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Mahindra University Hyderabad

École Centrale School of Engineering

End-semester Regular Examination, June 2024 (2022 Batch)

Program: B. Tech. **Branch: CM** **Year: II** **Semester: Spring**

Subject: Functional Analysis (MA2212)

Date: 03/06/2024 **Start Time: 10:00 AM**

Time Duration: 3 Hours **Max. Marks: 100**

Instructions:

1) Each question carries 20 marks.
2) All questions are compulsory.

Q 1: **20 marks**

(a) Define a convergent and a Cauchy sequence in a metric space (X, d). Let $X = C[0,1]$ be the space of all continuous functions defined on $[0,1]$ with the metric d_1 given by

$$d_1(f,g) = \int_0^1 |f(x) - g(x)|\, dx \quad \text{for all} \quad f, g \in C[0,1].$$

Show that the sequence $(f_n)_{n\geq 2}$ defined as

$$f_n(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{1}{2} - \frac{1}{n}, \\ nt - \frac{n}{2} + 1 & \text{if } \frac{1}{2} - \frac{1}{n} < t \leq \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} < t \leq 1 \end{cases}$$

is Cauchy in $C[0,1]$.

[10 marks]

(b) Define a norm on a real vector space. For any $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, define a function $\|\cdot\|_p$ by

$$\|x\|_p = \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p}.$$

Show that $\|\cdot\|_p$ is a norm on $\mathbb{R}^n$. Also, write the metric induced by $\|\cdot\|_p$. [10 marks]

Q 2: **20 marks**

(a) Define equivalent norms. Prove that the following norms are equivalent on the space $\mathbb{R}^2$:

$$\|x\|_1 = |x_1| + |x_2|, \quad \|x\|_2 = (|x_1|^2 + |x_2|^2)^{1/2}, \quad \|x\|_\infty = \max\{|x_1|, |x_2|\}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$.

[10 marks]

(b) Show that a closed unit ball in an infinite-dimensional space is not necessarily compact. [10 marks]

(OR)

(b) Let $(X, \|\cdot\|)$ be any normed linear space. Show that the convergence of $\|y_1\| + \|y_2\| + \|y_3\| + \ldots$ may not imply the convergence of $y_1 + y_2 + y_3 + \ldots$ in general.

[10 marks]

20 marks

Q 3:

(a) Define a linear and bounded operator. Show that the left shift operator, $L : l^2 \to l^2$, defined as

$$L(a_1, a_2, a_3, \dots) = (a_2, a_3, \dots), \ (a_n)_{n\in\mathbb{N}} \in l^2$$

is linear and bounded with respect to the norm $\|x\|_2 = \left(\sum_{k=1}^{\infty} |x_k|^2\right)^{1/2}$. What is the operator norm of L?

[10 marks]

(b) Show that the functional defined on $C[a, b]$ by

$$f(x) = \int_a^b x(t) y_0(t)\, dt, \quad (y_0 \in C[a, b] \text{ is fixed})$$

is linear and bounded with respect to the norm $\|x\|_\infty = \max_{t\in[a,b]} |x(t)|$. What is the norm of f?

[10 marks]

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20 marks

Q 4:

(a) Let X be a normed space. Write the difference between the algebraic dual (X^*) and the dual (X'). Find the dual basis $\{f_1, f_2, f_3\}$ of the basis $\{(2, 5, 3), (1, 1, 1), (4, -2, 0)\}$ for $\mathbb{R}^3$. [10 marks]

(b) What do you mean by a Hilbert space? Show that the space l^p with $p \neq 2$ is not an inner product space, however, l^2 is an inner product space. [10 marks]

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20 marks

Q 5:

(a) Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space. Prove the Schwarz inequality

$$|\langle x, y\rangle| \le \|x\|\|y\| \quad \text{for all} \quad x, y \in X$$

where the equality sign holds if and only if $\{x, y\}$ is a linearly dependent set. [10 marks]

(b) Define an orthogonal and orthonormal set in an inner product space X. Let $X = C[0, 2\pi]$ be the inner product space of all real-valued continuous functions defined on $[0, 2\pi]$ with the inner product defined by

$$\langle f, g\rangle = \int_0^{2\pi} f(t)g(t)\, dt \quad \text{for all} \quad f, g \in C[0, 2\pi].$$

Prove that the sequences $\left\{\frac{\cos nt}{\sqrt{\pi}}\right\}_{n=1}^{\infty}$ and $\left\{\frac{\sin nt}{\sqrt{\pi}}\right\}_{n=1}^{\infty}$ are orthonormal in $C[0, 2\pi]$. [10 marks]
