Mahindra University

Mahindra University École Centrale School of Engineering Hyderabad

End Semester Examination

Program: B. Tech./Pre-PhD Branch: open elective

Year: III Semester: spring

Subject: Mathematical Modeling (MA 3231/MA 6012)

Date: 02-06-2025 **Time: 10 a.m - 1 p.m**

Time Duration: 180 Minutes **Max. Marks: 100**

Instructions:

1. There are 5 questions, all of which are compulsory. Use of calculator is allowed.
2. The order of answers should be same as the order of questions.
3. Justify your answer wherever required. Guesswork will not be considered in evaluation.

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1. Explain the difference between discrete and continuous mathematical models in detail. **20 M**
Two competing species follow the discrete nonlinear model:

$$x_{n+1} = x_n(1 + r_1 - ax_n - bx_ny_n),$$
$$y_{n+1} = y_n(1 + r_2 - cx_n - dx_ny_n),$$

where r_1, r_2 are intrinsic growth rates, and a, b, c, d are interaction coefficients. Find the equilibrium points. Analyze their stability of equilibrium points using linear stability analysis for different values of a, b, c, d.

2. Consider the following system of nonlinear ordinary differential equations describing the dynamics of a prey–predator interaction: **20 M**

$$\frac{dx}{dt} = \alpha x\left(1 - \frac{x}{K}\right) - \frac{\beta xy}{1 + hx},$$
$$\frac{dy}{dt} = \delta \cdot \frac{\beta xy}{1 + hx} - \gamma y,$$

where: $x(t)$: population of prey, $y(t)$: population of predators, α: intrinsic growth rate of prey, K: carrying capacity of the prey, β: predation rate, h: handling time (Holling type II response), δ: efficiency of converting consumed prey into predator births, γ: predator death rate.

a) Perform a nondimensionalization of the system by introducing appropriate scaled variables:

$$u = \frac{x}{K}, \quad v = \frac{y}{Y}, \quad \tau = \alpha t,$$

where Y is a suitable characteristic predator population to be determined.

b) Derive the dimensionless system of equations in terms of $u(\tau)$ and $v(\tau)$.

c) Identify the equilibrium points of the dimensionless system.

d) Perform a linear stability analysis by computing the Jacobian matrix evaluated at the equilibrium points.

e) Determine the local stability of the non-trivial equilibrium point (i.e., where both $u \neq 0$ and $v \neq 0$) using the eigenvalues of the Jacobian.

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3. Define a traveling wave solution Consider the nonlinear partial differential equation: **20 M**

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} + u(u-1)(u-2),$$

where $u(x;t)$ represents a scalar field (e.g., velocity), and $K > 0$ is the viscosity coefficient. Assume a traveling wave solution of the form $u(x,t) = U(\xi)$, where $\xi = x - ct$, and c is the wave speed. Substitute this ansatz into the PDE to derive an ordinary differential equation (ODE) for $U(\xi)$. Solve the resulting ODE subject to the boundary conditions:

$$U(\xi) \to 0 \quad \text{as} \quad \xi \to \infty, \quad U(\xi) \to U_0 \quad \text{as} \quad \xi \to -\infty,$$

where $U_0 > 0$ is a constant. Interpret the physical meaning of the solution and the role of K in shaping the wave profile.

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4. a) The Princeton Forest Ecosystem is modeled using three main vegetation types: Hardwood (H), Pine (P), Grassland (G). Based on long-term ecological studies, the annual transition probabilities among these vegetation types are given by the following Markov transition matrix: **20 M**

$$P = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}$$

Each row corresponds to the current state, and each column corresponds to the next year's state. For example, the entry in row 1, column 2 represents the probability that a Hardwood area becomes Pine in one year.

- Draw the corresponding graphical representation of the forest.
- Calculate the normalized eigenvector associated with eigenvalue 1 and give the prediction of growth of the trees. (5)

b) Explain the noise model and random walk model in details. (10)

c) Explain the linear second order difference stochastic equation in detail. (5)

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5. a) Given the matrix **20 M**

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix},$$

perform the Singular Value Decomposition (SVD) of A. (10)

b) Explain the differences between Singular Value Decomposition (SVD) and Dynamic mode decomposition in detail. (10)

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