Notes of

Real and Functional Analysis

for the Master in Mathematical Engineering
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Indice

Parte I

Introduction

Capitolo 0

Course structure

This course is splitted in two parts:

1. Real Analysis $\rightsquigarrow$ measure and integration theory, in particular:
   - Collections and sequences of sets
   - Measurable space, measure, outer measure
   - Generation of an outer measure
   - Carathéodory's condition, measure induced by an outer measure
   - Lebesgue's measure on $\mathbb{R}^n$
   - Measurable functions
   - The Lebesgue integral
   - Abstract integration
   - Monotone convergence theorem, Fatou's Lemma, Lebesgue's dominated convergence theorem
   - Comparison between the Lebesgue and Riemann integrals
   - Different types of convergence
   - Derivative of a measure and the Radon-Nikodym theorem
   - Product measures and the Fubini-Tonelli theorem
   - Functions of bounded variation and absolutely continuous functions
2. Functional Analysis $\rightsquigarrow$ infinte dimensional linear algebra, in particular:
   - Metric spaces, completeness, separability, compactness
   - Normed spaces and Banach spaces
   - Spaces of integrable functions
   - Linear operators
   - Uniform boundedness theorem, open mapping theorem, closed graph theorem
   - Dual spaces and the Hahn-Banach theorem
   - Reflexivity
   - Weak and weak* convergences
   - Banach-Alaoglu theorem
   - Compact operators
   - Hilbert spaces
   - Projection theorem, Riesz representation theorem
   - Orthonormal basis, abstract Fourier series

- Spectral theorem for compact symmetric operators
- Fredholm alternativ

The foundation of this theory is the *Set Theory*, that is going to be explained in the next chapter. Enjoy!

NB: this page will be updated with more details and maybe the list of proofs.

Capitolo 1

Set Theory

Equipotent, finite/infinite, countable/uncountable sets, cardinality of continoum

Let X, Y be sets.

Def — Equipotent sets.
X, Y are equipotent if there exists a bijection $f : X \to Y$ (1-1 injective + onto surjective).

If X, Y are equipotent, then they have the same cardinality. On the other hand, X has cardinality $\geq$ than Y if there exists $f : X \to Y$ onto. For example, for

$$X = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad Y = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

exists $f : X \to Y$ s.t. $\forall\, y \in Y\ \exists\, x \in X$ s.t. $f(x) = y$ (f *takes* all the elements of the codomain), but doesn't exist $g : Y \to X$ s.t. $\forall\, x \in X\ \exists\, y \in Y$ s.t. $g(y) = x$ (g doesn't *take* all the elements of the codomain).

Def — Finite/infinite sets.
X is finite if it is equipotent to $Y = \{1, 2, ..., k\}$ for some $k \in \mathbb{N}$. X is infinite otherwise.

Prop. X is infinite iff it is equipotent to a proper subset, i.e. if exists a bijection between X and one of his subsets.

For example, between the integers set $\mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$ and the even integers set $\{0, \pm 2, \pm 4, ...\}$ there exists f s.t. $f(z) = 2z$ which is a bijection.

Def — Countable/uncountable (infinite) sets.
X inifinite is countable if it is equipotent to $\mathbb{N}$. It is uncountable otherwise, in which case is more than countable (countable sets are the `smallest` among infinite sets).

Def — Cardinality of continoum.
X has the cardinality of continoum if it is equipotent to $\mathbb{R}$. Any such set is uncountable.

For example:

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are countable
- $\mathbb{R}, \mathbb{R}^N, (0, 1), (0, 1)^N$ have the cardinality of continoum
- countable unions of countable sets are countable

Parte II

Real Analysis

Capitolo 2

Capitolo 3

Capitolo 4

Capitolo 5

Capitolo 6

Capitolo 7

Capitolo 8

Capitolo 9

Capitolo 10

Parte III

Functional Analysis

Capitolo 11

Capitolo 12

Capitolo 13

Capitolo 14

Capitolo 15

Capitolo 16

Capitolo 17

Capitolo 18

Parte IV

Esercitazioni

Capitolo 19

Capitolo 20

Capitolo 21

Capitolo 22

Capitolo 23

Capitolo 24

Capitolo 25

Capitolo 26

Capitolo 27

Capitolo 28