UNSW Mathematics Society Presents

MATH1081 Workshop

Presented by Fred Yan, Ronald Chiang, and Jay Liang

Overview I

1. Sets and Functions
2. Integers, Modular Arithmetic and Relations
3. Enumeration and Probability
4. Logic and Proofs
5. Graphs

1. Sets and Functions

2018 T1 Q1 (i)

(a) Suppose that A and B are sets with $|A| = 2$ and $|B| = 3$.
Find $|P(A) \times P(B)|$ and $|P(A \times B)|$.

(b) Is it true that

$$|P(X) \times P(Y)| \leq |P(X \times Y)|$$

for all non-empty finite sets X, Y? Explain your answer.

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The cardinality of a power set $|P(A)|$ is given by $2^{|A|}$.

Size of cartesian product

The cardinality of a cartesian product $|A \times B|$ is given by $|A| \times |B|$.

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Suppose that A and B are sets with $|A| = 2$ and $|B| = 3$.
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$$\begin{aligned} |P(A) \times P(B)| &= |P(A)| \times |P(B)| \\ &= 2^{|A|} \times 2^{|B|} \\ &= 2^2 \times 2^3 \\ &= 32. \end{aligned}$$

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$$\begin{aligned} |P(A \times B)| &= 2^{|A \times B|} \\ &= 2^{|A| \times |B|} \\ &= 2^{2 \times 3} \\ &= 64. \end{aligned}$$

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Investigation:

$$\begin{aligned} |P(X) \times P(Y)| \leq |P(X \times Y)| &\iff |P(X)| \times |P(Y)| \leq 2^{|X \times Y|} \\ &\iff 2^{|X|} \times 2^{|Y|} \leq 2^{|X| \times |Y|} \\ &\iff 2^{|X|+|Y|} \leq 2^{|X| \times |Y|} \\ &\iff |X| + |Y| \leq |X| \times |Y|. \end{aligned}$$

What happens when $|X| = |Y| = 1$?

b) Proof

Is it true that

$$|P(X) \times P(Y)| \leq |P(X \times Y)|$$

for all non-empty finite sets X, Y? Explain your answer.

The statement is false and we'll provide a counterexample:
Let $|X| = |Y| = 1$. Then we have

$$\begin{aligned} |P(X) \times P(Y)| &= |P(X)| \times |P(Y)| \\ &= 2^{|X|} \times 2^{|Y|} \\ &= 2^{|X|+|Y|} \\ &= 2^{1+1} \\ &= 4. \end{aligned}$$

b) Proof

Is it true that

$$|P(X) \times P(Y)| \leq |P(X \times Y)|$$

for all non-empty finite sets X, Y? Explain your answer.

$$\begin{aligned} |P(X \times Y)| &= 2^{|X \times Y|} \\ &= 2^{|X| \times |Y|} \\ &= 2^{|X| \times |Y|} \\ &= |X| \times |Y| \\ &= 1 \times 1 \\ &= 1. \end{aligned}$$

Now since $4 > 1$, then we have $|P(X) \times P(Y)| > |P(X \times Y)|$ and the statement is false. $\square$

2017 T1 Q1 (ii)*

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$$\begin{aligned}
(A \cap (A \cap B)^c) \cup B^c &\equiv (A \cap (A^c \cup B^c)) \cup B^c \text{ (De Morgan's Law)} \\
&\equiv (A \cap A^c) \cup (A \cap B^c) \cup B^c \text{ (Distributive Law)} \\
&\equiv \varnothing \cup (A \cap B^c) \cup B^c \text{ (Intersection w/ complement)} \\
&\equiv (A \cap B^c) \cup B^c \text{ (Identity Law)} \\
&\equiv B^c \text{ (Absorption Law)}
\end{aligned}$$

2016 S2 Q1 (v)

a) If X and Y are sets with $|X| = 2$ and $|Y| = n$, how many injective functions are there from X to Y? Explain your answer.
b) If $|X| = n$ and $|Y| = 2$, how many surjective functions are there from X to Y? Explain your answer.

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Injective Function Definition

A function $f : A \to B$ is injective if and only if for any x_1 and x_2 in A, $f(x_1) = f(x_2)$ implies that $x_1 = x_2$.
In other words, if the outputs are equal then the inputs are equal.

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Surjective Function Definition

A function $f : A \to B$ is surjective if and only if for any y in B, there exists x in A such that $f(x) = y$.

In other words, every element in the codomain is a function value, i.e. f of something in A.

a) Solution*

If X and Y are sets with $|X| = 2$ and $|Y| = n$, how many injective functions are there from X to Y? Explain your answer.

b) Solution*

If $|X| = n$ and $|Y| = 2$, how many surjective functions are there from X to Y? Explain your answer.

Inverse Functions*

Theorem

A function $g : Y \to X$ is an inverse of $f : X \to Y$ if and only if

$$g \circ f = i_x \text{ and } f \circ g = i_y.$$

2017 T1 Q1 (iv)

Let $f : [1, \infty) \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \ln x + 1 \quad \text{for all } x \in [1, \infty),\ g(y) = e^{y-1} \quad \text{for all } y \in \mathbb{R}.$$

a) Write down the domain and codomain of $g \circ f$.
b) Prove that $g \circ f$ is the identity map on $[1, \infty)$.
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a) Since the codomain of f is a subset of the domain of g, $g \circ f$ is defined. $\text{Dom}(g \circ f) = [1, \infty)$. $\text{Codom}(g \circ f) = \mathbb{R}$.

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b) We know that $g \circ f : [1, \infty) \to \mathbb{R}$ is defined. So

$$(g \circ f)(x) = g(f(x))$$

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Since $(g \circ f)(x) = x$ for $x \in [1, \infty)$, $g \circ f$ is the identity map on $[1, \infty)$.

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c)We have proved in b) that $g \circ f = i_x$ for all $x \in [1, \infty)$. For f to be the inverse of g, we also need $f \circ g = i_y$ for all $y \in \mathbb{R}$.

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Notice that the range of g is $(0, \infty)$ which is not a subset of the domain of f.

Hence, $f \circ g$ is not defined and f is not the inverse of g.

2018 T2 Q1 (ii)*

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x, y) = x^2 + y^2$.
a) Suppose that S is a unit square $[0, 1] \times [0, 1]$. Find $f(S)$.
b) Find and sketch $f^{-1}([1, 2])$.

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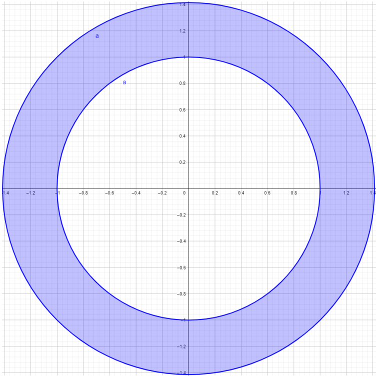
a) $f(S) = [0, 2]$
b) $f(S) = \{(x, y) | 1 \leq x^2 + y^2 \leq 2\}$



2020 T2*

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Let $b : \mathbb{R} \to \mathbb{R}$ be define by $b(x) = 14x^{13} + 11x^3 + 19x - 16$.
Show that b is bijective.

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Show that b is bijective.

To prove b is bijective, we have to prove it's both injective and surjective.

Notice $b'(x) = 182x^{12} + 33x^2 + 19 > 0$. Since $b'(x) > 0$ for all $x \in \mathbb{R}$, b is a strictly increasing function. Hence b is injective.

Notice that $b(x) \to \infty$ as $x \to \infty$ and $b(x) \to -\infty$ as $x \to -\infty$. Hence, for all $y \in R$, there exists $M > 0$ such that $x > M \implies b(x) > y$ and $x < -M \implies b(x) < y$. Thus, by the intermediate value theorem, there exists $x_0 \in [-M, M]$ such that $b(x_0) = y$ for all $y \in \mathbb{R}$.
Hence $\text{range}(b) = \mathbb{R} = \text{codom}(b)$. Thus, b is surjective.
Hence b is both injective and surjective and thus, bijective. $\square$

2. Integers, Modular Arithmetic and Relations

2015 T1 Q2 (i)

Evaluate

$$2015^{1082} \pmod{11}.$$

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Calculator: [2015÷11] = [-183] = [×11] =

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Now let's try a power of 2 where

$$2015^2 \equiv 2^2 \equiv 4 \pmod{11},$$

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Now let's try a power of 2 where

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since, if $a \equiv b \pmod{m}$, then $a^n \equiv b^n \pmod{m}$.

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$$\begin{array}{r|r}
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2015^2 \equiv 4 \pmod{11} & 2015^7 \equiv 7 \pmod{11} \\
2015^3 \equiv 8 \pmod{11} & 2015^8 \equiv 8 \pmod{11} \\
2015^4 \equiv 5 \pmod{11} & 2015^9 \equiv 6 \pmod{11} \\
2015^5 \equiv 10 \pmod{11} & 2015^{10} \equiv {\color{red}1} \pmod{11}
\end{array}$$

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By observing more powers of 2015,

$2015^1 \equiv 2 \pmod{11}$	$2015^6 \equiv 9 \pmod{11}$
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we can see that there is a cycle every 10 powers.

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we can see that there is a cycle every 10 powers.
Consider

$$1082 \equiv 2 \pmod{10}.$$

Hence,

$$2015^{1082} \equiv 2015^2 \equiv 4 \pmod{11}.$$

2015 T1 Q2 (ii)

a) Find gcd(105, 342).
b) Solve, or explain why there is no solution to:
  α) $105x \equiv 9 \pmod{342}$
  β) $105x \equiv 8 \pmod{342}$.

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$$\begin{aligned} 342 &= 105 \times 3 + 27, \\ 105 &= 27 \times 3 + 24, \\ 27 &= 24 \times 1 + 3, \\ 24 &= 3 \times 3. \end{aligned}$$

Thus

$$\gcd(105, 342) = \gcd(27, 105) = \gcd(24, 27) = \gcd(3, 24) = 3.$$

2015 T2 Q2 (ii)

b) Solve, or explain why there is no solution to:

α) $105x \equiv 9 \pmod{342}$.

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2016 T2 Q2 (i)

Bézout's Identity

Consider

$$ax + by = c,$$

where a, b, and c are integers, with a and b not both zero.

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To check solubility, we refer to Bézout's Identity.

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Bézout's Identity

Consider

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where a, b, and c are integers, with a and b not both zero.
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To check solubility, we refer to Bézout's Identity.

Note:

$$ax + by = c \Leftrightarrow ax \equiv c \pmod{\text{b}}.$$

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We know

$$105x \equiv 9 \pmod{342} \Leftrightarrow 105x + 342y = 9$$

for some integer y.

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We know

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for some integer y.

Since $\gcd(105, 342) = 3$ divides 9, there are 3 integer solutions for x. (From Bézout's Identity)

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Dividing both side by the greatest common divisor,

$$105x + 342y = 9 \Rightarrow 35x + 114y = 3.$$

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Dividing both side by the greatest common divisor,

$$105x + 342y = 9 \Rightarrow 35x + 114y = 3.$$

Through the Euclidean Algorithm:

$$\begin{aligned} 114 &= 3(35) + 9, \\ 35 &= 3(9) + 8, \\ 9 &= 1(8) + 1, \\ 8 &= 8(1). \end{aligned}$$

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$$\begin{aligned} 1 &= 9 - 8, \\ &= 9 - (35 - 3(9)), \\ &= (114 - 3(35)) - (35 - 3(114 - 3(35))), \end{aligned}$$

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$$\begin{aligned} 1 &= 9 - 8, \\ &= 9 - (35 - 3(9)), \\ &= (114 - 3(35)) - (35 - 3(114 - 3(35))), \\ &= 35(-13) + 114(4). \end{aligned}$$

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Multiplying both sides by 3,

$$3 = 35(-39) + 114(12) = 35x + 114y.$$

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However, solutions must be $0 \leq x < 342$. (342 from original mod)

We can find more solutions by adding 114 to our x.

Thus,

$$x = -39 + 114, -39 + 2(114), -39 + 3(114) = 75, 189, 303.$$

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We know

$$105x \equiv 8 \pmod{342} \Leftrightarrow 105x + 342y = 8,$$

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for some integer y, but $\gcd(105, 342) = 3$ does not divide 8.

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for some integer y, but $\gcd(105, 342) = 3$ does not divide 8.

Hence, by Bézout's Identity, there are no solutions for

$$105x \equiv 8 \pmod{342}.$$

2016 T2 Q1 (vii)

Suppose $\gcd(a, m) = d$.
Show that if x is a solution of $ax \equiv b \pmod{m}$, then so is $x + m/d$.

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2012 T2 Q2 (ii)

Given that $1039^{-1} \equiv 31 \pmod{2013}$:

a) Find integers k and l such that $1039k + 2013l = 1$.

b) What is the general solution to $1039x + 2013y = 4$?

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We know that

$$1039 \times 31 \equiv 1 \pmod{2013} \Leftrightarrow 1039(31) + 2013y = 1$$

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Solving for y we see that $y = \dfrac{1 - 1039(31)}{2013} = -16$.
Therefore

$$1039(31) + 2013(-16) = 1 = 1031k + 2013l.$$

Comparing coefficients, we see that $k = 31$ and $l = -16$.

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Given that $1039^{-1} \equiv 31 \pmod{2013}$:

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From a), we can multiply both sides by 4 to find that

$$1039(31) + 2013(-16) = 1 \Rightarrow 1039(124) + 2013(-64) = 4.$$

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Thus, possible solutions are $x = 124$ and $y = -64$.

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Hence, the general solution is $x = 124 + 2013z$ for some integer z.

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Hence, from equating coefficients, the general solution is $x = 124 + 2013z$ and $y = -64 - 1039z$ for some integer z.

2017 T1 Q2 (iii)

Let $\sim$ be the relation on the set $\mathbb{R} \times \mathbb{R}$ defined by

$$(a_1, a_2) \sim (b_1, b_2) \text{ if and only if } a_1^2 + a_2^2 = b_1^2 + b_2^2.$$

a) Prove that $\sim$ is an equivalence relation.
b) Give a geometric description of the equivalence class with (1,0).
c) Is it true that all equivalence classes are infinite in size? Explain.

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2017 T1 Q2 (iii)

Properties and Types of Relations

Let R be a relation defined on the set A

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(i) Reflexive: All elements of the relation relate to themselves.

ie. $x\mathrm{R}x$ for all $x \in A$.

2017 T1 Q2 (iii)

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ie. If $x\mathrm{R}y$ and $y\mathrm{R}x$, then $x = y$ for all $x, y \in A$.

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(iv) Transitive: If $x\mathrm{R}y$ and $y\mathrm{R}z$, then $x\mathrm{R}z$ for all $x, y, z \in A$.

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(iv) Transitive: If $x\mathrm{R}y$ and $y\mathrm{R}z$, then $x\mathrm{R}z$ for all $x, y, z \in A$.

Equivalence Relation: Reflexive, Symmetrical, Transitive.

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ie. If $x\text{R}y$, then $y\text{R}x$ for all $x, y \in A$.
(iii) Anti-Symm: Elements only relate both ways with themselves.
ie. If $x\text{R}y$ and $y\text{R}x$, then $x = y$ for all $x, y \in A$.
(iv) Transitive: If $x\text{R}y$ and $y\text{R}z$, then $x\text{R}z$ for all $x, y, z \in A$.

Equivalence Relation: Reflexive, Symmetrical, Transitive.
Partial Order: Reflexive, Anti-Symmetrical, Transitive.

2017 T1 Q2 (iii)

Let $\sim$ be the relation on the set $\mathbb{R} \times \mathbb{R}$ defined by

$$(a_1, a_2) \sim (b_1, b_2) \text{ if and only if } a_1^2 + a_2^2 = b_1^2 + b_2^2.$$

a) Prove that $\sim$ is an equivalence relation.

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Therefore

$$(a_1, a_2) \sim (a_1, a_2).$$

where (a_1, a_2) is a part of the set $\mathbb{R} \times \mathbb{R}$.

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To prove reflexivity: $x\mathrm{R}x$ for all $x \in A$.

Let a_1 and a_2 be real numbers. Consider

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Therefore

$$(a_1, a_2) \sim (a_1, a_2).$$

where (a_1, a_2) is a part of the set $\mathbb{R} \times \mathbb{R}$.

Hence, the relation $\sim$ is reflexive.

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Let a_1, a_2, b_1 and b_2 be real numbers.

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To prove symmetry: If $x\mathrm{R}y$, then $y\mathrm{R}x$ for all $x, y \in A$.

Let a_1, a_2, b_1 and b_2 be real numbers. Suppose that $(a_1, a_2) \sim (b_1, b_2)$.

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Let a_1, a_2, b_1 and b_2 be real numbers. Suppose that $(a_1, a_2) \sim (b_1, b_2)$. By definition

$$a_1^2 + a_2^2 = b_1^2 + b_2^2$$

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$$a_1^2 + a_2^2 = b_1^2 + b_2^2 \Rightarrow b_1^2 + b_2^2 = a_1^2 + a_2^2.$$

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Let a_1, a_2, b_1 and b_2 be real numbers. Suppose that $(a_1, a_2) \sim (b_1, b_2)$. By definition

$$a_1^2 + a_2^2 = b_1^2 + b_2^2 \Rightarrow b_1^2 + b_2^2 = a_1^2 + a_2^2.$$

Therefore $(b_1, b_2) \sim (a_1, a_2)$, where (a_1, a_2) and (b_1, b_2) are in $\mathbb{R} \times \mathbb{R}$.

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a) Prove that $\sim$ is an equivalence relation.

To prove symmetry: If $x\mathrm{R}y$, then $y\mathrm{R}x$ for all $x, y \in A$.

Let a_1, a_2, b_1 and b_2 be real numbers. Suppose that $(a_1, a_2) \sim (b_1, b_2)$. By definition

$$a_1^2 + a_2^2 = b_1^2 + b_2^2 \Rightarrow b_1^2 + b_2^2 = a_1^2 + a_2^2.$$

Therefore $(b_1, b_2) \sim (a_1, a_2)$, where (a_1, a_2) and (b_1, b_2) are in $\mathbb{R} \times \mathbb{R}$.

Hence, the relation $\sim$ is symmetrical.

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a) Prove that $\sim$ is an equivalence relation.

To prove transitivity: If $x\mathrm{R}y$ and $y\mathrm{R}z$, then $x\mathrm{R}z$ for all $x, y, z \in A$.

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a) Prove that $\sim$ is an equivalence relation.

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Let a_1, a_2, b_1, b_2, c_1 and c_2 be real numbers.

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Suppose that $(a_1, a_2) \sim (b_1, b_2)$ and $(b_1, b_2) \sim (c_1, c_2)$.

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$$a_1^2 + a_2^2 = b_1^2 + b_2^2 \text{ and } b_1^2 + b_2^2 = c_1^2 + c_2^2.$$

Therefore $(a_1, a_2) \sim (c_1, c_2)$, since $a_1^2 + a_2^2 = b_1^2 + b_2^2 = c_1^2 + c_2^2$.

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Suppose that $(a_1, a_2) \sim (b_1, b_2)$ and $(b_1, b_2) \sim (c_1, c_2)$. By definition,

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Therefore $(a_1, a_2) \sim (c_1, c_2)$, since $a_1^2 + a_2^2 = b_1^2 + b_2^2 = c_1^2 + c_2^2$.

Hence, the relation $\sim$ is transitive.

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a) Prove that $\sim$ is an equivalence relation.

We know that the relation $\sim$ is reflexive, symmetrical and transitive.

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b) Give a geometric description of the equivalence class with (1,0).

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$$[(1,0)] = \{(a_1, a_2) \in \mathbb{R} \times \mathbb{R} \mid (a_1, a_2) \sim (1,0)\}.$$

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By definition, $a_1^2 + a_2^2 = 1^2 + 0^2 = 1$ for the real numbers a_1 and a_1.

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By definition, $a_1^2 + a_2^2 = 1^2 + 0^2 = 1$ for the real numbers a_1 and a_1.
Hint: Let $a_1 = x$ and $a_2 = y$. Thus, $x^2 + y^2 = 1$.

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By definition, $a_1^2 + a_2^2 = 1^2 + 0^2 = 1$ for the real numbers a_1 and a_1.
Hint: Let $a_1 = x$ and $a_2 = y$. Thus, $x^2 + y^2 = 1$.

The equivalence class with (1,0) is the set of ordered pairs containing all the coordinates on the unit circle.

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This is not true. Consider the equivalence class $[(0, 0)]$.

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c) Is it true that all equivalence classes are infinite in size? Explain.

This is not true. Consider the equivalence class $[(0, 0)]$.
By definition, $a_1^2 + a_2^2 = 0^2 + 0^2 = 0$ for some real numbers a_1 and a_2.

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This is not true. Consider the equivalence class $[(0, 0)]$.
By definition, $a_1^2 + a_2^2 = 0^2 + 0^2 = 0$ for some real numbers a_1 and a_2.
Thus, $a_1^2 = -a_2^2$ where $a_1^2 \geq 0$ and $a_2^2 \geq 0$ since squares are non-negative.

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If $a_1^2 > 0$, then $-a_2^2 > 0$.

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If $a_1^2 > 0$, then $-a_2^2 > 0$. Therefore, $a_2^2 < 0$.

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If $a_1^2 > 0$, then $-a_2^2 > 0$. Therefore, $a_2^2 < 0$. This contradicts $a_2^2 \geq 0$.

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Thus, $a_1^2 = -a_2^2$ where $a_1^2 \geq 0$ and $a_2^2 \geq 0$ since squares are non-negative.

If $a_1^2 > 0$, then $-a_2^2 > 0$. Therefore, $a_2^2 < 0$. This contradicts $a_2^2 \geq 0$.
We can also use this argument for the case that $a_2^2 > 0$.

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Let $\sim$ be the relation on the set $\mathbb{R} \times \mathbb{R}$ defined by

$$(a_1, a_2) \sim (b_1, b_2) \text{ if and only if } a_1^2 + a_2^2 = b_1^2 + b_2^2.$$

c) Is it true that all equivalence classes are infinite in size? Explain.

This is not true. Consider the equivalence class $[(0, 0)]$.
By definition, $a_1^2 + a_2^2 = 0^2 + 0^2 = 0$ for some real numbers a_1 and a_2.
Thus, $a_1^2 = -a_2^2$ where $a_1^2 \geq 0$ and $a_2^2 \geq 0$ since squares are non-negative.

If $a_1^2 > 0$, then $-a_2^2 > 0$. Therefore, $a_2^2 < 0$. This contradicts $a_2^2 \geq 0$.
We can also use this argument for the case that $a_2^2 > 0$.

Thus, $a_1^2 = a_2^2 = 0$.

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Thus, $a_1^2 = a_2^2 = 0$. Therefore, there is only 1 element for $[(0, 0)]$: $(0, 0)$.
Hence, there exists an equivalence class which has a size of 1.

2016 T2 Q2 (ii)

Let $\star$ be the relation on the set of integers $\mathbb{Z}$ defined by

$$a \star b \text{ if and only if } a^2 \equiv b^2 \pmod{4}.$$

Given that $\star$ is an equivalence relation, find its equivalence classes.

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As we are in the world of modulo 4, all integers can be represented by 0, 1, 2 and 3. We just need to check each of these.

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For $x \equiv 1 \pmod 4 \Rightarrow x^2 \equiv 1 \pmod 4$

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For $x \equiv 1 \pmod 4 \Rightarrow x^2 \equiv 1 \pmod 4$ Thus all integers where $a \equiv 1 \pmod 4$ are in $[1]$. These are all the numbers in the form 1+4n, where n is some integer ie. ..., -7, -3, 1, 5,...

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For $x \equiv 2 \pmod{4} \Rightarrow x^2 \equiv 4 \equiv 0 \pmod{4}$.

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Since $[0]$ contains all even integers and $[1]$ contains all odd integers, $[0] \cup [1] = \mathbb{Z}$.

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Since $[0]$ contains all even integers and $[1]$ contains all odd integers, $[0] \cup [1] = \mathbb{Z}$. Hence, $[0]$ and $[1]$ are all possible equivalence classes.

2013 T1 Q2 (i)

Suppose that $A = \{2, 3, 5, 6, 15, 30, 35, 70, 105, 210\}$.
Consider the relation $|$ on set A defined by $a|b$ if and only if a divides b.
a) Prove that $|$ is a partial order.
b) Draw a Hasse diagram for the partially ordered set $(A, |)$.
c) Find, if they exist, all

- α) minimal elements.
- β) least elements.
- γ) maximal elements.
- δ) greatest elements.

d) Find two elements of A that do not have a greatest lower bound. Explain why that is.

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The number x is known to divide x as $x \neq 0$, since $0 \notin A$.

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Hence, the relation $|$ is reflexive.

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By definition, x divides y, meaning $x \leq y$.

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By definition, x divides y, meaning $x \leq y$. Similarly, since y divides x, $y \leq x$.

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By definition, x divides y, meaning $x \leq y$. Similarly, since y divides x, $y \leq x$. Combining these inequalities gives us: $x \leq y \leq x$.

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Therefore by sandwich theorem, $x = y$ where $x, y \in A$.

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Hence, the relation $|$ is anti-symmetrical.

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By definition, $y = ax$ and $z = by$ for some integers a and b.
Substituting this value of y into the second equation gives us $z = abx$.

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Since ab is an integer, it is shown that $x|z$.

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Since ab is an integer, it is shown that $x|z$.

Hence, the relation $|$ is transitive.

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Suppose that $A = \{2, 3, 5, 6, 15, 30, 35, 70, 105, 210\}$.
Consider the relation $|$ on set A defined by $a|b$ if and only if a divides b.
a) Prove that $|$ is a partial order.

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Thus, the relation $|$ is reflexive, anti-symmetrical and transitive.

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Let's draw this up!

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Suppose that $A = \{2, 3, 5, 6, 15, 30, 35, 70, 105, 210\}$.
Consider the relation $|$ on set A defined by $a|b$ if and only if a divides b.
c) Find, if they exist, all

- α) minimal elements.
- β) least elements.
- γ) maximal elements.
- δ) greatest elements.

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Suppose that $A = \{2, 3, 5, 6, 15, 30, 35, 70, 105, 210\}$.
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Minimal Elements: There exists no element that relates to the minimal elements other than itself.

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In this case, the minimal elements are the primes in A: 2, 3, 5.

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From the diagram, we know that all composites have an element that relate to them.

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Minimal Elements: There exists no element that relates to the minimal elements other than itself.

In this case, the minimal elements are the primes in A: 2, 3, 5.

From the diagram, we know that all composites have an element that relate to them. Since primes only have the factors 1 and itself, there are no elements that divide 2, 3 and 5 in A other than itself.

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Suppose that $A = \{2, 3, 5, 6, 15, 30, 35, 70, 105, 210\}$.
Consider the relation $|$ on set A defined by $a|b$ if and only if a divides b.
c) α) Find, if they exist, all minimal elements.

Minimal Elements: There exists no element that relates to the minimal elements other than itself.

In this case, the minimal elements are the primes in A: 2, 3, 5.

From the diagram, we know that all composites have an element that relate to them. Since primes only have the factors 1 and itself, there are no elements that divide 2, 3 and 5 in A other than itself. This is because 1 is not in A.

2013 T1 Q2 (i)

Suppose that $A = \{2, 3, 5, 6, 15, 30, 35, 70, 105, 210\}$.
Consider the relation $|$ on set A defined by $a|b$ if and only if a divides b.
c) β) Find, if they exist, all least elements.

2013 T1 Q2 (i)

Suppose that $A = \{2, 3, 5, 6, 15, 30, 35, 70, 105, 210\}$.
Consider the relation $|$ on set A defined by $a|b$ if and only if a divides b.
c) β) Find, if they exist, all least elements.

Least Elements: The element must relate to every other element.

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Least Elements: The element must relate to every other element.

Let a be the least element.

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Consider the relation $|$ on set A defined by $a|b$ if and only if a divides b.
c) β) Find, if they exist, all least elements.

Least Elements: The element must relate to every other element.

Let a be the least element. Thus, $a|2$ and $a|3$, since 2 and 3 are elements of A.

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Least Elements: The element must relate to every other element.

Let a be the least element. Thus, $a|2$ and $a|3$, since 2 and 3 are elements of A. Since 2 and 3 are coprime, a must be 1, but 1 is not in A.

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Least Elements: The element must relate to every other element.

Let a be the least element. Thus, $a|2$ and $a|3$, since 2 and 3 are elements of A. Since 2 and 3 are coprime, a must be 1, but 1 is not in A. Therefore there exists no element of A that relates to 2 and 3.

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Suppose that $A = \{2, 3, 5, 6, 15, 30, 35, 70, 105, 210\}$.
Consider the relation $|$ on set A defined by $a|b$ if and only if a divides b.
c) β) Find, if they exist, all least elements.

Least Elements: The element must relate to every other element.

Let a be the least element. Thus, $a|2$ and $a|3$, since 2 and 3 are elements of A. Since 2 and 3 are coprime, a must be 1, but 1 is not in A. Therefore there exists no element of A that relates to 2 and 3.

Hence, there are no least elements.

2013 T1 Q2 (i)

Suppose that $A = \{2, 3, 5, 6, 15, 30, 35, 70, 105, 210\}$.
Consider the relation $|$ on set A defined by $a|b$ if and only if a divides b.
c) γ) Find, if they exist, all maximal elements.

2013 T1 Q2 (i)

Suppose that $A = \{2, 3, 5, 6, 15, 30, 35, 70, 105, 210\}$.
Consider the relation $|$ on set A defined by $a|b$ if and only if a divides b.
c) γ) Find, if they exist, all maximal elements.

Maximal Elements: There are no elements that the maximal elements relate to other than itself.

2013 T1 Q2 (i)

Suppose that $A = \{2, 3, 5, 6, 15, 30, 35, 70, 105, 210\}$.
Consider the relation $|$ on set A defined by $a|b$ if and only if a divides b.
c) γ) Find, if they exist, all maximal elements.

Maximal Elements: There are no elements that the maximal elements relate to other than itself.

From the diagram we see that 210 is the only element that does not relate to another.

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Suppose that $A = \{2, 3, 5, 6, 15, 30, 35, 70, 105, 210\}$.
Consider the relation $|$ on set A defined by $a|b$ if and only if a divides b.
c) γ) Find, if they exist, all maximal elements.

Maximal Elements: There are no elements that the maximal elements relate to other than itself.

From the diagram we see that 210 is the only element that does not relate to another.

Hence, the only maximal element is 210.

2013 T1 Q2 (i)

Suppose that $A = \{2, 3, 5, 6, 15, 30, 35, 70, 105, 210\}$.
Consider the relation $|$ on set A defined by $a|b$ if and only if a divides b.
c) δ) Find, if they exist, all greatest elements.

2013 T1 Q2 (i)

Suppose that $A = \{2, 3, 5, 6, 15, 30, 35, 70, 105, 210\}$.
Consider the relation $|$ on set A defined by $a|b$ if and only if a divides b.
c) δ) Find, if they exist, all greatest elements.

Greatest Elements: Every element relates to the greatest element.

2013 T1 Q2 (i)

Suppose that $A = \{2, 3, 5, 6, 15, 30, 35, 70, 105, 210\}$.
Consider the relation $|$ on set A defined by $a|b$ if and only if a divides b.
c) δ) Find, if they exist, all greatest elements.

Greatest Elements: Every element relates to the greatest element.

We know that every element in A divides 210. Thus every element relates to 210.

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Suppose that $A = \{2, 3, 5, 6, 15, 30, 35, 70, 105, 210\}$.
Consider the relation $|$ on set A defined by $a|b$ if and only if a divides b.
c) δ) Find, if they exist, all greatest elements.

Greatest Elements: Every element relates to the greatest element.

We know that every element in A divides 210. Thus every element relates to 210.

Hence, the greatest element is 210.

2013 T1 Q2 (i)

Suppose that $A = \{2, 3, 5, 6, 15, 30, 35, 70, 105, 210\}$.
Consider the relation $|$ on set A defined by $a|b$ if and only if a divides b.
d) Find two elements of A that do not have a greatest lower bound.
Explain why that is.

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Suppose that $A = \{2, 3, 5, 6, 15, 30, 35, 70, 105, 210\}$.
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Consider the relation $|$ on set A defined by $a|b$ if and only if a divides b.
d) Find two elements of A that do not have a greatest lower bound. Explain why that is.

Given two elements, the lower bounds are all the elements that relate to both. Thus, the greatest lower bound is the greatest element from these lower bounds.

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Given two elements, the lower bounds are all the elements that relate to both. Thus, the greatest lower bound is the greatest element from these lower bounds.

Consider the greatest lower bound of 2 and 3.

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Given two elements, the lower bounds are all the elements that relate to both. Thus, the greatest lower bound is the greatest element from these lower bounds.

Consider the greatest lower bound of 2 and 3. This would be an element that relates to both i.e. dividing both 2 and 3.

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Given two elements, the lower bounds are all the elements that relate to both. Thus, the greatest lower bound is the greatest element from these lower bounds.

Consider the greatest lower bound of 2 and 3. This would be an element that relates to both i.e. dividing both 2 and 3. Since 2 and 3 are coprime, the only number possible which could relate to both 2 and 3 is 1.

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Consider the greatest lower bound of 2 and 3. This would be an element that relates to both i.e. dividing both 2 and 3. Since 2 and 3 are coprime, the only number possible which could relate to both 2 and 3 is 1. We know that 1 is not in A and therefore there are no elements which relate to both 2 and 3.

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Suppose that $A = \{2, 3, 5, 6, 15, 30, 35, 70, 105, 210\}$.
Consider the relation $|$ on set A defined by $a|b$ if and only if a divides b.
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Given two elements, the lower bounds are all the elements that relate to both. Thus, the greatest lower bound is the greatest element from these lower bounds.

Consider the greatest lower bound of 2 and 3. This would be an element that relates to both i.e. dividing both 2 and 3. Since 2 and 3 are coprime, the only number possible which could relate to both 2 and 3 is 1. We know that 1 is not in A and therefore there are no elements which relate to both 2 and 3.

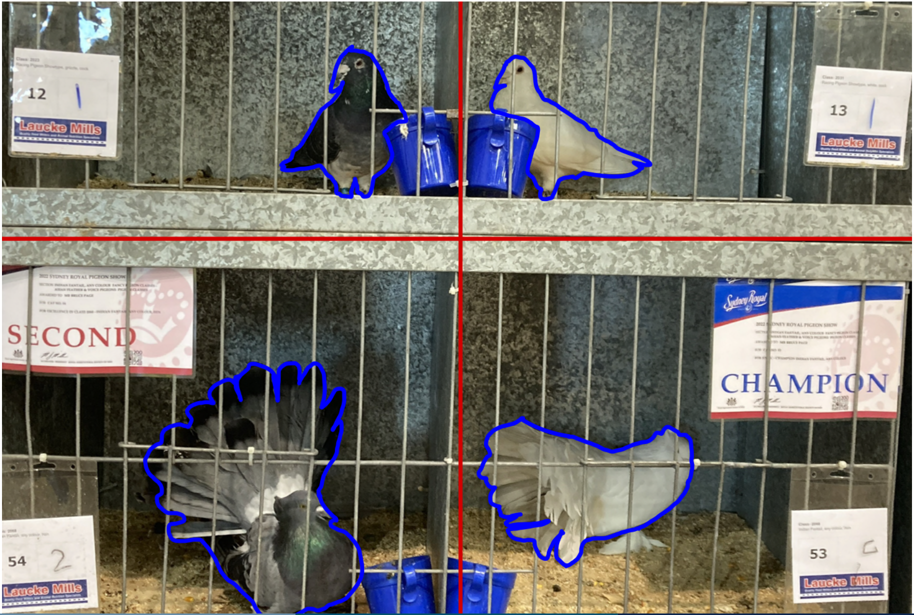
Since no lower bounds exist, 2 and 3 can't have a greatest lower bound.

3. Enumeration and Probability

2016 T2 Q4 v)

Let (x_i, y_i) for $i = 1, 2, 3, 4, 5$ be 5 distinct points in the plane with integer coordinates. Prove that at least one midpoint of two such points has integer coordinates.

2016 T2 Q4 v)


12
Laucke Mills
13
Laucke Mills
SECOND
Sydney Royal
CHAMPION
54
2
Laucke Mills
53
Laucke Mills

2016 T2 Q4 v)

Pigeonhole Principle

Given that there are $n + 1$ pigeons sorted across n pigeonholes then there is at least one pigeonhole with two or more pigeons.

2016 T2 Q4 v)

Pigeonhole Principle

Given that there are $n+1$ pigeons sorted across n pigeonholes then there is at least one pigeonhole with two or more pigeons.

Generalised Pigeonhole Principle:
With n pigeons and k pigeonholes, there is at least one pigeonhole with $\left\lceil \frac{n}{k} \right\rceil$ or more pigeons.

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Let (x_i, y_i) for $i = 1, 2, 3, 4, 5$ be 5 distinct points in the plane with integer coordinates. Prove that at least one midpoint of two such points has integer coordinates.

The midpoint of (a, b) and (c, d) is

$$\left(\frac{a+c}{2}, \frac{b+d}{2}\right).$$

Let (x_i, y_i) for $i = 1, 2, 3, 4, 5$ be 5 distinct points in the plane with integer coordinates. Prove that at least one midpoint of two such points has integer coordinates.

The midpoint of (a, b) and (c, d) is

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To have the average of two integers be an integer, they must both either be odd or both be even.

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To have the average of two integers be an integer, they must both either be odd or both be even.

Therefore, we need to have a and c with b and d be the same parity.

2016 T2 Q4 v)

Let (x_i, y_i) for $i = 1, 2, 3, 4, 5$ be 5 distinct points in the plane with integer coordinates. Prove that at least one midpoint of two such points has integer coordinates.

2016 T2 Q4 v)

Let (x_i, y_i) for $i = 1, 2, 3, 4, 5$ be 5 distinct points in the plane with integer coordinates. Prove that at least one midpoint of two such points has integer coordinates.

Coordinates are represented by two numbers, x and y.

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Let (x_i, y_i) for $i = 1, 2, 3, 4, 5$ be 5 distinct points in the plane with integer coordinates. Prove that at least one midpoint of two such points has integer coordinates.

Coordinates are represented by two numbers, x and y.

All possible combinations of parity for x and y are given by

$$\text{(odd, odd), (even, odd), (even, even), (odd, even).}$$

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Since there are 5 distinct points and only 4 possible cases of parity, by pigeonhole principle, there must exist 2 points which share parity for both coordinates.

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Coordinates are represented by two numbers, x and y.

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(odd, odd), (even, odd), (even, even), (odd, even).

Since there are 5 distinct points and only 4 possible cases of parity, by pigeonhole principle, there must exist 2 points which share parity for both coordinates.

These two points would make a midpoint with integer coordinates.

2019 T3 Q3 i)

How many different ways are there to distribute 80 identical balls and 40 distinct lollipops among 8 children:

a) with no further restriction?

b) with each child getting at least one ball and exactly 5 lollipops?

2019 T3 Q3 i)

How many different ways are there to distribute 80 identical balls and 40 distinct lollipops among 8 children:
a) with no further restriction?

2019 T3 Q3 i)

How many different ways are there to distribute 80 identical balls and 40 distinct lollipops among 8 children:
a) with no further restriction?

Two problems:
α) Distribute 80 identical balls among 8 children.

2019 T3 Q3 i)

How many different ways are there to distribute 80 identical balls and 40 distinct lollipops among 8 children:
a) with no further restriction?

Two problems:
α) Distribute 80 identical balls among 8 children.
β) Distribute 40 distinct lollipops among 8 children.

2019 T3 Q3 i)

How many different ways are there to distribute 80 identical balls and 40 distinct lollipops among 8 children:
a) with no further restriction?

Two problems:
α) Distribute 80 identical balls among 8 children.
β) Distribute 40 distinct lollipops among 8 children.

We can get the answer to a) by multiplying the answers of each sub-problem since it would cover all combinations of distributing 80 identical balls and 40 distinct lollipops among 9 children.

2019 T3 Q3 i)

α) Distribute 80 identical balls among 8 (distinct) children.

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We can represent the amount of balls each child gets by x_i.

2019 T3 Q3 i)

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We can represent the amount of balls each child gets by x_i. Thus

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 80,$$

2019 T3 Q3 i)

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We can represent the amount of balls each child gets by x_i. Thus

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where all x_i is a non-negative integer.

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The general solution for n identical objects and r distinct boxes is

$$\binom{n+r-1}{r-1}$$

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The general solution for n identical objects and r distinct boxes is

$$\binom{n+r-1}{r-1} \Rightarrow \binom{80+8-1}{8-1} = \binom{87}{7},$$

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The general solution for n identical objects and r distinct boxes is

$$\binom{n+r-1}{r-1} \Rightarrow \binom{80+8-1}{8-1} = \binom{87}{7},$$

where there are $r = 8$ children (boxes) and $n = 80$ balls (objects).

2019 T3 Q3 i)

α) Distribute 80 identical balls among 8 (distinct) children.

We can represent the amount of balls each child gets by x_i. Thus

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 80,$$

where all x_i is a non-negative integer.

The general solution for n identical objects and r distinct boxes is

$$\binom{n+r-1}{r-1} \Rightarrow \binom{80+8-1}{8-1} = \binom{87}{7},$$

where there are $r = 8$ children (boxes) and $n = 80$ balls (objects).

This formula only works when $x_i \geq 0$ i.e. can be 0.

2019 T3 Q3 i)

β) Distribute 40 distinct lollipops among 8 children.

2019 T3 Q3 i)

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The general solution for n distinct objects and r distinct boxes is

$$r^n$$

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2019 T3 Q3 i)

β) Distribute 40 distinct lollipops among 8 children.

The general solution for n distinct objects and r distinct boxes is

$$r^n \Rightarrow 8^{40}$$

where there are $r = 8$ children (boxes) and $n = 40$ lollipops (objects).

2019 T3 Q3 i)

β) Distribute 40 distinct lollipops among 8 children.

The general solution for n distinct objects and r distinct boxes is

$$r^n \Rightarrow 8^{40}$$

where there are $r = 8$ children (boxes) and $n = 40$ lollipops (objects).

The reasoning behind this is the fact that each distinct object has r choices to go to.

2019 T3 Q3 i)

β) Distribute 40 distinct lollipops among 8 children.

The general solution for n distinct objects and r distinct boxes is

$$r^n \Rightarrow 8^{40}$$

where there are $r = 8$ children (boxes) and $n = 40$ lollipops (objects).

The reasoning behind this is the fact that each distinct object has r choices to go to. In this case, each lollipop has a choice from 8 children to go to.

2019 T3 Q3 i)

β) Distribute 40 distinct lollipops among 8 children.

The general solution for n distinct objects and r distinct boxes is

$$r^n \Rightarrow 8^{40}$$

where there are $r = 8$ children (boxes) and $n = 40$ lollipops (objects).

The reasoning behind this is the fact that each distinct object has r choices to go to. In this case, each lollipop has a choice from 8 children to go to. 40 of these lollipops gives us 8^{40} by multiplication principle.

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How many different ways are there to distribute 80 identical balls and 40 distinct lollipops among 8 children:
a) with no further restriction?

2019 T3 Q3 i)

How many different ways are there to distribute 80 identical balls and 40 distinct lollipops among 8 children:
a) with no further restriction?

Since we know that α) has

$$\binom{87}{7}$$

different ways to distribute the 80 identical balls

2019 T3 Q3 i)

How many different ways are there to distribute 80 identical balls and 40 distinct lollipops among 8 children:
a) with no further restriction?

Since we know that α) has

$$\binom{87}{7}$$

different ways to distribute the 80 identical balls and β) has

$$8^{40}$$

different ways to distribute the 40 distinct lollipops.

2019 T3 Q3 i)

How many different ways are there to distribute 80 identical balls and 40 distinct lollipops among 8 children:
a) with no further restriction?

Since we know that α) has

$$\binom{87}{7}$$

different ways to distribute the 80 identical balls and β) has

$$8^{40}$$

different ways to distribute the 40 distinct lollipops.

Hence, there are

$$\binom{87}{7} \times 8^{40}$$

different ways to distribute the objects among 8 children.

2019 T3 Q3 i)

How many different ways are there to distribute 80 identical balls and 40 distinct lollipops among 8 children:
b) with each child getting at least one ball and exactly 5 lollipops?

2019 T3 Q3 i)

How many different ways are there to distribute 80 identical balls and 40 distinct lollipops among 8 children:
b) with each child getting at least one ball and exactly 5 lollipops?

2019 T3 Q3 i)

How many different ways are there to distribute 80 identical balls and 40 distinct lollipops among 8 children:
b) with each child getting at least one ball and exactly 5 lollipops?

Two problems:
α) Distribute 80 identical balls among 8 children

2019 T3 Q3 i)

How many different ways are there to distribute 80 identical balls and 40 distinct lollipops among 8 children:
b) with each child getting at least one ball and exactly 5 lollipops?

Two problems:

α) Distribute 80 identical balls among 8 children
AND each child must get at least one ball.

2019 T3 Q3 i)

How many different ways are there to distribute 80 identical balls and 40 distinct lollipops among 8 children:
b) with each child getting at least one ball and exactly 5 lollipops?

Two problems:

α) Distribute 80 identical balls among 8 children
AND each child must get at least one ball.

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2019 T3 Q3 i)

How many different ways are there to distribute 80 identical balls and 40 distinct lollipops among 8 children:
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Again, we can get the answer to b) by multiplying α) and β).

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We can represent the amount of balls each child gets by x_i. Thus

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 80,$$

where all x_i is a positive integer.

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The general solution for n identical objects and r distinct boxes is

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where there are $r = 8$ children (boxes) and $n = 80$ balls (objects).

This formula only works when x_i must be positive i.e. can't be 0.

2019 T3 Q3 i)

β) Distribute 40 distinct lollipops among 8 children
AND each child gets exactly 5 lollipops.

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We can have each child choose 5 lollipops from the pile.

2019 T3 Q3 i)

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This means that the first child has ${}^{40}C_5$ ways to choose.

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i.e. how many ways you can distribute 40 lollipops so each child gets 5.

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&= \frac{40!}{\cancel{35!}5!} \times \frac{\cancel{35!}}{\cancel{30!}5!} \times ... \times \frac{\cancel{10!}}{\cancel{5!}5!} \times \frac{\cancel{5!}}{5!}
\end{aligned}$$

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&= \frac{40!}{\cancel{35!}5!} \times \frac{\cancel{35!}}{\cancel{30!}5!} \times \dots \times \frac{\cancel{10!}}{\cancel{5!}5!} \times \frac{\cancel{5!}}{5!} \\
&= \frac{40!}{(5!)^8} \text{ ways to distribute.}
\end{aligned}$$

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different ways to distribute the 40 distinct lollipops.
Hence, there are

$$\binom{79}{7} \times \frac{40!}{(5!)^8}$$

different ways to distribute the objects among 8 children.

2017 T1 Q4 iii)

Consider the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 36,$$

where $x_1, x_2, x_3, x_4, x_5, x_6$ are non-negative integers.
How many solutions are there with all the x_i being odd and $x_i \leq 7$?

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To fix this, we can rewrite $x_i = 2y_i + 1$, where y_i is an integer.

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Since $x_i \leq 7$, $2y_i + 1 \leq 7$. Hence, $y_i \leq 3$.

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Using this substitution of $x_i = 2y_i + 1$,

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 + x_6 &= 36 \\ (2y_1 + 1) + (2y_2 + 1) + ... + (2y_6 + 1) &= \end{aligned}$$

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$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 + x_6 &= 36 \\ (2y_1 + 1) + (2y_2 + 1) + ... + (2y_6 + 1) &= \\ 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6) + 6 &= 36 \end{aligned}$$

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$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 + x_6 &= 36 \\ (2y_1 + 1) + (2y_2 + 1) + ... + (2y_6 + 1) &= \\ 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6) + 6 &= 36 \\ y_1 + y_2 + y_3 + y_4 + y_5 + y_6 &= 15. \end{aligned}$$

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$$\Downarrow$$

Consider the equation

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 15,$$

where $y_i \geq 0$ is an integer. How many solutions for all $y_i \leq 3$?

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Instead we can look at the Total - Complement.

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Instead we can look at the Total - Complement. We know that the amount of solutions for all $y_i \leq 3$ is equivalent to the total number of solutions minus the solutions where some $y_i > 3 \Rightarrow y_i \geq 4$.

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From the formula before, with no restrictions there are a total of

$$\binom{15+6-1}{6-1} = \binom{20}{5}$$

solutions for the equation above.

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To find all the solutions for the complement, where some $y_i \geq 4$

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where $y_i \geq 0$ is an integer. How many solutions for all $y_i \leq 3$?

To find all the solutions for the complement, where some $y_i \geq 4$, we need to find out when there is one such y_i, there are two such y_i and there are three such y_i.

2017 T1 Q4 iii)

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To find all the solutions for the complement, where some $y_i \geq 4$, we need to find out when there is one such y_i, there are two such y_i and there are three such y_i. There reason there cannot be four is because there are only 15 objects so there cannot be four or more $y_i \geq 4$.

2017 T1 Q4 iii)

Consider the equation

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where $y_i \geq 0$ is an integer. How many solutions for all $y_i \leq 3$?

For one such y_i, we can consider the case that $y_1 \geq 4$.

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Consider the equation

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where $y_i \geq 0$ is an integer. How many solutions for all $y_i \leq 3$?

For one such y_i, we can consider the case that $y_1 \geq 4$. Let's make the substitution that $y_1 = a_1 + 4$ and for every other $y_i = a_i$.

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$$\begin{aligned} y_1 + y_2 + y_3 + y_4 + y_5 + y_6 &= 15 \\ (a_1 + 4) + a_2 + a_3 + a_4 + a_5 + a_6 &= 15 \end{aligned}$$

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Consider the equation

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For one such y_i, we can consider the case that $y_1 \geq 4$. Let's make the substitution that $y_1 = a_1 + 4$ and for every other $y_i = a_i$. This give us

$$\begin{aligned} y_1 + y_2 + y_3 + y_4 + y_5 + y_6 &= 15 \\ (a_1 + 4) + a_2 + a_3 + a_4 + a_5 + a_6 &= 15 \\ a_1 + a_2 + a_3 + a_4 + a_5 + a_6 &= 11, \end{aligned}$$

where $a_i \geq 0$.

2017 T1 Q4 iii)

Consider the equation

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 15,$$

where $y_i \geq 0$ is an integer. How many solutions for all $y_i \leq 3$?

For one such y_i, we can consider the case that $y_1 \geq 4$. Let's make the substitution that $y_1 = a_1 + 4$ and for every other $y_i = a_i$. This give us

$$\begin{aligned} y_1 + y_2 + y_3 + y_4 + y_5 + y_6 &= 15 \\ (a_1 + 4) + a_2 + a_3 + a_4 + a_5 + a_6 &= 15 \\ a_1 + a_2 + a_3 + a_4 + a_5 + a_6 &= 11, \end{aligned}$$

where $a_i \geq 0$. Therefore the total amount of solutions is ${}^{16}\mathrm{C}_5$.

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Consider the equation

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 15,$$

where $y_i \geq 0$ is an integer. How many solutions for all $y_i \leq 3$?

For two such y_i, we can consider the case that $y_1, y_2 \geq 4$.

2017 T1 Q4 iii)

Consider the equation

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$$\begin{aligned} y_1 + y_2 + y_3 + y_4 + y_5 + y_6 &= 15 \\ (a_1 + 4) + (a_2 + 4) + a_3 + a_4 + a_5 + a_6 &= 15 \end{aligned}$$

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$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 15,$$

where $y_i \geq 0$ is an integer. How many solutions for all $y_i \leq 3$?

For two such y_i, we can consider the case that $y_1, y_2 \geq 4$. Let's make the substitution that $y_1 = a_1 + 4$, $y_2 = a_2 + 4$ and for every other $y_i = a_i$. This gives us

$$\begin{aligned} y_1 + y_2 + y_3 + y_4 + y_5 + y_6 &= 15 \\ (a_1 + 4) + (a_2 + 4) + a_3 + a_4 + a_5 + a_6 &= 15 \\ a_1 + a_2 + a_3 + a_4 + a_5 + a_6 &= 7, \end{aligned}$$

where $a_i \geq 0$.

2017 T1 Q4 iii)

Consider the equation

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 15,$$

where $y_i \geq 0$ is an integer. How many solutions for all $y_i \leq 3$?

For two such y_i, we can consider the case that $y_1, y_2 \geq 4$. Let's make the substitution that $y_1 = a_1 + 4$, $y_2 = a_2 + 4$ and for every other $y_i = a_i$. This gives us

$$\begin{aligned} y_1 + y_2 + y_3 + y_4 + y_5 + y_6 &= 15 \\ (a_1 + 4) + (a_2 + 4) + a_3 + a_4 + a_5 + a_6 &= 15 \\ a_1 + a_2 + a_3 + a_4 + a_5 + a_6 &= 7, \end{aligned}$$

where $a_i \geq 0$. Therefore the total amount of solutions is ${}^{12}\mathrm{C}_5$.

2017 T1 Q4 iii)

Consider the equation

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 15,$$

where $y_i \geq 0$ is an integer. How many solutions for all $y_i \leq 3$?

For three such y_i, we can consider the case that $y_1, y_2, y_3 \geq 4$.

2017 T1 Q4 iii)

Consider the equation

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 15,$$

where $y_i \geq 0$ is an integer. How many solutions for all $y_i \leq 3$?

For three such y_i, we can consider the case that $y_1, y_2, y_3 \geq 4$. Let's make the substitution that $y_1 = a_1 + 4$, $y_2 = a_2 + 4$, $y_3 = a_3 + 4$ and for every other $y_i = a_i$.

2017 T1 Q4 iii)

Consider the equation

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 15,$$

where $y_i \geq 0$ is an integer. How many solutions for all $y_i \leq 3$?

For three such y_i, we can consider the case that $y_1, y_2, y_3 \geq 4$. Let's make the substitution that $y_1 = a_1 + 4$, $y_2 = a_2 + 4$, $y_3 = a_3 + 4$ and for every other $y_i = a_i$. This gives us

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 15$$

2017 T1 Q4 iii)

Consider the equation

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 15,$$

where $y_i \geq 0$ is an integer. How many solutions for all $y_i \leq 3$?

For three such y_i, we can consider the case that $y_1, y_2, y_3 \geq 4$. Let's make the substitution that $y_1 = a_1 + 4$, $y_2 = a_2 + 4$, $y_3 = a_3 + 4$ and for every other $y_i = a_i$. This gives us

$$\begin{aligned} y_1 + y_2 + y_3 + y_4 + y_5 + y_6 &= 15 \\ (a_1 + 4) + (a_2 + 4) + (a_3 + 4) + a_4 + a_5 + a_6 &= 15 \end{aligned}$$

2017 T1 Q4 iii)

Consider the equation

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 15,$$

where $y_i \geq 0$ is an integer. How many solutions for all $y_i \leq 3$?

For three such y_i, we can consider the case that $y_1, y_2, y_3 \geq 4$. Let's make the substitution that $y_1 = a_1 + 4$, $y_2 = a_2 + 4$, $y_3 = a_3 + 4$ and for every other $y_i = a_i$. This gives us

$$\begin{aligned} y_1 + y_2 + y_3 + y_4 + y_5 + y_6 &= 15 \\ (a_1 + 4) + (a_2 + 4) + (a_3 + 4) + a_4 + a_5 + a_6 &= 15 \\ a_1 + a_2 + a_3 + a_4 + a_5 + a_6 &= 3, \end{aligned}$$

where $a_i \geq 0$.

2017 T1 Q4 iii)

Consider the equation

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 15,$$

where $y_i \geq 0$ is an integer. How many solutions for all $y_i \leq 3$?

For three such y_i, we can consider the case that $y_1, y_2, y_3 \geq 4$. Let's make the substitution that $y_1 = a_1 + 4$, $y_2 = a_2 + 4$, $y_3 = a_3 + 4$ and for every other $y_i = a_i$. This gives us

$$\begin{aligned} y_1 + y_2 + y_3 + y_4 + y_5 + y_6 &= 15 \\ (a_1 + 4) + (a_2 + 4) + (a_3 + 4) + a_4 + a_5 + a_6 &= 15 \\ a_1 + a_2 + a_3 + a_4 + a_5 + a_6 &= 3, \end{aligned}$$

where $a_i \geq 0$. Therefore the total amount of solutions is ${}^{8}\mathrm{C}_5$.

2017 T1 Q4 iii)

The Inclusion-Exclusion Principle

$$\begin{aligned}|A_1 \cup A_2 \cup \dots \cup A_n| = |A_1| + |A_2| + \dots + |A_n| \\ - |A_1 \cap A_2| - |A_1 \cap A_3| - \dots - |A_{n-1} \cap A_n| \\ + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + \dots + |A_{n-2} \cap A_{n-1} \cap A_n| + \dots \\ \dots \\ + (-1)^{n+1}|A_1 \cap A_2 \cap \dots \cap A_n|\end{aligned}$$

2017 T1 Q4 iii)

The Inclusion-Exclusion Principle

$$
\begin{aligned}
|A_1 \cup A_2 \cup \ldots \cup A_n| = {} & |A_1| + |A_2| + \ldots + |A_n| \\
& - |A_1 \cap A_2| - |A_1 \cap A_3| - \ldots - |A_{n-1} \cap A_n| \\
& + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + \ldots + |A_{n-2} \cap A_{n-1} \cap A_n| + \ldots \\
& \ldots \\
& + (-1)^{n+1}|A_1 \cap A_2 \cap \ldots \cap A_n|
\end{aligned}
$$

In our case, A_i represents when $y_i \geq 4$.

2017 T1 Q4 iii)

The Inclusion-Exclusion Principle

$$
\begin{aligned}
|A_1 \cup A_2 \cup ... \cup A_n| = |A_1| + |A_2| + ... + |A_n| \\
&- |A_1 \cap A_2| - |A_1 \cap A_3| - ... - |A_{n-1} \cap A_n| \\
&+ |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + ... + |A_{n-2} \cap A_{n-1} \cap A_n| + ... \\
&... \\
&+ (-1)^{n+1}|A_1 \cap A_2 \cap ... \cap A_n|
\end{aligned}
$$

In our case, A_i represents when $y_i \geq 4$. So A_1 is when $y_1 \geq 4$

2017 T1 Q4 iii)

The Inclusion-Exclusion Principle

$$
\begin{aligned}
|A_1 \cup A_2 \cup ... \cup A_n| = |A_1| &+ |A_2| + ... + |A_n| \\
&- |A_1 \cap A_2| - |A_1 \cap A_3| - ... - |A_{n-1} \cap A_n| \\
&+ |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + ... + |A_{n-2} \cap A_{n-1} \cap A_n| + ... \\
&... \\
&+ (-1)^{n+1}|A_1 \cap A_2 \cap ... \cap A_n|
\end{aligned}
$$

In our case, A_i represents when $y_i \geq 4$. So A_1 is when $y_1 \geq 4$ and $A_1 \cap A_2$ is when both $y_1, y_2 \geq 4$.

2017 T1 Q4 iii)

The Inclusion-Exclusion Principle

$$
\begin{aligned}
|A_1 \cup A_2 \cup \ldots \cup A_n| = {} & |A_1| + |A_2| + \ldots + |A_n| \\
& - |A_1 \cap A_2| - |A_1 \cap A_3| - \ldots - |A_{n-1} \cap A_n| \\
& + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + \ldots + |A_{n-2} \cap A_{n-1} \cap A_n| + \ldots \\
& \ldots \\
& + (-1)^{n+1}|A_1 \cap A_2 \cap \ldots \cap A_n|
\end{aligned}
$$

In our case, A_i represents when $y_i \geq 4$. So A_1 is when $y_1 \geq 4$ and $A_1 \cap A_2$ is when both $y_1, y_2 \geq 4$.

For us, $|A_1| = |A_2|$

2017 T1 Q4 iii)

The Inclusion-Exclusion Principle

$$\begin{aligned}|A_1 \cup A_2 \cup \ldots \cup A_n| = {} & |A_1| + |A_2| + \ldots + |A_n| \\ & - |A_1 \cap A_2| - |A_1 \cap A_3| - \ldots - |A_{n-1} \cap A_n| \\ & + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + \ldots + |A_{n-2} \cap A_{n-1} \cap A_n| + \ldots \\ & \ldots \\ & + (-1)^{n+1}|A_1 \cap A_2 \cap \ldots \cap A_n|\end{aligned}$$

In our case, A_i represents when $y_i \geq 4$. So A_1 is when $y_1 \geq 4$ and $A_1 \cap A_2$ is when both $y_1, y_2 \geq 4$.

For us, $|A_1| = |A_2|$ since we can follow the same process for A_i that we did with A_1 to derive the same amount of solutions.

2017 T1 Q4 iii)

The Inclusion-Exclusion Principle

$$\begin{aligned}|A_1 \cup A_2 \cup \ldots \cup A_n| = {} & |A_1| + |A_2| + \ldots + |A_n| \\ & - |A_1 \cap A_2| - |A_1 \cap A_3| - \ldots - |A_{n-1} \cap A_n| \\ & + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + \ldots + |A_{n-2} \cap A_{n-1} \cap A_n| + \ldots \\ & \ldots \\ & + (-1)^{n+1}|A_1 \cap A_2 \cap \ldots \cap A_n|\end{aligned}$$

In our case, A_i represents when $y_i \geq 4$. So A_1 is when $y_1 \geq 4$ and $A_1 \cap A_2$ is when both $y_1, y_2 \geq 4$.

For us, $|A_1| = |A_2|$ since we can follow the same process for A_i that we did with A_1 to derive the same amount of solutions. We can also do the same for $A_i \cap A_j$ and $A_i \cap A_j \cap A_k$.

2017 T1 Q4 iii)

Consider the equation

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 15,$$

where $y_i \geq 0$ is an integer. How many solutions for all $y_i \leq 3$?

2017 T1 Q4 iii)

Consider the equation

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 15,$$

where $y_i \geq 0$ is an integer. How many solutions for all $y_i \leq 3$?

For all cases of some $y_i \geq 4$, from inclusion-exclusion principle, we have

2017 T1 Q4 iii)

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For all cases of some $y_i \geq 4$, from inclusion-exclusion principle, we have

$$\begin{aligned} |A_1 \cup A_2 \cup ... \cup A_6| &= |A_1| + |A_2| + ... + |A_6| \\ &- |A_1 \cap A_2| - |A_1 \cap A_3| - ... - |A_5 \cap A_6| \\ &+ |A_1 \cap A_2 \cap A_3| + ... + |A_4 \cap A_5 \cap A_6| \end{aligned}$$

2017 T1 Q4 iii)

Consider the equation

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 15,$$

where $y_i \geq 0$ is an integer. How many solutions for all $y_i \leq 3$?

For all cases of some $y_i \geq 4$, from inclusion-exclusion principle, we have

$$\begin{aligned}
|A_1 \cup A_2 \cup \ldots \cup A_6| &= |A_1| + |A_2| + \ldots + |A_6| \\
&- |A_1 \cap A_2| - |A_1 \cap A_3| - \ldots - |A_5 \cap A_6| \\
&+ |A_1 \cap A_2 \cap A_3| + \ldots + |A_4 \cap A_5 \cap A_6| \\
&= {}^6\mathrm{C}_1|A_1| - {}^6\mathrm{C}_2|A_1 \cap A_2| + {}^6\mathrm{C}_3|A_1 \cap A_2 \cap A_3|
\end{aligned}$$

2017 T1 Q4 iii)

Consider the equation

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 15,$$

where $y_i \geq 0$ is an integer. How many solutions for all $y_i \leq 3$?

For all cases of some $y_i \geq 4$, from inclusion-exclusion principle, we have

$$\begin{aligned}
|A_1 \cup A_2 \cup \ldots \cup A_6| &= |A_1| + |A_2| + \ldots + |A_6| \\
&- |A_1 \cap A_2| - |A_1 \cap A_3| - \ldots - |A_5 \cap A_6| \\
&+ |A_1 \cap A_2 \cap A_3| + \ldots + |A_4 \cap A_5 \cap A_6| \\
&= {}^6\mathrm{C}_1|A_1| - {}^6\mathrm{C}_2|A_1 \cap A_2| + {}^6\mathrm{C}_3|A_1 \cap A_2 \cap A_3| \\
&= {}^6\mathrm{C}_1{}^16\mathrm{C}_5 - {}^6\mathrm{C}_2{}^12\mathrm{C}_5 + {}^6\mathrm{C}_3{}^8\mathrm{C}_5
\end{aligned}$$

2017 T1 Q4 iii)

Consider the equation

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 15,$$

where $y_i \geq 0$ is an integer. How many solutions for all $y_i \leq 3$?

For all cases of some $y_i \geq 4$, from inclusion-exclusion principle, we have

$$\begin{aligned}
|A_1 \cup A_2 \cup ... \cup A_6| &= |A_1| + |A_2| + ... + |A_6| \\
&- |A_1 \cap A_2| - |A_1 \cap A_3| - ... - |A_5 \cap A_6| \\
&+ |A_1 \cap A_2 \cap A_3| + ... + |A_4 \cap A_5 \cap A_6| \\
&= {}^6\mathrm{C}_1 |A_1| - {}^6\mathrm{C}_2 |A_1 \cap A_2| + {}^6\mathrm{C}_3 |A_1 \cap A_2 \cap A_3| \\
&= {}^6\mathrm{C}_1 {}^16\mathrm{C}_5 - {}^6\mathrm{C}_2 {}^12\mathrm{C}_5 + {}^6\mathrm{C}_3 {}^8\mathrm{C}_5
\end{aligned}$$

Hence, we have ${}^{20}\mathrm{C}_5$

2017 T1 Q4 iii)

Consider the equation

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 15,$$

where $y_i \geq 0$ is an integer. How many solutions for all $y_i \leq 3$?

For all cases of some $y_i \geq 4$, from inclusion-exclusion principle, we have

$$\begin{aligned}
|A_1 \cup A_2 \cup \ldots \cup A_6| &= |A_1| + |A_2| + \ldots + |A_6| \\
&- |A_1 \cap A_2| - |A_1 \cap A_3| - \ldots - |A_5 \cap A_6| \\
&+ |A_1 \cap A_2 \cap A_3| + \ldots + |A_4 \cap A_5 \cap A_6| \\
&= {}^6\mathrm{C}_1 |A_1| - {}^6\mathrm{C}_2 |A_1 \cap A_2| + {}^6\mathrm{C}_3 |A_1 \cap A_2 \cap A_3| \\
&= {}^6\mathrm{C}_1 {}^16\mathrm{C}_5 - {}^6\mathrm{C}_2 {}^12\mathrm{C}_5 + {}^6\mathrm{C}_3 {}^8\mathrm{C}_5
\end{aligned}$$

Hence, we have ${}^{20}\mathrm{C}_5 - ({}^6\mathrm{C}_1 {}^16\mathrm{C}_5 - {}^6\mathrm{C}_2 {}^12\mathrm{C}_5 + {}^6\mathrm{C}_3 {}^8\mathrm{C}_5) = 86$ solutions.

2013 T1 Q4 ii)

Let S be the set of all 14-letter words from the 26 letters of the English alphabet, where no letter is used more than once.
a) How many such words are there altogether?
b) How many words in S contain the sub-words JULIA or TONY?
c) How many words in S contain all five vowels?

2013 T1 Q4 ii)

Let S be the set of all 14-letter words from the 26 letters of the English alphabet, where no letter is used more than once.
a) How many such words are there altogether?

2013 T1 Q4 ii)

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2013 T1 Q4 ii)

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a) How many such words are there altogether?

We have 26 letters where repetition is not allowed.

2013 T1 Q4 ii)

Let S be the set of all 14-letter words from the 26 letters of the English alphabet, where no letter is used more than once.
a) How many such words are there altogether?

We have 26 letters where repetition is not allowed. This is just choosing 14 letters from the 26 and then ordering them.

2013 T1 Q4 ii)

Let S be the set of all 14-letter words from the 26 letters of the English alphabet, where no letter is used more than once.
a) How many such words are there altogether?

We have 26 letters where repetition is not allowed. This is just choosing 14 letters from the 26 and then ordering them.

Hence, within the set S, there are ${}^{26}\mathrm{P}_{14}$ words altogether.

2013 T1 Q4 ii)

Let S be the set of all 14-letter words from the 26 letters of the English alphabet, where no letter is used more than once.
b) How many words in S contain the sub-words JULIA or TONY?

2013 T1 Q4 ii)

Let S be the set of all 14-letter words from the 26 letters of the English alphabet, where no letter is used more than once.
b) How many words in S contain the sub-words JULIA or TONY?

2013 T1 Q4 ii)

Let S be the set of all 14-letter words from the 26 letters of the English alphabet, where no letter is used more than once.
b) How many words in S contain the sub-words JULIA or TONY?

These words hint at the use of inclusion-exclusion

2013 T1 Q4 ii)

Let S be the set of all 14-letter words from the 26 letters of the English alphabet, where no letter is used more than once.
b) How many words in S contain the sub-words JULIA or TONY?

These words hint at the use of inclusion-exclusion such that

$$|A_{JULIA} \cup A_{TONY}| = |A_{JULIA}| + |A_{TONY}| - |A_{JULIA} \cap A_{TONY}|$$

2013 T1 Q4 ii)

Let S be the set of all 14-letter words from the 26 letters of the English alphabet, where no letter is used more than once.
b) How many words in S contain the sub-words JULIA or TONY?

These words hint at the use of inclusion-exclusion such that

$$|A_{JULIA} \cup A_{TONY}| = |A_{JULIA}| + |A_{TONY}| - |A_{JULIA} \cap A_{TONY}|$$

where A_{JULIA} are the words that contain JULIA and A_{TONY} are the words that contain TONY.

2013 T1 Q4 ii)

Let S be the set of all 14-letter words from the 26 letters of the English alphabet, where no letter is used more than once.
b) How many words in S contain the sub-words JULIA or TONY?

These words hint at the use of inclusion-exclusion such that

$$|A_{JULIA} \cup A_{TONY}| = |A_{JULIA}| + |A_{TONY}| - |A_{JULIA} \cap A_{TONY}|$$

where A_{JULIA} are the words that contain JULIA and A_{TONY} are the words that contain TONY.

Thus, to find how many words with JULIA or TONY

2013 T1 Q4 ii)

Let S be the set of all 14-letter words from the 26 letters of the English alphabet, where no letter is used more than once.
b) How many words in S contain the sub-words JULIA or TONY?

These words hint at the use of inclusion-exclusion such that

$$|A_{JULIA} \cup A_{TONY}| = |A_{JULIA}| + |A_{TONY}| - |A_{JULIA} \cap A_{TONY}|$$

where A_{JULIA} are the words that contain JULIA and A_{TONY} are the words that contain TONY.

Thus, to find how many words with JULIA or TONY, we need all the words with JULIA

2013 T1 Q4 ii)

Let S be the set of all 14-letter words from the 26 letters of the English alphabet, where no letter is used more than once.
b) How many words in S contain the sub-words JULIA or TONY?

These words hint at the use of inclusion-exclusion such that

$$|A_{JULIA} \cup A_{TONY}| = |A_{JULIA}| + |A_{TONY}| - |A_{JULIA} \cap A_{TONY}|$$

where A_{JULIA} are the words that contain JULIA and A_{TONY} are the words that contain TONY.

Thus, to find how many words with JULIA or TONY, we need all the words with JULIA, all the with with TONY

Let S be the set of all 14-letter words from the 26 letters of the English alphabet, where no letter is used more than once.
b) How many words in S contain the sub-words JULIA or TONY?

These words hint at the use of inclusion-exclusion such that

$$|A_{JULIA} \cup A_{TONY}| = |A_{JULIA}| + |A_{TONY}| - |A_{JULIA} \cap A_{TONY}|$$

where A_{JULIA} are the words that contain JULIA and A_{TONY} are the words that contain TONY.

Thus, to find how many words with JULIA or TONY, we need all the words with JULIA, all the with with TONY and all the words that have both JULIA and TONY as subwords.

2013 T1 Q4 ii)

Let S be the set of all 14-letter words from the 26 letters of the English alphabet, where no letter is used more than once.
b) How many words in S contain the sub-words JULIA or TONY?

2013 T1 Q4 ii)

Let S be the set of all 14-letter words from the 26 letters of the English alphabet, where no letter is used more than once.
b) How many words in S contain the sub-words JULIA or TONY?

To find the words with JULIA, consider

$$[\ \underline{J}\ \underline{U}\ \underline{L}\ \underline{I}\ \underline{A}\ ]\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_$$

2013 T1 Q4 ii)

Let S be the set of all 14-letter words from the 26 letters of the English alphabet, where no letter is used more than once.
b) How many words in S contain the sub-words JULIA or TONY?

To find the words with JULIA, consider

$$[\ \underline{\text{J}}\ \underline{\text{U}}\ \underline{\text{L}}\ \underline{\text{I}}\ \underline{\text{A}}\ ]\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_$$

where the letters for JULIA have been chosen, so 9 spaces are left

2013 T1 Q4 ii)

Let S be the set of all 14-letter words from the 26 letters of the English alphabet, where no letter is used more than once.
b) How many words in S contain the sub-words JULIA or TONY?

To find the words with JULIA, consider

$$[\ \underline{J}\ \underline{U}\ \underline{L}\ \underline{I}\ \underline{A}\ ]\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_$$

where the letters for JULIA have been chosen, so 9 spaces are left from the remaining 21 letters in the alphabet (no repetition).

2013 T1 Q4 ii)

Let S be the set of all 14-letter words from the 26 letters of the English alphabet, where no letter is used more than once.
b) How many words in S contain the sub-words JULIA or TONY?

To find the words with JULIA, consider

$$[\ \underline{J}\ \underline{U}\ \underline{L}\ \underline{I}\ \underline{A}\ ]\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_$$

where the letters for JULIA have been chosen, so 9 spaces are left from the remaining 21 letters in the alphabet (no repetition). This gives us ${}^{21}\mathrm{C}_9$ ways to choose these letters.

To arrange these, we want to preserve to order of JULIA

2013 T1 Q4 ii)

Let S be the set of all 14-letter words from the 26 letters of the English alphabet, where no letter is used more than once.
b) How many words in S contain the sub-words JULIA or TONY?

To find the words with JULIA, consider

$$[\ \underline{J}\ \underline{U}\ \underline{L}\ \underline{I}\ \underline{A}\ ]\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_$$

where the letters for JULIA have been chosen, so 9 spaces are left from the remaining 21 letters in the alphabet (no repetition). This gives us ${}^{21}C_9$ ways to choose these letters.

To arrange these, we want to preserve to order of JULIA, so we can treat this as a letter

2013 T1 Q4 ii)

Let S be the set of all 14-letter words from the 26 letters of the English alphabet, where no letter is used more than once.
b) How many words in S contain the sub-words JULIA or TONY?

To find the words with JULIA, consider

$$[\ \underline{J}\ \underline{U}\ \underline{L}\ \underline{I}\ \underline{A}\ ]\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_$$

where the letters for JULIA have been chosen, so 9 spaces are left from the remaining 21 letters in the alphabet (no repetition). This gives us ${}^{21}C_9$ ways to choose these letters.

To arrange these, we want to preserve to order of JULIA, so we can treat this as a letter, giving us 10 objects and 10! arrangements.

2013 T1 Q4 ii)

Let S be the set of all 14-letter words from the 26 letters of the English alphabet, where no letter is used more than once.
b) How many words in S contain the sub-words JULIA or TONY?

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Hence, there are ${}^{21}C_9 \times 10!$ words with JULIA.

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$$[\ \underline{T}\ \underline{O}\ \underline{N}\ \underline{Y}\ ]\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_$$

where the letters for TONY have been chosen, so 10 spaces are left from the remaining 22 letters in the alphabet (no repetition). This gives us ${}^{22}\mathrm{C}_{10}$ ways to choose these letters.

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To arrange these, we want to preserve to order of TONY, so we can treat this as a letter, giving us 11 objects and 11! arrangements.

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where the letters for JULIA and TONY have been chosen, so 5 spaces are left from the remaining 17 letters in the alphabet (no repetition). This gives us ${}^{17}\mathrm{C}_5$ ways to choose these letters.

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We want to preserve both JULIA and TONY

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where the letters for JULIA and TONY have been chosen, so 5 spaces are left from the remaining 17 letters in the alphabet (no repetition). This gives us ${}^{17}C_5$ ways to choose these letters.

We want to preserve both JULIA and TONY so they are both individual letters.

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We want to preserve both JULIA and TONY so they are both individual letters. This gives us 7 objects and 7! arrangements.

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We want to preserve both JULIA and TONY so they are both individual letters. This gives us 7 objects and 7! arrangements.

Hence, there are ${}^{17}\mathrm{C}_5 \times 7!$ words with JULIA and TONY.

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From inclusion-exclusion principle,

$$|A_{JULIA} \cup A_{TONY}| \quad = \quad |A_{JULIA}| + |A_{TONY}| - |A_{JULIA} \cap A_{TONY}|$$

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where A_{JULIA} are the words that contain JULIA and A_{TONY} are the words that contain TONY.

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where A_{JULIA} are the words that contain JULIA and A_{TONY} are the words that contain TONY.

Hence, there are ${}^{21}\mathrm{C}_9 \times 10! + {}^{22}\mathrm{C}_{10} \times 11! - {}^{17}\mathrm{C}_5 \times 7!$ such words.

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Let S be the set of all 14-letter words from the 26 letters of the English alphabet, where no letter is used more than once.
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Suppose that all 5 vowels have already been chosen for the word.

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Suppose that all 5 vowels have already been chosen for the word. This leaves us with 9 more letters for the 14-letter word and a choice from 21 letters left over, since there is no repetition.

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Thus, there are only ${}^{21}\mathrm{C}_9$ ways to choose the rest of the letters.

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Now, we just need to arrange the 14 letters that have been chosen, which gives us 14! arrangements. This is because the vowels don't need to be in a particular order.

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Hence, there are ${}^{21}\mathrm{C}_9 \times 14!$ words in S with all 5 vowels.

2012 T1 Q4 iii)

Four friends, June, Alex, Wayne, Joe, are playing a hand of bridge. A standard pack of 52 playing cards is dealt, giving each player 13 cards.

a) Find the probability that June and Wayne are dealt all 13 spades.

b) Find the probability that the cards dealt to June and Wayne contain at least one complete suite. (13 spades/hearts/clubs/diamonds)

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Probability: $\dfrac{\text{Number of Cases with Requirement}}{\text{Total Cases}}$

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Across June and Wayne, there are 26 cards dealt.

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Across June and Wayne, there are 26 cards dealt. Thus, the total cases is given by choosing 26 cards from the deck of 52: ${}^{52}C_{26}$.

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The cases where all 13 spades are dealt means that 13 of the 26 slots are taken by the spades.

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The cases where all 13 spades are dealt means that 13 of the 26 slots are taken by the spades. Thus, there are 13 more cards to be chosen from the 39 remaining cards. Meaning the number of cases with 13 spades chosen is given by ${}^{39}\mathrm{C}_{13}$.

Hence the probability is given by ${}^{39}\mathrm{C}_{13}/{}^{52}\mathrm{C}_{26}$.

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Four friends, June, Alex, Wayne, Joe, are playing a hand of bridge. A standard pack of 52 playing cards is dealt, giving each player 13 cards.

b) Find the probability that the cards dealt to June and Wayne contain at least one complete suite. (13 spades/hearts/clubs/diamonds)

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This is another application of inclusion-exclusion principle.

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There is only one way to get 13 spades and hearts dealt since there are only 26 cards between June and Wayne.

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There is only one way to get 13 spades and hearts dealt since there are only 26 cards between June and Wayne.

Since there are only 26 cards, there is no way to get 3 or more complete suites as that would requite 39 or more cards to be dealt between June and Wayne.

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b) Find the probability that the cards dealt to June and Wayne contain at least one complete suite. (13 spades/hearts/clubs/diamonds)

Now we can apply the inclusion-exclusion principle:

$$\begin{aligned} |A_S \cup A_H \cup A_C \cup A_D| &= |A_S| + |A_H| + |A_C| + |A_D| \\ &- |A_S \cap A_H| - \ldots - |A_C \cap A_D| \end{aligned}$$

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&- |A_S \cap A_H| - \ldots - |A_C \cap A_D| \\
&= {}^4\mathrm{C}_1 \times |A_S| - {}^4\mathrm{C}_2 \times |A_S \cap A_H| \\
&= {}^4\mathrm{C}_1 \times {}^{39}\mathrm{C}_{13} - {}^4\mathrm{C}_2 \times 1 \\
&= 4 \times {}^{39}\mathrm{C}_{13} - {}^4\mathrm{C}_2,
\end{aligned}$$

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&= {}^4\mathrm{C}_1 \times |A_S| - {}^4\mathrm{C}_2 \times |A_S \cap A_H| \\
&= {}^4\mathrm{C}_1 \times {}^{39}\mathrm{C}_{13} - {}^4\mathrm{C}_2 \times 1 \\
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\end{aligned}$$

where A_S represents where there is a complete suite of spades etc.

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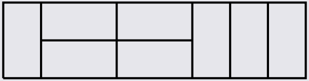
&= {}^4\mathrm{C}_1 \times |A_S| - {}^4\mathrm{C}_2 \times |A_S \cap A_H| \\
&= {}^4\mathrm{C}_1 \times {}^{39}\mathrm{C}_{13} - {}^4\mathrm{C}_2 \times 1 \\
&= 4 \times {}^{39}\mathrm{C}_{13} - {}^4\mathrm{C}_2,
\end{aligned}$$

where A_S represents where there is a complete suite of spades etc.

Hence, the probability is $(4 \times {}^{39}\mathrm{C}_{13} - {}^4\mathrm{C}_2)/{}^{52}\mathrm{C}_{26}$

A straight path of width 2 units is to be laid using 1-unit by 2-unit paving slabs. Here is an example of a path of length 8 units.



Let a_n be the number of ways to lay a path of width 2 and length n.
a) Find a_1, a_2 and a_3.
b) Obtain a recurrence relation for a_n. Explain your answer.
(You do NOT need to solve this recurrence relation.)

2018 T2 Q4 v)

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Let's draw this up!

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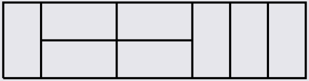

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There is actually another way to make a path of length n which is adding two horizontal blocks to make a 2 by 2 and adding it to all $n-2$ cases. Hence, the a_{n-2}.

2019 T3 Q3 iv)

Obtain the general solution to the recurrence relation

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2019 T3 Q3 iv)

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A general recurrence relation of order k has the form

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Solutions for Linear, Homogeneous, Order 1: $a_n = ra_{n-1}$
The solution is $a_n = Ar^n$ for some constant A.

2019 T3 Q3 iv)

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A guess does not work if there are shared terms in h_n and p_n. If that is the case, multiply p_n by n to form a new guess.

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Hence, the general solution h_n is given by $A(3^n) + Bn(3^n)$ since the solutions of the r quadratic are the same.

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Hence, the general solution is

$$a_n = A(3^n) + Bn(3^n) + n^2\left(\frac{5n}{6} + \frac{5}{2}\right)3^n.$$

4. Logic and Proofs

Mathematical Induction

Prove by mathematical induction that $3^n > n^3$ for all integers $n \geq 4$.

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Proof: Let $P(n)$ be the statement $3^n > n^3$ where $n \in \mathbb{Z}^+$ and $n \geq 4$.
Step 1: We have to check our base case $P(4)$ is true.

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We have that $3^4 = 81$ and $4^3 = 64$.

Prove by mathematical induction that $3^n > n^3$ for all integers $n \geq 4$.

Proof: Let $P(n)$ be the statement $3^n > n^3$ where $n \in \mathbb{Z}^+$ and $n \geq 4$.
Step 1: We have to check our base case $P(4)$ is true.

We have that $3^4 = 81$ and $4^3 = 64$.
So $3^4 > 4^3$ and $P(4)$ is true.

Mathematical Induction

Prove by mathematical induction that $3^n > n^3$ for all integers $n \geq 4$.

Step 2:
Assume $P(k)$ is true for all integers $k \geq 4$.
i.e $3^k > k^3$.

Prove by mathematical induction that $3^n > n^3$ for all integers $n \geq 4$.

Step 2:
Assume $P(k)$ is true for all integers $k \geq 4$.
i.e $3^k > k^3$.

We now have to prove $P(k+1)$ is also true.
i.e $3^{k+1} > (k+1)^3$.

Mathematical Induction*

Prove by mathematical induction that $3^n > n^3$ for all integers $n \geq 4$.

Assume $3^k > k^3$.
RTP: $3^{k+1} > (k+1)^3 = k^3 + 3k^2 + 3k + 1$.

Mathematical Induction

Prove by mathematical induction that $3^n > n^3$ for all integers $n \geq 4$.

$$\begin{aligned}
3^{k+1} &= 3(3^k) \\
&> 3k^3 \text{ (by induction hypothesis)} \\
&= k^3 + 2k^3 \\
&\geq k^3 + 8k^2 \text{ (since } k \geq 4) \\
&= k^3 + 3k^2 + 5k^2 \\
&\geq k^3 + 3k^2 + 20k \text{ (since } k \geq 4) \\
&= k^3 + 3k^2 + 3k + 17k \\
&\geq k^3 + 3k^2 + 3k + 1 \text{ (since } k \geq 4) \\
&= (k+1)^3.
\end{aligned}$$

Hence $P(k+1)$ is true if $P(k)$ is true. Thus, by the principle of mathematical induction, $P(n)$ is true for all integers $n \geq 4$. $\square$

Contradiction

Prove $\sqrt[3]{3} + \sqrt{2}$ is irrational.

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Assume for a contradiction that $\sqrt[3]{3} + \sqrt{2}$ is rational.
Let $\sqrt[3]{3} + \sqrt{2} = \frac{a}{b}$ where $a, b \in \mathbb{Z}^+$. Then we have

$$\sqrt[3]{3} = \frac{a}{b} - \sqrt{2}$$

Contradiction

Prove $\sqrt[3]{3}+\sqrt{2}$ is irrational.

Assume for a contradiction that $\sqrt[3]{3}+\sqrt{2}$ is rational.
Let $\sqrt[3]{3}+\sqrt{2}=\frac{a}{b}$ where $a,b\in\mathbb{Z}^+$. Then we have

$$\begin{aligned}\sqrt[3]{3} &= \frac{a}{b}-\sqrt{2}\\ 3 &= \left(\frac{a}{b}-\sqrt{2}\right)^3\end{aligned}$$

Contradiction

Prove $\sqrt[3]{3} + \sqrt{2}$ is irrational.

Assume for a contradiction that $\sqrt[3]{3} + \sqrt{2}$ is rational.
Let $\sqrt[3]{3} + \sqrt{2} = \frac{a}{b}$ where $a, b \in \mathbb{Z}^+$. Then we have

$$\begin{aligned}
\sqrt[3]{3} &= \frac{a}{b} - \sqrt{2} \\
3 &= \left(\frac{a}{b} - \sqrt{2}\right)^3 \\
3 &= \frac{a^3}{b^3} - \frac{3\sqrt{2}a^2}{b^2} + \frac{12a}{b} - 2\sqrt{2}
\end{aligned}$$

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\end{aligned}$$

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3 &= \frac{a^3}{b^3} - \frac{3\sqrt{2}a^2}{b^2} + \frac{12a}{b} - 2\sqrt{2} \\
3b^3 &= a^3 - 3\sqrt{2}a^2b + 12ab^2 - 2\sqrt{2}b^3 \\
\sqrt{2}(3a^2b + 2b^3) &= a^3 - 3b^3 + 12ab^2
\end{aligned}$$

Contradiction

Prove $\sqrt[3]{3}+\sqrt{2}$ is irrational.

Assume for a contradiction that $\sqrt[3]{3}+\sqrt{2}$ is rational.
Let $\sqrt[3]{3}+\sqrt{2}=\frac{a}{b}$ where $a,b\in\mathbb{Z}^+$. Then we have

$$\begin{aligned}
\sqrt[3]{3} &= \frac{a}{b}-\sqrt{2}\\
3 &= \left(\frac{a}{b}-\sqrt{2}\right)^3\\
3 &= \frac{a^3}{b^3}-\frac{3\sqrt{2}a^2}{b^2}+\frac{12a}{b}-2\sqrt{2}\\
3b^3 &= a^3-3\sqrt{2}a^2b+12ab^2-2\sqrt{2}b^3\\
\sqrt{2}(3a^2b+2b^3) &= a^3-3b^3+12ab^2\\
\sqrt{2} &= \frac{a^3-3b^3+12ab^2}{3a^2b+2b^3}
\end{aligned}$$

Contradiction

Prove $\sqrt[3]{3} + \sqrt{2}$ is irrational.

$$\sqrt{2} = \frac{a^3 - 3b^3 + 12ab^2}{3a^2b + 2b^3}$$

$$\sqrt{2} = \frac{A}{B} \text{ (where A and B are integers).}$$

Contradiction

Prove $\sqrt[3]{3} + \sqrt{2}$ is irrational.

$$\sqrt{2} = \frac{a^3 - 3b^3 + 12ab^2}{3a^2b + 2b^3}$$
$$\sqrt{2} = \frac{A}{B} \text{ (where A and B are integers).}$$

Hence $\sqrt{2}$ is rational.

Contradiction

Prove $\sqrt[3]{3} + \sqrt{2}$ is irrational.

$$\sqrt{2} = \frac{a^3 - 3b^3 + 12ab^2}{3a^2b + 2b^3}$$
$$\sqrt{2} = \frac{A}{B} \text{ (where A and B are integers).}$$

Hence $\sqrt{2}$ is rational.

But this is a contradiction as $\sqrt{2}$ is irrational and our assumption is false. Thus, $\sqrt[3]{3} + \sqrt{2}$ is irrational. $\square$

Lemma

$\sqrt{2}$ is irrational.

Contrapositive

Definition

$$p \to q \equiv \sim q \to \sim p.$$

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Contrapositive

For all $n \in \mathbb{Z}^+$, prove that if n is not prime, then $2^n - 1$ is not prime.

Let $n \in \mathbb{Z}^+$. Suppose n is not prime. Then n is either 1 or composite.

<u>Case 1:</u> $n = 1$.

Contrapositive

Definition

$$p \to q \equiv \sim q \to \sim p.$$

For all $n \in \mathbb{Z}^+$, prove that if $2^n - 1$ is prime, then n is prime.

Contrapositive

For all $n \in \mathbb{Z}^+$, prove that if n is not prime, then $2^n - 1$ is not prime.

Let $n \in \mathbb{Z}^+$. Suppose n is not prime. Then n is either 1 or composite.

Case 1: $n = 1$.
If $n = 1$ then we have $2^n - 1 = 2^1 - 1 = 1$ which is not prime. Hence, $2^n - 1$ is not prime for $n = 1$.

Contrapositive

For all $n \in \mathbb{Z}^+$, prove that if $2^n - 1$ is prime, then n is prime.

Case 2: $n = ab$ where $a, b \in Z^+$ and $a, b > 1$.

Contrapositive

For all $n \in \mathbb{Z}^+$, prove that if $2^n - 1$ is prime, then n is prime.

<u>Case 2:</u> $n = ab$ where $a, b \in Z^+$ and $a, b > 1$.
If $n = ab$, then we have

$$2^n - 1 = 2^{ab} - 1$$

Contrapositive

For all $n \in \mathbb{Z}^+$, prove that if $2^n - 1$ is prime, then n is prime.

<u>Case 2:</u> $n = ab$ where $a, b \in Z^+$ and $a, b > 1$.
If $n = ab$, then we have

$$\begin{aligned} 2^n - 1 &= 2^{ab} - 1 \\ &= (2^a)^b - 1 \end{aligned}$$

Contrapositive

For all $n \in \mathbb{Z}^+$, prove that if $2^n - 1$ is prime, then n is prime.

Case 2: $n = ab$ where $a, b \in Z^+$ and $a, b > 1$.
If $n = ab$, then we have

$$\begin{aligned} 2^n - 1 &= 2^{ab} - 1 \\ &= (2^a)^b - 1 \\ &= (2^a - 1)(1 + 2^a + (2^a)^2 + ... + (2^a)^{b-1}) \end{aligned}$$

Contrapositive

For all $n \in \mathbb{Z}^+$, prove that if $2^n - 1$ is prime, then n is prime.

<u>Case 2:</u> $n = ab$ where $a, b \in Z^+$ and $a, b > 1$.
If $n = ab$, then we have

$$\begin{aligned} 2^n - 1 &= 2^{ab} - 1 \\ &= (2^a)^b - 1 \\ &= (2^a - 1)(1 + 2^a + (2^a)^2 + ... + (2^a)^{b-1}) \\ &= AB \end{aligned}$$

Where $A, B \in \mathbb{Z}^+$ and $A, B > 1$.

Contrapositive

For all $n \in \mathbb{Z}^+$, prove that if $2^n - 1$ is prime, then n is prime.

Case 2: $n = ab$ where $a, b \in Z^+$ and $a, b > 1$.
If $n = ab$, then we have

$$\begin{aligned} 2^n - 1 &= 2^{ab} - 1 \\ &= (2^a)^b - 1 \\ &= (2^a - 1)(1 + 2^a + (2^a)^2 + ... + (2^a)^{b-1}) \\ &= AB \end{aligned}$$

Where $A, B \in \mathbb{Z}^+$ and $A, B > 1$.
Hence when n is composite, $2^n - 1$ is not prime.

Contrapositive

For all $n \in \mathbb{Z}^+$, prove that if $2^n - 1$ is prime, then n is prime.

<u>Case 2:</u> $n = ab$ where $a, b \in Z^+$ and $a, b > 1$.
If $n = ab$, then we have

$$\begin{aligned} 2^n - 1 &= 2^{ab} - 1 \\ &= (2^a)^b - 1 \\ &= (2^a - 1)(1 + 2^a + (2^a)^2 + ... + (2^a)^{b-1}) \\ &= AB \end{aligned}$$

Where $A, B \in \mathbb{Z}^+$ and $A, B > 1$.
Hence when n is composite, $2^n - 1$ is not prime.

Thus in both cases, the contrapositive statement is true. Hence the original statement is true for all $n \in \mathbb{Z}^+$. $\square$

2017 T1 Q3 (iv)

Absolute value function

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$

2017 T1 Q3 (iv)

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Prove that if x is a real number then

$$x|x-3| \leq 2x^2 - 9x + 12.$$

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The absolute value is a strong indicator that we need cases!!!

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The absolute value is a strong indicator that we need cases!!! Notice that

$$|x-3| = \begin{cases} x-3 & x \geq 3 \\ -(x-3) & x < 3. \end{cases}$$

2017 T1 Q3 (iv)

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The absolute value is a strong indicator that we need cases!!! Notice that

$$|x-3| = \begin{cases} x-3 & x \geq 3 \\ -(x-3) & x < 3. \end{cases}$$

So lets take cases when $x \geq 3$ and $x < 3$.

2017 T1 3) (iv)*

Prove that if x is a real number then

$$x|x-3| \leq 2x^2 - 9x + 12.$$

2017 T1 3) (iv)*

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2017 T1 3) (iv)

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Suppose $x \in \mathbb{R}$.
Case 1: For the first case, we take $x \geq 3$. Now consider

$$\begin{aligned} 2x^2 - 9x + 12 - x|x-3| &= 2x^2 - 9x + 12 - x(x-3) \\ &= 2x^2 - 9x + 12 - x^2 + 3x \\ &= x^2 - 6x + 12 \\ &= (x-3)^2 + 3 \\ &\geq 0 \end{aligned}$$

Hence for $x \geq 3$, $2x^2 - 9x + 12 - x|x-3| \geq 0$ implying that

$$x|x-3| \leq 2x^2 - 9x + 12.$$

2017 T1 3) (iv)

Prove that if x is a real number then

$$x|x-3| \leq 2x^2 - 9x + 12.$$

Case 2: For the second case, we take $x < 3$. Now consider

$$\begin{aligned} 2x^2 - 9x + 12 - x|x-3| &= 2x^2 - 9x + 12 + x(x-3) \\ &= 2x^2 - 9x + 12 + x^2 - 3x \\ &= 3x^2 - 12x + 12 \\ &= 3(x-2)^2 \\ &\geq 0 \end{aligned}$$

Hence for $x < 3$, $2x^2 - 9x + 12 - x|x-3| \geq 0$ implying that

$$x|x-3| \leq 2x^2 - 9x + 12.$$

Thus, $x|x-3| \leq 2x^2 - 9x + 12$. for all $x \in \mathbb{R}$.

2020 T2

For a sequence of integers $\{a_n\}_{n=1}^{\infty}$, we say that a_n is eventually even if and only if

$$\exists M \in \mathbb{Z} \quad \forall n > M \quad a_n \text{ is even.}$$

a) Write symbolically the statement "a_n is not eventually even", simplifying your answer so that the negation symbols are not used.

b) Prove that the sequence given by

$$a_n = 475 + \left\lfloor \frac{28n + 995}{n} \right\rfloor$$

is not eventually even.

a) Solution

For a sequence of integers $\{a_n\}_{n=1}^{\infty}$, we say that a_n is eventually even if and only if

$$\exists M \in \mathbb{Z} \quad \forall n > M \quad a_n \text{ is even.}$$

a) Write symbolically the statement "a_n is not eventually even", simplifying your answer so that the negation symbols are not used.

Let P be the statement "a_n is eventually even".

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Let P be the statement "a_n is eventually even". The statement "a_n is not eventually even" can be represented by

$$\sim P \;\equiv\; \sim (\exists M \in \mathbb{Z} \quad \forall n > M \quad a_n \text{ is even})$$

a) Solution

For a sequence of integers $\{a_n\}_{n=1}^{\infty}$, we say that a_n is eventually even if and only if

$$\exists M \in \mathbb{Z} \quad \forall n > M \quad a_n \text{ is even.}$$

a) Write symbolically the statement "a_n is not eventually even", simplifying your answer so that the negation symbols are not used.

Let P be the statement "a_n is eventually even". The statement "a_n is not eventually even" can be represented by

$$\begin{aligned} \sim P \; &\equiv \; \sim(\exists M \in \mathbb{Z} \quad \forall n > M \quad a_n \text{ is even}) \\ &\equiv \quad \forall M \in \mathbb{Z} \quad \exists n > M \quad a_n \text{ is odd.} \end{aligned}$$

b) Proof*

b) Prove that the sequence given by

$$a_n = 475 + \left\lfloor \frac{28n + 995}{n} \right\rfloor$$

is not eventually even. $\sim P \;\equiv\; \forall M \in \mathbb{Z} \quad \exists n > M \quad a_n$ is odd.

b) Proof*

b) Prove that the sequence given by

$$a_n = 475 + \left\lfloor \frac{28n + 995}{n} \right\rfloor$$

is not eventually even. $\sim P \quad \equiv \quad \forall M \in \mathbb{Z} \quad \exists n > M \quad a_n$ is odd.

b) Proof

For a sequence of integers $\{a_n\}_{n=1}^{\infty}$, we say that a_n is eventually even if and only if

$$\exists M \in \mathbb{Z} \quad \forall n > M \quad a_n \text{ is even.}$$

b) Prove that the sequence given by

$$a_n = 475 + \left\lfloor \frac{28n + 995}{n} \right\rfloor$$

is not eventually even.

To prove a_n is not eventually even, we have to prove $\sim P$ is true.
We are required to prove $\forall M \in \mathbb{Z} \quad \exists n > M \quad a_n$ is odd.
Notice $a_n = 475 + \left\lfloor 28 + \frac{995}{n} \right\rfloor$. We can make $a_n = 503$ which is odd if $0 \leq \frac{995}{n} < 1$. So we need $n > 995$ and $n > M$. So we can choose $n = M^2 + 996$ which will guarantee $n > 995$ and $n > M$.

b) Proof

<u>Proof:</u> Let $M \in \mathbb{Z}$. Choose $n = M^2 + 996 > M$. Then we have

$$\begin{aligned}
a_n &= 475 + \left\lfloor \frac{28n + 995}{n} \right\rfloor \\
&= 475 + \left\lfloor 28 + \frac{995}{n} \right\rfloor \\
&= 475 + \left\lfloor 28 + \frac{995}{M^2 + 996} \right\rfloor \\
&= 475 + 28 \text{ (since } 0 < \frac{995}{M^2 + 996} < 1) \\
&= 503.
\end{aligned}$$

Hence, a_n is odd and $\sim P$ is true. Thus, a_n is not eventually even. $\square$

2016 T1 Q1*

i) Construct a truth table for

$$(p \wedge q) \to (q \to p).$$

Is this a tautology, contradiction or contingency? Give reasons.

2016 T1 Q1*

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$$(p \wedge q) \to (q \to p).$$

Is this a tautology, contradiction or contingency? Give reasons.

p	q	$p \wedge q$	$q \to p$	$(p \wedge q) \to (q \to p)$
T	T	T	T	T
T	F	F	T	T
F	T	F	F	T
F	F	F	T	T

Since $(p \wedge q) \to (q \to p)$ is always true, this is a tautology.

2016 T1 Q1*

ii) Suppose you have a complete truth table for n propositions $p_1, p_2, ..., p_n$.
In how many of the 2^n lines of the truth table is $p_1 \to p_n$ true?

2016 T1 Q1*

ii) Suppose you have a complete truth table for n propositions $p_1, p_2, ..., p_n$.
In how many of the 2^n lines of the truth table is $p_1 \to p_n$ true?

Notice initially, half of p_1 is true and the other half is false.

Case 1: p_1 is false.
If p_1 is false, then $p_1 \to p_n$ will be true regardless of p_n. This can occur in $\frac{2^n}{2} = 2(2^{n-2})$ ways.

Case 2: p_1 is true.
If p_1 is true, then $p_1 \to p_n$ is only true when p_n is false. Notice that within the true p_1, half of them will correspond to a true p_n. Hence, this can occur in $\frac{\frac{2^n}{2}}{2} = 2^{n-2}$ ways.

Hence the total number of ways is $2(2^{n-2}) + 2^{n-2} = 3(2^{n-2})$ ways.

2016 T1 3) (iii)*

Consider the following arguments:

(1) I will have a good holiday only if I have both time and money.

(2) If I work then I will have no time.

(3) If I do not work, then I will have no money.

(4) Therefore, I will not have a good holiday.

a) Express this argument in symbolic logical form using the labels:

$$g = \text{"I will have a good holiday"}$$
$$t = \text{"I have time"}$$
$$m = \text{"I have money"}$$
$$w = \text{"I work"}.$$

2016 T1 3) (iii)

a) Solution

Consider the following arguments:

(1) I will have a good holiday only if I have both time and money.

(2) If I work then I will have no time.

(3) If I do not work, then I will have no money.

(4) Therefore, I will not have a good holiday.

$$\begin{aligned} &P_1 \;\; g \to (t \wedge m) \\ &P_2 \;\; w \to \sim t \\ &P_3 \sim w \to \sim m \\ &P_4 \sim g \end{aligned}$$

2016 T1 3) (iii)*

b) Show that P_4 can be logically deduced from the previous statements.

P_1 $g \to (t \wedge m)$
P_2 $w \to \sim t$
P_3 $\sim w \to \sim m$
P_4 $\sim g$

2016 T1 3) (iii)*

b) Show that P_4 can be logically deduced from the previous statements.

$P_1 \;\; g \to (t \wedge m)$
$P_2 \;\; w \to \; \sim t$
$P_3 \sim w \to \; \sim m$
$P_4 \sim g$
Consider $P_1 \wedge P_2 \wedge P_3$
$\equiv (w \to \; \sim t) \wedge (\sim w \to \; \sim m) \wedge (g \to (t \wedge m))$
$\equiv (w \vee \sim w) \wedge (w \to \; \sim t) \wedge (\sim w \to \; \sim m) \wedge (g \to (t \wedge m))$ (Union)
$\implies \; (\sim t \vee \sim m) \wedge (g \to (t \wedge m))$ (Case exhaustion)
$\equiv (\sim t \vee \sim m) \wedge ((\sim t \; \vee \sim m) \to \; \sim g)$ (Contrapositive)
$\implies \sim g$ (Modus ponens)
$\equiv P_4.$

Hence, P_4 can be logically deduced.

2017 T2 Q3 (v)*

Use induction to show that for all integers $n \geq 4$, there exists $a, b \in \mathbb{N}$ such that $n = 2a + 5b$.

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2017 T2 Q3 (v)*

Use induction to show that for all integers $n \geq 4$, there exists $a, b \in \mathbb{N}$ such that $n = 2a + 5b$.

Let $P(n)$ be the statement $n = 2a + 5b$ for some $a, b \in \mathbb{N}$.

Step 1: Prove $P(4)$ and $P(5)$ is true.

For $P(4)$, we have $P(4) = 4 = 2(2) + 5(0)$ where $0, 2 \in \mathbb{N}$. Hence, $P(4)$ is true.

For $P(5)$, we have $P(5) = 5 = 2(0) + 5(1)$ where $0, 1 \in \mathbb{N}$. Hence, $P(5)$ is true.

2017 T2 Q3 (v)

Use induction to show that for all integers $n \geq 4$, there exists $a, b \in \mathbb{N}$ such that $n = 2a + 5b$.

Step 2:

Assume $P(k)$ is true for all integers $k \geq 4$.
i.e There exists $a, b \in \mathbb{N}$ such that $k = 2a + 5b$.

We are now required to prove $P(k+2)$ is also true.
i.e There exists $A, B \in \mathbb{N}$ such that $k + 2 = 2A + 5B$.

$$\begin{aligned} k + 2 &= 2a + 5b + 2 \text{ (from induction hypothesis)} \\ &= 2(a+1) + 5b \\ &= 2A + 5B \end{aligned}$$

where $A = a + 1 \in \mathbb{N}$ and $B = b \in \mathbb{N}$. Hence, $P(k+2)$ is true.

2017 T2 Q3 (v)

Use induction to show that for all integers $n \geq 4$, there exists $a, b \in \mathbb{N}$ such that $n = 2a + 5b$.

Step 3:

Hence, because $P(4)$ and $P(5)$ is true from step 1 and $P(k+2)$ is true if $P(k)$ is true from step 2, then by the principle of mathematical induction, $P(n)$ is true for all integers $n \geq 4$. $\square$

5. Graphs

Types of paths and circuits

Definition: Euler Path

A path that passes through every edge of the graph G exactly once.
A connected graph G has an Euler path if and only if there are exactly 2 vertices of odd degree.

Definition: Euler Circuit

A circuit that passes through every edge of the graph G exactly once.
A connected graph G has an Euler circuit if and only if there are only even vertex degrees.

Types of paths and circuits

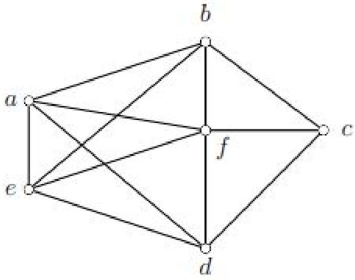
Definition: Hamilton Path

A path that passes through every vertex of the graph G exactly once.
No easy criteria for Hamilton's.
A Hamilton path or circuit uses at most 2 edges incident to each vertex.

Definition: Hamilton Circuit

A circuit that passes through every vertex of the graph G exactly once except for the first and last vertex.
If a graph G has a Hamilton circuit, then the circuit must include all edges incident with vertices of degree 2.
Let G be a connected and simple graph with $n \geq 3$ vertices, such that each vertex has degree at least $n/2$. Then G has a Hamilton circuit.

Example 1



2018 S2 Q2 iii)

Let G denote the graph shown above.

a) Does G have an Euler path? Explain your answer.

b) Does G have a Hamilton circuit? Explain your answer.

c) Is G bipartite? Explain your answer.

Example 1 Solutions

a)

G does have an Euler path. Why?

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G is connected.

G has exactly two odd vertices at c and f.

Example 1 Solutions

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b)
G does have a Hamilton circuit. Why?

Example 1 Solutions

a)

G does have an Euler path. Why?

G is connected.

G has exactly two odd vertices at c and f.

b)

G does have a Hamilton circuit. Why?

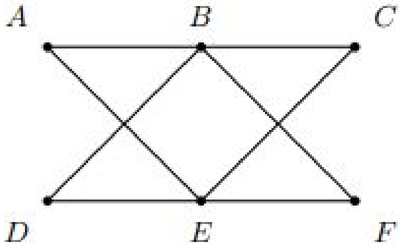
G is connected, simple and has more than 3 vertices.

G has 6 vertices and each vertex has degree greater than $\frac{6}{2} = 3$.

c)

We will return to c) later.

Example 2



2012 S2 Q2 iii) c) and d)

Let the graph above be K.

c) Show that K has an Euler circuit.

d) Show that K has no Hamiltonian circuit.

Example 2 Solutions

c)
K is connected.
K has only vertices with even degrees.

Example 2 Solutions

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d)
To show this, we shall combine two different results, as well as a proof by contradiction.

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To show this, we shall combine two different results, as well as a proof by contradiction.
Firstly, we have that, if K did contain a Hamiltonian circuit, it should use up all edges incident with vertices with degree of 2. In K, this would be the vertices: A, C, D, F.
This means edges AB, DB, FB, CB must be used.

Example 2 Solutions

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K is connected.
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To show this, we shall combine two different results, as well as a proof by contradiction.
Firstly, we have that, if K did contain a Hamiltonian circuit, it should use up all edges incident with vertices with degree of 2. In K, this would be the vertices: A, C, D, F.
This means edges AB, DB, FB, CB must be used.
Secondly, we have that a Hamiltonian circuit uses at most 2 edges incident to any vertex.
Thus, the vertex B leads to a contradiction as B has all 4 of its edges being used up.

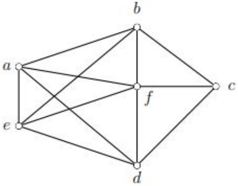
Bipartite Graph

Definition: Bipartite Graph

A graph G is bipartite if there are two disjoint sets V_1 and V_2 such that $V_1 \cup V_2 = V$ and every edge of G connects a vertex in V_1 to a vertex in V_2.

In loose terms, a graph is bipartite if every vertex in one subgroup only 'bridges' over to vertices in the other subgroup and not to its own.

Example 3



Return to 2018 S2 Q2 iii) c)

Let G denote the graph shown above.

a) Does G have an Euler path? Explain your answer.

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c) Is G bipartite? Explain your answer.

Example 3 Solutions

c)

G cannot be bipartite. Why?

Example 3 Solutions

c)

G cannot be bipartite. Why?

Let's try splitting the vertices into two groups!

Planar Graphs

Definition: Planar Graph

A graph is planar if you can draw it (in 2D!) without crossing edges.

Theorems for Planar Graphs

If G is a connected planar graph with e edges and $v \geq 3$ vertices, then
(1) $e \leq 3v - 6$,
(2) $e \leq 2v - 4$ (if G has no cycles of length 3).

Other Useful Theorems/Lemmas

Definition: Euler's Formula

If G is a connected planar graph with e edges and v vertices, and if r is the number of regions in a planar representation of G, then,

$$v - e + r = 2.$$

Definition: Handshaking Lemma

For any graph with vertices V and edges E:

$$2|E| = \sum_{v \in V} \deg(v).$$

For any planar graph with vertices V and regions R:

$$2|E| = \sum_{r \in R} \deg(r).$$

Example 4

2014 S2 Q2 ii)

A connected planar graph has vertices of degrees 6, 5, 5, 2, 2, 1, 1.

a) How many regions does the graph have?

b) Draw an example of such a planar graph.

Example 4 Solutions

a)
We are given the number of vertices and degrees of each vertex. We are trying to find number of regions. What can we do?

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Example 4 Solutions

a)

We are given the number of vertices and degrees of each vertex. We are trying to find number of regions. What can we do?

Notice we have most components of Euler's formula, and furthermore, Handshaking Lemma can be used to determine the missing piece, the number of edges.

By Handshaking Lemma, $2e = 6 + 5 + 5 + 2 + 2 + 1 + 1 \implies e = 11$.

Finally, by Euler's formula, $7 - 11 + r = 2 \implies r = 6$.

Example 4 Solutions

b)

Let's try draw it out!

Example 5

2014 S1 Q2 iv)

a) State Euler's Formula for a connected planar graph having v vertices, e edges and r regions.

b) Show that if G is a connected planar simple graph with $v \geq 3$ vertices, then $e \leq 3v - 6$.

c) Hence show that a connected planar simple graph with $v \geq 3$ vertices has at least one vertex of degree less than or equal to 6.

Example 5 Solutions

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$v - e + r = 2.$

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b)
The proof uses a few niche results. I will briefly describe it here, but it is more so knowing how to use the $e \leq 3v - 6$ theorem that it more important.
Typically, the theorem will be used to prove that a graph is **not** planar!

Example 5 Solutions

a)

$v - e + r = 2.$

b)

The proof uses a few niche results. I will briefly describe it here, but it is more so knowing how to use the $e \leq 3v - 6$ theorem that it more important.

Typically, the theorem will be used to prove that a graph is **not** planar! G is simple, connected, planar and has $v \geq 3$, therefore, each of its regions must have at least a degree of 3.

Handshaking Lemma tells us that $2e =$ total region degrees.

So, $2e \geq 3r \implies 2e \geq 3(2 + e - v)$.

Example 5 Solutions

c)

Clearly, trying to brute force this will be rather hard considering that 'at least' accounts for a lot of possibilities. Instead, we should consider proof by contradiction, as this turns the 'at least' statement into 'all vertices have degree greater than 6'.

Example 5 Solutions

c)

Clearly, trying to brute force this will be rather hard considering that 'at least' accounts for a lot of possibilities. Instead, we should consider proof by contradiction, as this turns the 'at least' statement into 'all vertices have degree greater than 6'.

Recall the other Handshaking Lemma. $2e =$ total vertex degrees.

Thus, by our assumption that 'all vertices have degree greater than 6', $2e \geq 6v \implies e \geq 3v$.

This leads to a contradiction, as $e \leq 3v - 6$.

Isomorphism and Homeomorphism

Definition: Isomorphism

Let G and G' be graphs with vertices V and V', and edges E and E'. Then G is isomorphic to G', if and only if there are two bijections $f : V \to V'$ and $g : E \to E'$, such that e is incident with v in G if and only if $g(e)$ is incident with $f(v)$ in G'.

Loosely speaking, this means that the two graphs are the same except for how the vertices are labelled.

Important Property of Isomorphic Graphs

If two graphs are isomorphic, they must have the same invariant properties.
Examples of invariant properties include: number of vertices and edges, total degree, number of vertices of a certain degree, etc.

Isomorphism and Homeomorphism

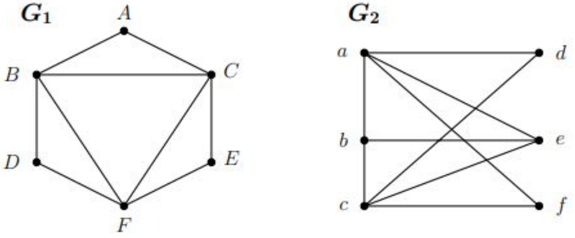
Definition: Homeomorphism

Two graphs are homeomorphic if and only if they can both be obtained from a common graph by elementary subdivisions.

Definition: Kuratowski's Theorem

A graph is planar if and only if it does not contain a subgraph homeomorphic to $K_{3,3}$ or K_5.

Example 6



2018 S1 Q2 iv) c)

c) Are G_1 and G_2 isomorphic? Explain why.

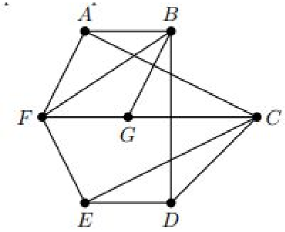
Example 6 Solutions

c)
Usually, unless its trivial to determine the isomorphism, you should begin by trying to **disprove** the existence of an isomorphism.

Example 6 Solutions

c)
Usually, unless its trivial to determine the isomorphism, you should begin by trying to **disprove** the existence of an isomorphism.
Run through some common invariants and see if there's any discrepancy between the two graphs.
In this case, notice that in G_2, vertices B has degree 3, whereas on G_1, no vertices have degree 3.

Example 7



2012 S2 Q2 iv)

a) State Kuratowski's theorem characterising non-planar graphs.

b) Show that the above graph is not planar.

Example 7 Solutions

a)

A graph is planar if and only if it does not contain a subgraph homeomorphic to $K_{3,3}$ or K_5 (as shown in lecture slides).

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Taking its contrapositive we get, a graph is **non-planar** if and only if it **does contain** a subgraph homeomorphic to $K_{3,3}$ or K_5.
b)
The idea would be to determine whether the above graph is a subgraph of $K_{3,3}$ or K_5.
This means we should in a sense, do the opposite of elementary subdivisions.
This is because rather than taking $K_{3,3}$ or K_5 and breaking it down to a subgraph, we are starting at the potential subgraph and are trying to build it back up to being a $K_{3,3}$ or K_5.

Example 7 Solutions

b)

Let's note some key features of $K_{3,3}$ and K_5. A $K_{3,3}$ has 6 vertices and 9 edges, whereas a K_5 has 5 vertices and 10 edges.

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The graph we have right now has 7 vertices and 11 edges. It seems it would be easier to attain the $K_{3,3}$ graph from our graph, than to try attain K_5.

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The graph we have right now has 7 vertices and 11 edges. It seems it would be easier to attain the $K_{3,3}$ graph from our graph, than to try attain K_5.

We are trying to seek a change to our graph that leads to a decrease of 1 vertex and 2 edges. Let's try remove vertex E, and replace the detour path FED with just FD.

Let's draw it out and see if we can form a $K_{3,3}$!

Adjacency Matrix

Definition: Adjacency Matrix

Let G be a graph with an ordered listing of vertices $v_1, v_2, ..., v_n$. The adjacency matrix of G is the $n \times n$ matrix $A = [a_{ij}]$ with a_{ij} being the number of edges connecting v_i and v_j.

Powers of the Adjacency Matrix

Given the A^k matrix, the ith row and jth column value of this matrix graphically represents the number of walks between v_i and v_j that is of length k.

Example 8

2013 S1 Q2 ii)

A graph H on the vertices A, B, C, D (in that order) has adjacency matrix:

$$M = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

a) Draw the graph H.

b) M^3 has the number 12 in position 1, 2 (that is first row, second column). What does that mean in terms of the graph?

Example 8 Solutions

a)
Let's try draw it out!

Example 8 Solutions

b)

Recall that the nth power of an adjacency matrix conveys information regarding the number of possible walks from one vertex to another that passes through n edges.

Example 8 Solutions

b)
Recall that the nth power of an adjacency matrix conveys information regarding the number of possible walks from one vertex to another that passes through n edges.
So for H, position 1, 2 refers to vertices A, B. This means that there are 12 possible walks from A to B that has length of 3.
Let's head back to our graph and see what these walks may look like.

Example 9

2012 S1 Q2 iii)

Consider the graph G consisting of the vertices of a cube and its edges.

a) Draw a planar representation of this graph, labelled with vertices 1, 2, 3, 4, 5, 6, 7, 8 so that the path 123456781 is a Hamiltonian circuit, and so that 1 and 6 correspond to opposite vertices in the cube.

b) Is the graph G bipartite?

c) With the above labelling, write down the first four rows of the adjacency matrix A of the graph.

d) Without computing the matrix product A^3 , compute the (1, 3) entry and the (1, 6) entry of A^3.

Example 9 Solutions

a)
Let's try draw it! Keep in mind that 'Hamiltonian' means passing through all vertices exactly once.

b)
Try this yourselves.

Example 9 Solutions

c)
Let's first define the adjacency matrix to have rows and columns ordered 1, 2, 3, ..., 8.
Refer back to our graph drawn above. We can fill in the entries one by one.

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{pmatrix}$$

Example 9 Solutions

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Let's first define the adjacency matrix to have rows and columns ordered 1, 2, 3, ..., 8.
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$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Example 9 Solutions

d)

Again, recall what the nth power of a matrix means.

Rather than calculating the powers of the 8 by 8 matrix, we can instead do this question more intuitively and graphically.

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Rather than calculating the powers of the 8 by 8 matrix, we can instead do this question more intuitively and graphically.

We are aiming to find the number of possible walks for vertex 1 to 3 and vertex 1 to 6 that are exactly of length 3.

Let's refer back to our graph and see if we can visually determine the solution.

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Rather than calculating the powers of the 8 by 8 matrix, we can instead do this question more intuitively and graphically.

We are aiming to find the number of possible walks for vertex 1 to 3 and vertex 1 to 6 that are exactly of length 3.

Let's refer back to our graph and see if we can visually determine the solution.

Turns out the (1, 3) entry is 0 and the (1, 6) entry is 6.

Weighted Trees

Definition: Tree

A tree is a simple connected graph with no circuits.

Definition: Spanning Tree

A spanning tree is a sub-tree that contains every vertex of graph.

Definition: Weighted Tree

A weighted tree is a tree whose edges have been given numbers called weights. The weight of an edge e is denoted by w(e).

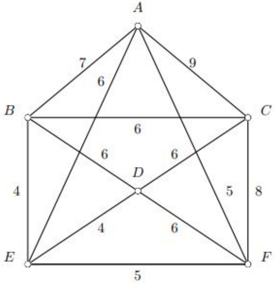
Kruskal's Algorithm

Definition: Kruskal's Algorithm

Kruskal's Algorithm can be used to find the minimal spanning tree. This means the spanning tree with lowest sum of weights.

1. Start with T, the empty tree.
2. Sort the edges by increasing weight.
3. Going down the list, add an edge to T if and only if it does not form a circuit with the existing edges.
4. Continue until T has $n-1$ edges.

Example 10



2013 S2 Q2 v)

a) Use Kruskal's algorithm to construct a minimal spanning tree T for the above weighted graph. Make a table showing the details of each step in your application of the algorithm.

Example 10 Solutions

a)

Recall Kruskal's Algorithm. We shall begin with an empty tree and build our way up to a minimal spanning tree by constructing a table with the weighted edges.

Edge	Weight	Chosen?
BE	4	
DE	4	
AF	5	
EF	5	
AE	6	
BC	6	
BD	6	
CD	6	
DF	6	

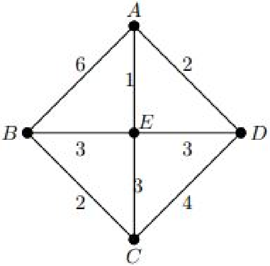
Dijkstra's Algorithm

Definition: Dijkstra's Algorithm

Dijkstra's Algorithm can be used to find the minimal spanning tree with respect to a particular vertex. This means, we can for example obtain the tree such that all the walks from a particular vertex A to any other vertex is the least in terms of weight.

1. Start with the subgraph T, contaning only the given vertex, A.
2. Consider all edges e incident to A.
3. Of these edges, choose an edge e with the lowest weight and note down the vertex, B, it leads to.
4. Add edge e and B to T.
5. Continue until T contains all the vertices of G.

Example 11



2014 S1 Q2 v)

a) Use Dijkstra's algorithm to find a spanning tree that gives the shortest paths from A to every other vertex of the graph. Make a table showing the details of each step in your application of the algorithm.

Example 11 Solutions

a)

Recall Dijkstra's algorithm. We start at the assigned vertex, in this case, A, and we build up the tree by constructing a table to consider the potential edge candidates.

Edge candidates	Next edge	Next vertex
$AE(1), AD(2), AB(6)$		
$AB(6), AD(2), EB(4), ED(4), EC(4)$		
$AB(6), DC(6), EB(4), EC(4)$		
$BC(6), DC(6), EC(4)$		