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Game Theory (CS4187)
Lecture 1

Date: 08/08/2024

Instructor: Gourav Saha

NOT THE FINAL VERSION!

Contents of this lecture

1. **What is a game theory?**

2. Let's play some games!

3. Need for rigor when analyzing rational players.

    - Braess's Paradox

What is game theory?

- In the course "Optimizing techniques for AI", you have learned how to **formulate** and **solve** optimization problems:

$$\max_{X} \quad f(X)$$

$$\text{subject to:}$$

$$g(X) \leq \theta$$

where $X \in \mathbb{R}^N$ and $\theta \in \mathbb{R}^M$ (N decision variables and M constraints).

What is game theory?

Let's consider two setups:

Setup 1: Consider an "optimization setup" with two scalar decision variables x_1 and x_2, where x_1 and x_2 can be adjusted by a single individual.

$$\max_{x_1, x_2} \quad f(x_1, x_2)$$

Setup 2: Consider an "optimization setup" with two scalar decision variables x_1 and x_2, where x_1 and x_2 are adjusted by two different individuals each having their own objective function.

$$\max_{x_1} \quad f_1(x_1, x_2) \qquad\qquad \max_{x_2} \quad f_2(x_1, x_2)$$

Individual 1 **Individual 2**

What is game theory?

Setup 2: Consider an “optimization setup” with two scalar decision variables x_1 and x_2, where x_1 and x_2 are adjusted by two different individuals each having their own objective function.

$$\max_{x_1} \quad f_1\left(x_1, x_2\right) \qquad\qquad \max_{x_2} \quad f_2\left(x_1, x_2\right)$$

Individual 1 **Individual 2**

➢ In setup 2, the decision variable of individual 1, i.e. x_1, effects the objective function of individual 2 and vice versa.

What is game theory?

<u>EXAMPLE (Relative order preserving marks update)</u>:

1. Towards the end of a course the marks of the students are low.

2. Suppose the instructor decided to give an **additional assignment** in the end of the course to increase the marks of the students so that they get better grades.

3. The instructor also decided that since the assignment was not planned from the beginning the marks increase due to this additional assignment **should not change the relative ordering** of the student based on their original marks.

4. Hence, it was decided that everyone's **marks will be increased by the same amount**. One possible way to do so is to **calculate the average score** of all the students **for this additional assignment** and **add it to their original marks**.

What is game theory?

<u>EXAMPLE (Relative order preserving marks update)</u>:

- This game-theoretic setup because the marks of one student will effect the marks of the other student (because of the average). Student are the individuals.

- For this setup, the objective function of the $\boldsymbol{i^{th}}$ **student** is

$$f_i\left(x_1, x_2, \ldots, x_N\right) = s_i + \frac{1}{N}\left(\sum_{n=1}^{N} x_n\right) - \rho\left(x_i\right)$$

where s_i is the original marks of the i^{th} student, x_i is the marks of the i^{th} student in the additional assignment, and $\rho(x_i)$ is a monotonic increasing function of x_i that captures the effort put by the i^{th} student for the additional assignment. The second term of the equation is the average marks of the students for the additional assignment.

What is game theory?

Definition: Game theory is the study of decision making in a multi-player setup where:

1. The players are rational (“rational” means that the player will try to maximize their objective).

2. Each player has their own objective.

3. Decision of one players effects the objective of other players and vice-versa.

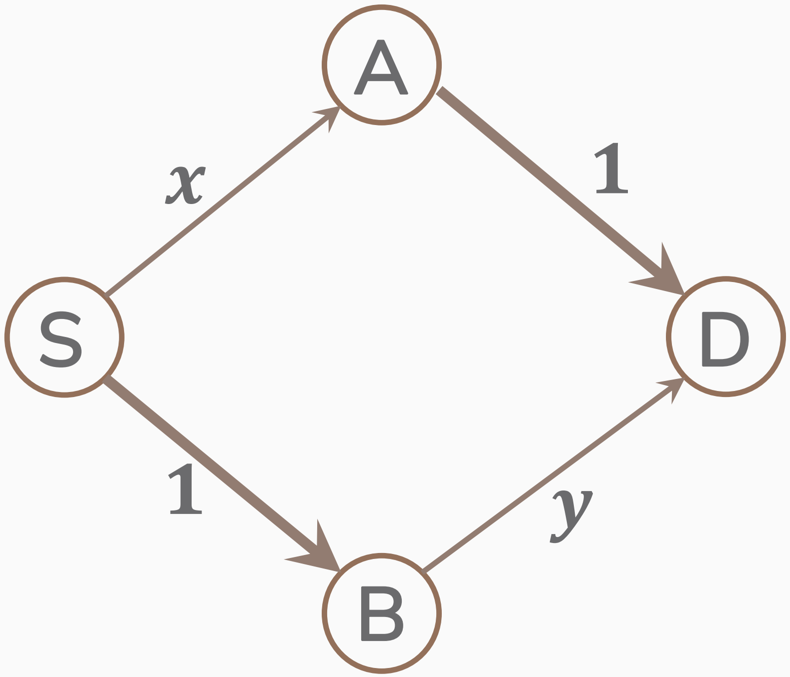
Contents of this lecture

1. What is a game theory?

2. **Let’s play some games!**

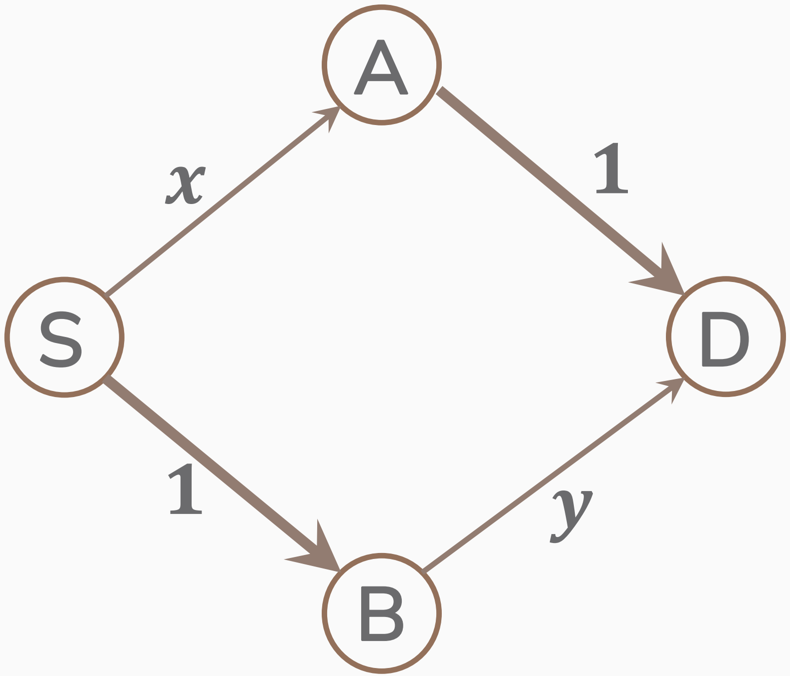
3. Need for rigor when analyzing rational players.
    - Braess’s Paradox

Game 1: Select the road (Version 1)



- The figure shows a road network.
- All of you have to drive from source, S, to destination, D. There are two paths, one via A and the other via B. There are two options:
  - Option 1: SA -> AD
  - Option 2: SB -> BD
- The edge weights represents the time (in hours) taken for the corresponding road segment. x and y are the fraction of the total vehicles that choose options 1 and 2 respectively.

Game 1: Select the road (Version 1)



Option 1: SA -> AD

Option 2: SB -> BD

Choose between options 1 and 2 to minimize our total travel time. While making your decision you can assume:

- Drivers knows the road network.
- Drivers knows the travel time in each road as a function of the fraction of users.

Game 1: Select the road (Version 1)

https://forms.gle/G7eGDXkaHkFeuHN68

Option 1: SA -> AD

Option 2: SB -> BD

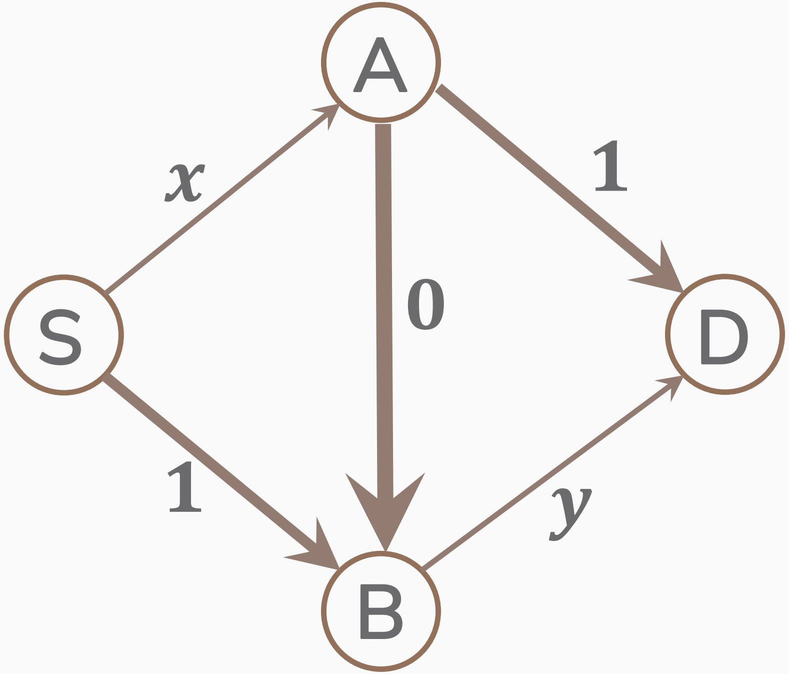
Choose between options 1 and 2 to minimize our total travel time. While making your decision you can assume:

- Drivers knows the road network.
- Drivers knows the travel time in each road as a function of the fraction of users.

Game 1: Select the road (Version 1)

Round	# Option 1	# Option 2	Time Option 1	Time Option 2
1	5	6	1.45	1.54
2				
3				
4				
5				

Game 2: Select the road (Version 2)



- Everything same as version 1 except that Elon Musk installed a Hyperloop tunnel between A to B which has **zero travel time** not matter the amount of traffic.

Option 1: SA -> AD

Option 2: SB -> BD

Option 3: SA -> AB -> BD

Choose between options 1, 2, and 3 to minimize our total travel time. You can make the same assumptions as version 1.

Game 2: Select the road (Version 2)

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- Everything same as version 1 except that Elon Musk installed a Hyperloop tunnel between A to B which has **zero travel time** not matter the amount of traffic.

Option 1: SA -> AD

Option 2: SB -> BD

Option 3: SA -> AB -> BD

Choose between options 1, 2, and 3 to minimize our total travel time. You can make the same assumptions as version 1.

Game 2: Select the road (Version 2)

Round	# Option 1	# Option 2	# Option 3	Time Option 1	Time Option 2	Time Option 3
1	6	1	4	1.54	1.45	1
2	5	2	4	1.45	1.54	1
3	3	1	7	1.27	1.72	1
4						
5						

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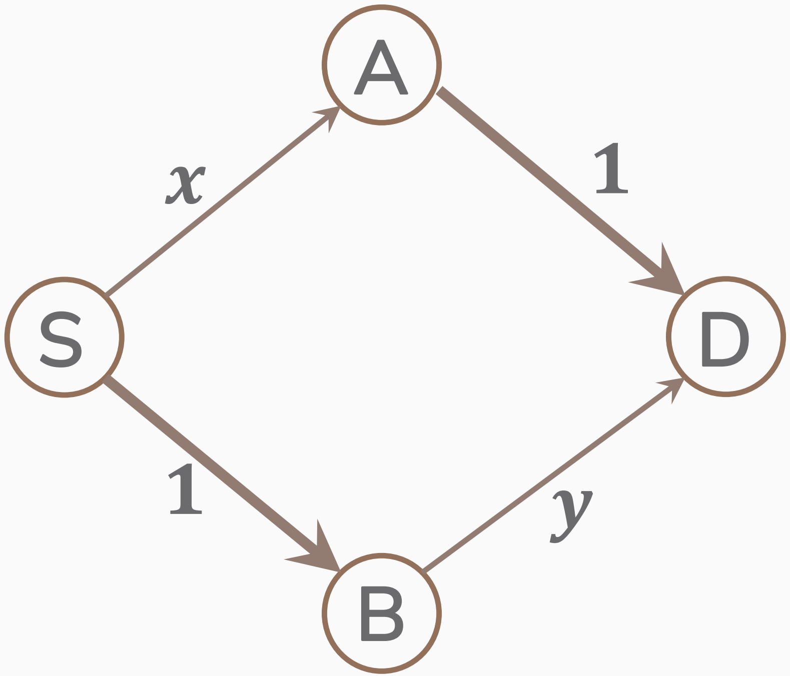
3. **Need for rigor when analyzing rational players.**
   - **Braess’s Paradox**

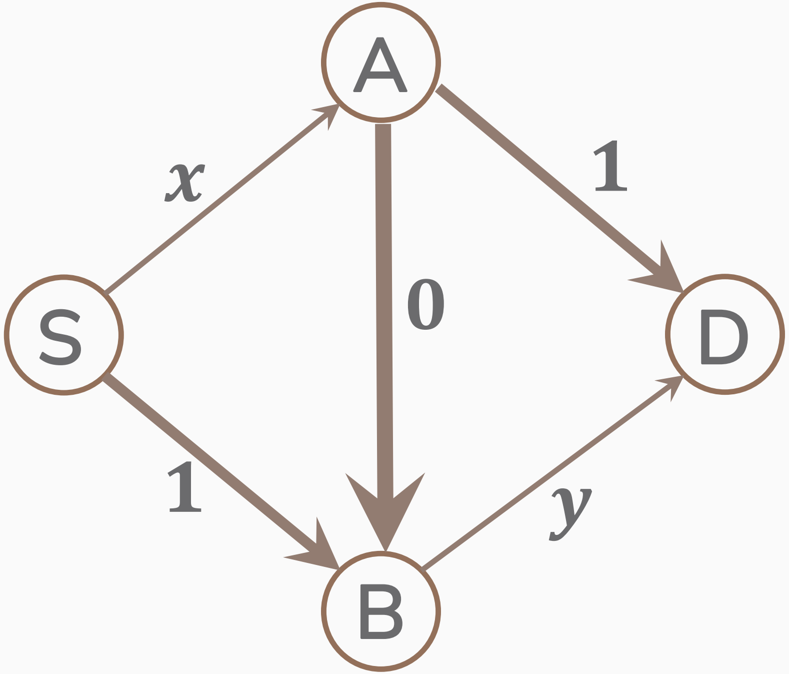
Need of rigor in game theory



We will not be rigorous today because we don't have all the necessary skillsets yet! The example is just suppose to be a wake-up call.

- Let's analyze Game 1.
- What is the minimum travel time of a driver?
  - Both options are symmetric in terms of travel time.
  - Hence, it is only logical that there will be a 50-50 split of traffic, i.e. $x = y = 0.5$.
  - So the minimum travel time of a driver is 1.5.

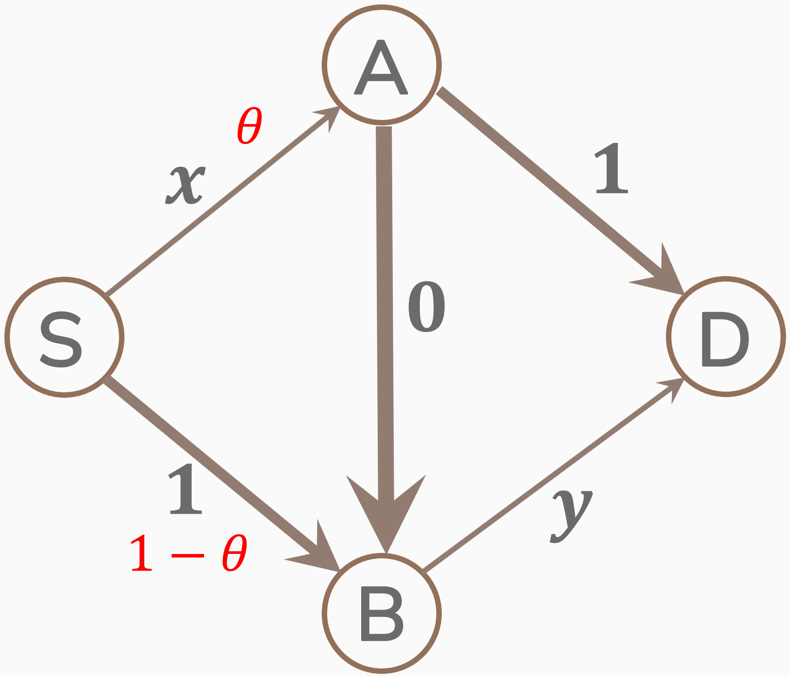
Need of rigor in game theory



- Let's analyze Game 2.
- What is the minimum travel time of a driver?

Only logical analysis for the time being.
Nothing rigorous for now.

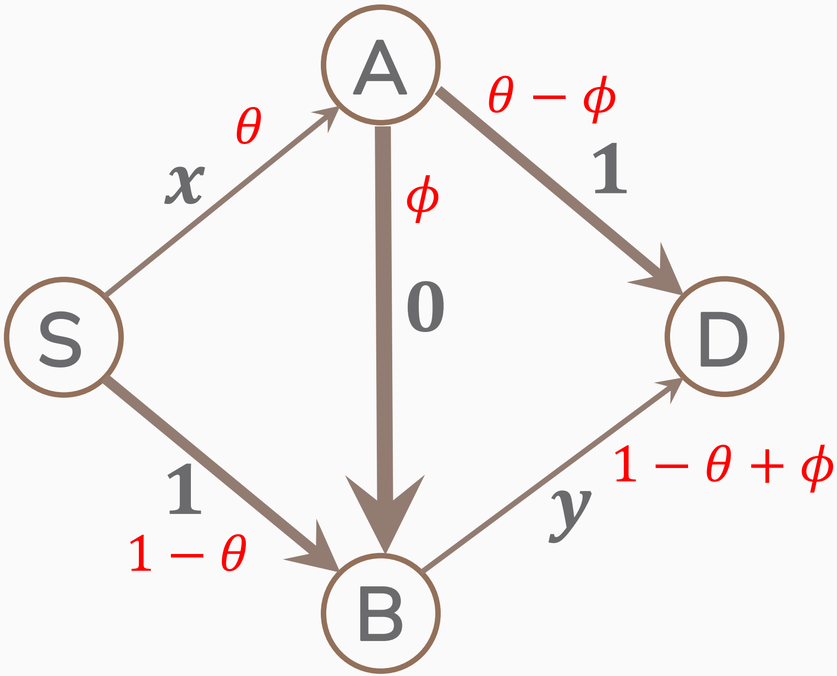
Need of rigor in game theory



- Let's analyze Game 2.

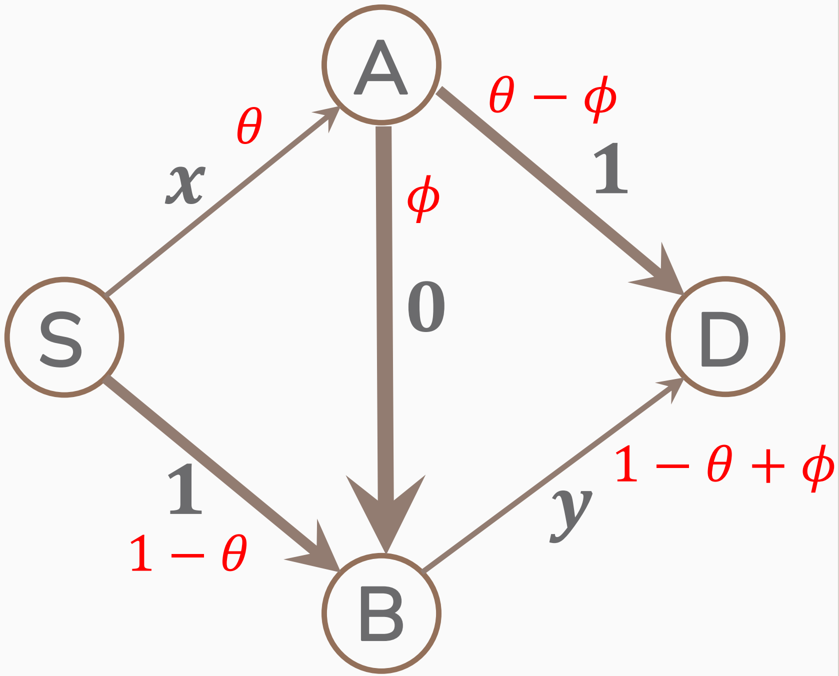
- What is the minimum travel time of a driver?
  - Consider the split of traffic as θ and $1-\theta$ in road segments SA and SB resp.

Need of rigor in game theory



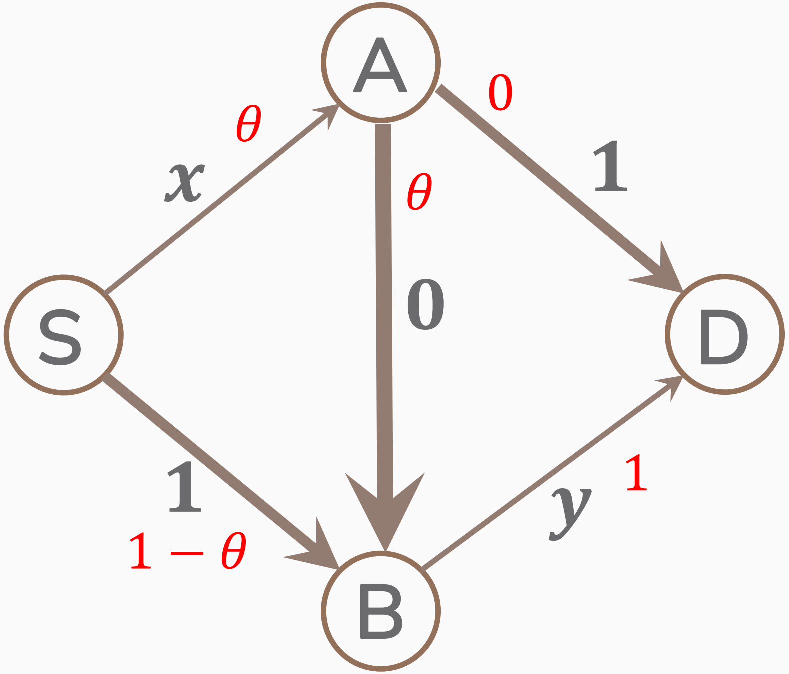
- Let's analyze Game 2.
- What is the minimum travel time of a driver?
  - Consider the split of traffic as θ and $1 - \theta$, where $\theta \in [0,1]$, in road segments SA and SB resp.
  - Out of θ fraction of traffic in SA, ϕ and $\theta - \phi$ fraction of the traffic went to road segment AB and AD. Off course, $\phi \in [0, \theta]$.
  - So the fraction of traffic in segment BD is $1 - \theta + \phi$.

Need of rigor in game theory



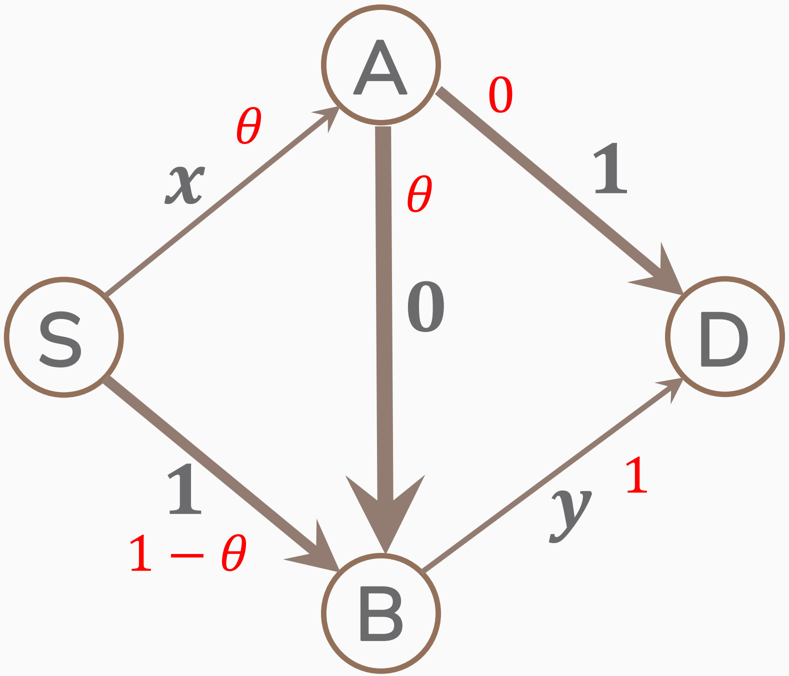
- Let's analyze Game 2.
- What is the minimum travel time of a driver?
  - The travel time for option 1 (SA -> AD) is $\boldsymbol{\theta + 1}$.
  - The travel time for option 3 (SA -> AB -> BD) is $\theta + (1 - \theta + \phi) = \boldsymbol{1 + \phi}$.
  - Since $\phi \leq \theta$, $\boldsymbol{1 + \phi \leq \theta + 1}$ and hence **option 3 is better than option 1**.

Need of rigor in game theory



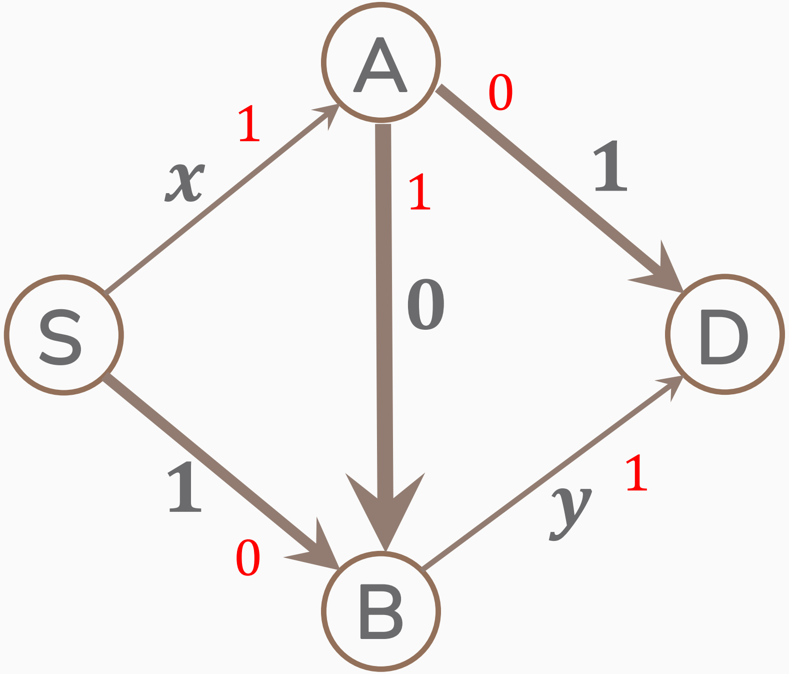
- Let's analyze Game 2.
- What is the minimum travel time of a driver?
  - The travel time for option 1 (SA -> AD) is $\boldsymbol{\theta + 1}$.
  - The travel time for option 3 (SA -> AB -> BD) is $\theta + (1 - \theta + \phi) = \boldsymbol{1 + \phi}$.
  - Since $\phi \leq \theta$, $\boldsymbol{1 + \phi \leq \theta + 1}$ and hence **option 3 is better than option 1**.
  - Therefore, no rational driver will choose option 1. Hence $\boldsymbol{\phi = \theta}$ (correspondingly the fraction of traffic in AD, AB, and BD are shown in the figure).

Need of rigor in game theory



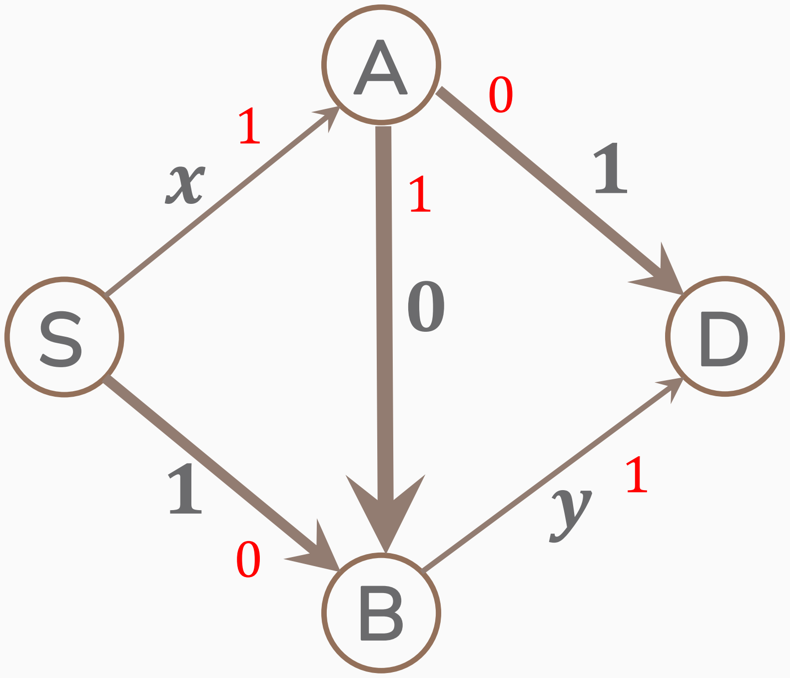
- Let's analyze Game 2.
- What is the minimum travel time of a driver?
  - Now we will compare options 2 and 3.
  - The travel time for option 2 (SB -> BD) is $\mathbf{1+1=2}$.
  - The travel time for option 3 (SA -> AB -> BD) is $\boldsymbol{\theta+0+1=\theta+1}$.
  - Off course, $\boldsymbol{\theta+1\leq 2}$ and hence **option 3 is better than option 2**.

Need of rigor in game theory



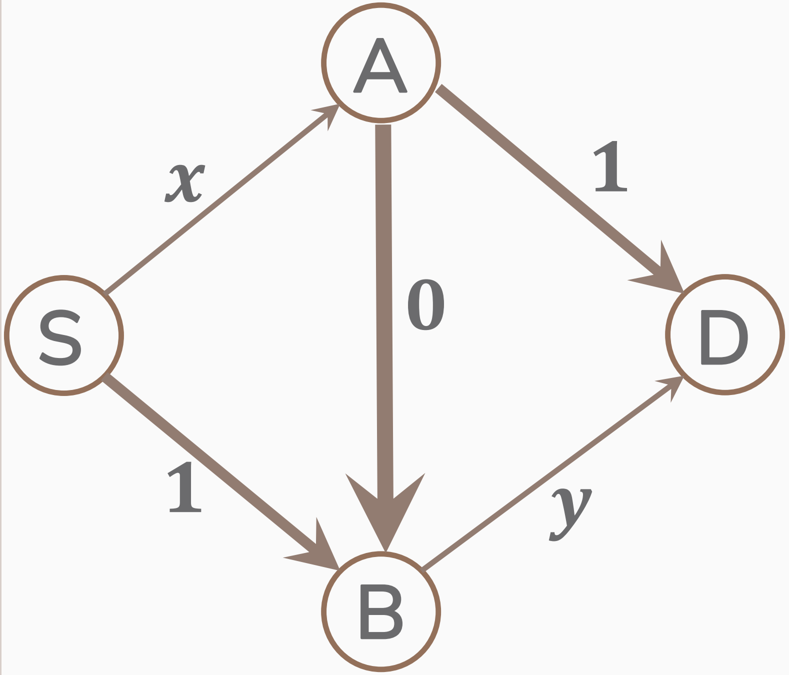
- Let's analyze Game 2.
- What is the minimum travel time of a driver?
  - Now we will compare options 2 and 3.
  - The travel time for option 2 (SB -> BD) is $\mathbf{1+1=2}$.
  - The travel time for option 3 (SA -> AB -> BD) is $\boldsymbol{\theta+0+1=\theta+1}$.
  - Off course, $\boldsymbol{\theta+1\leq 2}$ and hence **option 3 is better than option 2**.
  - Therefore, no rational driver will choose option 2. Corresponding traffic is shown in the figure.

Need of rigor in game theory



- Let's analyze Game 2.
- What is the minimum travel time of a driver?
  - So, the minimum travel time with **rational drivers** is 2.

Need of rigor in game theory



Braess's Paradox

- Now here is a rather counterintuitive observation. Compared to Game 1, Game 2 has an additional road and that too with 0 travel time.
- So, even though game 2 has more resources (an extra road), its travel time is more! This is called the Braess's paradox.
- We will encounter such counterintuitive phenomenons while analyzing a game. Therefore, when dealing with game-theoretic setup, be rigorous with your analysis. Don't make assumptions!

Thank You!