Mahindra University
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Game Theory (CS4187)

Lectures 16 and 17

Date: 26/09/2024
28/09/2024

Instructor: Gourav Saha

Broad Idea of Today’s Lecture

- The topics of this lecture is:
  1. Correlated equilibrium.

NOTE:

- Don’t follow “the textbooks” for this topic. The textbook by Narahari has this topic but introduces it from the perspective of cooperative game theory which we are not deadling in this course. The textbook by Harrington doesn’t have this topic.

- So possible references that you can follow for this topic is: (i) NPTEL lectures of the course “Introduction to Game Theory and Mechanism Design” by Prof. Swaparva Nath, and (ii) The book titled “Algorithmic Game Theory” b Noam Nisan and Tim Roughgarden.

Correlated Equilibrium

The overall idea of correlated equilibrium is as follows:

- ➢ The game (SFG) has a coordinator. This coordinator samples a strategy profile $s = (s_1, s_2, \cdots, s_n)$ where **$\boldsymbol{s}$ is sampled from a joint probability distribution $\boldsymbol{\sigma}$**, i.e. the probability of sampling s is $\sigma(s)$. Off course, $\sigma(s) \geq 0$ and $\sum_s \sigma(s) = 1$.

- ➢ The coordinator **suggests** player i to play strategy s_i where s_i is the strategy of player i corresponding to the strategy profile s sampled in the previous step. To be noted:
  - **Player $\boldsymbol{i}$ don't know the strategy $\boldsymbol{s_{-i}}$** of other players corresponding to sample s.
  - We assume that all the **players knows the joint probability distribution $\boldsymbol{\sigma}$**. *Caution: Knowing probability distribution σ and knowing s are completely different concepts.*

- ➢ The joint probability distribution $\boldsymbol{\sigma}$ is a correlated equilibrium if no player has an incentive to **unilaterally** not play the strategy suggested to it by the coordinator.

Correlated Equilibrium

To convert the idea mentioned in the previous slide into a mathematical definition, we need to find the following quantity:

What is the expected utility of player i to play action a_i given that the suggested strategy by the coordinator is s_i and assuming that all other players will play the strategy suggested by the coordinator?

Correlated Equilibrium

To convert the idea mentioned in the previous slide into a mathematical definition, we need to find the following quantity:

What is the expected utility of player i to play action a_i given that the suggested strategy by the coordinator is s_i and assuming that all other players will play the strategy suggested by the coordinator?

$$\begin{aligned}
E_\sigma\left[u_i\left(a_i, X_{-i}\right) | X_i = s_i\right] &= \sum_{s_{-i}} E_\sigma\left[u_i\left(a_i, X_{-i}\right) | X_i = s_i, X_{-i} = s_{-i}\right] P\left[X_{-i} = s_{-i} | X_i = s_i\right] \\
&= \sum_{s_{-i}} u_i\left(a_i, s_{-i}\right) P\left[X_{-i} = s_{-i} | X_i = s_i\right] \\
&= \sum_{s_{-i}} u_i\left(a_i, s_{-i}\right) \frac{P\left[X_{-i} = s_{-i}, X_i = s_i\right]}{P\left[X_i = s_i\right]} \\
&= \frac{\sum_{s_{-i}} \sigma\left(s_i, s_{-i}\right) u_i\left(a_i, s_{-i}\right)}{P\left[X_i = s_i\right]}
\end{aligned} \qquad \text{(E.1.)}$$

Correlated Equilibrium

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$$\begin{aligned} E_\sigma\left[u_i\left(a_i, X_{-i}\right) \mid X_i = s_i\right] &= \sum_{s_{-i}} E_\sigma\left[u_i\left(a_i, X_{-i}\right) \mid X_i = s_i, X_{-i} = s_{-i}\right] P\left[X_{-i} = s_{-i} \mid X_i = s_i\right] \\ &= \sum_{s_{-i}} u_i\left(a_i, s_{-i}\right) P\left[X_{-i} = s_{-i} \mid X_i = s_i\right] \\ &= \sum_{s_{-i}} u_i\left(a_i, s_{-i}\right) \frac{P\left[X_{-i} = s_{-i}, X_i = s_i\right]}{P\left[X_i = s_i\right]} \\ &= \frac{\sum_{s_{-i}} \sigma\left(s_i, s_{-i}\right) u_i\left(a_i, s_{-i}\right)}{P\left[X_i = s_i\right]} \end{aligned} \qquad \text{(E.1.)}$$

X_i and X_{-i} are the **random variables** corresponding to the strategy profile of player i and other players resp. that is sampled by the coordinator. s_i and s_{-i} are instance of the Random variables X_i and X_{-i}.

Correlated Equilibrium

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Why X_{-i} here?
Because we player i assumes that the other players plays the strategy suggested by the coordinator.

Correlated Equilibrium

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E_\sigma\left[u_i\left(a_i, X_{-i}\right)|X_i = s_i\right] &= \sum_{s_{-i}} E_\sigma\left[u_i\left(a_i, X_{-i}\right)|X_i = s_i, X_{-i} = s_{-i}\right] P\left[X_{-i} = s_{-i}|X_i = s_i\right] \\
&= \sum_{s_{-i}} u_i\left(a_i, s_{-i}\right) P\left[X_{-i} = s_{-i}|X_i = s_i\right] \\
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&= \frac{\sum_{s_{-i}} \sigma\left(s_i, s_{-i}\right) u_i\left(a_i, s_{-i}\right)}{P\left[X_i = s_i\right]}
\end{aligned} \quad \text{(E.1.)}$$

Using law of total expectation.

Correlated Equilibrium

- To convert the idea mentioned in the previous slide into a mathematical definition, we need to find the following quantity:

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E_\sigma\left[u_i\left(a_i, X_{-i}\right)|X_i = s_i\right] &= \sum_{s_{-i}} E_\sigma\left[u_i\left(a_i, X_{-i}\right)|X_i = s_i, \underline{X_{-i} = s_{-i}}\right] P\left[X_{-i} = s_{-i}|X_i = s_i\right] \\
&= \sum_{s_{-i}} u_i\left(a_i, s_{-i}\right) P\left[X_{-i} = s_{-i}|X_i = s_i\right] \\
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&= \frac{\sum_{s_{-i}} \sigma\left(s_i, s_{-i}\right) u_i\left(a_i, s_{-i}\right)}{P\left[X_i = s_i\right]}
\end{aligned} \quad \text{(E.1.)}$$

Just substituted $X_{-i} = s_{-i}$ because of this.

Correlated Equilibrium

To convert the idea mentioned in the previous slide into a mathematical definition, we need to find the following quantity:

What is the expected utility of player i to play action a_i given that the suggested strategy by the coordinator is s_i and assuming that all other players will play the strategy suggested by the coordinator?

$$
\begin{aligned}
E_\sigma\left[u_i\left(a_i, X_{-i}\right) \mid X_i=s_i\right] &= \sum_{s_{-i}} E_\sigma\left[u_i\left(a_i, X_{-i}\right) \mid X_i=s_i, X_{-i}=s_{-i}\right] P\left[X_{-i}=s_{-i} \mid X_i=s_i\right] \\
&= \sum_{s_{-i}} u_i\left(a_i, s_{-i}\right) P\left[X_{-i}=s_{-i} \mid X_i=s_i\right] \\
&= \sum_{s_{-i}} u_i\left(a_i, s_{-i}\right) \frac{P\left[X_{-i}=s_{-i}, X_i=s_i\right]}{P\left[X_i=s_i\right]} \quad \text{Bayes' rule.} \\
&= \frac{\sum_{s_{-i}} \sigma\left(s_i, s_{-i}\right) u_i\left(a_i, s_{-i}\right)}{P\left[X_i=s_i\right]} \qquad \text{(E.1.)}
\end{aligned}
$$

Correlated Equilibrium

- To convert the idea mentioned in the previous slide into a mathematical definition, we need to find the following quantity:

What is the expected utility of player i to play action a_i given that the suggested strategy by the coordinator is s_i and assuming that all other players will play the strategy suggested by the coordinator?

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&= \sum_{s_{-i}} u_i\left(a_i, s_{-i}\right) P\left[X_{-i} = s_{-i} | X_i = s_i\right] \\
&= \sum_{s_{-i}} u_i\left(a_i, s_{-i}\right) \frac{P\left[X_{-i} = s_{-i}, X_i = s_i\right]}{P\left[X_i = s_i\right]} \\
&= \frac{\sum_{s_{-i}} \sigma\left(s_i, s_{-i}\right) u_i\left(a_i, s_{-i}\right)}{P\left[X_i = s_i\right]}
\end{aligned} \qquad \text{(E.1.)}$$

These two terms are same by definition. Also,

$$\sigma(s) = \sigma(s_i, s_{-i})$$

Correlated Equilibrium

- Based on the overall idea of correlated equilibrium mentioned in this slide, player i will not have incentive to **unilaterally** change its strategy but play the strategy s_i suggested by the coordinator if and only if, **for all** $s_i' \in S_i$ the following condition is satisfied,

$$\frac{\sum_{s_{-i}} \sigma\left(s_i, s_{-i}\right) u_i\left(s_i, s_{-i}\right)}{P\left[X_i = s_i\right]} \geq \frac{\sum_{s_{-i}} \sigma\left(s_i, s_{-i}\right) u_i\left(s_i^{'}, s_{-i}\right)}{P\left[X_i = s_i\right]}$$

which is equivalent to,

$$\sum_{s_{-i}} \sigma\left(s_i, s_{-i}\right) u_i\left(s_i, s_{-i}\right) \geq \sum_{s_{-i}} \sigma\left(s_i, s_{-i}\right) u_i\left(s_i^{'}, s_{-i}\right) \qquad \textbf{(E.2.)}$$

Correlated Equilibrium: Definition

- Based on the discussion in the previous page, here is the formal definition of correlated equilibrium **(CE)**.

Definition 1 (Correlated Equilibrium): A joint probability distribution σ over the strategy set $S = S_1 \times S_2 \times \cdots \times S_n$ of all the players is called a correlated equilibrium if and only if,

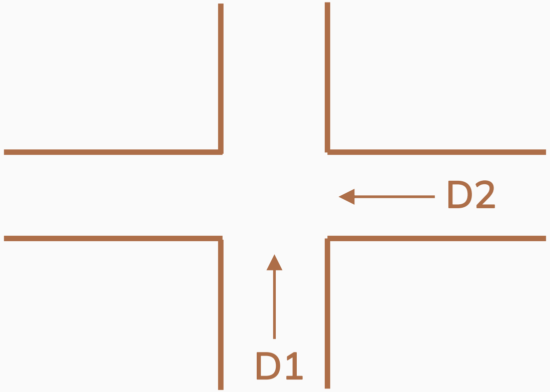
$$\sum_{s_{-i}} \sigma\left(s_i, s_{-i}\right) u_i\left(s_i, s_{-i}\right) \geq \sum_{s_{-i}} \sigma\left(s_i, s_{-i}\right) u_i\left(s_i^{'}, s_{-i}\right) \text{ , } \forall s_i, s_i^{'} \in S_i \text{ , } \forall i \in N \qquad \text{(E.3.)}$$

NOTE: (i) In (E.3.), (s_i, s_{-i}) is equivalent to the strategy profile sampled by the coordinator.

(ii) LHS of (E.3.) is proportional (not equal) to the expected utility of player i if it plays the coordinator's suggestion s_i assuming that other players are playing the coordinator's strategy.

(iii) RHS of (E.3.) is proportional (not equal) to the expected utility of player i if it plays a strategy s_i' while other players are playing the coordinator's strategy.

Correlated Equilibrium: Relevance in Real-World Situation

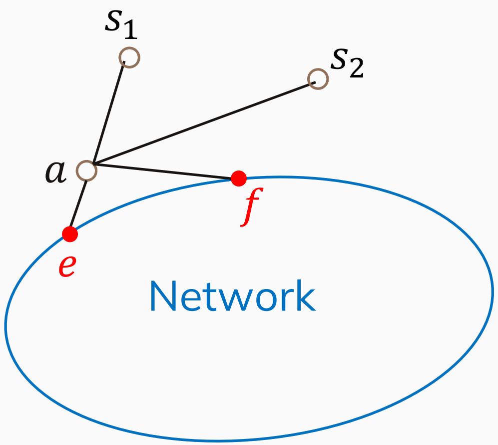


		D2	
		Wait	Go
D1	Wait	0, 0	1, 2
	Go	2, 1	-10, -10

Traffic Coordination:

- This is a classic example that justifies the practical relevance of CE.
- PSNE: (Go,Wait) and (Wait, Go).
- MSNE: D1: (0.85, 0.15)
  D2: (0.85, 0.15)
- In MSNE, there is a finite chance of accident (Go, Go). This is weird!
- Who is the game coordinator? Answer: Traffic light

Correlated Equilibrium: Relevance in Real-World Situation

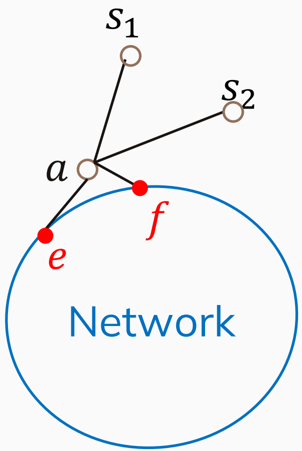


		s_2	
		e	f
s_1	e	-5, -5	-1, -2
	f	-2, -1	-6, -6

Network Congestion Games:

- We encountered this example back in module 1.
- PSNE: (e, f) and (f, e).
- MSNE: s_1: (0.625, 0.375)
  s_2 : (0.625, 0.375)
- Who is the game coordinator? Answer: **Node $\boldsymbol{a}$**. It is very much practical to think that node a, which is an intermediate node between s_1 (and s_2) and the network, can act like a coordinator for nodes s_1 and s_2.

Computing CE: Example



		s_2	
		e	f
s_1	e	-5, -5	-1, -2
	f	-2, -1	-6, -6

Network Congestion Games:

$$\sum_{s_{-i}} \sigma\left(s_i, s_{-i}\right) u_i\left(s_i, s_{-i}\right) \geq \sum_{s_{-i}} \sigma\left(s_i, s_{-i}\right) u_i\left(s_i^{'}, s_{-i}\right), \forall s_i, s_i^{'} \in S_i, \forall i = 1, 2, \ldots, n$$

For player $i = 1$:

1. $s_1 = e$, $s_1' = e$:

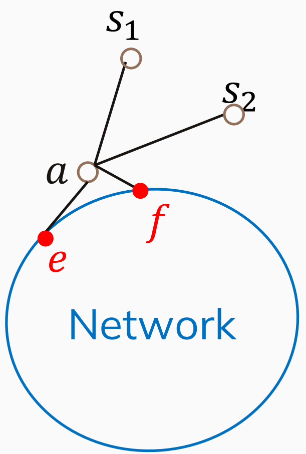
$$\sum_{s_{-1} \in \{e, f\}} \sigma\left(e, s_{-1}\right) u_1\left(e, s_{-1}\right) \geq \sum_{s_{-1} \in \{e, f\}} \sigma\left(e, s_{-1}\right) u_1\left(e, s_{-1}\right)$$

$$\sigma(e, e) u_1(e, e) + \sigma(e, f) u_1(e, f) \geq \sigma(e, e) u_1(e, e) + \sigma(e, f) u_1(e, f)$$

$$-5\sigma(e, e) - \sigma(e, f) \geq -5\sigma(e, e) - \sigma(e, f)$$

Trivial inequality. LHS and RHS are same.

Computing CE: Example



		s_2	
		e	f
s_1	e	-5, -5	-1, -2
	f	-2, -1	-6, -6

Network Congestion Games:

$$\sum_{s_{-i}} \sigma\left(s_i, s_{-i}\right) u_i\left(s_i, s_{-i}\right) \geq \sum_{s_{-i}} \sigma\left(s_i, s_{-i}\right) u_i\left(s_i^{'}, s_{-i}\right), \forall s_i, s_i^{'} \in S_i, \forall i = 1, 2, \ldots, n$$

For player $i = 1$:

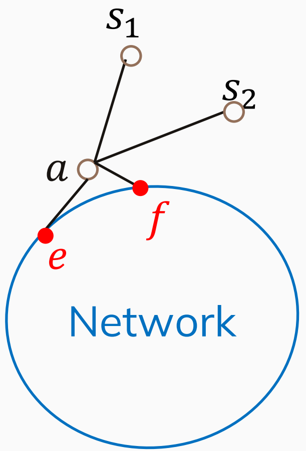
2. $s_1 = e, s_1' = f$:

$$\sum_{s_{-1} \in \{e, f\}} \sigma\left(e, s_{-1}\right) u_1\left(e, s_{-1}\right) \geq \sum_{s_{-1} \in \{e, f\}} \sigma\left(e, s_{-1}\right) u_1\left(f, s_{-1}\right)$$

$$\sigma(e, e) u_1(e, e) + \sigma(e, f) u_1(e, f) \geq \sigma(e, e) u_1(f, e) + \sigma(e, f) u_1(f, f)$$

$$-5\sigma(e, e) - \sigma(e, f) \geq -2\sigma(e, e) - 6\sigma(e, f)$$

Computing CE: Example



		s_2	
		e	f
s_1	e	-5, -5	-1, -2
	f	-2, -1	-6, -6

Network Congestion Games:

$$\sum_{s_{-i}} \sigma(s_i, s_{-i})\, u_i(s_i, s_{-i}) \geq \sum_{s_{-i}} \sigma(s_i, s_{-i})\, u_i\left(s_i^{'}, s_{-i}\right), \forall s_i, s_i^{'} \in S_i, \forall i = 1, 2, \ldots, n$$

For player $i = 1$:

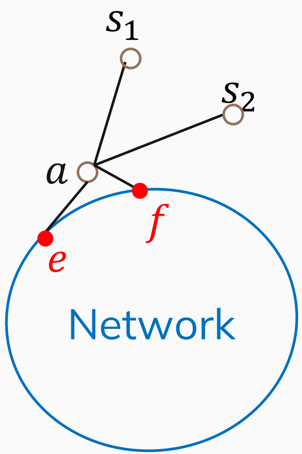
3. $s_1 = f, s_1' = e$:

$$\sum_{s_{-1}\in\{e,f\}} \sigma(f, s_{-1})\, u_1(f, s_{-1}) \geq \sum_{s_{-1}\in\{e,f\}} \sigma(f, s_{-1})\, u_1(e, s_{-1})$$

$$\sigma(f,e)\, u_1(f,e) + \sigma(f,f)\, u_1(f,f) \geq \sigma(f,e)\, u_1(e,e) + \sigma(f,f)\, u_1(e,f)$$

$$-2\sigma(f,e) - 6\sigma(f,f) \geq -5\sigma(f,e) - 1\sigma(f,f)$$

Computing CE: Example



		s_2	
		e	f
s_1	e	-5, -5	-1, -2
	f	-2, -1	-6, -6

Network Congestion Games:

$$\sum_{s_{-i}} \sigma(s_i, s_{-i}) u_i(s_i, s_{-i}) \geq \sum_{s_{-i}} \sigma(s_i, s_{-i}) u_i\left(s_i^{'}, s_{-i}\right), \forall s_i, s_i^{'} \in S_i, \forall i = 1, 2, \ldots, n$$

For player $i = 1$:

4. $s_1 = f, s_1' = f$:

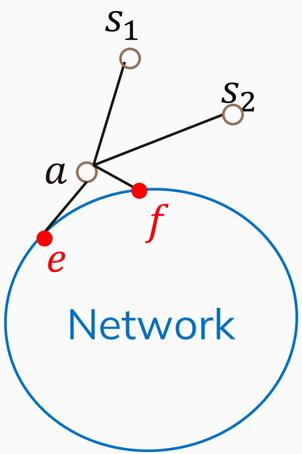
$$\sum_{s_{-1} \in \{e,f\}} \sigma(f, s_{-1}) u_1(f, s_{-1}) \geq \sum_{s_{-1} \in \{e,f\}} \sigma(f, s_{-1}) u_1(f, s_{-1})$$

$$\sigma(f, e) u_1(f, e) + \sigma(f, f) u_1(f, f) \geq \sigma(f, e) u_1(f, e) + \sigma(f, f) u_1(f, f)$$

$$-2\sigma(f, e) - 6\sigma(f, f) \geq -2\sigma(f, e) - 6\sigma(f, f)$$

Trivial inequality. LHS and RHS are same.

Computing CE: Example



		s_2	
		e	f
s_1	e	-5, -5	-1, -2
	f	-2, -1	-6, -6

Network Congestion Games:

$$\sum_{s_{-i}} \sigma\left(s_i, s_{-i}\right) u_i\left(s_i, s_{-i}\right) \geq \sum_{s_{-i}} \sigma\left(s_i, s_{-i}\right) u_i\left(s_i^{'}, s_{-i}\right), \forall s_i, s_i^{'} \in S_i, \forall i = 1, 2, \ldots, n$$

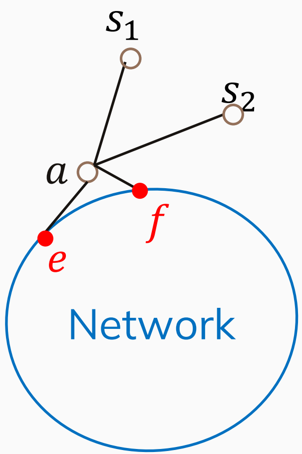
For player $i = 1$:

$$-5\sigma(e,e) - \sigma(e,f) \geq -2\sigma(e,e) - 6\sigma(e,f)$$

$$-2\sigma(f,e) - 6\sigma(f,f) \geq -5\sigma(f,e) - 1\sigma(f,f)$$

We don’t include trivial inequalities.

Computing CE: Example



		s_2	
		e	f
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Network Congestion Games:

$$\sum_{s_{-i}} \sigma(s_i, s_{-i})\, u_i(s_i, s_{-i}) \geq \sum_{s_{-i}} \sigma(s_i, s_{-i})\, u_i\left(s_i^{'}, s_{-i}\right), \forall s_i, s_i^{'} \in S_i, \forall i = 1, 2, \ldots, n$$

For player $i = 2$:

1. $s_2 = e, s_2' = e$:

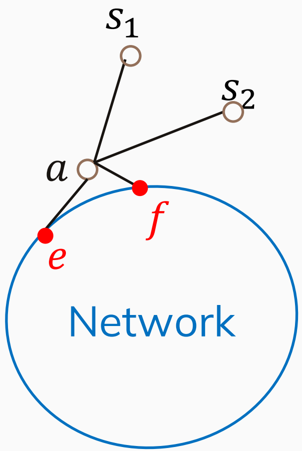
$$\sum_{s_{-2} \in \{e,f\}} \sigma(e, s_{-2})\, u_2(e, s_{-2}) \geq \sum_{s_{-2} \in \{e,f\}} \sigma(e, s_{-2})\, u_2(e, s_{-2})$$

$$\sigma(e,e)\, u_2(e,e) + \sigma(e,f)\, u_2(e,f) \geq \sigma(e,e)\, u_2(e,e) + \sigma(e,f)\, u_2(e,f)$$

$$-5\sigma(e,e) - 2\sigma(e,f) \geq -5\sigma(e,e) - 2\sigma(e,f)$$

Trivial inequality. LHS and RHS are same.

Computing CE: Example



		s_2	
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s_1	e	-5, -5	-1, -2
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Network Congestion Games:

$$\sum_{s_{-i}} \sigma(s_i, s_{-i})\, u_i(s_i, s_{-i}) \geq \sum_{s_{-i}} \sigma(s_i, s_{-i})\, u_i\left(s_i^{'}, s_{-i}\right), \forall s_i, s_i^{'} \in S_i, \forall i = 1, 2, \ldots, n$$

For player $i = 2$:

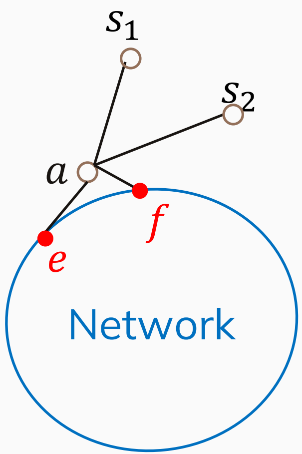
2. $s_2 = e$, $s_2' = f$:

$$\sum_{s_{-2}\in\{e,f\}} \sigma(e, s_{-2})\, u_2(e, s_{-2}) \geq \sum_{s_{-2}\in\{e,f\}} \sigma(e, s_{-2})\, u_2(f, s_{-2})$$

$$\sigma(e,e)\, u_2(e,e) + \sigma(e,f)\, u_2(e,f) \geq \sigma(e,e)\, u_2(f,e) + \sigma(e,f)\, u_2(f,f)$$

$$-5\sigma(e,e) - 2\sigma(e,f) \geq -\sigma(e,e) - 6\sigma(e,f)$$

Computing CE: Example



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Network Congestion Games:

$$\sum_{s_{-i}} \sigma(s_i, s_{-i})\, u_i(s_i, s_{-i}) \geq \sum_{s_{-i}} \sigma(s_i, s_{-i})\, u_i\left(s_i^{'}, s_{-i}\right), \forall s_i, s_i^{'} \in S_i, \forall i = 1, 2, \ldots, n$$

For player $i = 2$:

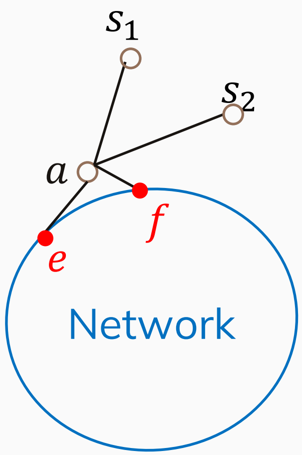
3. $s_2 = f, s_2' = e$:

$$\sum_{s_{-2} \in \{e,f\}} \sigma(f, s_{-2})\, u_2(f, s_{-2}) \geq \sum_{s_{-2} \in \{e,f\}} \sigma(f, s_{-2})\, u_2(e, s_{-2})$$

$$\sigma(f,e)\, u_2(f,e) + \sigma(f,f)\, u_2(f,f) \geq \sigma(f,e)\, u_2(e,e) + \sigma(f,f)\, u_2(e,f)$$

$$-\sigma(f,e) - 6\sigma(f,f) \geq -5\sigma(f,e) - 2\sigma(f,f)$$

Computing CE: Example



s_2

		e	f
s_1	e	-5, -5	-1, -2
	f	-2, -1	-6, -6

Network Congestion Games:

$$\sum_{s_{-i}} \sigma\left(s_i, s_{-i}\right) u_i\left(s_i, s_{-i}\right) \geq \sum_{s_{-i}} \sigma\left(s_i, s_{-i}\right) u_i\left(s_i^{'}, s_{-i}\right), \forall s_i, s_i^{'} \in S_i, \forall i = 1, 2, \ldots, n$$

For player $i = 2$:

4. $s_2 = f, s_2' = f$:

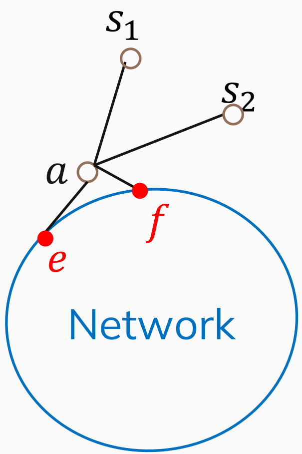
$$\sum_{s_{-2}\in\{e,f\}} \sigma\left(f, s_{-2}\right) u_2\left(f, s_{-2}\right) \geq \sum_{s_{-2}\in\{e,f\}} \sigma\left(f, s_{-2}\right) u_2\left(f, s_{-2}\right)$$

$$\sigma\left(f, e\right) u_2\left(f, e\right) + \sigma\left(f, f\right) u_2\left(f, f\right) \geq \sigma\left(f, e\right) u_2\left(f, e\right) + \sigma\left(f, f\right) u_2\left(f, f\right)$$

$$-\sigma\left(f, e\right) - 6\sigma\left(f, f\right) \geq -\sigma\left(f, e\right) - 6\sigma\left(f, f\right)$$

Trivial inequality. LHS and RHS are same.

Computing CE: Example



		s_2	
		e	f
s_1	e	-5, -5	-1, -2
	f	-2, -1	-6, -6

Network Congestion Games:

$$\sum_{s_{-i}} \sigma\left(s_i, s_{-i}\right) u_i\left(s_i, s_{-i}\right) \geq \sum_{s_{-i}} \sigma\left(s_i, s_{-i}\right) u_i\left(s_i^{'}, s_{-i}\right), \forall s_i, s_i^{'} \in S_i, \forall i = 1, 2, \ldots, n$$

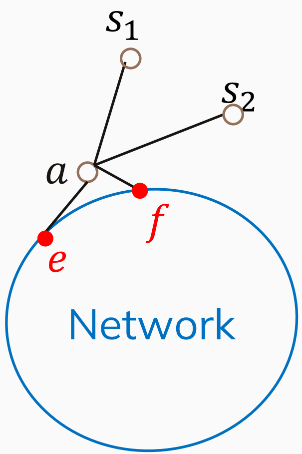
For player $i = 2$:

$$-5\sigma(e, e) - 2\sigma(e, f) \quad \geq \quad -\sigma(e, e) - 6\sigma(e, f)$$

$$-\sigma(f, e) - 6\sigma(f, f) \quad \geq \quad -5\sigma(f, e) - 2\sigma(f, f)$$

We don’t include trivial inequalities.

Computing CE: Example



		s_2	
		e	f
s_1	e	-5, -5	-1, -2
	f	-2, -1	-6, -6

Network Congestion Games:

$$-5\sigma(e,e) - \sigma(e,f) \geq -2\sigma(e,e) - 6\sigma(e,f)$$

$$-2\sigma(f,e) - 6\sigma(f,f) \geq -5\sigma(f,e) - 1\sigma(f,f)$$

$$-5\sigma(e,e) - 2\sigma(e,f) \geq -\sigma(e,e) - 6\sigma(e,f)$$

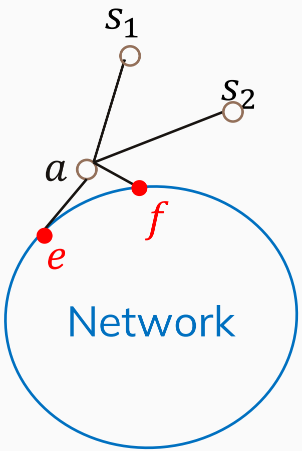
$$-\sigma(f,e) - 6\sigma(f,f) \geq -5\sigma(f,e) - 2\sigma(f,f)$$

$$\sigma(e,e) + \sigma(e,f) + \sigma(f,e) + \sigma(f,f) = 1$$

$$\sigma(e,e), \sigma(e,f), \sigma(f,e), \sigma(f,f) \geq 0$$

- ➢ Combining the non-trivial inequalities corresponding to players s_1 and s_2 we get the above. The last two equality/inequality just implies that σ is a probability distribution and hence its elements should add to 1 and its elements has to be greater than zero.

Computing CE: Example



		s_2	
		e	f
s_1	e	-5, -5	-1, -2
	f	-2, -1	-6, -6

Network Congestion Games:

$$-5\sigma(e,e) - \sigma(e,f) \geq -2\sigma(e,e) - 6\sigma(e,f)$$

$$-2\sigma(f,e) - 6\sigma(f,f) \geq -5\sigma(f,e) - 1\sigma(f,f)$$

$$-5\sigma(e,e) - 2\sigma(e,f) \geq -\sigma(e,e) - 6\sigma(e,f)$$

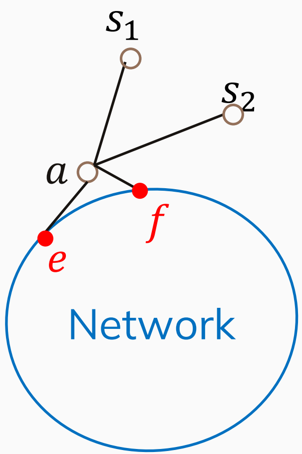
$$-\sigma(f,e) - 6\sigma(f,f) \geq -5\sigma(f,e) - 2\sigma(f,f)$$

$$\sigma(e,e) + \sigma(e,f) + \sigma(f,e) + \sigma(f,f) = 1$$

$$\sigma(e,e), \sigma(e,f), \sigma(f,e), \sigma(f,f) \geq 0$$

- The above can be solved using linear programming with $\sigma(e,e)$, $\sigma(e,f)$, $\sigma(f,e)$, and $\sigma(f,f)$ as optimization variables (just think of $\sigma(e,e)$, $\sigma(e,f)$, $\sigma(f,e)$, and $\sigma(f,f)$ as x, y, z, and w respectively to aid the thinking process). The objective function is zero (or any constant).

Computing CE: Example



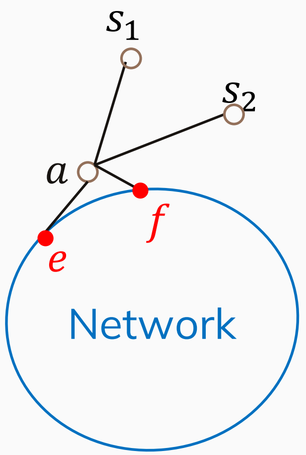
		s_2	
		e	f
s_1	e	-5, -5	-1, -2
	f	-2, -1	-6, -6

Network Congestion Games:

- Can there be an objective function?
- One possible choice is expected utility of all the players:

$$\sum_{i\in N}\left(\sum_{s_{-i}}\sigma\left(s_i,s_{-i}\right)u_i\left(s_i,s_{-i}\right)\right)$$

Computing CE: Example



		s_2	
		e	f
s_1	e	-5, -5	-1, -2
	f	-2, -1	-6, -6

Network Congestion Games:

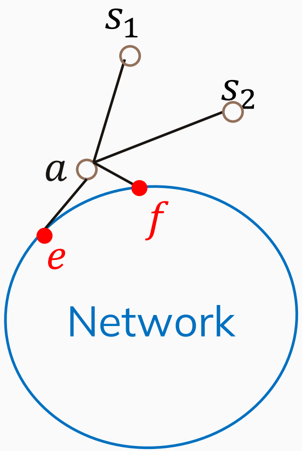
- ➢ Can there be an objective function?
- ➢ One possible choice is expected utility of all the players:

$$\sum_{i \in N} \left(\sum_{s_{-i}} \sigma\left(s_i, s_{-i}\right) u_i\left(s_i, s_{-i}\right) \right)$$

$$\begin{aligned} = \quad & -5\sigma(e,e) - \sigma(e,f) - 2\sigma(f,e) - 6\sigma(f,f) \\ & -5\sigma(e,e) - 2\sigma(e,f) - \sigma(f,e) - 6\sigma(f,f) \end{aligned}$$

$$= \quad -10\sigma(e,e) - 3(e,f) - 3\sigma(f,e) - 12\sigma(f,f)$$

Computing CE: Example



		s_2	
		e	f
s_1	e	-5, -5	-1, -2
	f	-2, -1	-6, -6

$$\max\ -10\sigma(e,e) - 3(e,f) - 3\sigma(f,e) - 12\sigma(f,f)$$

subject to:

$$\begin{aligned}
-5\sigma(e,e) - \sigma(e,f) &\geq -2\sigma(e,e) - 6\sigma(e,f)\\
-2\sigma(f,e) - 6\sigma(f,f) &\geq -5\sigma(f,e) - 1\sigma(f,f)\\
-5\sigma(e,e) - 2\sigma(e,f) &\geq -\sigma(e,e) - 6\sigma(e,f)\\
-\sigma(f,e) - 6\sigma(f,f) &\geq -5\sigma(f,e) - 2\sigma(f,f)\\
\sigma(e,e) + \sigma(e,f) + \sigma(f,e) + \sigma(f,f) &= 1\\
\sigma(e,e), \sigma(e,f), \sigma(f,e), \sigma(f,f) &\geq 0
\end{aligned}$$

Computing CE: Optimization Problem in General

$$\max \sum_{i \in N} \left(\sum_{s_{-i}} \sigma(s)\, u_i(s) \right)$$

subject to:

$$\sum_{s_{-i}} \sigma(s) \left(u_i(s_i, s_{-i}) - u_i\left(s_i^{'}, s_{-i}\right)\right) \geq 0\,,\ \forall s_i, s_i^{'} \in S_i\ ,\ \forall i \in N$$

$$\sum_{s \in S} \sigma(s) = 1$$

$$\sigma(s) \geq 0\ ,\forall s \in S$$

Check next two slides for explanation.

Computing CE: Optimization Problem in General

$$\max \sum_{i \in N} \left(\sum_{s_{-i}} \sigma\left(s\right) u_i\left(s\right) \right)$$

subject to:

$$\sum_{s_{-i}} \sigma\left(s\right) \left(u_i\left(s_i, s_{-i}\right) - u_i\left(s_i^{'}, s_{-i}\right) \right) \geq 0\,,\ \forall s_i, s_i^{'} \in S_i\ ,\ \forall i \in N$$

$$\sum_{s \in S} \sigma\left(s\right) = 1$$

$$\sigma\left(s\right) \geq 0\ , \forall s \in S$$

Remember, $s = (s_i\ , s_{-i})$.

Computing CE: Optimization Problem in General

$$\max \sum_{i \in N} \left(\sum_{s_{-i}} \sigma(s)\, u_i(s) \right)$$

subject to:

$$\sum_{s_{-i}} \sigma(s) \left(u_i(s_i, s_{-i}) - u_i\left(s_i^{'}, s_{-i}\right) \right) \geq 0\,,\ \forall s_i, s_i^{'} \in S_i\ ,\ \forall i \in N$$

} EQUIVALENT to equation (E.3) in this slide (click here).

$$\sum_{s \in S} \sigma(s) = 1$$

$$\sigma(s) \geq 0\,, \forall s \in S$$

Computing CE: Optimization Problem in General

$$\sum_{s_{-i}} \sigma\left(s\right)\left(u_i\left(s_i, s_{-i}\right) - u_i\left(s_i^{'}, s_{-i}\right)\right) \geq 0\,,\ \forall s_i, s_i^{'} \in S_i\ ,\ \forall i \in N$$

$$\sum_{s \in S} \sigma\left(s\right) = 1$$

$$\sigma\left(s\right) \geq 0\ , \forall s \in S$$

We can also compute CE without an objective function in which case it is just a bunch of linear constraints.

Relation between MSNE and CE

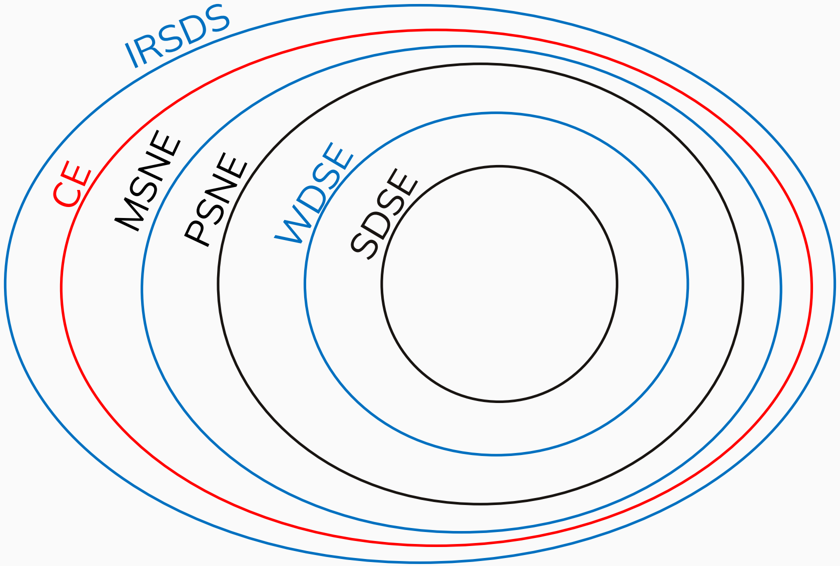
Theorem: An MSNE of a game is also CE of the game.

Proof: The proof is given in proof.pdf of this folder. Essentially, we show that if $\sigma^* = (\sigma_1^*, \sigma_2^*, \cdots, \sigma_n^*)$ is an MSNE and for all (s_i, s_{-i}) we set,

$$\begin{aligned} \sigma(s_i, s_{-i}) &= \sigma_i^\star(s_i) \cdot \sigma_{-i}^\star(s_{-i}) \\ &= \sigma_i^\star(s_i) \cdot \prod_{j \neq i} \sigma_j^\star(s_j) \\ &= \prod_{j \in N} \sigma_j^\star(s_j) \qquad \text{(E.4.)} \end{aligned}$$

Then, $\sigma(s_i, s_{-i})$ given by (E.4.) satisfies inequality E.3. (check this slide) and is hence a CE.

Relation between MSNE and CE



- The theorem in the previous slides implies that **MSNE of SFG is a subset of CE of a SFG**. This is shown in the Venn diagram in the left.

Relation between MSNE and CE

CE

$$\sum_{s_{-i}} \sigma(s)\left(u_i(s_i, s_{-i}) - u_i\left(s_i^{'}, s_{-i}\right)\right) \geq 0\,,\ \forall s_i, s_i^{'} \in S_i\,,\ \forall i \in N$$

$$\sum_{s \in S} \sigma(s) = 1$$

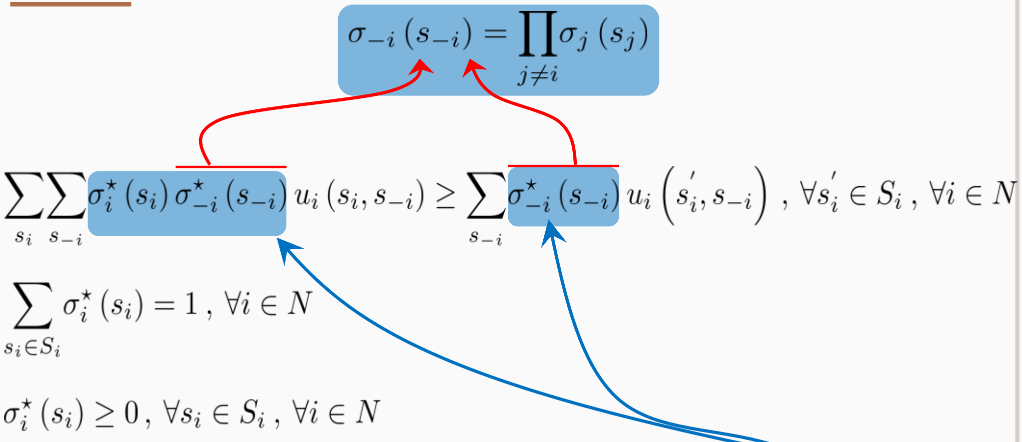
$$\sigma(s) \geq 0\,, \forall s \in S$$

- Now lets talk about the relation between MSNE and CE as far as computation is concerned.

- Computing CE amounts to solving a bunch of **linear constraints**. The decision variables are $\sigma(s)$ for all $s \in S$.

Relation between MSNE and CE

MSNE



$$\sigma_{-i}\left(s_{-i}\right) = \prod_{j\neq i}\sigma_j\left(s_j\right)$$

$$\sum_{s_i}\sum_{s_{-i}}\sigma_i^\star\left(s_i\right)\sigma_{-i}^\star\left(s_{-i}\right)u_i\left(s_i,s_{-i}\right) \geq \sum_{s_{-i}}\sigma_{-i}^\star\left(s_{-i}\right)u_i\left(s_i^{'},s_{-i}\right)\ ,\ \forall s_i^{'}\in S_i\ ,\ \forall i\in N$$

$$\sum_{s_i\in S_i}\sigma_i^\star\left(s_i\right)=1\ ,\ \forall i\in N$$

$$\sigma_i^\star\left(s_i\right)\geq 0,\ \forall s_i\in S_i\ ,\ \forall i\in N$$

- Now lets talk about the relation between MSNE and CE as far as computation is concerned.
- Computing CE amounts to solving a bunch of **linear constraints**. The decision variables are $\sigma(s)$ for all $s \in S$.
- Computing MSNE amounts to solving a bunch of **non-linear constraints**. The decision variables are $\sigma_i^*(s_i)$ for all $s_i \in S_i$ and all $i \in N$. We are dealing with non-linear constraints because **these** terms are **products of the decision variables**.

Relation between MSNE and CE

CE

$$\sum_{s_{-i}} \sigma(s)\left(u_i(s_i, s_{-i}) - u_i\left(s_i^{'}, s_{-i}\right)\right) \geq 0\,,\ \forall s_i, s_i^{'} \in S_i\ ,\ \forall i \in N$$

$$\sum_{s \in S} \sigma(s) = 1$$

$$\sigma(s) \geq 0\ , \forall s \in S$$

MSNE

$$\sum_{s_i}\sum_{s_{-i}} \sigma_i^{\star}(s_i)\, \sigma_{-i}^{\star}(s_{-i})\, u_i(s_i, s_{-i}) \geq \sum_{s_{-i}} \sigma_{-i}^{\star}(s_{-i})\, u_i\left(s_i^{'}, s_{-i}\right)\ ,\ \forall s_i^{'} \in S_i\ ,\ \forall i \in N$$

$$\sum_{s_i \in S_i} \sigma_i^{\star}(s_i) = 1\,,\ \forall i \in N$$

$$\sigma_i^{\star}(s_i) \geq 0\,,\ \forall s_i \in S_i\ ,\ \forall i \in N$$

- Essentially, one of the conditions for a CE to be a MSNE (not the only condition) is that the probability distribution $\sigma(s)$ has to be independent across players. Mathematically, $\sigma(s)$ must satisfy,

$$\sigma(s) = \prod_{i \in N} \sigma_i(s_i) \qquad \textbf{(E.5.)}$$

where $\sigma_i(s_i)$ is the probability that game coordinator chooses action s_i for player i given that the joint probability distribution is $\sigma(s)$. Mathematically,

$$\sigma_i(s_i) = \sum_{s_{-i} \in S_{-i}} \sigma(s_i, s_{-i})$$

Relation between MSNE and CE

CE

$$\sum_{s_{-i}} \sigma(s)\left(u_i(s_i, s_{-i}) - u_i\left(s_i^{'}, s_{-i}\right)\right) \geq 0\,,\ \forall s_i, s_i^{'} \in S_i\,,\ \forall i \in N$$

$$\sum_{s \in S} \sigma(s) = 1$$

$$\sigma(s) \geq 0\,, \forall s \in S$$

MSNE

$$\sum_{s_i}\sum_{s_{-i}} \sigma_i^{\star}(s_i)\,\sigma_{-i}^{\star}(s_{-i})\,u_i(s_i, s_{-i}) \geq \sum_{s_{-i}} \sigma_{-i}^{\star}(s_{-i})\,u_i\left(s_i^{'}, s_{-i}\right)\,,\ \forall s_i^{'} \in S_i\,,\ \forall i \in N$$

$$\sum_{s_i \in S_i} \sigma_i^{\star}(s_i) = 1\,,\ \forall i \in N$$

$$\sigma_i^{\star}(s_i) \geq 0\,,\ \forall s_i \in S_i\,,\ \forall i \in N$$

- Equation (E.5.) is an additional constraint on $\sigma(s)$ for it be a MSNE. This additional constraint explains why MSNE is a CE but not vice-versa.
- Equation (E.5.) is a non-linear constraint. Hence, MSNE involves solving non-linear constraints.

Relation between MSNE and CE

CE

$$\sum_{s_{-i}} \sigma\left(s\right)\left(u_i\left(s_i, s_{-i}\right) - u_i\left(s_i^{'}, s_{-i}\right)\right) \geq 0\,,\ \forall s_i, s_i^{'} \in S_i\ ,\ \forall i \in N$$

$$\sum_{s \in S} \sigma\left(s\right) = 1$$

$$\sigma\left(s\right) \geq 0\ , \forall s \in S$$

MSNE

$$\sum_{s_i}\sum_{s_{-i}} \sigma_i^{\star}\left(s_i\right)\sigma_{-i}^{\star}\left(s_{-i}\right)u_i\left(s_i, s_{-i}\right) \geq \sum_{s_{-i}} \sigma_{-i}^{\star}\left(s_{-i}\right)u_i\left(s_i^{'}, s_{-i}\right)\ ,\ \forall s_i^{'} \in S_i\ ,\ \forall i \in N$$

$$\sum_{s_i \in S_i} \sigma_i^{\star}\left(s_i\right) = 1\,,\ \forall i \in N$$

$$\sigma_i^{\star}\left(s_i\right) \geq 0\,,\ \forall s_i \in S_i\ ,\ \forall i \in N$$

- To this end, finding CE can be formulated as a linear optimization problem while finding MSNE is a non-linear optimization problem.

- But what about the **number of variables** and the **number of constraints**?

Relation between MSNE and CE

CE

$$\sum_{s_{-i}} \sigma(s)\left(u_i(s_i, s_{-i}) - u_i\left(s_i^{'}, s_{-i}\right)\right) \geq 0\,,\ \forall s_i, s_i^{'} \in S_i\ ,\ \forall i \in N$$

$$\sum_{s \in S} \sigma(s) = 1$$

$$\sigma(s) \geq 0\ , \forall s \in S$$

MSNE

$$\sum_{s_i}\sum_{s_{-i}} \sigma_i^{\star}(s_i)\,\sigma_{-i}^{\star}(s_{-i})\,u_i(s_i, s_{-i}) \geq \sum_{s_{-i}} \sigma_{-i}^{\star}(s_{-i})\,u_i\left(s_i^{'}, s_{-i}\right)\ ,\ \forall s_i^{'} \in S_i\ ,\ \forall i \in N$$

$$\sum_{s_i \in S_i} \sigma_i^{\star}(s_i) = 1\,,\ \forall i \in N$$

$$\sigma_i^{\star}(s_i) \geq 0\,,\ \forall s_i \in S_i\ ,\ \forall i \in N$$

Number of variables

CE:

- The decision variables $\sigma(s)$ for all $s \in S$.
- So, the number of decision variables is equal to all possible values of s. And the number of possible value of s is,

$$|S| = \prod_{i \in N} |S_i| \qquad \textbf{(E.6.)}$$

Relation between MSNE and CE

CE

$$\sum_{s_{-i}} \sigma(s)\left(u_i(s_i, s_{-i}) - u_i\left(s_i^{'}, s_{-i}\right)\right) \geq 0, \; \forall s_i, s_i^{'} \in S_i \;, \; \forall i \in N$$

$$\sum_{s \in S} \sigma(s) = 1$$

$$\sigma(s) \geq 0 \;, \forall s \in S$$

MSNE

$$\sum_{s_i}\sum_{s_{-i}} \sigma_i^{\star}(s_i)\, \sigma_{-i}^{\star}(s_{-i})\, u_i(s_i, s_{-i}) \geq \sum_{s_{-i}} \sigma_{-i}^{\star}(s_{-i})\, u_i\left(s_i^{'}, s_{-i}\right) \;, \; \forall s_i^{'} \in S_i \;, \; \forall i \in N$$

$$\sum_{s_i \in S_i} \sigma_i^{\star}(s_i) = 1 \;, \; \forall i \in N$$

$$\sigma_i^{\star}(s_i) \geq 0 \;, \; \forall s_i \in S_i \;, \; \forall i \in N$$

Number of variables

MSNE

- The decision variables are $\sigma_i^*(s_i)$ for all $s_i \in S_i$ and all $i \in N$.
- The number of variables due to $\sigma_i^*(s_i)$ for all $s_i \in S_i$ is $|S_i|$. So, to get the total number of decision variables, we have to sum over $|S_i|$ for all $i \in N$. We get,

$$\sum_{i \in N} |S_i| \qquad \text{(E.7.)}$$

- Comparing (E.6.) and (E.7.) we get that MSNE has lesser number of variables.

Relation between MSNE and CE

CE

$$\sum_{s_{-i}} \sigma(s)\left(u_i(s_i, s_{-i}) - u_i\left(s_i^{'}, s_{-i}\right)\right) \geq 0\,,\ \forall s_i, s_i^{'} \in S_i\,,\ \forall i \in N$$

$$\sum_{i \in N} |S_i|^2$$

$$\sum_{s \in S} \sigma(s) = 1 \qquad 1$$

$$\sigma(s) \geq 0\,, \forall s \in S \qquad \prod_{i \in N} |S_i|$$

MSNE

$$\sum_{s_i}\sum_{s_{-i}} \sigma_i^{\star}(s_i)\, \sigma_{-i}^{\star}(s_{-i})\, u_i(s_i, s_{-i}) \geq \sum_{s_{-i}} \sigma_{-i}^{\star}(s_{-i})\, u_i\left(s_i^{'}, s_{-i}\right)\,,\ \forall s_i^{'} \in S_i\,,\ \forall i \in N$$

$$\sum_{i \in N} |S_i|$$

$$\sum_{s_i \in S_i} \sigma_i^{\star}(s_i) = 1\,,\ \forall i \in N \qquad n$$

$$\sigma_i^{\star}(s_i) \geq 0\,,\ \forall s_i \in S_i\,,\ \forall i \in N \qquad \sum_{i \in N} |S_i|$$

Number of constraints

CE

- The number of constraints corresponding to each of the constraints are written in red. So, the total number of constraints is,

$$\sum_{i \in N} |S_i|^2 + 1 + \prod_{i \in N} |S_i|$$

MSNE

- The number of constraints corresponding to each of the constraints are written in red. So, the total number of constraints is,

$$2\sum_{i \in N} |S_i| + n$$

- Which one has more constraints depends on the values of n and $|S_i|$.

Thank You!