Game Theory (CS 4187)

Lecture 35 (21/11/2024)

Lecture 36 (25/11/2024)

Lecture 37 (26/11/2024)

- In the previous lecture, we talked about a domain restriction called single peaked preference.

- In this lecture, we will mainly focus on another domain restriction called quasi-linear utility functions.

Quasi-Linear environment

- Recall that social choice function (SCF), f, is a mapping from type, θ, to outcome, x. We have, $f : \Theta \to X$.

- In quasi-linear environment, the outcome $x = (a, p)$ where:
  - a is the allocation. We have, $a \in A$.

  - $p = (p_1, p_2, \dots, p_n)$ where p_i is the price that player i has to pay. We have, $p_i \in \mathbb{R}$. This is why quasi-linear environments are mainly associated with mechanism design with money.

Quasi-Linear environment

- The utility function of player i in quasi-linear env. is a quasi-linear function with the following structure,

Linear in p_i

$$U_i((a,p),\theta_i) = V_i(a,\theta_i) - p_i$$

Note: Quasi-linear functions are those that are linear in atleast one of the variables.

Book uses +. This leads to minor differences between my and the books approach.

Quasi-Linear Utility Function is a Domain Restriction

> Consider two outcomes,

$$x = (a, (P_i, P_{-i}))$$

Same Different Same

$$x' = (a, (P_i', P_{-i}))$$

where $P_i \succ P_i'$.

> If $R_i = P$ in GS theorem for quasi-linear utility function, then as θ_i varies in Θ_i, BOTH these conditions must hold:

- There exist a $\theta_i \in \Theta_i$, such that $x \succ x'$.
- There exist a $\theta_i \in \Theta_i$, such that $x' \succ x$.

Quasi-Linear Utility Function is a Domain Restriction

- We have,

$$U_i((\overbrace{a,(p_i,p_{-i})}^{x}),\theta_i) = V_i(a,\theta_i) - p_i$$

$$U_i((\overbrace{a,(p_i',p_{-i})}^{x'}),\theta_i) = V_i(a,\theta_i) - p_i'$$

- No matter what θ_i is, if $p_i > p_i'$, then

$$U_i((\overbrace{a,(p_i,p_{-i})}^{x}),\theta_i) < U_i((\overbrace{a,(p_i',p_{-i})}^{x'}),\theta_i)$$

and hence,

$$x' \succ x \quad \forall\, \theta_i \in \Theta_i$$

- $x \succ x'$ is NOT possible for any θ_i and hence the quasi-linear utility function is a domain restriction.

Decomposition of Social Choice Function

- Since $x = (a, p)$, the SCF $f(\theta)$ can be interpreted as,

$$f(\theta) = \left(a(\theta), p(\theta)\right)$$

where,

$a(\theta)$ is the allocation rule.

$p(\theta) = \left(p_1(\theta), p_2(\theta), \ldots, p_2(\theta)\right)$ is the pricing rule.

<u>Note</u>: This decomposition of $f(\theta)$ in $a(\theta)$ and $p(\theta)$ is only in HINDSIGHT. As such, whenever we are tryin to solve an optimization problem in say x and y, we can't optimize x and y separately.

Desirable properties of Pricing Rule

- Weak Budget Balance (WBB):

$$\sum_{i \in N} p_i(\theta) \geq 0 \; ; \quad \forall \theta \in \Theta$$

- Strong Budget Balance (SBB):

$$\sum_{i \in N} p_i(\theta) = 0 \; ; \quad \forall \theta \in \Theta$$

By default, the phrase "budget balance" means SBB.

- $p_i(\theta) \geq 0 \; ; \quad \forall \theta \in \Theta$

Examples of Allocation Rule

1) **Dictatorial rule**: Consider a player $d \in N$. Then dictatorial allocation rule is,

$$a(\theta_d, \theta_{-d}) = \underset{a \in A}{\operatorname{argmax}} \; V_d(a, \theta_d)$$

2) **Allocation efficient rule**:

$$a(\theta) = \underset{a \in A}{\operatorname{argmax}} \sum_{i \in N} V_i(a, \theta_i)$$

3) **Max-min rule (fair rule)**:

$$a(\theta) = \underset{a \in A}{\operatorname{argmax}} \left(\min_{i \in N} \; V_i(a, \theta_i) \right)$$

Desirable Properties of SCF

The social choice function $f(\theta) = (a(\theta), p(\theta))$ should have the following properties:

- Dominant strategy incentive compatible (DSIC).
- Ex-post efficient (EPE). → Also called Pareto-optimality.
  - Not same as allocative efficiency (AE).
- Non-dictatorial. We will use DI for dictatorial.
- Weak budget balance (WBB).

Assumption

> We assume that the set of outcome is limited to those that can result from pricing rules that are WBB.

$$\bar{X} = \{(a, p_1, p_2, \ldots, p_n) : a \in A, \sum_{i \in N} p_i \geq 0\}$$

Lemma 1 : If there are atleast two players, then NO SCF in quasi-linear environment that is DI.

Lemma 2 : A SCF is EPE in quasi-linear environment if and only if AE and budget balanced (BB). ← means SBB

- In the next 9 pages, there is a ROUGH proof of Lemma 1 and Lemma 2 that we discussed during lecture.
- The proofs are there in chapter 18 of the book (lemma 18.1 and 18.2). Be aware that book uses

$$U_i((a,P),\theta_i) = V_i(a,\theta_i) + P_i$$

Our definition uses minus (−).

This leads to minor changes in the proof.

Lemma 1 : If there are atleast two players, then NO SCF in quasi-linear environment that is DI.

Proof:

$$u_i((a,p),\theta_i) = v_i(a,a_i) - p_i$$

$$d \in N \Rightarrow \theta_d \Rightarrow (a_d, p_d)$$

$$v_i(a,\theta_i) - p_i$$

<u>Lemma 2</u> : A SCF is EPE in quasi-linear environment if and only if AE and budget balanced (BB). ← means SBB

<u>Proof</u> : AE and BB $\Rightarrow$ EPE.

Consider $\theta \in \Theta$,

$$\sum_{i \in N} u_i(f(\theta), \theta_i) = \sum_{i \in N} \Big(v_i(a(\theta), \theta_i) - p_i(\theta) \Big)$$

$$= \sum_{i \in N} v_i(a(\theta), \theta_i) - \overbrace{\sum_{i \in N} p_i(\theta)}^{0}$$

<u>Lemma 2</u> : A SCF is EPE in quasi-linear environment if and only if AE and budget balanced (BB). ← means SBB

<u>Proof</u> : AE and BB $\Rightarrow$ EPE.

Consider $\theta \in \Theta$,

$$\sum_{i \in N} u_i(f(\theta), \theta_i) = \sum_{i \in N} v_i(a(\theta), \theta_i)$$

$$\geq \sum_{i \in N} v_i(a, \theta_i) \quad \forall a \in A$$

$$\geq \sum_{i \in N} v_i(a, \theta_i) - \sum_{i \in N} p_i \; ; \; \forall a \in A, \; \forall p_i$$

Lemma 2 : A SCF is EPE in quasi-linear environment if and only if AE and budget balanced (BB). means SBB

Proof : AE and BB $\Rightarrow$ EPE.

Consider $\theta \in \Theta$,

$$\sum_{i \in N} u_i(f(\theta), \theta_i) \geq \sum_{i \in N} v_i(a, \theta_i) - \sum_{i \in N} p_i \; ; \forall a \in A, \forall p_i$$

$$\geq \sum_{i \in N} u_i(\overbrace{(a, p)}^{x}, \theta_i) \; ; \forall a \in A, \forall p_i$$

Lemma 2 : A SCF is EPE in quasi-linear environment if and only if AE and budget balanced (BB). ← means SBB

Proof : EPE $\Rightarrow$ AE and BB

This is same as proving !(AE and BB) $\Rightarrow$! EPE.

! AE or ! BB $\Rightarrow$! EPE

$\boxed{\begin{array}{l} !AE \Rightarrow !EPE \\ !BB \Rightarrow !EPE \end{array}}$

Lemma 2 : A SCF is EPE in quasi-linear environment if and only if AE and budget balanced (BB). ← means SBB

Proof : $\neg AE \Rightarrow \neg EPE$

If $\neg AE$, then there exist $\theta \in \Theta$ and $b \in A$

$$\sum_{i \in N} v_i(b, \theta_i) > \sum_{i \in N} v_i(a(\theta), \theta_i)$$

Define,

$$\varepsilon = \sum_{i \in N} v_i(b, \theta_i) - \sum_{i \in N} v_i(a(\theta), \theta_i)$$

<u>Lemma 2</u> : A SCF is EPE in quasi-linear environment if and only if AE and budget balanced (BB). ← means SBB

<u>Proof</u> : $!AE \Rightarrow !EPE$

$$\varepsilon = \sum_{i \in N} v_i(b, \theta_i) - \sum_{i \in N} v_i(a(\theta), \theta_i)$$

$$a'(\theta) = a(\theta)$$

$$p_i'(\theta) = p_i(\theta) + v_i(b, \theta_i) - v_i(a(\theta), \theta_i) - \frac{\varepsilon}{n}$$

$$U_i((a(\theta), p(\theta)), \theta_i) = V_i(a(\theta), \theta_i) - P_i(\theta)$$

$$U_i((b, p'(\theta)), \theta_i) = V_i(b, \theta_i) - \left(P_i(\theta) + V_i(b, \theta_i) - V_i(a(\theta), \theta_i) - \frac{\varepsilon}{n}\right)$$

$$= \cancel{V_i(b, \theta_i)} - \cancel{V_i(b, \theta_i)} + V_i(a(\theta), \theta_i) - P_i(\theta) + \frac{\varepsilon}{n}$$

Lemma 2 : A SCF is EPE in quasi-linear environment if and only if AE and budget balanced (BB). ← means SBB

Proof: $\neg BB \Rightarrow \neg EPE$

- Lemmas 1 and 2 addresses two of the desirable properties: i) EPE, and ii) Non-DI.

- We still have to address DSIC. This is where Groves mechanism comes into picture (name Groves is from VCG mechanisms; 2^{nd} price auctions is a special case of VCG mechanisms).

Groves Mechanism

- Recall that outcome is $x = (a, p)$. Consequently, the SCF

  Allocation, Pricing

  $f(\theta) = (a(\theta), p(\theta))$

  Allocation rule, Pricing Rule

- We want to find $a(\theta)$ and $p(\theta)$ such that the SCF is DSIC.

- Say that we fix our allocation rule as allocation efficient rule, i.e.

$$a^*(\theta) = \underset{a \in A}{\operatorname{argmax}} \sum_{i \in N} V_i(a, \theta_i)$$

- Then we ask if there is a pricing rule $p(\theta)$ such that the SCF $f(\theta) = (a^*(\theta), p(\theta))$ is DSIC.

- Why did we choose allocation efficient rule?

  Answer: It is one of the two conditions that is required for EPE (EPE is one of the desirable properties of an SCF in quasi-linear env.). The other condition is budget balance.

- Now we find $p(\theta)$ so that $f(\theta) = (a^*(\theta), p(\theta))$ is DSIC.

- Now $P(\theta) = (P_1(\theta), P_2(\theta), \ldots, P_n(\theta))$. Consider the following payment rule for player i,

$$P_i^*(\theta) = -\sum_{j \neq i} V_j(a^*(\theta), \theta_j) + h_i(\theta_{-i})$$

where $h_i : \Theta_{-i} \to \mathbb{R}$ is an arbitrary function and $a^*(\theta)$ is the allocation efficient rule.

- Groves Class of mechanisms,

$$a^*(\theta) = \underset{a \in A}{\operatorname{argmax}} \sum_{i \in N} V_i(a, \theta_i)$$

$$P_i^*(\theta) = -\sum_{j \neq i} V_j(a^*(\theta), \theta_j) + h_i(\theta_{-i})$$

Important Proof.

Theorem 1: Groves class of mechanisms are DSIC.

Proof: The condition for DSIC is that $\forall \theta_i, \forall \theta_i', \forall \theta_{-i}, \forall i$,

$$u_i(\theta, f(\theta_i, \theta_{-i})) \geq u_i(\theta, f(\theta_i', \theta_{-i}))$$

For quasi-linear env. if we use Grove's mechanism, proving above inequality will be same as proving the following $\forall \theta_i, \forall \theta_i', \forall \theta_{-i}, \forall i$,

$$v_i(a^*(\theta_i, \theta_{-i}), \theta_i) - p_i^*(\theta_i, \theta_{-i}) \geq v_i(a^*(\theta_i', \theta_{-i}), \theta_i) - p_i^*(\theta_i', \theta_{-i})$$

Now, by substituting the expressions for $p_i^*(\theta) = p_i^*(\theta_i, \theta_{-i})$

in the LHS of the last equation in the previous page we get,

$$v_i(a^*(\theta_i, \theta_{-i}), \theta_i) + \sum_{j \neq i} v_j(a^*(\theta_i, \theta_{-i}), \theta_j) - h_i(\theta_{-i})$$

$$= \sum_{j \in N} v_j(a^*(\theta_i, \theta_{-i}), \theta_j) - h_i(\theta_{-i})$$

By the definition of allocation eff. mechanism $a^*(\theta)$ we have,

$$\sum_{j \in N} v_j(a^*(\theta_i, \theta_{-i}), \theta_j) \geq \sum_{j \in N} v_j(a^*(\theta_i', \theta_{-i}), \theta_j)$$

in the LHS of the last equation in the previous page we get,

$$v_i(a^*(\theta_i, \theta_{-i}), \theta_i) + \sum_{j \neq i} v_j(a^*(\theta_i, \theta_{-i}), \theta_j) - h_i(\theta_{-i})$$

$$= \sum_{j \in N} v_j(a^*(\theta_i, \theta_{-i}), \theta_j) - h_i(\theta_{-i})$$

$$\geq \sum_{j \in N} v_j(a^*(\theta_i', \theta_{-i}), \theta_j) - h_i(\theta_{-i})$$ (By def. of allocation eff. rule.)

$$= v_i(a^*(\theta_i', \theta_{-i}), \theta_i) - \underbrace{\left(-\sum_{j \neq i} v_j(a^*(\theta_i', \theta_{-i}), \theta_j) + h_i(\theta_{-i})\right)}_{p_i^*(\theta_i', \theta_{-i})}$$

Vickery
Clark's
Groves

VCG Mechanisms

Groves class of mechanisms,

$$a^*(\theta) = \underset{a \in A}{\operatorname{argmax}} \sum_{i \in N} v_i(a, \theta_i)$$

$$p_i^*(\theta) = -\sum_{j \neq i} v_j(a^*(\theta), \theta_j) + h_i(\theta_{-i})$$

- Groves class of mehanisms are called so because based on the $h_i(\theta_{-i})$ choosen, we can get different mechanisms.
- One of such mechanisms is called VCG mechanisms.

<u>Note</u>: In book it is called Clarke's Pivotal mechanism.

- In VCG mechanism, we choose $h_i(\theta_{-i})$ as follows,

$$h_i(\theta_{-i}) = \max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j)$$

- So essentially the VCG mechanism is,

$$\boxed{\begin{aligned} a^*(\theta) &= \underset{a \in A}{\operatorname{argmax}} \sum_{i \in N} v_i(a, \theta_i) \\ p_i^*(\theta) &= \left(\max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j) \right) - \sum_{j \neq i} v_j(a^*(\theta), \theta_j) \end{aligned}}$$

$$a^*(\theta) = \underset{a \in A}{\operatorname{argmax}} \sum_{i \in N} v_i(a, \theta_i)$$

$$p_i^*(\theta) = \overbrace{\left(\max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j) \right)}^{\text{Term-1}} - \overbrace{\sum_{j \neq i} v_j(a^*(\theta), \theta_j)}^{\text{Term-2}}$$

> **Intuition**: From player i's perspective:

- Term-1: Maximum welfare of other players when player i is NOT participating.
- Term-2: Welfare of other players when player i is participating.
- So, Term-1 minus Term-2 is the loss of welfare of other players is player i participates. Player i has to pay $p_i^*(\theta)$ for this loss.

Weak Budget Balance of VCG Mechanism

$$a^*(\theta) = \underset{a \in A}{\operatorname{argmax}} \sum_{i \in N} V_i(a, \theta_i)$$

$$P_i^*(\theta) = \left(\max_{a \in A} \sum_{j \neq i} V_j(a, \theta_j) \right) - \sum_{j \neq i} V_j(a^*(\theta), \theta_j)$$

Two of the desirable properties of Pricing rule are:

1) Weak Budget Balance (WBB): $\sum_{i \in N} P_i(\theta) \geq 0 \;;\; \forall \theta \in \Theta$

2) $P_i(\theta) \geq 0 \;;\; \forall \theta \in \Theta,\; \forall i \in N$

Do VCG mechanism satisfy these two properties?

Answer in next page?

Weak Budget Balance of VCG Mechanism

$$a^*(\theta) = \underset{a \in A}{\operatorname{argmax}} \sum_{i \in N} v_i(a, \theta_i)$$

$$p_i^*(\theta) = \underline{\left(\max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j)\right)} - \underline{\sum_{j \neq i} v_j(a^*(\theta), \theta_j)}$$

Is $p_i(\theta) \geq 0$; $\forall \theta \in \Theta$, $\forall i \in N$?

Answer: This term will definitely be greater than or equal to this term because because maximum of a function $f(x)$ is always greater than the value of $f(x)$ for any x.

Weak Budget Balance of VCG Mechanism

$$a^*(\theta) = \underset{a \in A}{\operatorname{argmax}} \sum_{i \in N} v_i(a, \theta_i)$$

$$p_i^*(\theta) = \left(\max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j) \right) - \sum_{j \neq i} v_j(a^*(\theta), \theta_j)$$

Is the payment rule WBB?

Answer: Since $p_i(\theta) \geq 0$ for all $i \in N$ and all $\theta \in \Theta$ then obviously,

$$\sum_{i \in N} p_i(\theta) \geq 0 \; ; \; \forall \theta \in \Theta$$

Hence, WBB holds for VCG mechanism.

Generality of Quasi-Linear Environment

- Mostly used in mechanism design using money.
- Outcome $x = (a, p)$

$$x = (\underbrace{a}_{\text{Allocation}}, \underbrace{p}_{\text{Pricing}})$$

- Quasi-Linear Utility:

$$U_i((a,p), \theta_i) = V_i(a, \theta_i) - p_i$$

- Generic utility function in mechanism design:

$$U_i(x, \theta_i)$$

Generality of Quasi-Linear Environment

Groves class of mechanisms,

$$a^*(\theta) = \underset{a \in A}{\operatorname{argmax}} \sum_{i \in N} v_i(a, \theta_i)$$

$$p_i^*(\theta) = -\sum_{j \neq i} v_j(a^*(\theta), \theta_j) + h_i(\theta_{-i})$$

VCG Mechanism

$$a^*(\theta) = \underset{a \in A}{\operatorname{argmax}} \sum_{i \in N} v_i(a, \theta_i)$$

$$p_i^*(\theta) = \left(\max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j) \right) - \sum_{j \neq i} v_j(a^*(\theta), \theta_j)$$

Generality of Quasi-Linear Environment

- Outcome $x = (\underset{\text{Allocation}}{a}, \underset{\text{Pricing}}{p})$

- Main point:

i) a which is called "allocation" should NOT be interpreted literally, i.e. we are trying to allocate some goods. a is nothing but an outcome.

ii) The mehanism designer decides the outcome a for individual type profile.

iii) Pricing rule makes the mechanism DSIC.

Example of VCG Mechanism

- We will consider the example of auctioning a single, indivisible item. We will show that in this case VCG mechanism reduces to second-price auctions.

- Players: The bidders. There are n bidders who are bidding for a single item.

- Type: θ_i which is the value that bidder i is TRULY willing to pay this item. Any price over θ_i, then bidder i is definitely NOT interested. Player i's bid is b_i. Auctioneer knows θ_i through b_i (direct mechanism).

Example of VCG Mechanism

- **Outcome:** Let $y_i \in \{0,1\}$ implies whether bidder i is allocated the item. Let y be a vector whose i^{th} element is y_i.
  Let $p_i \in \mathbb{R}$ be the amount bidder i has to pay. Let p be vector whose i^{th} element is p_i.

$$X = \left\{ (y,p) : y_i \in \{0,1\}, \forall i; \sum_{i=1}^{n} y_i \leq 1; \; p_i \in \mathbb{R}, \forall i \right\}$$

- **Utility Function:**

$$U_i\big(\overbrace{(y,p)}^{x}, \theta_i\big) = \underbrace{\theta_i y_i}_{v_i(a,\theta_i)} - p_i$$

Example of VCG Mechanism

> **Allocation Rule:**

$$a^*(\theta) = \underset{a \in A}{\operatorname{argmax}} \sum_{i \in N} v_i(a, \theta_i)$$

For single indivisible item auction, the above formula becomes,

$$y^*(\theta) = \underset{y}{\operatorname{argmax}} \sum_{i=1}^{n} \theta_i y_i$$

> The above optimization problem simplifies to:

$$y_i^*(\theta) = \begin{cases} 1 \ ; & i = \underset{i \in N}{\operatorname{argmax}} \ \theta_i \\ 0 \ ; & \text{otherwise} \end{cases}$$

} Allocate to highest θ_i (or highest b_i if DSIC).

Example of VCG Mechanism

- Pricing Rule:

$$p_i^*(\theta) = \left(\max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j)\right) - \sum_{j \neq i} v_j(a^*(\theta), \theta_j)$$

For single indivisible item auction, the above formula becomes,

$$p_i^*(\theta) = \overbrace{\left(\max_{y} \sum_{j \neq i} \theta_j y_j\right)}^{\text{Term-1}} - \overbrace{\sum_{j \neq i} \theta_j y_j^*(\theta)}^{\text{Term-2}}$$

Example of VCG Mechanism

- **Pricing Rule:**

$$p_i^*(\theta) = \overbrace{\left(\max_{y} \sum_{j \neq i} \theta_j y_j \right)}^{\text{Term-1}} - \overbrace{\sum_{j \neq i} \theta_j y_j^*(\theta)}^{\text{Term-2}}$$

Now, there are two cases: Case-1 (Player i is allocated the item)

- $y_i^*(\theta) = 1$ and $y_j^*(\theta) = 0 \ \forall j \neq i$. Hence, Term-2 is ZERO.

- Player i has the highest bid. Hence, the maximum value of Term-1 is the second highest bid (Second highest θ_i if DSIC).

- So, Term-1 minus Term-2 is the second highest bid which Player i has to pay if it is allocated the item.

Example of VCG Mechanism

> **Pricing Rule:**

$$p_i^*(\theta) = \overbrace{\left(\max_{y} \sum_{j \neq i} \theta_j y_j \right)}^{\text{Term-1}} - \overbrace{\sum_{j \neq i} \theta_j y_j^*(\theta)}^{\text{Term-2}}$$

Now, there are two cases: Case-2 (Player i is NOT allocated the item)

- $y_i^*(\theta) = 0$. But there exist a player $j \neq i$ who has the highest θ_j and hence $y_j^*(\theta) = 1$. This implies that Term-2 is the highest θ_j.

- For Term-1, Player i is NOT participating. But player i didn't have the highest θ_i anyway (thats why $y_i^*(\theta) = 0$). Hence Term-1, is also the highest θ_j. Term-1 minus Term-2 is ZERO.