Mahindra University Hyderabad
École Centrale School of Engineering
End Semester Examinations,
December-2024

Program: B. Tech. **Branch:** Computation & Mathematics **Year:** II **Semester:** I
Subject: Algebra (MA 2106)

Date: 19/12/2024 **Start Time:** 10.00 AM
Time Duration: 3 Hours **Max. Marks:** 100

Instructions:

1. All questions are compulsory. The credit for each question is mentioned at the end of the question.
2. Justify your answer wherever required.
3. All the notations are standard as used in class lectures.

1. Find the following: [3×13]

(i) The number of units in the ring $\mathbb{Z}_{16}$.

(ii) The order of the field $\frac{\mathbb{Z}_5[x]}{\langle x^3+x+1 \rangle}$.

(iii) The multiplicative inverse of $3+4i$ in the ring $\mathbb{Z}_7[i]$.

(iv) The order of the factor group $\frac{\mathbb{Z}_7 \times \mathbb{Z}_{35}}{\langle (1,5) \rangle}$.

(v) Inverse of $(\bar{9}, \bar{5})$ in $U(13) \times U(16)$.

(vi) The number of left cosets of $\{(1), (123), (132)\}$ in S_5.

(vii) The number of abelian groups of order 1800.

(viii) The quotient and remainder upon dividing $6x^5 + x^3 + 4x + 7$ by $2x^2 + 5x + 9$ in $\mathbb{Z}_{11}[x]$.

(ix) A generator of $U(14)$, if it is cyclic.

(x) The Kernel of the homomorphism $f : \mathbb{Z}_{28} \mapsto \mathbb{Z}_{35}$ defined by $f([t]) = [5t]$.

(xi) A subgroup H of S_4 such that $|H| = 6$.

(xii) The smallest subgroup of S_4 which contains all the elements of order 3.

(xiii) The number of elements in the ring $\frac{\mathbb{Z}[i]}{\langle 5+i \rangle}$.

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2. (i) Construct a field of order 343. [04]

(ii) Show that $x^5 + 39x^3 + 78x^2 + 507$ is irreducible over $\mathbb{Q}$. [04]

(iii) Show that $x^5 + 7x^2 + 49$ is irreducible over $\mathbb{Q}$. [7]

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3. (i) Let $S = \{a + bi \mid a, b \in \mathbb{Z}, b \text{ is even}\}$. Prove or disprove that S is an ideal of $\mathbb{Z}[i]$. [5]

(ii) Let $\mathbb{F}$ be a field and $p(x) \in \mathbb{F}[x]$. Show that $p(x)$ is irreducible if and only if the ideal $\langle p(x) \rangle$ is maximal. [10]

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4. (i) Show that the multiplicative group of the field $\frac{\mathbb{Z}_3[x]}{\langle x^2+1 \rangle}$ is cyclic. [6]

(ii) Show that every non-zero element in $\mathbb{Z}_7[\sqrt{3}] = \{a + b\sqrt{3} \mid a, b \in \mathbb{Z}_7\}$ has a multiplicative inverse. [10]

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5. Let

$$R = \left\{ \begin{pmatrix} a & 3b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}.$$

(i) Show that R is a subring of $M_2(\mathbb{Z})$. [04]

(ii) Let $\phi : \mathbb{Z}[\sqrt{3}] \to R$ be the map defined by

$$\phi(a + b\sqrt{3}) = \begin{pmatrix} a & 3b \\ b & a \end{pmatrix}.$$

Prove or disprove that ϕ is a ring isomorphism. [11]
