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Game Theory (CS4187)
Lectures 3 and 4

Date: 13/08/2024 and 19/08/2024
Instructor: Gourav Saha

NOT THE FINAL VERSION!

Do take lecture notes! No additional lecture notes will be provided other than the slides.

Overall Agenda

- The idea of today’s lecture is simultaneous move games.

- The main agenda of today’s lecture is to formulate simultaneous move games.

Contents of this lecture

1. The Prisoner’s dilemma.
2. Definition of strategic form games.
3. Examples of simultaneous move games.
4. Toy examples are their practical relevance.

The Prisoner's Dilemma

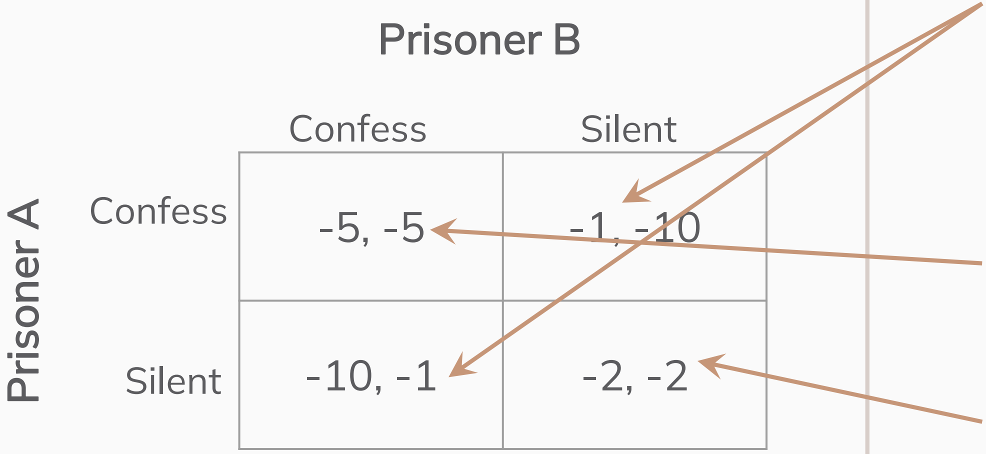
- Prisoner's dilemma is perhaps the most popular game. It's setup is as follows:

There are two prisoner's A and B who have **allegedly** committed a crime. An interrogator **privately** tells each of the prisoners that:

1. If he is the only one to confess, he will get a light sentence of 1 year while the other prisoner gets 10 years.
2. If both prisoner's confess, they each get 5 years jail time.
3. If neither prisoner's confess, they each get 2 years jail time.
4. The interrogator told these same rules to the other prisoner (why this?).

- The prisoners have to **simultaneously decide** (a prisoner has to decide its action without knowing the decision of the other prisoner) whether they want to **confess** or **stay silent**.

The Prisoner’s Dilemma



		Prisoner B	
		Confess	Silent
Prisoner A	Confess	-5, -5	-1, -10
	Silent	-10, -1	-2, -2

- The important rules:
  - i. If he is the only one to confess, he will get a light sentence of 1 year while the other prisoner gets 10 years.
  - ii. If both prisoner’s confess, they each get 5 years jail time.
  - iii. If neither prisoner’s confess, they each get 2 years jail time.
- In the figure shown, the utilities of the player is written in a matrix form.

The Prisoner’s Dilemma

		Prisoner B	
		Confess	Silent
Prisoner A	Confess	-5, -5	-1, -10
	Silent	-10, -1	-2, -2

The few important things to note:

- **Minus sign** is included because we want to **maximize payoff**.
- When writing the payoffs in “matrix form”, **row player’s payoff comes first**, followed by the payoff of the column player.

What are Strategic Form Games?

- Strategic form games are basically **simultaneous move games** where all the players **act simultaneously** **without the knowledge of the decisions/actions of the other players**.

- The are also called **static games**, **normal form games**.

- So, **(i)** simultaneous move games, **(ii)** static games, **(iii)** strategic form games, and **(iv)** normal form games are all **synonymous**.

What are Strategic Form Games?

Definition: A strategic form game (SFG) Γ is a tuple, $\left\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \right\rangle$ where,

1. $\mathrm{N} = \{1, 2, \cdots, \mathrm{n}\}$ is the **set of players**.
2. $\mathrm{S_i}$ is the **set of actions** of the $\mathrm{i^{th}}$ player. The action of the $\mathrm{i^{th}}$ player, denoted by s_i belongs to set $\mathrm{S_i}$, i.e. $s_i \in \mathrm{S_i}$.
3. $\mathrm{u_i} : \mathrm{S_1} \times \mathrm{S_2} \times \cdots \times \mathrm{S_n} \rightarrow \mathbb{R}$ is the utility function of the $\mathrm{i^{th}}$ player. To elaborate, the payoff the $\mathrm{i^{th}}$ player is $u_i(s_1, s_2, \cdots, s_n)$.

- The **outcome** (also called **strategy profile**) is $\mathrm{s} = (s_1, s_2, \cdots, s_n)$, i.e. the decision made by all the players.

What are Strategic Form Games?

Example: For the prisoner's dilemma:

1. Set of players is $N = \{Prioner\ A, Prisoner\ B\}$.
2. Set of actions for Prisoner i, where $i \in \{A, B\}$, is $S_i = \{confess, silent\}$.
3. The utility function is:

$$\underbrace{\begin{gathered} u_A(confess, confess) = -5 \\ u_A(confess, silent) = -1 \\ u_A(silent, confess) = -10 \\ u_A(silent, silent) = -2 \end{gathered}}_{\text{Player A}} \qquad \underbrace{\begin{gathered} u_B(confess, confess) = -5 \\ u_B(confess, confess) = -10 \\ u_B(confess, confess) = -1 \\ u_B(confess, confess) = -2 \end{gathered}}_{\text{Player B}}$$

Examples of SFG: Pollution Game

The step of the game is as follows:

- There are N countries. Each of these N countries have to **simultaneously decide** (a country has to decide its action without knowing the decision of the other countries) whether **to pass law for climate control** or to **do nothing about climate control**.
- To **pass law for climate control**, it costs a country **3 units** (like 3 billion dollars per year).
- Also, each country that **does nothing about climate control** adds a cost of **1 unit** (healthcare cost) **to all countries including itself**.
- Each country knows the associated cost (not Bayesian).
- Each country assumes that the other countries knows the rules of the game (similar to point iv mentioned in prisoner's dilemma in page 5).

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In many cases this is clear just based on the reading of the problem statements.

Assume if not mentioned.

Examples of SFG: Pollution Game

The step of the game is as follows:

- The set of players is $Q = \{1, 2, \cdots, N\}$ (**NOTE:** I have not used the usual symbol N for the set of players because the symbol N is used for the number of countries. You can use whatever notations you want but you **must define it**).
- Set of actions for country i, where $i \in Q$, is $S_i = \{C, NC\}$ where C implies climate control and NC implies no climate control.
- The payoff is as follows:
  - Say that k of the N countries decided not to do anything about climate control.
  - The cost of those countries which decided to pass laws for climate control is $3 + k$.
  - The cost of those countries which decided to do nothing about climate control is k.

Examples of SFG: Pollution Game

The step of the game is as follows:

- The set of players is $Q = \{1, 2, \cdots, N\}$ (**NOTE:** I have not used the usual symbol N for the set of players because the symbol N is used for the number of countries. You can use whatever notations you want but you **must define it**).

- Set of actions for country i, where $i \in Q$, is $S_i = \{0, 1\}$ where 0 implies climate control and 1 implies no climate control.

- The payoff is as follows: Let s_i denote the decision of country i. The utility of country i is,

$$u_i(s_1, s_2, \ldots, s_N) = \begin{cases} -\left(3 + \sum_{i \in Q} s_i\right) & ; s_i = 0 \\ -\sum_{i \in Q} s_i & ; s_i = 1 \end{cases}$$

This is an example that shows that not all games can be written in matrix form. In fact, **only two player games can be written in matrix form.**

Examples of SFG: Pollution Game

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$$u_i\left(s_1, s_2, \ldots, s_N\right) = \begin{cases} -\left(3 + \sum\limits_{i \in Q} s_i\right) & ; s_i = 0 \\ -\sum\limits_{i \in Q} s_i & ; s_i = 1 \end{cases}$$

This is equal to k, the number of countries that does nothing about climate control.

Examples of SFG: Pollution Game

The step of the game is as follows:

- The set of players is $\mathcal{Q} = \{1, 2, \cdots, N\}$ (**NOTE:** I have not used the usual symbol N for the set of players because the symbol N is used for the number of countries. You can use whatever notations you want but you **must define it**).
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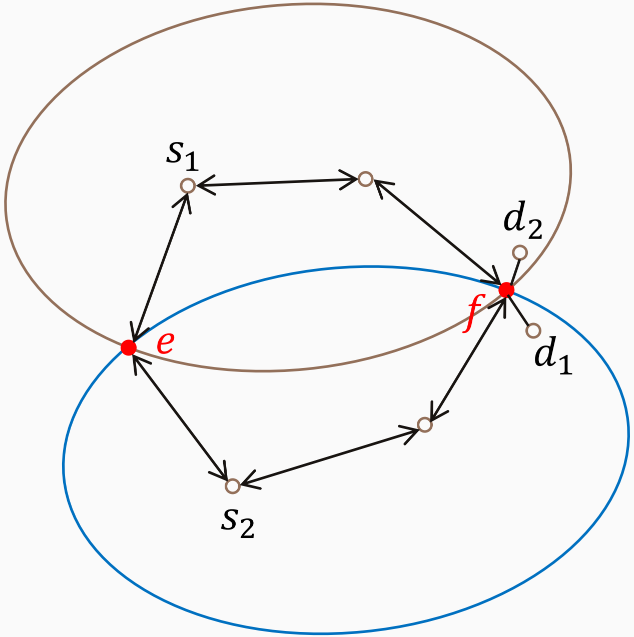
$$u_i\left(s_1, s_2, \ldots, s_N\right) = \begin{cases} -\left(3 + \sum_{i \in \mathcal{Q}} s_i\right) & ; s_i = 0 \\ -\sum_{i \in \mathcal{Q}} s_i & ; s_i = 1 \end{cases}$$

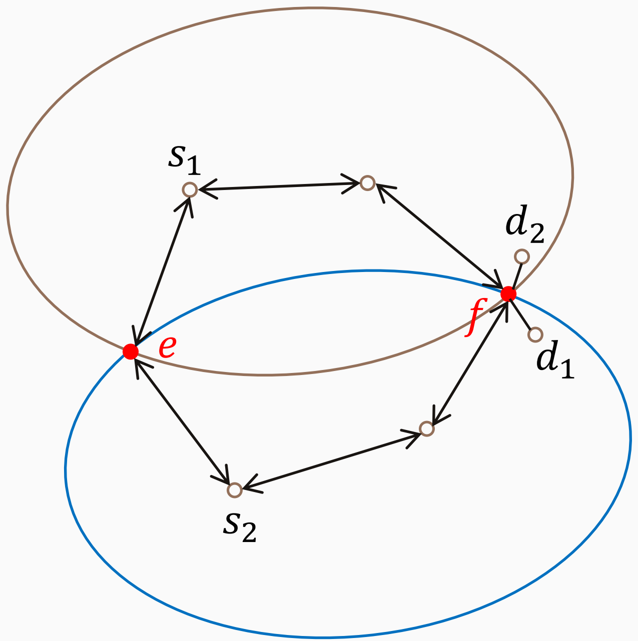
The definition of $s_i = 0$ and $s_i = 1$ were “smartly” chosen to simplify the expression for k.

Examples of SFG: ISP Routing



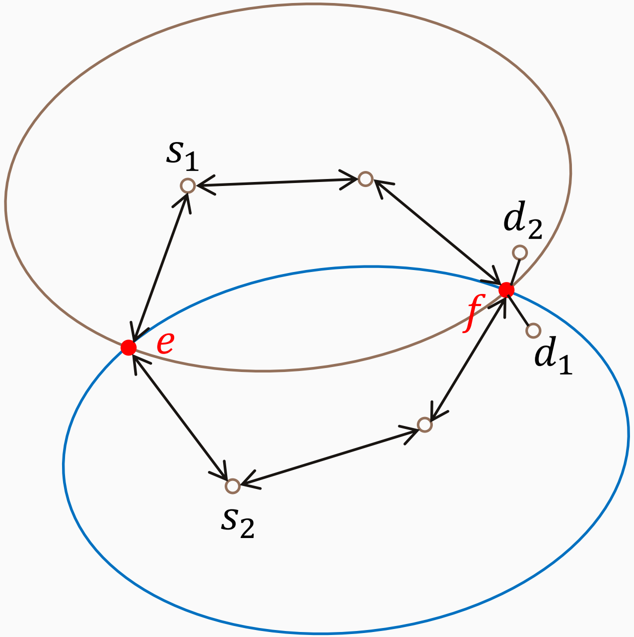
- Consider two internet service providers, ISPs 1 and 2. Each have their own networks shown in the figure using the brown and the blue ovals.
- Sometimes a node in one ISPs network has to send to a node in another ISPs network:
  - Node s_1 of ISP-1 wants to send a packet to node d_1 of ISP-2.
  - Node s_2 of ISP-2 wants to send a packet to node d_2 of ISP-1.
- These exchanges of packets happens across **internet exchange points** e and f.

Examples of SFG: ISP Routing



- Both s_1 and s_2 have packets to send to d_1 and d_2 respectively. ISP-1 routes for s_1 and ISP-2 routes for s_2.
- The packets for both s_1 and s_2 can be transmitted through either exchange points e or f. The ISPs have to **simultaneously decide** which exchange point to choose.
- Once a packet enters an ISPs network, it is the responsibility of that ISP to route the packet to the destination node. The **cost incurred by a ISP is the number of links it has to route a packet**.
  - Since d_1 and d_2 are very close to f, we will NOT count the transmission delay for links $f \rightarrow d_1$ and $f \rightarrow d_2$.

Examples of SFG: ISP Routing

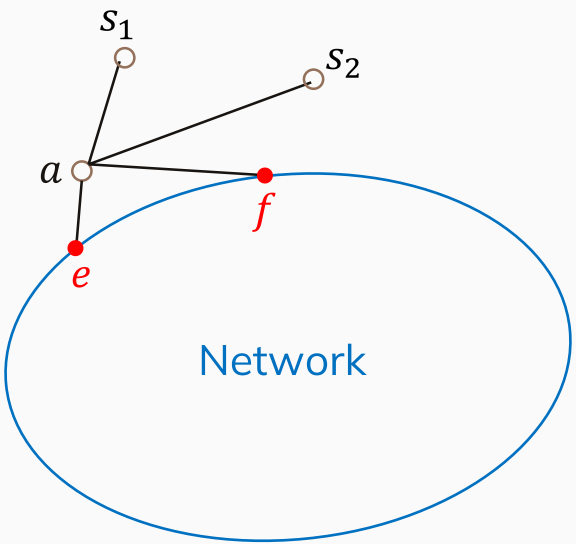


- This game can be expressed in matrix form as follows:

		ISP-2	
		e	f
ISP-1	e	-4, -4	-1, -5
	f	-5, -1	-2, -2

In order to write each of the 8 payoff values, consider each of the outcome/strategy profile and count the number of links that an ISP has to use **to send its own packet** AND the **packet of the other ISP**.

Examples of SFG: Network Congestion Games



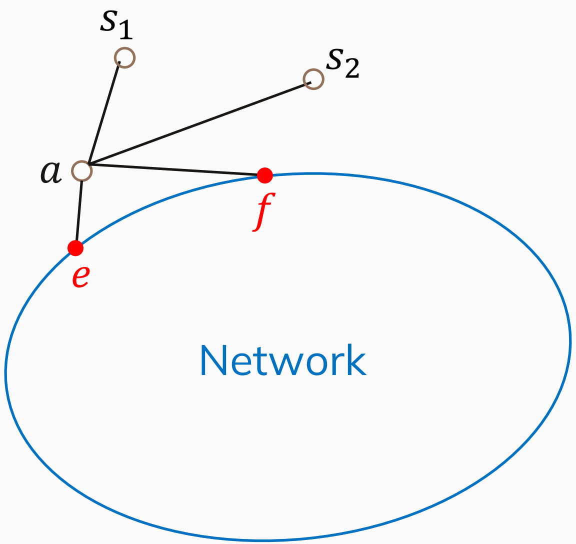
- There are two sources s_1 and s_2. They want to send packets to some nodes in the remainder of the network.

- The packets can enter the network through either nodes e or f.

- The links $a-e$ and $a-f$ can get easily congested. Hence, if both s_1 and s_2 send their packets through $a-e$ or $a-f$, their will be higher packet transmission delay.
  - Links s_1-a and s_2-a does not get congested easily due to overall lower traffic.

- Link $a-e$ in short, hence lower delay.

Examples of SFG: Network Congestion Games



- Both s_1 and s_2 has to send a packet. They both have to decide between nodes e and f **simulataneously**.
- This game can be expressed in matrix form as follows:

$$\begin{array}{c|c|c|c}
 & & \multicolumn{2}{c}{\boldsymbol{s_2}} \\
 & & e & f \\
\hline
\boldsymbol{s_1} & e & \text{-5 ms, -5 ms} & \text{-1 ms, -2 ms} \\
\hline
 & f & \text{-2 ms, -1 ms} & \text{-6 ms, -6 ms}
\end{array}$$

Explanation of the payoff matrix given during lecture.

Examples of SFG: Competing coffee shops

		Starbucks	
		3 dollars	6 dollars
CCD	WiFi	$\mathbf{0.5p - \theta, 1.5}$	
	No WiFi		

This game shows that the **set of actions** for the involved players **may not be same.**

- Café Coffee Day (CCD) and Starbucks are competing for customers.
- CCD is deciding whether to **provide WiFi or not**. Starbucks is thinking of whether to charge **3 dollars** or **6 dollars** for its coffee.
- The following are the cases:
  - **Case 1 (WiFi, 3 dollars)**: In this case, both the shops have 50% market share. The payoff of CCD is $\mathbf{0.5p - \theta}$ where p is the price of CCDs coffee and θ is the cost to provide WiFi. Payoff of starbucks is $\mathbf{0.5 \cdot 3}$.

Examples of SFG: Competing coffee shops

		Starbucks	
		3 dollars	6 dollars
CCD	WiFi	$0.5p - \theta, 1.5$	$0.8p - \theta, 1.2$
	No WiFi		

This game shows that the **set of actions** for the involved players **may not be same.**

- Café Coffee Day (CCD) and Starbucks are competing for customers.
- CCD is deciding whether to **provide WiFi or not**. Starbucks is thinking of whether to charge **3 dollars** or **6 dollars** for its coffee.
- The following are the cases:
  - **Case 2 (WiFi, 6 dollars)**: In this case, CCD have 80% market share. The payoff of CCD is $\mathbf{0.8p - \theta}$ where p is the price of CCDs coffee and θ is the cost to provide WiFi. Payoff of starbucks is $\mathbf{0.2 \cdot 6}$.

Examples of SFG: Competing coffee shops

Starbucks

CCD	3 dollars	6 dollars
WiFi	$0.5p - \theta, 1.5$	$0.8p - \theta, 1.2$
No WiFi	$0.2p, 2.4$	

This game shows that the **set of actions** for the involved players **may not be same.**

- Café Coffee Day (CCD) and Starbucks are competing for customers.
- CCD is deciding whether to **provide WiFi or not**. Starbucks is thinking of whether to charge **3 dollars** or **6 dollars** for its coffee.
- The following are the cases:
  - **Case 3 (NO WiFi, 3 dollars)**: In this case, starbucks have 80% market share. The payoff of CCD is $0.2p$ where p is the price of CCDs coffee. Payoff of starbucks is $0.8 \cdot 3$.

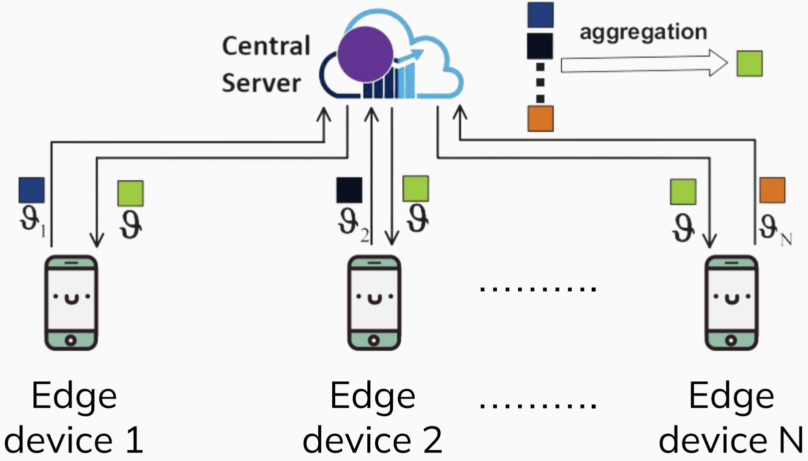
Examples of SFG: Competing coffee shops

		Starbucks	
		3 dollars	6 dollars
CCD	WiFi	$0.5p - \theta, 1.5$	$0.8p - \theta, 1.2$
	No WiFi	$0.2\mathrm{p}, 2.4$	$0.5\mathrm{p}, 3$

This game shows that the **set of actions** for the involved players **may not be same.**

- Café Coffee Day (CCD) and Starbucks are competing for customers.
- CCD is deciding whether to **provide WiFi or not**. Starbucks is thinking of whether to charge **3 dollars** or **6 dollars** for its coffee.
- The following are the cases:
  - **Case 3 (NO WiFi, 6 dollars)**: In this case, both the shops have 50% market share. The payoff of CCD is $0.5p$ where p is the price of CCDs coffee. Payoff of starbucks is $0.5 \cdot 6$.

Examples of SFG: Federated Learning



Edge device 1

Edge device 2

....

Edge device N

The above diagram and the example is adapted from the following paper:
[1] Y, Zhan, Peng Li, Z. Qu, D. Zeng, and S. Guo, "A Learning-Based Incentive Mechanism for Federated Learning", vol. 7, IEEE Internet of Things Journal, 2020.

Examples of SFG

Other than the games discussed in this lecture slide, I have also discussed the following two games in lecture 1:

- The “extra assignment” game.
- The game related to Braess’s paradox.

It is your job to formulate those as a SFG. While formulating the game for Braess’s paradox, assume that the number of vehicles (the players of the game) is a finite number.

Mandatory Reading

- Chapter 4 (Strategic form games) of the book by Y. Narahari. It has many examples of SFG that we did not discuss during the lecture hours.

Thank You!