1 Proof the MSNE is a CE

Say that $\sigma^\star = \left(\sigma_i^\star, \sigma_{-i}^\star\right)$ is a MSNE of an SFG. Then according to definition 6 of MSNE in lectures 12 to 15, for all $i \in N$,

$$U_i\left(s_i, \sigma_{-i}^\star\right) \text{ is same for all } s_i \in \delta\left(\sigma_i^\star\right)$$
$$U_i\left(s_i, \sigma_{-i}^\star\right) \geq U_i\left(s_i^i, \sigma_{-i}^\star\right) \ , \ \forall s_i \in \delta\left(\sigma_i^\star\right), s_i^{'} \notin \delta\left(\sigma_i^\star\right)$$

Writting the above two equations in expanded form we get,

$$\sum_{s_{-i}} \sigma_{-i}^\star\left(s_{-i}\right) u_i\left(s_i, s_{-i}\right) = \sum_{s_{-i}} \sigma_{-i}^\star\left(s_{-i}\right) u_i\left(s_i^{'}, s_{-i}\right) \ , \ \forall s_i \in \delta\left(\sigma_i^\star\right), s_i^{'} \in \delta\left(\sigma_i^\star\right) \ , \ \forall i \in N \tag{1}$$

$$\sum_{s_{-i}} \sigma_{-i}^\star\left(s_{-i}\right) u_i\left(s_i, s_{-i}\right) \geq \sum_{s_{-i}} \sigma_{-i}^\star\left(s_{-i}\right) u_i\left(s_i^{'}, s_{-i}\right) \ , \ \forall s_i \in \delta\left(\sigma_i^\star\right), s_i^{'} \notin \delta\left(\sigma_i^\star\right) \ , \ \forall i \in N \tag{2}$$

Equations (1) and (2) implies,

$$\sum_{s_{-i}} \sigma_{-i}^\star\left(s_{-i}\right) u_i\left(s_i, s_{-i}\right) \geq \sum_{s_{-i}} \sigma_{-i}^\star\left(s_{-i}\right) u_i\left(s_i^{'}, s_{-i}\right) \ , \ \forall s_i \in \delta\left(\sigma_i^\star\right), s_i^{'} \in S_i \ , \ \forall i \in N \tag{3}$$

We want to prove that if we set

$$\sigma\left(s_i, s_{-i}\right) = \sigma_i^\star\left(s_i\right) \sigma_{-i}^\star\left(s_{-i}\right) \ , \ \forall s_i \in S_i \ , \ s_{-i} \in S_{-i} \ , \ \forall i \in N \tag{4}$$

then $\sigma\left(s_i, s_{-i}\right)$ will be a CE. In other words $\sigma\left(s_i, s_{-i}\right)$ given by equation (4) will satisfy,

$$\sum_{s_{-i}} \sigma\left(s_i, s_{-i}\right) u_i\left(s_i, s_{-i}\right) \quad \geq \quad \sum_{s_{-i}} \sigma\left(s_i, s_{-i}\right) u_i\left(s_i^{'}, s_{-i}\right) \ , \ \forall s_i, s_i^{'} \in S_i \ , \ \forall i \in N$$

So essentially, we want to prove that,

$$\sum_{s_{-i}} \sigma_i^\star\left(s_i\right) \sigma_{-i}^\star\left(s_{-i}\right) u_i\left(s_i, s_{-i}\right) \quad \geq \quad \sum_{s_{-i}} \sigma_i^\star\left(s_i\right) \sigma_{-i}^\star\left(s_{-i}\right) u_i\left(s_i^{'}, s_{-i}\right) \ , \ \forall s_i, s_i^{'} \in S_i \ , \ \forall i \in N \tag{5}$$

We divide the proof into two cases.

CASE-1 ($s_i \notin \delta\left(\sigma_i^\star\right)$):

Since $s_i \notin \delta\left(\sigma_i^\star\right)$, then by definition $\sigma_i^\star\left(s_i\right) = 0$. In this case, both LHS and RHS of inequality (5) are zero and hence inequality (5) is true.

CASE-2 ($s_i \in \delta\left(\sigma_i^\star\right)$):

Consider the LHS of inequality (5). We have,

$$\begin{aligned}
\sum_{s_{-i}} \sigma_i^\star\left(s_i\right) \sigma_{-i}^\star\left(s_{-i}\right) u_i\left(s_i, s_{-i}\right) \quad &= \quad \sigma_i^\star\left(s_i\right) \sum_{s_{-i}} \sigma_{-i}^\star\left(s_{-i}\right) u_i\left(s_i, s_{-i}\right) \\
&\geq \quad \sigma_i^\star\left(s_i\right) \sum_{s_{-i}} \sigma_{-i}^\star\left(s_{-i}\right) u_i\left(s_i^{'}, s_{-i}\right) \qquad \text{using (3)} \\
&= \quad \sum_{s_{-i}} \sigma_i^\star\left(s_i\right) \sigma_{-i}^\star\left(s_{-i}\right) u_i\left(s_i^{'}, s_{-i}\right)
\end{aligned} \tag{6}$$

This concludes the proof.