Game Theory (CS4187)

Lectures 18

Date: 30/09/2024

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Broad Idea of Today's Lecture

Till now we have discussed the following solutions concepts:

1. Dominant strategy.
2. IRSDS and dominance solvability.
3. PSNE.
4. MSNE.
5. Correlated equilibrium.

The topics of this lecture is:

1. Maxmin strategy (brief discussion only).
2. Computing MSNE of Two Player Zero Sum game.
   - Relation with maxmin strategy.

Maxmin Strategy: Motivation

		Player 2	
		a	b
Player 1	A	2, 1	2, -20
	B	3, 0	-10, 1
	C	-100, 2	3, 3

Example 1:

- Consider the SFG captured by the payoff matrix in the left. **Directly adapted from Game Theory by Michael Maschler.**
- You can verify that the PSNE of this game is $(\boldsymbol{C}, \boldsymbol{b})$.
- The above PSNE says that player 2 will play action $\boldsymbol{b}$. But say due one (or both) of the following reasons, player 2 plays action $\boldsymbol{a}$:
  - Player 2 is irrational.
  - Player 2 actually wanted to play b but due to mistake played a.
- So if player 2 plays a and player 1 plays C, player 1's utility will be the worst possible, i.e. -100!

Maxmin Strategy: Motivation

Player 2

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Example 1:

- In such situations, players have to think from security/adversarial perspective. To put it more precisely:

What action should a player take such that its **minimum payoff against all possible actions of other players is maximized**?

- Player 1's action?

- Player 2's action?

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Example 1:

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HOMEWORK: Let's say that the probability of such a mistake is ε. Convert this game into an EFG. Then convert this EFG to and SFG. Finally, compute the PSNE of this SFG.

Maxmin Strategy: Definition

Definition (Maxmin in Pure Strategy): Maxmin strategy of player i in its pure strategy set S_i, is a strategy in S_i that maximizes the worst case/minimum payoff of player i against all possible pure strategy profile s_{-i} of other players. Mathematically,

$$x_i^{maxmin} \in \arg\max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$$

$$\underline{v_i} = \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$$

Definition (Maxmin in Mixed Strategy): Maxmin strategy of player i in its mixed strategy set $\Delta(S_i)$, is a strategy in $\Delta(S_i)$ that maximizes the worst case/minimum payoff of player i against all possible pure strategy profile σ_{-i} of other players. Mathematically,

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Question: Can a strictly dominant strategy of a player be its maxmin strategy?

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Question: Can $\underline{v_i}$ for pure and mixed strategy be different?

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NOTE: Just like dominant strategy, pure strategy of a player only depend on its own utility function.

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NOTE: Maxmin strategy is important in those cases where all the players are not trying to maximize there payoff but rather trying to minimize yours.

Zero-Sum Games

- In a zero-sum game, the sum of utility of all players sums to zero for all strategy profiles.

$$\sum_{i \in N} u_i(s) = 0 \ , \ \forall s \in S$$

- In a two player zero sum game,

$$u_1(s_1, s_2) + u_2(s_1, s_2) = 0 \ , \ \forall s_1 \in S_1, \forall s_2 \in S_2$$

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Question: A constant sum game is where the sum of utility of all players sums to a constant c for all strategy profiles.

$$\sum_{i \in N} u_i(s) = c \ , \ \forall s \in S$$

Can we convert a constant sum game to a zero sum game?

Zero-Sum Games

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- In a two player zero sum game,

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Question: Does the first equation imply the following

$$\sum_{i \in N} U_i(\sigma) = 0 \ , \ \forall \sigma \in \Delta(S)$$

Matrix Games

		Player 2	
		a	b
Player 1	A	2, -2	2, -2
	B	3, -3	-10, 10
	C	-100, 100	-3, 3

- Two player zero-sum games are also called matrix games. Why?
- Matrix form of writing equality.
- Notes are not complete. Read chapter 9. Specifically, section 9.1 and 9.4 of the book by Narahari. You can google the phrase "Row Player's Linear Program" and "Column Player's Linear Program" to fid the main result.

Thank You!