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Abel integral equation in the reconstruction of the helical magnetic field structure by the polarized synchrotron radiation of AGN jet.

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In honorable memory
of Nikolai Nikolayevich Nefedov

Abstract— This paper is devoted to the mathematical side of the inverse problem of reconstructing the magnetic structure in AGN jets from data on the rotation measure for polarization angle of synchrotron emission from relativistic electrons. Based on the azimuthal symmetry of the helical field, we show how the reconstruction problem reduces to the integral Abel equation. By giving a solution to this equation, we formulate the constraints on the magnetic field components and thermal electron density under which the problem has a single and stable solution. These results may be useful to astrophysicists and specialists in integral equations and inverse problems.

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1. INTRODUCTION

One of the important components of N.N. Nefedov's multifaceted scientific activity was the development of the connection between the theory of small parameters and methods for solving inverse problems of mathematical physics. In this area, there are many problems, important for applications that require serious mathematical development. In this paper, we consider one such problem which arises in the study of the magnetism of celestial bodies, especially galaxies. This topic always attracted the keen attention of Nikolai Nikolaevich.

Investigating magnetic fields of all celestial bodies (except the Earth) we somehow arrive to an inverse problem, since we are forced to judge the corresponding magnetic field from the observations of an outside observer. This is particularly acute when dealing with magnetic fields of galaxies. In fact, observations of the magnetic fields of stars primarily make use of the Zeeman effect, in which the splitting of spectral lines occurs in the same region where the radiation of interest is produced. For galaxies, the Zeeman splitting of spectral lines is too weak compared to their broadening caused by random motions of the medium, so one has to use another magneto-optical effect – the Faraday effect of the polarization plane rotation in the propagation of an electromagnetic wave in a mirror-asymmetric, e.g. magnetized, medium. Originally this phenomenon was discovered by Faraday for sugar solution and used in the so-called saccharimetry, and at the beginning of the development of radio astronomy it was proposed by V.L. Ginzburg for the study of magnetic fields in extragalactic astronomy (e.g. [1]). This method provided the most important radio data available to modern science on the structure of magnetic fields in galaxies (e.g. [2]).

The Faraday rotation is recruited in all parts of the beam from the place of formation of the radio emission to the observer, so naturally the inverse problem arises for some cumbersome integral equation, which, however, in practically important cases manages to reduce to the Abel integral equation. As is well known, the Abel equation is one of the rare integral equations for which substantial progress can be made by analytical methods. However, in order to couple these analytical approaches with the available observational data, which are initially in the form of some discrete table, it is necessary to determine precisely the conditions under which these methods lead to a stable solution. This is the focus of the present work.

2. DIRECT PROBLEM

The Faraday effect arises from the difference in the phase velocities for right- and left-polarized radio waves, u_+ and u_-, which leads to a rotation of the polarization plane by an angle

$$d\psi = \frac{1}{2}\omega(u_+^{-1} - u_-^{-1})dr.$$

Summarizing it along the observational beam and substituting expressions for velocities, see, e.g., [2], we obtain the polarization rotation proportional to the square of the wavelength λ of the synchrotron radiation from relativistic electrons:

$$\Delta\psi = \psi(r) - \psi_0(r) = R(r)\lambda^2. \tag{1}$$

This angle, which increases as the radiation propagates from a point source located at a point r inside the magnetically active region to a point r_0 at its exit, is defined by the integral of the product of the longitudinal magnetic field $\mathbf{B}$ and thermal electron density n along the entire path, see, e.g., [3], given by

$$R(r) = -K\int_{r_0}^{r} n\mathbf{B}d\mathbf{r}, \quad \text{where in CGS system we use the constant } \ K = e^3/2\pi m^2 c^4. \tag{2}$$

On the mathematical side of the question, we are considering only the case of the thin beam, whereas in the applied case it would be necessary to take into account the resolution power, which would complicate the problem in some way by transforming one-dimensional integrals into three-dimensional ones.

Now, assuming that the radiation sources are continuously distributed along the observation beam passing through the magnetically active region, and setting the intrinsic radiation intensity as $\varepsilon(r)$, we calculate the final polarization by integration of the relative complex polarizations from each point along the beam:

$$P/P_0 = \int_{r_0}^{r} \varepsilon(r)\exp\left(2i\psi(r)\right)dr \Bigg/ \int_{r_0}^{r}\varepsilon(r)dr. \tag{3}$$

Considering a cylindrical beam, commonly called an AGN jet in astronomy, introduce, for convenience, an orthogonal Cartesian coordinate system, fixing it to the beam so that the axis Oz coincides with the axis of the cylindrical structure. Inside, we define a symmetric, usually called helical, magnetic field with azimuthal B_φ and longitudinal B_z components in the form of

$$\mathbf{B} = (B_\varphi \sin\varphi, -B_\varphi\cos\varphi, -B_z). \tag{4}$$

Assuming azimuthal symmetry, consider the magnetic field components that depend only on the distance to the jet center $\rho = (x^2+y^2)^{1/2}$. Let us fix the target distance x by directing the observation beam parallel to the plane Oyz at an angle θ to the Oz axis. Note that all field components are now expressed by the single variable y, since the azimuthal angle $\varphi(y)$, is also uniquely determined by the y coordinate as $\cos\varphi = x/(x^2+y^2)^{1/2}$. Introduce the radius of the jet as ρ_0 and restrict the symmetric region of values for y to the segment $[-l,l]$, where $l = (\rho_0^2 - x^2)^{1/2}$.

Calculating the integrals in the expression (3), we pass from the integration along the observation ray dr to the integration along the axis Oy. The variable replacement is made by the relation $y = r\sin\theta$, from which it follows that $d\mathbf{r} = (0, dy, dy\,\mathrm{ctg}\theta)$, and the integrals in the complex polarization (3) can be written as follows

$$P/P_0 = \int_{-l}^{l}\varepsilon(y)\exp\left(2i\left(\psi_0(y) + R(y)\lambda^2\right)\right)dy \Bigg/ \int_{-l}^{l}\varepsilon(y)dy. \tag{5}$$

Simplify the expression by assuming parity of the function $\varepsilon(y)$ and reduce it to an integration over the segment $[0,l]$. To do this, we divide the integral into two parts and replace y *by* $-y$ in one of them, thus obtaining

$$P/P_0 = \int_0^l \varepsilon(y)\left(\exp(2i(\psi_0(y)+R(y)\lambda^2)) + \exp(2i(\psi_0(-y)+R(-y)\lambda^2))\right)dy \Bigg/ \int_0^l 2\varepsilon(y)dy. \tag{6}$$

Rewrite the functions $R(y)$ and $R(-y)$ in a more convenient form, taking into account the parity of the integrand expression $-\mathbf{B}d\mathbf{r} = (B_\varphi\cos\varphi + B_z\mathrm{ctg}\theta)dy$:

$$R(y) = \phi(l) + \phi(y) \quad \text{and} \quad R(-y) = \phi(l) - \phi(y).$$

Here, following the classic idea of, see, e.g., [3], we have introduced a function called the Faraday depth:

$$\phi(y) = K\int_0^y (nB_\varphi \cos\varphi + nB_z \mathrm{ctg}\theta)\, dy. \tag{7}$$

Finally, we notice that the initial polarization angle $\psi_0(y)$, when defined as the angle between the fixed direction $\mathbf{N} = (1,0,0)$ and the oscillation direction of the electric induction vector $\mathbf{E} \sim [\mathbf{B}, d\mathbf{r}]$, is an odd function: $\psi_0(-y) = -\psi_0(y)$. This oddity is a direct consequence of the fact that the direction of the magnetic field changes sign when passing through $y = 0$, and of the fact that the cosine of the initial angle is an even function, so that

$$\cos\psi_0 = \frac{B_z \sin\theta - B_\varphi \cos\varphi\cos\theta}{((B_\varphi \sin\varphi)^2 + (B_z\sin\theta - B_\varphi\cos\varphi\cos\theta)^2)^{1/2}}.$$

Substitution of the derived expressions into the integral (6) after simple arithmetic transformations gives us the following expression

$$P/P_0 = \int_0^l \varepsilon(y)\cos\left(2\psi_0(y) + 2\phi(y)\lambda^2\right) dy \exp\left(2i\phi(l)\lambda^2\right) \Big/ \int_0^l \varepsilon(y) dy,$$

written in the algebraic form of a complex number. Now, using in traditional way the expression $P/P_0 = \Pi\exp(2i\Psi)$, we rewrite the half of the polarization argument Ψ and the polarization modulus Π in the form

$$\Psi = \phi(l)\lambda^2 \quad \text{and} \quad \Pi = \int_0^l \frac{\varepsilon(y)}{\int_0^l \varepsilon(y)dy}\cos\left(2\psi_0(y) + 2\phi(y)\lambda^2\right) dy. \tag{8}$$

These formulas determine the angle and degree of polarization that can hypothetically be obtained from observations of AGN jets with a helical magnetic field at different wavelengths λ. These formulas can also be used to formulate the inverse problem of reconstructing the helical magnetic structure from the synchrotron polarization data.

3. INVERSE PROBLEM

Let's consider the inverse problem of reconstructing the helical field profile in the AGN jet from the complex polarization data. Noting the direct proportionality between the polarization angle Ψ and the square of the wavelength λ^2, we can introduce the proportionality coefficient $RM = \Psi/\lambda^2$, traditionally called the rotation measure. Obviously, the calculation of this coefficient requires a measurement of the polarization at a single wavelength (we recall here that the polarization angle is measured from the axis of the jet). This fact distinguishes the geometry of the jet from the geometry of a flat layer, where such a calculation would require at least observations at several wavelengths. Increasing the number of measurements at different wavelengths in turn increases the accuracy of the RM calculation. Assuming a sufficient spectral resolution of the radio telescope, we know *a priori* the dependence of this coefficient on the target distance $RM(x)$. Thus, the problem is to recover the internal structure of the magnetic helicoidal field using the known dependence of $RM(x)$, as well as the known geometry of the problem (the angle of inclination of the jet to the observation beam θ and the radius of the jet ρ_0).

Rewrite the expression for the profile of the Faraday rotation measure with the substitution of $\cos\varphi$:

$$RM(x) = K\int_0^l \left(\frac{xnB_\varphi}{(x^2+y^2)^{1/2}} + nB_z\mathrm{ctg}\theta\right) dy \tag{9}$$

and note that only the longitudinal component of the magnetic field is responsible for the even part of the $RM(x)$ profile, while the azimuthal component determines the odd part of $RM(x)$ relative to the jet center. Let us introduce the even and odd parts of the Faraday rotation measure as

$$RM_e(x) = (RM(x) + RM(-x))/2 \quad \text{and} \quad RM_o(x) = (RM(x) - RM(-x))/2 \tag{10}$$

and associate them with the field components

$$RM_e(x) = K\mathrm{ctg}\theta\int_0^l nB_z\, dy \quad \text{and} \quad RM_o(x) = K\int_0^l \frac{xnB_\varphi\, dy}{(x^2+y^2)^{1/2}}. \tag{11}$$

Remembering that the integrand expressions depend only on $\rho = (x^2 + y^2)^{1/2}$, and after replacing the integration variable $y \to \rho$, rewrite these expressions in the form of the Volterra integral equation:

$$\frac{RM_e(x)}{K\mathrm{ctg}\theta} = \int_x^{\rho_0} \frac{\rho n B_z(\rho)}{(\rho^2 - x^2)^{1/2}} d\rho \quad \text{and} \quad \frac{RM_o(x)}{Kx} = \int_x^{\rho_0} \frac{n B_\varphi(\rho)}{(\rho^2 - x^2)^{1/2}} d\rho. \tag{12}$$

This equation is one of the possible forms of the well known Abel equation, [4], however better known form can be obtained after substitutions: $t = 1 - (x/\rho_0)^2$ and $s = 1 - (\rho/\rho_0)^2$, then we get

$$\frac{2RM_e(t)}{\rho_0 K\mathrm{ctg}\theta} = \int_0^t \frac{nB_z(s)}{(t-s)^{1/2}} ds \quad \text{and} \quad \frac{2RM_o(t)}{\rho_0 K(1-t)^{1/2}} = \int_0^t \frac{nB_\varphi(s)(1-s)^{-1/2}}{(t-s)^{1/2}} ds. \tag{13}$$

One of the amazing properties of the Abel equation is that the corresponding integral operator can be inverted so

$$g(t) = \int_0^t \frac{f(s)}{(t-s)^{1/2}} ds \quad \text{can be solved by the inversion} \quad f(s) = \frac{1}{\pi}\frac{d}{ds}\int_0^s \frac{g(t)}{(s-t)^{1/2}} dt. \tag{14}$$

Returning to the old variables and notations, we obtain the complete solution of the inverse reconstruction problem in the form of two integrals

$$nB_z(\rho) = \frac{-2}{\pi K\mathrm{ctg}\theta}\frac{d}{\rho d\rho}\int_\rho^{\rho_0} \frac{RM_e(x)}{(x^2-\rho^2)^{1/2}} x dx = \frac{-2}{\pi K\mathrm{ctg}\theta}\int_\rho^{\rho_0} \frac{\partial_x RM_e(x)}{(x^2-\rho^2)^{1/2}} dx, \tag{15}$$

$$nB_\varphi(\rho) = \frac{-2}{\pi K}\frac{d}{d\rho}\int_\rho^{\rho_0} \frac{RM_o(x)}{(x^2-\rho^2)^{1/2}} dx = \frac{-2\rho}{\pi K}\int_\rho^{\rho_0} \frac{\partial_x (RM_o(x)/x)}{(x^2-\rho^2)^{1/2}} dx. \tag{16}$$

Obtained solutions look very promising for use in astrophysical applications, however, it should be noted that the reconstruction inverse problem has many specific details that are not obvious at first sight. Exactly therefore, this point should be emphasized.

Equation (14) was obtained by Abel in 1823, finding the curve along which a frictionless sliding particle will reach the lowest position in the same time regardless of its initial position. Subsequently, it turned out that equations of this type lead to problems from very different fields of physics: optics, seismology, plasma physics, gas dynamics, and astrophysics, as we see in this paper. So what is the problem with this equation and why it has been studied so extensively, both from a theoretical point of view, e.g. existence, uniqueness and stability of solutions, see, e.g., [5, 6], and from an applied point of view, e.g. convergence of numerical methods [7]? The main difficult moment of the Abel equation is its ill-posed nature. This is reflected in the fact that the equation may simply not have a solution in the desired function space, or this solution may be unstable with respect to the perturbation of the function on the left-hand side (in our case it is the measurement of the profile of the Faraday rotation measure). For this purpose, a lot of numerical methods are developed, most of which using the formulas of inversion involve approximation of the measured functions by smooth polynomials, splines or piecewise polynomial functions [8, 9, 13]. Another approaches imply solving the problem with the help of iterative methods for integral equations with Tikhonov regularization [11]. Our purpose is not to list all previous results of Abel equation investigations, but focus on two questions: existance of solution and its stability.

Talking about the solvability of Abel equation, it is necessary, at first, to denote the functional class of solutions, in our case, of magnetic filed profiles and rotation measure functions. Thus it is easy to show, for example, that the Abel transformation is a continuous transformation acting from $L^2(0,1)$ to $L^2(0,1)$. However, it is simultaneously possible to check by formula (12) that the smooth constant profile of the even part of the Faraday rotation measure $RM_e = \mathrm{const}$ could lead to a singularity in the longitudinal component of the magnetic field $B_z \sim s^{-1/2}$ or $B_z \sim (\rho_0^2 - \rho^2)^{-1/2}$. In other words, for a continuous profile of the Faraday measure there may not exist a continuous profile of the magnetic field. There is the classical result of solvability obtained in 1928 year by Tonelli [14]. Using this result, we can formulate the following theorem.

Theorem 1. *If the function $RM_e(t)$, where $t = 1 - (x/\rho_0)^2$, is absolutely continuous on $t \in [0,1]$ (for every $\varepsilon > 0$ there is $\delta > 0$ such that whenever a finite sequence of pairwise disjoint sub-intervals $[t_k^{'}, t_k^{''}]$ satisfy $\sum |t_k^{''} - t_k^{'}| < \delta$ the corresponding sum $\sum |RM_e(t_k^{''}) - RM_e(t_k^{'})| < \varepsilon$), then there exists the unique solution of the inverse problem (12) for the component $nB_z \in L^1(0,1)$ defined by formula (15).*

In other words, we cannot claim that the reconstructed magnetic field profiles will be continuous, but we can claim that they will at least be integrable. Moreover, the existance result in the space $L^1(0,1)$ can be strengthened by the requirement not of absolute continuity, but simply of bounded variation and continuity from the right ($t = 1$ or $x = 0$) of the Faraday rotation measure function $RM_e(t)$, see [5]. The similar result will be true for the azimuthal component nB_φ, but the requirement for absolute continuity will be imposed on odd part of Faraday rotation function $RM_o(t)/(1-t)^{1/2}$ or on $RM_o(x)/x$.

The sustainability problem is also easy to demonstrate with a simple example. Just consider, e.g., a sequence of Faraday rotation measures $RM_{e,n}(t) = \sin(\pi n t)/n^{1/2}$, which converges to zero by norm in $C[0,1]$ or $L^{1,2}[0,1]$. Using formula (15) get an expression for the magnetic field component $nB_z(s)$, which (it can be easily calculated, see, e.g., [15]) does not converge to zero by the norm either in $C[0,1]$ or in $L^{1,2}[0,1]$. That clearly means the discontinuity of the inverse Abel integral operator – which is easily explained by compactness of the Abel operator – therefore, a little noise in the Faraday rotation measures can lead to the large errors in the reconstructed magnetic fields. To circumvent this problem, some a priori information about the magnetic field is needed to distinguish a compact in $L^2(0,1)$ space. For example, one of the classical results obtained in the paper [16] can be reformulated for our problem as follows:

Theorem 2. *Let the function $nB_z(s)$ be a continuous, bounded and non-increasing solution of the inverse problem (12) with the exact left-hand side $RM_e(t)$. Furthermore, the function $nB_z^n(s)$ be an approximate solution of the problem with the noise rotation measure $RM_e^\delta(t)$:*

$$||RM_e(t) - RM_e^\delta(t)||_{L^2} < \delta_n, \quad \textit{and} \quad \delta_n \to 0 \;\; \textit{for } n \to \infty.$$

Then at each segment $[t^{'}, t^{''}] \subset [0,1]$ the approximate solution $nB_z^n(s)$ converges uniformly to the exact solution $nB_z(s)$ for $n \to \infty$, i.e. the reconstruction problem is stable.

The same result is true for the azimuthal component of the magnetic field, but at the same time the continuity and monotonicity requirements are imposed on the function $nB_\varphi(s)/(1-s)^{1/2}$. The result about uniform convergence can be extended to the whole segment $s \in [0,1]$, but there appears an additional condition that at the edges of the segment the value of the approximate Faraday measure coincides with the value of the exact one. Moreover, as shown in the paper, the a priori information about the considered magnetic field profiles functional space can be not necessarily continuous and monotonicity, but also convexity or monotonic convexity – it is only necessary to select some compact in the corresponding L^2 space.

4. DISCUSSION AND CONCLUSIONS

We formulate conditions which enable to give a unique and stable inversion of reconstruction for magnetic field in galactic jets. The results can be compared with corresponding results for galactic discs, [17], lo learn that the disc geometry is more friendly for such investigations rather the disc one. The requirements obtained like continuity, monotonic behavior etc are to some extent some sort of a physical idealization, however, looks quite reasonable and in our opinion can be considered as a promising physical model.

Practical application of the method suggested obviously requires some development of observational methods. Current progress in the extragalactic radioastronomy looks promising in this respect, e.g., [18].

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CONFLICTS OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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