Notes of

Real and Functional Analysis

for the Master in Mathematical Engineering
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Parte I

Introduction

Capitolo 0

Course structure

This course is splitted in two parts:

1. Real Analysis $\rightsquigarrow$ measure and integration theory, in particular:
   - Collections and sequences of sets
   - Measurable space, measure, outer measure
   - Generation of an outer measure
   - Carathéodory's condition, measure induced by an outer measure
   - Lebesgue's measure on $\mathbb{R}^n$
   - Measurable functions
   - The Lebesgue integral
   - Abstract integration
   - Monotone convergence theorem, Fatou's Lemma, Lebesgue's dominated convergence theorem
   - Comparison between the Lebesgue and Riemann integrals
   - Different types of convergence
   - Derivative of a measure and the Radon-Nikodym theorem
   - Product measures and the Fubini-Tonelli theorem
   - Functions of bounded variation and absolutely continuous functions
2. Functional Analysis $\rightsquigarrow$ infinte dimensional linear algebra, in particular:
   - Metric spaces, completeness, separability, compactness
   - Normed spaces and Banach spaces
   - Spaces of integrable functions
   - Linear operators
   - Uniform boundedness theorem, open mapping theorem, closed graph theorem
   - Dual spaces and the Hahn-Banach theorem
   - Reflexivity
   - Weak and weak* convergences
   - Banach-Alaoglu theorem
   - Compact operators
   - Hilbert spaces
   - Projection theorem, Riesz representation theorem
   - Orthonormal basis, abstract Fourier series
   - Spectral theorem for compact symmetric operators

- Fredholm alternativ

The foundation of this theory is the *Set Theory*, that is going to be explained in the next chapter. Enjoy!

NB: this page will be updated with more details and maybe the list of proofs.

Capitolo 1

Set Theory

1.1 Equipotent, finite/infinite, countable/uncountable sets, cardinality of continoum

Let X, Y be sets.

Def — Equipotent sets.
X, Y are equipotent if there exists a bijection $f : X \to Y$ (1-1 injective + onto surjective).

If X, Y are equipotent, then they have the same cardinality. On the other hand, X has cardinality $\geq$ than Y if there exists $f : X \to Y$ onto. For example, for

$$X = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad Y = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

exists $f : X \to Y$ s.t. $\forall y \in Y \; \exists x \in X$ s.t. $f(x) = y$ (f *takes* all the elements of the codomain), but doesn't exist $g : Y \to X$ s.t. $\forall x \in X \; \exists y \in Y$ s.t. $g(y) = x$ (g doesn't *take* all the elements of the codomain).

Def — Finite/infinite sets.
X is finite if it is equipotent to $Y = \{1, 2, ..., k\}$ for some $k \in \mathbb{N}$. X is infinite otherwise.

Prop. X is infinite iff it is equipotent to a proper subset, i.e. if exists a bijection between X and one of his subsets.

For example, between the integers set $\mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$ and the even integers set $\{0, \pm 2, \pm 4, ...\}$ there exists f s.t. $f(z) = 2z$ which is a bijection.

Def — Countable/uncountable (infinite) sets.
X inifinite is countable if it is equipotent to $\mathbb{N}$. It is uncountable otherwise, in which case is more than countable (countable sets are the "smallest" among infinite sets).

Def — Cardinality of continoum.
X has the cardinality of continoum if it is equipotent to $\mathbb{R}$. Any such set is uncountable.

For example:

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are countable
- $\mathbb{R}, \mathbb{R}^N, (0,1), (0,1)^N$ have the cardinality of continoum
- countable unions of countable sets are countable

1.2 Families of subsets

Let X be a set.

Def — Power set.
The power set of X, i.e. the set of all subsets of X, is

$$\mathcal{P}(X) = \{Y : Y \subset X\}$$

It is sometimes denoted as 2^X.

The power set has cardinality strictly bigger than X. For example, $\mathcal{P}(\mathbb{N})$ has the cardinality of continoum.

Def — Family of subsets.
A family, or collection, of subsets of X is just $\mathcal{C} \subset \mathcal{P}(X)$. Tipically, a family of subsets (induced by $I \subset \mathbb{R}$ set of indexes) is $\mathcal{C} = \{E_i\}_{i \in I}$ where $E_i \subset X \;\forall\, i \in I$.

For example, $\{E_1, E_2, E_3\}$ is a family of subsets.

Def — Union and intersection.
Given a family of sets $\{E_i\}_{i \in I} \subset \mathcal{P}(X)$, will often be considered

$$\bigcup_{i \in I} E_i = \{x \in X : \exists\, i \in I \text{ s.t. } x \in E_i\}$$
$$\bigcap_{i \in I} E_i = \{x \in X : x \in E_i \;\forall\, i \in I\}$$

$\{E_i\}$ is said to be (pairwise) disjoint if $E_i \cap E_j = \varnothing \;\forall\, i \neq j$.

Ex — Standard topology of $\mathbb{R}$.
Given $X = \mathbb{R}$ $\left(\text{or } \mathbb{R}^N\right)$, the standard/euclidian topology of $\mathbb{R}$ $\left(\text{or } \mathbb{R}^N\right)$ is $\mathcal{T} = \{E \subset X : E \text{ is open}\}$, i.e. it is the family of all open subsets of X.
More generally, this can be defined in metric spaces (X, d) where X is a set and d a distance between $x, y \in X$.

Some properties of $\mathcal{T}$:

- $\varnothing, X \in \mathcal{T}$
- finite intersection of open sets is open [$\circledast$]
- any (finite/infinite, countable/uncountable, ...) union of open sets is open [$\circledcirc$]

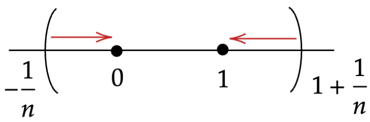
Def — Covering and subcovering.
$\{E_i\}_{i \in I}$ is a covering of X if $X = \bigcup_{i \in I} E_i$. Any subfamily $\{E_i\}_{i \in J, J \subset I}$ is a subcovering if it is a covering.

1.3 Sequences of sets

A sequence is just a family of subsets where $I \equiv \mathbb{N}$, e.g. $\{E_n\}_{n\in\mathbb{N}}$.

Def — Monotone sequences.
$\{E_n\}$ is increasing (not decreasing), $\{E_n\} \nearrow$, if $E_n \subset E_{n+1}\ \forall\, n \in \mathbb{N}$. On the other hand, $\{E_n\}$ is decreasing (not increasing), $\{E_n\} \searrow$, if $E_{n+1} \subset E_n\ \forall\, n \in \mathbb{N}$. If $\{E_n\}$ is increasing/decreasing then it is monotone.

For example, given $X = \mathbb{R}$ and $E_n = \left(-\dfrac{1}{n}, 1 + \dfrac{1}{n}\right)$ for $n \geq 1$, we can say that E_n is a monotone decreasing sequence:



But what is $\bigcap_{n=1}^{\infty} E_n$? We know that

$$\bigcap_{n=1}^{\infty} E_n = [0, 1]$$

and this is an infinite intersection of open sets (this does not disagree with the prop ⊛). This type of intersection is called "Gδ-set": a countable intersection of open sets.

Similarly, $E_n = \left[a + \dfrac{1}{n}, b - \dfrac{1}{n}\right]$, a<b, is increasing and

$$\bigcup_{n=1}^{\infty} E_n = (a, b)$$

is called "Fσ-set": a countable union of closed sets (doesn't disagree with ⊚).

Def — lim sup and lim inf.
Let $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{P}$. We define

$$\limsup_n E_n := \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} E_k\right) \qquad \liminf_n E_n := \bigcup_{n=1}^{\infty}\left(\bigcap_{k=n}^{\infty} E_k\right)$$

If these two sets are equal

$$\limsup_n E_n = \liminf_n E_n = \lim_n E_n = F$$

then F is the limit of the succession.

Take note that $\{E_n\} \nearrow (\searrow) \implies \exists\ \lim_n E_n = \bigcup_n E_n\ (\bigcap_n E_n)$.

Parte II

Real Analysis

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Parte III

Functional Analysis

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