Game Theory (CS4187)

Lectures 12
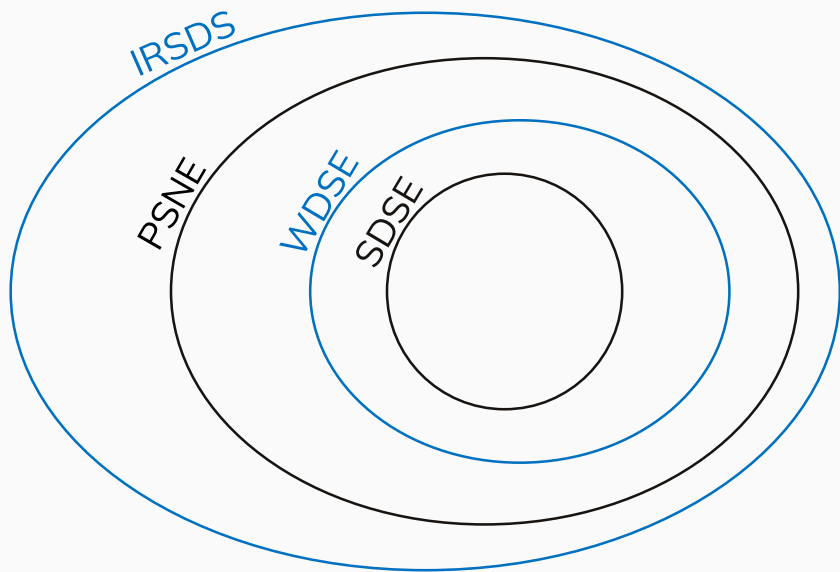
Date: 16/09/2024

19/09/2024

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Instructor: Gourav Saha

Broad Idea of Today's Lecture

➢ The topics for today's lectures are:

1. Does PSNE always exist?
2. Mixed Strategies
3. Mixed Strategy Nash Equilibrium (MSNE).
4. Properties for MSNE.
   A. Alternate definitions of MSNE.
5. Domination by Mixed Strategy.

Recap

IRSDS

PSNE

WDSE

SDSE

Till now we learned four kind of **solution concepts**:

➤ Strictly dominant strategy equilibrium (SDSE).

➤ Weakly dominant strategy equilibrium (WDSE).

➤ Iterated Removal of Strictly Dominated Strategy (IRSDS).
  - IRSDS is NOT really a solution concept unless the game is dominance solvable.

➤ Pure Strategy Nash Equilibrium (PSNE).

Recap
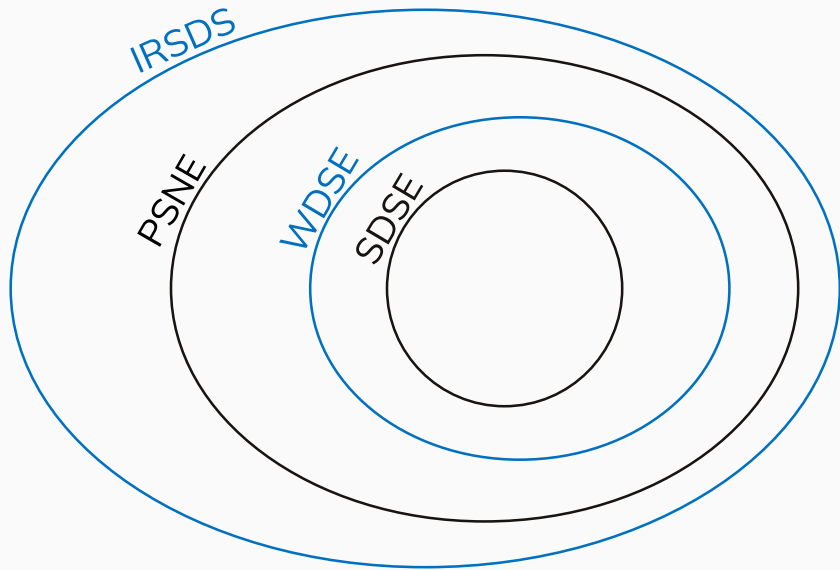
We can prove that:

➢ Every SDSE of a game is a WDSE, PSNE, and will survive IRSDS.

➢ Every WDSE of a game is a PSNE and will survive IRSDS.

➢ Every PSNE survives IRSDS.

Homework: Prove the following (all of them are easy):

(a) SDSE of a game is also its WDSE.

(b) SDSE/WDSE of a game is also its PSNE.

(c) SDSE/WDSE can't be eliminated using IRSDS.

Does PSNE always exist?

➢ Well, **it does not!** That's why I am asking the question 😊 .

In order to prove that all games does not have PSNE, all we need is one example for which the game does not have PSNE. So, let's discuss three!

Does PSNE always exist?

Player 2

	Rock	Paper	Scissors
Rock	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0

Player 1

Example 1 (Rock, Papers, Scissors):

➤ I hope you know the rules of this game 😊.

➤ The payoff matrix is made based on these rules.
  - Rock crushes scissors. Hence, if **player 1 plays scissors** and **player 2 plays rock**, then player 1 gets −1 and player 2 gets +1.
  - Scissors cut paper. Hence, if **player 1 plays scissors** and **player 2 plays paper**, then player 1 gets +1 and player 2 gets −1.

➤ Does this game have any PSNE?

Does PSNE always exist?

Student

	Learn	Don't learn
Quiz	7, 7	2, -5
Don't quiz	10, -3	0, 0

Instructor

Example 2 (Conducting quiz):

➢ Let's first explain the payoff matrix.

➢ The payoff of the instructor can be understood as follows:

- Instructor's payoff is based on (i) encouraging students to learn, (ii) effort to conduct quiz.

- It takes effort to conduct quiz. So, if students learn but instructor does not quiz, professor's payoff is more compared to if student learns and instructor conducts quiz. This explains the first column.

Does PSNE always exist?

Student

	Learn	Don't learn
Quiz	7, 7	2, -5
Don't quiz	10, -3	0, 0

Instructor

Example 2 (Conducting quiz):

➢ Let's first explain the payoff matrix.

➢ The **payoff of the instructor** can be understood as follows:

- Student's payoff is based on (i) marks, (ii) effort to learn.

- If the **instructor conducts quiz**, and the student does not learn, the student will not score marks. Hence the payoff will be much worse compared to if it learned *(assuming that score marks is worth the effort)*. This explains the first row.

Does PSNE always exist?

Student

	Learn	Don't learn
Quiz	7, 7	2, -5
Don't quiz	10, -3	0, 0

Instructor

Example 2 (Conducting quiz):

➢ Let's first explain the payoff matrix.

➢ The **payoff of the instructor** can be understood as follows:

- Student's payoff is based on (i) marks, (ii) effort to learn.

- If the **instructor don't conduct quiz**, the student will put effort to learn for no reason. Hence not learning has higher payoff when instructor does not conduct quiz. This explains the second row.

Does PSNE always exist?

Student

	Learn	Don't learn
Quiz	7, 7	2, -5
Don't quiz	10, -3	0, 0

Instructor

Example 2 (Conducting quiz):

➢ Let's first explain the payoff matrix.

➢ Does this game have PSNE?

Does PSNE always exist?

Example 3 (Guess a number lottery):

➢ There is a lottery going on. There are N participants. These participants have to **simultaneously** write a integer between 1 to M, in a piece of paper and give it to the lottery organizer. Here are the rules:

- **Rule 1:** If there is **a participant whose number is equal to ANY another participant**, than that participant has to give **10 rupees** to the lottery organizer.

- **Rule 2:** Among those **participants whose guess is not equal to anyone**, the participant with the highest guess gets **100 rupees**.

➢ Does this game have PSNE when:

  a) $N = 2$?

  b) For any N? What role does M plays to determine the existence of PSNE? (I have to work this one out!)

Does PSNE always exist?

➢ Consider all the examples that you have seen till now. If we think about it logically, the strategy that the players have to use in order to "win the game" must have some **random** component:

- • **Example 1:** Nobody has one single strategy to win rock-paper-scissors! We make random decisions.

- • **Example 2:** There is a reason why **"surprise test"** is such an useful pedagogical tool.

- • **Example 3:** It is called lottery for a reason! If there was one single strategy, the outcome will be deterministic and lottery will not have any meaning.

So, even logically speaking, the **strategies of the player have to be randomized**.

➢ There are other games where PSNE exists but real life evidence shows that the outcome of PSNE is not really followed; the player in such games in real life behaves randomly.

- • Example: **"Volunteers' dilemma and bystander effect"**, **mandatory reading** from the book by **J.E. Harrington**.

Mixed Strategy

Student

	Learn	Don't learn
Quiz	7, 7	2, -5
Don't quiz	10, -3	0, 0

Instructor

➢ **Mixed strategy** and **randomized strategy** are synonymous. **Mixed strategy is more common in game theory** and we will use it throughout this course.

➢ First, let's understand the concept of mixed strategies using "conducting quiz" example.

- Strategy set of student $S_{student} = \{learn, \ don't \ learn\}$.

- $\sigma_{student}(learn) = 0.3$.

- $\sigma_{student}(don't \ learn) = 0.7$.

Mixed strategy of student.

Mixed Strategy

Student

	Learn	Don't learn
Quiz	7, 7	2, -5
Don't quiz	10, -3	0, 0

Instructor

➢ **Mixed strategy** and **randomized strategy** are synonymous. **Mixed strategy is more common in game theory** and we will use it throughout this course.

➢ First, let's understand the concept of mixed strategies using "conducting quiz" example.

- Strategy set of instructor $S_{instructor} = \{quiz,\ don't\ quiz\}$.

- $\sigma_{instructor}(quiz) = 0.5$.

- $\sigma_{instructor}(don't\ quiz) = 0.5$.

Mixed strategy of instructor.

Mixed Strategy

Definition 1 (Mixed Strategy): The mixed strategy of player i with pure strategy set S_i is a probability distribution over all strategies in S_i, i.e. $\sigma_i : S_i \rightarrow [0, 1]$ such that,

$$\sum_{s_i \in S_i} \sigma_i\left(s_i\right) = 1$$

Mixed Strategy

Definition 1 (Mixed Strategy): The mixed strategy of player i with pure strategy set S_i is a probability distribution over all strategies in S_i, i.e. $\sigma_i : S_i \to [0,1]$ such that,

$$\sum_{s_i \in S_i} \sigma_i\,(s_i) = 1$$

➢ The **set of mixed strategy** of player i is denoted using $\Delta(S_i)$. To be precise, let $S_i = \{s_{i,1}, s_{i,2}, \cdots, s_{i,m_i}\}$. Let $\sigma_i(s_{i,j}) = \sigma_{i,j}$. Then,

$$\Delta\,(S_i) = \left\{ (\sigma_{i,1}, \sigma_{i,2}, \ldots, \sigma_{i,m_i}) : \sigma_{i,j} \geq 0\,, \forall j = 1, 2, \ldots, m_i \quad \text{and} \quad \sum_{j=1}^{m_i} \sigma_{i,j} = 1 \right\}$$

Mixed Strategy

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$$\Delta(S_i) = \left\{ (\sigma_{i,1}, \sigma_{i,2}, \ldots, \sigma_{i,m_i}) : \sigma_{i,j} \geq 0 , \forall j = 1, 2, \ldots, m_i \quad \text{and} \quad \sum_{j=1}^{m_i} \sigma_{i,j} = 1 \right\}$$

➤ **NOTE:** Set of pure strategy of a player is a subset of its set of mixed strategy. **How?**

Mixed Strategy

Instructor

Student

	Learn	Don't learn
Quiz	7, 7	2, -5
Don't quiz	10, -3	0, 0

Mixed Strategy and Set of Mixed Strategy

Example (Conducting quiz):

➢ Let player 1 and player 2 be instructor and student respectively.

➢ For player 1 (instructor):

- $s_{1,1} = Quiz$ and $s_{1,2} = Don't\ quiz$. Pure strategy is $S_1 = \{s_{1,1}, s_{1,2}\}$.
- $\sigma_{1,1}$ is the probability of choosing $s_{1,1}$ (Quiz).
- $\sigma_{1,2}$ is the probability of choosing $s_{1,2}$ (Don't quiz).
- Mixed strategy is $\sigma_1 = (\sigma_{1,1}, \sigma_{1,2})$ where $\sigma_{1,1}, \sigma_{1,2} \geq 0$ and $\sigma_{1,1} + \sigma_{1,2} = 1$.
- Set of mixed strategy is,
$$\Delta(S_1) = \{(\sigma_{1,1}, \sigma_{1,2}) : \sigma_{1,1}, \sigma_{1,2} \geq 0, \sigma_{1,1} + \sigma_{1,2} = 1\}$$

Mixed Strategy

Student

<table>
<tr><td rowspan="2">Instructor</td><td></td><td>Learn</td><td>Don't learn</td></tr>
<tr><td>Quiz</td><td>7, 7</td><td>2, -5</td></tr>
<tr><td></td><td>Don't quiz</td><td>10, -3</td><td>0, 0</td></tr>
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Mixed Strategy and Set of Mixed Strategy

Example (Conducting quiz):

➤ Let player 1 and player 2 be instructor and student respectively.

➤ For player 2 (student):

- $s_{2,1} = Learn$ and $s_{2,2} = Don't\ learn$. Pure strategy is $S_2 = \{s_{2,1}, s_{2,2}\}$.

- $\sigma_{2,1}$ is the probability of choosing $s_{2,1}$ (Learn).

- $\sigma_{2,2}$ is the probability of choosing $s_{2,2}$ (Don't learn).

- Mixed strategy is $\sigma_2 = (\sigma_{2,1}, \sigma_{2,2})$ where $\sigma_{2,1}, \sigma_{2,2} \geq 0$ and $\sigma_{2,1} + \sigma_{2,2} = 1$.

- Set of mixed strategy is,
$$\Delta(S_2) = \{(\sigma_{2,1}, \sigma_{2,2}) : \sigma_{2,1}, \sigma_{2,2} \geq 0, \sigma_{2,1} + \sigma_{2,2} = 1\}$$

Mixed Strategy

Utilities of a player in Mixed Strategy

➢ Utility of player i for mixed strategy profile $\sigma = [\sigma_1, \sigma_2, \cdots, \sigma_N]$, where σ_j is the mixed strategy of player j, is,

$$U_i(\sigma_1, \sigma_2, \ldots, \sigma_n) = \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \cdots \sum_{s_n \in S_n} \sigma_1(s_1)\, \sigma_2(s_2) \cdots \sigma_n(s_n)\, u_i(s_1, s_2, \ldots, s_n)$$

$$= \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \cdots \sum_{s_n \in S_n} \left(\prod_{j=1}^{n} \sigma_j(s_j) \right) u_i(s_1, s_2, \ldots, s_n)$$
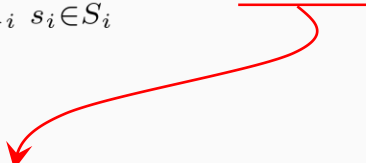
Mixed Strategy

Utilities of a player in Mixed Strategy

➢ Just like for pure strategies, we can represent a strategy profile $\sigma = [\sigma_1, \sigma_2, \cdots, \sigma_N]$ as $\boldsymbol{\sigma} = (\boldsymbol{\sigma_i}, \boldsymbol{\sigma_{-i}})$ or $\boldsymbol{\sigma} = (\boldsymbol{\sigma_{-i}}, \boldsymbol{\sigma_i})$ where $\boldsymbol{\sigma_{-i}}$ is the strategy profile of all players but player i. And hence, $U_i(\sigma_1, \sigma_2, \cdots, \sigma_N)$ (in previous slide) can be equivalently written as,

$$U_i(\sigma_i, \sigma_{-i}) = \sum_{s_{-i} \in S_{-i}} \sum_{s_i \in S_i} \sigma_i(s_i) \sigma_{-i}(s_{-i}) u_i(s_i, s_{-i})$$

where,

$$\sigma_{-i}(s_{-i}) = \prod_{j \neq i} \sigma_j(s_j)$$

Mixed Strategy

Utilities of a player in Mixed Strategy

➢ By **changing the order of summation** of $U_i(\sigma_i, \sigma_{-i})$ from previous slide we get,

$$U_i(\sigma_i, \sigma_{-i}) = \sum_{s_i \in S_i} \sigma_i(s_i) \left(\sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i}) u_i(s_i, s_{-i}) \right)$$

$$= \sum_{s_i \in S_i} \sigma_i(s_i) \boxed{U_i(s_i, \sigma_{-i})}$$

where,

$$\boxed{U_i(s_i, \sigma_{-i})} = \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i}) u_i(s_i, s_{-i})$$

$U_i(s_i, \sigma_{-i})$ is a **short-hand notation** that represent the utility of player i for the strategy profile (s_i, σ_{-i}). In this strategy profile, player i is playing pure strategy s_i while the players are playing mixed strategy σ_{-i}.

Mixed Strategy

Utilities of a player in Mixed Strategy

➢ Utility of player i for mixed strategy profile $\sigma = [\sigma_1, \sigma_2, \cdots, \sigma_N]$, where σ_j is the mixed strategy of player j, is,

$$U_i(\sigma_1, \sigma_2, \ldots, \sigma_N) = \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \cdots \sum_{s_N \in S_N} \sigma_1(s_1) \sigma_2(s_2) \cdots \sigma_N(s_N) u_i(s_1, s_2, \ldots, s_N)$$

$$= \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \cdots \sum_{s_N \in S_N} \left(\prod_{j=1}^{N} \sigma_j(s_j) \right) u_i(s_1, s_2, \ldots, s_N)$$

➢ Note that we have used **upper case U** for denoting utility function corresponding to **mixed strategy** and **lower case u** for denoting utility function corresponding to **pure strategy**. But sometimes, we may use the same case to represent both because it is easy to distinguish between them based on the arguments of the function. Examples:

a) $u_i(\sigma_i, \sigma_{-i})$ means utility of player i all the players are using mixed strategy.

b) $u_i(s_i, \sigma_{-i})$ means utility of player i when player i is using pure strategy s_i while rest of the players are using mixed strategy profile σ_{-i}.

Mixed Strategy

Student

	Learn	Don't learn
Quiz	7, 7	2, -5
Don't quiz	10, -3	0, 0

Instructor

Utilities of a player in Mixed Strategy

Example (Conducting quiz):

➢ Utility of player 1 (instructor) under mixed strategy $\sigma_1 = (\sigma_{1,1}, \sigma_{1,2})$ and $\sigma_2 = (\sigma_{2,1}, \sigma_{2,2})$ of players 1 and 2 resp. is,

$$
\begin{aligned}
U_1(\sigma_1, \sigma_2) &= \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \sigma_1(s_1) \sigma_2(s_2) u_1(s_1, s_2) \\
&= \sigma_1(s_{1,1}) \sigma_2(s_{2,1}) u_1(s_{1,1}, s_{2,1}) + \sigma_1(s_{1,1}) \sigma_2(s_{2,2}) u_1(s_{1,1}, s_{2,2}) \\
&\quad + \sigma_1(s_{1,2}) \sigma_2(s_{2,1}) u_1(s_{1,2}, s_{2,1}) + \sigma_1(s_{1,2}) \sigma_2(s_{2,2}) u_1(s_{1,2}, s_{2,2}) \\
&= 7\sigma_{1,1}\sigma_{2,1} + 2\sigma_{1,1}\sigma_{2,2} + 10\sigma_{1,2}\sigma_{2,1} + 0\sigma_{1,2}\sigma_{2,2}
\end{aligned}
$$

Mixed Strategy

Student

	Learn	Don't learn
Quiz	7, 7	2, -5
Don't quiz	10, -3	0, 0

Instructor

Utilities of a player in Mixed Strategy

Example (Conducting quiz):

➤ Utility of player 2 (student) under mixed strategy $\sigma_1 = (\sigma_{1,1}, \sigma_{1,2})$ and $\sigma_2 = (\sigma_{2,1}, \sigma_{2,2})$ of players 1 and 2 resp. is,

$$
\begin{aligned}
U_2(\sigma_1, \sigma_2) &= \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \sigma_1(s_1) \sigma_2(s_2) u_2(s_1, s_2) \\
&= \sigma_1(s_{1,1}) \sigma_2(s_{2,1}) u_2(s_{1,1}, s_{2,1}) + \sigma_1(s_{1,1}) \sigma_2(s_{2,2}) u_2(s_{1,1}, s_{2,2}) \\
&\quad + \sigma_1(s_{1,2}) \sigma_2(s_{2,1}) u_2(s_{1,2}, s_{2,1}) + \sigma_1(s_{1,2}) \sigma_2(s_{2,2}) u_2(s_{1,2}, s_{2,2}) \\
&= 7\sigma_{1,1}\sigma_{2,1} - 5\sigma_{1,1}\sigma_{2,2} - 3\sigma_{1,2}\sigma_{2,1} + 0\sigma_{1,2}\sigma_{2,2}
\end{aligned}
$$

An Important Property of Mixed Strategy

Theorem 1: Consider the set of **best response** of player i against strategy σ_{-i} of other players,

$$B_i\left(\sigma_{-i}\right) \in \arg\max_{\sigma_i \in \Delta(S_i)} u_i\left(\sigma_i, \sigma_{-i}\right)$$

where $S_i = \{s_{i,1}, s_{i,2}, \cdots, s_{i,m_i}\}$ is the set of pure strategies of player i. Following conditions are true:

Condition 1: There exist a pure strategy of player i which is the best response to strategy σ_{-i} of other players. In other words, there exists a pure strategy of player i in $B_i(s_{-i})$. Mathematically this is equivalent to,

$$\max_{\sigma_i \in \Delta(S_i)} u_i\left(\sigma_i, \sigma_{-i}\right) = \max_{s_i \in S_i} u_i\left(s_i, \sigma_{-i}\right)$$

Condition 2: Define the set of pure strategies of player i which is the best response of player i against σ_{-i} of other players,

$$B_i^{\text{pure}}\left(\sigma_{-i}\right) \in \arg\max_{s_i \in S_i} u_i\left(s_i, \sigma_{-i}\right)$$

For all mixed strategies $\sigma_i \in B_i(\sigma_{-i})$, the probability of player i to take any action $s_{i,j} \in S_i$ is zero if $s_{i,j} \notin B_i^{pure}(\sigma_{-i})$. Mathematically,

$$\sigma_i\left(s_{i,j}\right) = 0 \ , \forall s_{i,j} \notin B_i^{\text{pure}}\left(\sigma_{-i}\right)$$

An Important Property of Mixed Strategy

Proof of Theorem 1 (Intuition):

$$U_i\left(\sigma_i, \sigma_{-i}\right) = \sum_{s_{-i} \in S_{-i}} \sum_{s_i \in S_i} \sigma_i\left(s_i\right) \sigma_{-i}\left(s_{-i}\right) u_i\left(s_i, s_{-i}\right)$$

$$\sigma_{-i}\left(s_{-i}\right) = \prod_{j \neq i} \sigma_j\left(s_j\right)$$

$$U_1\left(\sigma_1, \sigma_2\right) \quad = \quad 7\sigma_{1,1}\sigma_{2,1} + 2\sigma_{1,1}\sigma_{2,2} + 10\sigma_{1,2}\sigma_{2,1} + 0\sigma_{1,2}\sigma_{2,2}$$

Mixed Strategy Nash Equilibrium (MSNE)

➢ Now that we know what mixed strategies are, let's look into mixed strategy Nash equilibrium (MSNE).

➢ The definition of MSNE is same as PSNE with the only difference that pure strategy profile $s^* = (s_i^*, s_{-i}^*)$ gets replaced with mixed strategy profile $\sigma^* = (\sigma_i^*, \sigma_{-i}^*)$. Everything else remains the same.

Definition 2 (MSNE): A strategy profile $\sigma^* = (\sigma_i^*, \sigma_{-i}^*)$ is called a MSNE if

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*) \ \ \forall \sigma_i \in \Delta(S_i), \forall i = 1,2, \dots, n$$

Definition 3 (MSNE): A strategy profile $\sigma^* = (\sigma_i^*, \sigma_{-i}^*)$ is called a MSNE if

$$\sigma_i^* \in B_i(\sigma_{-i}^*), \forall i = 1,2, \dots, n$$

where $B_i(\sigma_{-i}^*)$ is the set of **best response** of player i against strategy σ_{-i}^* of other players.

Equivalent definitions (Just like in PSNE).

Mixed Strategy Nash Equilibrium (MSNE)

➢ The problems with Definition 2 is that unlike PSNE, we **can't even use them to verify** if a given mixed strategy profile σ^* is a MSNE. This is because even for one player, say player i, we have to check **infinite number of $\sigma_i \in \Delta(S_i)$** to check if the inequality given in definition 2 holds.

➢ Since definitions 2 and 3 are **tautology**, the above argument applies for definition 3 as well.

➢ Since **verification of a solution** is a easier problem compared to **finding the solution**, we can definitely not use definitions 2 and 3 to find MSNE.

Mixed Strategy Nash Equilibrium (MSNE)

Definition 2 (MSNE): A strategy profile $\sigma^* = (\sigma_i^*, \sigma_{-i}^*)$ is called a MSNE if

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*) \ \ \forall \sigma_i \in \Delta(S_i), \forall i = 1,2, \dots, n$$

Definition 4 (MSNE): A strategy profile $\sigma^* = (\sigma_i^*, \sigma_{-i}^*)$ is called a MSNE if

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*) \ \ \forall s_i \in S_i, \forall i = 1,2, \dots, n$$

Proof behind definition 4: Direct consequence of condition 1 of Theorem 1.

Mixed Strategy Nash Equilibrium (MSNE)

➤ Let's see if definition 4 can be used to **verify** if a given strategy profile σ^* is MSNE.

(S1): For every player $i \in N$

(S2):   For every pure strategy $s_i \in S_i$

(S3):   if $u_i(\sigma_i^*, \sigma_{-i}^*) < u_i(s_i, \sigma_{-i}^*)$:

(S4):   return "σ^* is NOT MSNE."

(S5): return "σ^* is MSNE."

n times

$|S_i|$ times

$|S_1| \cdot |S_2| \cdot \cdots |S_n|$ times

Mixed Strategy Nash Equilibrium (MSNE)

➢ Let's see if definition 4 can be used to **find an MSNE**. Let's understand using the quiz example:

Utility of players:

$$U_1\left(\sigma_1, \sigma_2\right) = 7\sigma_{1,1}\sigma_{2,1} + 2\sigma_{1,1}\sigma_{2,2} + 10\sigma_{1,2}\sigma_{2,1}$$

$$U_2\left(\sigma_1, \sigma_2\right) = 7\sigma_{1,1}\sigma_{2,1} - 5\sigma_{1,1}\sigma_{2,2} - 3\sigma_{1,2}\sigma_{2,1}$$

MSNE check corresponding to player 1 (instructor):

$$7\sigma_{1,1}^\star\sigma_{2,1}^\star + 2\sigma_{1,1}^\star\sigma_{2,2}^\star + 10\sigma_{1,2}^\star\sigma_{2,1}^\star \geq 7\sigma_{2,1}^\star + 2\sigma_{2,2}^\star$$

$$7\sigma_{1,1}^\star\sigma_{2,1}^\star + 2\sigma_{1,1}^\star\sigma_{2,2}^\star + 10\sigma_{1,2}^\star\sigma_{2,1}^\star \geq 10\sigma_{2,1}^\star$$

Mixed Strategy Nash Equilibrium (MSNE)

➢ Let's see if definition 4 can be used to **find an MSNE**. Let's understand using the qquiz example:

Utility of players:

$$U_1\left(\sigma_1, \sigma_2\right) \;=\; 7\sigma_{1,1}\sigma_{2,1} + 2\sigma_{1,1}\sigma_{2,2} + 10\sigma_{1,2}\sigma_{2,1}$$

$$U_2\left(\sigma_1, \sigma_2\right) \;=\; 7\sigma_{1,1}\sigma_{2,1} - 5\sigma_{1,1}\sigma_{2,2} - 3\sigma_{1,2}\sigma_{2,1}$$

MSNE check corresponding to player 2 (student):

$$7\sigma_{1,1}^{\star}\sigma_{2,1}^{\star} - 5\sigma_{1,1}^{\star}\sigma_{2,2}^{\star} - 3\sigma_{1,2}^{\star}\sigma_{2,1}^{\star} \;\geq\; 7\sigma_{1,1}^{\star} - 3\sigma_{1,2}^{\star}$$

$$7\sigma_{1,1}^{\star}\sigma_{2,1}^{\star} - 5\sigma_{1,1}^{\star}\sigma_{2,2}^{\star} - 3\sigma_{1,2}^{\star}\sigma_{2,1}^{\star} \;\geq\; -5\sigma_{1,1}^{\star}$$

Mixed Strategy Nash Equilibrium (MSNE)

➢ Let's see if definition 4 can be used to **find an MSNE**. Let's understand using the qquiz example:

All the involved inequalities are:

$$7\sigma^\star_{1,1}\sigma^\star_{2,1} - 5\sigma^\star_{1,1}\sigma^\star_{2,2} - 3\sigma^\star_{1,2}\sigma^\star_{2,1} \geq 7\sigma^\star_{1,1} - 3\sigma^\star_{1,2}$$

$$7\sigma^\star_{1,1}\sigma^\star_{2,1} - 5\sigma^\star_{1,1}\sigma^\star_{2,2} - 3\sigma^\star_{1,2}\sigma^\star_{2,1} \geq -5\sigma^\star_{1,1}$$

$$7\sigma^\star_{1,1}\sigma^\star_{2,1} + 2\sigma^\star_{1,1}\sigma^\star_{2,2} + 10\sigma^\star_{1,2}\sigma^\star_{2,1} \geq 7\sigma^\star_{2,1} + 2\sigma^\star_{2,2}$$

$$7\sigma^\star_{1,1}\sigma^\star_{2,1} + 2\sigma^\star_{1,1}\sigma^\star_{2,2} + 10\sigma^\star_{1,2}\sigma^\star_{2,1} \geq 10\sigma^\star_{2,1}$$

Mixed Strategy Nash Equilibrium (MSNE)

➤ Let's see if definition 4 can be used to **find an MSNE**. The general case:

$$U_i\left(\sigma_i^\star, \sigma_{-i}^\star\right) \;\geq\; U_i\left(s_i, \sigma_{-i}^\star\right) \;;\; \forall s_i \in S_i, \; \forall i \in N$$

➤ We now expand the above equation to make it more clear. We have,

$$U_i\left(\sigma_i^\star, \sigma_{-i}^\star\right) \;=\; \sum_{s_i \in S_1} \sum_{s_{-i} \in S_{-i}} \sigma_i\left(s_i\right) \sigma_{-i}\left(s_{-i}\right) u_i\left(s_i, s_{-i}\right)$$

$$\sigma_{-i}\left(s_{-i}\right) = \prod_{j \neq i} \sigma_j\left(s_j\right)$$

And,

$$U_i\left(s_i, \sigma_{-i}^\star\right) \;=\; \sum_{s_{-i} \in S_{-i}} \sigma_{-i}\left(s_{-i}\right) u_i\left(s_i, s_{-i}\right)$$

Mixed Strategy Nash Equilibrium (MSNE)

➤ Let's see if definition 4 can be used to **find an MSNE**. The general case:

$$U_i\left(\sigma_i^\star, \sigma_{-i}^\star\right) \quad \geq \quad U_i\left(s_i, \sigma_{-i}^\star\right) \; ; \; \forall s_i \in S_i, \; \forall i \in N$$

➤ Based the equations discussed in the previous slide, the above equation can be written in expanded form as,

$$\sum_{s_i \in S_1} \sum_{s_{-i} \in S_{-i}} \sigma_i\left(s_i\right) \left(\prod_{j \neq i} \sigma_j\left(s_j\right)\right) u_i\left(s_i, s_{-i}\right) \geq \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j\left(s_j\right)\right) u_i\left(s_i, s_{-i}\right) ; \forall s_i \in S_i, i \in N$$

➤ So, in the general case, we have to write the above inequality **for the players** and **all the pure strategy of the player**. There we be $\prod_{j \in N} |S_i|$ such inequalities.

➤ The variable of these inequalities are $\sigma_j(s_j)$'s. We have to solve these inequalities for $\sigma_j(s_j)$'s.

Mixed Strategy Nash Equilibrium (MSNE)

Definition 5 (Support of mixed strategy): The support of the mixed strategy profile σ_i of player i is the set of all pure strategies of player i which has non-zero probability is σ_i.

$$\delta\left(\sigma_i\right) = \left\{s_i \in S_i \ : \ \sigma_i\left(s_i\right) > 0\right\}$$

Definition 6 (MSNE): A strategy profile $\sigma^* = (\sigma_i^*, \sigma_{-i}^*)$ is called a MSNE if and only if,

<u>Condition 1</u>: $U_i(s_i, \sigma_{-i}^*)$ is same for all $s_i \in \delta(\sigma_i^*)$.

<u>Condition 2</u>: $U_i(s_i, \sigma_{-i}^*) \geq U_i(s_i', \sigma_{-i}^*)$ for all $s_i \in \delta(\sigma_i^*)$ and $s_i' \notin \delta(\sigma_i^*)$.

Mixed Strategy Nash Equilibrium (MSNE)

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Intuition:

$$U_1(\sigma_1, \sigma_2) \;\; = \;\; 7\sigma_{1,1}\sigma_{2,1} + 2\sigma_{1,1}\sigma_{2,2} + 10\sigma_{1,2}\sigma_{2,1} + 0\sigma_{1,2}\sigma_{2,2}$$

Mixed Strategy Nash Equilibrium (MSNE)

Definition 6 (MSNE): A strategy profile $\sigma^* = (\sigma_i^*, \sigma_{-i}^*)$ is called a MSNE if and only if,

<u>Condition 1</u>: $U_i(s_i, \sigma_{-i}^*)$ is same for all $s_i \in \delta(\sigma_i^*)$.

<u>Condition 2</u>: $U_i(s_i, \sigma_{-i}^*) \geq U_i(s_i', \sigma_{-i}^*)$ for all $s_i \in \delta(\sigma_i^*)$ and $s_i' \notin \delta(\sigma_i^*)$.

➤ Just like we used definition 4 to find an approach to compute MSNE, we will now use definition 6 to compute MSNE.

- Definition 6 is particularly useful for **two-player** games because the **resulting equations are linear in σ_i^*'s**.

- Not so useful for **more than two players** games because the **resulting equations are non-linear in σ_i^*'s**. For more than two players, definition 4 leads to a **seemingly** simpler approach.

Mixed Strategy Nash Equilibrium (MSNE)

Definition 6 (MSNE): A strategy profile $\sigma^* = (\sigma_i^*, \sigma_{-i}^*)$ is called a MSNE if and only if,

<u>Condition 1</u>: $U_i(s_i, \sigma_{-i}^*)$ is same for all $s_i \in \delta(\sigma_i^*)$.

<u>Condition 2</u>: $U_i(s_i, \sigma_{-i}^*) \geq U_i(s_i', \sigma_{-i}^*)$ for all $s_i \in \delta(\sigma_i^*)$ and $s_i' \notin \delta(\sigma_i^*)$.

➢ The solution approach is:

- **Step 1: Fix support $\boldsymbol{\delta(\sigma_i^*)}$** for all $i \in N$. Remember: Support cannot be a null set. Why?
- **Step 2:** For the chosen support in step 1, find the equalities due to condition 1 and inequalities due to condition 2. Find a solution of these equalities and inequalities (if a solution exists).

  Go to step 1 and repeat for **another** support.

➢ This approach is discussed in details in chapter 11 of the book and also examples are solved in chapter 7 of the book.

Mixed Strategy Nash Equilibrium (MSNE)

Student

	s_{21} Learn	s_{22} Don't learn
s_{11} Quiz	7, 7	2, -5
s_{12} Don't quiz	10, -3	0, 0

Instructor

<u>Condition 1</u>: $U_i(s_i, \sigma_{-i}^*)$ is same for all $s_i \in \delta(\sigma_i^*)$.

<u>Condition 2</u>: $U_i(s_i, \sigma_{-i}^*) \geq U_i(s_i', \sigma_{-i}^*)$ for all $s_i \in \delta(\sigma_i^*)$ and $s_i' \notin \delta(\sigma_i^*)$.

Example-1 (Quiz example)

➤ Let $X_i = \delta(\sigma_i^*)$ be the support of player i. This notation is introduced for simplicity.

➤ Set containing possible values of X_1 is $\{\{s_{11}\}, \{s_{12}\}, \{s_{11}, s_{12}\}\}$.

➤ Set containing possible values of X_2 is $\{\{s_{21}\}, \{s_{22}\}, \{s_{21}, s_{22}\}\}$.

➤ Total number of supports? General case?

Mixed Strategy Nash Equilibrium (MSNE)

Student

	s_{21} Learn	s_{22} Don't learn
s_{11} Quiz	7, 7	2, -5
s_{12} Don't quiz	10, -3	0, 0

Instructor

Condition 1: $U_i(s_i, \sigma^*_{-i})$ is same for all $s_i \in \delta(\sigma^*_i)$.

Condition 2: $U_i(s_i, \sigma^*_{-i}) \geq U_i(s'_i, \sigma^*_{-i})$ for all $s_i \in \delta(\sigma^*_i)$ and $s'_i \notin \delta(\sigma^*_i)$.

Example-1 (Quiz example)

➢ From <u>this slide (click here)</u> and the slide after that, we get the following utility functions of the players in mixed strategy,

$$U_1(\sigma_1, \sigma_2) = 7\sigma_{1,1}\sigma_{2,1} + 2\sigma_{1,1}\sigma_{2,2} + 10\sigma_{1,2}\sigma_{2,1}$$
$$U_2(\sigma_1, \sigma_2) = 7\sigma_{1,1}\sigma_{2,1} - 5\sigma_{1,1}\sigma_{2,2} - 3\sigma_{1,2}\sigma_{2,1}$$

➢ **Step 1:** Let $X_1 = \{s_{11}, s_{12}\}$, $X_2 = \{s_{21}, s_{22}\}$.

Mixed Strategy Nash Equilibrium (MSNE)

Student

	s_{21} Learn	s_{22} Don't learn
s_{11} Quiz	7, 7	2, -5
s_{12} Don't quiz	10, -3	0, 0

Instructor

<u>Condition 1</u>: $U_i(s_i, \sigma_{-i}^*)$ is same for all $s_i \in \delta(\sigma_i^*)$.

<u>Condition 2</u>: $U_i(s_i, \sigma_{-i}^*) \geq U_i(s_i', \sigma_{-i}^*)$ for all $s_i \in \delta(\sigma_i^*)$ and $s_i' \notin \delta(\sigma_i^*)$.

Example-1 (Quiz example)

➤ From <u>this slide (click here)</u> and the slide after that, we get the following utility functions of the players in mixed strategy,

$$U_1(\sigma_1, \sigma_2) = 7\sigma_{1,1}\sigma_{2,1} + 2\sigma_{1,1}\sigma_{2,2} + 10\sigma_{1,2}\sigma_{2,1}$$
$$U_2(\sigma_1, \sigma_2) = 7\sigma_{1,1}\sigma_{2,1} - 5\sigma_{1,1}\sigma_{2,2} - 3\sigma_{1,2}\sigma_{2,1}$$

➤ **Step 2:** For player 1, condition 1 implies,

$$U_1(s_{11}, \sigma_2^\star) = U_1(s_{12}, \sigma_2^\star)$$
$$\Rightarrow 7\sigma_{2,1}^\star + 2\sigma_{2,2}^\star = 10\sigma_{2,1}^\star \qquad \text{(E.1)}$$

For player 2, condition 1 implies,

$$U_2(\sigma_1^\star, s_{21}) = U_2(\sigma_1^\star, s_{22})$$
$$\Rightarrow 7\sigma_{1,1}^\star - 3\sigma_{1,2}^\star = -5\sigma_{1,1}^\star \qquad \text{(E.2)}$$

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Mixed Strategy Nash Equilibrium (MSNE)

Student

	s_{21} Learn	s_{22} Don't learn
s_{11} Quiz	7, 7	2, -5
s_{12} Don't quiz	10, -3	0, 0

Instructor (left vertical label)

Condition 1: $U_i(s_i, \sigma_{-i}^*)$ is same for all $s_i \in \delta(\sigma_i^*)$.

Condition 2: $U_i(s_i, \sigma_{-i}^*) \geq U_i(s_i', \sigma_{-i}^*)$ for all $s_i \in \delta(\sigma_i^*)$ and $s_i' \notin \delta(\sigma_i^*)$.

Example-1 (Quiz example)

➤ From this slide (click here) and the slide after that, we get the following utility functions of the players in mixed strategy,

$$U_1(\sigma_1, \sigma_2) = 7\sigma_{1,1}\sigma_{2,1} + 2\sigma_{1,1}\sigma_{2,2} + 10\sigma_{1,2}\sigma_{2,1}$$
$$U_2(\sigma_1, \sigma_2) = 7\sigma_{1,1}\sigma_{2,1} - 5\sigma_{1,1}\sigma_{2,2} - 3\sigma_{1,2}\sigma_{2,1}$$

➤ **Step 2:** For player 1 and player 2, condition 2 does not exist for the chosen support $X_1 = \{s_{11}, s_{12}\}$ and $X_2 = \{s_{21}, s_{22}\}$, because there does not exists any s_i' such that $s_i' \notin X_i$ for $i = 1$ and 2..

X_i is same as $\delta(\sigma_i^*)$.

Mixed Strategy Nash Equilibrium (MSNE)

Student

	s_{21} Learn	s_{22} Don't learn
s_{11} Quiz	7, 7	2, -5
s_{12} Don't quiz	10, -3	0, 0

Instructor

<u>Condition 1</u>: $U_i(s_i, \sigma^*_{-i})$ is same for all $s_i \in \delta(\sigma^*_i)$.

<u>Condition 2</u>: $U_i(s_i, \sigma^*_{-i}) \geq U_i(s'_i, \sigma^*_{-i})$ for all $s_i \in \delta(\sigma^*_i)$ and $s'_i \notin \delta(\sigma^*_i)$.

Example-1 (Quiz example)

➤ So, the set of equations for the chosen support $X_1 = \{s_{11}, s_{12}\}$ and $X_2 = \{s_{21}, s_{22}\}$ are,

$$7\sigma^{\star}_{2,1} + 2\sigma^{\star}_{2,2} = 10\sigma^{\star}_{2,1}$$
$$7\sigma^{\star}_{1,1} - 3\sigma^{\star}_{1,2} = -5\sigma^{\star}_{1,1}$$

$$\sigma^{\star}_{1,1} + \sigma^{\star}_{1,2} = 1$$
$$\sigma^{\star}_{2,1} + \sigma^{\star}_{2,2} = 1$$

$$\sigma^{\star}_{1,1}\sigma^{\star}_{1,2} > 0$$
$$\sigma^{\star}_{2,1}, \sigma^{\star}_{2,2} > 0$$

Why STRICT inequality?

Mixed Strategy Nash Equilibrium (MSNE)

Student

	s_{21} Learn	s_{22} Don't learn
s_{11} Quiz	7, 7	2, -5
s_{12} Don't quiz	10, -3	0, 0

Instructor

<u>Condition 1</u>: $U_i(s_i, \sigma_{-i}^*)$ is same for all $s_i \in \delta(\sigma_i^*)$.

<u>Condition 2</u>: $U_i(s_i, \sigma_{-i}^*) \geq U_i(s_i', \sigma_{-i}^*)$ for all $s_i \in \delta(\sigma_i^*)$ and $s_i' \notin \delta(\sigma_i^*)$.

Example-1 (Quiz example)

➤ So, the set of equations for the chosen support $X_1 = \{s_{11}, s_{12}\}$ and $X_2 = \{s_{21}, s_{22}\}$ are,

$$7\sigma_{2,1}^\star + 2\sigma_{2,2}^\star = 10\sigma_{2,1}^\star$$
$$7\sigma_{1,1}^\star - 3\sigma_{1,2}^\star = -5\sigma_{1,1}^\star$$

If there are more than two pure strategies in the support X_1, then this equation is more conveniently written as,

$$\sigma_{1,1}^\star + \sigma_{1,2}^\star = 1$$
$$\sigma_{2,1}^\star + \sigma_{2,2}^\star = 1$$

$$w_1 = 7\sigma_{2,1}^\star + 2\sigma_{2,2}^\star$$
$$w_1 = 10\sigma_{2,1}^\star$$

$$\sigma_{1,1}^\star \sigma_{1,2}^\star > 0$$
$$\sigma_{2,1}^\star, \sigma_{2,2}^\star > 0$$

Mixed Strategy Nash Equilibrium (MSNE)

Student

	s_{21} Learn	s_{22} Don't learn
Instructor s_{11} Quiz	7, 7	2, -5
s_{12} Don't quiz	10, -3	0, 0

<u>Condition 1</u>: $U_i(s_i, \sigma_{-i}^*)$ is same for all $s_i \in \delta(\sigma_i^*)$.

<u>Condition 2</u>: $U_i(s_i, \sigma_{-i}^*) \geq U_i(s_i', \sigma_{-i}^*)$ for all $s_i \in \delta(\sigma_i^*)$ and $s_i' \notin \delta(\sigma_i^*)$.

Example-1 (Quiz example)

➤ So, the set of equations for the chosen support $X_1 = \{s_{11}, s_{12}\}$ and $X_2 = \{s_{21}, s_{22}\}$ are,

$$7\sigma_{2,1}^\star + 2\sigma_{2,2}^\star = 10\sigma_{2,1}^\star$$
$$7\sigma_{1,1}^\star - 3\sigma_{1,2}^\star = -5\sigma_{1,1}^\star$$

Similarly, if there are more than two pure strategies in the support X_2, then this equation is more conveniently written as,

$$\sigma_{1,1}^\star + \sigma_{1,2}^\star = 1$$
$$\sigma_{2,1}^\star + \sigma_{2,2}^\star = 1$$

$$w_2 = 7\sigma_{1,1}^\star - 3\sigma_{1,2}^\star$$
$$w_2 = -5\sigma_{1,1}^\star$$

$$\sigma_{1,1}^\star \sigma_{1,2}^\star > 0$$
$$\sigma_{2,1}^\star, \sigma_{2,2}^\star > 0$$

Mixed Strategy Nash Equilibrium (MSNE)

Student

	S_{21} Learn	S_{22} Don't learn
S_{11} Quiz	7, 7	2, -5
S_{12} Don't quiz	10, -3	0, 0

Instructor

<u>Condition 1</u>: $U_i(s_i, \sigma_{-i}^*)$ is same for all $s_i \in \delta(\sigma_i^*)$.

<u>Condition 2</u>: $U_i(s_i, \sigma_{-i}^*) \geq U_i(s_i', \sigma_{-i}^*)$ for all $s_i \in \delta(\sigma_i^*)$ and $s_i' \notin \delta(\sigma_i^*)$.

Example-1 (Quiz example)

➢ We now have to find a solution for the equations in the previous slide provide a solution exists.

➢ Since there are four variables and four equalities, we solve these equalities to find a solution and then check if the solution satisfies the inequalities. Solving the equalities we get,

$$
\begin{aligned}
\sigma_{1,1}^\star &= 0.8 \\
\sigma_{1,2}^\star &= 0.2 \\
\sigma_{2,1}^\star &= 0.4 \\
\sigma_{2,2}^\star &= 0.6
\end{aligned}
$$

This solution definitely satisfies the inequalities.

Mixed Strategy Nash Equilibrium (MSNE)

Student

	s_{21} Learn	s_{22} Don't learn
s_{11} Quiz	7, 7	2, -5
s_{12} Don't quiz	10, -3	0, 0

Instructor

Condition 1: $U_i(s_i, \sigma_{-i}^*)$ is same for all $s_i \in \delta(\sigma_i^*)$.

Condition 2: $U_i(s_i, \sigma_{-i}^*) \geq U_i(s_i', \sigma_{-i}^*)$ for all $s_i \in \delta(\sigma_i^*)$ and $s_i' \notin \delta(\sigma_i^*)$.

Example-1 (Quiz example)

➢ Hence one of the MSNE is (there may or may not be more),

$$\sigma_1^\star = [0.8, 0.2]$$
$$\sigma_2^\star = [0.4, 0.6]$$

➢ MSNE is also sometimes written as,

$$\sigma_1^\star = [0.8\,(\text{Quiz})\,, 0.2\,(\text{Don't quiz})]$$
$$\sigma_2^\star = [0.4\,(\text{Learn})\,, 0.6\,(\text{Don't learn})]$$

Mixed Strategy Nash Equilibrium (MSNE)

Student

	s_{21} Learn	s_{22} Don't learn
s_{11} Quiz	7, 7	2, -5
s_{12} Don't quiz	10, -3	0, 0

Instructor

Condition 1: $U_i(s_i, \sigma^*_{-i})$ is same for all $s_i \in \delta(\sigma^*_i)$.

Condition 2: $U_i(s_i, \sigma^*_{-i}) \geq U_i(s'_i, \sigma^*_{-i})$ for all $s_i \in \delta(\sigma^*_i)$ and $s'_i \notin \delta(\sigma^*_i)$.

Example-1 (Quiz example)

➤ From this slide (click here) and the slide after that, we get the following utility functions of the players in mixed strategy,

$$U_1(\sigma_1, \sigma_2) = 7\sigma_{1,1}\sigma_{2,1} + 2\sigma_{1,1}\sigma_{2,2} + 10\sigma_{1,2}\sigma_{2,1}$$
$$U_2(\sigma_1, \sigma_2) = 7\sigma_{1,1}\sigma_{2,1} - 5\sigma_{1,1}\sigma_{2,2} - 3\sigma_{1,2}\sigma_{2,1}$$

We will redo the steps in the previous slides for another support X_1 and X_2 .

➤ **Step 1:** Let $X_1 = \{s_{11}, s_{12}\}, X_2 = \{s_{21}\}$.

Mixed Strategy Nash Equilibrium (MSNE)

Student

	s_{21} Learn	s_{22} Don't learn
s_{11} Quiz	7, 7	2, -5
s_{12} Don't quiz	10, -3	0, 0

Instructor

<u>Condition 1</u>: $U_i(s_i, \sigma_{-i}^*)$ is same for all $s_i \in \delta(\sigma_i^*)$.

<u>Condition 2</u>: $U_i(s_i, \sigma_{-i}^*) \geq U_i(s_i', \sigma_{-i}^*)$ for all $s_i \in \delta(\sigma_i^*)$ and $s_i' \notin \delta(\sigma_i^*)$.

Example-1 (Quiz example)

➢ From <u>this slide (click here)</u> and the slide after that, we get the following utility functions of the players in mixed strategy,

$$
\begin{aligned}
U_1(\sigma_1, \sigma_2) &= 7\sigma_{1,1}\sigma_{2,1} + 2\sigma_{1,1}\sigma_{2,2} + 10\sigma_{1,2}\sigma_{2,1} \\
U_2(\sigma_1, \sigma_2) &= 7\sigma_{1,1}\sigma_{2,1} - 5\sigma_{1,1}\sigma_{2,2} - 3\sigma_{1,2}\sigma_{2,1}
\end{aligned}
$$

➢ **Step 2:** For player 1, condition 1 implies,

$$U_1(s_{11}, \sigma_2^\star) = U_1(s_{12}, \sigma_2^\star)$$

$$\Rightarrow 7\sigma_{2,1}^\star + 2\sigma_{2,2}^\star = 10\sigma_{2,1}^\star \qquad \text{(E.3)}$$

For player 2, condition 1 does not exists because is only one pure strategy in X_2 and we can't write an equation with just one strategy.

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Mixed Strategy Nash Equilibrium (MSNE)

Student

	S_{21} Learn	S_{22} Don't learn
S_{11} Quiz	7, 7	2, -5
S_{12} Don't quiz	10, -3	0, 0

Instructor

Condition 1: $U_i(s_i, \sigma_{-i}^*)$ is same for all $s_i \in \delta(\sigma_i^*)$.

Condition 2: $U_i(s_i, \sigma_{-i}^*) \geq U_i(s_i', \sigma_{-i}^*)$ for all $s_i \in \delta(\sigma_i^*)$ and $s_i' \notin \delta(\sigma_i^*)$.

Example-1 (Quiz example)

➢ From this slide (click here) and the slide after that, we get the following utility functions of the players in mixed strategy,

$$U_1(\sigma_1, \sigma_2) = 7\sigma_{1,1}\sigma_{2,1} + 2\sigma_{1,1}\sigma_{2,2} + 10\sigma_{1,2}\sigma_{2,1}$$
$$U_2(\sigma_1, \sigma_2) = 7\sigma_{1,1}\sigma_{2,1} - 5\sigma_{1,1}\sigma_{2,2} - 3\sigma_{1,2}\sigma_{2,1}$$

➢ **Step 2:** For player 1, condition 2 does not exist for the chosen support $X_1 = \{s_{11}, s_{12}\}$ because there does not exists any s_1' such that $s_1' \notin \delta(\sigma_1^*)$.

For player 2, condition 2 implies,

$$U_2(\sigma_1^\star, s_{21}) \geq U_2(\sigma_1^\star, s_{22})$$
$$\Rightarrow 7\sigma_{1,1}^\star - 3\sigma_{1,2}^\star \geq -5\sigma_{1,1}^\star$$

Mixed Strategy Nash Equilibrium (MSNE)

Student

	s_{21} Learn	s_{22} Don't learn
s_{11} Quiz	7, 7	2, -5
s_{12} Don't quiz	10, -3	0, 0

Instructor

<u>Condition 1</u>: $U_i(s_i, \sigma^*_{-i})$ is same for all $s_i \in \delta(\sigma^*_i)$.

<u>Condition 2</u>: $U_i(s_i, \sigma^*_{-i}) \geq U_i(s'_i, \sigma^*_{-i})$ for all $s_i \in \delta(\sigma^*_i)$ and $s'_i \notin \delta(\sigma^*_i)$.

Example-1 (Quiz example)

➢ So, the set of equations for the chosen support $X_1 = \{s_{11}, s_{12}\}$ and $X_2 = \{s_{21}\}$ are,

$$7\sigma^\star_{2,1} + 2\sigma^\star_{2,2} = 10\sigma^\star_{2,1}$$

$$7\sigma^\star_{1,1} - 3\sigma^\star_{1,2} \geq -5\sigma^\star_{1,1}$$

$$\sigma^\star_{1,1} + \sigma^\star_{1,2} = 1$$
$$\sigma^\star_{2,1} + \sigma^\star_{2,2} = 1$$

$$\sigma^\star_{1,1}\sigma^\star_{1,2} > 0$$
$$\sigma^\star_{2,1} > 0$$ ⎱ Why STRICT inequality?
$$\sigma^\star_{2,2} = 0$$ ⎱ Why ZERO?

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Mixed Strategy Nash Equilibrium (MSNE)

Student

	s_{21} Learn	s_{22} Don't learn
s_{11} Quiz	7, 7	2, -5
s_{12} Don't quiz	10, -3	0, 0

Instructor

Condition 1: $U_i(s_i, \sigma^*_{-i})$ is same for all $s_i \in \delta(\sigma^*_i)$.

Condition 2: $U_i(s_i, \sigma^*_{-i}) \geq U_i(s'_i, \sigma^*_{-i})$ for all $s_i \in \delta(\sigma^*_i)$ and $s'_i \notin \delta(\sigma^*_i)$.

Example-1 (Quiz example)

➤ So, the set of equations for the chosen support $X_1 = \{s_{11}, s_{12}\}$ and $X_2 = \{s_{21}\}$ are,

$$7\sigma^\star_{2,1} + 2\sigma^\star_{2,2} = 10\sigma^\star_{2,1}$$

$$7\sigma^\star_{1,1} - 3\sigma^\star_{1,2} \geq -5\sigma^\star_{1,1}$$

Similarly, if there are more than two pure strategies in the support X_2, then this equation is more conveniently written as,

$$\sigma^\star_{1,1} + \sigma^\star_{1,2} = 1$$
$$\sigma^\star_{2,1} + \sigma^\star_{2,2} = 1$$

$$w_2 = 7\sigma^\star_{1,1} - 3\sigma^\star_{1,2}$$
$$w_2 \geq -5\sigma^\star_{1,1}$$

$$\sigma^\star_{1,1}\sigma^\star_{1,2} > 0$$
$$\sigma^\star_{2,1} > 0$$
$$\sigma^\star_{2,2} = 0$$

Mixed Strategy Nash Equilibrium (MSNE)

Student

	s_{21} Learn	s_{22} Don't learn
s_{11} Quiz	7, 7	2, -5
s_{12} Don't quiz	10, -3	0, 0

Instructor

<u>Condition 1</u>: $U_i(s_i, \sigma^*_{-i})$ is same for all $s_i \in \delta(\sigma^*_i)$.

<u>Condition 2</u>: $U_i(s_i, \sigma^*_{-i}) \geq U_i(s'_i, \sigma^*_{-i})$ for all $s_i \in \delta(\sigma^*_i)$ and $s'_i \notin \delta(\sigma^*_i)$.

Example-1 (Quiz example)

➢ So, the set of equations for the chosen support $X_1 = \{s_{11}, s_{12}\}$ and $X_2 = \{s_{21}\}$ are,

$$7\sigma^\star_{2,1} + 2\sigma^\star_{2,2} = 10\sigma^\star_{2,1} \quad \Longleftarrow$$

$$7\sigma^\star_{1,1} - 3\sigma^\star_{1,2} \geq -5\sigma^\star_{1,1}$$

$$\sigma^\star_{1,1} + \sigma^\star_{1,2} = 1 \quad \Longleftarrow$$
$$\sigma^\star_{2,1} + \sigma^\star_{2,2} = 1 \quad \Longleftarrow$$

$$\sigma^\star_{1,1}\sigma^\star_{1,2} > 0$$
$$\sigma^\star_{2,1} > 0$$
$$\sigma^\star_{2,2} = 0 \quad \Longleftarrow$$

➢ Here also there are 4 variables and 4 equations. Equations are marked with arrow.

➢ Solve the equations first.

➢ Then check if the solutions obtained satisfies the inequalities. If it does, then solution exists. Else, solution does not exist for the chosen support.

For this case, solution does not exist!

Mixed Strategy Nash Equilibrium (MSNE)

Student

	S_{21} Learn	S_{22} Don't learn
S_{11} Quiz	7, 7	2, -5
S_{12} Don't quiz	10, -3	0, 0

Instructor

Condition 1: $U_i(s_i, \sigma_{-i}^*)$ is same for all $s_i \in \delta(\sigma_i^*)$.

Condition 2: $U_i(s_i, \sigma_{-i}^*) \geq U_i(s_i', \sigma_{-i}^*)$ for all $s_i \in \delta(\sigma_i^*)$ and $s_i' \notin \delta(\sigma_i^*)$.

Example-1 (Quiz example)

➤ We just did it for 2 of the 9 possible combinations of X_1 and X_2.

Mixed Strategy Nash Equilibrium (MSNE)

➢ We will now discuss a general approach to find MSNE using definition 6 for n player games.

➢ **Step 1:** For each $i \in N$, fix support $X_i \in 2^{S_i} - \emptyset$. **Note:** 2^{S_i} is the **power set** of S_i and $\emptyset$ is the null set. So $2^{S_i} - \emptyset$ is the power set of S_i excluding the null set (remember that support can't be a null set).

PTO

Mixed Strategy Nash Equilibrium (MSNE)

➤ We will now discuss a general approach to find MSNE using definition 6 for n player games.

➤ **Step 2:** For X_i's fixed in step 1, definition 6 of MSNE will lead to the following equations. Solve these equations (provided there exists a solution). **(refer to chapter 11 of the book for explanation)**

(Eqn. 1)
$$w_i = \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s_i, s_{-i}); \ \forall s_i \in X_i, \forall i \in N$$

Condition 1

(Eqn. 2)
$$w_i \geq \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s_i, s_{-i}); \ \forall s_i \in S_i \backslash X_i, \forall i \in N$$

Condition 2

(Eqn. 3)
$$\sum_{s_i \in S_i} \sigma_i(s_i) = 1; \ \forall i \in N$$

Probability should add to 1.

(Eqn. 4)
$$\sigma_i(s_i) > 0; \ \forall s_i \in X_i, \forall i \in N$$

Pure strategies in support should have non-zero probability.

(Eqn. 5)
$$\sigma_i(s_i) = 0; \ \forall s_i \in S_i \backslash X_i, \forall i \in N$$

Pure strategies that are NOT in support should have zero probability.

Mixed Strategy Nash Equilibrium (MSNE)

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(Eqn. 1)
$$w_i = \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s_i, s_{-i}); \; \forall s_i \in X_i, \forall i \in N$$
$$\sum_{i \in N} |X_i|$$

➤ The expressions in red are the **number of equations** corresponding to each of the equations.

(Eqn. 2)
$$w_i \geq \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s_i, s_{-i}); \; \forall s_i \in S_i \backslash X_i, \forall i \in N$$
$$\sum_{i \in N} (|S_i| - |X_i|)$$

(Eqn. 3)
$$\sum_{s_i \in S_i} \sigma_i(s_i) = 1; \; \forall i \in N$$
$$n$$

➤ Eqn. 1, 3, and 5 are the **equalities**. The number of equalities is,
$$n + \sum_{i \in N} |S_i|$$

(Eqn. 4)
$$\sigma_i(s_i) > 0; \; \forall s_i \in X_i, \forall i \in N$$
$$\sum_{i \in N} |X_i|$$

(Eqn. 5)
$$\sigma_i(s_i) = 0; \; \forall s_i \in S_i \backslash X_i, \forall i \in N$$
$$\sum_{i \in N} (|S_i| - |X_i|)$$

Mixed Strategy Nash Equilibrium (MSNE)

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$$w_i \geq \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s_i, s_{-i}); \ \forall s_i \in S_i \backslash X_i, \forall i \in N$$

(Eqn. 3)
$$\sum_{s_i \in S_i} \sigma_i(s_i) = 1; \ \forall i \in N$$

(Eqn. 4)
$$\sigma_i(s_i) > 0; \ \forall s_i \in X_i, \forall i \in N$$

(Eqn. 5)
$$\sigma_i(s_i) = 0; \ \forall s_i \in S_i \backslash X_i, \forall i \in N$$

➤ The number of variables in these equations is
$$n + \sum_{i \in N} |S_i|$$
where n is because of the variables w_i (there are n of them). σ_i attached with player i has $|S_i|$ variables.

➤ So, the **number of equations and the number of equalities are same**. ALWAYS! That is a good news!

Mixed Strategy Nash Equilibrium (MSNE)

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$$w_i \geq \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s_i, s_{-i}); \ \forall s_i \in S_i \setminus X_i, \forall i \in N$$

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$$\sum_{s_i \in S_i} \sigma_i(s_i) = 1; \ \forall i \in N$$

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$$\sigma_i(s_i) > 0; \ \forall s_i \in X_i, \forall i \in N$$

(Eqn. 5)
$$\sigma_i(s_i) = 0; \ \forall s_i \in S_i \setminus X_i, \forall i \in N$$

➤ We solve the equalities for a solution. The equalities are eqn. 1, 3, and 5.

➤ We then check of the obtained solution satisfies the inequalities. The inequalities are eqn. 2 and 4.

➤ If the solution satisfies the inequalities, then MSNE exist for the support chosen in step 1. Else it does not.

Mixed Strategy Nash Equilibrium (MSNE)

➤ We will now discuss a general approach to find MSNE using definition 6 for n player games.

➤ **Step 2:** For X_i's fixed in step 1, definition 6 of MSNE will lead to the following equations. Solve these equations (provided there exists a solution). **(refer to chapter 11 of the book for explanation)**

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$$w_i = \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s_i, s_{-i}); \ \forall s_i \in X_i, \forall i \in N$$

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$$w_i \geq \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s_i, s_{-i}); \ \forall s_i \in S_i \backslash X_i, \forall i \in N$$

(Eqn. 3)
$$\sum_{s_i \in S_i} \sigma_i(s_i) = 1; \ \forall i \in N$$

(Eqn. 4)
$$\sigma_i(s_i) > 0; \ \forall s_i \in X_i, \forall i \in N$$

(Eqn. 5)
$$\sigma_i(s_i) = 0; \ \forall s_i \in S_i \backslash X_i, \forall i \in N$$

➤ We solve the equalities for a solution**s**. The equalities are eqn. 1, 3, and 5. **Remember:** When there are more the two players, we have to solve a **set of non-linear equations** which can have **multiple solutions**.

➤ We then check of one of those solutions satisfies the inequalities. The inequalities are eqn. 2 and 4. If it satisfies, then MSNE exist for the support chosen in **step 1**. Else it does not.

Mixed Strategy Nash Equilibrium (MSNE)

➢ We will now discuss a general approach to find MSNE using definition 6 for n player games.

➢ **Step 2:** For X_i's fixed in step 1, definition 6 of MSNE will lead to the following equations. Solve these equations (provided there exists a solution). **(refer to chapter 11 of the book for explanation)**

(Eqn. 1)
$$w_i = \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s_i, s_{-i}) \, ; \, \forall s_i \in X_i, \forall i \in N$$

(Eqn. 2)
$$w_i \geq \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j) \right) u_i(s_i, s_{-i}) \, ; \, \forall s_i \in S_i \backslash X_i, \forall i \in N$$

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$$\sum_{s_i \in S_i} \sigma_i(s_i) = 1; \, \forall i \in N$$

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$$\sigma_i(s_i) > 0; \, \forall s_i \in X_i, \forall i \in N$$

(Eqn. 5)
$$\sigma_i(s_i) = 0; \, \forall s_i \in S_i \backslash X_i, \forall i \in N$$

VERY IMPORTANT: In chapter 11 of the book, there is one example that is solved. There is only one example so you don't have to hunt for it. In the beginning of this example, did a series of steps to remove supports for which there will no MSNEs. It is must that you look into this example because it demonstrates how to simplify the problem.

Domination by Mixed Strategy

➤ The approach to find MSNE using definitions 4 or 6 reduces to solving a nonlinear feasibility problem (which can be solved at least in theory by existing non-linear solvers). So finding MSNE is a hard problem from computational point of view.

➤ The number of inequality constraints to find MSNE using definitions 4 or 6 increases with the number of pure strategies, S_i, of individual players. Hence, to simplify the nonlinear feasibility problem, it is beneficial to remove the strictly dominated strategies of a player using IRSDS.

Domination by Mixed Strategy

➤ If a strategy $s_i \in S_i$ of player i is **strictly dominated** by another strategy of player i, than the probability that player i will play strategy s_i in MSNE is **zero**, i.e. $\boldsymbol{\sigma_i(s_i) = 0}$ **if** $\boldsymbol{s_i}$ **is strictly dominated**.

➤ In lectures 9 and 10, to find if a strategy s_i is **strictly dominated**, we just tried to find **if there exists a strategy** $s_i' \in S_i$ such that,

$$u_i(s_i, s_{-i}) < u_i(s_i', s_{-i}) \, , \forall s_{-i} \in S_{-i}$$

➤ If s_i is strictly dominated, we can remove s_i from S_i. This reduces the size of S_i which in turn will simplify the computation involved to find MSNE using definition 4.

➤ We just don't have to stop with removing the strictly dominated strategies of a player. We can go one step further and use **IRSDS** to iteratively remove dominated strategies of players in every iteration.

Domination by Mixed Strategy

➢ In lectures 9 and 10, to find if a strategy s_i is **strictly dominated**, we just tried to find **if there exists a strategy $s_i' \in S_i$** such that,

$$u_i(s_i, s_{-i}) < u_i(s_i', s_{-i}) \,, \forall s_{-i} \in S_{-i}$$

➢ But the above definition only checks if a strategy s_i of player i is **strictly dominated** by another **pure strategy of player i**. But is possible that s_i does not get dominated by a pure strategy s_i' of player i but gets dominated by a mixed strategy σ_i' of player i. **(Why?)**

• If s_i is strictly dominated by a mixed strategy σ_i' but NOT a pure strategy s_i', then it makes sense to extend the concept of dominated to mixed strategies also. By doing so, we can remove more pure strategies s_i from S_i therefore simplifying the process of finding MSNE.

Domination by Mixed Strategy

➤ Strict domination by mixed strategy:

A strategy s_i is **strictly dominated**, if there exists a (mixed) strategy $\sigma_i' \in \Delta(S_i)$ such that,

$$U_i(s_i, \sigma_{-i}) < U_i(\sigma_i', \sigma_{-i}), \forall \sigma_{-i} \in \Delta(S_{-i})$$

➤ The above definition is **not useful from practical point** of view because $\Delta(S_{-i})$ contains **infinite number of mixed strategies of other players** and we can't check the above inequality for infinite number of σ_{-i}.

Domination by Mixed Strategy

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A strategy s_i is **strictly dominated**, if there exists a (mixed) strategy $\sigma_i' \in \Delta(S_i)$ such that,

$$U_i(s_i, \sigma_{-i}) < U_i(\sigma_i', \sigma_{-i}), \forall \sigma_{-i} \in \Delta(S_{-i})$$

Theorem 2: If,

$$U_i(s_i, s_{-i}) < U_i(\sigma_i', s_{-i}), \forall s_{-i} \in S_{-i}$$

then,

$$U_i(s_i, \sigma_{-i}) < U_i(\sigma_i', \sigma_{-i}), \forall \sigma_{-i} \in \Delta(S_{-i})$$

➤ The importance of theorem 2 is that we can verify, if this inequality is valid by only checking this inequality. But this inequality has only finite number of s_{-i} to verify.

Domination by Mixed Strategy

Theorem 2: If,

$$U_i(s_i, s_{-i}) < U_i(\sigma_i', s_{-i}) \,, \forall s_{-i} \in S_{-i}$$

then,

$$U_i(s_i, \sigma_{-i}) < U_i(\sigma_i', \sigma_{-i}) \,, \forall \sigma_{-i} \in \Delta(S_{-i})$$

Intuition:

Domination by Mixed Strategy

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$$U_i(s_i, \sigma_{-i}) < U_i(\sigma_i', \sigma_{-i}), \forall \sigma_{-i} \in \Delta(S_{-i})$$

➢ Strict domination by mixed strategy (useful version):

A strategy s_i is **strictly dominated**, if there exists a (mixed) strategy $\sigma_i' \in \Delta(S_i)$ such that,

$$U_i(s_i, s_{-i}) < U_i(\sigma_i', s_{-i}), \forall s_{-i} \in S_{-i}$$

Domination by Mixed Strategy

➢ Strict domination by mixed strategy (useful version):

A strategy s_i is **strictly dominated**, if there exists a (mixed) strategy $\sigma_i' \in \Delta(S_i)$ such that,

$$U_i(s_i, s_{-i}) < U_i(\sigma_i', s_{-i}) \,, \forall s_{-i} \in S_{-i}$$

➢ Now,

$$U_i(s_i, s_{-i}) \quad < \quad U_i\left(\sigma_i', s_{-i}\right) \; ; \; \forall s_{-i} \in S_{-i}$$

$$U_i(s_i, s_{-i}) \quad < \quad \sum_{s_{i,j} \in S_i} \sigma_i'(s_{i,j}) \, U_i(s_{i,j}, s_{-i}) \; ; \; \forall s_{-i} \in S_{-i}$$

➢ To simplify,

$$U_i(s_i, s_{-i}) < \sum_{s_{i,j} \in S_i} p_j U_i(s_{i,j}, s_{-i}) \; ; \; \forall s_{-i} \in S_{-i}$$

Domination by Mixed Strategy

➢ To simplify,

$$U_i\left(s_i, s_{-i}\right) \ < \ \sum_{s_{i,j} \in S_i} p_j U_i\left(s_{i,j}, s_{-i}\right) \ ; \ \forall s_{-i} \in S_{-i}$$

$$p_j \ \geq \ 0 \ ; \ \forall j$$

$$\sum_{s_{i,j} \in S_i} p_j \ = \ 1$$

Domination by Mixed Strategy

➢ To simplify,

$$U_i\left(s_i, s_{-i}\right) < \sum_{s_{i,j} \in S_i} p_j U_i\left(s_{i,j}, s_{-i}\right) ; \forall s_{-i} \in S_{-i}$$

$$p_j \geq 0 ; \forall j$$

$$\sum_{s_{i,j} \in S_i} p_j = 1$$

➢ The linear programming version of the above inequality is,

$$\min_{\{p_j\}} \sum_{s_{i,j} \in S_i} p_j$$

subject to:

$$U_i\left(s_i, s_{-i}\right) < \sum_{s_{i,j} \in S_i} p_j U_i\left(s_{i,j}, s_{-i}\right) ; \forall s_{-i} \in S_{-i}$$

$$p_j \geq 0 ; \forall j$$

If the objective function obtained after solving the above optimization problem is **strictly less than 1**, then strategy s_i is strictly dominated.

Thank You!