Mahindra University Hyderabad
École Centrale School of Engineering
Minor II
Program: B. Tech. Branch: CM Year: 2nd Semester: Fall
Subject: Real Analysis (MA2104)

Date: 9/11/2023
Time Duration: 1.5 Hours

Start Time: 10.00 AM
Max. Marks: 30

Instructions:

1. Each question carries 10 marks.
2. All questions are compulsory.
3. Please start each answer on a separate page and make sure to clearly number the responses.
4. Justification is essential wherever asked.

Q 1: 10 marks

(a) Define (i) open sets, (ii) closed sets, (iii) dense sets, (iii) perfect sets, and (v) compact sets in the context of a metric space. Provide examples for each. [5 marks]

(b) Define a metric on a nonempty set X. For $x \in \mathbb{R}$ and $y \in \mathbb{R}$, define

$$d_1(x,y) = (x-y)^2,$$
$$d_2(x,y) = |x^2 - y^2|,$$
$$d_3(x,y) = |x - 2y|,$$

for each of these, determine with justification whether it is a metric or not.

[5 marks]

Q 2: **10 marks**

(a) Let $\mathbb{R}$ be the metric space with usual metric $d(x, y) = |x - y|$. Find interior points, limit points, and boundary points of the following subsets of $\mathbb{R}$

(i) $\mathbb{Q}$,

(ii) $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$,

(iii) $\bigcap_{n=1}^{\infty} [-3, \frac{1}{n}]$,

(iv) $[0, 1] \cup [2, 3]$. [6 marks]

(b) For any collection $\{G_\alpha\}$ of open sets in a metric space, show that $\bigcup_\alpha G_\alpha$ is open.

[2 marks]

(c) The union of infinitely many closed sets in $\mathbb{R}$ need not be closed. Prove or justify with an example. [2 marks]

Q 3: **10 marks**

(a) Define separated and connected sets. Check whether the following sets are separated or not in the metric space $\mathbb{R}$. Justify your answer.

(i) $A = \{x \in \mathbb{R} \, : \, x > 0\}$ and $B = \{x \in \mathbb{R} : x < 0\}$.

(ii) $A = (1, 2]$ and $B = (2, 3)$. [3 marks]

(b) Let $\mathbb{R}$ and $\mathbb{R}^2$ be metric spaces with usual metrics. Which of the following sets are compact? Justify your answer.
(i) $A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \cup \{0\}$, (ii) $A = \{(x, y) \in \mathbb{R}^2 : y = 0\}$. [3 marks]

(c) For every $n \in \mathbb{N}$, define (i) $s_n = \frac{1}{n}$, (ii) $s_n = n^2$, (iii) $s_n = 1 + \frac{(-1)^n}{n}$.
Find the range of the above sequences and determine with justification whether each of these is bounded or not. [4 marks]
