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Game Theory (CS4187)

Lectures 5, 6, 7, and 8

Date: 20/08/2024
22/08/2024
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29/08/2024

Instructor: Gourav Saha

Broad idea of sequential move games

- The idea of this series of lecture is **sequential move games**.
  - Also called **extended form games** (EFGs) and **dynamics games**.
  - We will hardly use the term dynamics games in this course.

- There are two kind of EFGs (the following definitions are **informal**):
  - Perfect information EFG (PIEFG): When a player/agent **knows the history of actions** by other players (including itself).
  - Imperfect information EFG (IIEFG): When a player **does not know <u>some</u>** or **<u>all</u> of the history of actions** by other players (including itself).

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IMPORTANT: Perfect/imperfect information is very different from complete/incomplete information:

- **Perfect/imperfect information** deals with whether the players knows the **history of actions** or not.
- **Complete/incomplete information** deals with whether the players knows the **utility function** of all the players (including itself) or not.

Broad idea of sequential move games

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  - Imperfect information EFG (IIEFG): When a player **does not know some** or **all of the history of actions** by other players (including itself).

- Needless to say, **IIEFG is the more general form**. This is because in IIEFG the player "does not know **some** of the history of actions". "some" may imply that all the history of actions is known to the player.

Broad idea of sequential move games

- Both form of EFGs may or may not have **“chance moves”**. “chance moves” are used to model the **stochastic** nature of a game.

- All the variations of EFGs (perfect or imperfect, with or without chance moves) are best abstracted using **decision trees**.

- In the following slides, we will understand all the variants of EFGs using a series of example. For each of these examples, we will draw the decision tree that models the example as an EFG.

Examples of EFG

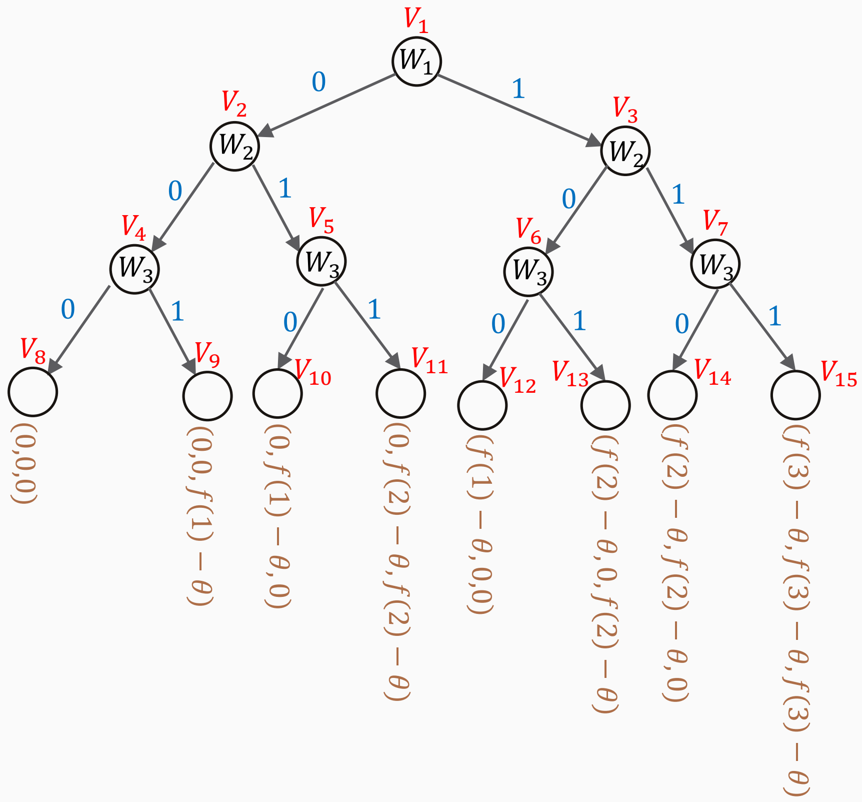
- We consider the **spectrum market** example discussed in lecture 2.
- There are **three wireless service providers** (WSPs) denoted W_1, W_2, and W_3.
- Government will be releasing a new spectrum band.
- If a WSP wants to use this new spectrum band, it has to "enter the market". "Entering the market" means to first **invest $\boldsymbol{\theta}$ units for infrastructure development** and then participating in spectrum auctions to win the spectrum band.
- If $k \leq 3$ WSPs enter the market, then the payoff/utility of a WSP who **enters** the market is $\boldsymbol{f(k) - \theta}$ where $\mathrm{f}(k)$ is **monotonic decreasing** w.r.t. k. The payoff/utility of a WSP who **does not enter** the market is $\mathbf{0}$. *Can you suggest a more general $f(\cdot)$?*
- The government will call each of the WSPs **in order** **(this is a simplification)** to know if they want to enter the market. The order is W_1 and then W_2 and then W_3.

Examples of EFG

- We consider the spectrum market example discussed in lecture 2.
- There are three wireless service providers (WSPs) denoted W_1, W_2, and W_3.
- Government will be releasing a new spectrum band.
- If a WSP wants to use this new spectrum band, it has to "enter the market". "Entering the market" means to first invest funds for infrastructure development and then participating in spectrum auctions to win the spectrum band.
- If $k \leq 3$ WSPs enter the market, then the payoff/utility of a WSP who enters the market is $f(k) - \theta$ where $f(k)$ is monotonic decreasing w.r.t k. The payoff/utility of a WSP who does not enter the market is 0.
- The government will call each of the WSPs in order (this is a simplification) to know if they want to enter the market. The order is W_1 and then W_2 and then W_3.

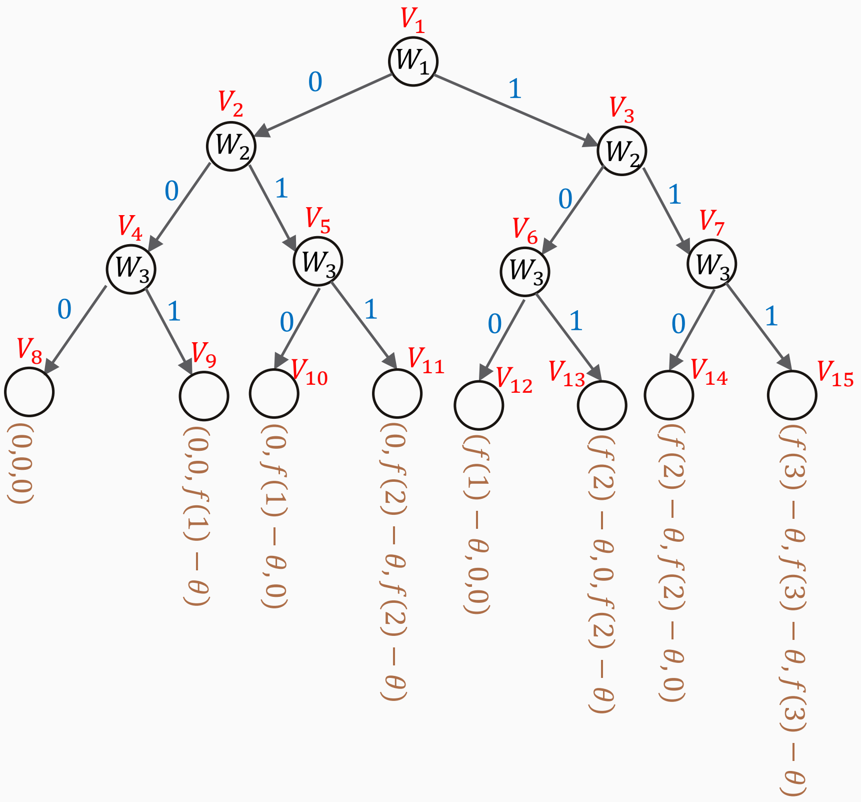
In the following slides we will discuss a few variants of this setup. However, all the points in this slide is true for all of these variants.

Examples of EFG: Variant 1



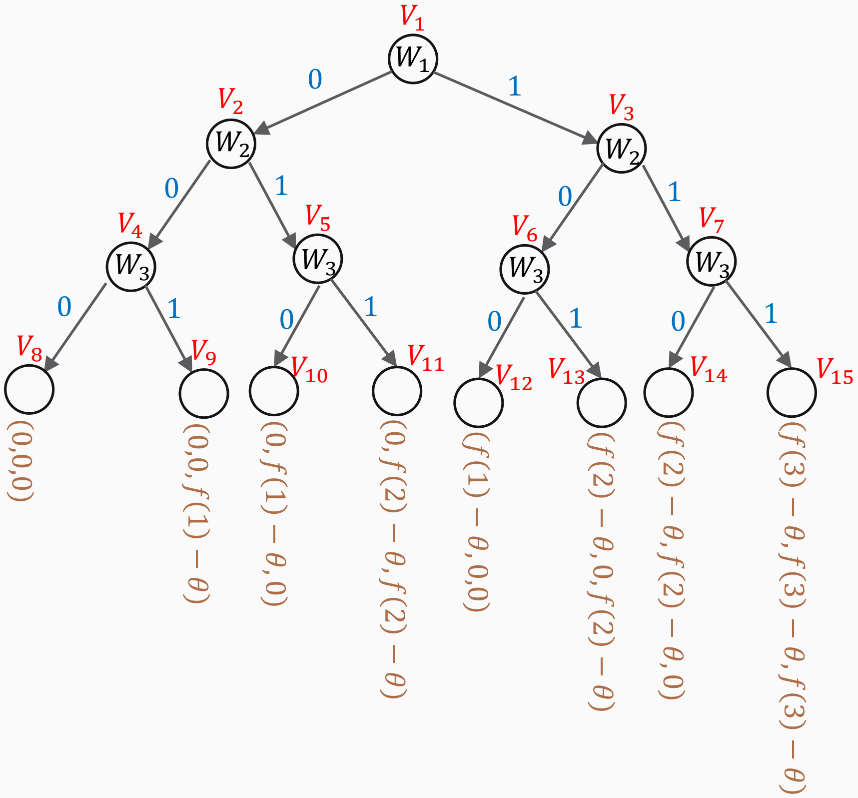
- In this variant, **as soon as a WSP makes a decision, the other WSPs gets to know about it**.
- This is an example of **PIEFG without chance moves** (because actions of all the players are known).
- The decision tree corresponding to this PIEFG is shown in the left.

Examples of EFG: Variant 1



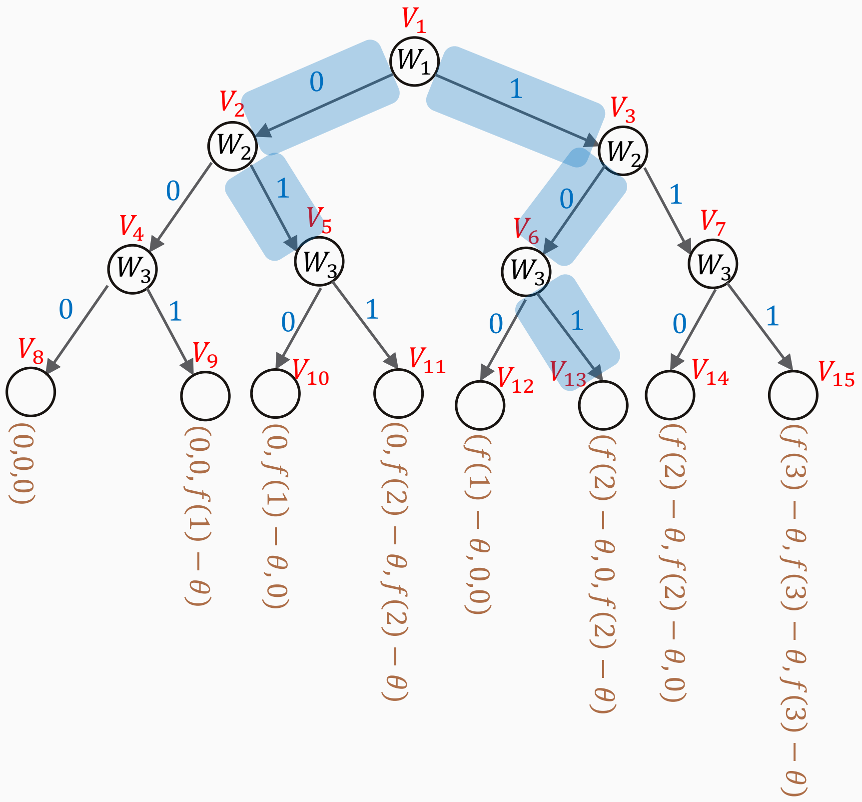
- In this variant, **as soon as a WSP makes a decision, the other WSPs gets to know about it**.
- This is an example of **PIEFG without chance moves**.
- The decision tree corresponding to this PIEFG is shown in the left.
    - The red text are the vertices of the tree.

Examples of EFG: Variant 1



- In this variant, **as soon as a WSP makes a decision, the other WSPs gets to know about it**.
- This is an example of **PIEFG without chance moves**.
- The decision tree corresponding to this PIEFG is shown in the left.
  - All **non-leaf vertices** have a **player W_i associated with it**. This player has to make a decision in that vertex.
  - The blue text shows all possible decisions that can be made in a non-leaf vertex.

Examples of EFG: Variant 1

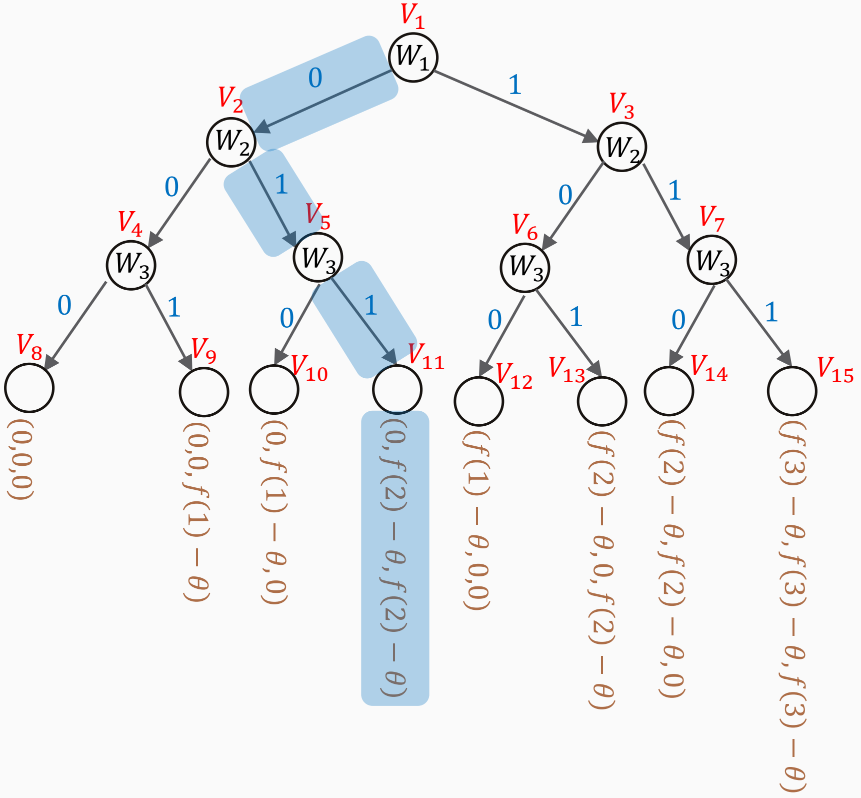


- In this variant, **as soon as a WSP makes a decision, the other WSPs gets to know about it**.
- This is an example of **PIEFG without chance moves**.
- The decision tree corresponding to this PIEFG is shown in the left.
  - Every vertex is associated with a history of actions. Example: The history of actions corresponding to:
    1. V_5 is $(0,1)$.
    2. V_{13} is $(1,0,1)$.
    3. V_1 is $\emptyset$.

The sequence of actions must be preserved.

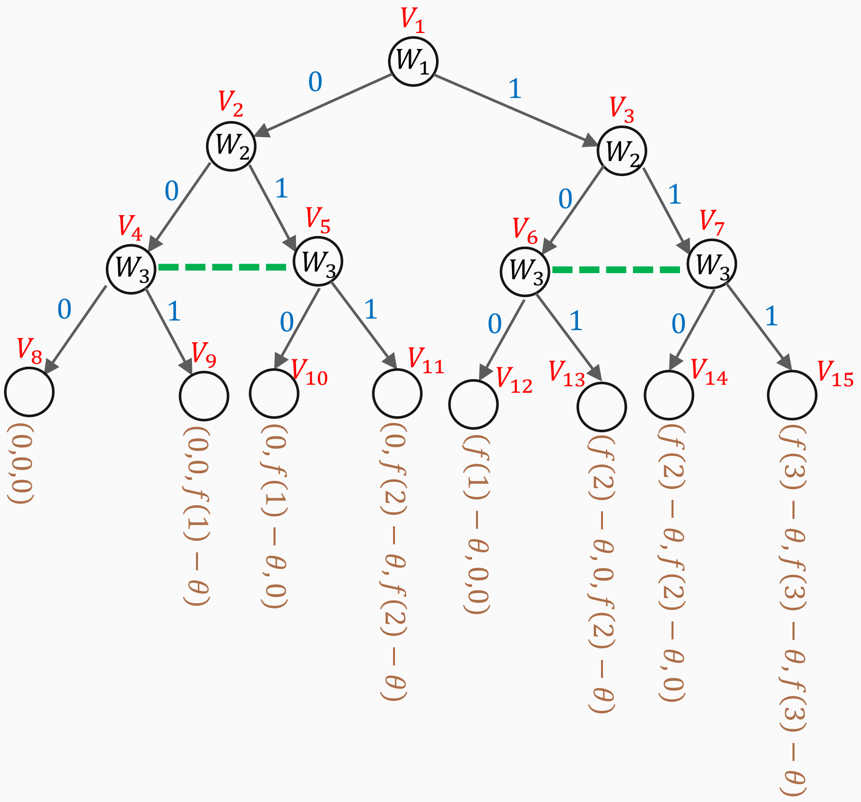
Null Set

Examples of EFG: Variant 1



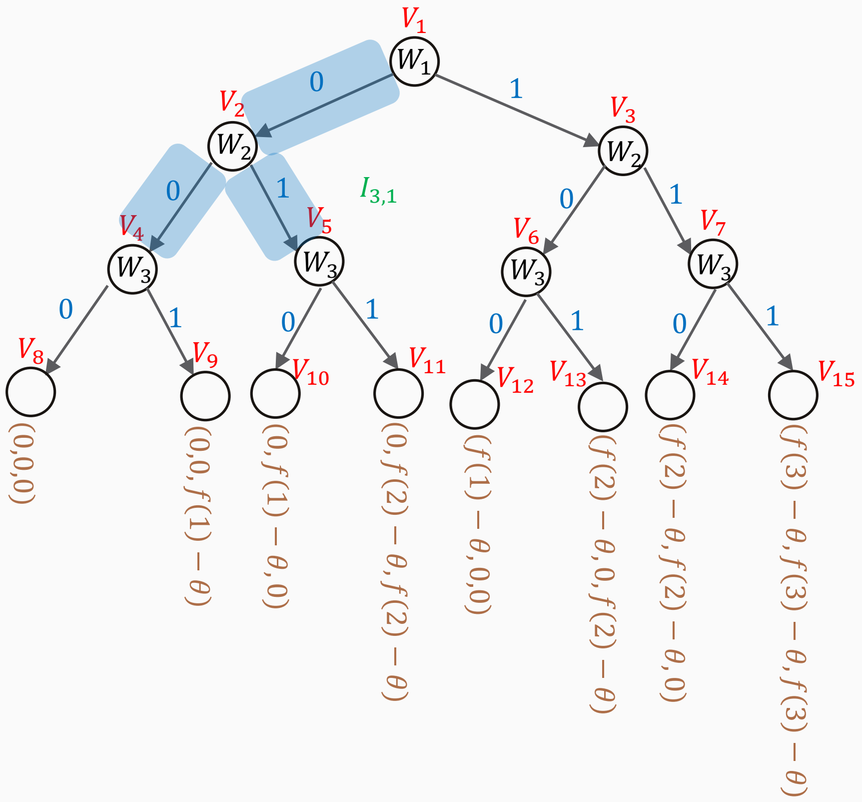
- In this variant, **as soon as a WSP makes a decision, the other WSPs gets to know about it**.
- This is an example of **PIEFG without chance moves**.
- The decision tree corresponding to this PIEFG is shown in the left.
  - All **leaf vertices** have **payoff/utility** of all the players associated with it. Example:
    1. For V_{11} only two players, W_2 and W_3, joined the market. Hence W_1's utility is 0 while W_2's and W_3's utility is $\mathrm{f}(2) - \theta$.

Examples of EFG: Variant 2



- In this variant, the **decision of W_1 and W_3 spreads** across the market as soon as it makes it however the **decision of W_2 does not spread**.
- This is an example of **IIEFG without chance moves** (because W_2's action is not known).
- The decision tree corresponding to this IIEFG is shown in the left.
  - Everything remains same as variant 1 but there are **dashed-green** lines. We will explain these dashed-green lines in the next few slides.

Examples of EFG: Variant 2

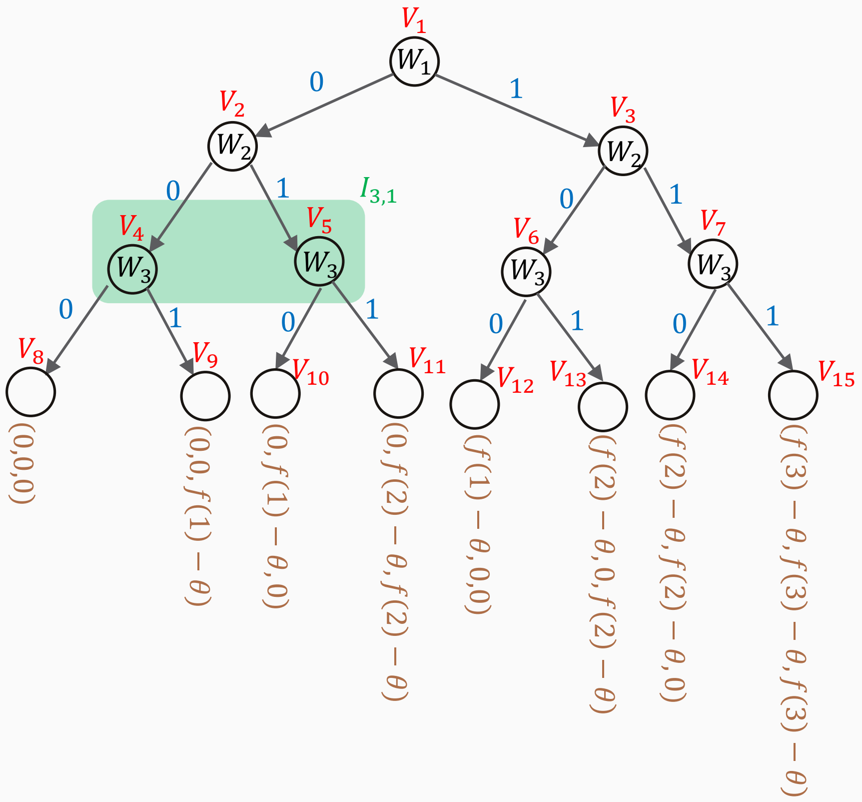


- In this variant, the **decision of W_1 and W_3 spreads** across the market as soon as it makes it however the **decision of W_2 does not spread**.

- This is an example of **IIEFG without chance moves**.

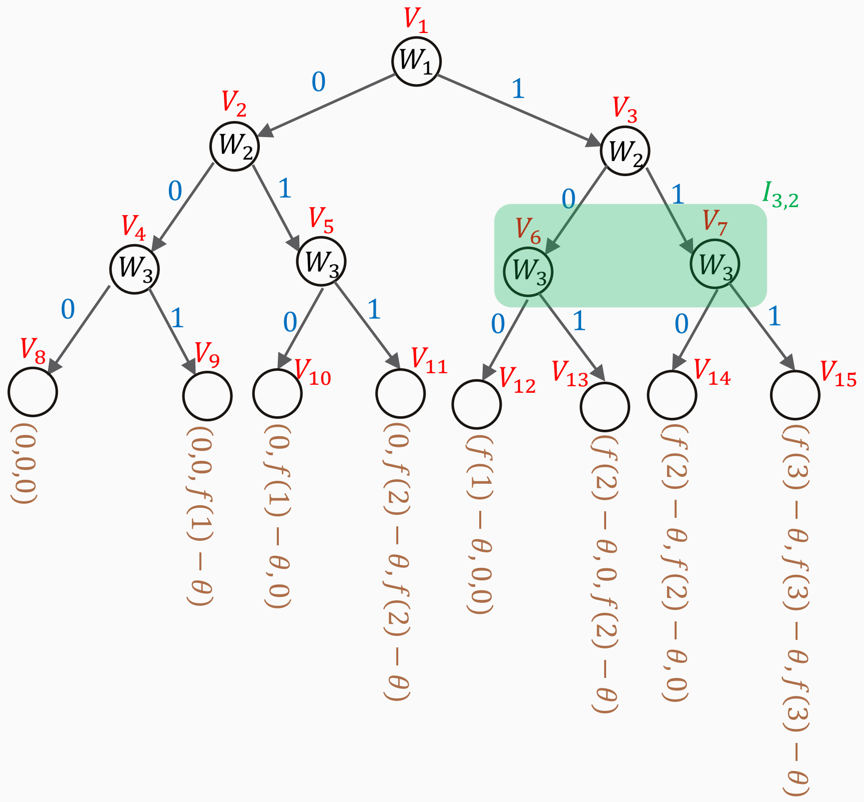
- The decision tree corresponding to this IIEFG is shown in the left.
  - Suppose W_1 decided to not join. So are at V_2. After that, since action of W_2 is not known, player **W_3 will not know whether it is at V_4 or V_5** (hence W_3 does not know if the history of actions is $(0,0)$ or $(0,1)$).

Examples of EFG: Variant 2



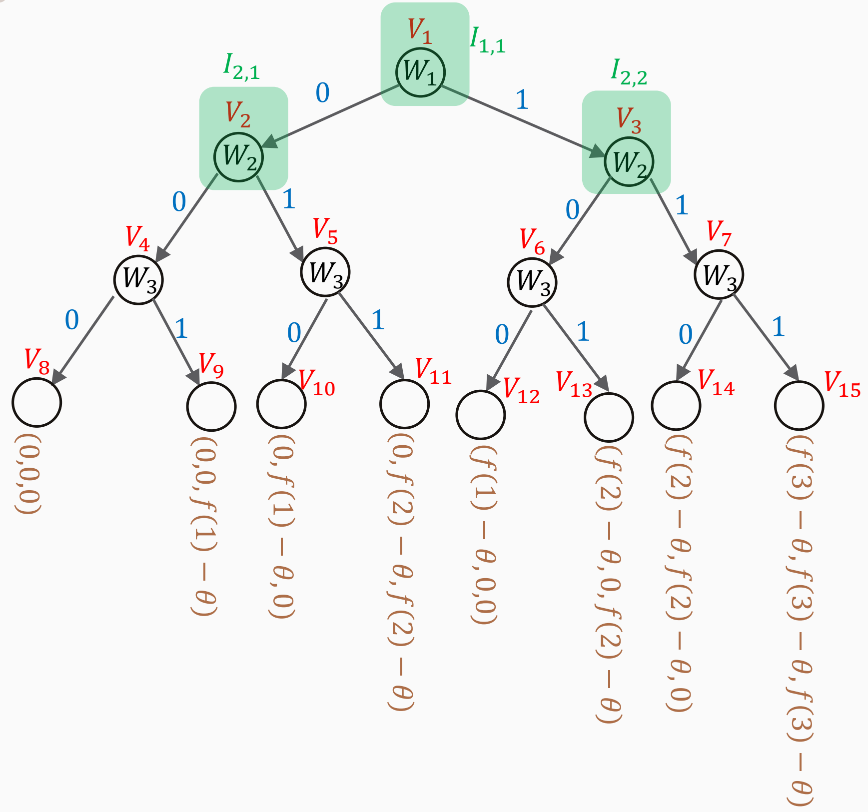
- In this variant, the **decision of W_1 and W_3 spreads** across the market as soon as it makes it however the **decision of W_2 does not spread**.
- This is an example of **IIEFG without chance moves**.
- The decision tree corresponding to this IIEFG is shown in the left.
  - Hence, we put V_4 and V_5 in a supernode, $I_{3,1}$, highlighted in green.
  - $I_{3,1} = \{V_4, V_5\}$ is one of the **information set** of W_3. The first subscript is the player index, and the second subscript is the index of the information set of a given player.

Examples of EFG: Variant 2



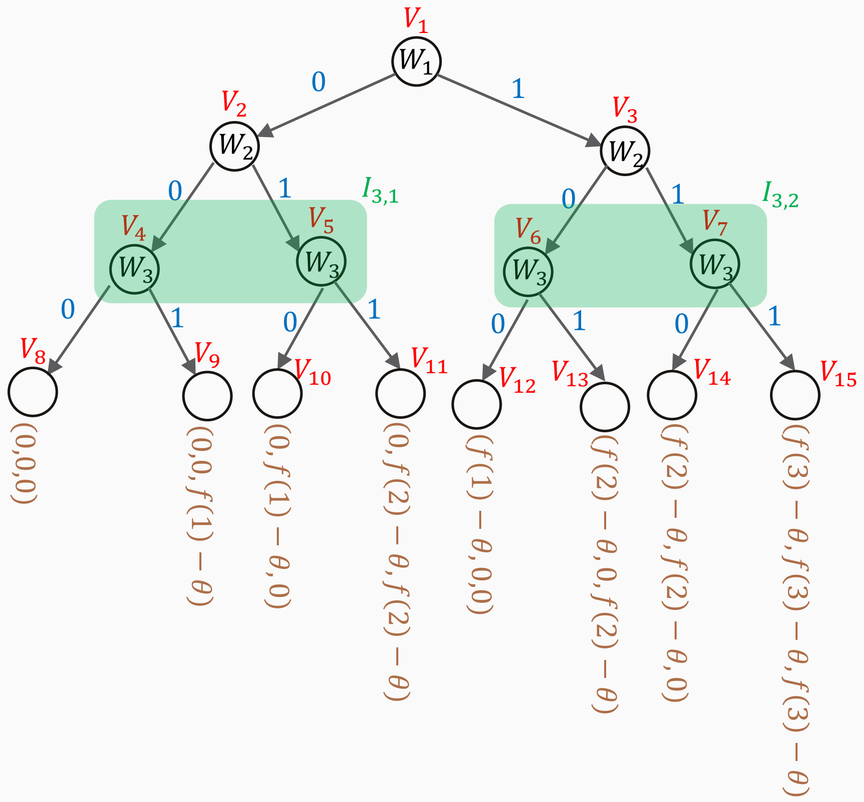
- In this variant, the **decision of W_1 and W_3 spreads** across the market as soon as it makes it however the **decision of W_2 does not spread**.
- This is an example of **IIEFG without chance moves**.
- The decision tree corresponding to this IIEFG is shown in the left.
  - Hence, we put V_4 and V_5 in a supernode, $I_{3,1}$, highlighted in green.
  - $I_{3,1} = \{V_4, V_5\}$ is one of the **information set** of W_3. $I_{3,2} = \{V_6, V_7\}$ is the second information set of W_3.

Examples of EFG: Variant 2



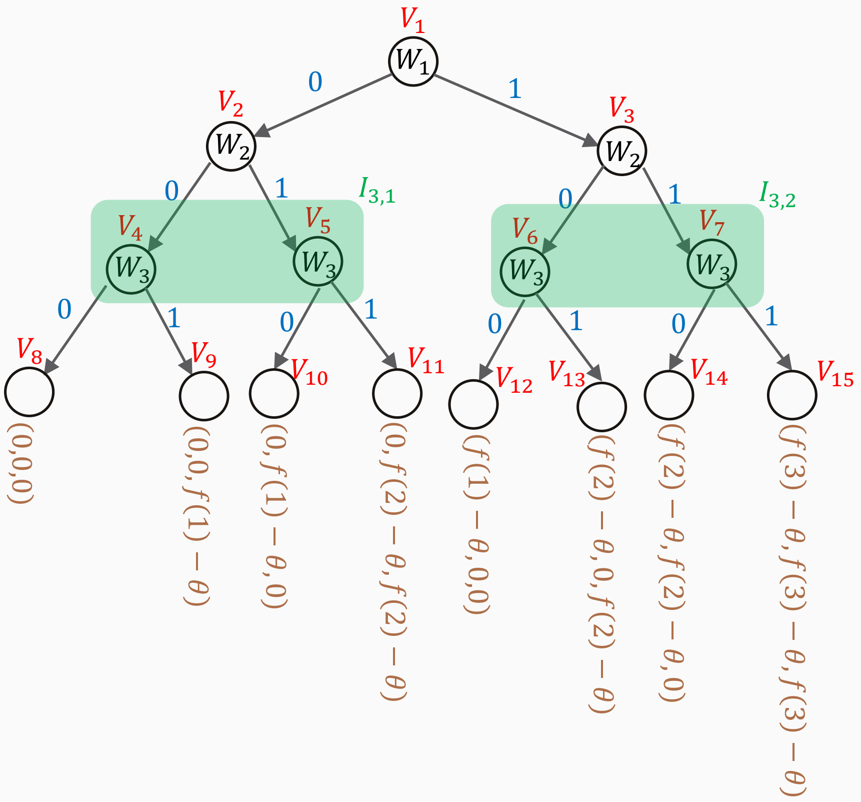
- In this variant, the **decision of W_1 and W_3 spreads** across the market as soon as it makes it however the **decision of W_2 does not spread**.
- This is an example of IIEFG **without chance moves**.
- The decision tree corresponding to this IIEFG is shown in the left.
  - Hence, we put V_4 and V_5 in a supernode, $I_{3,1}$, highlighted in green.
  - $I_{3,1} = \{V_4, V_5\}$ is one of the **information set** of W_3. $I_{3,2} = \{V_6, V_7\}$ is the second information set of W_3.
  - $I_{2,1} = \{V_2\}$, $I_{2,2} = \{V_3\}$, and $I_{1,1} = \emptyset$ are also information sets but "**trivial**".

Examples of EFG: Variant 2



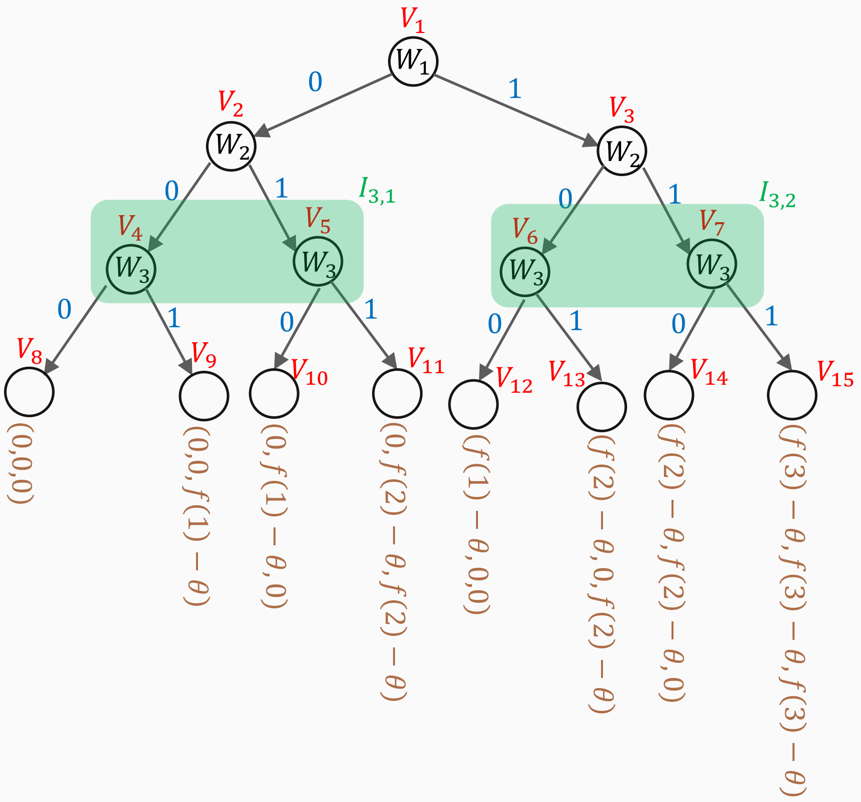
- In this variant, the **decision of W_1 and W_3 spreads** across the market as soon as it makes it however the **decision of W_2 does not spread**.
- This is an example of **IIEFG without chance moves**.
- The decision tree corresponding to this IIEFG is shown in the left.
  - Every information set is associated with a player and that player **can't distinguish** between the (non-leaf) vertices in an information set and hence the history of actions corresponding to these vertices.

Examples of EFG: Variant 2



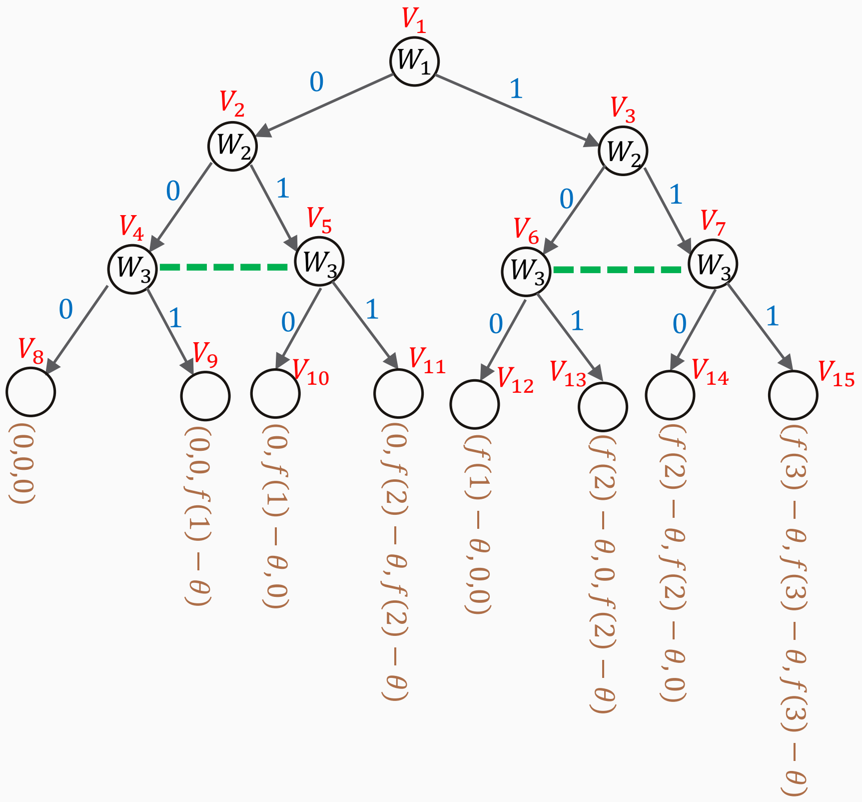
- In this variant, the **decision of W_1 and W_3 spreads** across the market as soon as it makes it however the **decision of W_2 does not spread**.
- This is an example of **IIEFG without chance moves**.
- The decision tree corresponding to this IIEFG is shown in the left.
  - Every information set is associated with a player and that player **can't distinguish** between the (non-leaf) vertices. Example: W_3 can't distinguish between **V_4 and V_5**, AND **V_6 and V_7**.

Examples of EFG: Variant 2



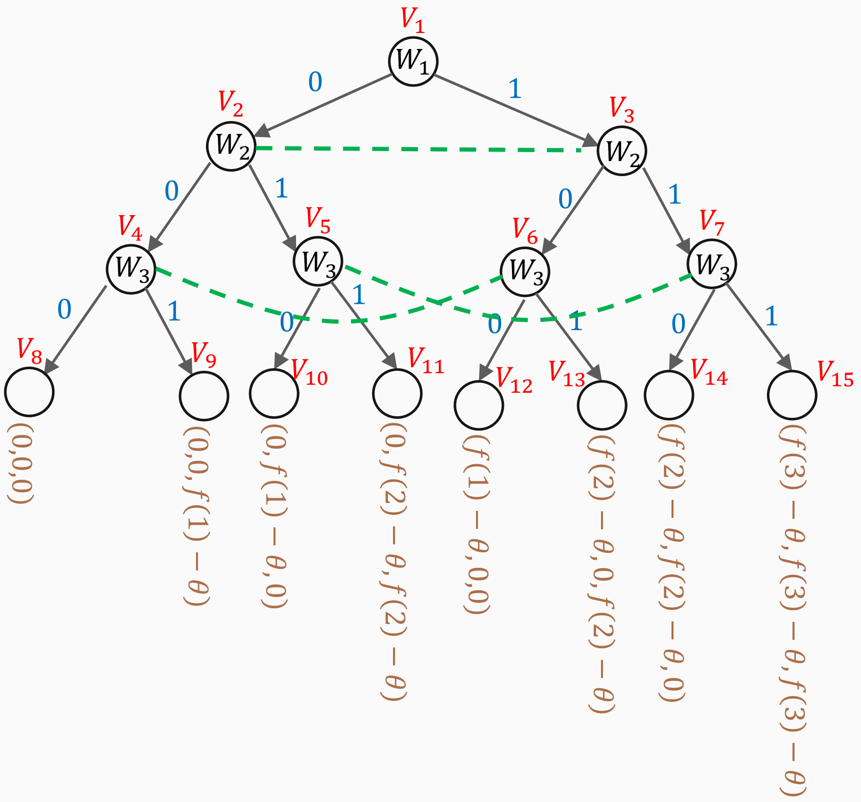
- In this variant, the **decision of W_1 and W_3 spreads** across the market as soon as it makes it however the **decision of W_2 does not spread**.
- This is an example of **IIEFG without chance moves**.
- The decision tree corresponding to this IIEFG is shown in the left.
  - The **payoffs will not change**.
  - This is because payoffs depends on the history of actions (at the leaf node).
  - Just because the history of actions is not known to a player does not change the said history of actions.

Examples of EFG: Variant 2



- In this variant, the **decision of W_1 and W_3 spreads** across the market as soon as it makes it however the **decision of W_2 does not spread**.
- This is an example of **IIEFG without chance moves**.
- The decision tree corresponding to this IIEFG is shown in the left.
    - Since it is difficult to draw such supernodes, we connect all the vertices corresponding to an information set using dashed lines.

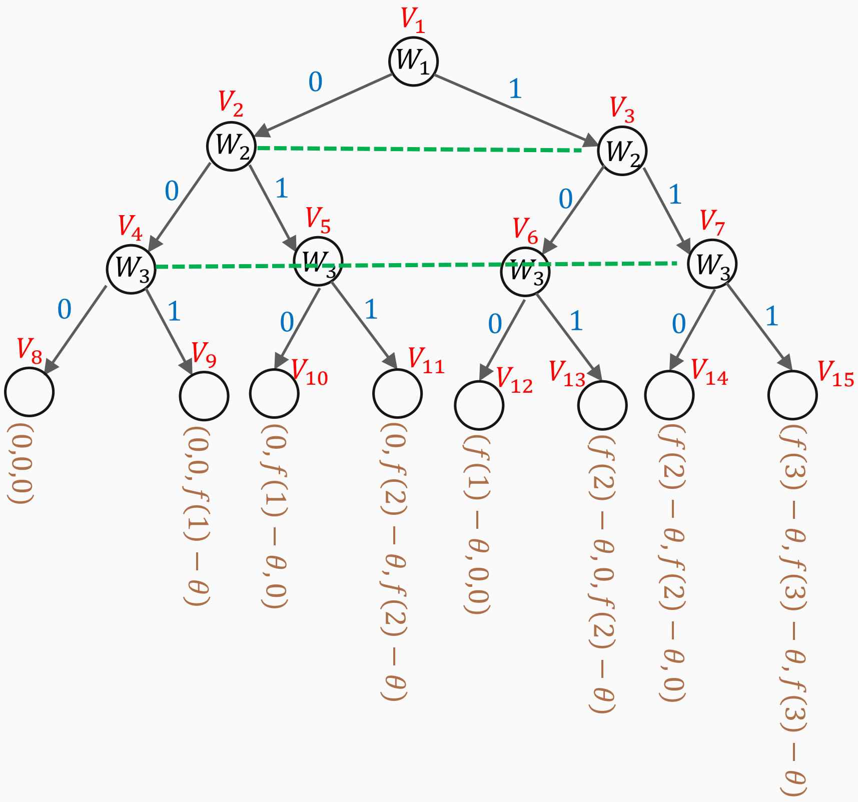
Examples of EFG: Variant 3



- In this variant, the **decision of W_2 and W_3 spreads** across the market as soon as it makes it however the **decision of W_1 does not spread**.
- This is an example of **IIEFG without chance moves**.
- The decision tree corresponding to this IIEFG is shown in the left.
    - Everything same as variant 2. Just that the action of W_1 is not known instead of W_2. Here the information sets are a bit complex and hence the need to device a **systematic approach** to find the information set.

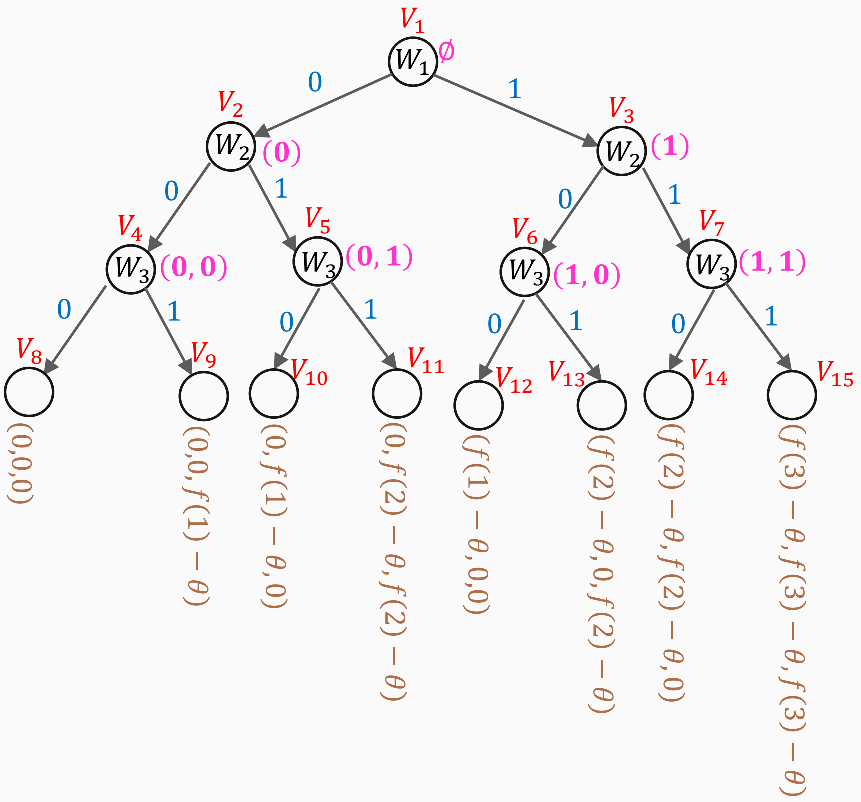
Discussed during lecture. In the process of writing it down.

Examples of EFG: Variant 4



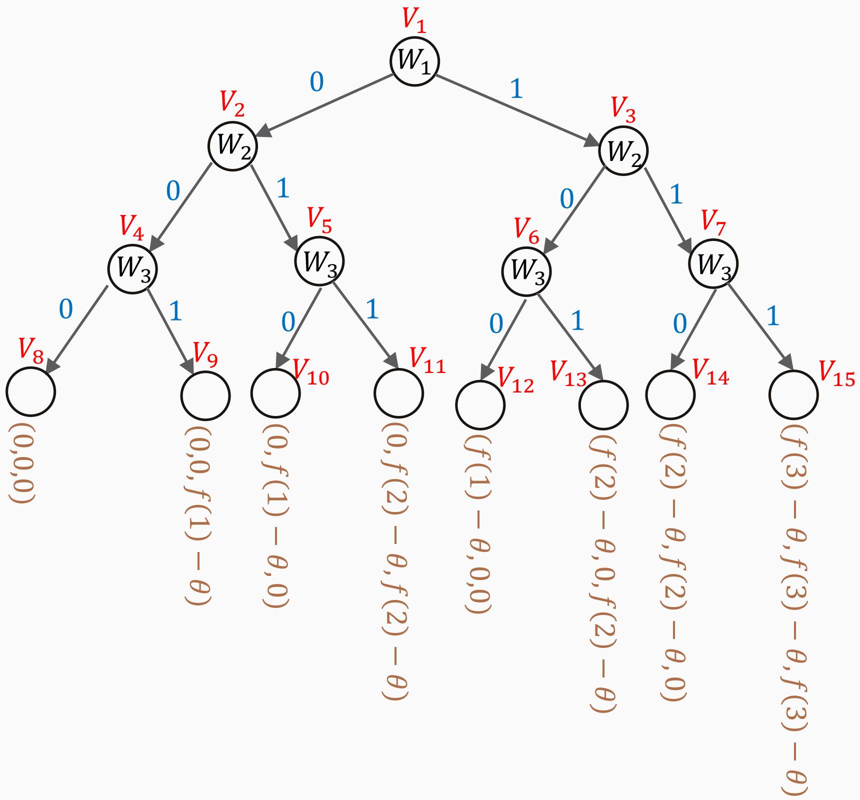
- In this variant, the **decision of W_3 spreads** across the market as soon as it makes it however the **decision of W_1 and W_2 does not spread**.
- This is an example of **IIEFG without chance moves**.
- The decision tree corresponding to this IIEFG is shown in the left.

Examples of EFG: Variant 4



- In this variant, the **decision of W_3 spreads** across the market as soon as it makes it however the **decision of W_1 and W_2 does not spread**.
- This is an example of **IIEFG without chance moves**.
- The decision tree corresponding to this IIEFG is shown in the left.

Examples of EFG: Variant 5



- In this variant, the **decision of W_1 and W_2 spreads** across the market as soon as it makes it however the **decision of W_3 does not spread**.

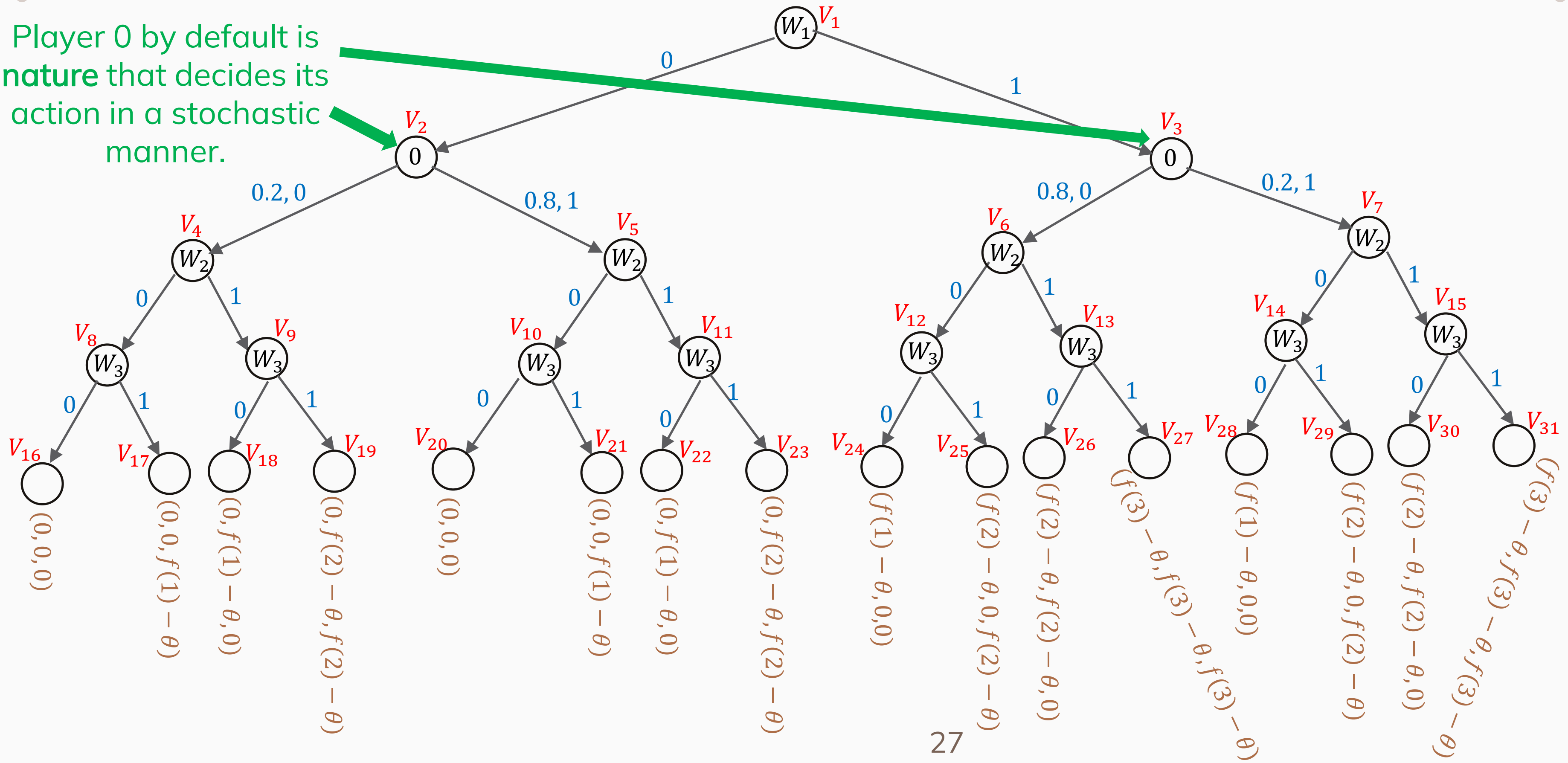
- This is also an example of IIEFG (**without chance moves**) but it looks like PIEFG **(but strictly speaking it is not PIEFG)** because W_3 is deciding it's action in the last step. Hence, the knowledge of it's action will not help other players to make their. Therefore it does not matter whether it's action is known or not.

Examples of EFG: Variant 6

Decision tree in next slide because of space shortage.

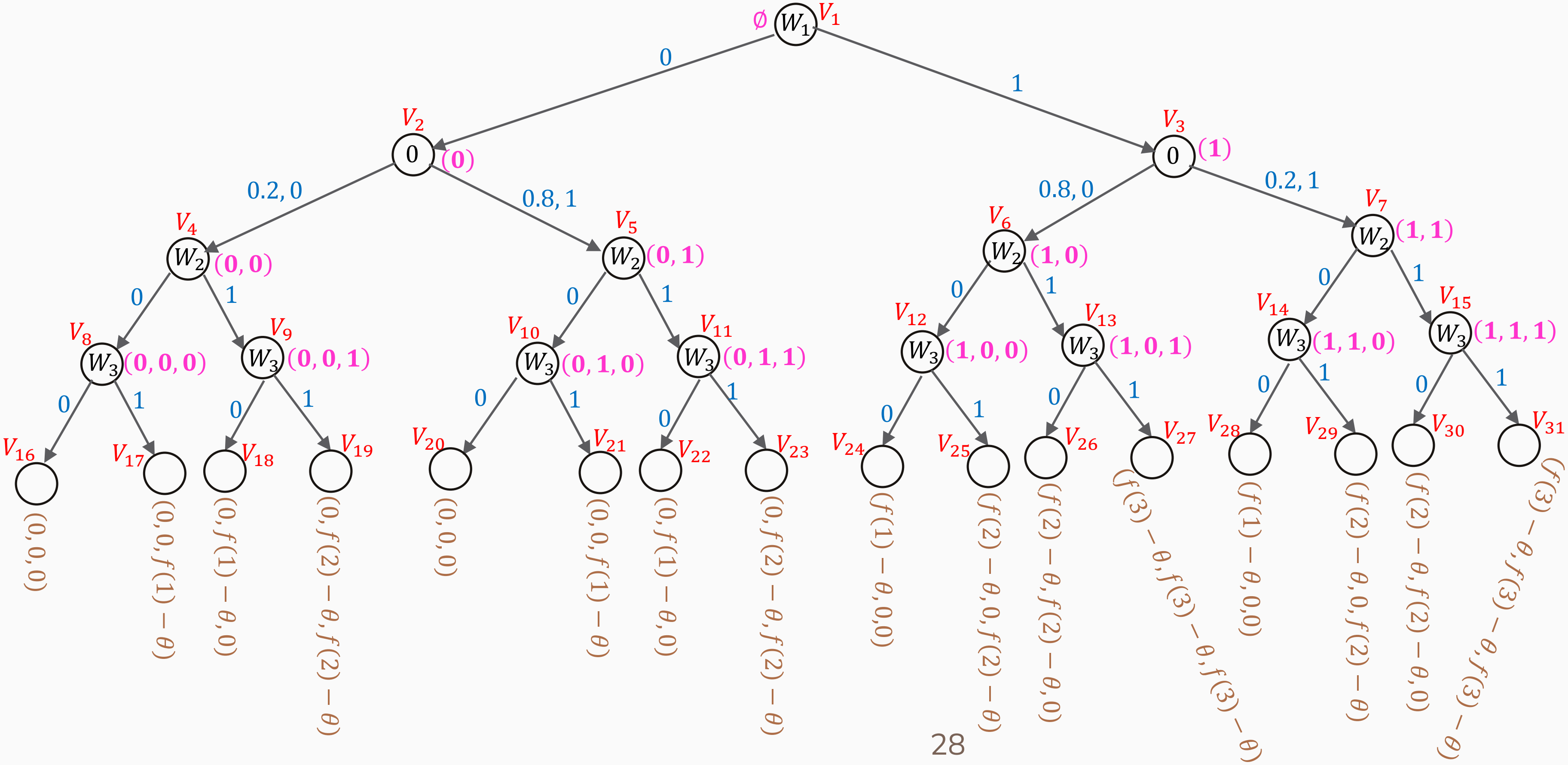
- In this variant, **as soon as a WSP makes a decision, the other WSPs gets to know about it**. However, **the probability that the correct decision of W_1 is known is 0.2**.
- This is an example of IIEFG **with chance moves**.
- The decision tree corresponding to this IIEFG is shown in the next page.

Examples of EFG: Variant 6

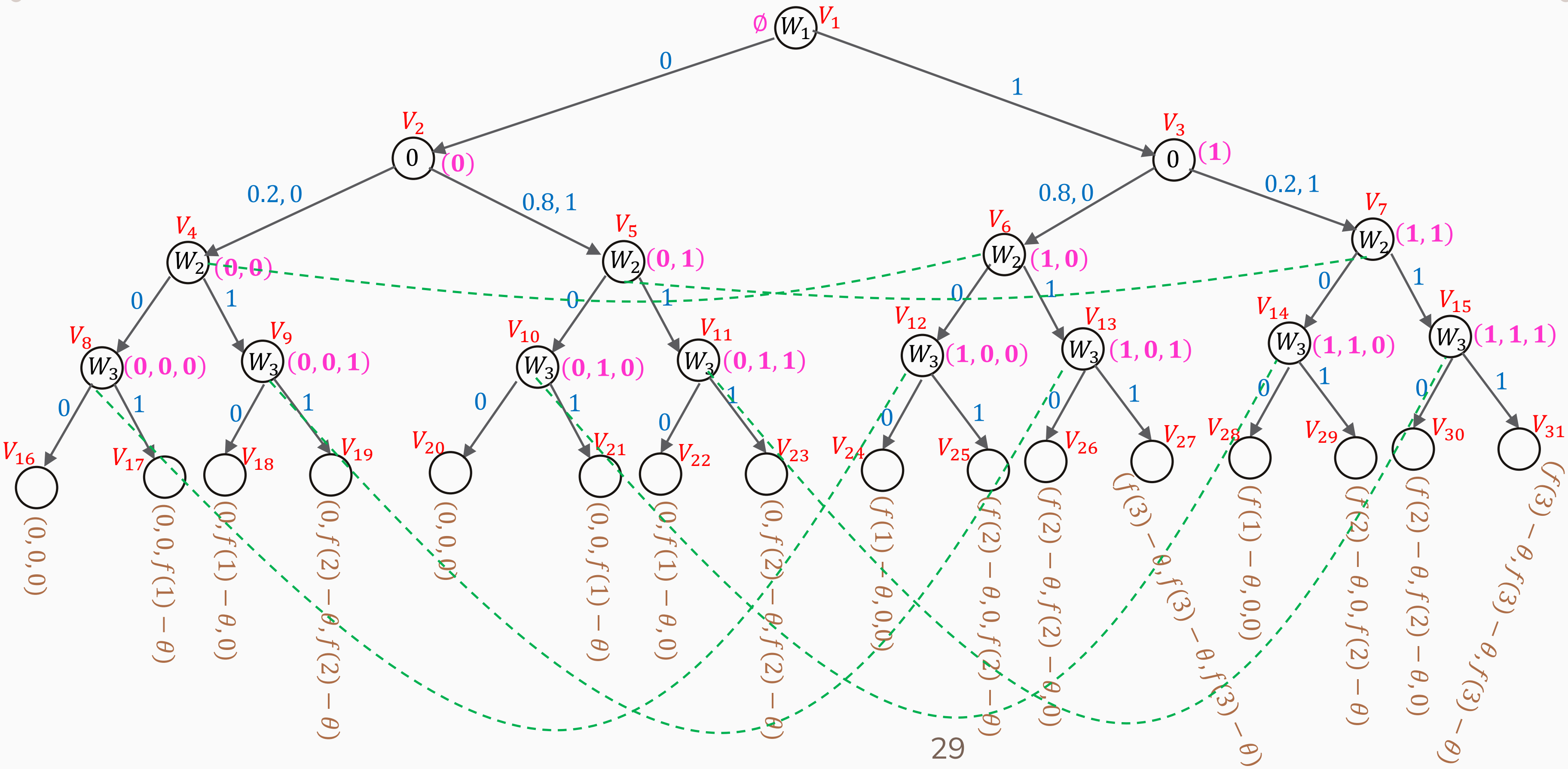


Player 0 by default is **nature** that decides its action in a stochastic manner.

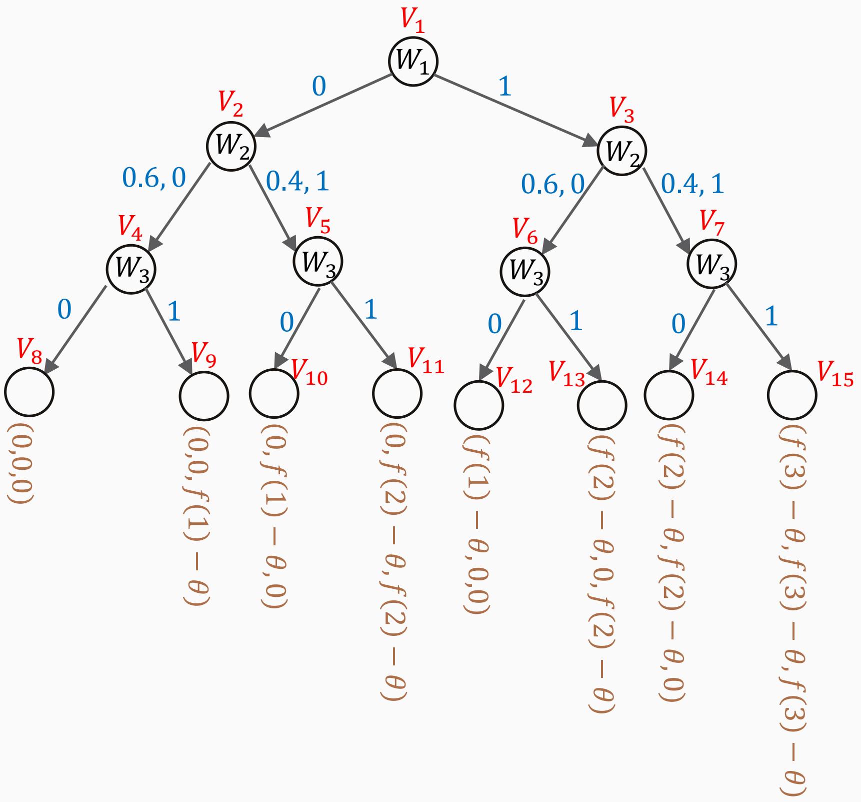
Examples of EFG: Variant 6



Examples of EFG: Variant 6

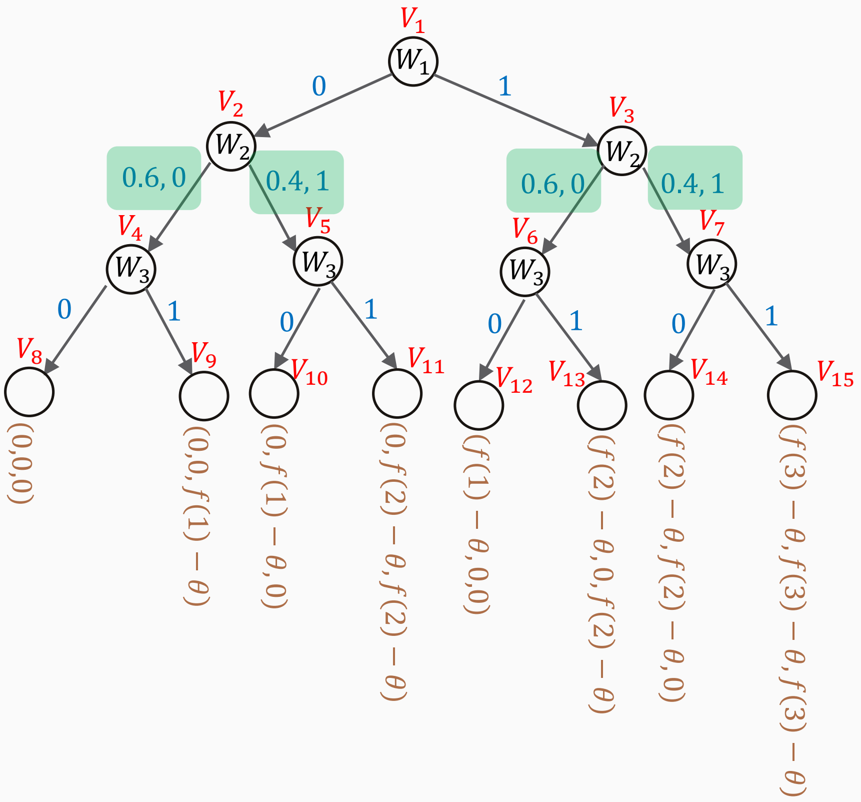


Examples of EFG: Variant 7



- In this variant, **W_2 is NOT a rational player. It decides to enters market with probability 0.4** **(not a practical example).** W_1 and W_3 are rational players. As soon as any WSP decides to enter the market, the other WSPs gets to know about it.
- This is an example of **PIEFG with chance moves**.
- The decision tree corresponding to this PIEFG is shown in the left.

Examples of EFG: Variant 7



- In this variant, **W_2 is NOT a rational player. It decides to enters market with probability 0.4 (not a practical example).** W_1 and W_3 are rational players. As soon as any WSP decides to enter the market, the other WSPs gets to know about it.
- This is an example of **PIEFG with chance moves**.
- The decision tree corresponding to this PIEFG is shown in the left.
  - Edges with probability and the corresponding actions are the only difference.

Definition of IIEFG with Chance Moves

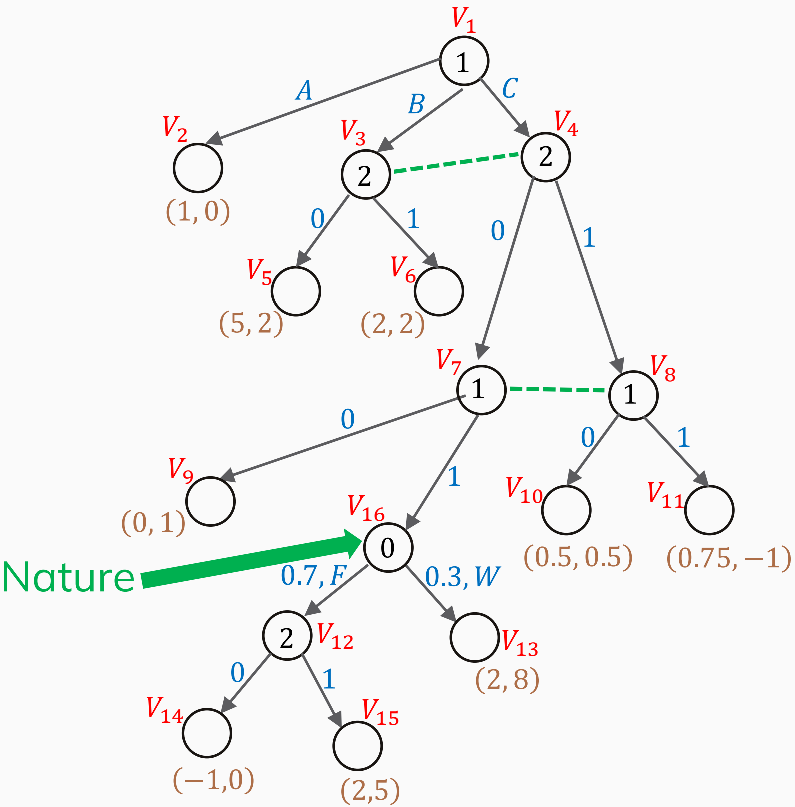
- Consider the following combinations:
  1. PIEFG without chance moves.
  2. PIEFG with chance moves.
  3. IIEFG without chance moves.
  4. IIEFG with chance moves.

  Out of the above four, **IIEFG with chance moves is the most general**. Other three are special cases of it. Hence, in the coming slides we will **formally define IIEFG** with chance moves* and also relate the definition with an **example**. Later, we will show how the other three is a special case of it.

* This definition is taken from the book: Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations by Yoav Shoham and Kevin Leyton-Brown.

I didn't follow our textbook because the definition was not very general. Also, the explanation was not enough. That said, even the above book did not account for chance moves. This is adapted from the book: Game Theory by M. Maschler, E. Solan, and S. Zamir.

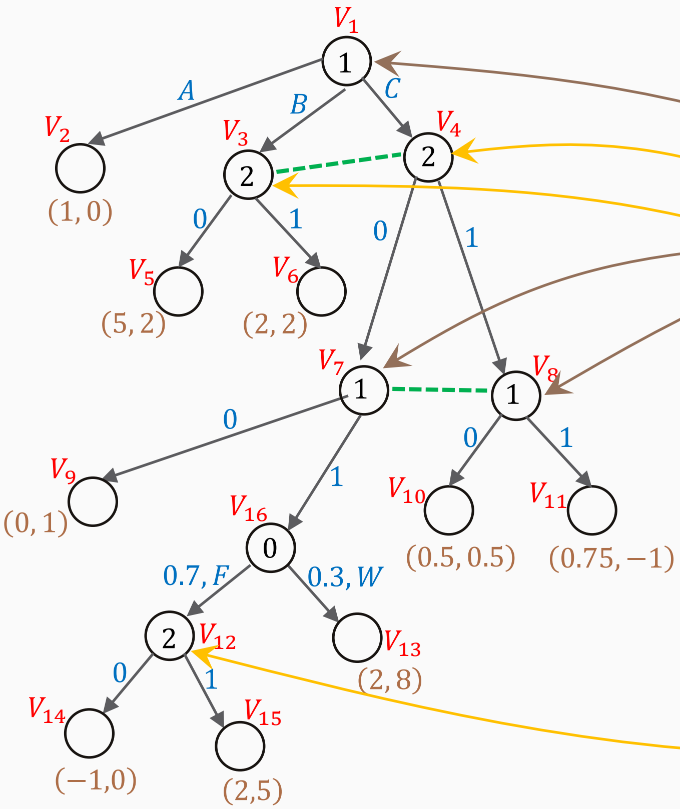
Definition of IIEFG with Chance Moves



- The decision tree with of a **fictitious** IIEFG with chance moves is shown in the left.

- Few things to note before moving forward:
  - A decision tree of an EFG completely describes the game. During exams and during class, the decision tree of a fictitious decision tree may be given with no practical justifications.

Definition of IIEFG with Chance Moves

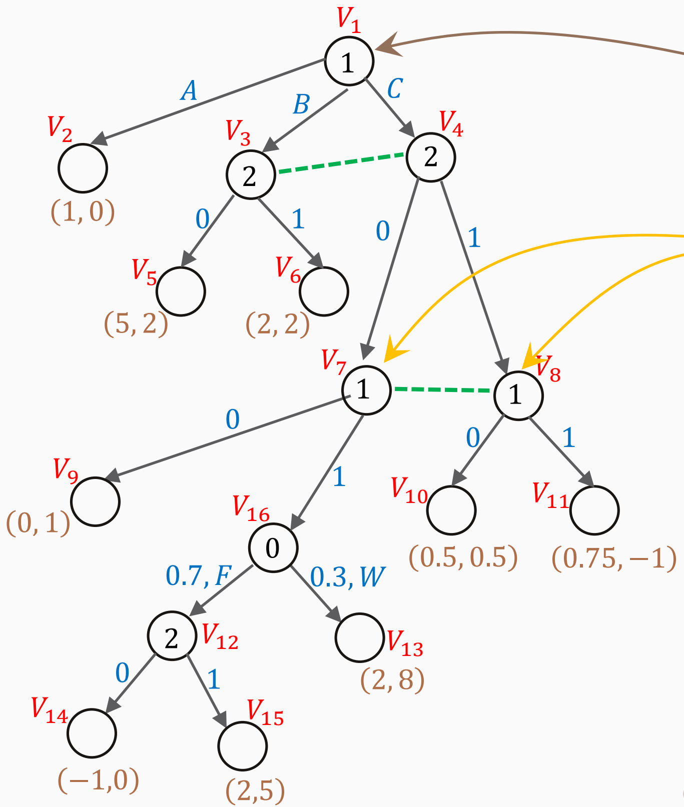


➢ The decision tree with of a **fictitious** IIEFG with chance moves is shown in the left.

➢ Few things to note before moving forward:

- A decision tree of an EFG completely describes the game. During exams and during class, the decision tree of a fictitious decision tree may be given with no practical justifications.
- In sequential games, it is possible that each player can make multiple decisions.
  1. Player 1 makes decisions here and here (makes decision two times).
  2. Player 2 makes decisions here and here (makes decision two times).

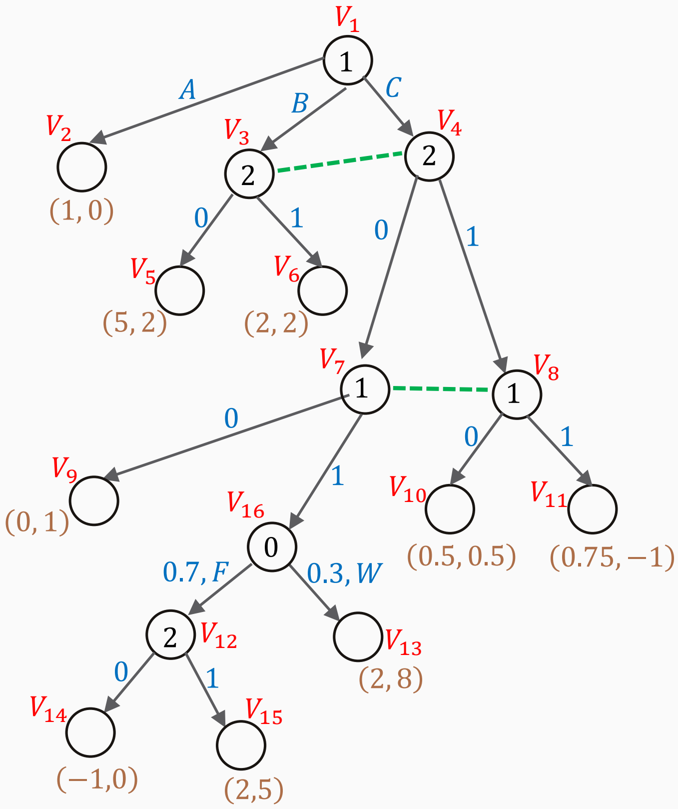
Definition of IIEFG with Chance Moves



- The decision tree with of a **fictitious** IIEFG with chance moves is shown in the left.

- Few things to note before moving forward:
  - The set of possible actions can be different each time a player makes decision. Example:
    1. When player 1 makes decision here, the set of possible actions are $\{A, B, C\}$.
    2. When player 1 makes decision here, the set of possible actions are $\{0,1\}$.

Definition of IIEFG with Chance Moves



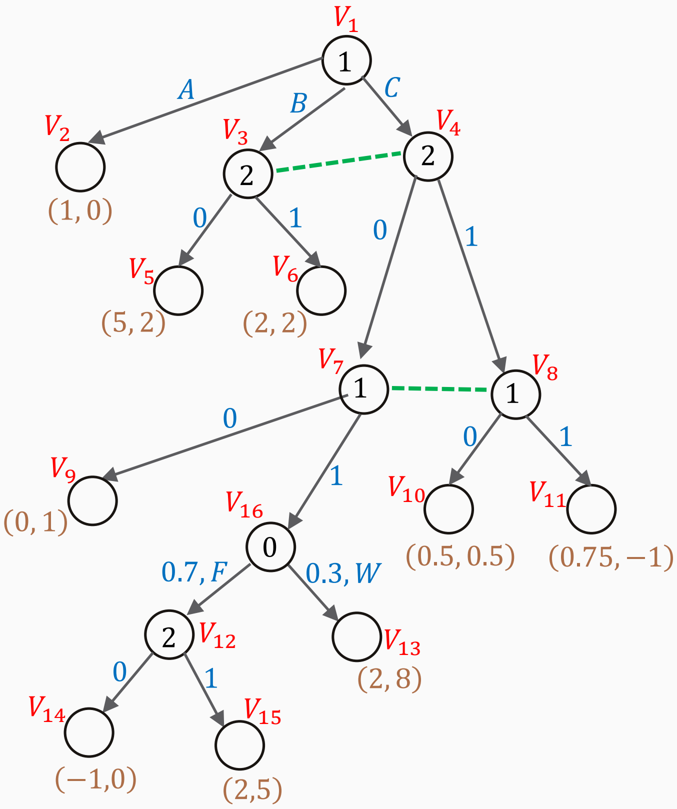
Definition (IIEFG with Chance Moves): An IIEFG with chance moves is a tuple $\langle N, A, H, Z, \rho, \mathcal{X}, \sigma, u, (p_i)_{i \in H_0}, I \rangle$ where:

- N is the **set of players**. The set of players also include **nature** which is usually denoted using $\mathbf{0}$.

Example: For the IIEFG shown in the left:

$$N = \{0, 1, 2\}$$

Definition of IIEFG with Chance Moves



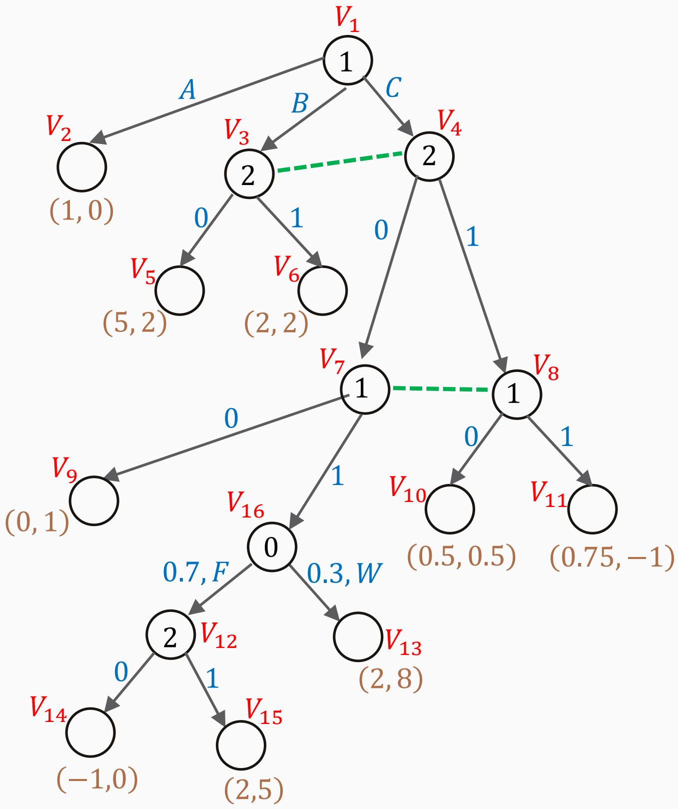
Definition (IIEFG with Chance Moves): An IIEFG with chance moves is a tuple $\langle N, A, H, Z, \rho, \mathcal{X}, \sigma, u, (p_i)_{i \in H_0}, I \rangle$ where:

- A is the **set of actions**. Set A contains the actions of **all the players**, across **all the decision instances**.

Example: For the IIEFG shown in the left:

$$A = \{A, B, C, 0,1, F, W\}$$

Definition of IIEFG with Chance Moves



Definition (IIEFG with Chance Moves): An IIEFG with chance moves is a tuple $\langle N, A, H, Z, \rho, \mathcal{X}, \sigma, u, (p_i)_{i \in H_0}, I \rangle$ where:

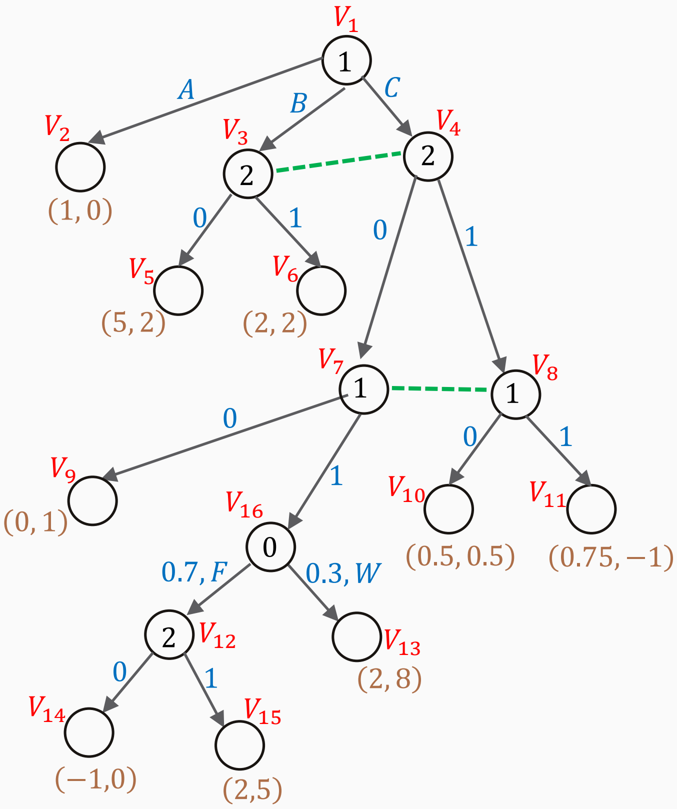
- H is the **set of non-terminal vertices**.
- Z is the **set of terminal vertices**. Sets H and Z are **disjoint**.

Example: For the IIEFG shown in the left:

$$H = \{V_1, V_3, V_4, V_7, V_8, V_{12}, V_{16}\}$$

$$Z = \{V_2, V_5, V_6, V_9, V_{10}, V_{11}, V_{13}, V_{14}, V_{15}\}$$

Definition of IIEFG with Chance Moves



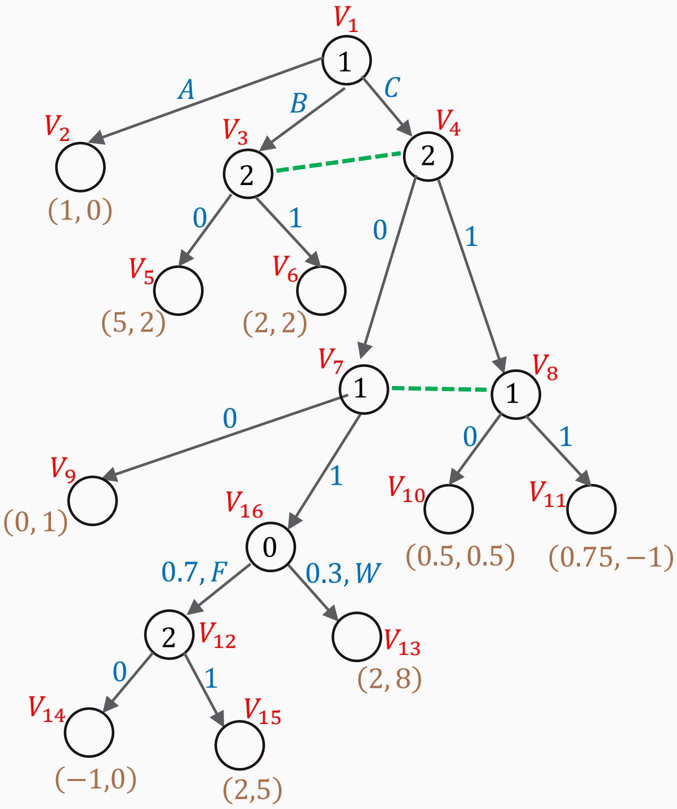
Definition (IIEFG with Chance Moves): An IIEFG with chance moves is a tuple $\langle N, A, H, Z, \rho, \mathcal{X}, \sigma, u, (p_i)_{i \in H_0}, I \rangle$ where:

- $\rho : H \rightarrow N$ is the **player function**. It assigns a player in set N to each non-terminal vertices in H.

Example: For the IIEFG shown in the left:

$$\rho(V_1) = 1$$
$$\rho(V_3) = 2$$
$$\rho(V_4) = 2$$
$$\rho(V_7) = 1$$
$$\rho(V_8) = 1$$
$$\rho(V_{12}) = 2$$
$$\rho(V_{16}) = 0$$

Definition of IIEFG with Chance Moves



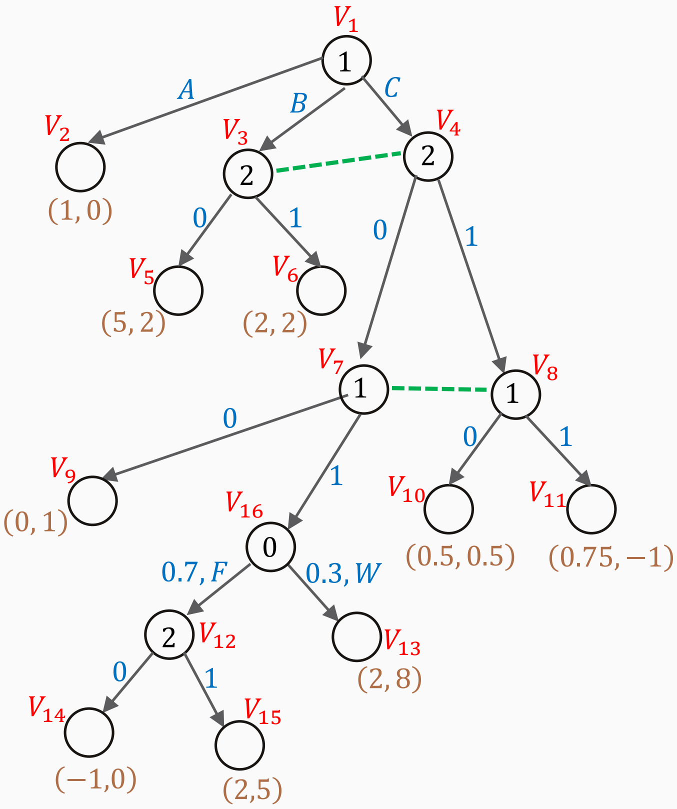
Definition (IIEFG with Chance Moves): An IIEFG with chance moves is a tuple $\langle N, A, H, Z, \rho, \mathcal{X}, \sigma, u, (p_i)_{i\in H_0}, I \rangle$ where:

- $\mathcal{X} : H \rightarrow 2^A$ is the **action function**. It assigns a subset of actions from set A to each non-terminal vertices in H. Remember: 2^A is the power set of A. Power set of A contains all subsets of A.

Example: For the IIEFG shown in the left:

$$\mathcal{X}(V_1) = \{A, B, C\}$$
$$\mathcal{X}(V_3) = \{0,1\}$$
$$\mathcal{X}(V_4) = \{0,1\}$$
$$\mathcal{X}(V_7) = \{0,1\}$$
$$\mathcal{X}(V_8) = \{0,1\}$$
$$\mathcal{X}(V_{12}) = \{0,1\}$$
$$\mathcal{X}(V_{16}) = \{F, W\}$$

Definition of IIEFG with Chance Moves



Definition (IIEFG with Chance Moves): An IIEFG with chance moves is a tuple $\langle N, A, H, Z, \rho, \mathcal{X}, \sigma, u, (p_i)_{i \in H_0}, I \rangle$ where:

- $\sigma: H \times A \rightarrow H \cup Z$ is the **successor function**. It maps a non-terminal vertex and an action to a vertex (either terminal or non-terminal).

Example: For the IIEFG shown in the left:

$$\sigma(V_1, A) = V_2$$

$$\sigma(V_1, B) = V_3$$

$$\sigma(V_1, C) = V_4$$

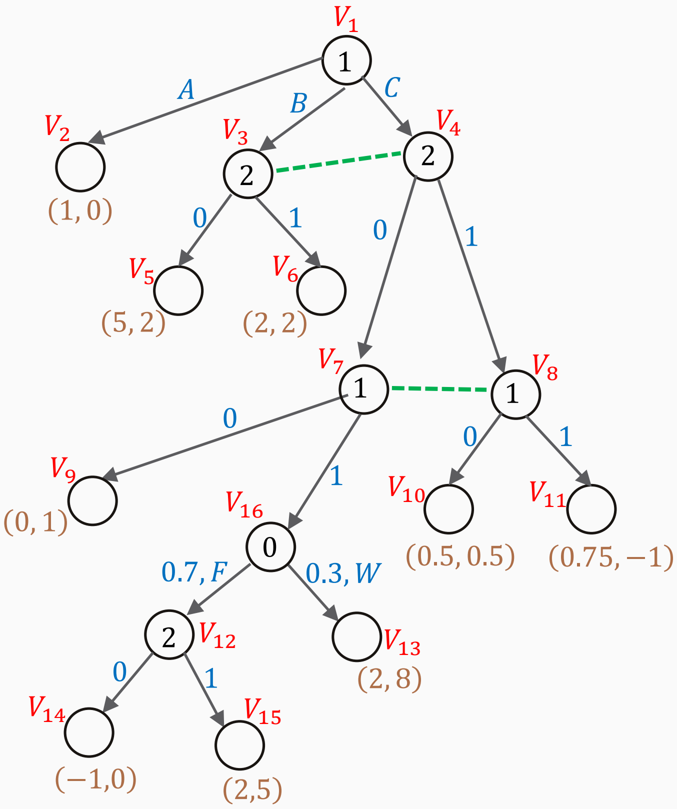
$$\sigma(V_{16}, W) = V_{13}$$

$$\sigma(V_8, 0) = V_{10}$$

$$\sigma(V_{12}, 1) = V_{15}$$

I could not list all, there are too many!

Definition of IIEFG with Chance Moves



Definition (IIEFG with Chance Moves): An IIEFG with chance moves is a tuple $\langle N, A, H, Z, \rho, \mathcal{X}, \sigma, u, (p_i)_{i \in H_0}, I \rangle$ where:

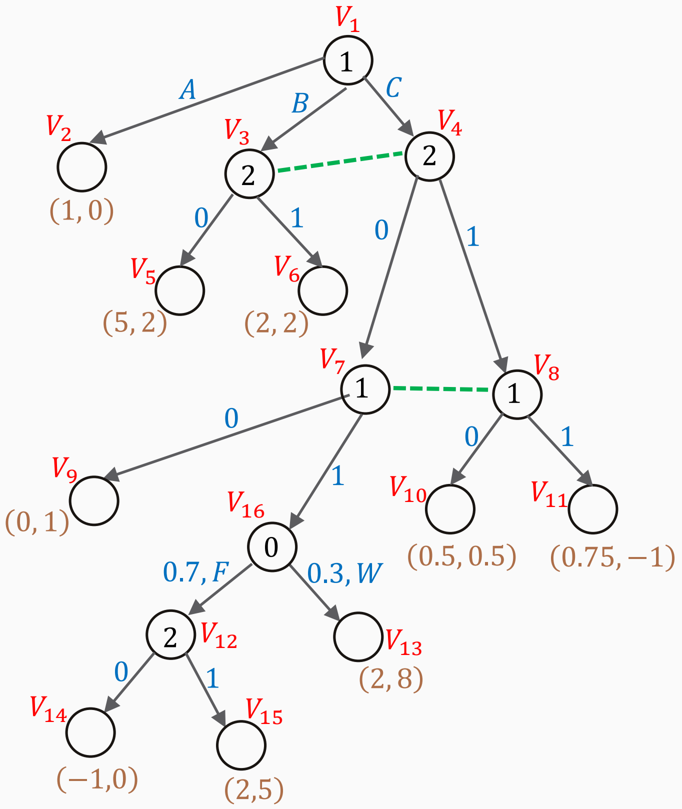
- ➢ $u = (u_n)_{n \in N \setminus \{0\}}$, where $u_n : Z \rightarrow \mathbb{R}$ is the **utility function** of player n (not nature). u_n maps a terminal vertex in Z with the utility of player n corresponding to that vertex.

Example: For the IIEFG shown in the left:

$$u_1(V_2) = 1, \qquad u_2(V_2) = 0$$
$$u_1(V_5) = 5, \qquad u_2(V_5) = 2$$
$$u_1(V_{11}) = 0.75, \qquad u_2(V_{11}) = -1$$
$$u_1(V_{13}) = 2, \qquad u_2(V_{13}) = 8$$
$$u_1(V_{14}) = -1, \qquad u_2(V_{14}) = 0$$

I could not list all, there are still a few!

Definition of IIEFG with Chance Moves



Definition (IIEFG with Chance Moves): An IIEFG with chance moves is a tuple $\langle N, A, H, Z, \rho, \mathcal{X}, \sigma, u, (p_i)_{i \in H_0}, I \rangle$ where:

- $H_0 = \{i \in H : \rho(i) = 0\}$ is the **set of non-terminal vertices for which the player is nature**. For every $i \in H_0$, $\boldsymbol{p_i}$ **is the probability distribution** across actions in set $\mathcal{X}(i)$.

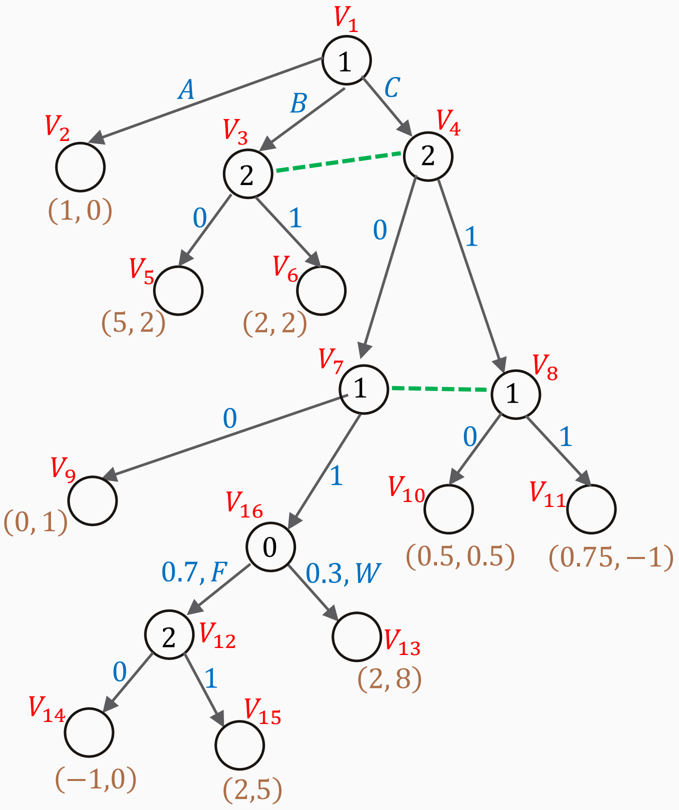
Example: For the IIEFG shown in the left:

$$H_0 = \{V_{16}\}$$

$$\mathcal{X}(V_{16}) = \{F, W\}$$

$$p_{V_{16}} = (0.7, 0.3)$$

Definition of IIEFG with Chance Moves



Definition (IIEFG with Chance Moves): An IIEFG with chance moves is a tuple $\langle N, A, H, Z, \rho, \mathcal{X}, \sigma, u, (p_i)_{i \in H_0}, I \rangle$ where:

- $I = (I_n)_{n \in N \setminus \{0\}}$ where $I_n = \{I_{n,1}, I_{n,2}, \cdots, I_{n,k_n}\}$ is the **set of information sets** of player n (not nature) and $I_{n,j}$ is the j^{th} information set of player n. $I_{n,j}$'s satisfies the following properties:

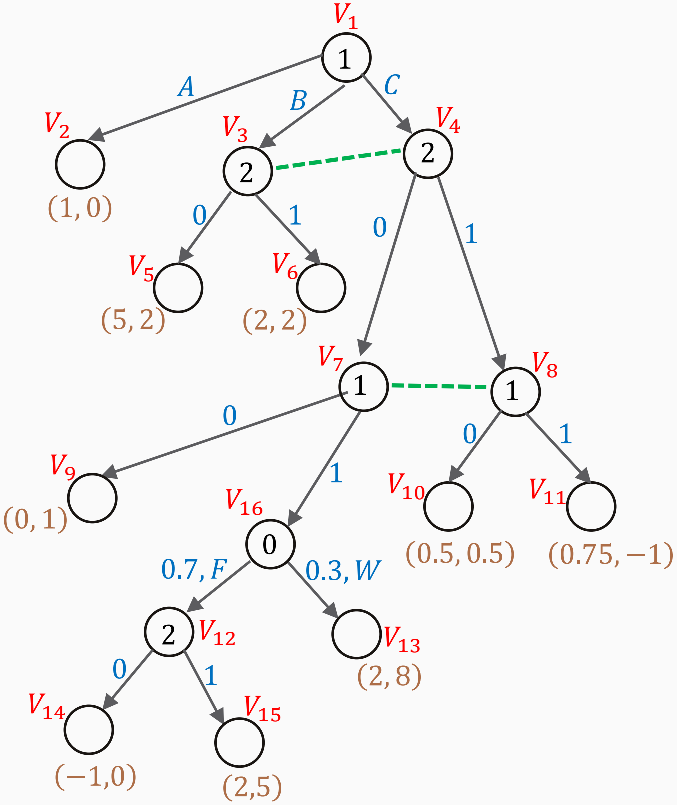
  i. $I_{n,j}$'s are **disjoint** sets.

  ii. For player n, $\bigcup_{j=1}^{k} \boldsymbol{I_{n,j}} = \boldsymbol{H_n}$ where $H_n = \{i \in H : \rho(i) = n\}$.

  iii. For any $h \in I_{n,j}$ and $h' \in I_{n,j}$, $\rho(h) = \rho(h') = n$. This basically says that **an information set is associated with one player** only (obviously!).

  iv. For any $h \in I_{n,j}$ and $h' \in I_{n,j}$, $\mathcal{X}(h) = \mathcal{X}(h')$. This basically says that the set of actions for all vertices in an information set should be the same. **Otherwise, the player will get to know which vertex it is in!**

Definition of IIEFG with Chance Moves



Definition (IIEFG with Chance Moves): An IIEFG with chance moves is a tuple $\langle N, A, H, Z, \rho, \mathcal{X}, \sigma, u, (p_i)_{i\in H_0}, I \rangle$ where:

- $I = (I_n)_{n\in N\setminus\{0\}}$ where $I_n = \{I_{n,1}, I_{n,2}, \cdots, I_{n,k_n}\}$ is the **set of information sets** of player n (not nature) and $I_{n,j}$ is the j^{th} information set of player n.

Example: For the IIEFG shown in the left:

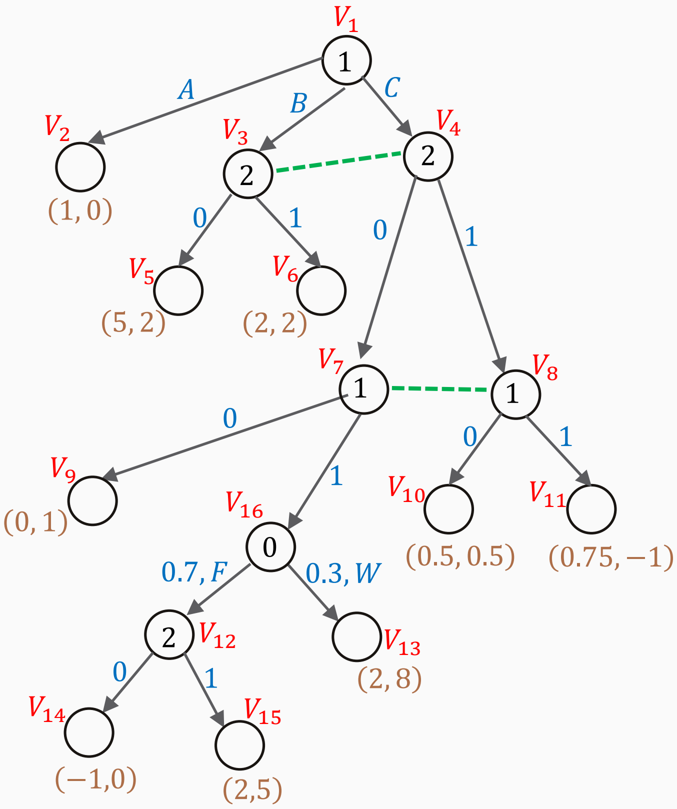
$$I_1 = \{\underbrace{\{V_1\}}_{I_{1,1}}, \underbrace{\{V_7, V_8\}}_{I_{1,2}}\}$$

Note: $H_1 = \{V_1, V_7, V_8\}$

$$I_2 = \{\underbrace{\{V_3, V_4\}}_{I_{2,1}}, \underbrace{\{V_{12}\}}_{I_{2,2}}\}$$

Note: $H_2 = \{V_3, V_4, V_{12}\}$

Definition of IIEFG with Chance Moves



Definition (IIEFG with Chance Moves): An IIEFG with chance moves is a tuple $\langle N, A, H, Z, \rho, \mathcal{X}, \sigma, u, (p_i)_{i \in H_0}, I \rangle$.

- As mentioned before, $h \in I_{n,j}$ and $h' \in I_{n,j}$, $\rho(h) = \rho(h') = n$.
- Therefore, the **player function** ρ can be instead thought of as a **mapping from an information set to a player**. To elaborate, for every information set $\tilde{I} \in I_n$, $\rho(\tilde{I})$ is the player corresponding to information set $\tilde{I}$.

- As mentioned before, $h \in I_{n,j}$ and $h' \in I_{n,j}$, $\mathcal{X}(h) = \mathcal{X}(h')$.
- Therefore, the **action function** $\mathcal{X}$ can be instead thought of as a **mapping from an information set to a subset of action in A**. To elaborate, for every information set $\tilde{I} \in I_n$, $\mathcal{X}(\tilde{I})$ is the set of actions of player $\rho(\tilde{I})$ corresponding to information set $\tilde{I}$.

Definition of IIEFG with Chance Moves

- Now that we have discussed IIEFG with chance moves, let’s see how the followings are special cases of IIEFG with chance moves:
  1. PIEFG without chance moves.
  2. PIEFG with chance moves.
  3. IIEFG without chance moves.

Definition of PIEFG without Chance Moves

Definition (PIEFG without Chance Moves): A PIEFG without chance moves is a tuple $\langle N, A, H, Z, \rho, \mathcal{X}, \sigma, u \rangle$ where the definition of all the terms is same as that in IIEFG with chance moves.

- There is no player 0 (nature) because there is no chance moves. Hence, there is no associated probability distribution $(\boldsymbol{p_i})_{\boldsymbol{i \in H_0}}$ in the tuple.

- For games with perfect information (with or without chance moves), the **information sets are singleton sets**. In other words, **every non-terminal vertex is an information set** and hence there is no need to separately define an information set. For this reason, the set of information sets for all players, I, has been removed from the tuple.

Definition of PIEFG with Chance Moves

Definition (PIEFG with Chance Moves): A PIEFG with chance moves is a tuple $\langle N, A, H, Z, \rho, \mathcal{X}, \sigma, u, (p_i)_{i \in H_0} \rangle$ where the definition of all the terms is same as that in IIEFG with chance moves.

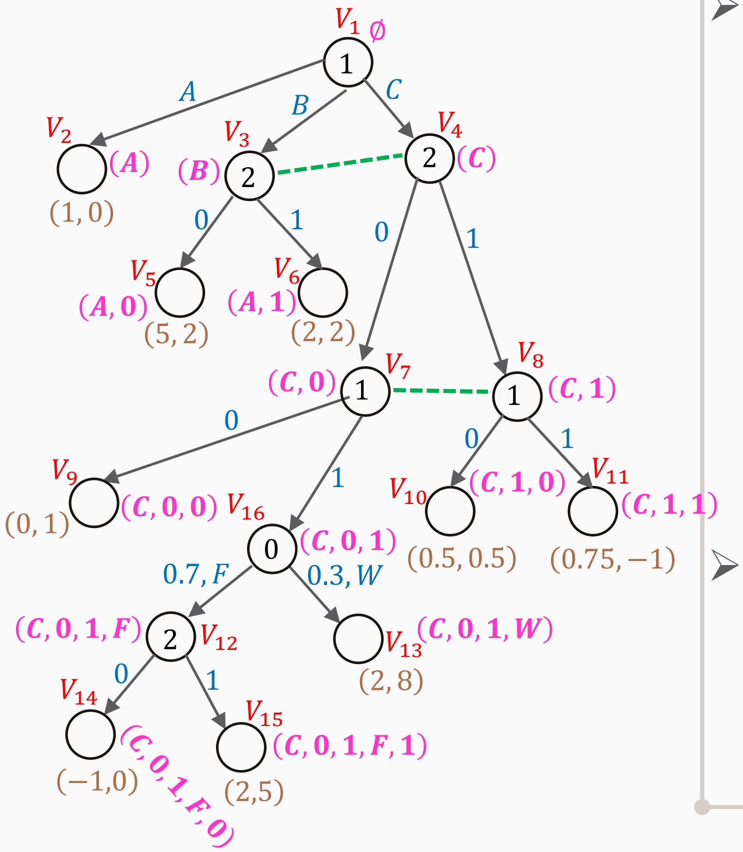
- No information set just like PIEFG without chance moves. But, it has $(\boldsymbol{p_i})_{\boldsymbol{i \in H_0}}$ to capture chance moves.

Definition of IIEFG without Chance Moves

Definition (IIEFG without Chance Moves): An IIEFG without chance moves is a tuple $\langle N, A, H, Z, \rho, \mathcal{X}, \sigma, u\ , I\rangle$ where the definition of all the terms is same as that in IIEFG with chance moves.

- There is no player 0 (nature) because there is no chance moves. Hence, there is no associated probability distribution $\boldsymbol{(p_i)_{i\in H_0}}$ in the tuple.

History Associated with a Vertex



- We have already seen in Slide 23 and Slide 30 that every vertex (both terminal and non-terminal) is associated with a history of actions.
  - Also called history instead of history of actions.
  - History of a vertex is a sequence. Let $\psi(h)$ denote the history of vertex h.

  In this slide we just discuss a systematic approach to find the history of a vertex (not that you don't know it by now; wanted to be rigorous).

- The history associated with root node is the null set. Starting from root node, the history of any non-root node can be found by iteratively using the following approach:

  For all action $a \in \mathcal{X}(h)$ of a non-terminal vertex h, the history of vertex $\sigma(h,a)$ is a appended to the end of history $\psi(h)$.

Converting an SFG to an EFG

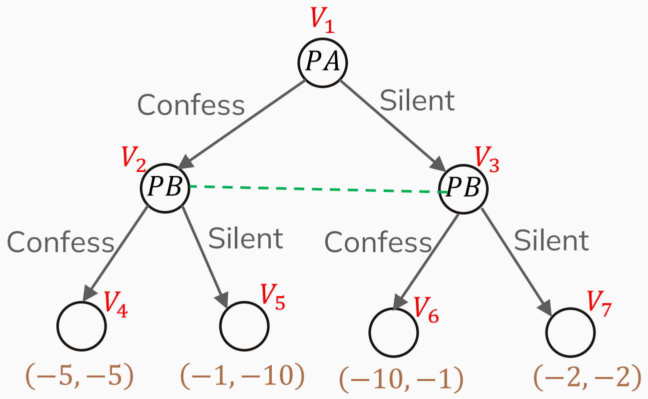
- Simultaneous moves can also be interpreted as sequential moves but with **imperfect information**. To elaborate, a n **players sequential move game** (SFG) is **equivalent** to a **sequential move game (EFG) where a player making a decision does not know the decisions of the players who made their decision before it**.

- The above equivalence is true because the only difference between SFGs and EFGs is that the players may get to know the decision of the players who made their decision before it and hence use this decision to make a more informed decision.

- The above equivalence also shows that SFG is a special case of EFG (more specifically IIEFG).

Converting an SFG to an EFG

		PB	
		Confess	Silent
PA	Confess	-5, -5	-1, -10
	Silent	-10, -1	-2, -2



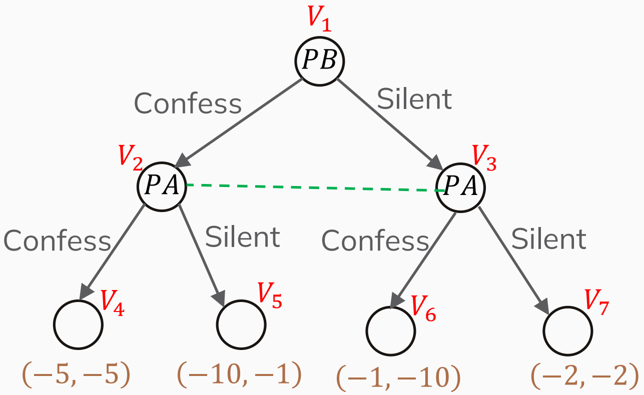
- Consider the prisoner’s dilemma game. Both the prisoners have to make their decisions simultaneously. Their payoff matrix is shown in the left.

- Prisoner’s dilemma can be equivalently modeled as a sequential move game (EFG) as follows:
  - Prisoner A **(PA)** makes decision first.
  - Then, prisoner B **(PB)** makes its decision without knowing the decision of prisoner A.

This is shown in the left.

Converting an SFG to an EFG

		PB	
		Confess	Silent
PA	Confess	-5, -5	-1, -10
	Silent	-10, -1	-2, -2



- Consider the prisoner's dilemma game. Both the prisoners have to make their decisions simultaneously. Their payoff matrix is shown in the left.
- Prisoner's dilemma can be equivalently modeled as a sequential move game (EFG) as follows:
  - Prisoner A **(PA)** makes decision first.
  - Then, prisoner B **(PB)** makes its decision without knowing the decision of prisoner A.

  This is shown in the left.
- We can also let prisoner B make decision first and then prisoner B. This is shown in the left.

EFG corresponding to a SFG is not unique!

Strategy of a player in EFG

- Informally, strategy of player is a **complete description** of what decision that player will make whenever it has to make a decision.

- Now, player n has to make decisions only at its **set of information sets**, I_n. Let $a_{n,j}$ denote the action that player n takes corresponding to $I_{n,j}$ *($I_{n,j}$ the j^{th} information set of player n)*. Then the strategy of player n is,

  $$s_n = \left(a_{n,1}, a_{n,2}, \cdots, a_{n,k_n}\right)$$

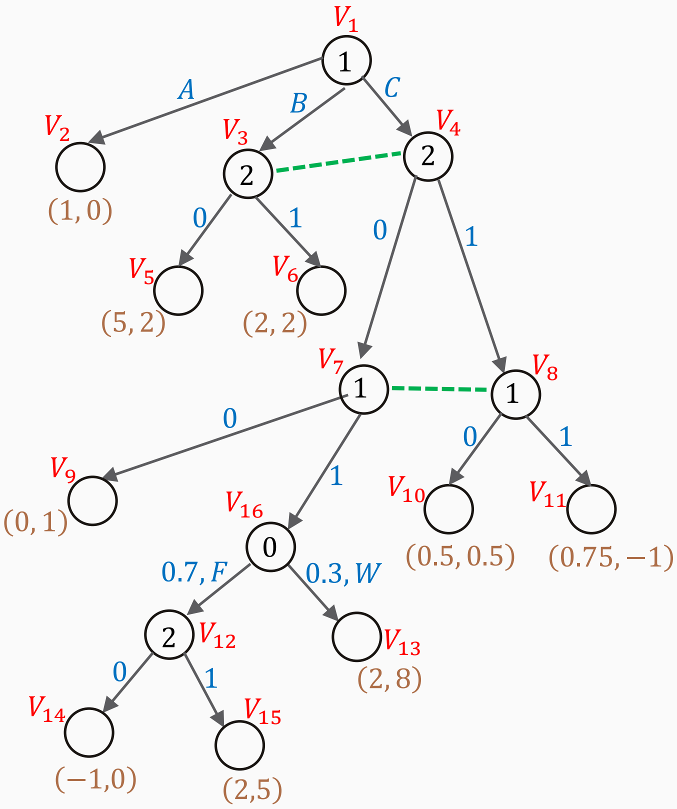
  s_n is a **complete description** of the decision player n will make in every one of its information set.

- The set of strategies at information set $I_{n,j}$ of player n is $\mathcal{X}\left(I_{n,j}\right)$. Hence, the **set of strategies of player n** is the cartesian product of all $\mathcal{X}\left(I_{n,j}\right)$,

  $$S_n = \mathcal{X}\left(I_{n,1}\right) \times \mathcal{X}\left(I_{n,2}\right) \times \cdots \times \mathcal{X}\left(I_{n,k_n}\right)$$

  We have $s_n \in S_n$.

Strategy of a player in EFG



Example:

$$S_n = \mathcal{X}(I_{n,1}) \times \mathcal{X}(I_{n,2}) \times \cdots \times \mathcal{X}(I_{n,k_n})$$

$$\begin{aligned} S_1 &= \mathcal{X}(\{V_1\}) \times \mathcal{X}(\{V_7, V_8\}) \\ &= \{A, B, C\} \times \{0,1\} \\ &= \{(A, 0), (A, 1), (B, 0), (B, 1), (C, 0), (C, 1)\} \end{aligned}$$

$$\begin{aligned} S_2 &= \mathcal{X}(\{V_3, V_4\}) \times \mathcal{X}(\{V_{12}\}) \\ &= \{0,1\} \times \{0,1\} \\ &= \{(0,0), (0,1), (1,0), (1,1)\} \end{aligned}$$

Converting an EFG to an SFG

- In the previous two slides we talked about the **strategy of a player** in EFG. Now that we have discussed strategy of a player, we can discuss about converting and EFG to its **equivalent** SFG.

- What is the physical interpretation of converting an EFG to a SFG?
  - Recall that strategy of a player in EFG is the complete description of what that player will do in each of its information sets. Informally, strategy of a player is the "**plan** of the player".
  - The players in EFG can make the plan in advance.
  - This "plan making" process can be thought of as a **SFG** where all the players **decide simultaneously** about the actions it will take at each of its information sets.

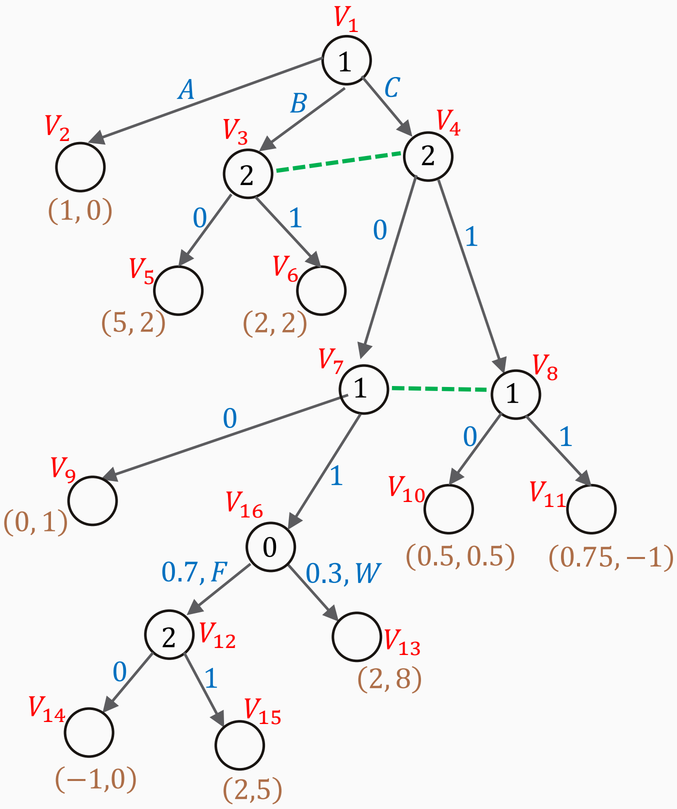
- SFG is simpler to solve compared to EFG. So if we convert EFG to SFG it will be easier to analyze the decisions made by players in EFG.

Converting an EFG to an SFG

- That said, the disadvantage of converting EFG to SFG is that it **does not capture the sequential nature** of the game. To elaborate:
  - In game theory, players has to make decisions while accounting for various decisions other players will make.
  - If the players are making a plan in advance (that is what converting an EFG to SFG means, i.e. to make a plan in advance about what to do in every information set), a player has to account for many more decisions other players can make compared to if the player reaches a particular information set and then decides what decision to take. This is because when a player reaches an information set, it **knows some of the decisions** made by other players (including itself). This reduces the **set of decision the other players can make**.

Because of this disadvantage, many of the solutions obtained by converting an EFG to SFG does not make logical sense.

Converting an EFG to an SFG: Example



$S_1 = \{(A,0),(A,1),(B,0),(B,1),(C,0),(C,1)\}$

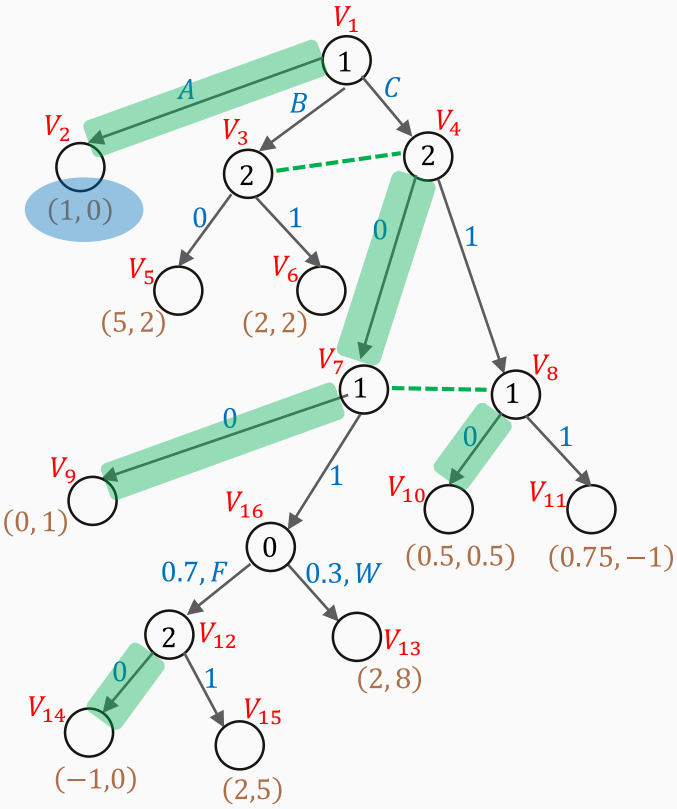
$S_2 = \{(0,0),(0,1),(1,0),(1,1)\}$

Player 2

Player 1

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(A,0)$				
$(A,1)$				
$(B,0)$				
$(B,1)$				
$(C,0)$				
$(C,1)$				

Converting an EFG to an SFG: Example



$S_1 = \{(A,0),(A,1),(B,0),(B,1),(C,0),(C,1)\}$

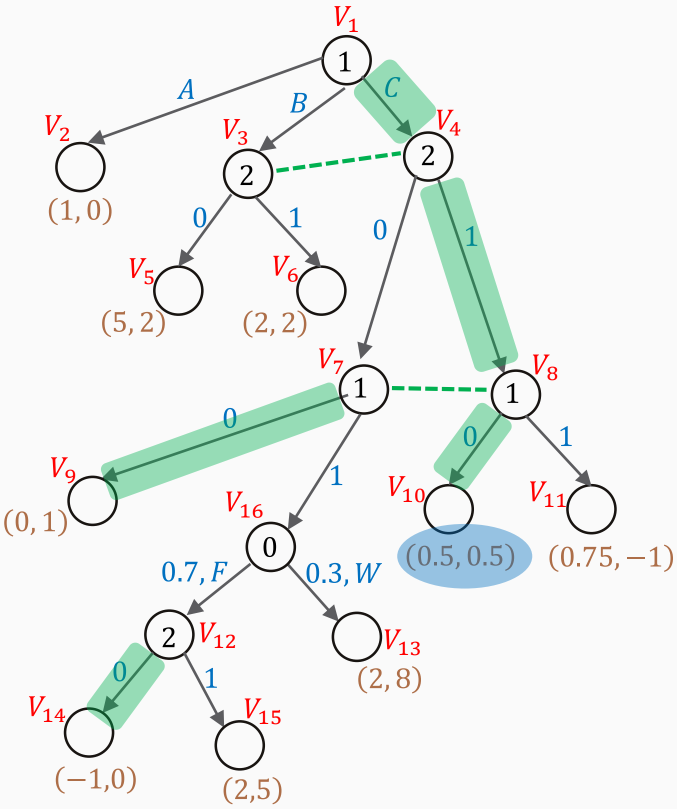
$S_2 = \{(0,0),(0,1),(1,0),(1,1)\}$

Player 2

Player 1

	(0,0)	(0,1)	(1,0)	(1,1)
$(A,0)$	(1,0)			
$(A,1)$				
$(B,0)$				
$(B,1)$				
$(C,0)$				
$(C,1)$				

Converting an EFG to an SFG: Example



$S_1 = \{(A,0),(A,1),(B,0),(B,1),(C,0),(C,1)\}$

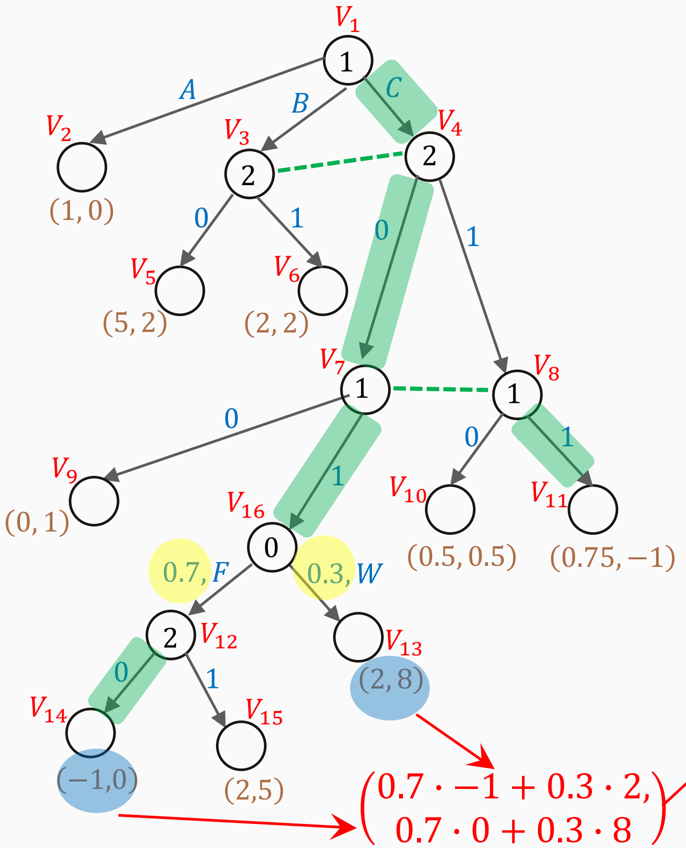
$S_2 = \{(0,0),(0,1),(1,0),(1,1)\}$

Player 2

Player 1

	(0,0)	(0,1)	(1,0)	(1,1)
$(A,0)$	(1,0)			
$(A,1)$				
$(B,0)$				
$(B,1)$				
$(C,0)$			(0.5,0.5)	
$(C,1)$				

Converting an EFG to an SFG: Example



$$\begin{pmatrix} 0.7 \cdot -1 + 0.3 \cdot 2, \\ 0.7 \cdot 0 + 0.3 \cdot 8 \end{pmatrix}$$

$S_1 = \{(A,0),(A,1),(B,0),(B,1),(C,0),(C,1)\}$

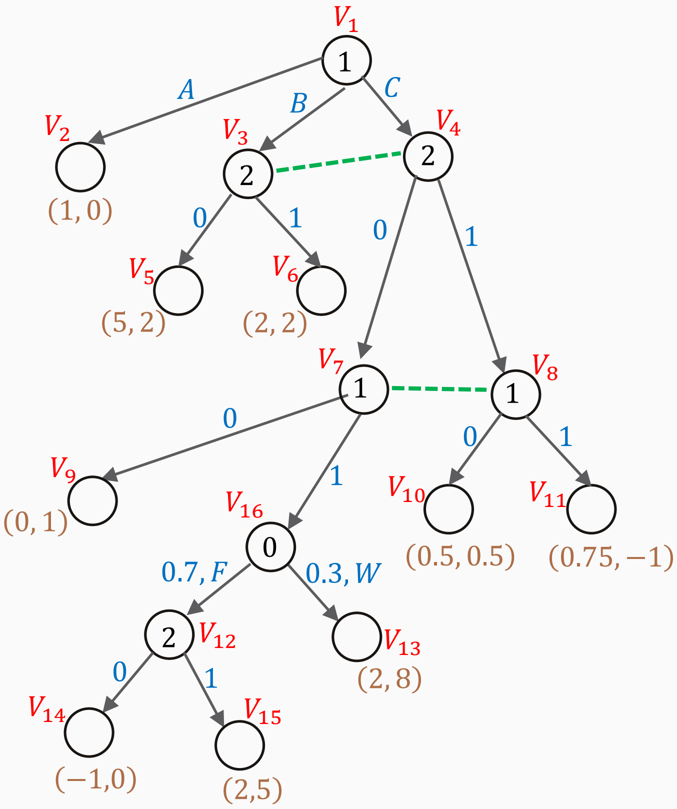
$S_2 = \{(0,0),(0,1),(1,0),(1,1)\}$

Player 2

Player 1

	(0,0)	(0,1)	(1,0)	(1,1)
$(A,0)$	(1,0)			
$(A,1)$				
$(B,0)$				
$(B,1)$				
$(C,0)$			(0.5,0.5)	
$(C,1)$	(−0.1,2.4)			

Converting an EFG to an SFG



$S_1 = \{(A,0),(A,1),(B,0),(B,1),(C,0),(C,1)\}$

$S_2 = \{(0,0),(0,1),(1,0),(1,1)\}$

		Player 2			
		$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
Player 1	$(A,0)$	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$
	$(A,1)$	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$
	$(B,0)$	$(5,2)$	$(5,2)$	$(2,2)$	$(2,2)$
	$(B,1)$	$(5,2)$	$(5,2)$	$(2,2)$	$(2,2)$
	$(C,0)$	$(0,1)$	$(0,1)$	$(0.5,0.5)$	$(0.5,0.5)$
	$(C,1)$	$(-0.1,2.4)$	$(2,5.9)$	$(0.75,-1)$	$(0.75,-1)$

Mandatory Reading

- Chapter 2 of the book by J.E. Harrington.

Thank You!