Mahindra University Hyderabad

École Centrale School of Engineering

End-semester Regular Examination, June 2023 (2021 Batch)

Program: B. Tech. Branch: CM Year: II Semester: Spring

Subject: Functional Analysis (MA2212)

Date: 13/06/2023 Start Time: 10:00 AM

Time Duration: 3 Hours Max. Marks: 100

Instructions:

1) Each question carries 20 marks.
2) All questions are compulsory.

Q 1: 20 marks

(a) (i) Consider two sets $X = \{1,2,3\}$ with discrete metric and $\mathbb{R}^2$ with Euclidean metric. Define a mapping $f : \{1,2,3\} \to \mathbb{R}^2$ by

$$f(1) = \left(\frac{1}{2}, 0\right),\ f(2) = \left(-\frac{1}{2}, 0\right),\ f(3) = \left(0, -\frac{\sqrt{3}}{2}\right).$$

Show that f is an isometry but not a bijection.

(ii) Define a norm on a real vector space. For any $x = (x_1, x_2, \ldots) \in l^p$ $(1 < p < \infty)$ define a function $||\cdot||_p$ by

$$||x||_p = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}.$$

Show that $||\cdot||_p$ is a norm on l^p. Also write a metric induced by $||\cdot||_p$.

(b) (i) Prove that a metric d induced by a norm $||\cdot||$ on a normed space X satisfies

$$d(x+z, y+z) = d(x,y) \quad \text{and} \quad d(\alpha x, \alpha y) = |\alpha| d(x,y)$$

for all $x, y, z \in X$ and for every scalar $\alpha \in \mathbb{R}$.

(ii) Check whether the following metric defined on s (set of all real sequences) for $x = (x_k) \in s,\ y = (y_k) \in s$

$$d(x,y) = \sum_{k=1}^{\infty} 2^{-k} \frac{|x_k - y_k|}{1 + |x_k - y_k|}$$

is induced by a norm or not?

Q 2: 20 marks

(a) Define convergence of a sequence in a normed space. Show that in a normed space X, vector addition and multiplication by scalars are continuous operations with respect to the norm; that is, the mappings defined by $(x,y) \mapsto x + y$ and $(a, x) \mapsto ax$ are continuous.

(b) (i) Define a convergent series in a normed space. Show that the convergence of $||y_1|| + ||y_2|| + ||y_3|| + \cdots$ may not imply the convergence of $y_1 + y_2 + y_3 + \cdots$.

(ii) Define a Schauder basis in a normed space. Show that (e_n), where $e_n = (\delta_{nj})$, is a Schauder basis for l^p, where $1 \le p < +\infty$.

20 marks

Q 3:

(a) Define equivalent norms. Show that on the space $\mathbb{R}^n$ the following norms

$$||x||_1 = |x_1| + |x_2| + \ldots + |x_n| \quad \text{and} \quad ||x||_2 = (|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2)^{1/2}.$$

are equivalent.

(b) Let $\{x_1, x_2, x_3\}$ be a linearly independent set of vectors in the normed space $\mathbb{R}^3$. Then there is a number $c > 0$ such that for every choice of scalars α_1, α_2, and α_3, we have

$$||\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3||_1 \geq c(|\alpha_1| + |\alpha_2| + |\alpha_3|). \tag{1}$$

What is the largest possible c in (1) if $x_1 = (1, 4, 7)$, $x_2 = (2, 5, 5)$ and $x_3 = (1, 2, 1)$?

(c) Define a linear and bounded operator. Let X be the normed space of all polynomials on $J = [0, 1]$ with norm given $||x|| = \max |x(t)|$, $t \in J$. Show that a differentiation operator T is defined on X by

$$Tx(t) = x'(t)$$

where the prime denotes differentiation with respect to t is linear but not bounded.

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20 marks

Q 4:

(a) Define a functional $f : \mathbb{R}^3 \to \mathbb{R}$ by means of

$$f(x) = x_1 a_1 + x_2 a_2 + x_3 a_3$$

where $a = (a_1, a_2, a_3) \in \mathbb{R}^3$ is fixed vector and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Show that f is a linear, bounded functional and find $||f||$ for $a = (3, 0, 4)$.

(b) Let X be a normed linear space. Write the difference between the algebraic dual (X^*) and the dual (X'). Find the dual basis $\{f_1, f_2, f_3\}$ of the basis $\{(1, 1, 0), (0, 1, 1), (1, 0, 1)\}$ for $\mathbb{R}^3$.

(c) Introduce a canonical mapping $C : X \to X^{**}$ and define an algebraically reflexive space. Show that C is linear.

real vector space

Q 5:

20 marks

(a) Define an inner product on a real vector space. Show the following functionals satisfy the axioms of the inner product (i) $\langle x, y \rangle = \sum_{k=1}^{n} x_k y_k$, where $x = (x_k)$, $y = (y_k) \in \mathbb{R}^n$ (ii) $\langle f, g \rangle = \int_a^b f(t)g(t)\,dt$, where $f,\ g \in C[a, b]$.

(b) (i) State parallelogram equality and show that every inner product space satisfies the parallelogram equality.
(ii) What do you mean by a Hilbert space? Show that the space $C[a, b]$ with norm $||x||_\infty = \max_{t \in [a,b]} |x(t)|$ is not an inner product space, hence not a Hilbert space.
