2-2-M2

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Mahindra University Hyderabad

École Centrale School of Engineering

Minor-II Exam

Program: B. Tech. **Branch: CM** **Year: II** **Semester: Spring**

Subject: Functional Analysis (MA2212)

Date: 16/04/2025 **Start Time: 10:00 AM**

Time Duration: 1.5 Hours **Max. Marks: 25**

Instructions:

1) All questions are compulsory.
2) Complete all parts of a question together; do not split them across different sections of the answer sheet.

Course outcomes (COs)

CO 1: Explain fundamental concepts of metric and normed spaces, including Banach spaces, with relevant examples.

CO 2: Apply the Banach fixed point theorem to solve problems in analysis and understand its significance in applications.

CO 3: Classify and examine bounded linear operators and functionals, and characterize the dual of common Banach spaces.

CO 4: Utilize key results such as the Schwarz inequality, projection theorem, and Riesz representation theorem to analyze Hilbert spaces and related operators.

Q.No.	Questions	Marks	CO	BL	PO	PI Code
1	**(a)** Prove that the function $\|\cdot\|_p : \ell^p \to \mathbb{R}$ defined by $\|x\|_p = \left(\sum_{j=1}^{\infty} \|x_j\|^p\right)^{\frac{1}{p}} \quad \forall\, x = (x_1, x_2, \ldots) \in \ell^p$ is a norm on ℓ^p for $1 \le p < \infty$. **(b)** Show that the sequence $x = \left(1, \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{3}}, \ldots, \sqrt{\frac{1}{n}}, \cdots\right)$ does not belong to ℓ^2, whereas the sequence $x = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots)$ belongs to ℓ^2. **(OR)** **(a)** Prove that a metric d induced by a norm on a normed space $(X, \|\cdot\|)$ satisfies (i) $d(x+\gamma, y+\gamma) = d(x,y)$ and (ii) $d(\alpha x, \alpha y) = \|\alpha\| d(x,y)$ for all $x, y, \gamma \in X$ and every scalar α. **(b)** Show that the space $C[0,1]$ is not complete, hence not Banach, with respect to the norm $\|f\|_1 = \int_0^1 \|f(t)\| dt.$	6	CO1	L3	PO1	1.2.1

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<tr><th>Q.No.</th><th>Questions</th><th>Marks</th><th>CO</th><th>BL</th><th>PO</th><th>PI Code</th></tr>
<tr><td>2</td><td>

(a) Show that the **closed unit ball** in the space ℓ^1:

$$\bar{B}(0;1) = \left\{ x = (x_n)_{n\in\mathbb{N}} \mid \|x\|_1 = \sum_{n=1}^{\infty} |x_n| \le 1 \right\}$$

is not compact.

(b) Define equivalent norms. If two norms $\|\cdot\|$ and $\|\cdot\|_0$ on a vector space X are equivalent, show that

$$\|x_n - x\| \to 0 \iff \|x_n - x\|_0 \to 0$$

as $n \to \infty$.

</td><td>7</td><td>CO1</td><td>L4</td><td>PO1</td><td>1.2.1</td></tr>
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State the **Banach** Fixed Point Theorem. Let $X = [1, \infty)$ be a metric space with the usual metric $d(x, y) = |x - y|$. Consider the mapping $T : X \to X$ defined by

$$T(x) = \frac{10}{11}\left(x + \frac{1}{x}\right) \quad \text{for all } x \in X.$$

Determine whether T is a contraction mapping. Also, determine whether T has a fixed point. If a fixed point exists, determine how many fixed points T has in X, and justify your answer.

</td><td>6</td><td>CO2</td><td>L3</td><td>PO1</td><td>1.2.1</td></tr>
<tr><td>4</td><td>

Consider the linear differential operator $D : C^1[0,1] \to C[0,1]$ defined by

$$(Df)(t) = f'(t), \quad t \in [0,1], \quad f \in C^1[0,1].$$

(a) Show that D is not bounded if both $C^1[0,1]$ and $C[0,1]$ are equipped with the supremum norm $\|f\|_\infty = \sup_{t\in[0,1]} |f(t)|$.

(b) Show that D is bounded if $C^1[0,1]$ is equipped with the norm

$$\|f\|_{1,\infty} := \sup_{t\in[0,1]} |f(t)| + \sup_{t\in[0,1]} |f'(t)|, \quad f \in C^1[0,1],$$

while $C[0,1]$ has the usual supremum norm $\|\cdot\|_\infty$.

(OR)

Define a bounded linear operator. Let $\mathbb{R}^n$ and $\mathbb{R}^m$ be equipped with the Euclidean norm $\|\cdot\|_2$, and let $A \in \mathbb{R}^{m\times n}$ be the matrix

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

Consider the linear operator $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T_A\mathbf{x} = A\mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^n$. Show that

$$\|T_A\mathbf{x}\|_2 \le \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2} \cdot \|\mathbf{x}\|_2.$$

</td><td>6</td><td>CO3</td><td>L4</td><td>PO1</td><td>1.2.1</td></tr>
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P.T.O.