Game Theory (CS4187)

Lecture 38 (28/11/2024)

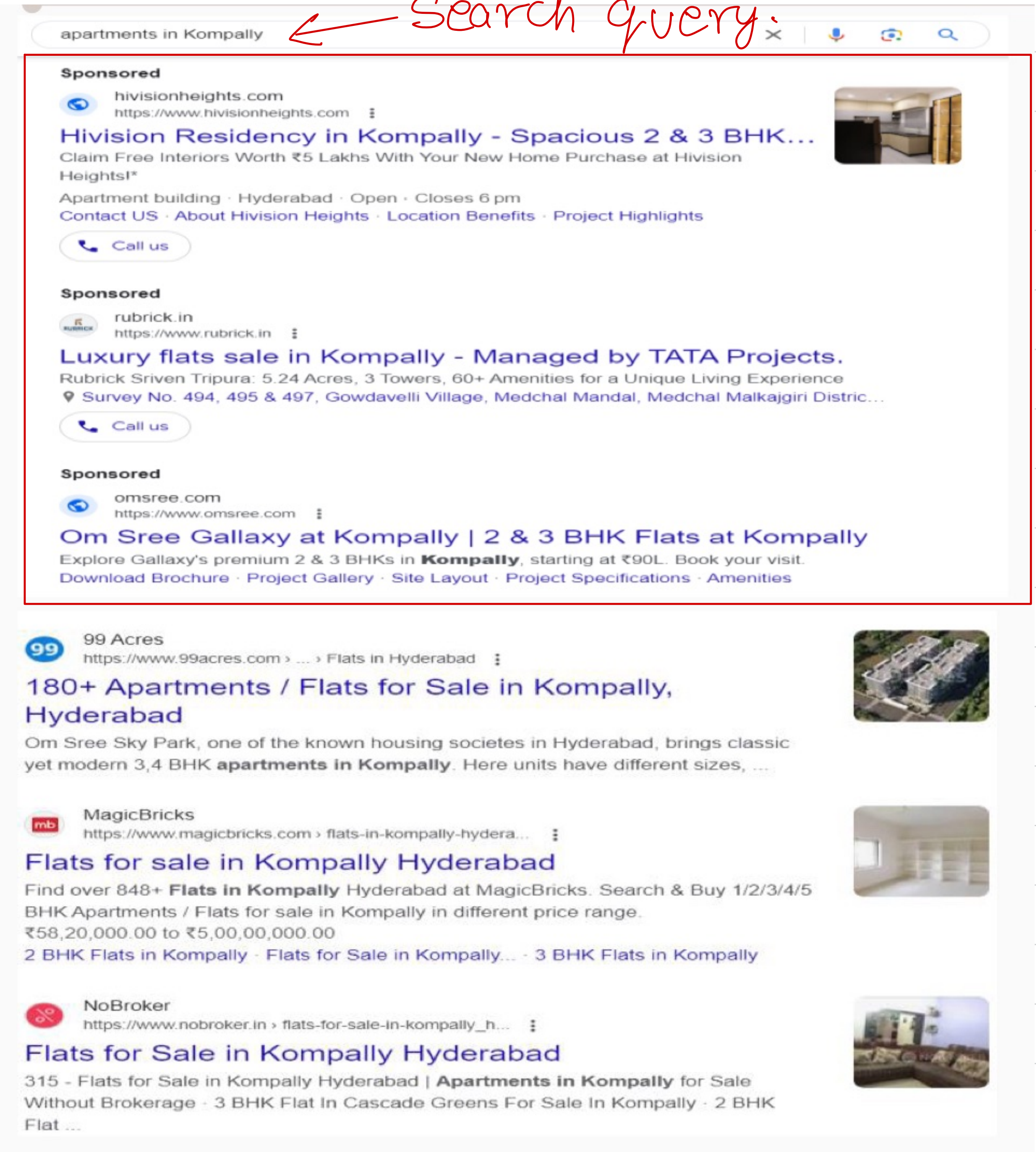
- I have recorded the following YouTube video for this lecture notes because I couldn't complete all the topics of this lecture in class:

https://youtu.be/y2JmkBuHBac

- Very important from exam perspective.

Sponsored Search Auctions

Search query:



Sponsored Ads.

Three Ad. Positions.

Sponsored Search Auctions

Setup:

- An user searches a query in a search engine.

- This search query gets linked to a set of advetisers,

$$N = \{1, 2, \dots, n\}$$

- There are m slots to advertise. Each adverliser can get atmost one slot.

- The search engine uses pay per click model, i.e. an advertiser has to pay only if its advertisement gets clicked; NOT for just showing the advertisement.

Sponsored Search Auctions

Setup:

- α_{ij} is the probability that a user will click on the i^{th} advertiser advertisement if it is in the j^{th} position ($j=1$ is the top-most position). For all $i \in N$, α_{ij} is non-increasing in j.

- When an user click the advertisement of i^{th} advertiser, it derives some value $\theta_i \in \mathbb{R}^+$ from it. θ_i may be the average money an user will spend on the advertiser's website. θ_i's are private information.

Sponsored Search Auctions

Setup:

> $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$. We have $\theta \in \Theta$. Also,

$$P(\Theta) = \prod_{i=1}^{n} P[\Theta_i]$$

Not needed for DSIC mechanisms. Why?

> **Assumptions**:

i) All clicks are valued equally no matter which position the advertisement is displayed, i.e. θ_i does not depend on advertisement position j. Valid?

Sponsored Search Auctions

Setup:

- $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$. We have $\theta \in \Theta$. Also,

$$P(\Theta) = \prod_{i=1}^{n} P[\Theta_i]$$

Not needed for DSIC mechanisms. Why?

- **Assumptions**:

  ii) α_{ij} are knowns to search engine. Valid?

  iii) α_{ij} does not depend on which other advertiser has been allocated to what other position.

VCG Mechanism For Sponsored Seach Auction

Formulation:

1) **Players:** Advertisers. Set of advertisers $N = \{1, 2, \dots, n\}$.

2) **Type:** θ_i which is the value to advertiser i if the advertisement is clicked.

The search engine does NOT know θ_i. Advertiser i bids b_i. If DSIC, then $b_i = \theta_i$.

VCG Mechanism For Sponsored Seach Auction

Formulation:

3) Outcome: Let $y_{i,j} \in \{0,1\}$ denote whether the j^{th} advertisement slot is allocated to the i^{th} user. Let Y be a matrix whose $(i,j)^{th}$ component is y_{ij}.

Let $p = (p_1, p_2, \ldots, p_n)$ where p_i is the payment of advertiser if an user clicks its advertisement.

$$X = \left\{ (Y, p) : y_{i,j} \in \{0,1\}, \forall i,j ; \sum_{i=1}^{n} y_{ij} \leq 1, \forall j ; \sum_{j=1}^{m} y_{ij} \leq 1, \forall i ; p_i \in \mathbb{R}, \forall i \right\}$$

VCG Mechanism For Sponsored Seach Auction

Formulation:

4) Utility Function:

$$U_i\big(\overbrace{(Y,P)}^{x},\ \theta_i\big) = \sum_{j=1}^{m}\Big(y_{ij}\,\alpha_{ij}\,(\theta_i - P_i)\Big)$$

Note: The utility function is NOT in quasi-linear form, i.e. $v_i(a, \theta_i) - P_i$! So, in order to use VCG mechanism, we have to first convert the utility function in quasi-linear form. That's what we do next.

VCG Mechanism For Sponsored Seach Auction

Formulation:

4) Utility Function:

$$U_i\big((\overset{x}{\overleftrightarrow{Y,P}}),\ \theta_i\big) = \sum_{j=1}^{m} \Big(y_{ij}\, \alpha_{ij}\, (\theta_i - P_i) \Big)$$

$$= \sum_{j=1}^{m} \Big(y_{ij}\, \alpha_{ij}\, \theta_i \Big) - \sum_{j=1}^{m} \Big(y_{ij}\, \alpha_{ij}\, P_i \Big)$$

$$= \theta_i \sum_{j=1}^{m} \big(y_{ij}\, \alpha_{ij} \big) - P_i \sum_{j=1}^{m} \big(y_{ij}\, \alpha_{ij} \big)$$

VCG Mechanism For Sponsored Seach Auction

Formulation:

4) Utility Function:

$$U_i(\overbrace{(Y,P)}^{x},\theta_i) = \theta_i \sum_{j=1}^{m} (y_{ij}\,\alpha_{ij}) - P_i \sum_{j=1}^{m} (y_{ij}\,\alpha_{ij})$$

$$= V_i(Y,\theta_i) - t_i \qquad \text{where}$$

$$\boxed{V_i(Y,\theta_i) = \theta_i \sum_{j=1}^{m} (y_{ij}\,\alpha_{ij})}\;, \quad \boxed{t_i = P_i \sum_{j=1}^{m} (y_{ij}\,\alpha_{ij})}$$

VCG Mechanism For Sponsored Seach Auction

Formulation:

4) Utility Function:

$$U_i(\overbrace{(Y, P)}^{x}, \theta_i) = V_i(Y, \theta_i) - t_i$$

The above utility function is in quasi-linear form. According to the above utility function, the outcome is Y and $t = (t_1, t_2, \ldots, t_n)$ and NOT Y and P. But wait! We finally need P_i's because P_i is what i^{th} advertiser has to pay. How do we get P_i?

VCG Mechanism For Sponsored Seach Auction

Formulation:

4) Utility Function:

$$U_i((\overbrace{Y, P}^{x}), \theta_i) = V_i(Y, \theta_i) - t_i$$

How do we get P_i?

We first compute Y and t_i. Then use the following to compute P_i:

$$t_i = P_i \sum_{j=1}^{m} (y_{ij} \, \alpha_{ij})$$

VCG Mechanism For Sponsored Seach Auction

Allocation Rule:

$$a^*(\theta) = \underset{a \in A}{\arg\max} \sum_{i \in N} v_i(a, \theta_i)$$

For sponsored search auction, the above formula becomes,

$$y^*(\theta) = \underset{y}{\arg\max} \sum_{i=1}^{n} v_i(y, \theta_i)$$

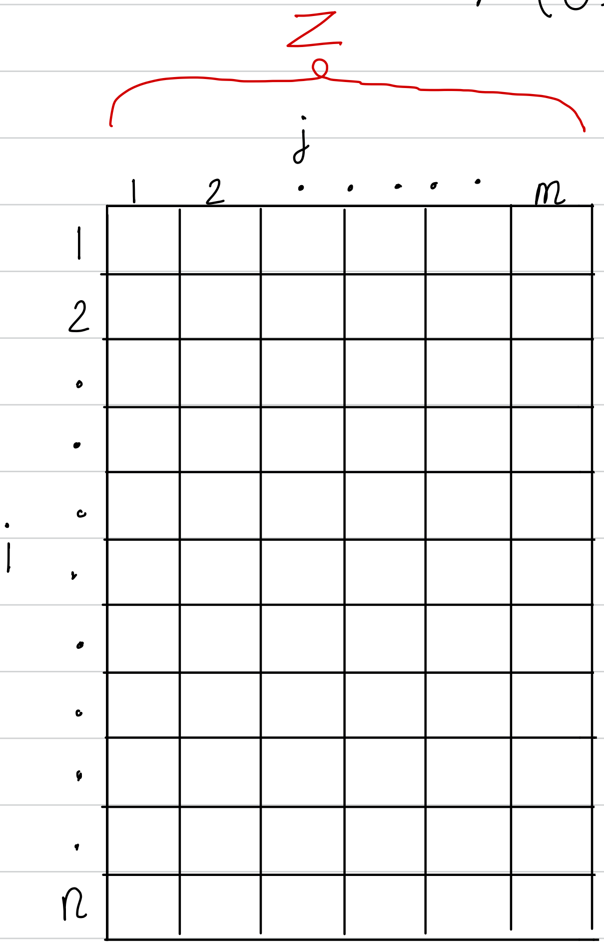
$$= \underset{y}{\arg\max} \sum_{i=1}^{n} \left(\theta_i \sum_{j=1}^{m} (y_{ij}\, \alpha_{ij}) \right)$$

$$= \underset{y}{\arg\max} \sum_{i=1}^{n} \sum_{j=1}^{m} (\theta_i \alpha_{ij})\, y_{ij}$$

VCG Mechanism For Sponsored Seach Auction

Allocation Rule:

$$Y^*(\theta) = \arg\max_{Y} \sum_{i=1}^{n} \sum_{j=1}^{m} (\theta_i \alpha_{ij})\, y_{ij}$$



Psuedocode:

1) Let Z be a matrix whose $(i,j)^{th}$ element $z_{i,j} = \theta_i \alpha_{ij}$. Set $y_{i,j} = 0,\ \forall i,j$.

2) For $j = 1, 2, \ldots, m$.

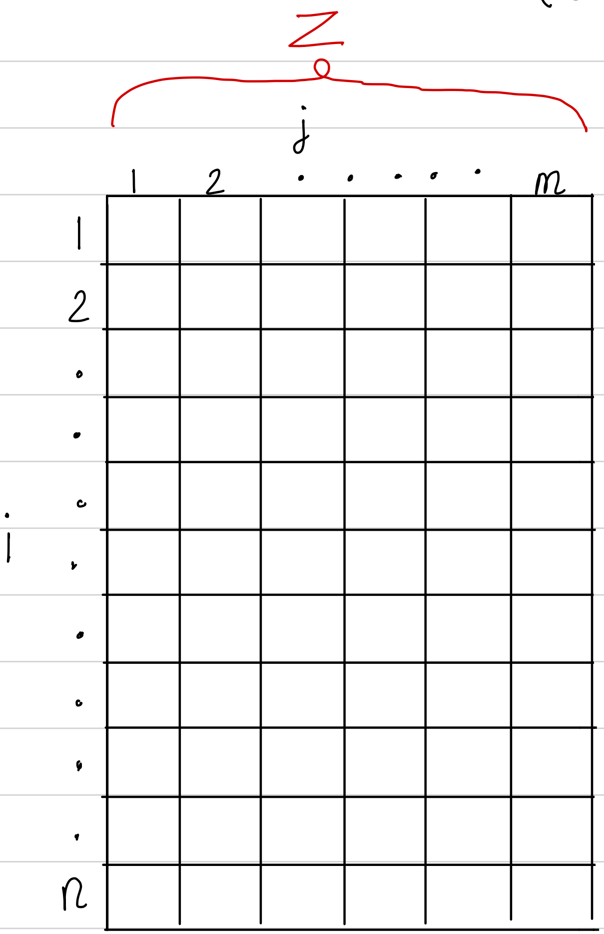
3) Find the largest element of Z. Let it be the element (i^*, j^*). Then set, $y_{i^*, j^*} = 1$.

4) Remove row i^* and column j^* from Z.

VCG Mechanism For Sponsored Seach Auction

Allocation Rule:

$$Y^*(\theta) = \underset{Y}{\arg\max} \sum_{i=1}^{n} \sum_{j=1}^{m} (\theta_i \alpha_{i,j})\, y_{i,j}$$



Remember: For all $i \in N$, $\alpha_{i,1} \geq \alpha_{i,2} \geq \cdots \geq \alpha_{i,m}$.

Psuedocode (better):

1) Set $S = N$. Set $y_{i,j} = 0$, $\forall i, j$.
2) For $j = 1, 2, \ldots, m$.
3) $\quad i^* = \underset{i \in S}{\arg\max}\ \theta_i \alpha_{i,j}$
4) $\quad$ Set $y_{i^*,j} = 1$. Also, $S = S \setminus \{i^*\}$.

VCG Mechanism For Sponsored Seach Auction

Allocation Rule:

$$Y^*(\theta) = \underset{Y}{\arg\max} \sum_{i=1}^{n} \sum_{j=1}^{m} (\theta_i \alpha_j)\, y_{ij}$$

> Special case: If $\alpha_{1,j} = \alpha_{2,j} = \cdots = \alpha_{n,j} = \alpha_j$, i.e. click probability just depend on the position and NOT on the advetiser.

Psuedocode (special case):

1) Sort θ_i's in descending order. Let $P(j)$ denote the advertiser in the j^{th} position of the sorted list.
2) Allocate j^{th} slot to $P(j)$.

VCG Mechanism For Sponsored Seach Auction

Allocation Rule:

$$Y^*(\theta) = \underset{Y}{\arg\max} \sum_{i=1}^{n} \sum_{j=1}^{m} (\theta_i \alpha_{ij}) y_{ij}$$

> Wait! we don't know θ_i's. Then how will any of these allocation rule work?

Answer: We use bids b_i's instead of θ_i's. Used θ_i because we are using direct mechanism.

VCG Mechanism For Sponsored Seach Auction

Pricing Rule:

$$p_i^*(\theta) = \left(\max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j) \right) - \sum_{j \neq i} v_j(a^*(\theta), \theta_j)$$

For sponsored search auction,

$$t_i^*(\theta) = \left(\max_{Y} \sum_{k \neq i} v_k(Y, \theta_k) \right) - \sum_{k \neq i} v_k(Y^*(\theta), \theta_k)$$

> Used k because j is used as ad position index.

VCG Mechanism For Sponsored Seach Auction

Pricing Rule:

$$t_i^*(\theta) = \left(\max_{Y} \sum_{k \neq i} V_k(Y, \theta_k) \right) - \sum_{k \neq i} V_k(Y^*(\theta), \theta_k)$$

Now, there can be two cases for player i:

1) **Case-1**: Player i is NOT allocated an ad slot.

2) **Case-2**: Player i is allocated an ad slot.

Let's deal with case-1 first.

VCG Mechanism For Sponsored Seach Auction

Pricing Rule:

$$t_i^*(\theta) = \overbrace{\left(\max_{Y} \sum_{k \neq i} V_k(Y, \theta_k) \right)}^{\text{Term-1}} - \overbrace{\sum_{k \neq i} V_k(Y^*(\theta), \theta_k)}^{\text{Term-2}}$$

Case-1: Player i is NOT allocated an ad slot.

- We prove in the next page that for case-1, term 1 and term 2 are equal.
- Hence, $t_i^*(\theta) = 0$. But, what about $P_i^*(\theta)$? According to,

$$t_i = P_i \sum_{j=1}^{m} (y_{ij} \alpha_{ij})$$

Any $P_i^*(\theta)$ works. Logically $P_i^*(\theta) = 0$.

$$\underbrace{\sum_{k \neq i} V_k(y^*(\theta), \theta_k)}_{\text{Term-2}} = \sum_{k \neq i} \sum_{j=1}^{m} (\theta_k a_{kj}) \, y^*_{kj}$$

For case-1, this term is ZERO.

$$= \sum_{k \neq i} \sum_{j=1}^{m} (\theta_k a_{kj}) \, y^*_{kj} + \overbrace{\underline{\sum_{j=1}^{m} (\theta_i a_{ij}) \, y^*_{ij}}}$$

$$= \sum_{k=1}^{n} V_k(y^*(\theta), \theta_k)$$

This is $\max_{Y} \sum_{k=1}^{n} V_k(Y, \theta_k)$ by definition of $Y^*(\theta)$.

$$= \max_{Y} \sum_{k=1}^{n} V_k(Y, \theta_k)$$

$$= \underbrace{\max_{Y} \sum_{k \neq i} V_k(Y, \theta_k)}_{\text{Term-1}}$$

Since the above term is ZERO.

VCG Mechanism For Sponsored Seach Auction

Pricing Rule:

$$t_i^*(\theta) = \underbrace{\left(\max_{Y} \sum_{k \neq i} v_k(Y, \theta_k) \right)}_{\text{Term-1}} - \underbrace{\sum_{k \neq i} v_k(Y^*(\theta), \theta_k)}_{\text{Term-2}}$$

Case-2: Player i is allocated an ad slot.

- When we are actually implementing a mechanism, we have to worry about its computational complexity.
- Computational complexity of the pricing rule is NOT high.
  - Once $Y^*(\theta)$ is computed, term-2 can be computed in $O(n)$ time.
  - To compute term-1, remove player i and solve the allocation problem again which has $O(nm)$ time complexity.

VCG Mechanism For Sponsored Seach Auction

Pricing Rule:

$$t_i^*(\theta) = \underbrace{\left(\max_{Y} \sum_{k \neq i} V_k(Y, \theta_k)\right)}_{\text{Term-1}} - \underbrace{\sum_{k \neq i} V_k(Y^*(\theta), \theta_k)}_{\text{Term-2}}$$

Case-2: Player i is allocated an ad slot.

- Here, and in exams we are interested in simplifying the payment rule (also allocation rule) as much as possible.
- Such simplification is only possible for the special case,

$$\alpha_{1,j} = \alpha_{2,j} = \cdots = \alpha_{n,j} = \alpha_j ,$$

- Recall, $P(j)$ denote the advertiser in the j^{th} position of the sorted list.

VCG Mechanism For Sponsored Seach Auction

Pricing Rule:

$$t_i^*(\theta) = \underbrace{\left(\max_{Y} \sum_{k \neq i} V_k(Y, \theta_k) \right)}_{\text{Term-1}} - \underbrace{\sum_{k \neq i} V_k(Y^*(\theta), \theta_k)}_{\text{Term-2}}$$

Case-2: Player i is allocated an ad slot.

> Then, for case-2, $P(\tilde{j}) = i$ for some $\tilde{j}$.

$$\sum_{k \neq i} V_k(Y^*(\theta), \theta_k) = \sum_{j=1}^{\tilde{j}-1} a_j \cdot \theta_{P(j)} + \boxed{\sum_{j=\tilde{j}+1}^{m} a_j \cdot \theta_{P(j)}}$$

Only valid when $m < n$. Think what changes when $m \geqslant n$.

$$\max_{Y} \sum_{k \neq i} V_k(Y, \theta_k) = \sum_{j=1}^{\tilde{j}-1} a_j \cdot \theta_{P(j)} + \boxed{\sum_{j=\tilde{j}}^{m} a_j \cdot \theta_{P(j+1)}}$$

VCG Mechanism For Sponsored Seach Auction

Pricing Rule:

$$t_i^*(\theta) = \overbrace{\left(\max_{Y} \sum_{k \neq i} V_k(Y, \theta_k) \right)}^{\text{Term-1}} - \overbrace{\sum_{k \neq i} V_k(Y^*(\theta), \theta_k)}^{\text{Term-2}}$$

Case-2: Player i is allocated an ad slot.

> Then, for case-2, $P(\tilde{j}) = i$ for some $\tilde{j}$.

$$\sum_{k \neq i} V_k(Y^*(\theta), \theta_k) = \sum_{j=1}^{\tilde{j}-1} a_j \cdot \theta_{P(j)} + \sum_{j=\tilde{j}+1}^{m} a_j \cdot \theta_{P(j)}$$

$$\max_{Y} \sum_{k \neq i} V_k(Y, \theta_k) = \sum_{j=1}^{\tilde{j}-1} a_j \cdot \theta_{P(j)} + \sum_{j=\tilde{j}}^{m} a_j \cdot \theta_{P(j+1)}$$

$P(\cdot)$ is for the original problem.
For $j \geq \tilde{j}$, bidders who were getting ad position $j+1$ will get ad position j when advertiser i is NOT there.

VCG Mechanism For Sponsored Seach Auction

Pricing Rule:

$$t_i^*(\theta) = \underbrace{\left(\max_{Y} \sum_{k \neq i} v_k(Y, \theta_k)\right)}_{\text{Term-1}} - \underbrace{\sum_{k \neq i} v_k(Y^*(\theta), \theta_k)}_{\text{Term-2}}$$

Case-2: Player i is allocated an ad slot.

> Then, for case-2, $P(\tilde{j}) = i$ for some $\tilde{j}$.

$$t_i^*(\theta) = \alpha_{\tilde{j}} \cdot \theta_{P(\tilde{j}+1)} + \sum_{j=\tilde{j}+1}^{m} \alpha_j \cdot \underbrace{\left(\theta_{P(j)} - \theta_{P(j+1)}\right)}_{\text{Positive quantity. Why?}}$$

$$p_i^*(\theta) = \frac{t_i^*(\theta)}{\alpha_{\tilde{j}}}$$

Why? Because $t_i = p_i \sum_{j=1}^{m} (y_{ij}\, \alpha_j)$.