1 Cournot competition with Incomplete Information

This example is obtained from the book Game Theory by Michael Maschler, Eilon Solan, Shmuel Zami.

1.1 Problem Setup

Consider a setup where multiple companies are selling a certain "good" to the market. There goods are homogeneous, i.e. goods produced by the companies are all the same from the perspective of the consumers. Each of these companies have to individually decide the amount of good to produce without knowing what the other company is producing. After the companies decides the amount of goods, the market decides the price, i.e. companies don't get to decide the price. This setup is called a Cournot competition. Just FYI, there is an alternate setup called Bertrand competition where companies decides the price and not the amount of good. You may want to check it out.

Consider Cournot competition between two companies 1 and 2 who has to decide the amount of good a_1 and a_2 that it wants to produce. a_1 and a_2 are continuous variables. Accordingly, the selling price per unit good is decided by the market and it is $f(a) = A - a$ where a is the total supply/amount of good. The per unit production cost of good is different for the two companies:

1. For company 1, we will normalize the production cost to 1.
2. For company 2, the production cost is either c^l or c^h, where $c_l < c_h$. The probability that production cost is c^l is p respectively.

$f(a)$, c_l , c_h, and p are common knowledge. Production cost of company 1 is known to company 2 but not vice-versa. Find the BNE of this game.

1.2 Prior probability

For this problem, the production cost is the type of the players. Let c_1 and c_2 be the production cost of companies 1 and 2 respectively. The prior probability of valuation is,

$$P[c_1, c_2] = \begin{cases} p_l & , c_1 = 1, c_2 = c^l \\ p_h & , c_1 = 1, c_2 = c^h \end{cases}$$

1.3 Utility function

Utility function of company i is,

$$u_i(c_1, c_2, a_1, a_2) \quad = \quad a_i(A - a_1 - a_2) - c_i a_i$$

1.4 Ex-interin Utilities of Players in Pure Strategy

Let's see if we can find BNE in pure strategies for both the companies.

Note the just like complete information case, the following is true for Bayesian games also:

1. There is no gaurantee that BNE in pure strategies exists.
2. Since we are dealing with continuous pure strategy space, it is possible that BNE in mixed strategies may not exists as well.

In general, ex-interim utility of player i in Bayesian game is,

$$U_i\left(s_i, s_{-i} \middle| \theta_i\right) = \sum_{\theta_{-i} \in \Theta_{-i}} P\left[\theta_{-i} \middle| \theta_i\right] u_i\left(\theta_i, \theta_{-i}, s_i(\theta_i), s_{-i}(\theta_{-i})\right) \tag{1}$$

Applying (1) in our setup we get the ex-interim utility of company 1 as,

$$\begin{aligned}
U_1\left(a_1, a_2 \middle| c_1 = 1\right) &= \sum_{c_2 \in \{c^l, c^h\}} P\left[c_2 \middle| c_1 = 1\right] u_1\left(c_1, c_2, a_{1,1}, a_{2,c_2}\right) \\
&= p u_1\left(1, c^l, a_{1,1}, a_{2,c^l}\right) + (1-p)\, u_1\left(1, c^h, a_{1,1}, a_{2,c^h}\right) \\
&= p\left(a_{1,1}\left(A - a_{1,1} - a_{2,c^l}\right) - c_1 a_{1,1}\right) \\
&\quad + (1-p)\left(a_{1,1}\left(A - a_{1,1} - a_{2,c^h}\right) - c_1 a_{1,1}\right) \\
&= a_{1,1}\left(A - a_{1,1}\right) - c_1 a_{1,1} - \left(p a_{2,c^l} + (1-p)\, a_{2,c^h}\right) \\
&= a_{1,1}\left(A - 2a_{1,1}\right) - a_{1,1}\left(p a_{2,c^l} + (1-p)\, a_{2,c^h}\right) \qquad (2)
\end{aligned}$$

Similarly, ex-interim utility of company 2 is,

$$\begin{aligned}
U_2\left(a_1, a_2 \middle| c_2 = c^l\right) &= \sum_{c_1 \in \{1\}} P\left[c_1 \middle| c_2 = c^l\right] u_2\left(c_1, c_2, a_{1,c_1}, a_{2,c^l}\right) \\
&= 1 \cdot u_2\left(1, c^l, a_{1,1}, a_{2,c^l}\right) \\
&= a_{2,c^l}\left(A - a_{1,1} - a_{2,c^l}\right) - c^l a_{2,c^l} \\
U_2\left(a_1, a_2 \middle| c_1 = c^h\right) &= \sum_{c_1 \in \{1\}} P\left[c_1 \middle| c_2 = c^h\right] u_2\left(c_1, c_2, a_{1,c_1}, a_{2,c^h}\right) \\
&= 1 \cdot u_2\left(1, c^h, a_{1,1}, a_{2,c^h}\right) \\
&= a_{2,c^h}\left(A - a_{1,1} - a_{2,c^h}\right) - c^h a_{2,c^h}
\end{aligned}$$

1.5 Computing BNE

Note the following:

1. For games where the set of types of the players are finite, compute BNE is equivalent to computing the NE for ex-ante game. This is true for our setup also because the type of company 2 is a finite set with only two elements. But, we choose to directly compute BNE instead of computing the NE for ex-ante game because it is simpler for this setup.
2. As far as manually calculating BNE (also NE in complete and incomplete information games) for continuous pure strategy space, using the best response version of the definition of BNE is simpler because high school calculus tools to find the best response.

In general, a strategy $s^\star = \left(s_i^\star, s_{-i}^\star\right)$ to be a BNE in pure strategy if,

$$s_{i,\theta_i}^\star \in B_i\left(s_{-i}^\star, \theta_i\right) \ , \forall \theta_i \in \Theta_i, \forall i \in N \qquad (3)$$

where,

$$B_i\left(s_{-i}, \theta_i\right) = \underset{s_{i,\theta_i} \in S_i}{\arg\max} U_i\left(s_i, s_{-i} | \theta_i\right) \qquad (4)$$

We now use this approach for our setup. The pure strategy BNE in our setup is $a^\star = (a_1^\star, a_2^\star)$ where $a_1^\star = \left(a_{1,1}^\star\right)$ and $a_2^\star = \left(a_{2,c^l}^\star, a_{2,c^h}^\star\right)$.

The RHS of (4) for company 1 is,

$$\begin{aligned}&\arg\max_{a_{1,1}\in\mathbb{R}^+} U_1\left(a_1, a_2|c_1 = 1\right)\\ \Rightarrow &\arg\max_{a_{1,1}\in\mathbb{R}^+}\left(a_{1,1}\left(A - 2a_{1,1}\right) - a_{1,1}\left(pa_{2,c^l} + (1-p)\,a_{2,c^h}\right)\right)\end{aligned} \tag{5}$$

We differentiate (5) with respect to $a_{1,1}$ and set the differential to zero. For the time being keep our finger's crossed that the system parameters are such that the optimal solution is positive! Differening the objective function in (5) w.r.t. $a_{1,1}$ and setting it to zero we get,

$$\begin{aligned}A - 4a_{1,1} - \left(pa_{2,c^l} + (1-p)\,a_{2,c^h}\right) &= 0\\ a_{1,1} &= \frac{A - \left(pa_{2,c^l} + (1-p)\,a_{2,c^h}\right)}{4}\end{aligned} \tag{6}$$

Now, according to (3),

$$a^\star_{1,1} = \frac{A - \left(pa^\star_{2,c^l} + (1-p)\,a^\star_{2,c^h}\right)}{4} \tag{7}$$

Points to ponder:

1. Is $a_{1,1}$ given by (6) a maxima or minima?
2. Qualitative significance if $a_{1,1}$ given by (6) is negative?

Similarly, the RHS of (4) for company 2 for $c_2 = c^l$ is,

$$\begin{aligned}&\arg\max_{a_{2,c^l}\in\mathbb{R}^+} U_2\left(a_1, a_2|c_2 = c^l\right)\\ \Rightarrow &\arg\max_{a_{2,c^l}\in\mathbb{R}^+}\left(a_{2,c^l}\left(A - a_{1,1} - a_{2,c^l}\right) - c^l a_{2,c^l}\right)\end{aligned} \tag{8}$$

We differentiate (8) with respect to a_{2,c^l} and set the differential to zero. For the time being keep our finger's crossed that the system parameters are such that the optimal solution is positive! Differening the objective function in (8) w.r.t. a_{2,c^l} and setting it to zero we get,

$$\begin{aligned}\left(A - a_{1,1} - c^l\right) - 2a_{2,c^l} &= 0\\ a_{2,c^l} &= \frac{A - a_{1,1} - c^l}{2}\end{aligned} \tag{9}$$

Now, according to (3),

$$a^\star_{2,c^l} = \frac{A - a^\star_{1,1} - c^l}{2} \tag{10}$$

We can repeat the same process for company 2 when $c_2 = c^h$ and we will get,

$$a^\star_{2,c^h} = \frac{A - a^\star_{1,1} - c^h}{2} \tag{11}$$

Now in order to find the pure strategy BNE, we have to solve (7), (10), and (11) which constitutes a system of linear equations. We roughly show the process below,

$$\begin{aligned}a^\star_{1,1} &= \frac{A - \left(pa^\star_{2,c^l} + (1-p)\,a^\star_{2,c^h}\right)}{4}\\ \Rightarrow a^\star_{1,1} &= \frac{A}{4} - \frac{p}{4}\left(\frac{A - a^\star_{1,1} - c^l}{2}\right) - \frac{(1-p)}{4}\left(\frac{A - a^\star_{1,1} - c^h}{2}\right)\end{aligned} \tag{12}$$

We solve (12) for $a^\star_{1,1}$ and then substitute it in (10) and (11) to get $a^\star_{2,c^l}$ and $a^\star_{2,c^h}$ respectively.