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Mahindra University Hyderabad

École Centrale School of Engineering

End-semester Examination (2023 Batch), May 2025

Program: B. Tech. **Branch: CM** **Year: II** **Semester: Spring**

Subject: Functional Analysis (MA2212)

Date: 28/05/2025 **Start Time: 10:00 AM**
Time Duration: 3 Hours **Max. Marks: 100**

Instructions:

1. All questions are compulsory.
2. Complete all parts of a question together; do not split them across different sections of the answer sheet.

Q 1: 20 marks

(i) Define a norm. Prove that the function $\|\cdot\|_p : \ell^p \to \mathbb{R}$ defined by

$$\|x\|_p = \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{\frac{1}{p}}, \quad \forall\, x = (x_1, x_2, \dots) \in \ell^p$$

is a norm on ℓ^p for $1 \le p < \infty$. [10 marks]

(ii) Define the Schauder basis in a normed linear space $(X, \|\cdot\|)$. Show that the sequence (e_n), where $e_n = (\delta_{nj})$, is a Schauder basis for $(\ell^2, \|\cdot\|_2)$, where

$$\|x\|_2 = \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2}, \quad \forall\, x = (x_1, x_2, \dots) \in \ell^2.$$

[10 marks]

Q 2: 20 marks

(i) Consider the sequence of functions $f_n(t) := \sqrt{2n+1}\, t^n$ in $C[0,1]$, defined for $t \in [0,1]$ and $n \in \mathbb{N}$. Prove that

$$\|f_n\|_2 = 1 \quad \text{and} \quad \|f_n\|_1 \to 0 \quad \text{as } n \to \infty,$$

where

$$\|f\|_p = \left(\int_0^1 |f(t)|^p\, dt\right)^{1/p} \quad \text{for } 1 \le p < \infty.$$

[10 marks]

P.T.O.

(ii) Define a convex set and a compact set in a normed linear space $(X, \|\cdot\|)$. Show that the *closed unit ball*

$$\bar{B}(0;1) = \{x = (x_1, x_2, \ldots) \in \ell^1 \mid \|x\|_1 \leq 1\}$$

is convex in ℓ^1, but not compact where

$$\|x\|_1 = \sum_{n=1}^{\infty} |x_n|, \quad \forall\, x = (x_1, x_2, \ldots) \in \ell^1.$$

[10 marks]

Q 3: 20 marks

(i) Let $X = [\sqrt{3}, \infty)$ be a metric space with the usual metric $d(x, y) = |x - y|$. Consider the mapping $T : X \to X$ defined by

$$T(x) = \frac{1}{2}\left(x + \frac{3}{x}\right), \quad \text{for all } x \in X.$$

(a) State the Banach fixed point theorem. [2 marks]

(b) Determine whether T is a contraction mapping. [6 marks]

(c) Use the Banach fixed point theorem to determine whether T has a fixed point in X or not. [2 marks]

(ii) Consider the set $CL(X, Y)$ of all continuous (bounded) linear operators from a normed space X to a normed space Y. That is, if $T \in CL(X, Y)$, then there exists a constant $M > 0$ such that

$$\|Tx\|_Y \leq M\|x\|_X \quad \text{for all } x \in X.$$

Then: [10 marks]

(a) Prove that $\|T\| \leq M$, where the operator norm $\|\cdot\|$ is defined by

$$\|T\| := \sup\{\|Tx\|_Y : x \in X,\ \|x\|_X \leq 1\}.$$

(b) Show that

$$\|Tx\|_Y \leq \|T\|\, \|x\|_X \quad \text{for all } x \in X.$$

Q 4: 20 marks

(i) Show that the right shift operator $R : \ell^1 \to \ell^1$, defined by

$$R(a_1, a_2, a_3, \ldots) := (0, a_1, a_2, \ldots), \quad \text{for } (a_n)_{n \in \mathbb{N}} \in \ell^1,$$

is a bounded linear operator on $(\ell^1, \|\cdot\|_1)$. [4 marks]

P.T.O.

(ii) Define a functional $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x) = x_1a_1 + x_2a_2 + x_3a_3,$$

where $a = (a_1, a_2, a_3) \in \mathbb{R}^3$ is a fixed vector and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Show that f is a linear, bounded functional on $(\mathbb{R}^3, \|\cdot\|_2)$, where

$$\|x\|_2 = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}, \quad \forall\, x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Also compute $\|f\|$ for $a = (3, 0, 4)$.

[10 marks]

(iii) Let $P_n[a, b]$ denote the vector space of all real polynomials of degree at most n. Let $x_0, x_1, x_2, \ldots, x_n$ be $(n+1)$ distinct fixed points in the interval $[a, b]$. Define a function $\langle \cdot, \cdot \rangle : P_n[a, b] \times P_n[a, b] \to \mathbb{R}$ by

$$\langle p, q \rangle = \sum_{i=0}^{n} p(x_i)\, q(x_i), \quad \forall\, p, q \in P_n[a, b].$$

Verify that $\langle \cdot, \cdot \rangle$ defines an inner product on $P_n[a, b]$.

[6 marks]

Q 5: 20 marks

(i) State the parallelogram equality, and show that every inner product space satisfies it.

[6 marks]

(ii) What is a Hilbert space? Show that the space $C[a, b]$, equipped with the norm

$$\|x\|_\infty = \max_{t \in [a,b]} |x(t)|,$$

is not an inner product space, and hence not a Hilbert space.

[8 marks]

(iii) Prove that in an inner product space, if $x_n \to x$ and $y_n \to y$, then

$$\langle x_n, y_n \rangle \to \langle x, y \rangle.$$

[6 marks]

— End of Question Paper —