Game Theory (CS4187)

Lecture 33 (14/11/2024)

> Notes are not complete. Refer chapter 17 of the book. You don't need to read the entire chapter. Just read till Theorem 17.1 (including it).

Dominant Strategy Incentive Compatibility (DSIC):

- A direct mechanism $D = (\Theta, f)$ is DSIC if,

$$u_i(\theta, f(\theta_i, \theta_{-i})) \geq u_i(\theta, f(\theta_i', \theta_{-i})) \quad , \forall \theta_i, \forall \theta_i', \forall \theta_{-i}, \forall i$$

- For a direct mechanism, SCF f is what characterizes it (by far and large because type Θ is specific to the system and not the mechanism). Hence, we can say that an SCF f is DSIC if,

same

$$u_i(\theta, f(\theta_i, \theta_{-i})) \geq u_i(\theta, f(\theta_i', \theta_{-i})) \quad , \forall \theta_i, \forall \theta_i', \forall \theta_{-i}, \forall i$$

- So DSIC is one of the properties of SCF f. This is because if f is DSIC we have a "better gauruntee" that a player will behave in a certain way.

- In what follows, we will describe a few other properties we want SCF to have.

Ex-Post Efficient:

- A SCF f is ex-post efficient if for all type profile θ, the outcome $f(\theta)$ is pareto-optimal. Mathematically, for all $\theta \in \Theta$, there does not exist an outcome $x \in X$ s.t.,

  - $u_i(x, \theta) \geq u_i(f(\theta), \theta)$, $\forall i \in N$

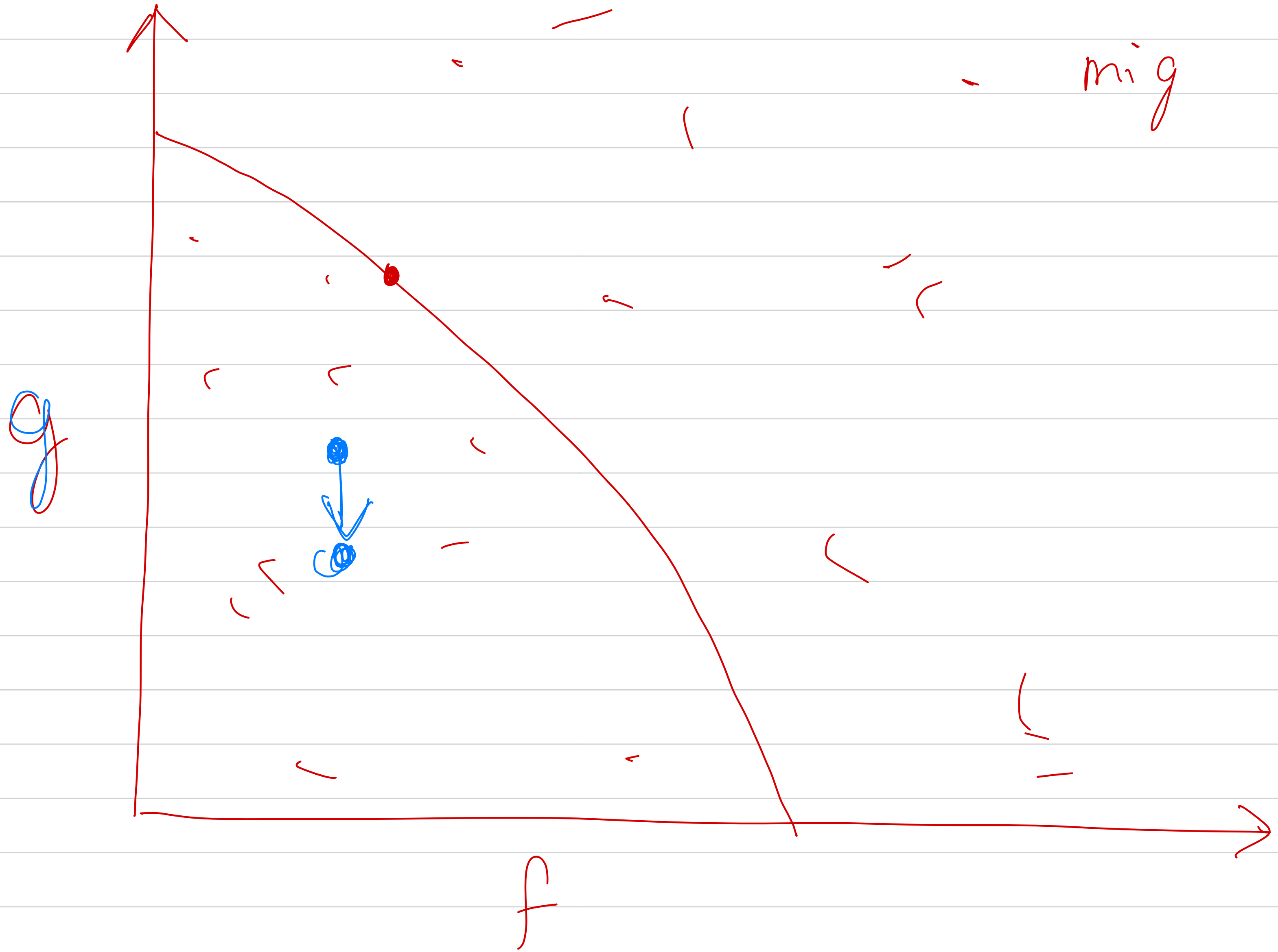
    AND

  - $u_i(x, \theta) > u_i(f(\theta), \theta)$ for some $i \in N$

- Relation with multi-objective optimization?

$\min f(\theta)$

$\min g(\theta)$



The following lemma gives a SUFFICIENT condition for ex-post efficiency.

Lemma: A SCF f is ex-post efficient if for all $\theta \in \Theta$,

$$\left.\sum_{i \in N} u_i(f(\theta), \theta) \geqslant \sum_{i \in N} u_i(x, \theta), \quad \forall x \in X \quad \right\} I_1$$

Proof: Recall the original definition of ex-post efficiency, f is ex-post efficient if for all $\theta \in \Theta$ there does not exist a $x \in X$ s.t.,

- $u_i(x, \theta) \geqslant u_i(f(\theta), \theta), \quad \forall i \in N$

AND

- $u_i(x, \theta) > u_i(f(\theta), \theta)$ for some $i \in N$

$\Big\} I_2$

Proof that I_1 implies I_2: We are given I_1. We prove I_2.

We prove this by contradiction, i.e. I_1 holds but I_2 does not hold. Since I_2 does not hold, there is an $\tilde{x} \in X$ s.t.

- $u_i(\tilde{x}, \theta) \geq u_i(f(\theta), \theta), \ \forall i \in N$

AND

- $u_i(\tilde{x}, \theta) > u_i(f(\theta), \theta)$ for some $i \in N$

The above two inequalities can be equivalently written as,

- $u_i(\tilde{x}, \theta) = u_i(f(\theta), \theta), \ \forall i \in \tilde{N}$ ($\tilde{N}$ is the subset of players for which equality holds)

AND

- $u_i(\tilde{x}, \theta) > u_i(f(\theta), \theta), \ \forall i \in N \setminus \tilde{N}$ ($N \setminus \tilde{N}$ is the set of players for which strict inequality holds)

Now we will prove that for $\tilde{x}$, I_1 does not hold and hence a contradiction. We have,

$$\sum_{i \in N} v_i(\tilde{x}, \theta) = \sum_{i \in \tilde{N}} v_i(\tilde{x}, \theta) + \sum_{i \in N \setminus \tilde{N}} v_i(\tilde{x}, \theta)$$

$$> \sum_{i \in \tilde{N}} v_i(f(\theta), \theta) + \sum_{i \in N \setminus \tilde{N}} v_i(f(\theta), \theta)$$

$$= \sum_{i \in N} v_i(f(\theta), \theta)$$

This establishes contradictions. Hence, I_1 implies I_2.

Non-Dictatorship

- Let's first define when SCF f is called a dictatorship.

- A SCF f is a dictatorship if there exist an agent $d \in N$ (the dictator) for which the outcome $f(\theta)$ maximizes it's payoff for all θ. Mathematically, SCF f is a dictatorship if there exist a $d \in N$ s.t.,

$$u_d(\theta, f(\theta)) \geq u_d(\theta, x), \quad \forall \theta \in \Theta, \ \forall x \in X$$

- A SCF f is non-dictatorship if the above inequality doesn't hold.

Important:

> Till now we have used a generic utility function $u_i(\theta, x)$.

> In this lecture, and for chain theorem of this lecture (GS theorem), the utility function is $u_i(\theta_j, x)$.

Rational Preference Relation

- Consider outcome set X. And, two outcomes $x, y \in X$. Then, the utility function $U_i(\cdot\,; \theta_i)$ induces a preference over outcome, i.e.,

$$x \succsim y \iff U_i(x, \theta_i) \geq U_i(y, \theta_i)$$

Rational Preference Relation

- A relation $\succsim$ on outcome set X is called rational preference relation (or :

  - Reflexivity: $\forall x \in X, \quad x \succsim x.$
  - Completeness: $\forall x, y \in X,$ either $x \succsim y$ or $y \succsim x.$
  - Transitivity: $\forall x, y, z \in X,$ if $x \succsim y$ and $y \succsim z$ then $x \succsim z.$

Strict - Total Preference Relation :

- If $x \succsim y$ and $y \succsim x$ for some $x, y \in X$ then it means that $x = y$.

- The set of all preference relation R.
  - ↳ R is going to be all permutations of the elements of X

- The set of all strict preference relation P.

Ordinal Preference Relation

$f : \Theta \to X$

$$R_i = \{ \succeq(\theta_i) : \theta_i \in \Theta_i \}$$

Theorem (GS Theorem): Consider SCF f s.t.

- $|X| \geq 3$.
- $R_i = P$; $\forall i \in N$
- f is an onto mapping.

f is DSIC if and only if f is dictatorial!