Game Theory (CS4187)

Lectures 9 and 10

Date: 03/09/2024
05/09/2024

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NOT COMPLETE!

Broad Idea of Today’s Lecture

- With this lecture we are **starting module 2**.

- The main topics for today’s lectures are:
  1. Dominant strategy.
  2. Iterated removal of strictly dominated strategy.
  3. Common knowledge.

Overview of Module 2

- In module 1 we learned how to model a practical scenarios in a game-theoretic way.
  - **Recap:** A game theoretic setup is characterized by:
    1. Multiple players.
    2. The utility (also called payoff) of a player is effected by the other players of the game.

- After modeling a problem as a game, we have to solve it. In **module 2**, we will learn how to **solve a game**.
  - Solving a game means to **find the "optimal" strategies** of the players.
  - A side objective is also to **analyze these solutions** as well.

Overview of Module 2

Belief vs Learning Based Approach to Solve a Game

To reiterate, in game theory, utility of a player is effected by the **decisions made by other players**. This leads to two different approaches to solve a game:

1. **Belief based approach:** In this approach, a player has a **belief of how other players will make decisions**. Based on this belief, it makes its decision to maximize its payoff.
    a. Different types of solutions of a game.
    b. Computing and analyzing these solutions.

2. **Learning based approach:** In this approach, a player makes its decision **based on the history of observation of other player's decisions** (including its own decision). Off course, the goal of a player is to maximize its payoff.

Overview of Module 2

Belief vs Learning Based Approach to Solve a Game

To reiterate, in game theory, utility of a player is effected by the **decisions made by other players**. This leads to two different approaches to solve a game:

1. Belief based approach: In this approach, a player has a **belief of how other players will make decisions**. Based on this belief, it makes its decision to maximize its payoff.
   a. Different types of solutions of a game. **5 lectures**
   b. Computing and analyzing these solutions. **4 lectures**

8-9 lectures

12 lectures

2. Learning based approach: In this approach, a player makes its decision **based on the history of observation of other player's decisions** (including its own decision). Off course, the goal of a player is to maximize its payoff. **3-4 lectures**

Solution concepts of SFG

- Over the next 3-4 lectures, we will discuss various **types of solutions** (also called **solution concepts**) associated with strategic form games (simultaneous moves). These solutions are:
  - Dominant strategy
  - Iterated removal of strictly dominanted strategy.
  - Pure strategy Nash equilibrium.
  - Mixed strategy Nash equilibrium.
  - Maxmin strategy.
  - Correlated equilibrium (maybe).

Belief based approach

- Only after that we will spend like 1-2 lectures on solution concepts for EFGs:
  - We need solution concepts of SFG to solve EFGs; imperfect information can be interpreted as SFG (discussed in lectures 5-8).

Solution concepts of SFG

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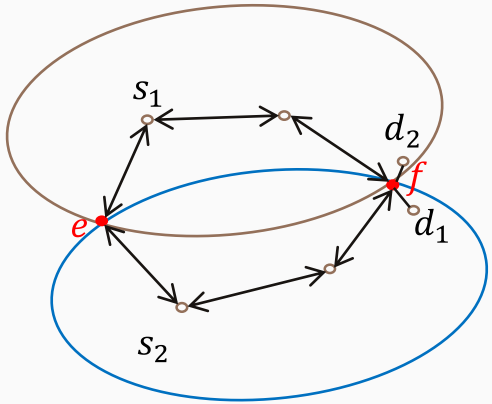
  - Dominant strategy.
  - Iterated removal of strictly dominant strategy.
  - Pure strategy Nash equilibrium.
  - Mixed strategy Nash equilibrium.
  - Maxmin strategy.
  - Correlated equilibrium (maybe).

Belief based approach

- Only after that we will spend like 1-2 lectures on solution concepts for EFGs.
  - We need solution concepts of SFG to solve EFGs; imperfect information can be interpreted as SFG (discussed in lectures 5-6).

So for the next few lectures assume that we are talking about solution concepts of SFGs by default. I will not keep repeating it.

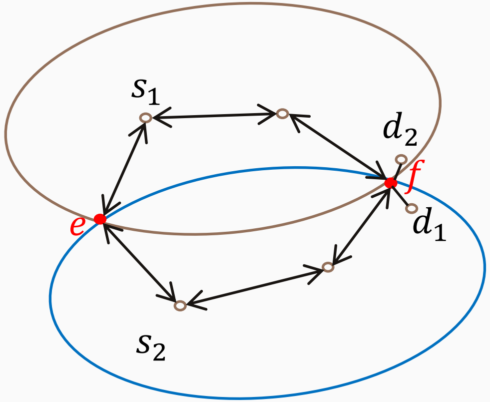
Example: Dominant Strategy



		ISP-2	
		e	f
ISP-1	e	-4, -4	-1, -5
	f	-5, -1	-2, -2

- Consider the ISP routing game that we discussed in lectures 3 and 4. The diagram and the payoff matrix is shown in the figure.

Example: Dominant Strategy

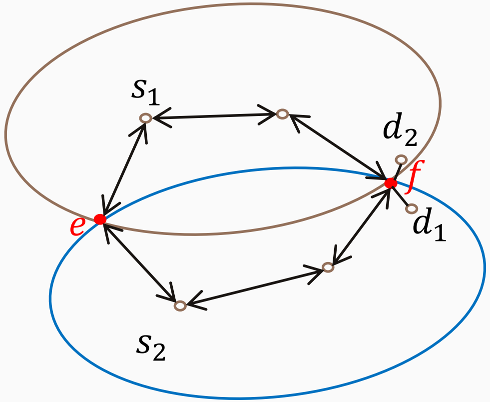


		ISP-2	
		e	f
ISP-1	e	-4	-1
	f	-5	-2

➢ Let's consider the utility/payoff of ISP-1 only.

- We will still consider the strategies of both ISP-1 and ISP-2.
- Just that we will focus on the payoff of the first player (so ignore the second element of the payoff).

Example: Dominant Strategy



		ISP-2	
		e	f
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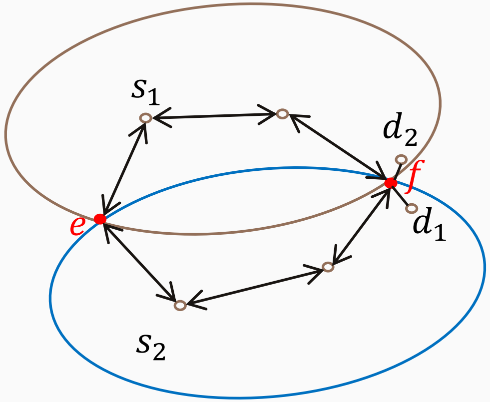
- Consider the ISP-2's decision is e. What decision will ISP-1 take if it is rational?

Example: Dominant Strategy



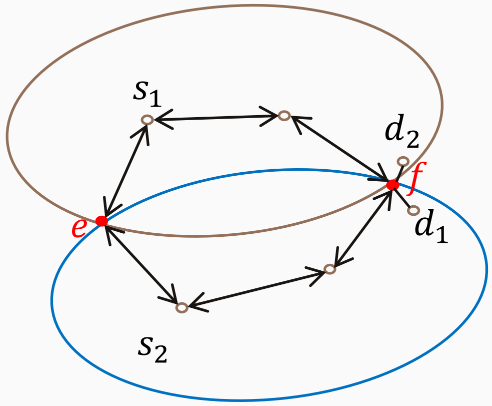
		ISP-2	
		e	f
ISP-1	e	-4	-1
	f	-5	-2

- Let’s consider the utility/payoff of ISP-1 only.
  - We will still consider the strategies of both ISP-1 and ISP-2.
  - Just that we will focus on the payoff of the first player (so ignore the second element of the payoff).

- Consider the ISP-2’s decision is e. What decision will ISP-1 take if it is rational?

Answer: e because $-4 > -5$.

Example: Dominant Strategy



ISP-2

ISP-1	e	f
e	-4	-1
f	-5	-2

- ➢ Let’s consider the utility/payoff of ISP-1 only.
  - We will still consider the strategies of both ISP-1 and ISP-2.
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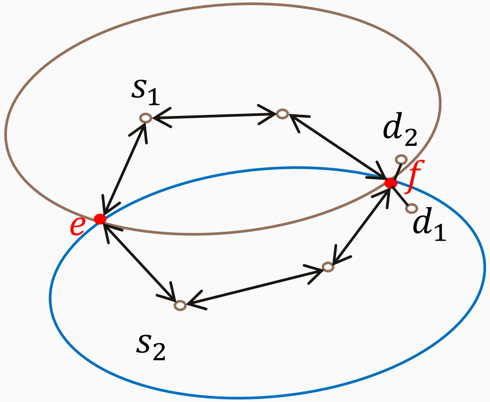
- ➢ Consider the ISP-2’s decision is f. What decision will ISP-1 take if it is rational?

Example: Dominant Strategy



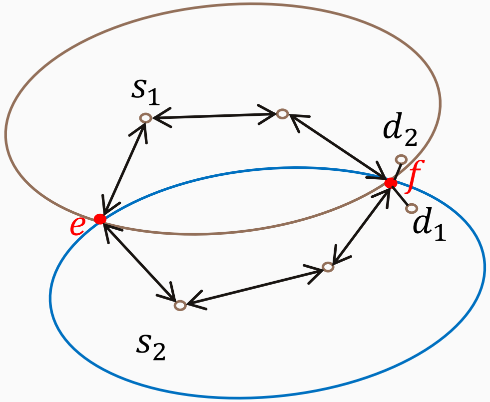
		ISP-2	
		e	f
ISP-1	e	-4	-1
	f	-5	-2

- Let's consider the utility/payoff of ISP-1 only.
  - We will still consider the strategies of both ISP-1 and ISP-2.
  - Just that we will focus on the payoff of the first player (so ignore the second element of the payoff).

- Consider the ISP-2's decision is f. What decision will ISP-1 take if it is rational?

Answer: e because $-1 > -2$.

Example: Dominant Strategy



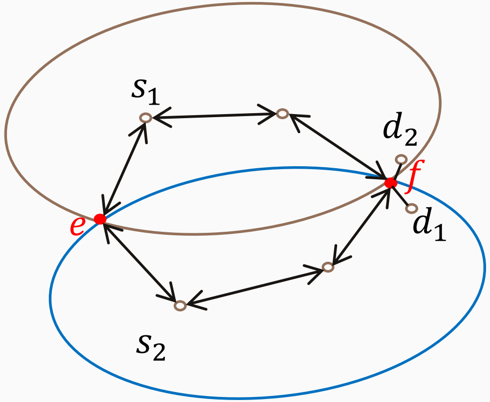
		ISP-2	
		e	f
ISP-1	e	-4	-1
	f	-5	-2

- Let’s consider the utility/payoff of ISP-1 only.
  - We will still consider the strategies of both ISP-1 and ISP-2.
  - Just that we will focus on the payoff of the first player (so ignore the second element of the payoff).

- This essentially means that no matter what ISP-2’s action is, e is the “better” action for ISP-1.

- In other words, for ISP-1, action e **dominates** f.
  - In other words, f is **dominated by** e for ISP-1.
  - e is the dominant strategy for ISP-1.

Example: Dominant Strategy



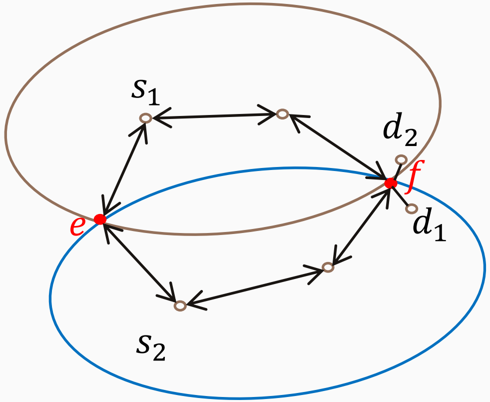
		ISP-2	
		e	f
ISP-1	e	-4	-5
	f	-1	-2

- Let's consider the utility/payoff of ISP-2 only.
- Consider the ISP-1's decision is e. What decision will ISP-2 take if it is rational?

  Answer: e because $-4 > -5$.
- Consider the ISP-1's decision is f. What decision will ISP-2 take if it is rational?

  Answer: e because $-1 > -2$.

Example: Dominant Strategy



		ISP-2	
		e	f
ISP-1	e	-4, -4	-1, -5
	f	-5, -1	-2, -2

DON’T GET USED TO COLOR CODING! IN EXAM, NO SUCH COLOR CODING. ALSO, YOU WILL SEE PAYOFF OF BOTH THE PALYERS SIMULTANOEUSLY WHICH CAN BE CONFUSING. **SO PRACTICE, PRACTICE, PRACTICE!**

Example: IRSDS

		ISP-2	
		e	f
ISP-1	e	-4, -4	-1, -5
	f	-5, -3	-2, -2

IRSDS: Iterated removal of strictly dominant strategy.

- Let’s consider a fictitious game where the payoff of ISP-2 for strategy profile (f, e) is -3 (instead of -1). Everything else remains the same.

Example: IRSDS

		ISP-2	
		e	f
ISP-1	e	-4	-1
	f	-5	-2

- Let’s consider the utility/payoff of ISP-1 only.
- The payoff of ISP-1 is same as the previous example.
- Hence, no matter what ISP-2’s action is, e is the “better” action for ISP-1. In other words:
  - e **dominates** f for ISP-1.
  - f is **dominated by** e for ISP-1.
  - e is the dominant strategy for ISP-1.

Example: IRSDS

		ISP-2	
		e	f
ISP-1	e	-4	-5
	f	-3	-2

- Let's consider the utility/payoff of ISP-2 only.
- Consider the ISP-1's decision is e. What decision will ISP-2 take if it is rational?

  Answer: e because $-4 > -5$.
- Consider the ISP-1's decision is f. What decision will ISP-2 take if it is rational?

  Answer: f because $-2 > -3$.

Example: IRSDS

		ISP-2	
		e	f
ISP-1	e	-4	-5
	f	-3	-2

- Let's consider the utility/payoff of ISP-2 only.
- Consider the ISP-1's decision is e. What decision will ISP-2 take if it is rational?

  Answer: e because $-4 > -5$.

Different

- Consider the ISP-1's decision is f. What decision will ISP-2 take if it is rational?

  Answer: f because $-2 > -3$.

- ISP-2 does not have a dominant strategy.

Example: IRSDS

		ISP-2	
		e	f
ISP-1	e	-4	-5
	f	-3	-2

- Let's consider the utility/payoff of ISP-2 only.
- Consider the ISP-1's decision is e. What decision will ISP-2 take if it is rational?

  Answer: e because $-4 > -5$.
- Consider the ISP-1's decision is f. What decision will ISP-2 take if it is rational?

  Answer: f because $-2 > -3$.
- But one minute! We know that ISP-1 will always take action e no matter what ISP-2 does. So, we can safely ignore this case. If we do so, the action of ISP-2 is going to be e.

Example: IRSDS

		ISP-2	
		e	f
ISP-1	e	-4	-5
	f	-3	-2

- Let's consider the utility/payoff of ISP-2 only.
- So essentially, in this case, ISP-2 had a dominant strategy (which is e), after eliminating the action f of ISP-1 because ISP-1 will never take action f as f is a dominated strategy.
  - So we **removed the dominated action f for ISP-1**. Hence the name, iterated removal of **strictly** dominated strategy. Why strictly?

Example: IRSDS

		ISP-2	
		e	f
ISP-1	e	-4	-5
	f	-3	-2

- Let’s consider the utility/payoff of ISP-2 only.

- **Critical observations:**
  - For IRSDS, ISP-2 could remove the chance that ISP-1 can take action f because **ISP-2 KNOWS that ISP-1 is rational**.
  - For dominant strategy, **we just need ISP-1 and ISP-2 to be rational**; **we don’t need each other to KNOW that the other player is rational**.

Recap: Definition of SFG

Definition: A strategic form game (SFG) Γ is a tuple, $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ where,

1. $\mathrm{N} = \{1, 2, \cdots, \mathrm{n}\}$ is the **set of players**.
2. $\mathrm{S_i}$ is the **set of actions** of the $\mathrm{i^{th}}$ player. The action of the $\mathrm{i^{th}}$ player, denoted by s_i belongs to set $\mathrm{S_i}$, i.e. $s_i \in \mathrm{S_i}$.
3. $\mathrm{u_i} : \mathrm{S_1} \times \mathrm{S_2} \times \cdots \times \mathrm{S_n} \rightarrow \mathbb{R}$ is the utility function of the $\mathrm{i^{th}}$ player. To elaborate, the payoff the $\mathrm{i^{th}}$ player is $u_i(s_1, s_2, \cdots, s_n)$.

- The **outcome** (also called **strategy profile**) is $\mathrm{s} = (s_1, s_2, \cdots, s_n)$, i.e. the decision made by all the players.

Notations

We will start discussing different kind of solutions of **strategic form game (SFG)**. Let's define a few important notations that are required for our discussion.

- s_i denotes the decision/action of the i^{th} player.
  - S_i denotes the **set of actions** of the i^{th} player. We have, $s_i \in S_i$.

- s_{-i} denotes the decision/action of the **all but the i^{th} player**. We have,

$$s_{-i} = (s_1, s_2, \cdots s_{i-1}, s_{i+1}, \cdots, s_n)$$

where n is the total number of players.

  - S_{-i} denotes the **set of actions** of all but the i^{th} player. We have,

$$S_{-i} = S_1 \times S_2 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n$$

Also, $s_{-i} \in S_{-i}$.

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- s_i denotes the decision/action of the i^{th} player.
  - S_i denotes the **set of actions** of the i^{th} player. We have, $s_i \in S_i$.
- $s = (s_1, s_2, \cdots, s_n)$ is strategy of all the players. Also called **outcome** or **strategy profile**. We can also write $\boldsymbol{s = (s_i, s_{-i})}$ or $\boldsymbol{s = (s_{-i}, s_i)}$.
  - S denotes the **set of actions** of all the players. We have, $s \in S$.

$$S = S_1 \times S_2 \times \cdots \times S_n$$

Also, $s \in S$.

Few Definitions

Definition (Strictly Dominated Strategy):

A strategy $s_i \in S_i$ of player i is strictly dominated by another strategy $s_i' \in S_i$ for player i such that for **every** strategy $s_{-i} \in S_{-i}$ of the other players,

$$u_i(s_i, s_{-i}) < u_i(s_i', s_{-i})$$

Point out:

1. Only one strategy s_i' is required.
2. Every strategy of other player.
3. Strict inequality.
4. Strictly or strongly.

Few Definitions

Definition (Weakly Dominated Strategy):

A strategy $s_i \in S_i$ of player i is weakly dominated by another strategy $s_i' \in S_i$ for player i such that for every strategy $s_{-i} \in S_{-i}$ of the other players,

$$u_i(s_i, s_{-i}) \leq u_i(s_i', s_{-i})$$

and there exists at least one strategy $\tilde{s}_{-i} \in S_{-i}$ of other players such that,

$$u_i(s_i, \tilde{s}_{-i}) < u_i(s_i', \tilde{s}_{-i})$$

Point out:

1. The importance of the additional clause.

Few Definitions

Definition 1 (Common Knowledge):

A **“fact”** is common knowledge if:

- all players know the fact.
- all players know that all players know the fact.
- all players know that all players know that all players know the fact.
- … ad infinitu

What is a fact?

1. Rationality of players.
2. Utility of other players (complete information game). Generally, this is an implicit assumption.

Few Definitions

Definition 2 (Common Knowledge):

A fact is common knowledge among the players if for any finite chain of players $i_1, i_2, \cdots, i_k$ the following holds: player i_1 knows that player i_2 knows that player i_3 knows . . . that player i_k knows the fact.

Dominant Strategy

Definition (Strictly Dominant Strategy):

A strategy $s_i \in S_i$ of player i is strictly dominant for player i is s_i strictly dominates all other $s_i' \in S_i \backslash \{s_i\}$.

Definition (Weakly Dominant Strategy):

A strategy $s_i \in S_i$ of player i is weakly dominant for player i is s_i weakly dominates all other $s_i' \in S_i \backslash \{s_i\}$.

Dominant Strategy

Definition (Strictly Dominant Strategy Equilibrium):

A strategy profile $(s_1, s_2, \cdots, s_n)$ is called a strictly dominant strategy equilibrium of the game if for all $i \in N$, s_i is a strongly dominant strategy for player i.

Definition (Weakly Dominant Strategy Equilibrium):

A strategy profile $(s_1, s_2, \cdots, s_n)$ is called a weakly dominant strategy equilibrium of the game if for all $i \in N$, s_i is a weakly dominant strategy for player i.

Dominant Strategy

Second price auctions

IRSDS

- In a game, a player may not always have a dominant strategy. Consequently, a game may not have a dominant strategy equilibrium.

- But the importance of the concept of dominant strategy and it’s allied topic strictly dominated strategy, helps to remove those strategies which a player will not play (given it is rational).

- By removing the strategies that a player will not play leads to a modified game. By removing the

IRSDS

		Player B			
		B_1	B_2	B_3	B_4
Player A	A_1	3, 2	4, 1	2, 3	0, 4
	A_2	4, 4	2, 5	1, 2	0, 4
	A_3	1, 3	3, 1	3, 1	4, 2
	A_4	5, 1	3, 1	2, 3	1, 4

- **Stage 0:** For each player $i \in N$, set $S_i^0 = S_i$.

Example:

s

IRSDS

		Player B			
		B_1	B_2	B_3	B_4
Player A	A_1	3, 2	4, 1	2, 3	0, 4
	A_2	4, 4	2, 5	1, 2	0, 4
	A_3	1, 3	3, 1	3, 1	4, 2
	A_4	5, 1	3, 1	2, 3	1, 4

➢ **Stage 1:** For each player $i \in N$,

- Find the set of **strictly** dominated strategy of player i for the current stage of the game,

$$\bar{S}_i^1 = \{i \in S_i^0 \mid \exists s_i' \in S_i^0 \; s.t.\, u_i(s_i, s_{-i}) < u_i(s_i', s_{-i}) \; \forall s_{-i} \in S_{-i}^0\}$$

- Find the set of strategy of player i that survives this stage of elimination,

$$S_i^1 = S_i^0 \backslash \bar{S}_i^1$$

- Check if there exists a player $i \in N$ for which $S_i^1 \neq S_i^0$. If it exists, then proceed to the next stage. Else, EXIT.

Example:

Common knowledge:

IRSDS

Player B

Player A

	B_1	B_2	B_3	B_4
A_1	3, 2	4, 1	2, 3	0, 4
A_2	4, 4	2, 5	1, 2	0, 4
A_3	1, 3	3, 1	3, 1	4, 2
A_4	5, 1	3, 1	2, 3	1, 4

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  - Find the set of strategy of player i that survives this stage of elimination,

$$S_i^1 = S_i^0 \backslash \bar{S}_i^1$$

  - Check if there exists a player $i \in N$ for which $S_i^1 \neq S_i^0$. If it exists, then proceed to the next stage. Else, EXIT.

- If S_i^1 is a **singleton** set for all $i \in N$, then the game has **strictly dominant strategy equilibrium**.

IRSDS

Player B

		B_1	B_2	B_4
Player A	A_1	3, 2	4, 1	0, 4
	A_3	1, 3	3, 1	4, 2
	A_4	5, 1	3, 1	1, 4

- **Stage 2:** For each player $i \in N$,
  - Find the set of **strictly** dominated strategy of player i for the current stage of the game,

$$\bar{S}_i^2 = \{i \in S_i^1 \mid \exists s_i' \in S_i^1 \; s.t. u_i(s_i, s_{-i}) < u_i(s_i', s_{-i}) \; \forall s_{-i} \in S_{-i}^1\}$$

  - Find the set of strategy of player i that survives stage 2 of elimination,

$$S_i^2 = S_i^1 \backslash \bar{S}_i^2$$

  - Check if there exists a player $i \in N$ for which $S_i^2 \neq S_i^1$. If it exists, then proceed to the next stage. Else, EXIT.

Example:

Common knowledge:

IRSDS

- **Stage $\boldsymbol{k}$:** For each player $i \in N$,
  - Find the set of **strictly** dominated strategy of player i for the current stage of the game,

    $$\bar{S}_i^k = \{i \in S_i^{k-1} \mid \exists s_i' \in S_i^{k-1}\ s.t.\, u_i(s_i, s_{-i}) < u_i(s_i', s_{-i})\ \forall s_{-i} \in S_{-i}^{k-1}\}$$

  - Find the set of strategy of player i that survives stage 2 of elimination,

    $$S_i^k = S_i^{k-1} \backslash \bar{S}_i^k$$

  - Check if there exists a player $i \in N$ for which $S_i^k \neq S_i^{k-1}$. If it exists, then proceed to the next stage. Else, EXIT.

IRSDS

		Player B	
		B_1	B_4
Player A	A_1	3, 2	0, 4
	A_3	1, 3	4, 2
	A_4	5, 1	1, 4

➢ **Stage 3:** For each player $i \in N$,

- Find the set of **strictly** dominated strategy of player i for the current stage of the game,

$$\bar{S}_i^3 = \{i \in S_i^2 \mid \exists s_i' \in S_i^2\ s.t.\ u_i(s_i, s_{-i}) < u_i(s_i', s_{-i})\ \forall s_{-i} \in S_{-i}^2\}$$

- Find the set of strategy of player i that survives stage 2 of elimination,

$$S_i^3 = S_i^2 \backslash \bar{S}_i^3$$

- Check if there exists a player $i \in N$ for which $S_i^3 \neq S_i^2$. If it exists, then proceed to the next stage. Else, EXIT.

Example:

Common knowledge:

IRSDS

	Player B	
Player A	B_1	B_4
A_3	1, 3	4, 2
A_4	5, 1	1, 4

- **Stage 4:** For each player $i \in N$,
  - Find the set of **strictly** dominated strategy of player i for the current stage of the game,

$$\bar{S}_i^4 = \{i \in S_i^3 \mid \exists s_i' \in S_i^3 \ s.t. u_i(s_i, s_{-i}) < u_i(s_i', s_{-i}) \ \forall s_{-i} \in S_{-i}^3\}$$

  - Find the set of strategy of player i that survives stage 2 of elimination,

$$S_i^4 = S_i^3 \backslash \bar{S}_i^4$$

  - Check if there exists a player $i \in N$ for which $S_i^4 \neq S_i^3$. If it exists, then proceed to the next stage. Else, EXIT.

Strategy set that survived IRSDS:

s

IRSDS

- As this example shows, the strategy set that survives IRSDS may not be a **singleton** set.

- If a strategy does not survive IRSDS, a rational player will never play it assuming common knowledge.
  - This also means that a **Nash Equilibrium** cannot include those strategies that does not survive IRSDS.

- If for each player strategy set after IRSDS is a singleton set, it is said to be **“dominance solvable”**.

- Since a game always may not be dominance solvable, **IRSDS is not really a solution concept**; it is more of an elimination strategy.

Thank You!