begin{array}{ccccccccc} a & + & b & + & c & + & d & = & 1 \\ 4a & + & 9b & + & 2c & + & 3d & = & 1 \\ & & b & & & + & d & = & 1. \end{array}$$

Continued

Question

2. Use Gaussian Elimination to solve the system of linear equations in part 1.
3. Are there zero, one, or infinitely many curves of the form $ax^2 + by^2 + cx + dy = 1$ which pass through the points A, B and C?
4. Using your answer from part 2, find the parabola of the form $y = \alpha x^2 + \beta x + \gamma$ which passes through A, B and C.

Solution

1. If the curve passes through $A(1,1)$, $B(2,3)$ and $C(0,1)$, then the coordinates of the points must satisfy the equation. We obtain the set of linear equations by substituting the coordinates of A, B and C into the equation $ax^2 + by^2 + cx + dy = 1$.
2. We can represent the system of linear equations as

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 4 & 9 & 2 & 3 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{array}\right) \xrightarrow{R_2 = R_2 - 4R_1} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 5 & -2 & -1 & -3 \\ 0 & 1 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 5 & -2 & -1 & -3 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 - 5R_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & -2 & -6 & -8 \end{array}\right)$$

Part 3, 4

Part 3, 4

Solution (cont.)

3. There are infinitely many solutions to the system of linear equations, and so there are infinitely many curves of the form $ax^2 + by^2 + cx + dy = 1$ passing through A, B and C.
4. Let $d = 1$. From the matrix, we have

$$\begin{aligned} -2c - 6d &= -8, \\ b + d &= 1 \text{ and} \\ a + b + c + d &= 1 \end{aligned}$$

from which we can deduce, using back substitution, that $c = 1$, $b = 0$ and $a = -1$.
That is, one curve passing through A, B and C is given by $(-1)x^2 + (0)y^2 + (1)x + (1)y = 1$. Hence, the parabola passing through is given by the equation $y = x^2 - x + 1$.

1141 2020 T1

Using the following Maple session, or otherwise, answer the questions below.

```
> with(LinearAlgebra):
> A := < < m, 1, 2 > | < 1, m, 1> | < 1, 1, 4*m > >:
> b := < -m^3-5*m^2-5*m+10, -m^2, -m >:
> M := < A | b >;
```

$$M := \begin{bmatrix} m & 1 & 1 & -m^3-5m^2-5m+10 \\ 1 & m & 1 & -m^2 \\ 2 & 1 & 4m & -m \end{bmatrix}$$

```
> M1 := RowOperation(M, [2, 1]):
> M2 := RowOperation(M1, [2, 1], -m):
> M3 := RowOperation(M2, [3, 1], -2);
```

$$M3 := \begin{bmatrix} 1 & m & 1 & -m^2 \\ 0 & 1-m^2 & 1-m & -5(m+2)(m-1) \\ 0 & 1-2m & 4m-2 & 2m^2-m \end{bmatrix}$$

```
> M4 := simplify(RowOperation(M3, 3, 1/(2*m - 1))):
> M5 := simplify(RowOperation(M4, 2, 1/(1 - m))):
> M6 := RowOperation(M5, [2, 3]):
> M7 := RowOperation(M6, [3, 2], m + 1);
```

$$M7 := \begin{bmatrix} 1 & m & 1 & -m^2 \\ 0 & -1 & 2 & m \\ 0 & 0 & 2m+3 & m^2+6m+10 \end{bmatrix}$$

1141 2020 T1 (cont.)

Question

1. For which real values of m, if any, does the system have no solution?
2. The system has infinitely many solutions when $m = 1$. For which other real value or values of m does the system have infinitely many solutions?
3. For which real value of values of m, if any, does the system have a unique solution?
4. For $m = 1$, the system has solution of the form $\mathbf{x} = \mathbf{a} + \lambda\mathbf{v}$. Find vectors $\mathbf{a}$ and $\mathbf{v}$.

Solution

1. When $m = -\frac{3}{2}$ *(the rightmost column becomes a leading column).*
2. When $m = \frac{1}{2}$ *(we test this value because we multiplied a row by $\frac{1}{2m-1}$, which means the system of linear equations represented by M_7 and M_3 differ when $m = \frac{1}{2}$).*
3. All $m \in \mathbb{R}$ where $m \neq 1, \frac{1}{2}, -\frac{3}{2}$.
4. Letting $m = 1$, we obtain, from M_3, the matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 1 \end{array}\right).$$

Solution (cont.)

4. From the matrix, we have

$$-x_2 + 2x_3 = 1,$$
$$x_1 + x_2 + x_3 = -1.$$

Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ represent the solution. We parameterise the variable x_3 corresponding to the non-leading column. That is, we let $x_3 = \lambda$. Then, $x_2 = 2\lambda - 1$ and $x_1 = -3\lambda$. Hence,

$$\mathbf{x} = \begin{pmatrix} -3\lambda \\ 2\lambda - 1 \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}.$$

That is, $\mathbf{a} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$.

Reminders

Calculating the Determinant of a 3×3 Matrix

Let $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$

1. $\det(A) = a_1(b_2c_3 - c_2b_3) - a_2(b_1c_3 - c_1b_3) + a_3(b_1c_2 - c_1b_2)$.
2. Suppose that the matrix A consisted of the row vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. The determinant of the matrix is equal to $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. That is, their scalar triple product.

Reminders (cont.)

Determinants and Solubility

Let A be an $n \times n$ matrix.

1. If $\det(A) \neq 0$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution and the solution is unique for all $\mathbf{b} \in \mathbb{R}^n$.
2. If $\det(A) = 0$, the equation $A\mathbf{x} = \mathbf{b}$ either has no solution or an infinite number of solutions for a given $\mathbf{b}$.

Reminders (cont.)

Some Properties of Determinants

Suppose that A and B are two $n \times n$ matrices. Then,

1. $\det(AB) = \det(A)\det(B)$.
2. A is an invertible matrix if and only if $\det(A) \neq 0$.
3. If a row (or column) of A is multiplied by a scalar, then the value of $\det(A)$ is multiplied by the same scalar. That is, if the matrix B is obtained from the matrix A by multiplying a row (or column) of A by the scalar λ, then $\det(B) = \lambda \det(A)$.

There are many more properties of determinants that are very useful to know.

MATH1141 June 2014

Examples

Consider the 2×2 complex matrix given as $A = \begin{pmatrix} 2 & i \\ 1+i & \alpha \end{pmatrix}$.

1. Find A^{-1} in the case where $\alpha \in \mathbb{R}$
2. Find all values of $\alpha \in \mathbb{C}$ for which $\det(A^2) = -1$.

MATH1141 June 2014

Solution

1. Inverting this matrix gives us:
$\frac{1}{\det(A)}\begin{pmatrix}\alpha & -i\\ -1-i & 2\end{pmatrix} = \frac{1}{2\alpha+1-i}\begin{pmatrix}\alpha & -i\\ -1-i & 2\end{pmatrix}.$
2. We are told that $\det(A^2) = -1$ to which we can then note that $[\det(A)]^2 = -1$ since $\det(A^2) = \det(AA) = \det(A)\det(A)$. Taking the square root of both sides:

$$\det(A) = \pm i.$$

Thus:

$$2\alpha + 1 - i = i \text{ or } 2\alpha + 1 - i = -i \implies \alpha = \frac{1}{2} \text{ or } \alpha = \frac{2i-1}{2}$$

MATH1131 June 2014 Q2 iv

Question

Consider an invertible matrix A

1. Simplify the matrix expression $(A^T A)^{-1}(A^T A)^T$

Q2 iv - Hand written solution

Solution

1. Considering that the inverse of the product is the reverse product of the inverses, and considering the fact that the transpose of the product is the reverse product of the transposes, we obtain:

$$\begin{aligned}(A^TA)^{-1}(A^TA)^T &= A^{-1}(A^T)^{-1}(A^T)(A^T)^T \\ &= A^{-1}I(A^T)^T \\ &= A^{-1}A \\ &= I.\end{aligned}$$

Extra Question

Question

Suppose that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has a determinant of 4.

1. Find the determinant of the matrix $\begin{pmatrix} 2a & 200b \\ c & 100d \end{pmatrix}$

Solution

1. We know that the determinant of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given as $ad - bc$.

   Hence the determinant of $\begin{pmatrix} 2a & 200b \\ c & 100d \end{pmatrix}$ is given as:

$$\begin{aligned} 2a \times 100d - 200b \times c &= 200(ad - bc) \\ &= 200(4) \\ &= 800. \end{aligned}$$

De Moivre's Theorem

Theorem

De Moivre's Theorem

For any real number θ and integer n,

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta),$$

or equivalently,

$$(e^{i\theta})^n = e^{in\theta}.$$

De Moivre's Theorem

Question

(i) Use de Moivre's Theorem to express $\cos(4\theta)$ as a polynomial in $\cos\theta$.

(ii) Hence, or otherwise, find the roots of

$$P(x) = 128x^4 - 32x^2 + 1.$$

(iii) By examining the roots of $P(x)$, find the exact value of $\sin(\frac{\pi}{8})\sin(\frac{3\pi}{8})$.

Solution.
(i) De Moivre's Theorem states that

$$\cos(4\theta) + i\sin(4\theta) = (\cos\theta + i\sin\theta)^4.$$

Applying the binomial theorem to the RHS, we have

$$\begin{aligned}\cos(4\theta) + i\sin(4\theta) &= \cos^4\theta + 4i\cos^3\theta\sin\theta + 6i^2\cos^2\theta\sin^2\theta \\ &\quad +4i^3\cos\theta\sin^3\theta + i^4\sin^4\theta \\ &= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) \\ &\quad +i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta).\end{aligned}$$

Now, by equating the real parts, we get

$$\cos(4\theta) = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.$$

Solution Continued

(i) Cont.
To get a polynomial in $\cos\theta$, we finally replace $\sin^2\theta$ with $1-\cos^2\theta$,

$$\begin{aligned}\cos(4\theta) &= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2, \\ &= 8\cos^4\theta - 8\cos^2\theta + 1.\end{aligned}$$

(ii) Substituting $x = \frac{1}{2}\cos\theta$ into $P(x)$, we get

$$P(\frac{1}{2}\cos\theta) = 8\cos^4\theta - 8\cos^2\theta + 1 = \cos(4\theta).$$

Factor theorem implies that if $\cos(4\theta) = 0$, then $x = \frac{1}{2}\cos\theta$ is a root of $P(x)$. Setting $\cos(4\theta) = 0$, we have

$$4\theta = \frac{2n+1}{2}\pi, \text{ where } n = 0, 1, 2, 3$$

Solution Continued

(ii) Cont.

$$\begin{aligned}\theta &= \frac{2n+1}{8}\pi, \text{ where } n = 0, 1, 2, 3 \\ &= \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}.\end{aligned}$$

Thus, the roots of $P(x) = 128x^4 - 32x^2 + 1$ are $\frac{1}{2}\cos(\frac{\pi}{8})$, $\frac{1}{2}\cos(\frac{3\pi}{8})$, $\frac{1}{2}\cos(\frac{\pi}{8})$ and $\frac{1}{2}\cos(\frac{7\pi}{8})$.
(iii) By Vieta's formula, we have

$$\frac{1}{2}\cos(\frac{\pi}{8}) \times \frac{1}{2}\cos(\frac{3\pi}{8}) \times \frac{1}{2}\cos(\frac{\pi}{8}) \times \frac{1}{2}\cos(\frac{7\pi}{8}) = \frac{1}{128}$$

$$\cos(\frac{\pi}{8})\cos(\frac{3\pi}{8})\cos(\frac{5\pi}{8})\cos(\frac{7\pi}{8}) = \frac{1}{8}.$$

Solution Continued

(iii) Cont.
But $\cos\theta = \sin(\frac{\pi}{2} - \theta)$, meaning that $\cos(\frac{\pi}{8}) = \sin(\frac{3\pi}{8})$, $\cos(\frac{3\pi}{8}) = \sin(\frac{\pi}{8})$, $\cos(\frac{5\pi}{8}) = \sin(\frac{-\pi}{8}) = -\sin(\frac{\pi}{8})$ and $\cos(\frac{7\pi}{8}) = \sin(\frac{-3\pi}{8}) = -\sin(\frac{3\pi}{8})$. Thus, we have

$$\sin(\frac{3\pi}{8})\sin(\frac{\pi}{8})(-\sin(\frac{\pi}{8}))(-\sin(\frac{3\pi}{8})) = \frac{1}{8}$$

$$(\sin(\frac{\pi}{8})\sin(\frac{3\pi}{8}))^2 = \frac{1}{8}.$$

Hence, we conclude that $\sin(\frac{\pi}{8})\sin(\frac{3\pi}{8}) = \frac{1}{2\sqrt{2}}$, since $\sin(\frac{\pi}{8}), \sin(\frac{3\pi}{8}) > 0$.

Roots of Unity

Question (from 2019Sem2 MATH1141 Paper)

By considering the seventh roots of unity, find the set of non-real solutions to the equation

$$(1+\omega)^7 = (1-\omega)^7.$$

Solution

Solution. Note that $\omega = 1$ is clearly not a solution to the equation, so we may divide both sides by $(1-\omega)^7$ to get

$$(\frac{1+\omega}{1-\omega})^7 = 1$$

Now, let $\frac{1+\omega}{1-\omega} = e^{i\theta}$ and rewrite 1 as $e^{i(2n\pi)}$, for $n \in \mathbb{Z}$. Then, by de Moivre's Theorem, our equation becomes

$$e^{i(7\theta)} = e^{i(2n\pi)}$$

Equating arguments, we have

$$7\theta = 2n\pi$$

$$\theta = \frac{2n\pi}{7}.$$

Solution Continued

Rearranging $\frac{1+\omega}{1-\omega} = e^{i\theta}$, we can isolate ω,

$$1 + \omega = e^{i\theta}(1 - \omega)$$

$$\omega(1 + e^{i\theta}) = e^{i\theta} - 1$$

$$\begin{aligned}
\omega &= \frac{e^{i\theta} - 1}{e^{i\theta} + 1} \\
&= \frac{e^{i\theta} - 1}{e^{i\theta} + 1} \times \frac{e^{-i\theta} + 1}{e^{-i\theta} + 1} \\
&= \frac{e^{0} + e^{i\theta} - e^{-i\theta} - 1}{e^{0} + e^{i\theta} + e^{-i\theta} + 1} \\
&= \frac{2i \sin\theta}{2 + 2\cos\theta} \\
&= \frac{i \sin\theta}{1 + \cos\theta}
\end{aligned}$$

Solution Continued

To find all the distinct roots, we restrict $0 \leq \theta < 2\pi$,

$$0 \leq \frac{2n\pi}{7} < 2\pi,$$

so $n = 0, 1, 2, 3, 4, 5, 6$. Note that $n = 0$ gives $\omega = 0$ which is real. However, we only care about the set of non-real solutions. Thus, we conclude that the set of non-real solutions is

$$S = \{\frac{i\sin(\frac{2n\pi}{7})}{1+\cos(\frac{2n\pi}{7})}, n = 1, 2, 3, 4, 5, 6\}.$$

2. Calculus

Question 1

Solve $|4x - 5| > 3$.

Case 1: $|4x-5|>3$

$4x-5>0$

$x>\frac{5}{4}$

$|4x-5| = 4x-5$

$4x-5>3$

$4x>8$

$x>2$ $\checkmark$

$x \in (-\infty, \frac{1}{2}) \cup (2, \infty)$

Case 2:

$4x-5<0$

$x<\frac{5}{4}$

$|4x-5| = -(4x+5)$

$-(4x+5)>3$

$4x+5<-3$

$x<\frac{1}{2}$ $\checkmark$

Solution

- Case 1:

$$\begin{aligned} 4x - 5 &\geq 0 \\ x &\geq \frac{5}{4} \end{aligned}$$

Now,

$$\begin{aligned} 4x - 5 &> 3 \\ 4x &> 8 \\ x &> 2 \end{aligned}$$

This fits within our given range for x.

Solution (Cont'd)

- Case 2:

$$\begin{aligned} 4x - 5 &< 0 \\ x &< \frac{5}{4} \end{aligned}$$

Now,

$$\begin{aligned} -(4x - 5) &> 3 \\ 4x - 5 &< -3 \\ 4x &< 2 \\ x &< \frac{1}{2} \end{aligned}$$

This fits within our given range for x. Therefore, the solution is $x \in \left(-\infty, \frac{1}{2}\right)$ and $(2, \infty)$.

Limits

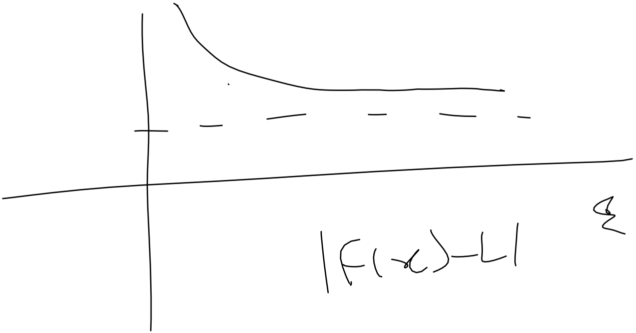
Definition (ϵ-M Definition of Limit at Infinity)

Suppose that L is a real number and f is a real-valued function defined on some interval (b, ∞). We say that $\lim_{x \to \infty} f(x) = L$ if for every positive real number ϵ, there is a real number M such that if $x > M$ then $|f(x) - L| < \epsilon$.

Question 2 *(MATH1131 Exam, June 2012)*

Use the ϵ-M definition of the limit to prove that:

$$\lim_{x \to \infty} \frac{x^2 - 2}{x^2 + 3} = 1.$$



$|f(x)-L|$ ε

$$\lim_{x \to \infty} \frac{x^2 - 2}{x^2+3} = 1 \qquad \varepsilon - M$$

$$|f(x) - L| = \left| \frac{x^2-2}{x^2+3} - 1 \right|$$

$$= \left| \frac{x^2-2-(x^2+3)}{x^2+3} \right|$$

$$= \left| \frac{-5}{x^2+3} \right| = \frac{5}{x^2+3}$$

$M(\varepsilon)$

$|f(x)-L| < \varepsilon$

for what x values is this is true?

$\frac{5}{x^2+3} < \varepsilon$

$\frac{5}{(\cancel{x^2+5})} < \frac{5}{x^2}$

$\frac{5}{x^2} < \varepsilon$, $\frac{5}{x^2+3} < \varepsilon$

$|f(x)-L| < \varepsilon$

$$\frac{5}{x^2} < \varepsilon$$

$$x > \sqrt{\frac{5}{\varepsilon}} \qquad \frac{5}{x^2} < \varepsilon \Rightarrow |f(x) - L| < \varepsilon$$

$$M = \sqrt{\frac{5}{\varepsilon}}$$

for $x > M$, $|f(x) - L| < \varepsilon$ for any ε

$$\therefore \lim_{x \to \infty} \frac{x^2 - 2}{x^2 + 3} = 1$$

Solution

Let $L = 1$, then

$$
\begin{aligned}
|f(x) - L| &= \left| \frac{x^2 - 2}{x^2 + 3} - 1 \right| \\
&= \left| \frac{-5}{x^2 + 3} \right| \\
&= \frac{5}{x^2 + 3} \\
&< \frac{5}{x^2}.
\end{aligned}
$$

Limits

Solution (Con't)

Let $\epsilon > 0$. Then $\frac{5}{x^2} < \epsilon$ if and only if $x > \sqrt{\frac{5}{\epsilon}}$

Let $M = \sqrt{\frac{5}{\epsilon}}$. Thus we have shown that for all $x > M$ then there exists an $\epsilon > 0$ such that $|f(x) - L| < \epsilon$, as required.

Question 3 *(MATH1131 Exam, June 2012)*

Evaluate the limit

$$\lim_{x \to 0} \frac{3x^2 + \sin(2x^2)}{x^2}.$$

$$\lim_{x\to 0}\left[\frac{3x^2+\sin(2x^2)}{x^2}\right]$$

$$\lim_{x\to 0}\left[\frac{3+\dfrac{\sin(2x^2)}{x^2}}{1}\right]$$

$$3+2\lim_{x\to 0}\left(\frac{\sin(2x^2)}{2x^2}\right) \qquad \lim_{x\to 0}\frac{\sin(x)}{x}=1$$

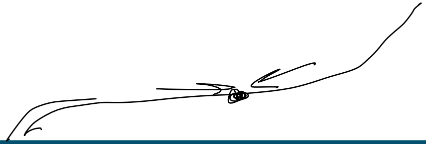
$$=3+1\cdot 2=5$$

Solution

Splitting the fraction,

$$\begin{aligned}
\lim_{x\to 0}\frac{3x^2+\sin\left(2x^2\right)}{x^2} &= \lim_{x\to 0}\left[\frac{3}{1}+\frac{\sin\left(2x^2\right)}{x^2}\right] \\
&= 3+\frac{2\sin\left(2x^2\right)}{2x^2} \\
&= 3+2\left(\frac{\sin\left(2x^2\right)}{2x^2}\right) \\
&= 3+2 \qquad\qquad \because \lim_{x\to 0}\frac{\sin x}{x}=1 \\
&= 5.
\end{aligned}$$

Properties of Continuous Functions



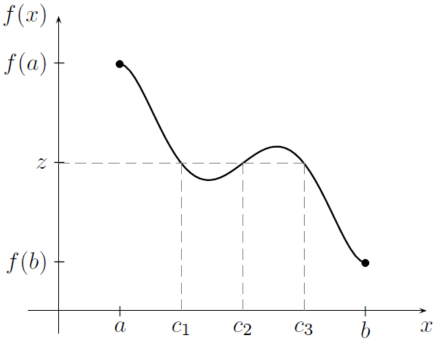
Definition (Continuity)

A function is continuous at a point $x = a$ if and only if $\lim\limits_{x \to a} f(x) = f(a)$.

Properties of Continuous Functions

Theorem (The Intermediate Value Theorem)

Suppose that f is continuous on the closed interval $[a, b]$. If z lies between $f(a)$ and $f(b)$ then there is at least one real number c in $[a, b]$ such that $f(c) = z$.



Properties of Continuous Functions

Question 4 *(MATH1141 Exam, Term 1 2019)*

Consider $f : A \to \mathbb{R}$ defined by

$$f(x) = \ln\left(\frac{2e^x - 1}{2 + \cos(\pi - x)}\right).$$

a) What is the maximal domain $A \subset \mathbb{R}$ of the function f?

b) By considering the **Intermediate Value Theorem**, explain carefully why $f(c) = 1$ for some $c \in A$.

$$f(x) = \ln\left(\frac{2e^{x}-1}{2+\cos(\pi - x)}\right)$$

what is A?

1. $2+\cos(\pi - x) \neq 0$

2. $\dfrac{2e^{x}-1}{\cancel{2+\cos(\pi-x)}} > 0$

$$-1 \leq \cos(\pi - x) \leq 1$$

$2+\cos(\pi - x)$ $\quad$ $1 \leq 2+\cos(\pi - x) \leq 3$

$$2e^{x} - 1 > 0$$

$$e^{x} > \frac{1}{2}$$

$$x > \ln \frac{1}{2}$$

$$x > -\ln 2$$

$$A = (-\ln 2, \infty)$$

$f(c) = 1 \quad c \in A$ using IVT

$\ln\left(\dfrac{2e^x - 1}{2 + \cos(\pi - x)}\right)$ is continuous (composition of cont. functions)

$x = 0 \quad f(0) = \ln\left(\dfrac{2e^0 - 1}{2 + \cos\pi}\right) = \ln\left(\frac{1}{1}\right) = 0$

$f(0) = 0 \qquad f(1) = \ln\left(\dfrac{2e - 1}{2 + \cos(\pi - 1)}\right) \approx 1.112$

$\therefore f(0) < 1,\ f(1) > 1$ by IVT there must exist c

Properties of Continuous Functions

s.t. $f(c) = 1$

Solution

a) Notice that

$$-1 \leq \cos(\pi - x) \leq 1.$$

Thus,

$$1 \leq 2 + \cos(\pi - x) \leq 3.$$

Therefore the denominator is always positive and non-zero.

Solution

a) Notice that

$$-1 \leq \cos(\pi - x) \leq 1.$$

Thus,

$$1 \leq 2 + \cos(\pi - x) \leq 3.$$

Therefore the denominator is always positive and non-zero.
The logarithm function needs its input to always be positive, thus:

$$\begin{aligned} 2e^x - 1 &> 0 \\ e^x &> \frac{1}{2} \\ x &> -\ln 2. \end{aligned}$$

Therefore $A = (-\ln 2, \infty)$.

Solution (Cont'd)
b) Notice that $f(0) = 0$ and that $f(1) \approx 1.112 > 1$. Because f is a composition of continuous functions, and the interval $(0, 1.112)$ lies within its natural domain, we can apply the **Intermediate Value Theorem**. Therefore there must exist a $c \in A$ where $f(c) = 1$.

Differentiable Functions

Question 6 *(MATH1131 Exam, June 2013)*

Consider the function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} ax^2 + 3x & \text{if } x \geq 1 \\ 2x + b & \text{if } x < 1 \end{cases}$$

Given that h is differentiable at $x = 1$, find the values of a and b.

$-\frac{1}{2}+3=\frac{5}{2}$

$$h(x) = \begin{cases} ax^2 + 3x & \text{for } x \geq 1 \\ 2x+b & \text{for } x < 1 \end{cases}$$

$x=1$

$2ax+3 = 2$

$2a+3=2$

$2a=-1$

$a=-\frac{1}{2}$

at $x=1$

f. is cont.

$\lim_{x \to 1^-} h(x) = 2+b = h(1)$

$2+b=\frac{5}{2}$, $b=\frac{1}{2}$

Solution

If h is differentiable at $x = 1$ then the derivative of $ax^2 + 3x$ must equal that of $2x + b$ at the point $x = 1$,

$$\begin{aligned} 2ax + 3 &= 2 \\ a &= -\frac{1}{2}. \end{aligned}$$

Differentiable Functions

Solution

If h is differentiable at $x = 1$ then the derivative of $ax^2 + 3x$ must equal that of $2x + b$ at the point $x = 1$,

$$\begin{aligned} 2ax + 3 &= 2 \\ a &= -\frac{1}{2}. \end{aligned}$$

If h is differentiable at $x = 1$, it is also continuous. Thus the left and right hand limits of h at $x = 1$ must be equal. Notice that:

$$\begin{aligned} \lim_{x \to 1^-} h(x) &= \lim_{x \to 1^-} 2x + b \\ &= 2 + b \\ \lim_{x \to 1^+} h(x) &= \lim_{x \to 1^+} ax^2 + 3x \\ &= a + 3 \end{aligned}$$

Solution (Con't)
Because the left and right hand limits are equal, we can say that

$$2 + b = a + 3.$$

Using $a = -\frac{1}{2}$, we deduce that $b = \frac{1}{2}$.

Mean Value Theorem

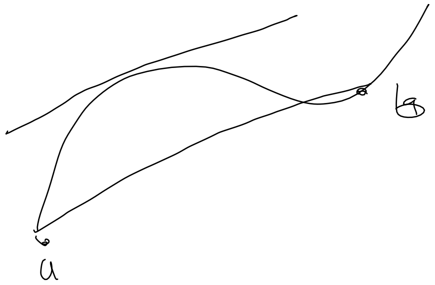
Theorem (Mean Value Theorem)

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). There exists at least one real number c in (a, b) such that:

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Question 7 *(MATH1141 Exam, June 2015)*

Assume that a differentiable function f on $\mathbb{R}$ is such that $f'(x) \leq 1$ for all $x \in \mathbb{R}$. Given that $f(2) = 2$, show that $f(x) \geq x$ for all $x \leq 2$.



a
b

$[x, 2]$

$\therefore$ cont, diff

$f(x) \geq x$

$$\frac{f(2) - f(x)}{2 - x} = f'(c) \quad \text{for sure}$$

$c \in (x, 2)$

$$\frac{f(2) - f(x)}{2 - x} \leq 1$$

$f(2) - f(x) \leq 2 - x$

$2 - f(x) \leq 2 - x$

$-f(x) \leq -x \quad \therefore f(x) \geq x$

$\forall x \leq 2$

Mean Value Theorem

Solution

Consider the interval $[x, 2]$ as we have specified $x \leq 2$. It is given that the function is continuous on this interval and differentiable on the closed interval $(x, 2)$, and f thus satisfies the conditions of the **Mean Value Theorem**. Consequently there exists $c \in (x, 2)$ such that

Mean Value Theorem

Solution

Consider the interval $[x, 2]$ as we have specified $x \leq 2$. It is given that the function is continuous on this interval and differentiable on the closed interval $(x, 2)$, and f thus satisfies the conditions of the **Mean Value Theorem**. Consequently there exists $c \in (x, 2)$ such that

$$
\begin{aligned}
\frac{f(2) - f(x)}{2 - x} &= f'(c) \\
\frac{f(2) - f(x)}{2 - x} &\leq 1, \qquad\qquad \because f'(x) \leq 1, \forall x \in \mathbb{R} \\
2 - f(x) &\leq 2 - x \\
-f(x) &\leq -x \\
\therefore f(x) &\geq x.
\end{aligned}
$$

Theorem (l'Hôpital's Rule)

Suppose that f and g are both differentiable functions and $a \in \mathbb{R}$. Suppose also that either 1 of the 2 following conditions hold:

- $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$;
- $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$;

$$\frac{f(x)}{g(x)} \to \frac{0}{0}, \frac{\infty}{\infty}$$

If

$$\lim_{x \to a} \frac{f'(x)}{g'(x)}$$

exists then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

$$\frac{f'(x)}{g'(x)}$$

Mean Value Theorem

Question 8 *(Adapted from MATH1141 Exam, June 2011)*

Evaluate the following limit using l'Hôpital's Rule

$$\lim_{h \to 0} \frac{e^{-1/h^2}}{h}. \quad = 0$$

Subsequently determine whether the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $x = 0$ where

$$f(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

$$\lim_{h \to 0} \frac{e^{-1/h^2}}{h} \to \frac{0}{0} \qquad e^{-\frac{1}{0}}$$

$$e^{-\infty} \to 0$$

$$= \lim_{h \to 0} \frac{\frac{2}{h^3} e^{-\frac{1}{h^2}}}{1}$$

$$= \lim_{h \to 0} \frac{2e^{-1/h^2}}{h^3} \to \frac{0}{0}$$

$$= \frac{ce^{-1/h^2}}{h^{-}}$$

$$\frac{e^{-1/h^2}}{h} = \frac{h^{-1}}{e^{1/h^2}}$$

$$= \lim_{h \to 0} \frac{-\frac{1}{h^2}}{-\frac{2}{h^3} e^{1/h^2}}$$

$$e^{\frac{1}{h^2}} \to e^{\infty} \to \infty$$

$$= \lim_{h \to 0} \frac{h}{e^{1/h^2}} \to \frac{0}{\infty} \to 0$$

$f(x)$, $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

Show that ~~$h''(0)$~~ exists
$f''(0)$

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h}$$

$$\lim_{h \to 0} \frac{e^{-1/h^2}}{h} = 0$$

$\therefore$ $f'(0)$ exists and equals 0

$\therefore$ $f(x)$ is differentiable at $x = 0$

Mean Value Theorem

Solution

$$
\begin{aligned}
\lim_{h\to 0} \frac{e^{-1/h^2}}{h} &= \lim_{h\to 0} \frac{h^{-1}}{e^{1/h^2}} \\
&= \lim_{h\to 0} \frac{-\frac{1}{h^2}}{-\frac{2}{h^3} e^{1/h^2}} && \text{by l'Hôpital's Rule} \\
&= \lim_{h\to 0} \frac{h}{2e^{1/h^2}} \\
&\to \frac{0}{\infty} \\
&\to 0
\end{aligned}
$$

Solution (Cont'd)
Check continuity:

$$\begin{aligned}
\lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} e^{-1/x^2} \\
&= 0 \\
&= f(0) \\
\lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} e^{-1/x^2} \\
&= 0 \\
&= f(0)
\end{aligned}$$

Solution (Cont'd)
Recall that $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$.
Check differentiability:

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{f(h) - 0}{h} \\ &= \lim_{h \to 0} \frac{e^{-1/h^2}}{h} \\ &= 0 \end{aligned}$$

Thus the function is differentiable at $x = 0$.

Inverse Functions

Theorem (Inverse Functions and One-to-oneness)

Suppose that f is a one-to-one function, then the inverse of f will be an unique function, f^{-1}, where the range(f^{-1}) = domain(f)

Theorem (Inverse Function Theorem)

Suppose that I is an open interval, $f : I \to \mathbb{R}$ is differentiable and $f'(x) \neq 0$ for all x in I. Then:

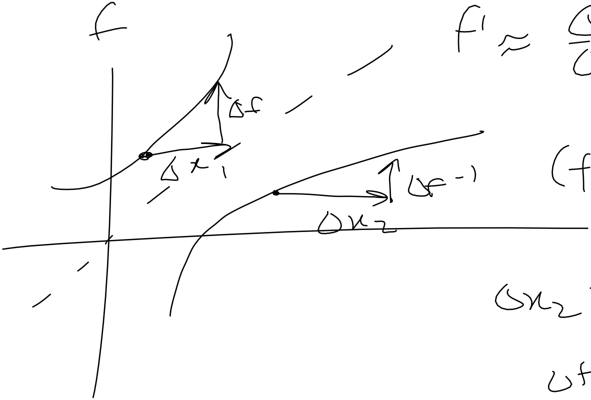
- f is one-to-one and has an inverse function, g: range(f) $\to$ domain(f)

  Invertible

- g is differentiable at all points in range(f) and
- the derivative of g is given by the formula

$$g'(x) = \frac{1}{f'(g(x))}$$

for all x in range (f)



$$f' \approx \frac{\Delta f}{\Delta x_1}$$

$$(f^{-1})' \sim \frac{\Delta f^{-1}}{\Delta x_2}$$

$$\Delta x_2 = \Delta f \quad \sim \frac{1}{f'}$$

$$\Delta f^{-1} = \Delta x_1$$

Inverse Functions

Alternative form of the Inverse Function Theorem
For a particular value of x (i.e. c), we have that:

$$(f^{-1})'(c) = \frac{1}{f'(f^{-1}(c))}$$

Question 9 *(MATH1141 Exam, June 2012)*

Consider the function $f : (0, 2\sqrt{\pi}] \to \mathbb{R}$ defined by

$$f(x) = x^2 + \cos(x^2)$$

a) Find all critical points of f and determine their nature.

b) Explain why f is invertible, state the domain of f^{-1} and find $f^{-1}(5\pi/2)$.

c) Where is f^{-1} differentiable?

critical point $f'(x)=0$ or point where $f(x)$ not diff...

$$f(x) = x^2 + \cos(x^2)$$

$$f'(x) = 2x + 2x(-\sin(x^2))$$

$$= 2x - 2x\sin(x^2)$$

$$= 2x(1-\sin(x^2))$$

$f'(x)=0$ $\qquad 2x(1-\sin(x^2))=0$

$$\sin(x^2)=1$$

$$x^2 = \frac{\pi}{2}, \frac{5\pi}{2}, \cancel{\frac{9\pi}{2}}, \ldots$$

$$0 < x \le 2\sqrt{\pi}$$

$$x^2 \le 4\pi$$

∴ we get critical points at $x = \sqrt{\frac{\pi}{2}}$, $x = \sqrt{\frac{5\pi}{2}}$

$x\sin(x^2))$

$$f'(x) = 2x - 2x\sin(x^2)$$

$$= 2 - 2\left(x \cdot 2x\overset{\cos}{\cancel{\sin}}(x^2) + \sin(x^2))\right)$$

$$= 2\left[1 - 2x^2\overset{\cos}{\cancel{\sin}}(x^2) - \sin(x^2)\right]$$

$$x^2 = \frac{\pi}{2}, \frac{5\pi}{2}$$

$$f''\left(\sqrt{\frac{\pi}{2}}\right) = 2(1 - 0 - 1) = 2(0) = 0$$

x	$\sqrt{\frac{\pi}{2}-\varepsilon}$	$\sqrt{\frac{\pi}{2}}$	$\sqrt{\frac{\pi}{2}}+\varepsilon$
$f''(x)$			

POI

$\sqrt{\frac{\pi}{2}}$ POI

$\therefore x = \sqrt{\frac{\pi}{2}}, \sqrt{\frac{5\pi}{2}}$

both are points of inflection.

$f(x) = x^2 + \cos(x^2)$

$f'(x) = 2x(1 - \sin(x^2))$ $\quad -1 \leq \sin(x^2) \leq 1$

$\geqslant 0$

$1 - \sin x^2 \geqslant 0$

$\therefore$ one-to-one, and thus invertible.

$f(x) = x^2 + \cos(x^2)$

$f(0) = 1$ $\quad f(2\sqrt{\pi}) = 4\pi + \cos(4\pi)$

$= 4\pi + 1$

$\therefore \mathrm{Ran}(f) = (1, 4\pi + 1]$

$\therefore \mathrm{Dom}(f^{-1}) = (1, 4\pi + 1]$

$$f(x) = \frac{5\pi}{2}$$

$$f^{-1}\left(\frac{5\pi}{2}\right)$$

$$x^2 + \cos(x^2) = \frac{5\pi}{2}$$

$$= \sqrt{\frac{5\pi}{2}}$$

$$u = x^2$$

$$u + \cos(u) = \frac{5\pi}{2}$$

$$u = \frac{5\pi}{2} \qquad x = \sqrt{\frac{5\pi}{2}}$$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

$(0, 2\sqrt{\pi}]$

$f'(f^{-1}(x)) = 0$

$f'(x) = 2\cancel{x}(1 - \sin(x^2))$

$\sin(u^2) = 1$

$\sin([f^{-1}(x)]^2) = 1$

$x \in (1, 4\pi + 1]$

$\sin$

$x^2 + \cos(x^2)$

$x \neq \cancel{\frac{\pi}{2}}, \frac{5\pi}{2}$

$f^{-1}(x) = \sqrt{\frac{\pi}{2}}, \sqrt{\frac{5\pi}{2}}$

$x = f(\sqrt{\frac{\pi}{2}}), f(\sqrt{\frac{5\pi}{2}})$

$x = \frac{\pi}{2}, \frac{5\pi}{2}$

Solution

a)

Notice $f'(x) = 2x - 2x\sin\left(x^2\right)$. Set $f'(x) = 0$ to find turning points. Thus,

$$\begin{aligned} 2x - 2x\sin\left(x^2\right) &= 0 \\ 2x\left(1 - \sin\left(x^2\right)\right) &= 0 \\ \sin\left(x^2\right) &= 1 \\ x^2 &= \frac{\pi}{2}, \frac{5\pi}{2} \\ x &= \frac{\sqrt{2\pi}}{2}, \frac{\sqrt{5\pi}}{2}. \end{aligned}$$

Nature: both are points of inflection. The other critical point is the endpoint at $x = 2\sqrt{\pi}$ which is the global maximum.

Solution Continued

b) From the previous part,

$$f'(x) = 2x - 2x\sin(x^2).$$

Notice that $\sin(x^2) \leq 1$, therefore $2x \geq 2x\sin(x^2)$, implying that

$$f'(x) = 2x - 2x\sin(x^2) \geq 0.$$

Solution Continued

b) From the previous part,

$$f'(x) = 2x - 2x\sin(x^2).$$

Notice that $\sin(x^2) \le 1$, therefore $2x \ge 2x\sin(x^2)$, implying that

$$f'(x) = 2x - 2x\sin(x^2) \ge 0.$$

Therefore, f is increasing in its domain and is one-to-one (injective). It is also continuous, as a composition of continuous functions. Therefore, f is invertible. $\text{Ran}(f) = (1, 4\pi + 1]$ and so $\text{Dom}(f^{-1}) = (1, 4\pi + 1]$.

Solution Continued

b)

Because $\cos\left(\frac{5\pi}{2}\right) = 0$, substituting $x = \sqrt{\frac{5\pi}{2}}$ into $f(x)$ demonstrates that $f(\sqrt{\frac{5\pi}{2}}) = \frac{5\pi}{2}$. Hence $f^{-1}(\frac{5\pi}{2}) = \sqrt{\frac{5\pi}{2}}$.

c)

Using the Inverse Function Theorem,

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

The derivative is not defined wherever $f'(f^{-1}(x)) = 0$. This is true whenever $f^{-1}(x) = \pi/2, 5\pi/2, ...$ or whenever $x = \sqrt{\pi/2}, \sqrt{5\pi/2}$ (all the solutions within the domain of f^{-1}).

Curve Sketching

Definition (Oblique Asymptotes)

Suppose that a and b are real numbers and that $a \neq 0$. We say that a straight line, given by the equation

$$y = ax + b,$$

is an oblique asymptote for a function f if

$$\lim_{n \to \infty} (f(x) - (ax + b)) = 0$$

$$\frac{\text{quadratic}}{\text{linear}} = \text{linear} + \underline{\quad} \to 0 \quad x \to \infty$$

Curve Sketching

Question 1

Sketch $\frac{3x^2-4}{x+2}$

Solution

1. Find intersections with the axes:
   When $x = 0$, $y = -2$.
   When $y = 0$, $x = \frac{\pm 2}{\sqrt{3}}$
2. Find vertical asymptote:
   This occurs when the denominator equals 0.
   Thus, the vertical asymptote exists at $x = -2$

$$\frac{3x^2-4}{x+2}$$

1. intercepts x/y int

2. asymptotes

3. functions +ve / -ve

4. turning points

y int $x=0$

y int:

$$\frac{3x^2-4}{x+2} = 0$$

$3x^2-4=0$

x int: $x = \frac{\pm 2}{\sqrt{3}}$

$\frac{-4}{2} = -2$ vertical asympt. $x=-2$

$$\frac{3x^2-4}{x+2}$$

$3x^2+6x-6x-12+8$

$= 3x(x+2)-6(x+2)+8$

$-ve \quad -2 \quad +ve \quad \overset{0}{-ve} \quad -ve \quad +\frac{2}{\sqrt{3}} \quad +ve$

$-\frac{2}{\sqrt{3}}$

Solution Continued

3.

Find oblique asymptote:

$\frac{3x^2-4}{x+2} = \frac{3x^2+6x-6x-12+8}{x+2}$

$\frac{3x^2-4}{x+2} \quad \begin{matrix} u \\ v \end{matrix}$

$\frac{vu' - uv'}{v^2}$

$= \frac{(x+2)(6x) - (3x^2-4)(1)}{(x+2)^2}$

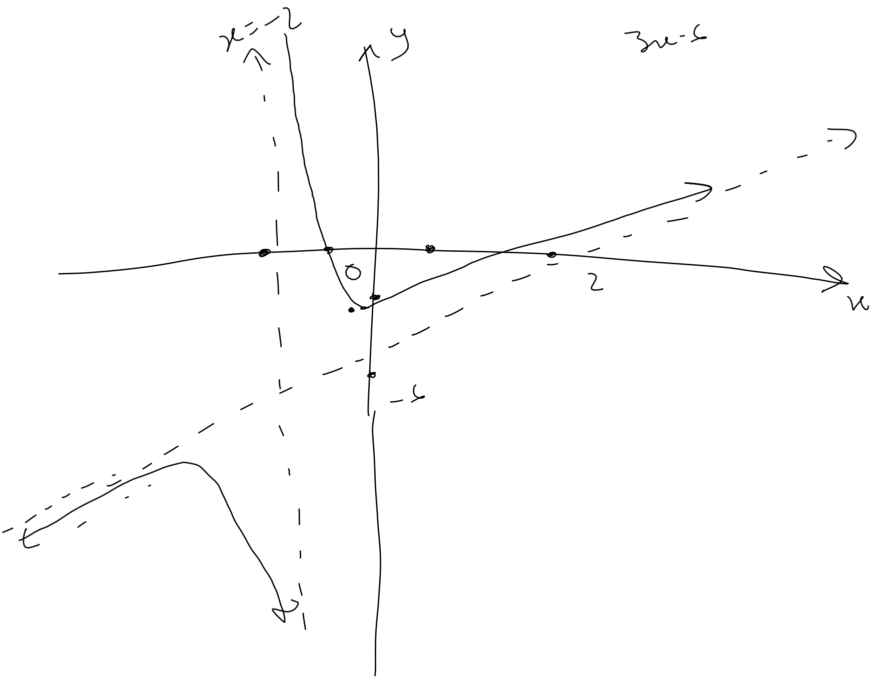
Then, we have $3x - 6 + \frac{8}{x+2}$,
Thus as we take $x \to \infty$, the oblique asymptote becomes $3x - 6$

4.

Then graph

$= \frac{3x^2+12x+4}{(x+2)^2}$

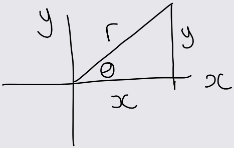
$x = -2 \pm \frac{2}{3}\sqrt{6} \approx -0.36, \ 3.63$


x = -2
y
3x-6
0
2
x
-6

Curve Sketching

Definition (Polar Coordinates)

Let P be every point in a plane
The pair of parameters (r, θ) defines the distance of the point P from the origin and the angle between OP and the positive horizontal axis respectively.



The Cartesian coordinates are defined as

$$x = rcos(\theta), y = rsin(\theta)$$

$\cos\theta = \frac{x}{r}$ $\sin\theta = \frac{y}{r}$

$\underline{x = r\cos\theta}$ $\underline{y = r\sin\theta}$

Question 2 *(MATH1131 JUNE 2011)*

a) Find the gradient, $\frac{dy}{dx}$, of $r = \underbrace{6\sin 2\theta}$ curve where $\theta = \frac{\pi}{6}$

Solution

$x = r\cos\theta$

$x = 6\sin 2\theta \cos\theta$

$\frac{dx}{d\theta} = 12\cos 2\theta \cos\theta + 6\sin 2\theta(-\sin\theta)$

$= \frac{3\sqrt{3}}{2}$ at $\theta = \frac{\pi}{6}$

$y = r\sin\theta$

$y = 6\sin(2\theta)\sin\theta$

$\frac{dy}{d\theta} = 12\cos(2\theta)\sin\theta + 6\sin(2\theta)\cos\theta$

$= \frac{15}{2}$

$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$

$= \frac{15}{2} \div \frac{3\sqrt{3}}{2} = \frac{15}{\cancel{2}} \times \frac{\cancel{2}}{3\sqrt{3}} = \frac{15}{3\sqrt{3}}$

$= \frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{3}$

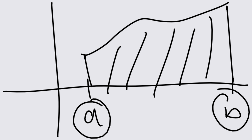
a) Solution provided in live stream

Integration

Theorem (The First Fundamental Theorem of Calculus)

If f is continuous function defined on $[a,b]$, then the function $F : [a,b] \to \mathbb{R}$, defined by

$$F(x) = \int_a^x f(x)\,dx$$



is continuous on $[a,b]$, differentiable on (a,b) and has derivative F' given by

$$F'(x) = f(x)$$

for all x in (a,b).

Integration

Theorem (The Second Fundamental Theorem of Calculus)

Suppose that f is a continuous function on $[a, b]$. If F is an antiderivative of f on $[a, b]$ then,

$$\int_a^b f(t)\, dt = F(b) - F(a).$$

Question 1 *(MATH1131 Exam, Semester 1 2011)*

Find:

$$\frac{d}{dx} \int_0^{sinh(x)} \frac{cos(t)}{\sqrt{1 + t^2}}\, dt.$$

Answer

$$= \left(\text{sub in upper limit}\right) \times \text{upper}' - \left(\text{sub in lower limit}\right) \times \text{lower}'$$

$$= \frac{\cos(\sinh(x))}{\sqrt{1+\sinh^2(x)}} \times \cosh(x) - \cancel{\frac{\cos(0)}{\sqrt{1+0^2}} \times 0}$$

$$= \frac{\cos(\sinh(x))}{\cancel{\cosh(x)}} \times \cancel{\cosh(x)}$$

$$\cosh^2(x) - \sinh^2(x) = 1$$

$$1 + \sinh^2(x) = \cosh^2(x)$$

Integration

Solution

Questions of this form follow the format:

$$\frac{d}{dx}\int_{l(x)}^{u(x)} f(t) = f(u(x))u'(x) - f(l(x))l'(x)\,dt.$$

In other words:

$$\frac{d}{dx}\int_{l(x)}^{u(x)} f(t) = (\text{Sub in upper limit})\cdot\text{upper'}-(\text{Sub in lower limit})\cdot\text{lower'}$$

Hence we obtain:

$$\frac{cos(sinh(x))}{\sqrt{1+sinh^2(x)}}cosh(x) - \frac{cos(0)}{\sqrt{1+0^2}}(0) = cos(sinh(x))$$

Since it is true that $\sqrt{1+sinh^2(x)} = cosh(x)$.

Integration

Integration by Parts

$$\int \underline{udv} = uv - \int vdu.$$

Question 2 (MATH1131 Exam, Term 2 2019)

Use integration by parts to evaluate $\int_0^{\sqrt{3}} 1 \tan^{-1}(x)\, dx$

Logs

Inverse trig ✓

Algebra ✓

Trig

Exponential

$u = \tan^{-1}(x)$

$\frac{du}{dx} = \frac{1}{1+x^2}$

$du = \frac{1}{1+x^2}\,dx$

$dv = 1\,dx$

$v = x$

$$I = \left[x\tan^{-1}(x)\right]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{x}{1+x^2}\,dx$$

$$= \frac{\sqrt{3}\pi}{3} - \frac{1}{2}\int_0^{\sqrt{3}} \frac{2x}{1+x^2}$$

$$= \frac{\sqrt{3}\pi}{3} - \frac{1}{2}\left[\ln(1+x^2)\right]_0^{\sqrt{3}}$$

$$= \frac{\sqrt{3}\pi}{3} - \frac{1}{2}\ln(4)$$

✓✓

Solution

To use the L.I.A.T.E rule, it is best to view this integral as:

$$I = \int_0^{\sqrt{3}} 1 \tan^{-1}(x)\, dx,$$

in which we can then set:

$$u = \tan^{-1}(x) \implies du = \frac{1}{1+x^2}dx$$

$$dv = 1dx \implies v = x$$

So, $I = [x \tan^{-1}(x)]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{x}{1+x^2}\, dx = \pi\frac{\sqrt{3}}{3} - \frac{1}{2}[ln(1+x^2)]_0^{\sqrt{3}}$.

Substituting in the upper and lower bounds gives:

$$I = \pi\frac{\sqrt{3}}{3} - \frac{1}{2}ln(4)$$

The comparison test

Suppose that f and g are integrable functions and that $0 \leq f(x) \leq g(x)$ whenever $x > a$.

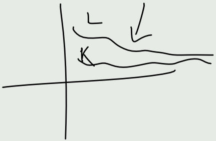
(i) If $\int_a^\infty g(x)\,dx$ converges then $\int_a^\infty f(x)\,dx$ converges.

(ii) If $\int_a^\infty f(x)\,dx$ diverges then $\int_a^\infty g(x)\,dx$ diverges.

Question 3

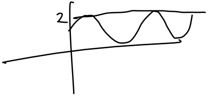
(MATH1131 Exam, Semester 2 2010)

Determine, whether the improper integral



$$K = \int_1^\infty \frac{1+\sin(x)}{3x^2}\,dx$$

converges or diverges.



$$L = \int_1^{\infty} \frac{2}{3x^2}$$

$$= \lim_{R \to \infty} \int_1^R \frac{2}{3x^2}\,dx$$

$$= \lim_{R \to \infty} \int_1^R \frac{2x^{-2}}{3}\,dx$$

$$= \lim_{R \to \infty} \left[\frac{2}{-3x}\right]_1^R$$

$$= \lim_{R \to \infty} \left[\frac{-2}{3R} + \frac{2}{3}\right]$$

$$= \frac{2}{3}$$ $\therefore$ L conerges

Solution

We know that $1 + \sin(x) \leq 2$ (verify by sketching the graph), hence we can consider:

$$\int_1^\infty \frac{2}{3x^2}\,dx = \frac{2}{3}\int_1^\infty \frac{1}{x^2}\,dx$$

and see whether this is a converging or diverging integral.

The improper integral $\int_a^\infty \frac{1}{x^p}\,dx$ is called a p integral and converges if $p > 1$ and diverges if $p \leq 1$. In our example, $p = 2$, hence this particular integral converges.

Thus, K converges since it is less than or equal to an already converging integral.

Log and Exponentials

Question 1 *(MATH1131 Exam, June 2012)*

Suppose that $y = x^{\sin(x)}$. Find $\frac{dy}{dx}$

Solution

By logging both sides, we get $ln(y) = \sin(x)ln(x)$.

Then by implicit differentiation, we get $\frac{1}{y}\frac{dy}{dx} = \cos(x)ln(x) + \frac{\sin(x)}{x}$.

Thus, $\frac{dy}{dx} = y(\cos(x)ln(x) + \frac{\sin(x)}{x})$.

And as $y = x^{\sin(x)}$,

$\frac{dy}{dx} = x^{\sin(x)}(\cos(x)ln(x) + \frac{\sin(x)}{x})$

$y = x^{\sin(x)}$ find $\frac{dy}{dx}$

$\ln(y) = \ln\left(x^{\sin x}\right)$

$\frac{d}{dx}\ln(y) = \frac{d}{dx}\sin(x)\ln(x)$

$\frac{d}{dy}\ln(y)\frac{dy}{dx} = \cos(x)\ln(x) + \frac{\sin x}{x}$

$\frac{1}{y}\frac{dy}{dx} = \cos(x)\ln(x) + \frac{\sin(x)}{x}$

$\frac{dy}{dx} = x^{\sin(x)}\left(\cos(x)\ln(x) + \frac{\sin(x)}{x}\right)$

Hyperbolic Functions

Definition (Hyperbolic Cosine)

The hyperbolic cosine function $cosh : \mathbb{R} \to \mathbb{R}$ is defined by the formula

$$coshx = \frac{1}{2}(e^x + e^{-x}) \quad \forall x \in \mathbb{R}$$

Definition (Hyperbolic Sine)

The hyperbolic sine function $sinh : \mathbb{R} \to \mathbb{R}$ is defined by the formula

$$sinhx = \frac{1}{2}(e^x - e^{-x}) \quad \forall x \in \mathbb{R}$$

Hyperbolic Functions

$$\cosh(x)+\sinh(x) = \frac{1}{2}(e^{x}+e^{-x}) + \frac{1}{2}(e^{x}-e^{-x})$$
$$= \frac{1}{2}e^{x} + \cancel{\frac{1}{2}e^{-x}} + \frac{1}{2}e^{x} - \cancel{\frac{1}{2}e^{-x}} = e^{x}$$

Question 1 *(MATH1131 Exam, Term 2 2019 - Adapted)*

a) Evaluate

$$\int_0^{ln(3)} \frac{1}{\cosh(x)+\sinh(x)}\,dx$$

$$I = \int_0^{\ln(3)} e^{-x}\,dx$$
$$= \left[-e^{-x}\right]_0^{\ln(3)} = -\frac{1}{3} - -1 = \frac{2}{3}$$

Solution

First note that from the definition,

$$\cosh(x) + \sinh(x) = e^x$$

hence $\int_0^{ln(3)} \frac{1}{\cosh(x)+\sinh(x)}\, dx = \int_0^{ln(3)} \frac{1}{e^x} = \int_0^{ln(3)} e^{-x}\, dx = \frac{2}{3}$