Game Theory (CS4187)

Lectures 19, 20, 28, 29, 30, 42

Date: 01/10/2024

03/10/2024

04/11/2024

05/11/2024

07/11/2024

Instructor: Gourav Saha

42 is a video lecture. No date.

Breakup of the lectures

This lecture slides happened over a huge span of time. So, I will give a broad idea of these lectures:

1. **Lectures 19 and 20:** These lectures happened towards the end of module 2. I covered perfect information extended form game (PIEFG). Also, I gave the broad idea of sequential equilibrium.

2. **Lectures 28, 29, and 30:** This lectures happened immediately after the fall break and just before minor 2. Before these lectures, we already covered module 3 and also started module 4. So don't get confused when going over the notes. In this lectures, we covered:

   - Subgame perfect Nash equilibrium for imperfect information extended for games (IIEFGS). This happened in lectures 28 and 29.

   - I again introduced sequential equilibrium towards the end of lecture 29. And, during lecture 30, we solved an example to compute sequential equilibrium (SE).

3. **Lectures 42:** Lecture 42 is a video lecture. Consider this as a substitute of one of the four lectures that I didn't take. In this lecture, I will again explain SE (because I am not making notes for it). Also, I will solve another example. **IMPORTANT:** I will also talk about computing mixed strategy SE which I did not do during lectures.

Game Theory (CS4187)

Lectures 19, 20

Date: 01/10/2024

03/10/2024

Broad Idea of Today's Lecture

Till now in module 2, we have discussed six solution concepts related to **strategic form game** (simultaneous move games).

The topics of this lecture is to extend few of the solution concepts of strategic form games to compute solutions for **extended form games** (sequential move games).
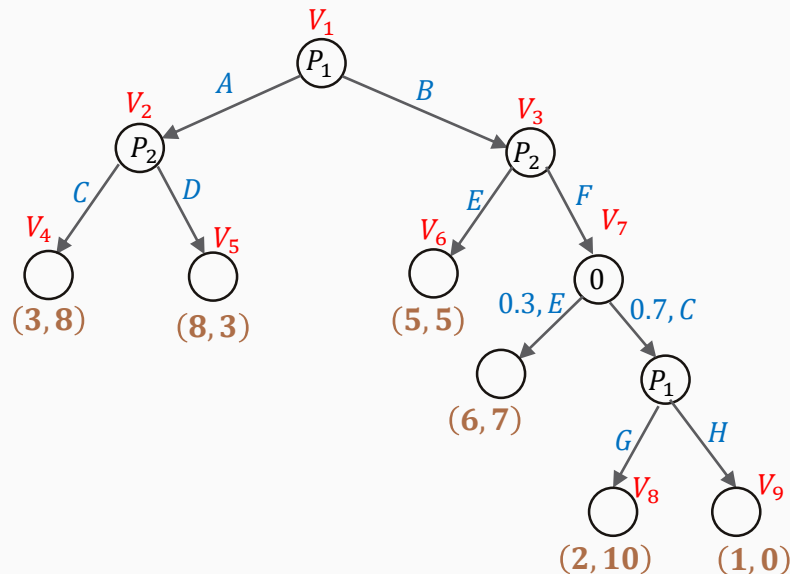
Strategies for PIEFG: Examples

Example 1
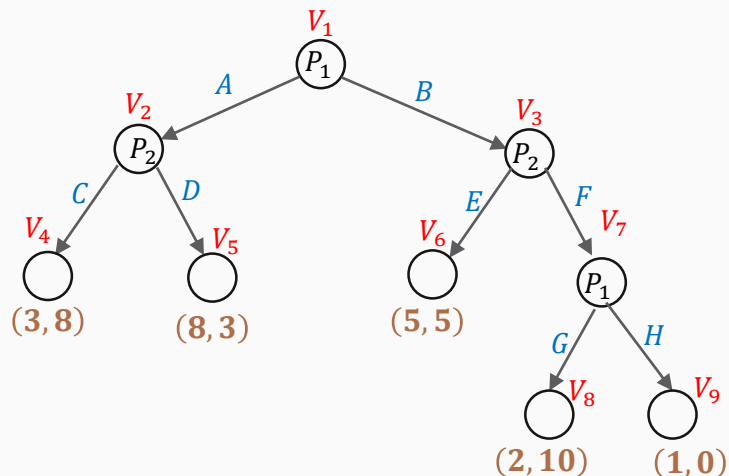
➢ Recall that EFGs are of two type: **perfect information** (PIEFG) and **imperfect information** (IIEFG).

- Both these types can be further sub-divided into games with **chances moves** and **without chance moves**.

➢ We start by discussing solution concepts of PIEFG, both with and without chance moves.

➢ To illustrate the concepts, we will use two examples shown in the left (for example 2, go to the next slide).

Strategies for PIEFG: Examples

Example 2

➢ Recall that EFGs are of two type: **perfect information** (PIEFG) and **imperfect information** (IIEFG).

  • Both these types can be further sub-divided into games with **chances moves** and **without chance moves**.

➢ We start by discussing solution concepts of PIEFG, both with and without chance moves.

➢ To illustrate the concepts, we will use two examples shown in the left (for example 2, go to the next slide).

Strategies for PIEFG

➢ Recall the definition of PIEFG with and without chance moves.

Definition (PIEFG without Chance Moves): A PIEFG without chance moves is a tuple $\Gamma = \langle N, A, H, Z, \rho, \mathcal{X}, \sigma, u \rangle$.

Definition (PIEFG with Chance Moves): A PIEFG without chance moves is a tuple $\Gamma = \langle N, A, H, Z, \rho, \mathcal{X}, \sigma, u, (p_i)_{i \in H_0} \rangle$.

- N is the set of players,

- A is the **set of actions**,

- H is the **set of non-terminal vertices**,

- Z is the **set of terminal vertices**,

- $\rho : H \to N$ is the **player function**,

- $\mathcal{X} : H \to 2^A$ is the **action function**,

- $\sigma : H \times A \to H \cup Z$ is the **successor function**,

- $u = (u_n)_{n \in N \setminus \{0\}}$, where $u_n : Z \to \mathbb{R}$ is the **utility function**,

- $H_0 = \{i \in H : \rho(i) = 0\}$ is the **set of non-terminal vertices for which the player is nature**. For every $i \in H_0$, p_i **is the probability distribution** across actions in set $\mathcal{X}(i)$.

Strategies for PIEFG

➤ A **pure strategy** for a player in PIEFG is a **complete specification** of which (deterministic) action to take at every vertices belonging to that player.

➤ Let,

$$H_i = \{v \in H : \rho(v) = i\}$$

is the set of non-terminal vertices where player i is playing. Let $a_{i,j}$ denote the action that player i takes corresponding to $H_{i,j}$ ($H_{i,j}$ the jth vertex in H_i). Then the **pure strategy of player i** is,

$$s_i = \left(a_{i,1}, a_{i,2}, \cdots, a_{i,|H_i|}\right)$$

➤ The **set of strategies of player i** is the cartesian product of all $\mathcal{X}(H_{i,j})$,

$$S_i = \mathcal{X}(H_{i,1}) \times \mathcal{X}(H_{i,2}) \times \cdots \times \mathcal{X}(H_{i,|H_i|})$$

We have $s_i \in S_i$.

Strategies for PIEFG

➢ Just like SFGs, in EFG too there is a concept of **strategy profile**. Strategy profile $s = (s_i, s_{-i})$ is the action of all the players.

- In EFG, each s_i is a **tuple**. This is one way in which strategy profile of SFGs and EFGs differ.
- Even in SFGs s_i can be a tuple.

➢ As we know, we can convert a EFG to and SFG. Let Γ be the EFG. Let $SFG(\Gamma)$ denote the SFG equivalent of Γ.

➢ Then all the solution concepts that we have learned about SFGs till now can be directly applied to $SFG(\Gamma)$ to find the solution of EFG Γ.

- The problem is that these solutions does not utilizes the sequential nature of the game and is hence may lead to illogical solutions.

PSNE of PIEFG

Example 1

V_1 P_1

A B

V_2 P_2 V_3 P_2

C D E F

V_4 V_5 V_6 V_7 P_1

$(3,8)$ $(8,3)$ $(5,5)$

G H

V_8 V_9

$(2,10)$ $(1,0)$

P2
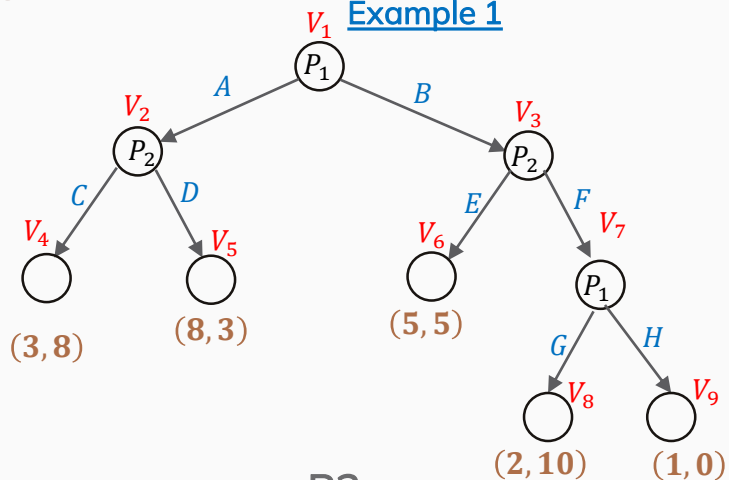
P1	(C,E)	(C,F)	(D,E)	(D,F)
(A,G)	3, 8	3, 8	8, 3	8, 3
(A,H)	3, 8	3, 8	8, 3	8, 3
(B,G)	5, 5	2, 10	5, 5	2, 10
(B,H)	5, 5	1, 0	5, 5	1, 0

➢ $SFG(\Gamma)$ for this game is shown in the table (directly adapted from Chapter 5 of the book Multiagent Systems by Shoham and Brown).

➢ Remember, PSNE in context of EFG is related to $SFG(\Gamma)$, i.e. PSNE is related to the SFG equivalent of EFG.

➢ PSNE of the game shown in the left is marked with the red circles. PSNEs are:

$$((A,G),(C,F))$$
$$((A,H),(C,F))$$
$$((B,H),(C,F))$$

➢ PSNE $((B,H),(C,E))$ is not logical! Why?

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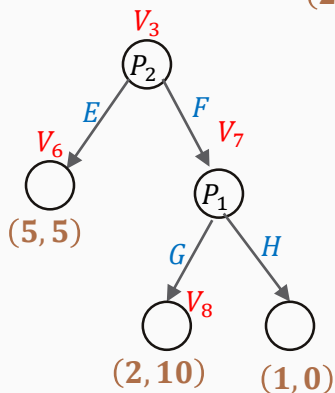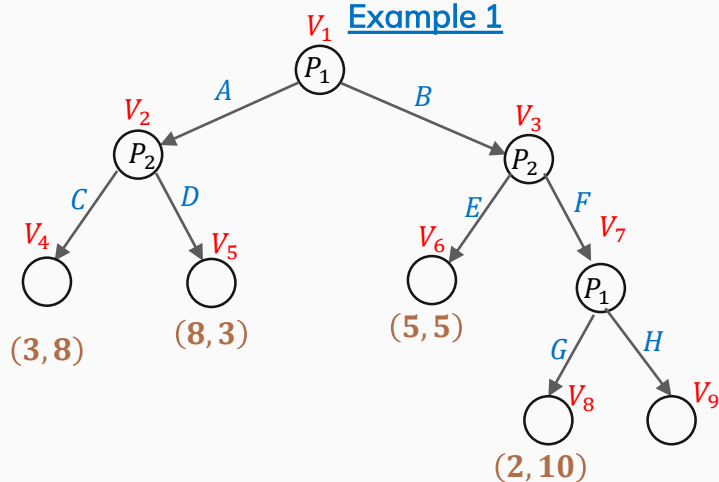
Subgame

Example 1

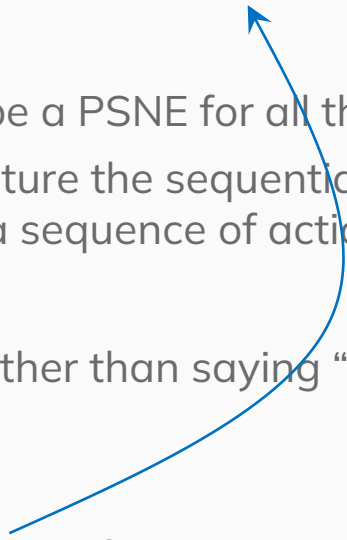**Definition (Subgame):** Given a EFG Γ, the subgame of Γ at non-terminal vertex h, denoted by $\Gamma(h)$, is the EFG described by the subtree rooted at vertex h.

➢ Example: $\Gamma(V_3)$ shown in the left.

Subgame Perfect Nash Equilibrium (SPNE)

Definition (Pure Strategy SPNE): Pure strategy SPNE of an EFG Γ are all strategy profiles $\boldsymbol{s}$ such that **for all $\boldsymbol{h} \in \boldsymbol{H}$**, the <u>restriction of $\boldsymbol{s}$ </u>to $\Gamma(h)$ is a PSNE of $SFG(\Gamma(h))$.

➢ Basically $\boldsymbol{s}$ should be a PSNE for all the subgames, then only $\boldsymbol{s}$ SPNE. **What is the logic?**
  - In order to capture the sequentially nature of the game, i.e. strategy s has to be rational after a sequence of actions was taken by other players.

➢ Most of the time, rather than saying "PSNE of $SFG(\Gamma(h))$", we will directly say "PSNE of $\Gamma(h)$".

➢ What does "restriction of $\boldsymbol{s}$"?

Backward Induction
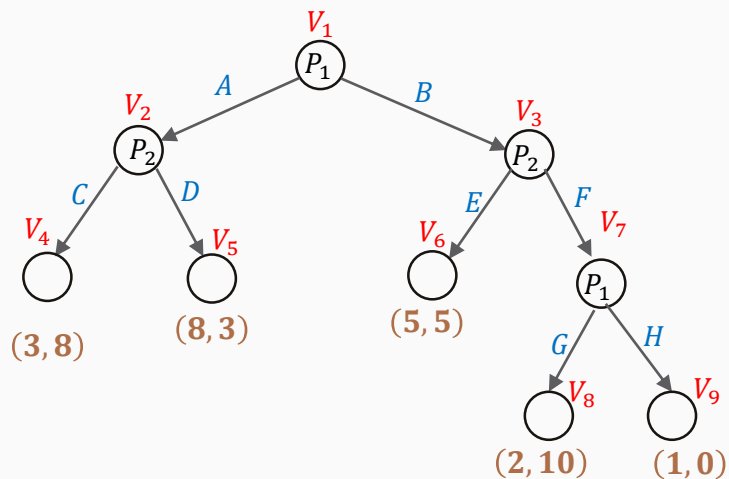
Example 1

- ➢ SPNE for PIEFG can be found using Backward induction. To learn about backward induction, look into chapter 8 of the book:

  Harrington, J.E., "**Games, Strategies, and Decision Making**", **2nd edition**.

- ➢ Compute SPNE of the PIEFG shown in the left. In the next few slides, we demonstrate backward induction.

Backward Induction

Example 1

➢ SPNE for PIEFG can be found using Backward induction. To learn about backward induction, look into chapter 8 of the book:

Harrington, J.E., "Games, Strategies, and Decision Making", 2nd edition.

➢ P1 is playing in the vertex with marked with the red arrow. Action G has a payoff of 2 and H has a payoff of 1. So, P1 chooses G.

Backward Induction

Example 1

➤ SPNE for PIEFG can be found using Backward induction. To learn about backward induction, look into chapter 8 of the book:

Harrington, J.E., "Games, Strategies, and Decision Making", 2nd edition.

➤ P2 is playing in the vertex with marked with the red arrow. Action F has a payoff of 10 and E has a payoff of 5. So, P2 chooses F.
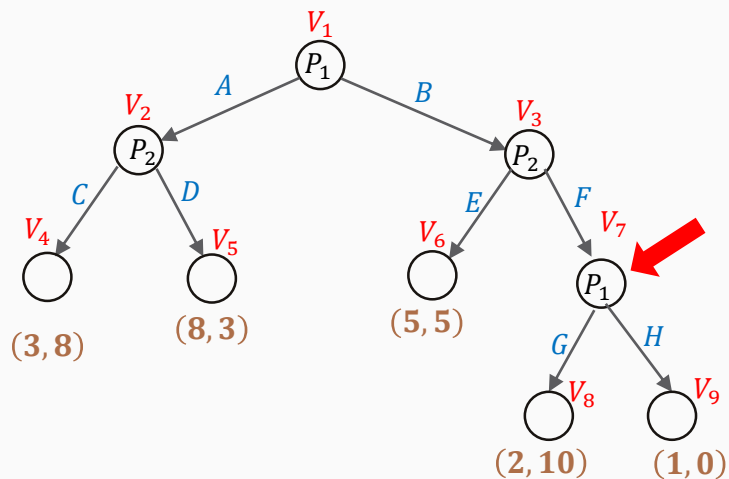
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Backward Induction

Example 1

➢ SPNE for PIEFG can be found using Backward induction. To learn about backward induction, look into chapter 8 of the book:

Harrington, J.E., "**Games, Strategies, and Decision Making**", **2nd edition**.

➢ P2 is playing in the vertex with marked with the red arrow. Action C has a payoff of 8 and C has a payoff of 3. So, P2 chooses C.
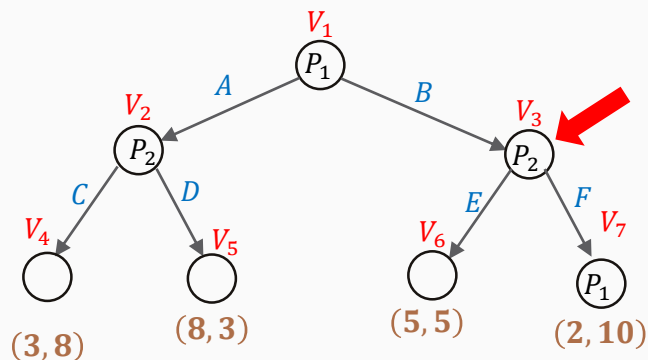
Backward Induction

Example 1

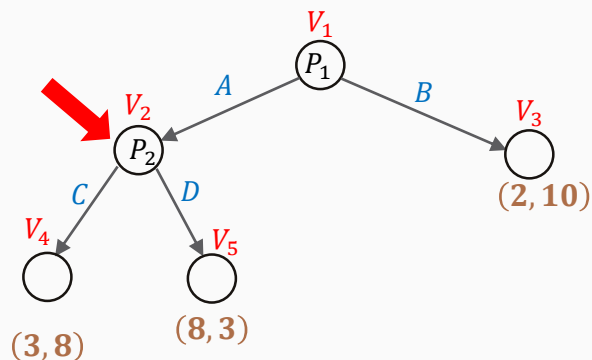> SPNE for PIEFG can be found using Backward induction. To learn about backward induction, look into chapter 8 of the book:

Harrington, J.E., "Games, Strategies, and Decision Making", 2nd edition.

> P1 is playing in the vertex with marked with the red arrow. Action A has a payoff of 3 and B has a payoff of 2. So, P1 chooses A.

Backward Induction

Example 1

➤ So, the SPNE is,

$$\big((A, G), (C, F)\big)$$

Backward Induction

Example 1

➢ So, the SPNE is,

$$\big((A, G), (C, F)\big)$$

➢ Is the computed SPNE one of the NEs that was discussed in <u>this slide (click here)</u>?

➢ Is it always true that an SPNE of an EFG Γ will be an NE of the induced SFG, $SFG(\Gamma)$?

Backward Induction
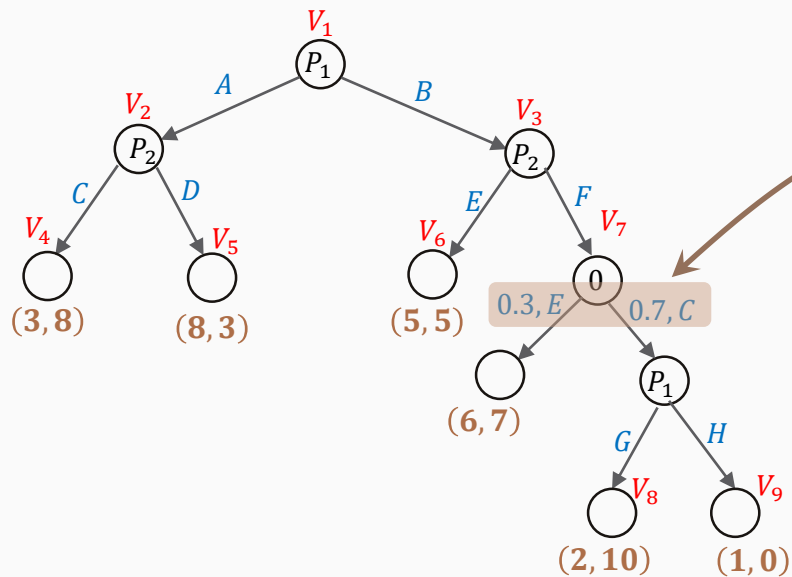
Example 2

V_1 — P_1

A B

V_2 — P_2 V_3 — P_2

C D E F

V_4 V_5 V_6 V_7

$(3, 8)$ $(8, 3)$ $(5, 5)$

0

$0.3, E$ $0.7, C$

$(6, 7)$ P_1

G H

V_8 V_9

$(2, 10)$ $(1, 0)$

➢ Compute SPNE of the PIEFG **with chance moves** shown in the left. In the next few slides, we demonstrate backward induction.

Backward Induction

Example 2

- ➢ Compute SPNE of the PIEFG **with chance moves** shown in the left. In the next few slides, we demonstrate backward induction.

- ➢ P1 is playing in the vertex with marked with the **red arrow**. Action G has a payoff of 2 and H has a payoff of 1. So, P1 chooses G.
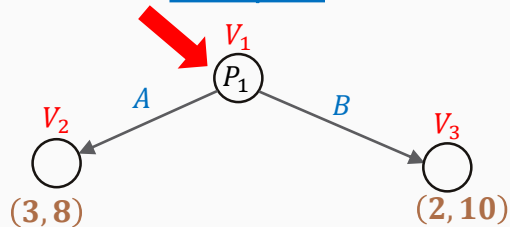
Backward Induction

Example 2

V_1

P_1

A B

V_2 V_3

P_2 P_2

C D E F

V_4 V_5 V_6 V_7

$(3, 8)$ $(8, 3)$ $(5, 5)$ 0

$0.3, E$ $0.7, C$

$(6, 7)$ $(2, 10)$

➢ Compute SPNE of the PIEFG **with chance moves** shown in the left. In the next few slides, we demonstrate backward induction.

➢ P2 is playing in the vertex with marked with the red arrow. Action E has a payoff of 5 and F has a payoff of:

$$0.3 * 7 + 0.7 * 10 = 9.1$$

The above payoff is for P2. The payoff of P1 for action F is,

$$0.3 * 6 + 0.7 * 2 = 3.2$$

P2 chooses F because 9.1 is greater than 5.

Backward Induction
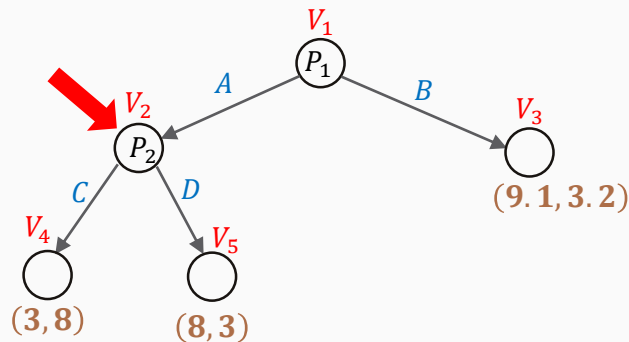
Example 2

➢ Compute SPNE of the PIEFG **with chance moves** shown in the left. In the next few slides, we demonstrate backward induction.

➢ P2 is playing in the vertex with marked with the red arrow. Action C has a payoff of 8 and C has a payoff of 3. So, P2 chooses C.

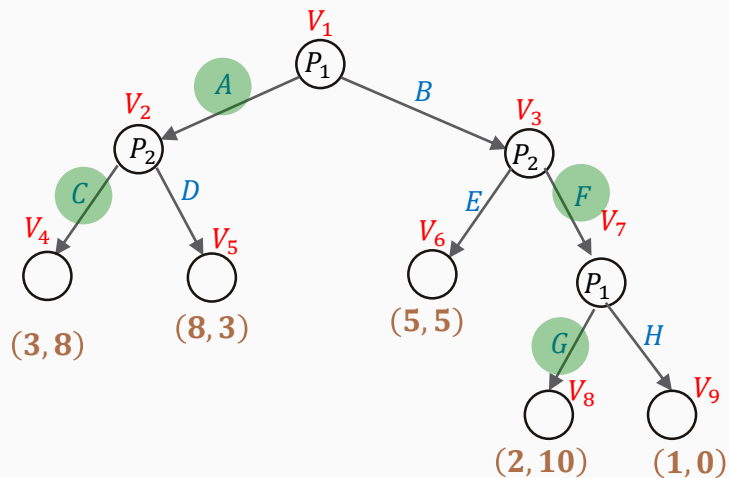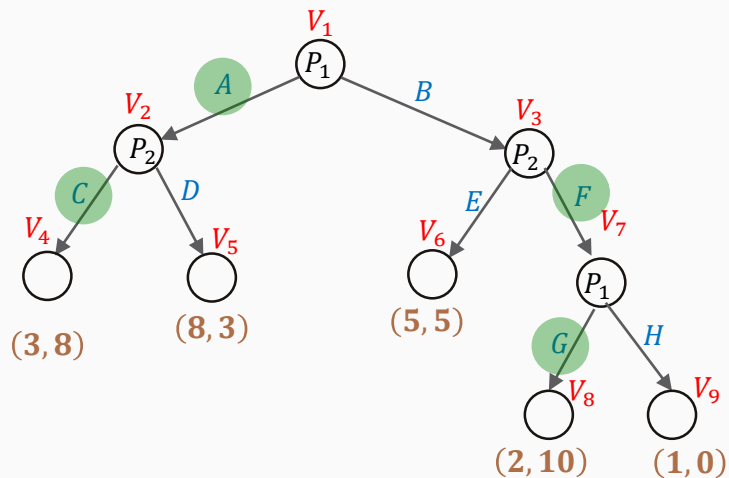# Backward Induction

Example 2

V_1

P_1

A B

V_2 V_3

$(\mathbf{3}, \mathbf{8})$ $(\mathbf{9.1}, \mathbf{3.2})$
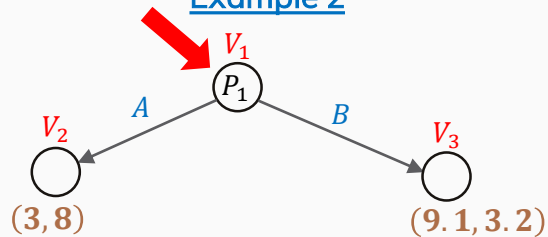
➤ Compute SPNE of the PIEFG **with chance moves** shown in the left. In the next few slides, we demonstrate backward induction.

➤ P1 is playing in the vertex with marked with the red arrow. Action B has a payoff of 9.1 and A has a payoff of 3. So, P1 chooses B.

Backward Induction

Example 2

➢ So, the SPNE is,

$$\big((B, G), (C, F)\big)$$

Game Theory (CS4187)

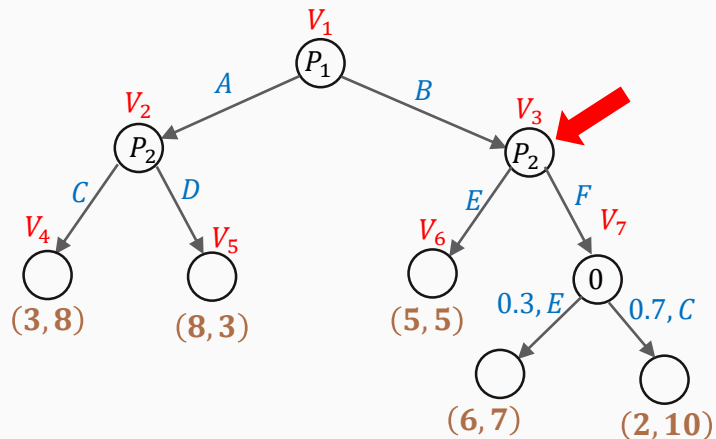# Lectures 28, 29

Date: 04/11/2024

05/11/2024

Solution Concepts for IIEFG

➢ For PIEFG, a pure strategy SPNEs/PSNEs always exists. This is a natural consequence of backward induction.

➢ But for IIEFG, it is not guaranteed that pure strategy SPNEs will exist. How?

- All SFGs can be expressed as EFGs.

- We know that there are SFGs for which PSNEs don't exist.

- Since SPNE is a subset of PSNE, SPNE may not exist.

➢ We will talk about the following solution concepts of IIEFGs:

  ➢ SPNEs.

  ➢ Sequential equilibrium (not coming for minor 2).

SPNE for IIEFGs
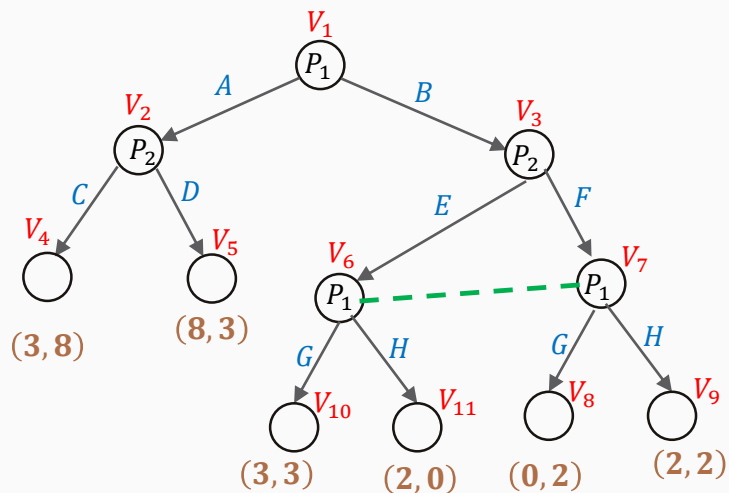
- ➢ We will demonstrate the concept using the example shown in the left.

- ➢ We can't solve imperfect information EFGs directly using backward induction. That said the overall idea of backward induction is still useful of IIEFGs.

- ➢ Backward induction can't be directly used because two nodes in the same information set can't have two different strategies as it is not possible to distinguish between the nodes in an information set. Example: P1 can take action G at vertex V6 and H at vertex V7 just because those actions gives better payoff in the corresponding vertices.

SPNE for IIEFGs
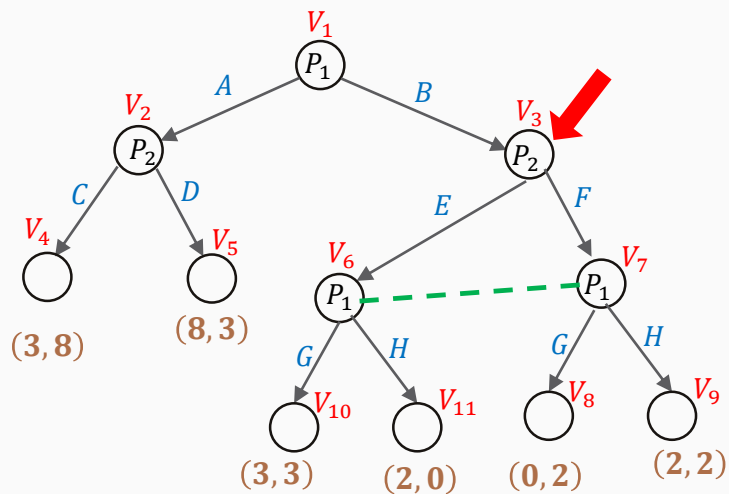
➤ We can see however that the subgame starting at vertex V3 (marked with red arrow) can definitely be solved. Infact, the subgame at V3 is a **simultaneous move game** between P1 and P2.

➤ Since, it is a simultaneous move game we can find it's NEs as we did for SFGs. This is shown in the next slide.

SPNE for IIEFGs
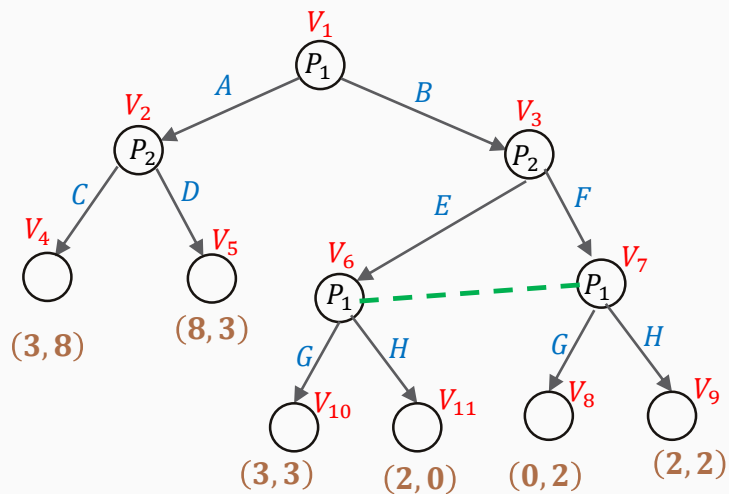
P2

	E	F
P1 G	3, 3	0, 2
H	2, 0	2, 2

➢ NEs are:

$$(G, E)$$
$$(H, F)$$

$\Big\}$ PSNEs

$$\left(\left(\frac{2}{3}\ (G), \frac{1}{3}\ (H)\right), \left(\frac{2}{3}\ (E), \frac{1}{3}\ (F)\right)\right)$$

$\Big\}$ MSNE

➢ For each of these NEs, we can do backward induction as before. We show it for one PSNE and one MSNE in the next few slides.

SPNE for IIEFGs
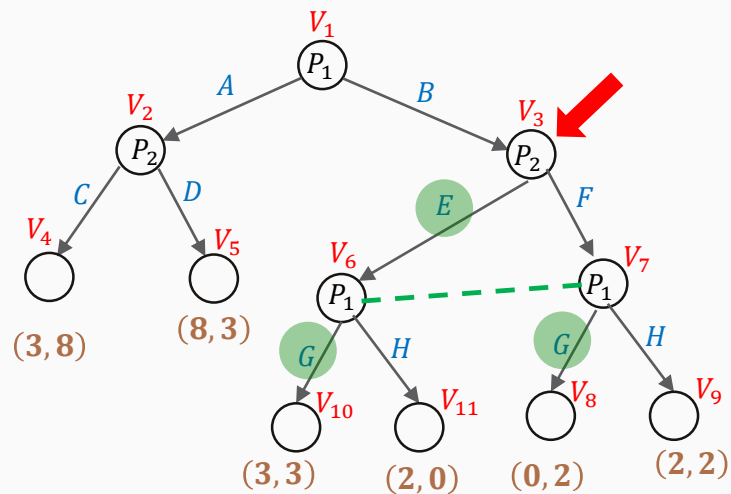
- ➢ Backward induction for NE (G, E).

- ➢ The actions are highlighted using green circles. The corresponding payoff is (3, 3).

SPNE for IIEFGs
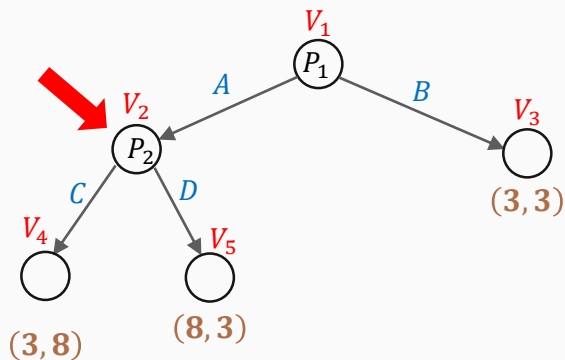
- ➤ Backward induction for NE (G, E).

- ➤ P2 is playing in the vertex with marked with the red arrow. Action C has a payoff of 8 and C has a payoff of 3. So, P2 chooses C.
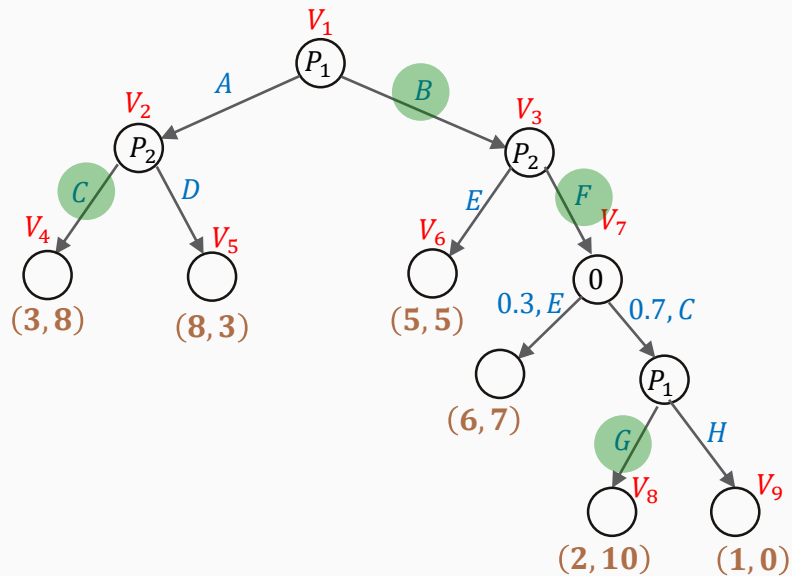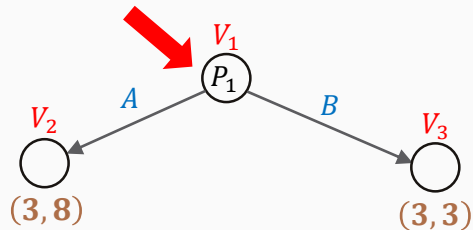
SPNE for IIEFGs

➤ Backward induction for NE (G, E).

➤ P2 is playing in the vertex with marked with the red arrow. Both actions A and B has the same payoffs. So both are possible. In the next two slides, we will show the SPNE for actions A and B separately.

SPNE for IIEFGs
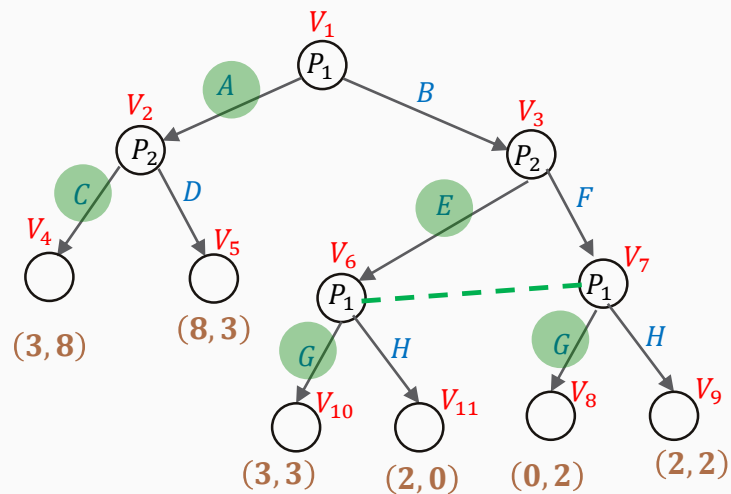
- ➤ Backward induction for NE (G, E).

- ➤ Let the action taken by P1 at vertex V1 be A.

- ➤ SPNE is:

$$\big((A, G), (C, E)\big)$$

SPNE for IIEFGs
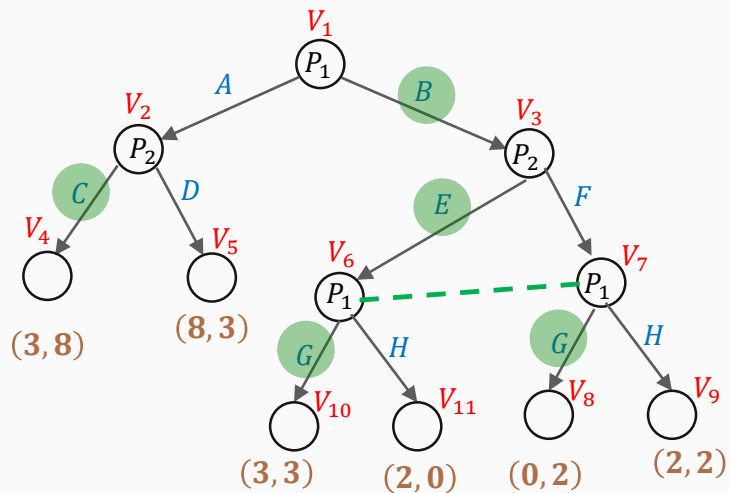
- ➤ Backward induction for NE (G, E).

- ➤ Let the action taken by P1 at vertex V1 be B.

- ➤ SPNE is:

$$((B, G), (C, E))$$

- ➤ So essentially, there are two SPNEs when the subgame starting at node V3 has NE (G, E). They are:

$$((A, G), (C, E))$$
$$((B, G), (C, E))$$

SPNE for IIEFGs
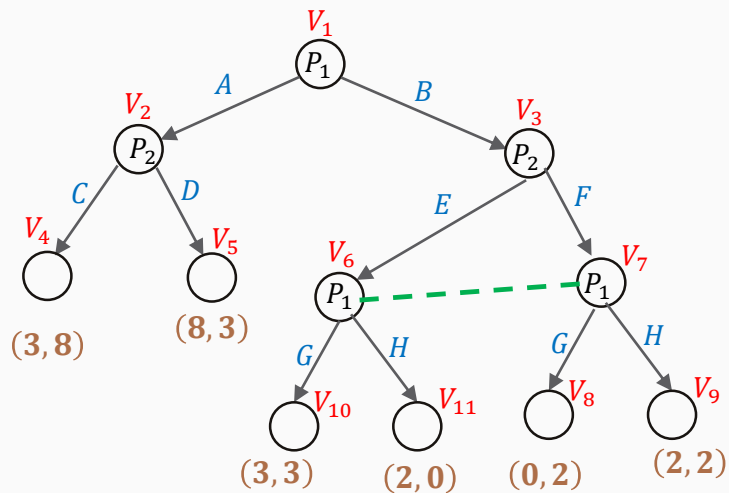
➤ Backward induction for NE

$$\left(\left(\frac{2}{3}\,(G),\frac{1}{3}\,(H)\right),\left(\frac{2}{3}\,(E),\frac{1}{3}\,(F)\right)\right)$$

➤ The payoff of the P1 and P2 for the above NE for the subgame at vertex V3 is

$$P1: \frac{2}{3}*\frac{2}{3}*3+\frac{2}{3}*\frac{1}{3}*2+\frac{1}{3}*\frac{2}{3}*0+\frac{1}{3}*\frac{1}{3}*2=2$$

$$P2: \frac{2}{3}*\frac{2}{3}*3+\frac{2}{3}*\frac{1}{3}*0+\frac{1}{3}*\frac{2}{3}*2+\frac{1}{3}*\frac{1}{3}*2=2$$

Game tree:

V_1: P_1
- Branch A → V_2: P_2
  - Branch C → V_4: $(3,8)$
  - Branch D → V_5: $(8,3)$
- Branch B → V_3: P_2
  - Branch E → V_6: P_1
    - Branch G → V_{10}: $(3,3)$
    - Branch H → V_{11}: $(2,0)$
  - Branch F → V_7: P_1
    - Branch G → V_8: $(0,2)$
    - Branch H → V_9: $(2,2)$
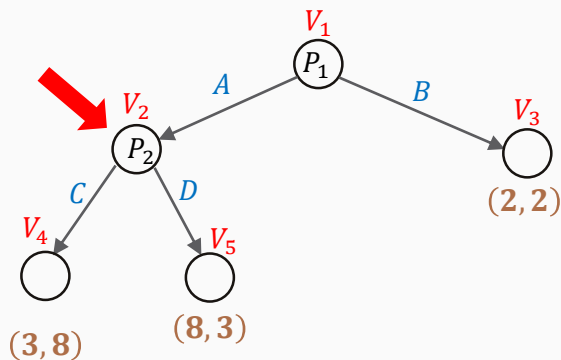
SPNE for IIEFGs

> ➢ Backward induction for NE

$$\left(\left(\frac{2}{3} \ (G), \frac{1}{3} \ (H) \right), \left(\frac{2}{3} \ (E), \frac{1}{3} \ (F) \right) \right)$$

> ➢ P2 is playing in the vertex with marked with the red arrow. Action C has a payoff of 8 and C has a payoff of 3. So, P2 chooses C.
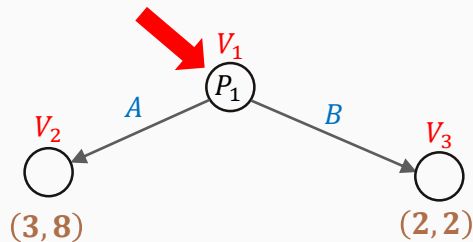
SPNE for IIEFGs

> Backward induction for NE

$$\left(\left(\frac{2}{3} \ (G), \frac{1}{3} \ (H) \right), \left(\frac{2}{3} \ (E), \frac{1}{3} \ (F) \right) \right)$$

> P1 is playing in the vertex with marked with the red arrow. Action A has a payoff of 3 and action B has a payoff of 2. P1 choose A.

SPNE for IIEFGs
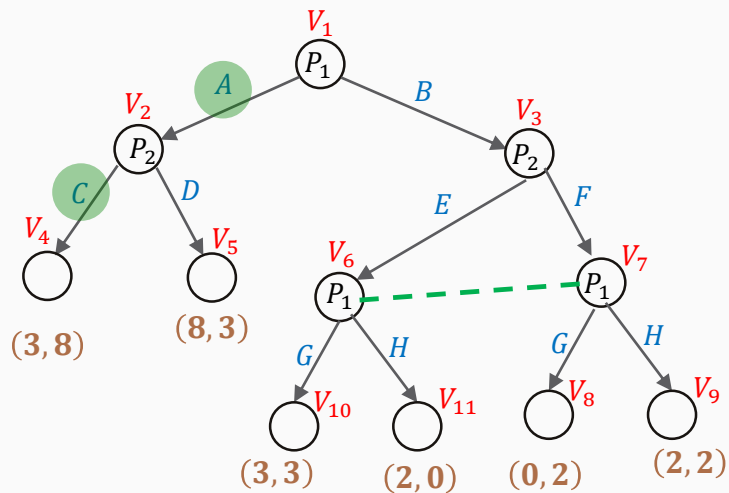
- ➤ Backward induction for NE

$$\left(\left(\frac{2}{3}\,(G), \frac{1}{3}\,(H) \right), \left(\frac{2}{3}\,(E), \frac{1}{3}\,(F) \right) \right)$$

- ➤ SPNE is,

$$\left(\left(A, \left(\frac{2}{3}\,(G), \frac{1}{3}\,(H) \right) \right), \left(C, \left(\frac{2}{3}\,(E), \frac{1}{3}\,(F) \right) \right) \right)$$

SPNE for IIEFGs
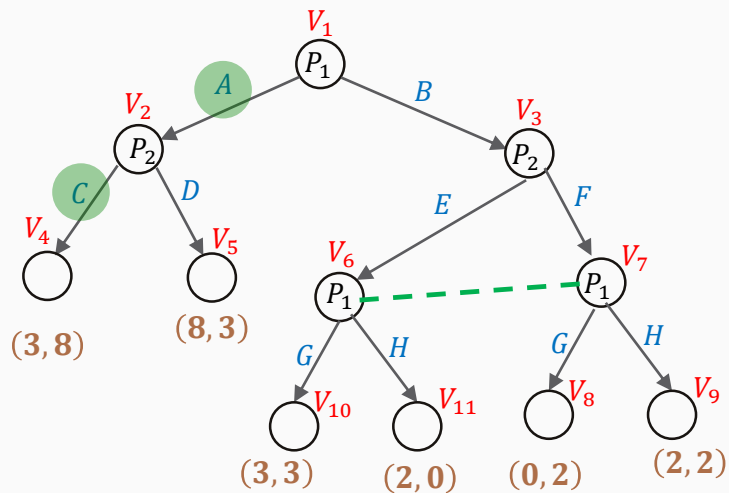
So the SPNEs that we found till now are:

$$((A, G), (C, E))$$

$$((B, G), (C, E))$$

$$\left(\left(A, \left(\frac{2}{3}\ (G), \frac{1}{3}\ (H)\right)\right), \left(C, \left(\frac{2}{3}\ (E), \frac{1}{3}\ (F)\right)\right)\right)$$

Find the SPNE when the NE for the subgame at vertex V3 is,

$$(H, F)$$

Solution Concepts for IIEFG

FIGURE 9.10

The Need for a Systematic Approach to Find Sub-Games
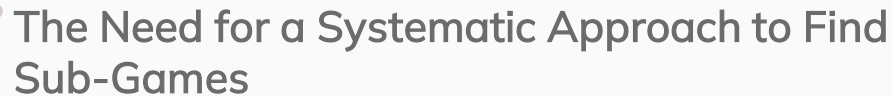
➢ For the problem that we discussed in the last few slides, the sub-game was easy to identify. This is always not so in case of imperfect information game.

➢ Consider the game shown in the left (directly adapted from the book by Harrington). What are the subgames?

Solution Concepts for IIEFG

FIGURE 9.10

The Need for a Systematic Approach to Find Sub-Games

Definition (Subtree): A sub-tree of T is a tree T' **consisting of a node in T** (called the **root of the sub-tree**) and **ALL of its descendants in T**, along with **ALL the edges that connect these nodes**.

Definition (Regular Subtree): A regular sub-tree is a subtree that contains all information sets that have at least one node in the subtree.

In other words, to be a regular subtree, a subtree that includes one node of an information set must include all nodes of that information set.

Game Theory (CS4187)

Lectures 29

Date: 05/11/2024

YouTube video:

https://youtu.be/huKiQ5mVlis

The video says that it is part 1 of lecture 42. This is because I recorded it along with lecture 42. But the topic covered in this video were the topics that we discussed in lecture 29. I have recorded the video because the lecture slides are not sufficient.

Solution Concepts for IIEFG
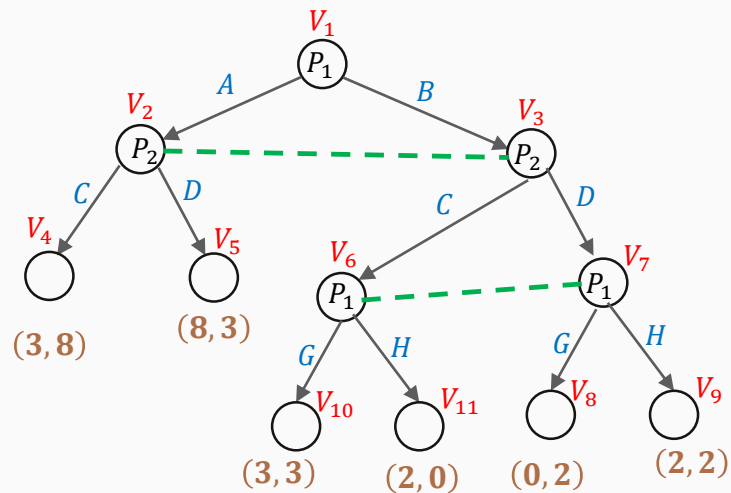
Sequential Equilibrium

➢ What is the need to come up with another equilibrium concept?

➢ Consider the EFG in the left. Player P1 in the information set {V6, V7} definitely knows that P1 played B in vertex V1.

- One may argue that a player (P1 here) will definitely remember its move before but the point conveyed above remains true if another player (say P3) was playing in information set {V6, V7}.

Solution Concepts for IIEFG
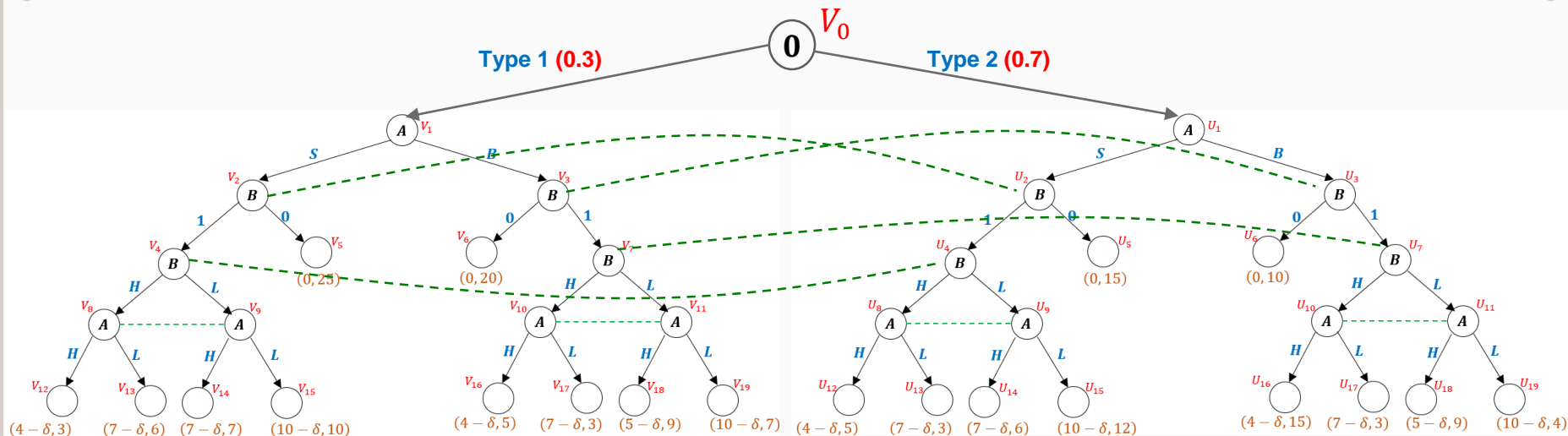
Sequential Equilibrium

➢ What is the need to come up with another equilibrium concept?

➢ For Bayesian sequential move games, for most case we will not have any sub-games!

Solution Concepts for IIEFG

Sequential Equilibrium

➢ A sequential equilibrium (SE) (s, μ) consists of:

- s is the strategy (pure or mixed) at every information set.

- μ is the belief. It containing belief vectors for every information set. Belief vector of an information set contains the belief the player corresponding to that information set holds about its probability of being in individual nodes of that information set.
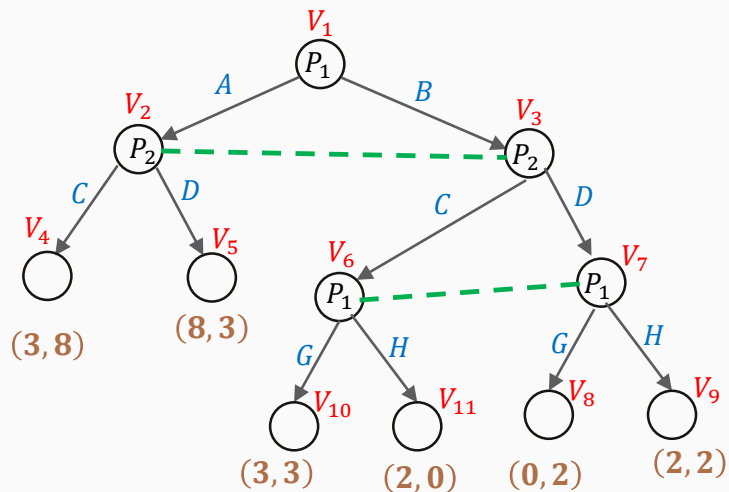
Solution Concepts for IIEFG
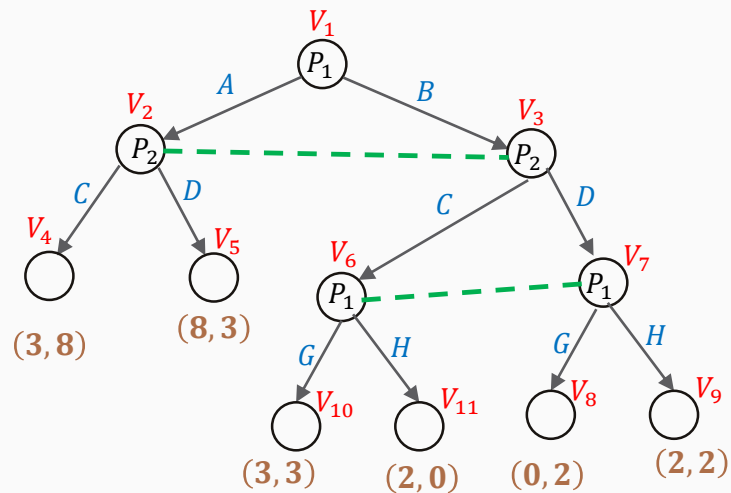
Sequential Equilibrium

➤ A sequential equilibrium (SE) (s, μ) must satisfy the following criteria:

Sequential rationality: Strategy s must be sequentially rationally given the belief μ.

Consistent belief: Belief μ must be consistent given the strategy s.

- Use **Bayes' rule** to compute μ.

Computing Sequential Equilibrium (Pure Strategy)

➤ **This was an example that was done during lecture 30. I am not going to solve it again here. But I will solve another example in the following slides.**
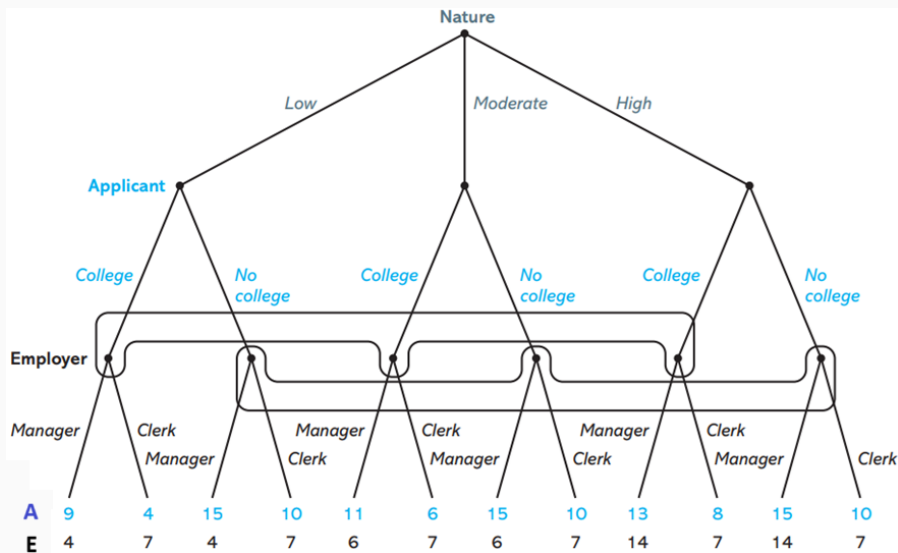
Game Theory (CS4187)

Lectures 30

Date: 07/11/2024

YouTube video:

https://youtu.be/XzOW04VuZJ4

The video says that it is part 2 of lecture 42. This is because I recorded it along with lecture 42. But the topic covered in this video were the topics that we discussed in lecture 30. I have recorded the video because the lecture slides are not sufficient.

Computing Pure Strategy Sequential Equilibrium
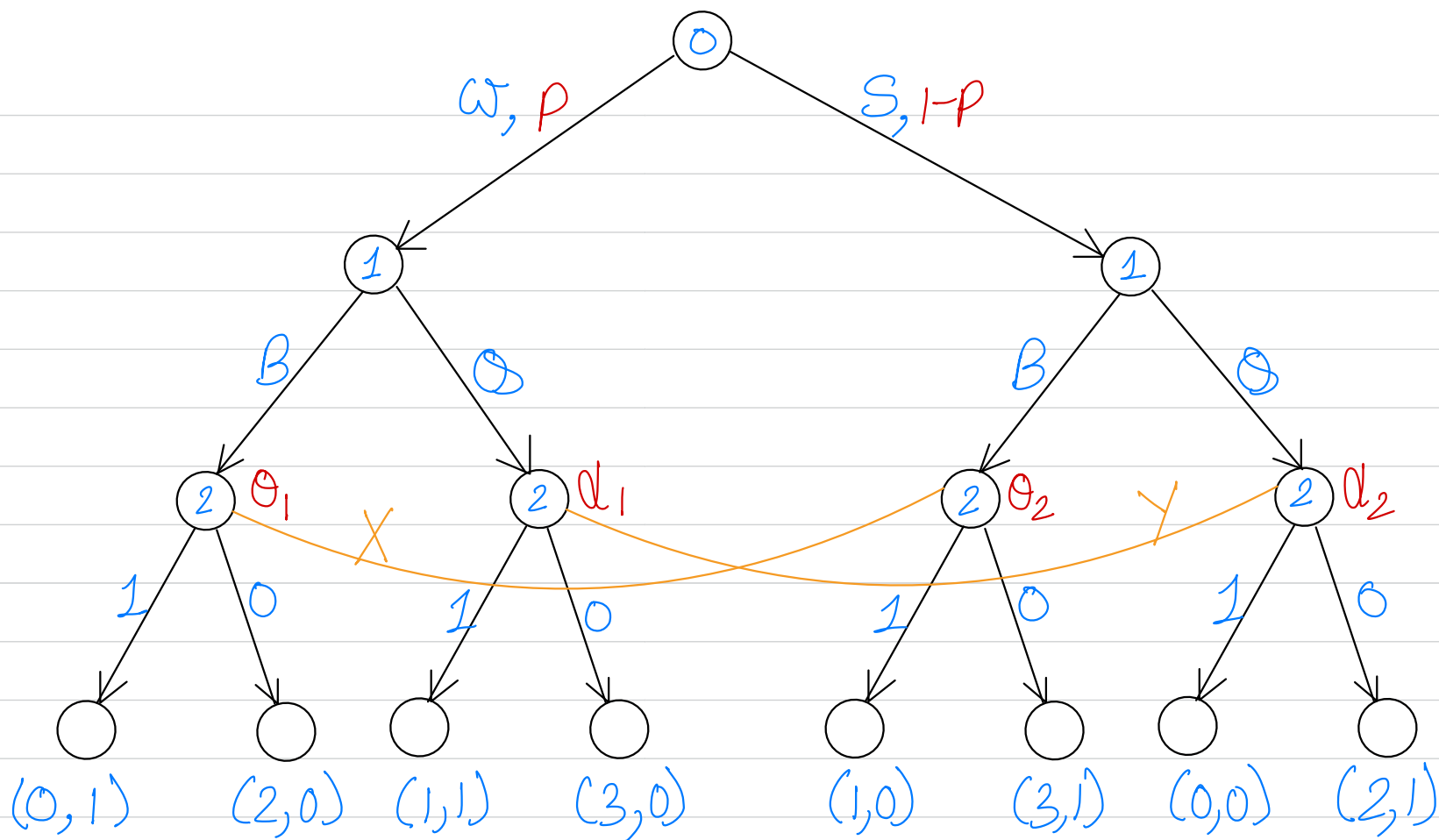
$$E[U_2^X(1)] = \theta_1$$

$$E[U_2^X(0)] = \theta_2$$

$$S_X = \begin{cases} 1 \; ; & \theta_1 > \theta_2 \\ 0 \; ; & \theta_1 \leq \theta_2 \end{cases}$$

$$E[U_2^Y(1)] = \alpha_1$$

$$E[U_2^Y(0)] = \alpha_2$$

$$S_Y = \begin{cases} 1 \; ; & \alpha_1 > \alpha_2 \\ 0 \; ; & \alpha_1 \leq \alpha_2 \end{cases}$$

Diagram (extensive form game tree):

Root node labeled 0.
- Left branch: ω, P → node Z_1 (labeled 1)
- Right branch: $S, 1-P$ → node Z_2 (labeled 1)

From Z_1 (node 1):
- B → node 2 (Θ_1)
- Θ → node 2 (α_1) [highlighted]

From Z_2 (node 1):
- B → node 2 (Θ_2) [highlighted]
- Θ → node 2 (α_2)

Information sets: X connects Θ_1 and Θ_2; Y connects α_1 and α_2.

Terminal nodes (left to right):
- From Θ_1: 1 → $(0,1)$ [highlighted], 0 → $(2,0)$
- From α_1: 1 → $(1,1)$ [highlighted], 0 → $(3,0)$
- From Θ_2: 1 → $(1,0)$ [highlighted], 0 → $(3,1)$
- From α_2: 1 → $(0,0)$ [highlighted], 0 → $(2,1)$

$\underline{Case-1}$

$S_X = 1 \; ; \; \Theta_1 > \Theta_2$

$S_Y = 1 \; ; \; \alpha_1 > \alpha_2$

$S_{Z_1} = \Theta$

$S_{Z_2} = B$

$\alpha_1 = \dfrac{P \cdot 1}{P \cdot 1 + (1-P) \cdot 0} = 1$

$\alpha_2 = 0$

$\Theta_2 = \dfrac{(1-P) \cdot 1}{(1-P) \cdot 1 + P \cdot 0} = 1$

$\Theta_1 = 0$

➤ IS $\Theta_1 > \Theta_2$? NO.

➤ Is $\alpha_1 > \alpha_2$? YES.

Following SE is NOT possible

$(\Theta, B, 1, 1, (0,1), (1,0))$

$\underbrace{\qquad\qquad}_{Strategy} \quad \underbrace{\qquad\qquad}_{Belief}$

Tree diagram with nodes:

- Root node: 0
- Left branch: ω, P to node 1 (labeled Z_1)
- Right branch: $S, 1-P$ to node 1 (labeled Z_2)

From node 1 (Z_1):
- $\varepsilon_1 \, B$ to node 2 (θ_1)
- $\theta \, 1-\varepsilon_1$ to node 2 (α_1)

From node 1 (Z_2):
- $\varepsilon_2 \, B$ to node 2 (θ_2)
- $\theta \, 1-\varepsilon_2$ to node 2 (α_2)

Information sets X and Y connecting the bottom nodes.

Terminal payoffs (left to right):
$(0,1) \quad (2,0) \quad (1,1) \quad (3,0) \quad (1,0) \quad (3,1) \quad (0,0) \quad (2,1)$

Branch labels: $1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0$
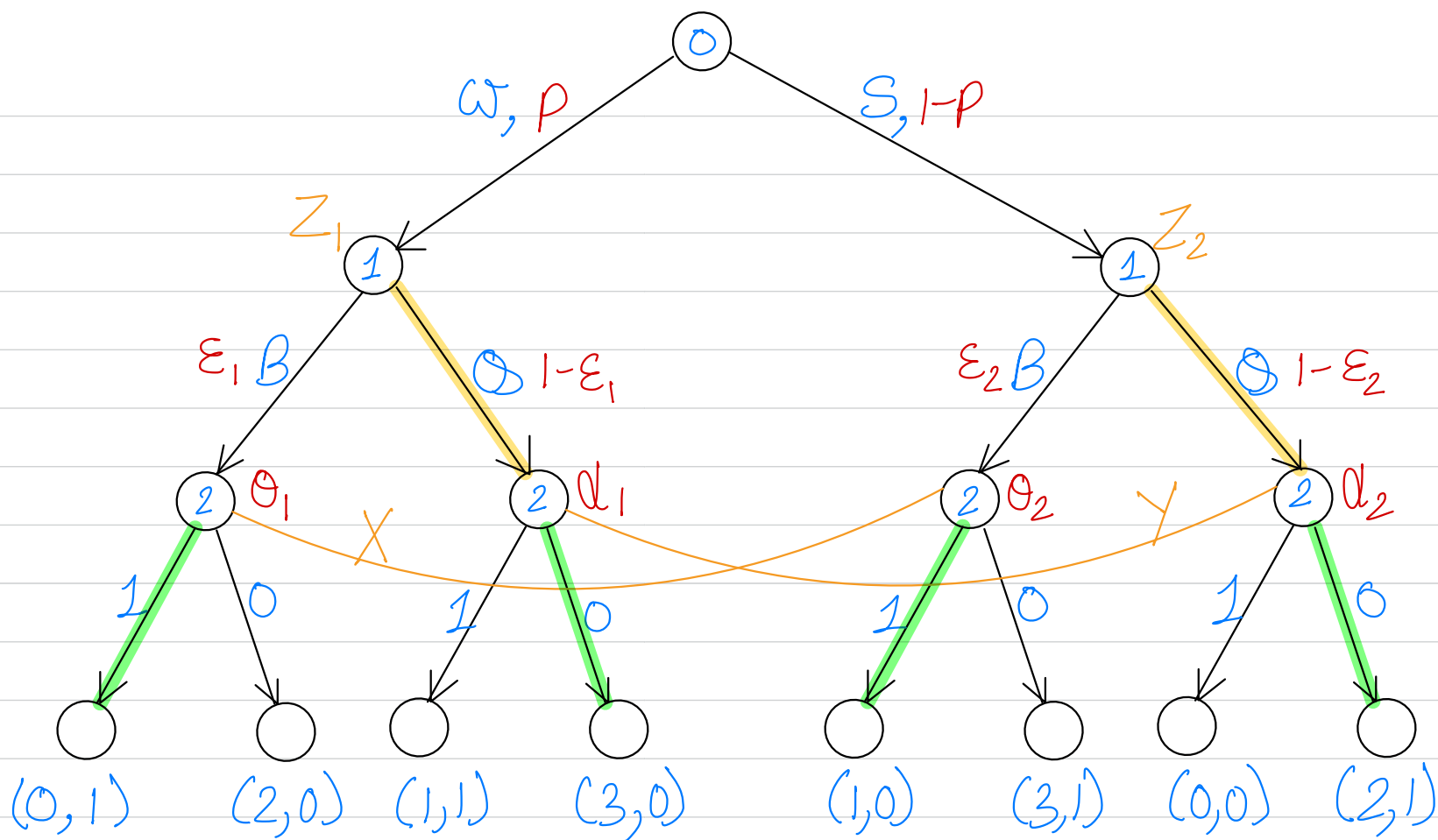
Case-2

$S_X = 1 \; ; \; \theta_1 > \theta_2$

$S_Y = 0 \; ; \; \alpha_1 \leq \alpha_2$

$S_{Z_1} = \theta$

$S_{Z_2} = \theta$

$\alpha_1 = \dfrac{P \cdot 1}{P \cdot 1 + (1-P) \cdot 1} = P$

$\alpha_2 = 1 - P$

$\theta_1 = \dfrac{P \cdot \varepsilon_1}{P \cdot \varepsilon_1 + (1-P)\varepsilon_2} \in [0,1]$

$\theta_2 \in [0,1]$

$\theta_1 + \theta_2 = 1$

- IS $\theta_1 > \theta_2$? $\theta_1 > 0.5$

- Is $\alpha_1 \leq \alpha_2$? $P \leq 1-P$
  $P \leq 0.5$

Following SE is possible if $P \leq 0.5$:

$\left(\theta, \theta, 1, 0, (\theta_1, 1-\theta_1), (P, 1-P) \right)$

for all $\theta_1 > 0.5$.

The tree diagram with labeled nodes and branches:

- Root node 0 with branches: ω, P (left) and $S, 1-P$ (right)
- Left node 1 labeled Z_1, right node 1 labeled Z_2
- Left node 1 branches: $1-\varepsilon_1$ B and $\emptyset, \varepsilon_1$
- Right node 1 branches: $1-\varepsilon_2$ B and $\emptyset, \varepsilon_2$
- Node 2 labeled Θ_1, node 2 labeled α_1, node 2 labeled Θ_2, node 2 labeled α_2
- Curves labeled X and Y
- Branches labeled 1, 0 for each node 2
- Terminal nodes: $(0,1)$, $(2,0)$, $(1,1)$, $(3,0)$, $(1,0)$, $(3,1)$, $(0,0)$, $(2,1)$
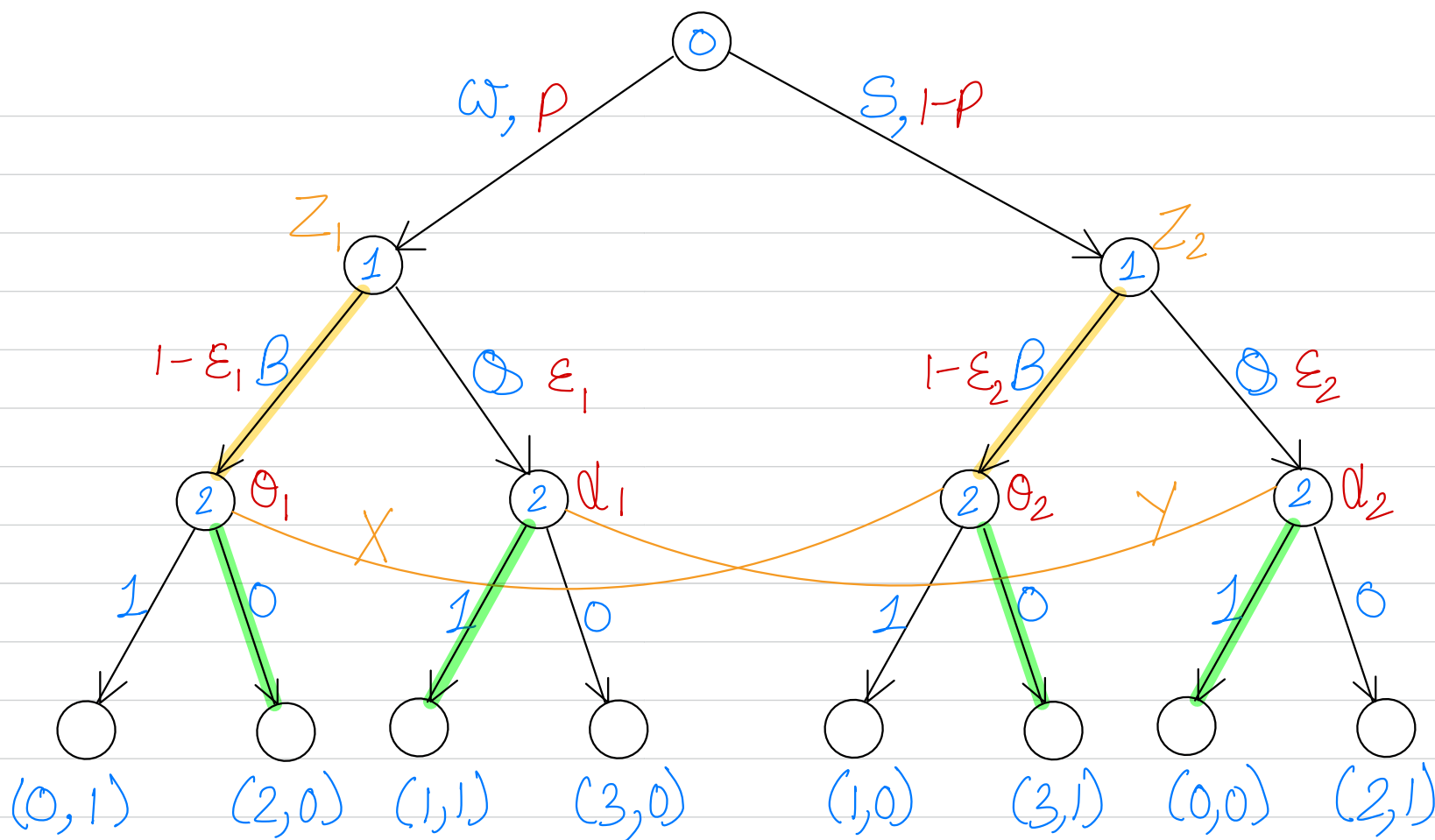
$\underline{\text{Case-3}}$

$S_X = 0 \; ; \; \Theta_1 \leq \Theta_2$

$S_Y = 1 \; ; \; \alpha_1 > \alpha_2$

$S_{Z_1} = B$

$S_{Z_2} = B$

$\alpha_1 = \dfrac{P \cdot \varepsilon_1}{P \cdot \varepsilon_1 + (1-P) \cdot \varepsilon_2} \in [0,1]$

$\alpha_2 \in [0,1]$ $\qquad \alpha_1 + \alpha_2 = 1$

$\Theta_1 = \dfrac{P \cdot 1}{P \cdot 1 + (1-P) \cdot 1} = P$

$\Theta_2 = 1 - P$

➤ IS $\Theta_1 \leq \Theta_2$? $\quad \begin{array}{c} P \leq 1-P \\ P \leq 0.5 \end{array}$

➤ Is $\alpha_1 > \alpha_2$? $\quad \alpha_1 > 0.5$

Following SE is possible if $P \leq 0.5$:

$(B, B, 0, 1, (P, 1-P), (\alpha_1, 1-\alpha_1))$

for all $\alpha_1 > 0.5$.

ω, P $S, 1-P$

Z_1 Z_2

B θ B θ

θ_1 X α_1 θ_2 Y α_2

1 0 1 0 1 0 1 0

$(0,1)$ $(2,0)$ $(1,1)$ $(3,0)$ $(1,0)$ $(3,1)$ $(0,0)$ $(2,1)$
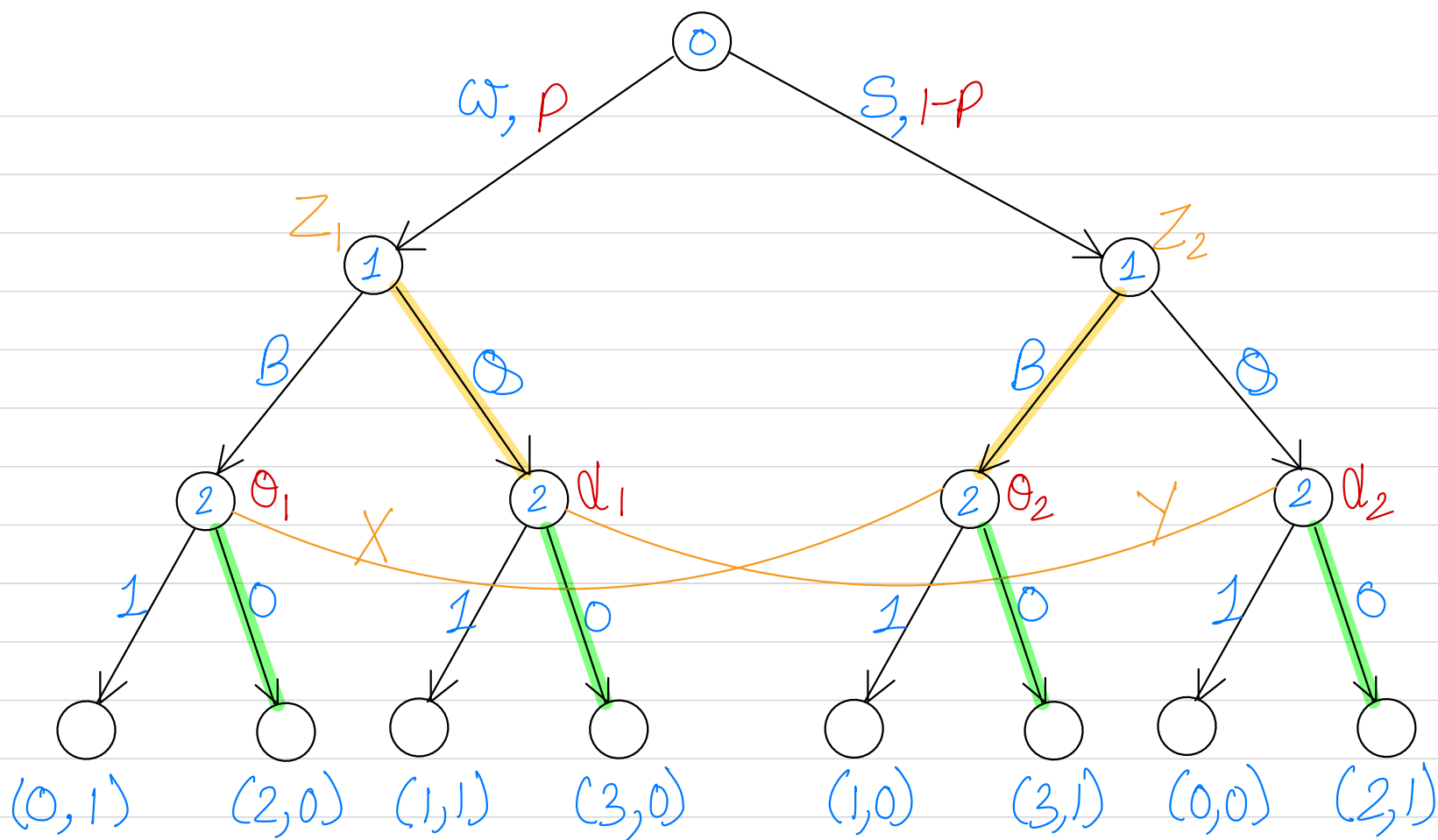
Case-4

$S_X = 0; \ \theta_1 \le \theta_2$

$S_Y = 0; \ \alpha_1 \le \alpha_2$

$S_{Z_1} = \theta$

$S_{Z_2} = B$

$\alpha_1 = \dfrac{P \cdot 1}{P \cdot 1 + (1-P) \cdot 0} = 1$

$\alpha_2 = 0$

$\theta_2 = \dfrac{(1-P) \cdot 1}{(1-P) \cdot 1 + P \cdot 0} = 1$

$\theta_1 = 0$

- IS $\theta_1 \le \theta_2$? YES.
- Is $\alpha_1 \le \alpha_2$? NO.

Following SE is NOT possible

$(\theta, B, 0, 0, (0,1), (1,0))$

Game Theory (CS4187)

Lectures 42

YouTube video:

https://youtu.be/xR3C82Iaibg

Computing Mixed Strategy Sequential Equilibrium
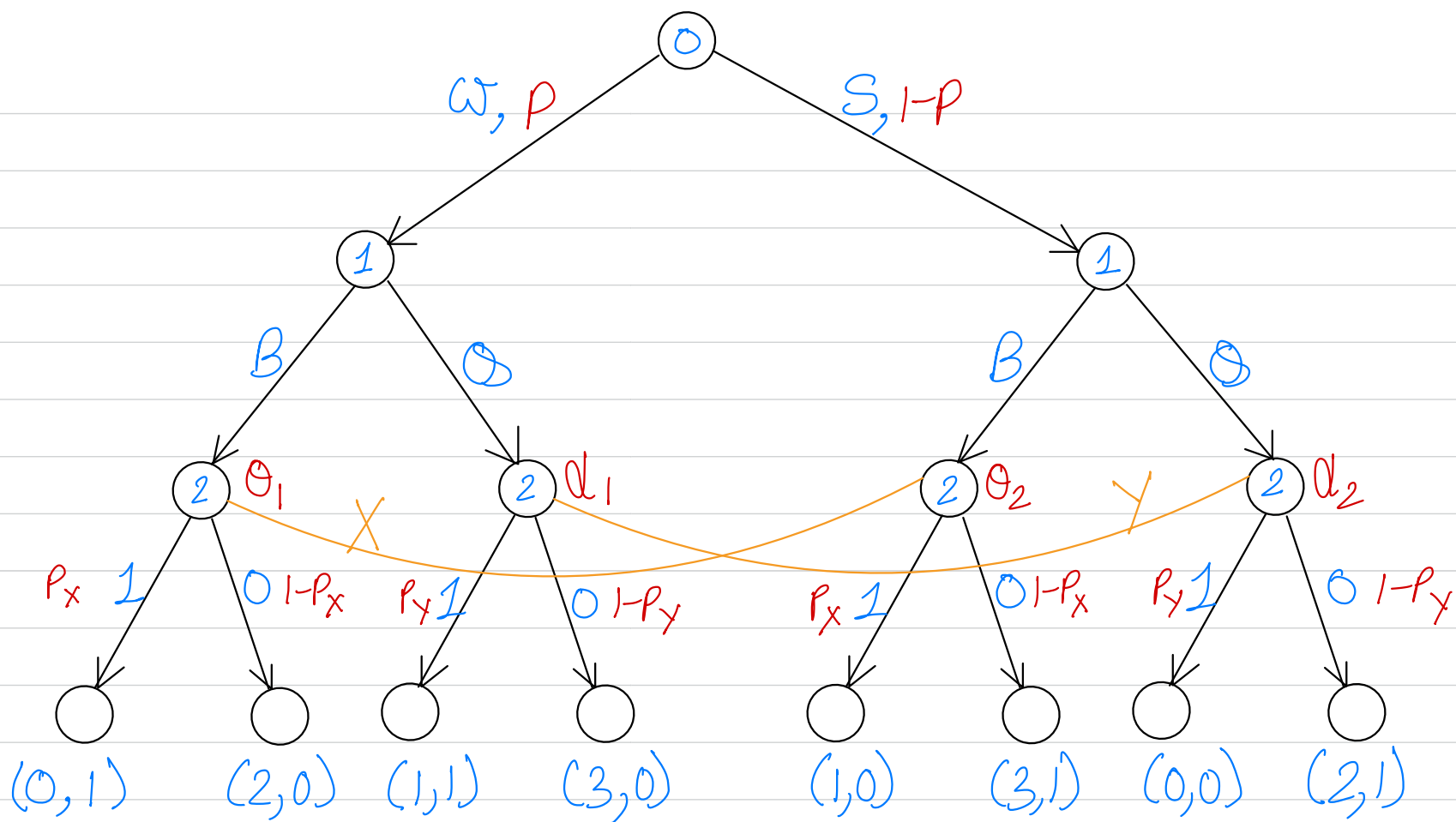
$$E\left[U_2^X\left((P_X(1), 1-P_X(0))\right)\right] = \theta_1 \cdot (P_X \cdot 1 + (1-P_X) \cdot 0) + \theta_2 \cdot (P_X \cdot 0 + (1-P_X) \cdot 1) = \theta_1 P_X + \theta_2(1-P_X)$$

P_X: Prob. that Player 2 will play 1 in information set X.

$$E\left[U_2^Y\left((P_Y(1), 1-P_Y(0))\right)\right] = \alpha_1 \cdot (P_Y \cdot 1 + (1-P_Y) \cdot 0) + \alpha_2 \cdot (P_Y \cdot 0 + (1-P_Y) \cdot 1) = \alpha_1 P_Y + \alpha_2(1-P_Y)$$

P_Y: Prob. that Player 2 will play 1 in information set Y.
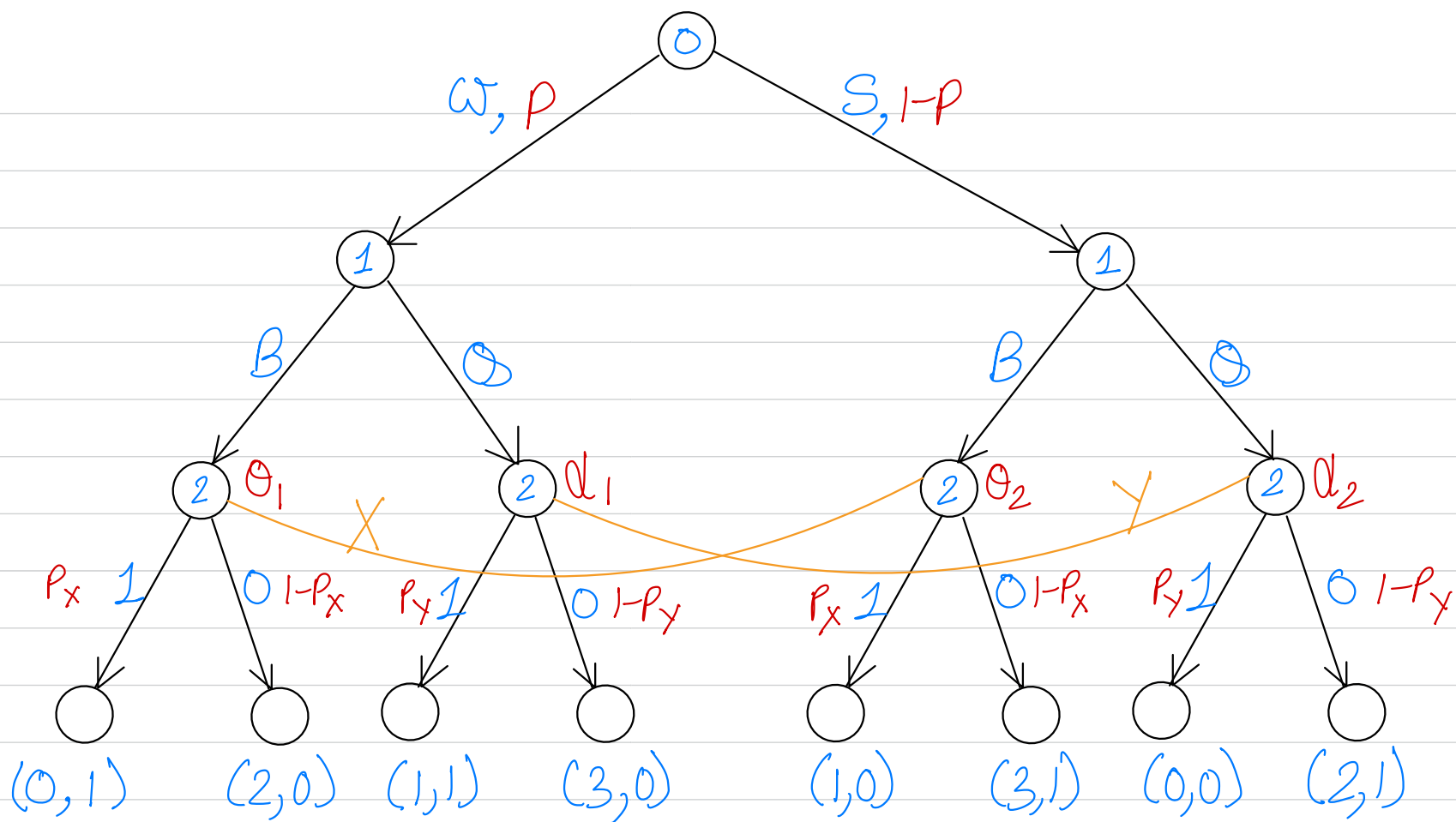
$$P_X^* = \arg\max_{P_X} E[U_2^X((P_X(1),\, 1-P_X(0)))]$$

$$= \arg\max_{P_X} \ \theta_1 P_X + \theta_2(1-P_X) = \begin{cases} 1 & ; \ \theta_1 > \theta_2 \\ 0 & ; \ \theta_1 < \theta_2 \\ \boxed{P_X} & ; \ \theta_1 = \theta_2 \end{cases} = \begin{cases} 1 & ; \ \theta_1 > 0.5 \\ 0 & ; \ \theta_1 < 0.5 \\ P_X & ; \ \theta_1 = 0.5 \end{cases}$$

Indifference: Same as def. 6 of MSNE.
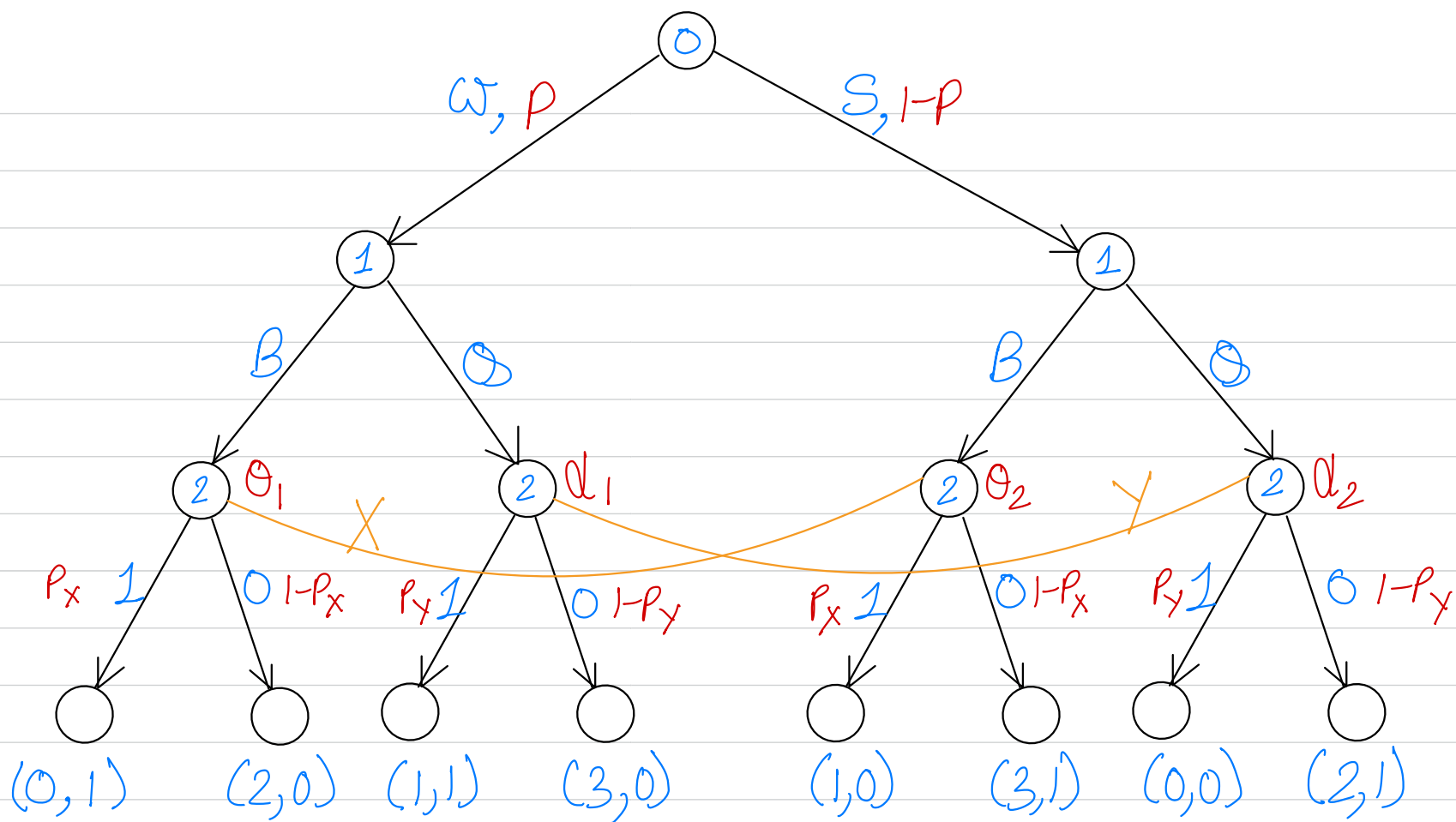
Game tree with root node 0. Left branch labeled ω, P; right branch labeled $S, 1-P$. Both lead to nodes labeled 1.

Left node 1: branches B (to node 2, θ_1) and $\emptyset$ (to node 2, α_1).
Right node 1: branches B (to node 2, θ_2) and $\emptyset$ (to node 2, α_2).

Node 2 (θ_1): P_x, 1 → $(0,1)$; 0, $1-P_x$ → $(2,0)$
Node 2 (α_1): P_y, 1 → $(1,1)$; 0, $1-P_y$ → $(3,0)$
Node 2 (θ_2): P_x, 1 → $(1,0)$; 0, $1-P_x$ → $(3,1)$
Node 2 (α_2): P_y, 1 → $(0,0)$; 0, $1-P_y$ → $(2,1)$

Information sets labeled x and y (orange curves).

$$P_y^* = \underset{P_y}{\arg\max}\ E[U_2^Y((P_y(1), 1-P_y(0)))]$$

$$= \underset{P_y}{\arg\max}\ \theta_1 P_y + \theta_2(1-P_y) = \begin{cases} 1 & ; \quad \alpha_1 > \alpha_2 \\ 0 & ; \quad \alpha_1 < \alpha_2 \\ \boxed{P_y} & ; \quad \alpha_1 = \alpha_2 \end{cases} = \begin{cases} 1 & ; \quad \alpha_1 > 0.5 \\ 0 & ; \quad \alpha_1 < 0.5 \\ P_y & ; \quad \alpha_1 = 0.5 \end{cases}$$

Indifference. Same as def. 6 of MSNE.

$$S_X = \begin{cases} 1 & ; \theta_1 > 0.5 \\ 0 & ; \theta_1 < 0.5 \\ (P_X(1), 1-P_X(0)) & ; \theta_1 = 0.5 \end{cases}$$
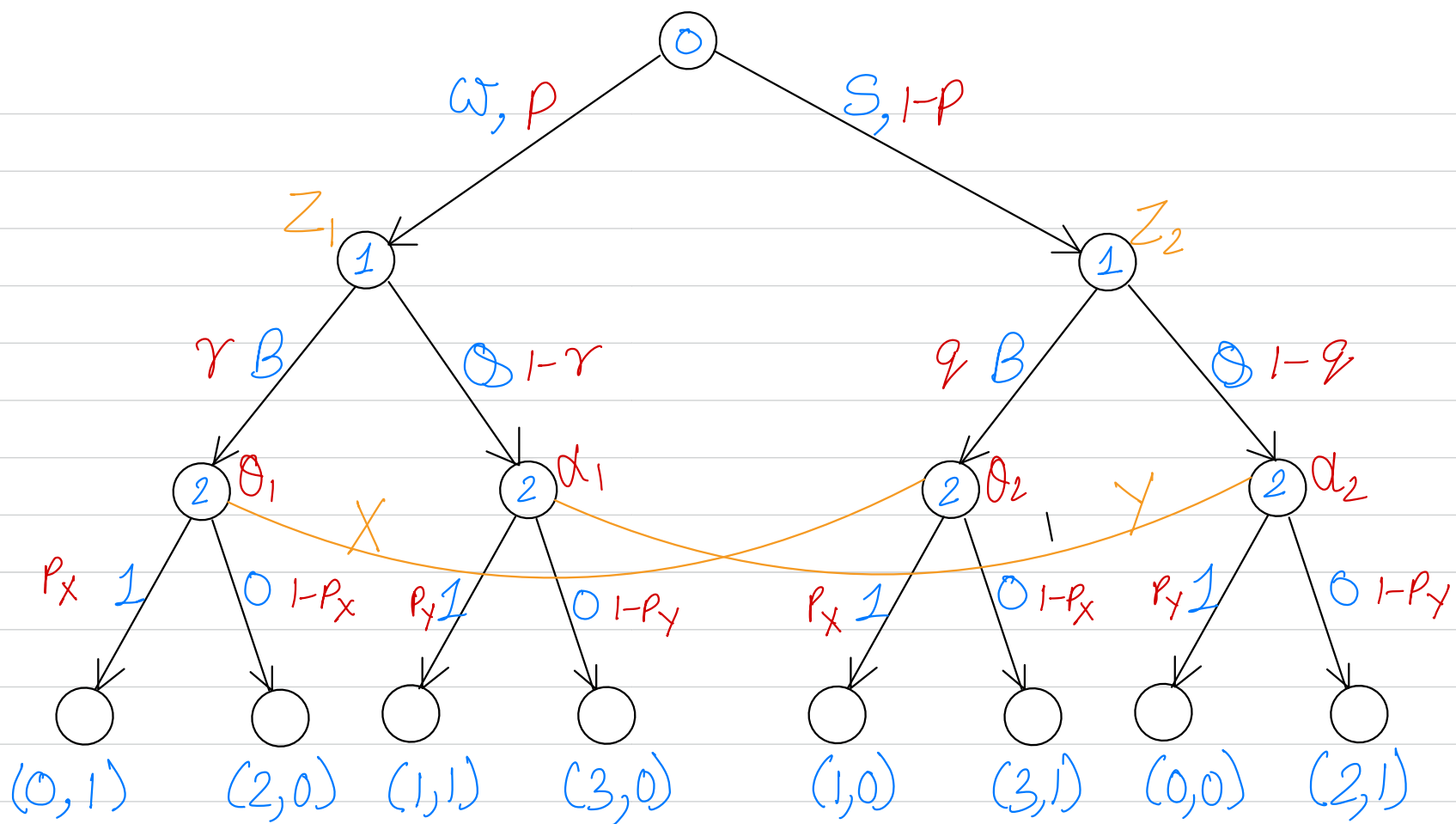
$$S_Y = \begin{cases} 1 & ; \alpha_1 > 0.5 \\ 0 & ; \alpha_1 < 0.5 \\ (P_Y(1), 1-P_Y(0)) & ; \alpha_1 = 0.5 \end{cases}$$

Tree diagram:

Root node 0, with two branches:
- Left: ω, P to node 1 (labeled Z_1)
- Right: $S, 1-P$ to node 1 (labeled Z_2)

From left node 1 (Z_1):
- Left: γ B to node 2 (θ_1)
- Right: θ $1-\gamma$ to node 2 (α_1)

From right node 1 (Z_2):
- Left: q B to node 2 (θ_2)
- Right: θ $1-q$ to node 2 (α_2)

From node 2 (θ_1):
- P_X 1 to $(0,1)$
- 0 $1-P_X$ to $(2,0)$

From node 2 (α_1):
- P_Y 1 to $(1,1)$
- 0 $1-P_Y$ to $(3,0)$

From node 2 (θ_2):
- P_X 1 to $(1,0)$
- 0 $1-P_X$ to $(3,1)$

From node 2 (α_2):
- P_Y 1 to $(0,0)$
- 0 $1-P_Y$ to $(2,1)$

Orange curves labeled X and Y.

$\underline{Case-5}$

$S_X = (P_X(1), 1-P_X(0))$

$\theta_1 = \theta_2 = 0.5$

$S_Y = (P_Y(1), 1-P_Y(0))$

$\alpha_1 = \alpha_2 = 0.5$

$E\left[U_1^{Z_1}\left((\gamma(B), 1-\gamma(\theta))\right)\right]$

$= \gamma \cdot P_X \cdot 0 + \gamma \cdot (1-P_X) \cdot 2$

$+ (1-\gamma) \cdot P_Y \cdot 1 + (1-\gamma) \cdot (1-P_Y) \cdot 3$

$= 2\gamma(1-P_X) + (1-\gamma)(3-2P_Y)$

$S_{Z_1} = \begin{cases} B & ; \ 2(1-P_X) > (3-2P_Y) \\ \theta & ; \ 2(1-P_X) < 3-2P_Y \\ (\gamma(B), 1-\gamma(\theta)) ; \ 2(1-P_X) = 3-2P_Y \end{cases}$

Tree edges and nodes:

- Root node: 0
  - Left edge: ω, P → node 1 labeled Z_1
  - Right edge: $S, 1-P$ → node 1 labeled Z_2

- Node Z_1 (left 1):
  - $\gamma\ B$ → node 2 labeled θ_1
  - $\emptyset\ 1-\gamma$ → node 2 labeled α_1

- Node Z_2 (right 1):
  - $q\ B$ → node 2 labeled θ_2
  - $\emptyset\ 1-q$ → node 2 labeled α_2

- Node θ_1:
  - $P_x\ 1$ → $(0,1)$
  - $0\ 1-P_x$ → $(2,0)$

- Node α_1:
  - $P_y\ 1$ → $(1,1)$
  - $0\ 1-P_y$ → $(3,0)$

- Node θ_2:
  - $P_x\ 1$ → $(1,0)$
  - $0\ 1-P_x$ → $(3,1)$

- Node α_2:
  - $P_y\ 1$ → $(0,0)$
  - $0\ 1-P_y$ → $(2,1)$

Information sets X and Y (orange connections).
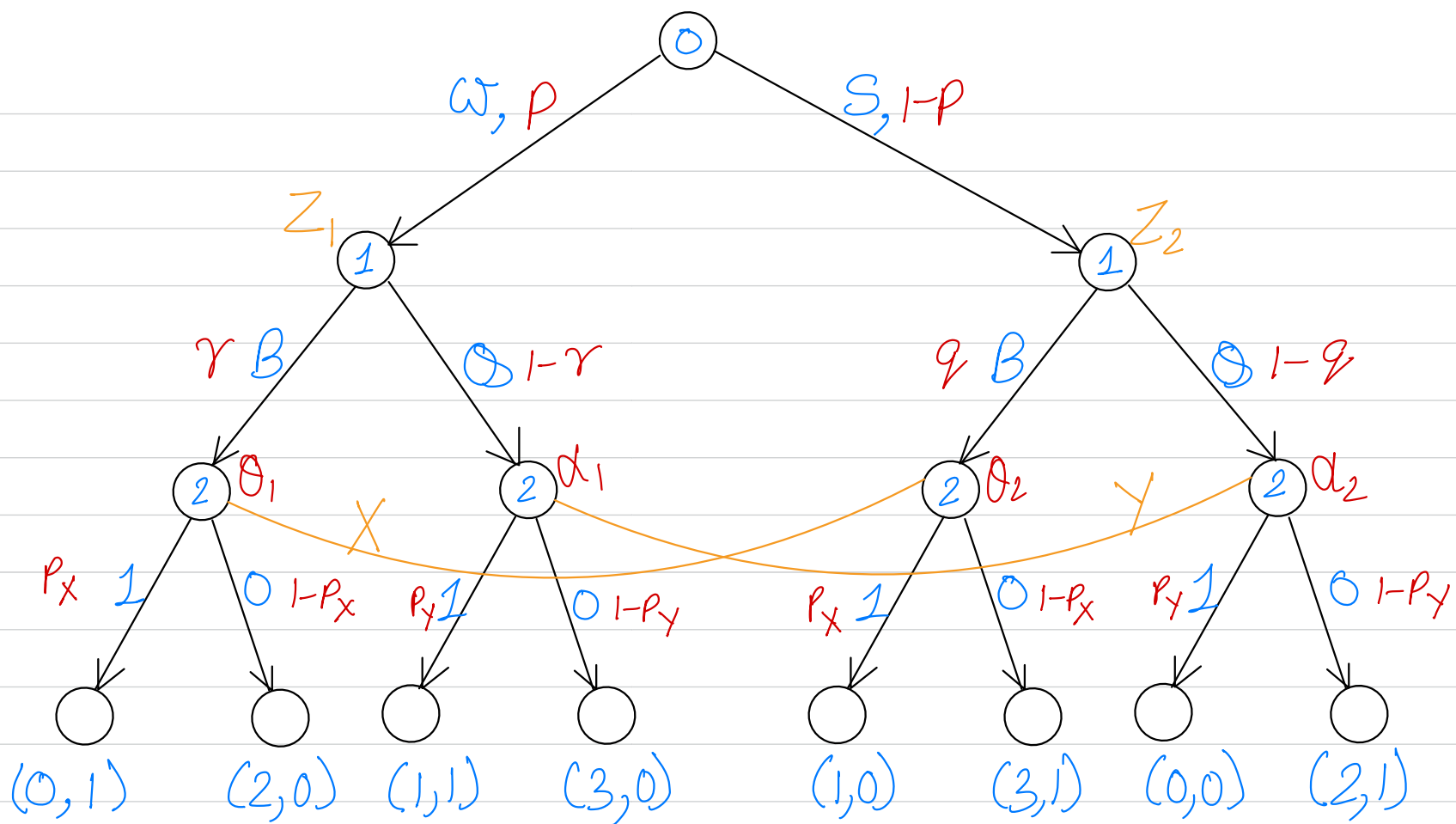
$\underline{Case-5}$

$S_X = (P_X(1), 1-P_X(0))$

$\theta_1 = \theta_2 = 0.5$

$S_Y = (P_Y(1), 1-P_Y(0))$

$\alpha_1 = \alpha_2 = 0.5$

$E\left[U_1^{Z_2}\left((q(B), 1-q(\emptyset)) \right) \right]$

$= q \cdot P_X \cdot 1 + q \cdot (1-P_X) \cdot 3$

$\quad + (1-q) \cdot P_Y \cdot 0 + (1-q) \cdot (1-P_Y) \cdot 2$

$= q \cdot (3-2P_X) + 2 \cdot (1-q) \cdot (1-P_Y)$

$S_{Z_2} = \begin{cases} B & ; (3-2P_X) > 2(1-P_Y) \\ \emptyset & ; (3-2P_X) < 2(1-P_Y) \\ (q(B), 1-q(\emptyset)) & ; (3-2P_X) = 2(1-P_Y) \end{cases}$

Tree diagram with root node labeled 0.

Edges from root: left ω, P to node 1 (labeled Z_1); right $S, 1-P$ to node 1 (labeled Z_2).

From left node 1: left γ B to node 2 (θ_1); right θ $1-\gamma$ to node 2 (d_1).

From right node 1: left q B to node 2 (θ_2); right θ $1-q$ to node 2 (d_2).

Orange arcs labeled X and Y connecting the level-2 nodes.

Leaf edges:
- P_X 1 and 0 $1-P_X$ from node θ_1
- P_Y 1 and 0 $1-P_Y$ from node d_1
- P_X 1 and 0 $1-P_X$ from node θ_2
- P_Y 1 and 0 $1-P_Y$ from node d_2

Leaves (left to right): $(0,1)$ $(2,0)$ $(1,1)$ $(3,0)$ $(1,0)$ $(3,1)$ $(0,0)$ $(2,1)$

$\underline{\text{Case-5}}$

$S_X = (P_X(1), 1-P_X(0))$

$\theta_1 = \theta_2 = 0.5$

$S_Y = (P_Y(1), 1-P_Y(0))$

$d_1 = d_2 = 0.5$

$\theta_1 = \dfrac{P\gamma}{P\gamma + (1-P)q} = 0.5 \qquad \Rightarrow \quad P\gamma = (1-P)q \qquad \text{—} \;①$

$d_1 = \dfrac{P(1-\gamma)}{P(1-\gamma) + (1-P)(1-q)} = 0.5 \qquad \Rightarrow \quad P(1-\gamma) = (1-P)(1-q) \qquad \text{—} \;②$

$\text{\textcolor{red}{Shortcut!}}$

Substitute $\gamma = \dfrac{1-P}{P}q$ from ① in ②

we get, $\boxed{P = 0.5}$

$$S_X = \begin{cases} 1 & ; \; \theta_1 > 0.5 \\ 0 & ; \; \theta_1 < 0.5 \\ (P_X(1), 1-P_X(0)) & ; \; \theta_1 = 0.5 \end{cases}$$

$$S_Y = \begin{cases} 1 & ; \; \alpha_1 > 0.5 \\ 0 & ; \; \alpha_1 < 0.5 \\ (P_Y(1), 1-P_Y(0)) & ; \; \alpha_1 = 0.5 \end{cases}$$

Tree diagram (top portion):

- Root node labeled O with two branches:
  - Left branch: ω, P (P in red)
  - Right branch: $S, 1-P$
- Left node labeled 1, marked Z_1:
  - Left branch: γ (red), B (blue)
  - Right branch: $\emptyset$, $1-\gamma$
- Right node labeled 1, marked Z_2:
  - Left branch: q (red), B (blue)
  - Right branch: $\emptyset$, $1-q$
- Nodes labeled 2:
  - First 2 marked θ_1 (red):
    - Left: P_X (red), 1 (blue)
    - Right: 0 (blue), $1-P_X$ (red)
  - Second 2 marked α_1 (red):
    - Left: 1 (red), 1 (blue)
    - Right: 0 (blue), 0 (red)
  - Third 2 marked θ_2 (red):
    - Left: P_X (red), 1 (blue)
    - Right: 0 (blue), $1-P_X$ (red)
  - Fourth 2 marked α_2 (red):
    - Left: 1 (red), 1 (blue)
    - Right: 0 (blue), 0 (red)
- Orange curves labeled X and Y.
- Leaf payoffs (blue): $(0,1)\quad(2,0)\quad(1,1)\quad(3,0)\quad(1,0)\quad(3,1)\quad(0,0)\quad(2,1)$

$\underline{Case\text{-}6}$

$S_X = (P_X(1), 1-P_X(0))$

$\theta_1 = \theta_2 = 0.5$

$S_Y = (1\,(1), 0\,(0))$

$\quad = 1$

$\alpha_1 > 0.5$

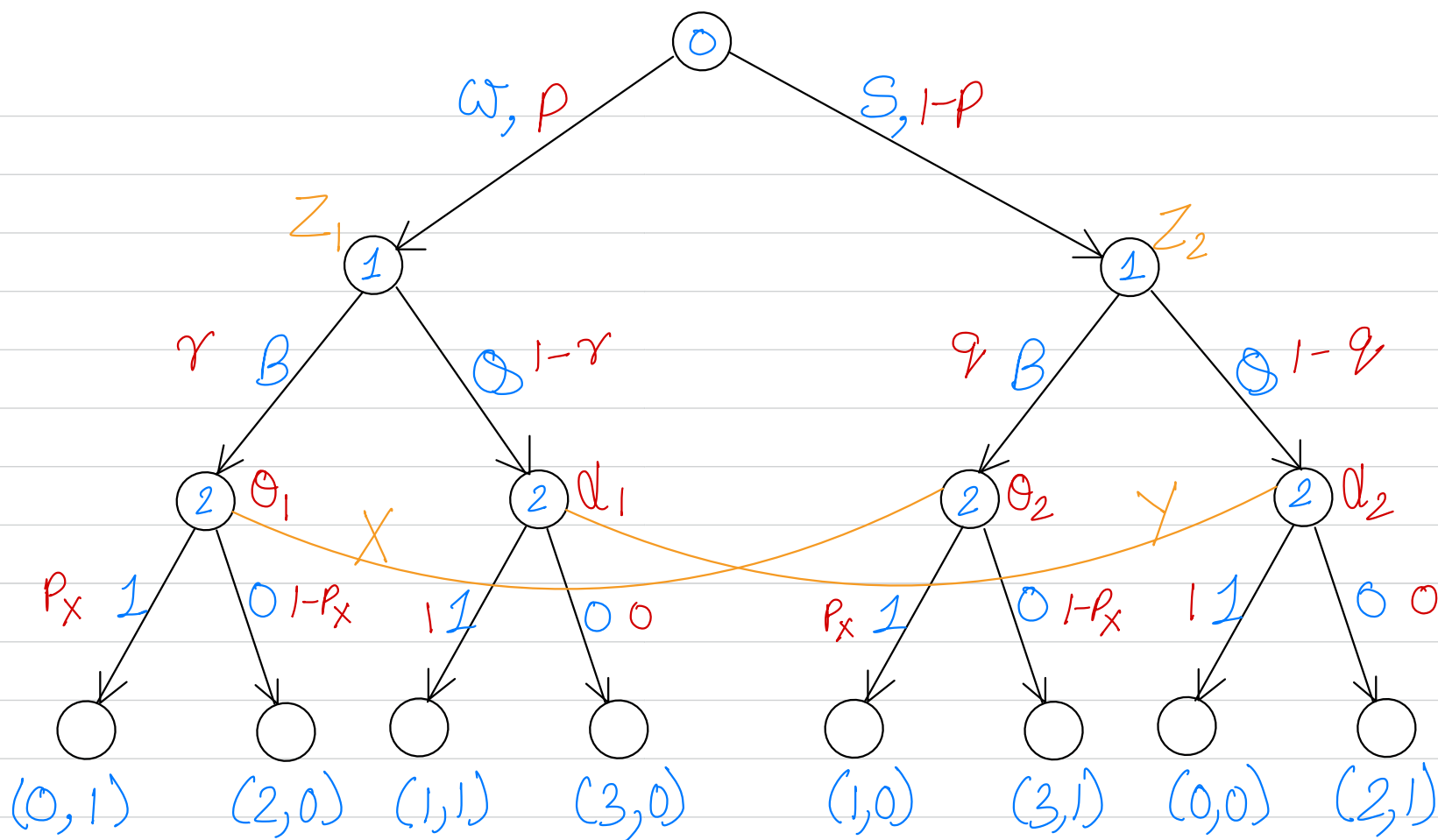$E\left[U_1^{Z_1}\left((\gamma(B), 1-\gamma(\emptyset))\right)\right]$

$= \gamma \cdot P_X \cdot 0 + \gamma \cdot (1-P_X) \cdot 2$

$\quad + (1-\gamma) \cdot 1 \cdot 1 + (1-\gamma) \cdot 0 \cdot 3$

$= 2\gamma(1-P_X) + (1-\gamma) \cdot 1$

$$S_{Z_1} = \begin{cases} B & ; \; 2(1-P_X) > 1 \\ \emptyset & ; \; 2(1-P_X) < 1 \\ (\gamma(B), 1-\gamma(\emptyset)) & ; \; 2(1-P_X) = 1 \end{cases}$$

The tree diagram contains the following labels:

Root node: 0

- Left branch: ω, P to node 1 labeled Z_1
- Right branch: $S, 1-P$ to node 1 labeled Z_2

From node 1 (Z_1):
- γ, B (left) to node 2 labeled θ_1
- θ, $1-\gamma$ (right) to node 2 labeled α_1

From node 1 (Z_2):
- q, B (left) to node 2 labeled θ_2
- θ, $1-q$ (right) to node 2 labeled α_2

From node 2 (θ_1):
- P_X, 1 (left) to leaf $(0,1)$
- 0, $1-P_X$ (right) to leaf $(2,0)$

From node 2 (α_1):
- 1, 1 (left) to leaf $(1,1)$
- 0, 0 (right) to leaf $(3,0)$

From node 2 (θ_2):
- P_X, 1 (left) to leaf $(1,0)$
- 0, $1-P_X$ (right) to leaf $(3,1)$

From node 2 (α_2):
- 1, 1 (left) to leaf $(0,0)$
- 0, 0 (right) to leaf $(2,1)$

Orange curves labeled X and Y connect the level-2 nodes.

$\underline{Case-6}$

$S_X = (P_X(1), 1-P_X(0))$

$\theta_1 = \theta_2 = 0.5$

$S_Y = (1(1), 0(0))$

$= 1$

$\alpha_1 > 0.5$

$E\left[U_1^{Z_2}\big((q(B), 1-q(\theta))\big)\right]$
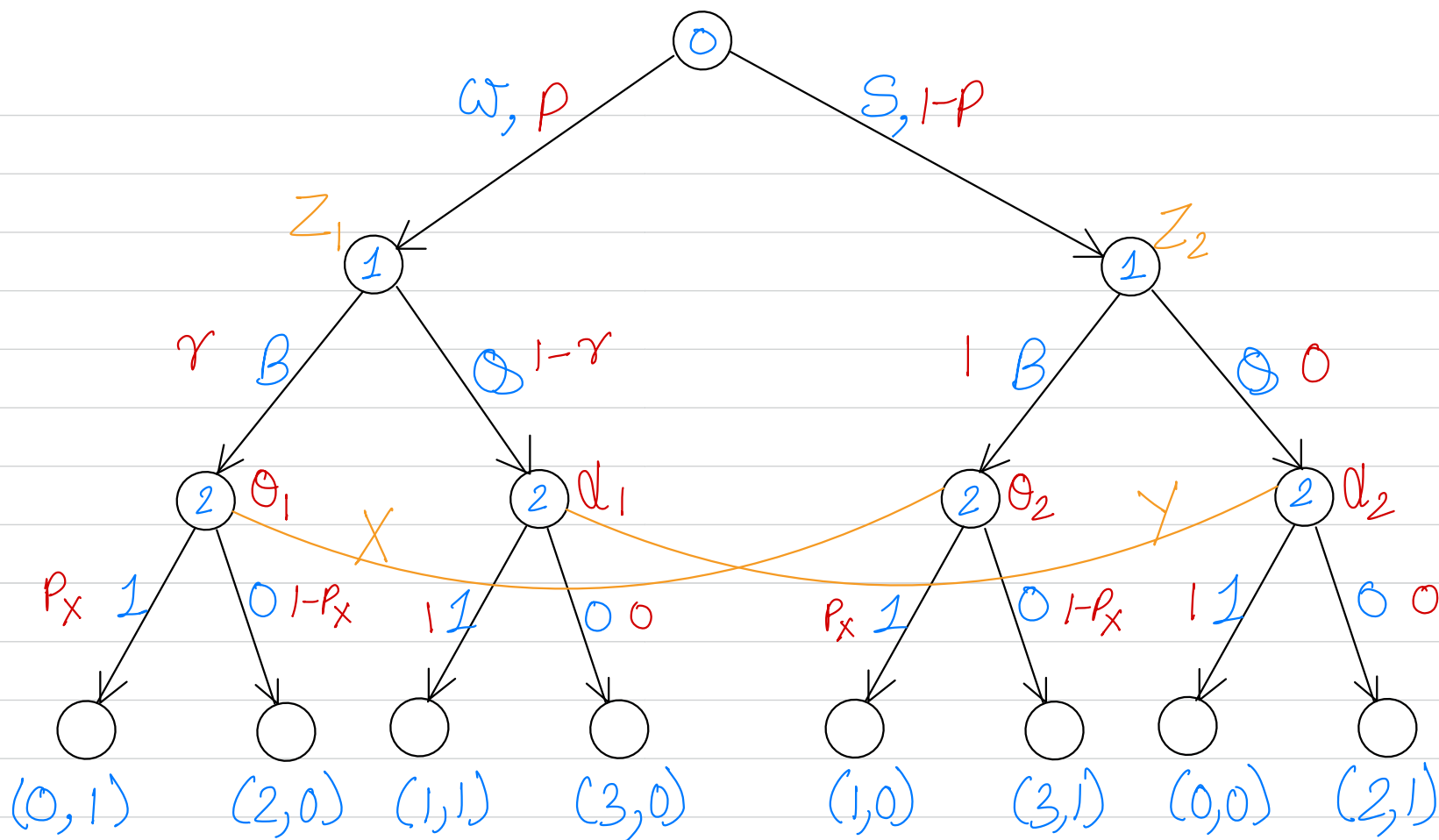
$= q \cdot P_X \cdot 1 + q \cdot (1-P_X) \cdot 3$

$\quad + (1-q) \cdot 1 \cdot 0 + (1-q) \cdot 0 \cdot 2$

$= 3q$

$S_{Z_2} = B$

Case-6

$S_X = (P_X(1), 1-P_X(0))$

$\theta_1 = \theta_2 = 0.5$

$S_Y = (1(1), 0(0))$

$= 1$

$d_1 > 0.5$

$$S_{Z_1} = \begin{cases} B & ; \ 2(1-P_X) > 1 \\ \emptyset & ; \ 2(1-P_X) < 1 \\ (\gamma(B), 1-\gamma(\emptyset)); \ 2(1-P_X) = 1 \end{cases}$$
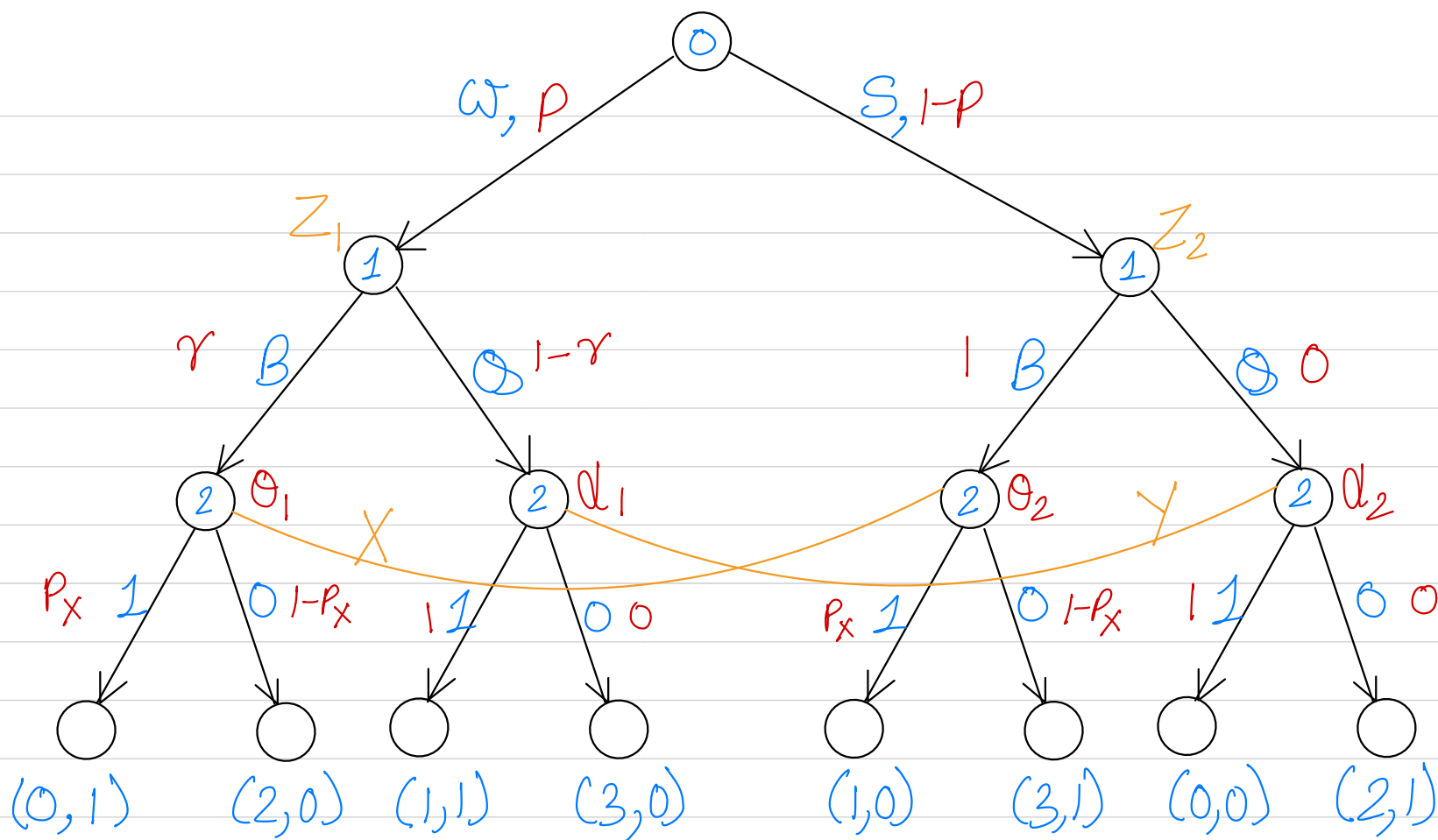
Case-A $(S_{Z_1} = B) \Rightarrow \gamma = 1$
$P_X < 0.5$

$\theta_1 = \dfrac{P \cdot 1}{P \cdot 1 + (1-P) \cdot 1} = P$

$d_1 = \dfrac{P \cdot \varepsilon_1}{P \cdot \varepsilon_1 + (1-P) \cdot \varepsilon_2} \in [0,1]$

Is $\theta_1 = 0.5$? If $P = 0.5$

Is $d_1 > 0.5$? Yes. Possible.

Tree diagram:

Root: 0
- Left edge: ω, P → node 1 (Z_1)
- Right edge: $S, 1-P$ → node 1 (Z_2)

From Z_1:
- γ B → node 2 (θ_1)
- $1-\gamma$ $\emptyset$ → node 2 (α_1)

From Z_2:
- 1 B → node 2 (θ_2)
- $\emptyset$ O → node 2 (α_2)

From θ_1: P_X 1 → $(0,1)$; 0 $1-P_X$ → $(2,0)$
From α_1: 1 1 → $(1,1)$; 0 0 → $(3,0)$
From θ_2: P_X 1 → $(1,0)$; 0 $1-P_X$ → $(3,1)$
From α_2: 1 1 → $(0,0)$; 0 0 → $(2,1)$

(X connects θ_1 and α_1; Y connects θ_2 and α_2)

$\underline{\text{Case-}6}$

$S_X = (P_X(1), 1-P_X(0))$

$\theta_1 = \theta_2 = 0.5$

$S_Y = (1\,(1), 0\,(0))$
$\quad = 1$

$\alpha_1 > 0.5$

$$S_{Z_1} = \begin{cases} B & ; \ 2(1-P_X) > 1 \\ \emptyset & ; \ 2(1-P_X) < 1 \\ (\gamma(B), 1-\gamma(\emptyset)) & ; \ 2(1-P_X) = 1 \end{cases}$$
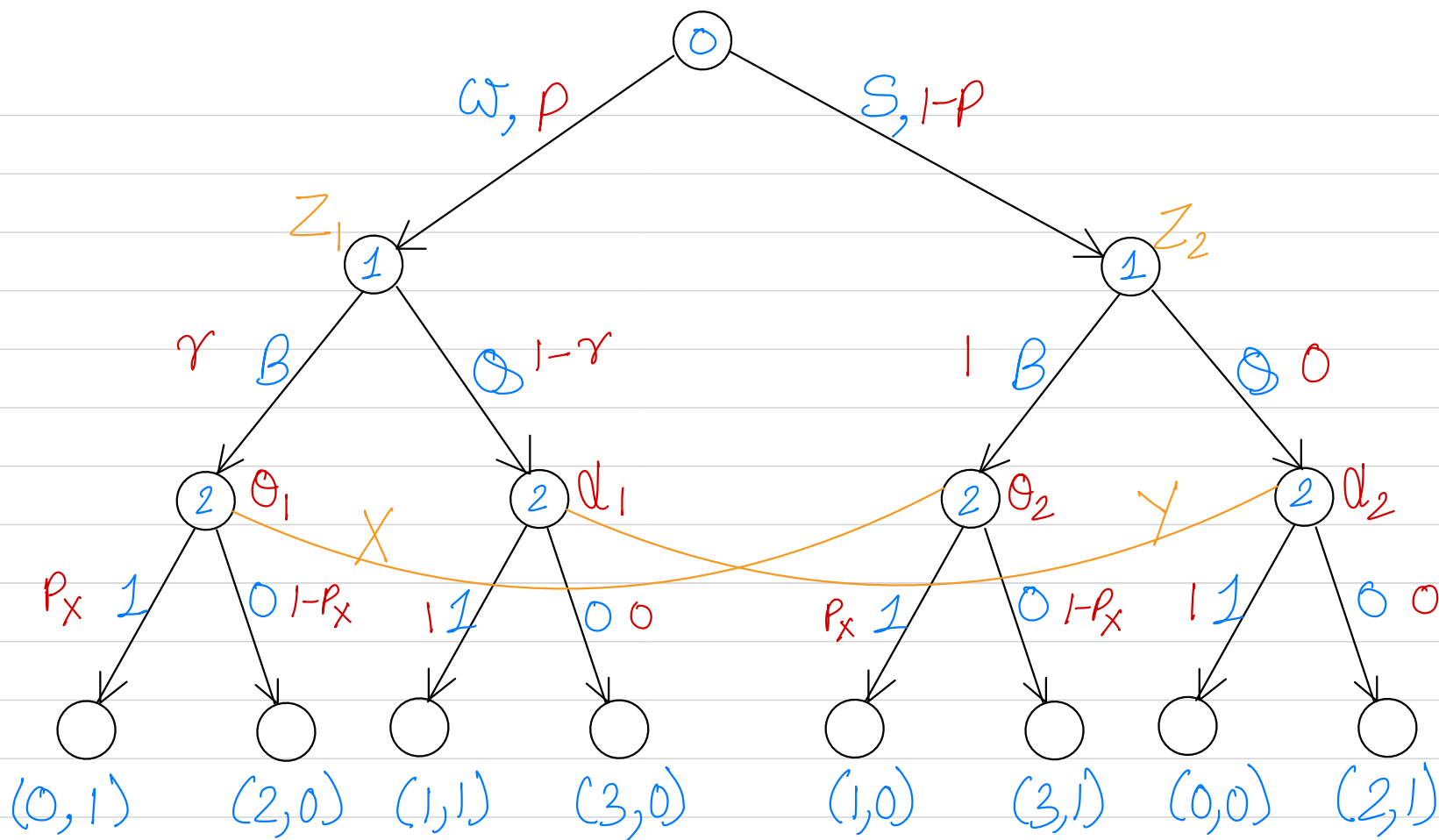

$\underline{\text{Case-}B\ (S_{Z_1} = \emptyset)} \Rightarrow \gamma = 0$

$P_X > 0.5$

$\theta_1 = \dfrac{P \cdot 0}{P \cdot 0 + (1-P) \cdot 1} = 0$

$\alpha_1 = \dfrac{P \cdot 1}{P \cdot 1 + (1-P) \cdot 0} = 1$

IS $\theta_1 = 0.5$? NO!

IS $\alpha_1 > 0.5$? Yes.

Case-6

$S_X = (P_X(1), 1-P_X(0))$

$\theta_1 = \theta_2 = 0.5$

$S_Y = (1(1), 0(0))$

$= 1$

$\alpha_1 > 0.5$

$$S_{Z_1} = \begin{cases} B & ; \ 2(1-P_X) > 1 \\ \oslash & ; \ 2(1-P_X) < 1 \\ (\gamma(B), 1-\gamma(\oslash)); \ 2(1-P_X) = 1 \end{cases}$$

Case-C $(S_{Z_1} = \oslash)$ $P_X = 0.5$
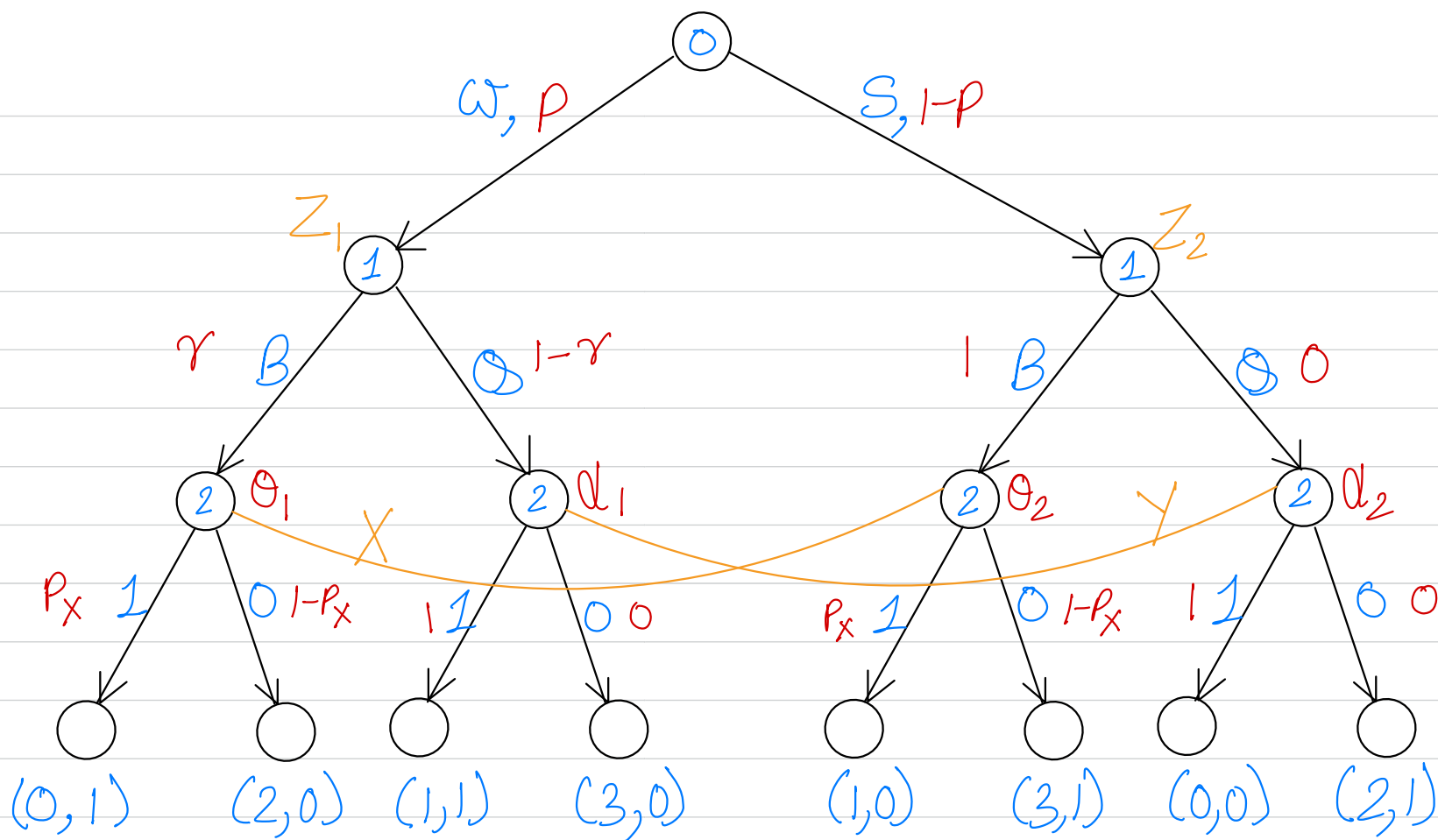
$\theta_1 = \dfrac{P \cdot \gamma}{P \cdot \gamma + (1-P) \cdot 1}$

$\alpha_1 = \dfrac{P \cdot (1-\gamma)}{P \cdot (1-\gamma) + (1-P) \cdot 0} = 1$

IS $\theta_1 = 0.5$? Set to 0.5. $\gamma = \dfrac{1-P}{P}$

IS $\alpha_1 > 0.5$? Yes.

Case-6

$S_X = (P_X(1), 1-P_X(0))$

$\theta_1 = \theta_2 = 0.5$

$S_Y = (1(1), 0(0)) = 1$

$\alpha_1 > 0.5$

Following SE is possible for $P \geqslant 0.5$:

$$\left(\left(\frac{1-P}{P}(B), \frac{2P-1}{P}(\theta)\right), B, (0.5(1), 0.5(0)), 1, (1), (1), (0.5, 0.5), (1,0)\right)$$

$\longleftarrow S \longrightarrow$ $\longleftarrow P \longrightarrow$

Thank You!