Mahindra University Hyderabad
École Centrale School of Engineering
Minor - I (2021 Batch), March 2023

Program: B. Tech. Branch: CM Year: II Semester: Spring
Subject: Functional Analysis (MA2212)

Date: 10/03/2023 **Start Time: 10:00 AM**
Time Duration: 1:30 Hours Max. Marks: 25

Instructions:

1) All questions are compulsory.

Q 1: 6 marks

(a) For any two positive real numbers α and β prove that

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q},$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

(b) Consider two sets $X = \{1, 2, 3\}$ with discrete metric and $\mathbb{R}^2$ with Euclidean metric. Define a mapping $f : \{1, 2, 3\} \to \mathbb{R}^2$ by

$$f(1) = \left(\frac{1}{2}, 0\right),\ f(2) = \left(-\frac{1}{2}, 0\right)\ f(3) = \left(0, -\frac{\sqrt{3}}{2}\right).$$

(i) Show that f is an isometry. (ii) Is it a homeomorphism? Justify.

Q 2: 6 marks

(a) On the space of all sequences of real numbers (s) define a function $d : s \times s \to \mathbb{R}$ by

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}$$

for any $x = (x_n) \in s$ and $y = (y_n) \in s$. Then prove that d is a metric on s.

(b) Can every metric on a vector space be obtained from a norm? Prove or give a counter-example.

Q 3: **6 marks**

(a) Let l^∞ be the space of all bounded sequences of real numbers with the following two metrics

$$d_\infty(x,y) = \sup_{1\le n<\infty} \{|x_n - y_n|\},$$

$$d_1(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|},$$

where $x = (x_1, x_2, x_3, \ldots,)$, $y = (y_1, y_2, y_3, \ldots)$ are elements of l^∞. Then prove or disprove that d_∞ and d_1 are equivalent.

(b) Find a sequence $x = (x_n)$ which is in l^p with $p > 2$ but $x \notin l^2$.

Q 4: **7 marks**

(a) Let $C[0,1]$ be a space of all continuous functions defined on $[0,1]$. Define a metric d_1 on $C[0,1]$ as follows

$$d_1(f,g) = \int_0^1 |f(x) - g(x)|\, dx$$

for all $f,\ g \in C[0,1]$. Prove that $(C[0,1], d_1)$ is not complete metric space.

(b) Let $X = \mathbb{R}^3$ be the vector space of all ordered triplets $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \ldots$ of real numbers. Define a function $||\cdot||_\infty : X \to \mathbb{R}$ as follows

$$||x||_\infty = \max\{|x_1|, |x_2|, |x_3|\}.$$

Is the function $||\cdot||_\infty$ a norm on X? Justify.
