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摘要

本文旨在讲述伯克科维奇空间和复动力系统退化之间的联系。我们将会解释两个例子:

(1) 有理函数双曲连通分支的两种性质的等价性 (嵌套朱利亚集与有界退化序列);

(2) 退化有理函数族的尺度变换极限。

我们通过伯克科维奇直线的动力系统 (非阿动力系统) 获得复动力系统上的性质。

关键词: 伯克科维奇空间, 嵌套朱利亚集, 有界退化序列, 尺度变换极限

Degeneration of complex dynamics

Zhou Qi (Mathematics and applied mathematics)
Directed by Prof. Xie Junyi

ABSTRACT

This thesis aims to illustrate some links between Berkovich spaces and degeneration of complex dynamics. We will illustrate examples:

(1) equivalence of two properties (nested Julia set and bounded escape sequence) of hyperbolic components of rational maps;

(2)rescaling limits for degenerated rational map families.

From above two examples, we can acquire complex dynamics properties from dynamics on Berkovich line(non-archimedean dynamics).

KEY WORDS: Bekovich space, nested Julia set, bounded escape sequence, rescaling limit

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第一章 Introduction

One-dimensional complex dynamical systems, understood as the global study of iteration of holomorphic mappings, has its roots in the early twentieth century with the work of Fatou and Julia. Local studies [1] was successfully attempted as the basis of holomorphic dynamics. Their greatest achievement, arguably, is their detailed description of the geometry and the dynamics of the connected components of the Fatou components. Their Classification Theorem asserts that every periodic Fatou component of a rational map(holomorphic map of the Riemann sphere) is either (i) a component of an immediate basin of attraction of some attracting or parabolic cycle or (ii) a rotation domain conformally equivalent to a disc or an annulus.

Their work also left many interesting open questions, such as Fatou's No Wandering Domains Conjecture, which states that all Fatou components are eventually periodic, and waited some 60 years for its resolution by Sullivan. Around 1980, two remarkable developments added to this area—computer graphics and quasiconformal mappings [2]. Sullivan realized there was a strong connection between holomorphic iteration and the actions of Kleinian groups, introducing what has become known as Sullivan's dictionary between these two subjects. Inspired by Henri Poincare's original perturbations of Fuchsian groups into quasi-Fuchsian groups, he injected the modern theory of quasiconformal mappings into complex iteration to solve Fatou's No Wandering Domains Conjecture. Therefore, one-dimensional complex dynamical systems get much further development once again.

We also gradually realized there was some link between complex dynamics and non-archimedean dynamics. Favre, Jonsson, Rivera-Latelier and other mathematicians use Berkovich spaces [3] to under complex dynamics. Along this method, we can also study dynamical properties on non-archimedean spaces, such as its Julia set, attracting or repelling points and periodic points. We can use Berkovich spaces to understand limiting properties of a sequence of rational maps and we can also study Berkovich space dynamics by properties of a sequence of rational maps. It has been verified that this method is useful and effective.

In this thesis, we attempt to illustrate some examples concerning complex dynamics, which we can get from Berkovich spaces. The fist example [4] illustrates how we should understand the “shape” of Julia set from dynamics on Berkovich hyperbolic line. In this example, we use a bounded escape hyperbolic rational map sequence to construct Berkovich dynamics with no repelling periodic points and we use this property to illustrate the shape of its Julia set is “nested”. The second example [5] comes from rescaling limits of rational maps. We use Berkovich projective line to give a bound on the number of the dynamical independent and postcritically infinite rescaling limits of degree d rational maps. Both examples illustrate that non-archimedean dynamics may be a promising and useful research method for complex dynamics and I think it is meaningful for myself to write this thesis.

第二章　Berkovich Line

In this chapter, we introduce the notion of Berkovich line. Tracing back to the history, Berkovich space introduced by Berkovich in 1900, is a version of an analytic space over a non-archimedean field, refining Tate's notion of a rigid analytic space. The 1-dimensional Berkovich space is called Berkovich line, and it is a locally compact(compact if projective), Hausdorff, and uniquely path-connected topological space. Therefore, Berkovich line is a $\mathbb{R}$-tree. We will pay our attention to Berkovich projective line and highlight the relationship between Berkovich projective line and complex dynamics. We will firstly introduce the notion of $\mathbb{R}$-tree, secondly give a strict definition of Berkovich projective line and thirdly explain its relationship(reduction map) with our usual dynamics. We will use our tools in this chapter to prove our main theorems in chapter 3 and chapter 4.

2.1　$\mathbb{R}$-trees and tree map

We firstly give the definition of $\mathbb{R}$-trees. In order to avoid technicalities, we shall only consider trees that are complete in the sense that they contain all their endpoints unless we give some particular explanations.

Definition 2.1.1. An interval structure on a set I is a partial order $\leq$ on I under which I becomes isomorphic (as a partially ordered set) to the real interval [0,1] or to the trivial real interval [0,0]={0}.

Let I be a set with an interval structure. A subinterval of I is a subset $J \subset I$ that becomes a subinterval of $[0,1]$ or $[0,0]$ under such an isomorphism. The opposite interval structure on I is obtained by reversing the partial ordering.

Definition 2.1.2. A tree is a set I together with the following data. For each $x, y \in X$, there exists a subset $[x,y] \subset X$ containing x and y and equipped with an interval structure. Furthermore, we have:

(1) $[x,x] = \{x\}$;

(2) if $x \neq y$, then $[x,y]$ and $[y,x]$ are equal as subsets of X but equipped with opposite interval structures; they have x and y as minimal elements, respectively;

(3) if $z \in [x,y]$ then $[x,z]$ and $[z,y]$ are subintervals of $[x,y]$ such that $[x,y] = [x,z] \cup [z,y]$ and $[x,z] \cap [z,y] = \{z\}$;

(4) for any $x, y, z \in X$ there exists a unique element $t \in [x, y]$ such that $[z, x] \cap [y, x] = [t, x]$ and $[z, y] \cap [x, y] = [t, y]$;

(5) if $x \in X$ and $(y_\alpha)_{\alpha \in A}$ is a net in X such that the segments $[x, y_\alpha]$ increase with α, then there exists $y \in X$ such that $\bigcup [x, y_\alpha[= [x, y[$.

In (5) we have used the convention $[x, y[:= [x, y] \setminus \{y\}$. Recall that a net is a sequence indexed by a directed (possibly uncountable) set. The subsets $[x, y]$ above will be called intervals or segments.

Remark 2.1.3. A tree as above carries a natural weak topology. Given a point $x \in X$, define two points $y, z \in X \setminus x$ to be equivalent if $]x, y] \cap]x, z] \neq \emptyset$. An equivalence class is called a tangent direction at x and the set of $y \in X$ representing a tangent direction $\vec{v}$ is denoted $U(\vec{v})$. The weak topology is generated by all such sets $U(\vec{v})$. Clearly X is arcwise connected and the connected components of $X \setminus x$ are exactly the sets $U(\vec{v})$ as $\vec{v}$ ranges over tangent directions at x. A tree is in fact uniquely arc connected in the sense that if $x \neq y$ and $\gamma : [0, 1] \to X$ is an injective continuous map with $\gamma(0) = x, \gamma(1) = y$, then the image of γ equals $[x, y]$. Since the sets $U(\vec{v})$ are connected, any point in X admits a basis of connected open neighorhoods. It is not very hard to illustrate X is compact in the weak topology [3].

If $\gamma = [x, y]$ is a nontrivial interval, then the annulus $A(\gamma) = A(x, y)$ is defined by $A(x, y) := U(\vec{v}_x) \cap U(\vec{v}_y)$, where $\vec{v}_x$ (resp.,$\vec{v}_y$) is the tangent direction at x containing y (resp., at y containing x).

An end of X is a point admitting a unique tangent direction. A branched point is a point having at least three tangent directions.

Example 2.1.4. The Paris metric makes the closed ball $D(0, 1) \subset \mathbb{R}^2$ into a $\mathbb{R}$-tree.

Definition 2.1.5. A subtree of a tree X is a subet $Y \subset X$ such that the intersection $[x, y] \cap Y$ is either empty or a closed subinterval of $[x, y]$ for any $x, y \in X$.

In particular, if $x, y \in Y$, then $[x, y] \subset Y$ and this interval is then equipped with the same interval structure as in X. The intersection of any collection of subtrees of X is a subtree (if nonempty). The convex hull of any subset $Z \subset X$ is the intersection of all subtrees containing Z. A subtree of X is finite if it is the convex hull of a finite set.

Definition 2.1.6. A metric tree is a tree X together with a choice of generalized metric on each interval $[x, y]$ in X such that whenever $[z, w] \subset [x, y]$, the inclusion $[z, w] \hookrightarrow [x, y]$ is an isometry in the obvious sense. Here a generalized metric means that:

Let I be a set with an interval structure. A generalized metric on I is a function $d : I \times I \to [0, +\infty]$ satisfying:

(1) $d(x, y) = d(y, x)$ for all x, y, and $d(x, y) = 0$ iff $x = y$;

(2) $d(x, y) = d(x, z) + d(z, y)$ whenever $x \leq z \leq y$;

(3) $d(x, y) < \infty$ if neither x nor y is an endpoint of I;

(4) if $0 < d(x, y) < \infty$, then for every $\epsilon > 0$ there exists $z \in I$ such that $x \leq z \leq y$ and $0 < d(x, z) < \epsilon$.

Definition 2.1.7. Hyperbolic space. Let X be a metric tree containing more than one point and let $x_0 \in X$ be a point that is not an end. Hyperbolic space $\mathcal{H}$ to be the set of points $x \in X$ having finite distance from x_0 (not depend on the choice of x_0).

Note that all points in $X \setminus \mathcal{H}$ are ends, but that some ends in X may be contained in $\mathcal{H}$. One can show that $\mathcal{H}$ is complete in this metric and that $\mathcal{H}$ is an $\mathbb{R}$-tree in the usual sense(not our simplified definition). In general, even if $\mathcal{H} = X$, the topology generated by the metric may be strictly stronger than the weak topology. This happens, for example, when there is a point with uncountable tangent space: such a point does not admit a countable basis of open neighborhoods.

Definition 2.1.8. Tree map. Let X and X' be $\mathbb{R}$-trees. We say that a continuous map $f : X \to X'$ is a tree map if it is open, surjective and finite in the sense that there exists a number d such that every point in X' has at most d preimages in X. The smallest such number d is called the topological degree of f.

Proposition 2.1.9. Let $f : X \to X'$ be a tree map of topological degree d.

(1) if $U \subset X$ is a connected open set, then so is $f(U)$ and $\partial f(U) \subset f(\partial U)$;

(2) if $U' \subset X'$ is a connected open set and U is a connected component of $f^{-1}(U')$, then $f(U) = U'$ and $f(\partial U) = \partial U'$; as a consequence, $f^{-1}(U')$ has at most d connected components;

(3) if $U \subset X$ is a connected open set and $U' = f(U)$, then U is a connected component of $f^{-1}(U')$ iff $f(\partial U) \subset \partial U'$ ($f(\partial U) = \partial U'$).

Proof: Left it to the reader. □

Corollary 2.1.10. Consider a point $x \in X$ and set $x' := f(x) \in X'$. Then there exists a connected open neighborhood V of x with the following properties:

(1) if $\vec{v}$ is a tangent direction at x, then there exists a tangent $\vec{v}'$ at x' such that $f(V \cap U(\vec{v})) \subset U(\vec{v}')$; furthermore, either $f(U(\vec{v})) = U(\vec{v}')$ or $f(U(\vec{v})) = X'$;

(2) if $\vec{v}'$ is a tangent direction at x' then there exists a tangent direction $\vec{v}$ at x such

that $f(V \cap U(\vec{v})) \subset U(\vec{v}')$.

Remark 2.1.11. From Corollary 2.1.10., we know $f(U(\vec{v})) = U(\vec{v}')$ or X'. In the case of the former, we call $\vec{v}$ a good direction; in the case of the latter, we call $\vec{v}$ a bad direction.

Definition 2.1.12. **Branched coverings between $\mathbb{R}$-trees**. Let $f : T_1 \to T_2$ be a continuous map between two metric $\mathbb{R}$-trees, we say f is a (tame) degree d branched covering if there is a finite subtree $S \subset T_1$ such that

(1) S is nowhere dense in T_1, and $f(S)$ is nowhere dense in T_2.

(2) For every $y \in T_2 \setminus f(S)$, there are exactly d preimages in T_1.

(3) For every $x \in T_1 \setminus S$, f is a local isometry.

(4) For every $x \in S$, and any sufficiently small neighborhood U of $f(x)$, $f : V \setminus f^{-1}(f(V \cap S)) \to U \setminus f(V \cap S)$ is an isometric covering, where V is the component of $f^{-1}(U)$ containing x.

2.2 Berkovich projective line

Definition 2.2.1. **Multiplicative norm**. Let A be a commutative ring containing 1, a multiplicative semi-norm on A is a function : $A \to \mathbb{R}_{\geq 0}$ satisfying [0]=0, [1]=1, $[fg]$=$[f][g]$ and $[f+g] \leq [f]$+$[g]$ for all $f, g \in A$.

For the rest of this thesis, we assume k is an algebraically closed, complete non-archimedean field. We denote its non-archimedean norm as $|\ |$. In this case, we use R_k represent its valuation ring, m_k represent the maximal ideal of its valuation ring and $\tilde{k} = R_k/m_k$ represent its residue field.

Definition 2.2.2. **Berkovich affine line**. We define Berkovich affine line $\mathbb{A}^1_{Berk}$ as the set of all of semi-norms [ ] over a polynomial ring $k[T]$, where T is an indeterminate, and [ ] restricted on k is just the non-archimedean norm on itself.

As a matter of fact, we can classify all semi-norms over a polynomial ring $k[T]$ as four kinds [6]–type I, type II, type III, and type IV.

Type I: if we choose $x \in k$, we denote $[\ ]_x$: $f \to f(x)$ for every $f \in k[T]$. In this case, we have an injective map i: $k \to \mathbb{A}^1_{Berk}$. We also call type I points classical or rigid points.

Type II and III: assume $r > 0$, we denote $B(a,r)$ as $\{x \in k \mid |x-a| < r\}$ and $D(a,r)$ as $\{x \in k \mid |x-a| \leq r\}$. We consider $[\ ]_{D(a,r)}$: $f \to \sup_{x \in D(a,r)} |f(x)|$. We can prove this is a multiplicative semi-norm, when k is non-archimedean. If $r \in |k^*|$, we call it type II. If $r \notin |k^*|$, we call it type III.

Type IV: we should note $B(a,r)$ and $D(a,r)$ are both open and closed sets of the non-archimedean field k. For any two balls $D(a,r)$ and $D(a^{'},r^{'})$, we know either they are disjoint with each other or one contains another. If we consider a decreasing nested sequence $D(a_i,r_i)$, we denote $D := \cap D(a_i,r_i)$ and we know D either is empty or some $D(a,r)$. We consider $[\ ]_{D(a_i,r_i)}$: $f \to \inf_i [f]_{D(a_i,r_i)}$. We can also prove this is a multiplicative semi-norm. If $D = \emptyset$, we call it type IV, otherwise, it will become type III, type II or type I.

Remark 2.2.3. I want to give some explanations about this classification. Berkovich affine line $\mathbb{A}^1_{Berk}$, partially ordered by the size of its multiplicative norms, is a $\mathbb{R}$-tree. We say one norm $[\ ]$ is bigger than another norm $[\ ]^{'}$ if we have $[f] \geq [f]^{'}$ for any $f \in k[T]$. Type I and type IV points are exactly end points of the tree, while type II points are exactly branched points of the tree. Note that some Berkovich affine line $\mathbb{A}^1_{Berk}$ may not have type III and type IV points if non-archimedean field k has some special properties (e.g. spherical complete).

Remark 2.2.4. k is neither connected nor locally compact, but $\mathbb{A}^1_{Berk}$ is connected and locally compact. We can consider its one point compactification, i.e. Berkovich projective line $\mathbb{P}^1_{Berk}$, which is locally compact and compact.

Definition 2.2.5. **Berkovich projective line**. $\mathbb{P}^1_{Berk}$. We can add an infinity norm $[\ ]_\infty$ to $\mathbb{A}^1_{Berk}$, which can be defined as $[f]_\infty = \infty$ for any non-constant polynomial of $k[T]$.

In this case, we have an injective map $i : \mathbb{P}^1_k \to \mathbb{P}^1_{Berk}$.

Remark 2.2.6. There is more than one method to define $\mathbb{P}^1_{Berk}$. We can also consider rational function field $k(T)$ over k and generalized multiplicative semi-norms $[\ ] : k(T) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$. What's more, we will also have four kinds of semi-norms and its situation is the same as affine line.

Extend polynomials and rational maps to Berkovich line

If g is a polynomial, we can extend it to Berkovich affine line $\mathbb{A}^1_{Berk}$ as follows:

Choose any $[\ ] \in \mathbb{A}^1_{Berk}$, we define $G([\ ])$ as a multiplicative norm $f \to [f(g)]$ for any polynomial in $k[T]$.

Of course, we can also extend g to Berkovich projective line $\mathbb{P}^1_{Berk}$, just to define G maps infinity norm to infinity norm.

If h is a rational function , we can extend it to $\mathbb{P}^1_k$ as follows:

Choose any generalized semi-norms $[\ ] \in \mathbb{P}^1_{Berk}$, we define $H([\ ])$ as a generalized norm $f \to [f(h)]$ for any rational function in $k(T)$.

Remark 2.2.7. Any such extension map is a tree map and preserves type of the points [6]. More explicitly, G or H is a continuous, surjective, degree d map (if degree of g or h is d) and it maps type I(II,III,IV) points to type I(II,III,IV) points.

Now we give the definition of hyperbolic space of the Berkovich projective line.

Definition 2.2.8. We define $\mathbb{H}_{Berk} := \{[\ ] \in \mathbb{P}^1_{Berk} \mid [\ ]$ is not of type I points $\}$, and call it hyperbolic space.

As a matter of fact, we can give a generalized metric [6] on $\mathbb{P}^1_{Berk}$ so that it becomes a metric tree and $\mathbb{H}_{Berk}$ is just the same defined as in Definition 2.1.7.. The metric can be described as follows:

If $[\ ]$ and $[\ ]'$ are all type II or III points, we denote them as $[\ ] = D(a, r)$ and $[\ ]' = D(a', r')$. Let $R = \max\{r, r', |a - a'|\}$, and we define $d([\ ], [\ ]') = \log(\frac{R^2}{rr'})$. In fact, we can also extend this distance to type IV points but I don't plan to clarify it here.

In fact, the topology induced by metric on $\mathbb{H}_{Berk}$ is strictly finer than weak topology as a $\mathbb{R}$-tree.

Remark 2.2.9. We can study local properties of f over $\mathbb{P}^1_k$ and have a notion-local degree $\deg_x(f)$ at any point $x \in \mathbb{P}^1_k$. As a matter of fact, we can extend the notion of local degree to Berkovich projective line $\mathbb{P}^1_{Berk}$, also denoted by $\deg_x(F)$. When we have this notion, we can exactly d preiamges if we count each point by its multiplicities. I don't plan to give a rigid definition of local degree of the Berkovich projective line, and the reader can acquire it from [6].

2.3 Reduction map

When we have a rational function f over $\mathbb{P}^1_k$, we can get a rational function $\tilde{f}$ (may be ∞) over $\mathbb{P}^1_{\tilde{k}}$ if we make all coefficients of f be in valuation ring of k but not all in maximal ideal and then get its image in $\tilde{k}$. More precisely:

(1) Write f as a quotient of polynomials P and Q in $R_k[x]$, so that at least one of the coefficients of these polynomials is a unit in R_k.

(2) Passing to the quotient $\tilde{k}[z]$, we obtain polynomials $\tilde{P}$ and $\tilde{Q}$ which may have a non-constant greatest common divisor H.

(3) Denote the roots of H by $\mathscr{H}(f)$, and we have $\tilde{f} = \frac{R}{S}$ where $R, S \in \tilde{k}[x]$ are such that $\tilde{P} = R \cdot H$ and $\tilde{Q} = S \cdot H$, under the agreement that if $S = 0$, then $\tilde{f} = \infty$.

We call the rational function $\tilde{f}$ reduction map. The following results illustrate the relationships between F and $\tilde{f}$.

We denote x_g as the Gauss point on $\mathbb{P}^1_{Berk}$, the type II point corresponding to $D(0,1)$. We can identify tangent space $T_{x_g}\mathbb{P}^1_{Berk}$ as $P^1_{\tilde{k}}$ [3]. Consider the reduction map $\rho : \mathbb{P}^1_k \to \mathbb{P}^1_{\tilde{k}}$. In fact, $U(\vec{v})$ is a direction of x_g iff it is the convex hull of $\rho^{-1}(z)$ in $\mathbb{P}^1_{Berk}$ for some $z \in \mathbb{P}^1_{\tilde{k}}$.

Proposition 2.3.1. Let $x \in \mathbb{H}_{Berk}$, and $\vec{v} \in T_x\mathbb{H}_{Berk}$.

(1) For all sufficiently small segment $\gamma = [x, y]$ representing $\vec{v}$, F maps γ homeomorphically to $F(\gamma)$ and expands by a factor of $m_{\vec{v}}F$.

(2) If $\vec{w} \in T_{F(x)}\mathbb{H}_{Berk}$, and $\vec{v}_1, ..., \vec{v}_k$ are the preimages of $\vec{w}$ in $T_x\mathbb{H}_{Berk}$, then

$$\sum_{i=1}^{k} m_{\vec{v}_i}F = \deg_x F$$

.

(3) If $\alpha : (0, \infty) \to \mathbb{H}_{Berk}$ is an end associated to $x \in \mathbb{P}^1_k$, then for all sufficiently large L, F maps $\alpha([L, \infty))$ homeomorphically to $F(\alpha([L, \infty)))$ and expands by a factor of $\deg_x(f)$.

Proof. We refer the reader to the note of [3] and the book of [6]. □

Proposition 2.3.2. Let $F : \mathbb{P}^1_{Berk} \to \mathbb{P}^1_{Berk}$ be a rational map of degree at least 1. Then F fixes the Gauss point x_g iff $\deg(\tilde{f}) \geq 1$.

Assume that $F(x_g) = x_g$. After identifying $T_{x_g}\mathbb{P}^1_{Berk}$ to $\mathbb{P}^1_{\tilde{k}}$ via reduction, the following hold:

(1) $\deg_{x_g}(F) = \deg(\tilde{f})$.

(2) $T_{x_g}F(w) = \tilde{f}(w)$ and $m_wF = \deg_w(\tilde{f})$ for all $w \in \mathbb{P}^1_{\tilde{k}}$ (Here we identify w to its representing direction).

(3) For all $w \in \mathscr{H}(f)$, we have that $F(U_w) = \mathbb{P}^1_{Berk}$(bad direction). Let $w' = \tilde{f}(w)$. Then there exists $\delta \geq 1$ such that every point in $U_{w'}$ has $\delta + \deg_w\tilde{f}$ preimages in U_w and every point not in $U_{w'}$ has δ preimages in U_w, counting multiplicities.

(4) For all $w \in \mathbb{P}^1_{\mathbb{C}} \backslash \mathscr{H}(f)$, we have that $F(U_w) = U_{w'}$(good direction) where $w' = \tilde{f}(w)$. The degree of $F : U_w \to U_{w'}$ is well defined and coincides with $\deg_w(\tilde{f})$.

Proof. We refer the reader to the work of [5] and [7]. □

Remark 2.3.3. The group PGL(2, k) acts transitively on $\mathbb{P}^1_k$(type I points), and also transitively on type II points of $\mathbb{P}^1_{Berk}$. In fact, we can choose an appropriate affine linear transformation mapping any $D(a, r)$ to another $D(a', r')$. However, we should notice PGL(2,k) may not act transitively on type III(IV) points. We may doubt the condition F fixes the Gauss point is too hard to satisfy, but it is not essential. Essentially, we can regard any type II point x as the Gauss point x_g and identify direction at x with $\mathbb{P}^1_{\tilde{k}}$. Namely, consider two elements $\eta, \gamma \in \mathrm{PGL}(2,k)$ and the points $x = \eta(x_g)$, $y = \gamma(x_g)$. Then $y = F(x)$ iff the rational map $F' = \gamma^{-1} F \eta$ fixes the Gauss point.

第三章 Nested Julia set

In this chapter, we will explain a work of Luo Yusheng [4] in 2021. He use the Berkovich space over complexified Robinson's field to acquire the equivalence of two properties of hyperbolic component of rational maps: nested Julia set and bounded escape sequence. "Bounded escape" firstly came from the MuMullen's example $f_n(z) = z^2 + \frac{1}{nz^2}$ and the example's Julia set has a topological feature—a Cantor set of circles. Luo found that this kind topological feature(nested Julia set) totally determines the existence of bounded escape sequence. And Luo proved that the topological feature can be acquired from "shape" of Julia set in corresponding Berkovich line. In section 3.1, we explain nested Julia set and bounded escaped sequence, and then introduce our main theorem 3.1.5. . From section 3.2 to 3.4, we construct the Berkovich projective line and make preparations for our main idea of proof. In section 3.5, we show how we should use our Berkovich space to get our one direction(from bounded sequence to nested Julia set), and as for the other direction, we need quasi-conformal surgery and we refer the reader to [2] and [4].

3.1 Nested Julia set

We use notation $\mathrm{Rat}_d(\mathbb{C})$ to represent the set of all rational functions of degree d. We can think $\mathrm{Rat}_d(\mathbb{C})$ as an Zariski open dense subset of $\mathbb{P}_{\mathbb{C}}^{2d+1}$, which is defined by resultant of two polynomials. Therefore, we sometimes denote $f_n \to \infty$ as $f_n \to \partial\mathrm{Rat}_d(\mathbb{C})$.

Let $M_d(\mathbb{C}) = \mathrm{Rat}_d(\mathbb{C})/\mathrm{PSL}(2,\mathbb{C})$ (conjugation action).

Markings and the length spectra. A conjugacy class of a rational map $[f]$ is called hyperbolic if the orbit of every critical point converges to some attracting periodic cycle. The space of hyperbolic rational maps is open in the moduli space $M_d(\mathbb{C})$, and a connected component $\mathcal{H}$ is called a hyperbolic component. For each hyperbolic component $\mathcal{H}$, there is a topological model

$$\sigma : J \to J$$

for the dynamics on the Julia set. That is, for any $[f] \in \mathcal{H}$, there is a homeomorphism

$$\phi(f) : J \to J(f)$$

which conjugates σ and f. A particular choice of such $\phi(f)$ will be called a marking of the Julia set.

Let $[f] \in \mathcal{H} \subset M_d$ be a hyperbolic rational map with a marking $\phi : J \to J(f)$. Let $\mathcal{J}$ be the space of periodic cycles of $\sigma : J \to J$. We define the length of a periodic cycle $C \in \mathcal{J}$ for $[f]$ by

$$L(C, [f]) = \log|(f^q)'(z)|,$$

where $q = |C|$ and $z \in \phi(C)$. The collection $(L(C, [f]) : C \in \mathcal{J}) \in \mathbb{R}_+^{\mathcal{J}}$ will be called the marked length spectrum of $[f]$.

Degeneracy with bounded length spectra. Let $\mathcal{H}$ be a hyperbolic component.

Definition 3.1.1. A hyperbolic component $\mathcal{H}$ is said to admit bounded escape if there exists a sequence $[f_n] \in \mathcal{H}$ with markings ϕ_n so that

(1) $[f_n]$ is degenerating;

(2) For any periodic cycle $C \in \mathcal{J}$,the sequence of lengths $L(C, [f_n])$ is bounded.

The first example comes from the McMullen family $f_n(z) = z^2 + \frac{1}{nz^2}$. The sequence f_n is contained in a hyperbolic component for all large n, and the Julia set J for this hyperbolic component is homeomorphic to a Cantor set of circles. In particular, any component of the Julia set separates the two points 0, ∞, and the Julia set is disconnected. We will show these two topology characteristics of the Julia set classify all examples admitting bounded escape.

Definition 3.1.2. Let $f \in \mathrm{Rat}_d(\mathbb{C})$ be a hyperbolic rational map. We say its Julia set $J(f)$ is nested if

(1) There are two points $p_1, p_2 \in \mathbb{P}^1_{\mathbb{C}}$ such that any component of $J(f)$ separates p_1 and p_2;

(2) $J(f)$ contains more than one component.

A hyperbolic component $\mathcal{H}$ is said to have nested Julia sets if the Julia set of any rational map in $\mathcal{H}$ is nested.

Remark 3.1.3. We know a hyperbolic component $\mathcal{H}$ is nested iff one of its elements is nested. According to [8] , there is a qusiconformal map ϕ defined on whole $\mathbb{P}_{\mathbb{C}}$ between arbitrary two different hyperbolic maps in the same hyperbolic component, and ϕ can conjugate dynamics of these two maps on their Julia sets. We should also notice that there may not be a canonical $\phi(f)$ for $[f] \in \mathcal{H}$.

If f is a hyperbolic rational map with nested Julia sets, and J is its Julia set, we know any Fatou component of f is either simply connected, or is isomorphic to an annulus because each component of J separates 2 points p_1 and p_2. We will call a simply connected Fatou component a disk Fatou component, and an annulus Fatou component a gap. A disk(gap) is called critical if it contains critical points of f.

We say a Julia component K of J is

(•) extremal if $J-K$ is contained in a component of $\mathbb{P}^1_{\mathbb{C}}-K$;

(•) buried if K does not intersect the boundary of any Fatou component;

(•) unburied if it is not buried.

Lemma 3.1.4. Let f be a hyperbolic rational map with nested Julia set J. Then

(1) The gaps are nested, backward invariant, and any gap is eventually mapped to a disk Fatou component.

(2) Any non-extremal periodic Julia component K is buried.

(3) Any unburied Julia component K is eventually mapped to an extremal Julia component by a degree $e=e(K)$ covering.

(4) Any buried Julia component K is a Jordan curve.

(5) A critical gap(disk) is mapped to a disk Fatou component whose boundary is contained in an extremal Julia component.

(6) The number of extremal Julia components is 2 and the extremal Julia components are mapped to extremal ones.

Proof. Since the Julia set is nested, the gaps are nested. Since no disk Fatou component is mapped to a gap, they are backward invariant. Let U be a periodic Fatou component. Since f is hyperbolic, U is attracting. Thus, U contains a critical point of the fist return map f^q. By Riemann-Hurwitz formula, U is a disk Fatou component. By no wandering domain theorem, every Fatou component is pre-periodic, so the statement (1) follows.

About (2)-(6), the reader can get from the work of [4]. □

Main Theorem 3.1.5. A hyperbolic component $\mathcal{H}$ admits bounded escape iff it has nested Julia sets.

Our main theorem connects these two different notions. If $\mathcal{H}$ is nested, we use quasi-conformal surgery to get a sequence of bounded escape. On the other hand, we use Berkovich hyperbolic line to deduce our $\mathcal{H}$ is nested.

3.2 Complexified Robinson's feield

Now, we make preparations for $\Rightarrow$.

We will study a complete, algebraically closed, non-archimedean field ${}^\rho\mathbb{C}$. This field is first introduced in the real setting by Robinson to formulate the non-standard analysis.

Definition 3.2.1. A subset $\omega \subset \mathscr{P}(N)$ of the power set of $\mathbb{N}$ is called an ultrafilter if

(1) If $A, B \in \omega$, then $A \cap B \in \omega$;

(2) If $A \in \omega$ and $A \subset B$, then $B \in \omega$;

(3) $\emptyset \notin \omega$;

(4) If $A \subset \mathbb{N}$, then either $A \in \omega$ or $\mathbb{N} - A \in \omega$.

We will call a set in ω as ω-big or simply big. Its complement is called ω-small or simply small. If a specific property is satisfied by a ω-big set, we will also say this property holds ω-almost surely.

Let $a \in \mathbb{N}$. We can construct an ultrafilter by

$$\omega_a := \{A \subset \mathscr{P}(N) \ : \ a \in A\}.$$

Any ultrafilter of the above type is called a principal ultrafilter. It can be verified that an ultrafilter is principal if and only if it contains a finite set. An ultrafilter that is not principal is called a non-principal ultrafilter. The existence of a non-principal ultrafilter is guaranteed by Zorn's lemma.

Let ω be a non-principal ultrafilter on $\mathbb{N}$. Let x_n be a sequence in a metric space (X, d) and $x \in X$. We say x is the ω-limit of x_n, denoted by

$$\lim_{\omega} x_n = x$$

if for every $\epsilon > 0$, the set $\{n : \ d(x_n, x) < \epsilon\}$ is ω-big.

It can be easily verified that [9]

(1) If the ω-limit exists, then it is unique.

(2) If x_n is contained in a compact set, then the ω-limit exists.

(3) If $x = \lim_{n\to\infty} x_n$ in the standard sense, then $x = \lim_{\omega} x_n$.

(4) If $x = \lim_{\omega} x_n$, then there exists a subsequence n_k such that $x = \lim_{k\to\infty} x_{n_k}$ in the standard sense.

From these properties, one should think of the non-principal ultrafilter ω as performing all the subsequence-selection in advance. From now on, we will fix a non-principal ultrafilter ω on $\mathbb{N}$.

Let $\mathbb{C}^{\mathbb{N}}$ be the set of all sequences in $\mathbb{C}$. We say two sequence (z_n) and (w_n) are equivalent if $z_n = w_n$ ω-almost surely. The set of equivalence classes will be denoted by

${}^*\mathbb{C}$.

The addition and multiplication are defined naturally: if $x, y \in {}^*\mathbb{C}$ are represented by (x_n) and (y_n), then $x+y$ and $x \cdot y$ are represented by $(x_n + y_n)$ and $(x_n \cdot y_n)$. This field is usually referred to as the ultrapower construction for $\mathbb{C}$.

Give $x, y \in^* \mathbb{C}$ represented by (x_n) and (y_n), we write $|x| \le |y|$ or $|x| < |y|$ if $|x_n| \le |y_n|$ or $|x_n| < |y_n|$ ω-almost surely.

We construct our ${}^\rho\mathbb{C}$ from ${}^*\mathbb{C}$.

Let $\rho_n \to 0$ be a positive sequence, which is regard as $\rho \in {}^*\mathbb{C}$. We construct

$$M_0 = \{t \in^* \mathbb{C} : \exists N \in \mathbb{N}, |t| < \rho^{-N}\}$$

and

$$M_1 = \{t \in^* \mathbb{C} : \forall N \in \mathbb{N}, |t| < \rho^{N}\}.$$

We remark that since $\rho_n \to 0$, M_0 consists of those sequences that are not growing to infinity too fast, while M_1 consists of those tending to 0 very fast. It is easy to show that M_0 forms a ring with respect to the addition and multiplication of ${}^*\mathbb{C}$. It can also be shown that M_1 is a maximal ideal of ring M_0. We define

$${}^\rho\mathbb{C} = M_0/M_1$$

as the quotient field. Note that $\mathbb{C}$ embeds into ${}^\rho\mathbb{C}$ via constant sequences.

We can define an equivalence relation on ${}^*\mathbb{C}$: $x \sim y$ if $x - y \in M_1$. Note that if $y \in M_0$, then $x \sim y$ iff $x \in [y]$ as a member of ${}^\rho\mathbb{C}$.

Non – archimedean norm on ${}^\rho\mathbb{C}$. We define a valuation for $[x] \in {}^\rho\mathbb{C}$ by

$$\nu([x]) := \lim_{\omega} \frac{\log|x_n|}{\log\rho_n}$$

and we briefly denote it as $\log_\rho|x|$. The reader can verify it is well-defined.

Notice that by definition, $\nu([\rho^t]) = t$ for $t \in \mathbb{R}$.

To simplify the notations, from now on, we will use a single roman letter to represent a number in ${}^\rho\mathbb{C}$, and drop the square bracket.

It can be easily verified that for $x, y \in {}^\rho\mathbb{C}$, we have

$$\nu(x \cdot y) = \nu(x) + \nu(y),$$

$$\nu(x + y) \ge \min(\nu(x), \nu(y)).$$

Hence, we get a non-archimedean norm via $|x|_\nu = e^{-\nu(x)}$. And the distance function is given by $d(x,y) = |x-y|_\nu$.

Recall that a metric space X is said to be spherically complete if for any nested sequence of closed balls $D_0 \supset D_1 \supset \ldots$, their intersection $\cap_j D_j$ is non-empty.

Proposition 3.2.2. The field $({}^\rho\mathbb{C}, d)$ is both spherically complete and algebraically closed.

Proof. Step1, $({}^\rho\mathbb{C}, d)$ is spherically complete.

Let $D_0 \supset D_1 \supset \ldots$ be a decreasing sequence of closed balls. We consider a decreasing sequence of open balls B_i so that $D_i \supset B_i \supset D_{i+1}$. We assume that B_i has radius r_i, and denote $q_i = -\log r_i$. Pick $\alpha_i \in B_i$, and assume that α_i is represented by $(a_{i,n})$. Since $B_j \subset B_i$ for all $j \geq i$, we know

$$\nu(\alpha_i, \alpha_j) = \lim_\omega \frac{\log|a_{i,n} - a_{j,n}|}{\log \rho_n} > q_i.$$

We can construct inductively a decreasing sequence $\mathbb{N} = N_0 \supset N_1 \supset \ldots$ such that

(1) N_k is ω-big;

(2) $\cap_{k=1}^\infty N_k = \emptyset$;

(3) For any $i \leq j \leq k$ and $l \in N_k$, we have

$$\nu_l(\alpha_{i,l}, \alpha_{j,l}) := \frac{\log|a_{i,l} - a_{j,l}|}{\log \rho_l} > q_i.$$

Indeed, we can set $N_0 = \mathbb{N}$ as the base case. Assume that N_k is constructed. To constructed N_{k+1}, we note that for any $i \leq k+1$,

$$\nu(\alpha_i, \alpha_{k+1}) = \lim_\omega \frac{\log|a_{i,n} - a_{k+1,n}|}{\log \rho_n} > q_i.$$

Hence, there exists an ω-big set N so that for all $i \leq k+1$ and $l \in N$,

$$\nu_l(\alpha_{i,l}, \alpha_{k+1,l}) > q_i.$$

We define $N_{k+1} = N \cap N_k \cap \{n : \ n \geq k+1\}$, then $N_{k+1} \subset N_k$ is still ω-big. Therefore, properties (1), (2), (3) are satisfied.

We now define the sequence $a_n = a_{k,j}$ for $j \in N_k - N_{k+1}$, and let $\alpha = (a_n)$. From our construction, we know $\nu(\alpha_i - \alpha) > q_i$, so $\alpha \in B_i$.

Since this holds for any i, we conclude that $\alpha \in \cap_i B_i$, so $\cap_i B_i \neq \emptyset$.

Step 2, ${}^\rho\mathbb{C}$ is algebraically closed.

Let $z^d + a_{d-1}z^{d-1} + \ldots + a_0$ be a monic polynomial with coefficients $a_n = (a_{n,k}) \in {}^\rho\mathbb{C}$. We assume that $M < \min(0, \nu(a_0), \ldots, \nu(a_{d-1}))$. Hence there is an ω-big set $N \subset \mathbb{N}$ so that for all $k \in N$ and $n = 0, ..., d-1$,

$$a_{n,k} < \rho_k^M.$$

Now let $f_k(z) = a_{d-1,k}z^{d-1,k} + ... + a_{0,k}$ and $g(z) = z^d$. Note that for any $k \in N$, on the circle centered at 0 of radius $d \cdot \rho_k^M$(note that $\rho_k^M > 1$ as $M < 0$)that

$$|f_k(z)| \leq |a_{d-1,k}| \cdot (d \cdot \rho_k^M)^{d-1} + ... + |a_{0,k}|$$

$< \rho_k^M \cdot d \cdot (d \cdot \rho_k^M)^{d-1} = |g(z)|$.

By Rouche's theorem, there are d solutions of $g+f_k(z) = 0$ in the ball $B(0, d\cdot\rho_k^M)$. Let x_k be such a root. Note that x_k is defined on an ω-big set N, so $x = (x_k)$ represents a point in ${}^\rho\mathbb{C}$ as $|x_k| < \rho_k^{M+1}$ for all $k \in N$. Moreover, x satisfies the equation $z^d+a_{d-1}z^{d-1}+...+a_0 = 0$. Therefore, ${}^\rho\mathbb{C}$ is algebraically closed. □

Corollary 3.2.3. The field $({}^\rho\mathbb{C}, d)$ is complete.

Corollary 3.2.4. The Berkovich hyperbolic space $\mathbb{H}_{Berk}({}^\rho\mathbb{C}, d)$ consists of only Type II points.

Remark 3.2.5. It is not hard to know the characteristic of residue field $\widetilde{{}^\rho\mathbb{C}}$ is 0. And the field $\mathbb{C}$ can be embeded into the residue field by taking constant sequence.

Given a sequence $f_n \in \mathrm{Rat}_d(\mathbb{C})$, we define $\mathbf{f} \in \mathrm{Rat}_d({}^\rho\mathbb{C})$ by taking its coefficients in most directly way. We usually denote F as the extension of $\mathbf{f}$ on $\mathbb{P}^1_{Berk}({}^\rho\mathbb{C})$ or on $\mathbb{H}_{Berk}({}^\rho\mathbb{C})$.

3.3 Approximating disks and annuli

We say a sequence of simply connected domains $D_n \subset \mathbb{C}$ approximates an open disk $B(a, r) \subset {}^\rho\mathbb{C}$ if

(1) any sequence $z_n \in D_n$ represents a point in the closed disk $D(a, r)$;

(2) any sequence $z_n \in \mathbb{P}^1_{\mathbb{C}} - D_n$ represents a point in $\mathbb{P}^1_{{}^\rho\mathbb{C}} - B(a, r)$.

Similarly, we say a sequence of annuli $A_n \subset \mathbb{C}$ approximate an open annulus $A = B(a, R) - D(a, r) \subset {}^\rho\mathbb{C}$ if

(1) any sequence $z_n \in A_n$ represents a point in the closed annulus $D(a, R) - B(a, r)$;

(2) any sequence $z_n \in \mathbb{P}^1_{\mathbb{C}} - A_n$ represents a point in $(\mathbb{P}^1_{{}^\rho\mathbb{C}} - B(a, R)) \cup D(a, r)$.

The following lemma gives the classification for approximating disks and annuli. To distinguish the disks in $\mathbb{C}$ and ${}^\rho\mathbb{C}$, we use $B_{\mathbb{C}}(a, r)$ to represent an open disk centered at a with radius r in $\mathbb{C}$.

Lemma 3.3.1. Let $B(0,1) \subset {}^{\rho}\mathbb{C}$ be an open disk. A sequence of simply connected domains $D_n \subset \mathbb{C}$ approximates $B(0,1)$ iff

$$B_{\mathbb{C}}(0, \rho_n^{-s}) \subset D_n \subset B_{\mathbb{C}}(0, \rho_n^{-S}) \quad \omega - \text{almost surely}$$

for all $s < 0 < S$.

Let $A = B(0,R) - D(0,r) \subset {}^{\rho}\mathbb{C}$ be an open annulus. Denote $A_{\mathbb{C}}(s,S) = B_{\mathbb{C}}(0,\rho_n^{-S}) - \overline{B_{\mathbb{C}}(0,\rho_n^{-s})} \subset \mathbb{C}$. A sequence of annuli $A_n \subset \mathbb{C}$ approximates A iff

$$A_{\mathbb{C}}(s_1, S_1) \subset A_n \subset A_{\mathbb{C}}(s_2, S_2) \quad \omega - \text{almost surely}$$

for all $s_2 < \log r < s_1$, $S_1 < \log R < S_2$.

Proof. We can use the definition to verify it. Left it to the reader. □

Recall that for $\vec{v} \in T_x\mathbb{H}_{Berk}({}^{\rho}\mathbb{C})$, $U_{\vec{v}}$ is identified with an open disk $B(a,r) \subset \mathbb{P}^1_{{}^{\rho}\mathbb{C}}$. We say a sequence of simply connected domains $D_n \subset \mathbb{P}^1_{\mathbb{C}}$ approximates $U_{\vec{v}}$ if after changing coordinates, $B(a,r) \subset {}^{\rho}\mathbb{C}$ and D_n approximates $B(a,r)$.

Given two points $x, y \in \mathbb{H}_{Berk}({}^{\rho}\mathbb{C})$, there is a unique component $U^{x,y}$ of $\mathbb{H}_{Berk}({}^{\rho}\mathbb{C}) - \{x, y\}$. We can identify it with an open annulus $A \subset \mathbb{P}^1_{{}^{\rho}\mathbb{C}}$. We define a sequence of $A_n \subset \mathbb{P}^1_{\mathbb{C}}$ approximates $U^{x,y}$ in a similar way.

It is important to understand how the limiting map acts on $U_{\vec{v}}$ and $U^{x,y}$. We first define a critical end as an end of $\mathbb{H}_{Berk}({}^{\rho}\mathbb{C})$ associates with a critical point in $\mathbb{P}^1_{{}^{\rho}\mathbb{C}}$. The critical tree C is defined as the convex hull of the critical ends. Since the residual characteristic of ${}^{\rho}\mathbb{C}$ is 0, any point not in the critical tree has local degree 1 [10].

Lemma 3.3.2. (1) Let $\vec{v} \in T_x\mathbb{H}_{Berk}({}^{\rho}\mathbb{C})$, and $\vec{w} = D_xF(\vec{v})$. Assume that $U_{\vec{v}}$ does not intersect the critical tree C. Then F is an isometric bijective from $U_{\vec{v}}$ to $U_{\vec{w}}$.

(2) Let $x, y \in \mathbb{H}_{Berk}({}^{\rho}\mathbb{C})$, and $x' = F(x)$, $y' = F(y)$. Assume that $U^{x,y}$ contains no critical ends. Then F is a branched covering map from $U^{x,y}$ to $U^{x',y'}$.

Proof. As for (1), we refer the reader to the work of [10] and [11].

As for (2), we can consider F acts on $\mathbb{P}^1_{Berk}({}^{\rho}\mathbb{C})$, and observe its action on Type I points. We just need to illustrate F maps the annulus to an annulus rather than a disk or sphere. This can be verified by Riemann-Hurwitz theorem. It is obvious that F is a branched covering map. □

Lemma 3.3.3. Let $x, y \in \mathbb{H}_{Berk}({}^{\rho}\mathbb{C})$, and $x' = F(x)$, $y' = F(y)$. Assume that $U^{x,y}$ contains no critical ends, so $F : U^{x,y} \to U^{x',y'}$ is a degree e branched covering. Then there exist sequences of annuli A_n and $A_n^{'}$ approximating $U^{x,y}$ and $U^{x^{'},y^{'}}$ such that

$$f_n : A_n \to A_n^{'}$$

is a degree e covering ω-almost surely.

Proof. This can be easily attained from our analysis before. □

Our next lemma can be found in [6], which will be used in our proof for main theorem.

Lemma 3.3.4. Let $\vec{v} \in T_x\mathbb{H}_{Berk}(^{\rho}\mathbb{C})$, and $\vec{w} = D_xF(\vec{v}) \in T_y\mathbb{H}_{Berk}(^{\rho}\mathbb{C})$. Let $y^{'} \in U_{\vec{w}}$. Then there exists $x^{'} \in U_{\vec{v}}$ such that F maps $[x, x^{'}]$ homeomorphically to $[y, y^{'}]$.

3.4 Marked length spectra and periodic ends

Let $f_n \to \infty$ in $\mathrm{Rat}_d(\mathbb{C})$. We construct dynamics on $\mathbb{H}_{Berk}(^{\rho}\mathbb{C})$ as follows:

(1) Use barycentric method in [12], we extend f_n to $\mathcal{E}f_n : \mathbb{H}^3 \to \mathbb{H}^3$, where $\mathbb{H}^3$ is the 3-dimensional hyperbolic space.

(2) Let $\rho_n = e^{-r_n} \to 0$, where $r_n = \max\limits_{y \in \mathcal{E}f_n^{-1}(\mathbf{0})} d_{\mathbb{H}^3}(\mathbf{0}, y) \to \infty$. We can get r_n from [12].

(3) Use ρ_n before, we construct $F : \mathbb{H}_{Berk}(^{\rho}\mathbb{C}) \to \mathbb{H}_{Berk}(^{\rho}\mathbb{C})$.

As for conjugacy class $[f]$ in $M_d(\mathbb{C})$, it can be proved $r([f]) := \inf\limits_{x \in \mathbb{H}^3} \max\limits_{y \in \mathcal{E}f_n^{-1}(x)} d_{\mathbb{H}^3}(x, y)$ is well defined for any representative f for $[f]$. And it goes to ∞ as $[f_n] \to \infty$ in $M_d(\mathbb{C})$. In this case, we choose an appropriate representative f_n for $[f_n]$ as follows [4]:

$$\max_{y \in \mathcal{E}f_n^{-1}(\mathbf{0})} d_{\mathbb{H}^3}(\mathbf{0}, y) \le r([f_n]) + 1.$$

It can be verified that different representatives satisfying this condition induce conjugated Berkovich dynamics [12].

Theorem 3.4.1. Let $[f_n] \to \infty$ in $\mathcal{M}_d$ and $r_n := r([f_n])$. Let f_n be a representative of $[f_n]$ satisfying condition before. Then the corresponding limiting map

$$F : \mathbb{H}_{Berk}(^{\rho}\mathbb{C}) \to \mathbb{H}_{Berk}(^{\rho}\mathbb{C})$$

is a branched covering of degree d with no totally invariant point.

Proof. We refer the reader to [4]. □.

Markings of periodic ends. Let $[f_n] \in \mathcal{H}$ be a degenerating sequence with markings $\phi_n : J \to J([f_n])$. We choose a representative f_n satisfying condition above, and let F be its dynamics on $\mathbb{H}_{Berk}(^{\rho}\mathbb{C})$. The sequence of markings $\phi_n : J \to J([f_n])$ naturally gives a map $\phi: J \to \mathbb{P}^1_{{}^\rho\mathbb{C}}$ by

$$\phi(t) = [(\phi_n(t))].$$

Note that if t is a periodic point of period q, then $f_n^q(\phi_n(t)) = \phi_n(t)$ for all n. Hence, $\phi(t)$ is a periodic point of $\mathbf{f} \in \mathrm{Rad}_d({}^\rho\mathbb{C})$ represented by f_n.

Let $\alpha : [0, \infty) \to \mathbb{H}_{Berk}({}^\rho\mathbb{C})$ represent an end. We show that the translation length

$$L(\alpha, F) = \lim_{x_i \to \alpha} d(x_i, x_g) - d(F(x_i), x_g)$$

is well defined.

If α is not a critical end, then F is an isometry on $\alpha([K, \infty))$ for a sufficiently large K. Hence, the translation length is a limit of a constant sequence.

If α is a critical end, then F is expanding with derivative $e \in \mathbb{N}_{\geq 2}$ on $\alpha([K, \infty))$ for a sufficiently large K. Hence, the translation length goes to $-\infty$.

If $C = \{\alpha_1, ..., \alpha_q\}$ is a cycle of periodic ends, we define

$$L(C, F) = \sum_{i=1}^{q} L(\alpha_i, F).$$

We say a periodic end C is attracting, indifferent or repelling if $L(C, F) < 0, = 0$ or > 0.

Theorem 3.4.2. Let $[f_n] \in \mathcal{H}$ be degenerating with markings ϕ_n, and let F be the associated limiting maps on $\mathbb{H}_{Berk}({}^\rho\mathbb{C})$. If $C \in \mathcal{J}$ is a periodic cycle, then

$$L(C, F) = \lim_{\omega} \frac{L(C, [f_n])}{r([f_n])}.$$

Moreover, every cycle of repelling periodic ends of F is represented by some cycle $C \in \mathcal{J}$.

Proof. By considering iterations if necessary, we may assume $C = t \in \mathcal{J}$ has period 1. By naturality, we may also assume $\phi_n(t) = 0$ for all n. Let $\mathbf{f} \in \mathrm{Rat}_d({}^\rho\mathbb{C})$ be represented by f_n. Note that $\mathbf{f}(0) = 0$.

We first claim that $0 \in {}^\rho\mathbb{C}$ is not a critical point of $\mathbf{f}$. Indeed, since $t \in J$, $|f_n'(\phi_n(t))| > 1$, so $|\mathbf{f}'(z)| \geq 1$. Hence, we can write

$$\mathbf{f}(z) = az(1 + \mathbf{h}(z)),$$

with $a = \mathbf{f}'(0)$ and $h(0) = 0$.

Let $M, L \in \mathrm{PSL}_2({}^\rho\mathbb{C})$ be $M(z) = kz$ and $L(z) = akz$. Note that if $|k|$ is sufficiently small,

$$L^{-1} \circ \mathbf{f} \circ M(z) = z(1 + \mathbf{h}(kz))$$

has non-constant reduction. Hence, we conclude that F sends $(M_n(\mathbf{0})) \in \mathbb{H}_{Berk}({}^{\rho}\mathbb{C})$ to $(L_n(\mathbf{0}))$ where M_n and L_n represent M and L respectively. Therefore, the translation length is

$$L(C, F) = \log|a|.$$

By choosing representatives, we have

$$\log|a| = -\nu(a) = -\lim_{\omega} \frac{\log|f_n'(0)|}{\log \rho_n} = \lim_{\omega} \frac{L(C, [f_n])}{r([f_n])}.$$

Thus, the result follows.

To show the moreover part, we note that any periodic point z_n not in the Julia set is non-repelling. The multiplier $|(f_n^q)'(z_n)| \leq 1$ has bounded norm. Hence $|(\mathbf{f}_n^q)'(z)| \leq 1$. But the total number of periodic points of period q for $\mathbf{f}$ is the same as that of f_n, so all repelling periodic points of $\mathbf{f}$ are represented by periodic points in the Julia sets. Therefore, every cycle of repelling periodic ends is represented by some $c \in \mathscr{J}$. □

3.5 Bounded escape ⇒ nested Julia sets

The proof consists of two steps. We first classify the limiting dynamics that can appear. We then use the limiting dynamics to deduce the topological properties on the Julia set.

Classification of limiting map. We can easily know F has no repelling periodic ends if f_n is a bounded escape. We shall first classify those limiting dynamics with no repelling periodic ends.

Lemma 3.5.1. Let $F: \mathbb{H}_{Berk}({}^{\rho}\mathbb{C}) \to \mathbb{H}_{Berk}({}^{\rho}\mathbb{C})$, and $x_0, x_1 \in \mathbb{H}_{Berk}({}^{\rho}\mathbb{C})$. Let $\vec{v}_0 \in T_{x_0}\mathbb{H}_{Berk}({}^{\rho}\mathbb{C})$ associated to x_1, and $\vec{v}_1 \in T_{x_1}\mathbb{H}_{Berk}({}^{\rho}\mathbb{C})$ such that

(1) $F(x_1) = x_0$;

(2) $D_{x_1}F(\vec{v}_1) = \vec{v}_0$;

(3) $U_{\vec{v}_1}$ does not intersect the critical tree nor contain x_0.

Then F has a repelling fixed end.

Proof. Since $U_{\vec{v}_1}$ does not intersect the critical tree, F is an isometric bijection from $U_{\vec{v}_1}$ to its image $U_{\vec{v}_0}$. Since $U_{\vec{v}_1}$ does not intersect x_0, $U_{\vec{v}_1} \subset U_{\vec{v}_0}$.

Let x_2 be the preimage of x_1 in $U_{\vec{v}_1}$. Then $d(x_0, x_1) = d(x_1, x_2)$, and $x_2 \in U_{\vec{v}_0}$. Hence, we can define x_n inductively by taking the preimage of x_{n-1} in $U_{\vec{v}_1}$. The union of the geodesic segments $\alpha := \cup_{k=0}^{\infty}[x_k, x_{k+1}]$ is an end which is fixed by F. It is repelling as $L(\alpha, F) = d(x_0, x_1) > 0$. □

The following lemma follows from our Theorem 7.3 and Theorem 10.83 in [6].

Lemma 3.5.2. Assume the limiting map $F : \mathbb{H}_{Berk}(^{\rho}\mathbb{C}) \to \mathbb{H}_{Berk}(^{\rho}\mathbb{C})$ has no repelling periodic ends. Then it has a fixed point $x \in \mathbb{H}_{Berk}(^{\rho}\mathbb{C})$ of local degree $\deg_x F \geq 2$.

Proposition 3.5.3. Assume the limiting map $F : \mathbb{H}_{Berk}(^{\rho}\mathbb{C}) \to \mathbb{H}_{Berk}(^{\rho}\mathbb{C})$ has no repelling periodic ends. Let $x \in \mathbb{H}_{Berk}(^{\rho}\mathbb{C})$ be a fixed point of multiplicity ≥ 2(which exists by Lemma 3.5.2.). Then the set

$$P = \bigcup_{i=0}^{\infty} F^{-i}(x)$$

is contained in a geodesic segment.

Proof. By Theorem 3.4.1., there is no totally invariant point. Hence, the preimage of x contains more than 1 point. Note it suffices to show P is contained in a line. Indeed, if we prove this, and P escapes to one end, then replace F by its second iterate if necessary, we get a repelling fixed end which is a contradiction.

We will now argue by contradiction to prove P is contained in a line. Suppose not. Then there are two points y, y' which are eventually mapped to x, and the convex hull of x, y, y' forms a 'tripod'. Replacing F by its iterate if necessary, we may assume that

$$F(y) = F(y') = x.$$

Let $\vec{v}$ be the tangent vector at x associated to y. There are two cases to consider:

Case(1): The preimage of $\vec{v}$ under $D_x F$ in $T_x \mathbb{H}_{Berk}(^{\rho}\mathbb{C})$ is infinite. Then we can construct a 'fan' as follows. Using Lemma 3.3.4., we construct $z_0 = y, z_1, ..., z_n, ...$ inductively so that F sends $[x, z_{n+1}]$ homeomorphically to $[x, z_n]$. Let $\vec{w}_k$ denote the tangent vector at x associated to z_k. Let $\vec{u}_{0,k}$ be the tangent vectors at $z_0 = y$ which is mapped to $\vec{w}_k$(there might be many such vectors, if that's the case, we just choose one).

Inductively, we let $\vec{u}_{n,k}$ be vector at z_n which is mapped to $\vec{u}_{n-1,k}$. Note that the components $U_{\vec{u}_{n-1,k}}$ are all disjoint. Since the critical tree for F is a finite tree, there is a K, such that for all $k \geq K$ and all n, the component $U_{\vec{u}_{n-1,k}}$ does not intersect the critical locus. Since $\vec{u}_{n,K}$ is mapped to $\vec{u}_{n-1,K}$, F^{K+1} is an isometric bijection from $U_{\vec{u}_{K,K}}$ to its image $U_{\vec{w}_K}$. Since the critical tree intersects $[x, y]$(as F is not injective on $[x, y]$),

$x \notin U_{\vec{u}_{K,K}}$. Now by Lemma 3.5.1., we conclude that there is a repelling periodic end of period $K+1$, which is a contradiction.

Case(2): The preimage of $\vec{v}$ under D_xF in $T_x\mathbb{H}_{Berk}(^{\rho}\mathbb{C})$ is finite. We remark that the idea is similar to the previous case, but the notations and indices become more complicated. Without loss of generality, we may assume $\vec{v}$ is totally invariant D_xF. Hence F is locally expanding in the direction $\vec{v}$. Let p be the branched point of the tripod $hull(x, y, y') \subset \mathbb{H}_{Berk}(^{\rho}\mathbb{C})$. Use Lemma 3.3.4., we choose $q \in U_{\vec{v}}$ so that F maps $[x, q]$ homeomorphically to $[x, p]$. We also choose z_1, z_1' so that F maps $[q, z_1]$ and $[q, z_1']$ homeomorphically to $[p, y]$ and $[p, y']$ respectively.

We claim that at least one of $[x, z_1]$ and $[x, z_1']$ branches off in $[x, p]$. Indeed, since F is (locally) expanding in the direction $\vec{v}$, $d(x, q) < d(x, p)$, so at least one of $[x, z_1]$ and $[x, z_1']$ must branch off $[x, p]$ (note that it may even happen before q). We assume $[x, z_1]$ branches off $[x, p]$.

We denote $z_0 = y$, q_1 as the branched point of hull(x, z_0, z_1). Note that $[x, F(q_1)] \subset [x, p] \subset [x, z_0]$. We construct a generalized 'fan' using Lemma 3.3.4. .

First, by inductively taking preimages of $F: [x, q_1] \to [x, F(q_1)]$, we construct $q_1, ..., q_n, ...$ so that F sends $[x, q_{n+1}]$ homeomorphically to $[x, q_n]$. Since F is (locally) expanding in the direction $\vec{v}$ and $[x, F(q_1)] \subset [x, p]$, $q_n \in [x, p]$ and $d(x, q_n) < d(x, q_{n-1})$ and $d(x, q_n) \to 0$.

We construct $z_0 = y, z_1, ..., z_n, ...$ inductively so that F sends $[q_{n+1}, z_{n+1}]$ homeomorphically to $[q_n, z_n]$. Let $\vec{w}_k$ denote the tangent vector at q_k associated to z_k. We define $q_{0,k}$ so that F maps $[z_0, q_{0,k}]$ homeomorphically to $[x, q_k]$. Let $\vec{u}_{0,k}$ be a tangent vector at $q_{0,k}$ which is mapped $\vec{w}_k$.

Since the critical tree is finite, for large n, the component $U_{\vec{w}_n}$ does not intersect the critical locus. Thus F is eventually an isometry between $[q_n, z_n]$ and $[q_{n-1}, z_{n-1}]$, so there exists $\epsilon > 0$ so that the length of $[q_n, z_n]$ is at least ϵ for all n. Since $d(x, q_k) \to 0$, for sufficiently large k, we can construct $q_{1,k} \in U_{\vec{w}_1}$ so that F maps $[z_1, q_{1,k}]$ homeomorphically to $[z_0, q_{0,k}]$. Inductively, for sufficiently large k, we define $q_{n,k} \in U_{\vec{w}_n}$ so that $[z_n, q_{n,k}]$ is mapped homeomorphically to $[z_{n-1}, q_{n-1,k}]$. We may assume that $q_{n,k} \in [z_n, q_{n,k-1}]$. Let $\vec{u}_{n,k}$ be a tangent vector at $q_{n,k}$ which is mapped $\vec{u}_{n-1,k}$.

The argument is now similar to Case (1). Note that the components $U_{\vec{u}_{n,k}}$ are disjoint. Since the critical tree for F is a finite tree, there is a K, such that for all $k \geq K$ and all n, the component $U_{\vec{u}_{n,k}}$ does not intersect the critical locus. Since $\vec{u}_{n,K}$ is mapped to $\vec{u}_{n-1,K}$, F^{K+1} is an isometry from $U_{\vec{u}_{K,K}}$ to its image $U_{\vec{w}_K}$. Since the critical tree intersect $[x, y]$, and $q_{K,K} \in U_{\vec{w}_K}$, so $q_K \notin U_{\vec{u}_{K,K}}$. Now by Lemma 3.5.1., we conclude that there exists a repelling periodic end of period $K + 1$, which is a contradiction. □

Let $I = [a, b]$ be the smallest geodesic segment containing P. Then F sends the boundary $\{a, b\}$ to the boundary $\{a, b\}$. As otherwise, we can find a point in P with preiamge outside of $[a, b]$, which is a contradiction.

Let $J \subset I$ be the closure of a component of $F^{-1}(\mathrm{Int}(I)) \cap I$. Then F maps J homeomorphically to I. Indeed, if the map is not injective, then there is a point $t \in \mathrm{Int}(J)$ with tangent vectors $\vec{v}_1, \vec{v}_2$ at t associated to a and b repsectively so that

$$D_t F(\vec{v}_1) = D_t F(\vec{v}_2).$$

But $D_t F$ is surjective , so there is a tangent vector $\vec{v}$ which is mapped to the tangent vector at $F(t)$ associated to either a and b. This means that P intersects non-trivially with $U_{\vec{v}}$, which is a contradiction. The map F is surjective by a similar argument: if F is not surjective and let $I' \subset I$ be the image. Note that I' is a subinterval of I, so P intersects $I - I'$ non-trivially. By Lemma 3.3.4., the preimage of $I - I'$ is not contained in I, which is a contradiction.

We also note that F has constant derivative on J. Indeed, if not, then we can find a point $t \in J$ with tangent vectors $\vec{v}_1, \vec{v}_2$ at t associated to a and b respectively so that the local degrees at $\vec{v}_1$ and $\vec{v}_2$ are different. Then, one of $D_t F(\vec{v}_i)$ has a preimage $\vec{v}$ in $T_t \mathbb{H}_{Berk}(^\rho\mathbb{C})$ other than $\vec{v}_i$. Then there is a point of P in $U_{\vec{v}}$ by Lemma 3.3.4., which is a contradiction.

By looking at the local degrees at the preimages of the point x, we conclude the sum of the derivatives on different components J equals d.

To summarize, we have the following proposition, which describes the limiting dynamics with no repelling periodic ends.

Proposition 3.5.4. Assume the limiting map $F : \mathbb{H}_{Berk}(^\rho\mathbb{C}) \to \mathbb{H}_{Berk}(^\rho\mathbb{C})$ has no repelling periodic ends. Let $x \in \mathbb{H}_{Berk}(^\rho\mathbb{C})$ be a fixed point with multiplicity ≥ 2 and $P = \cup_{i=0}^\infty F^{-i}(x)$. Let $I = [a, b]$ be the smallest geodesic segment that contains P. Then there exist $a = a_1 < b_1 \leq a_2 < b_2 \leq ... \leq a_k < b_k = b$ such that

(1) $F : [a_i, b_i] \to I$ is a linear isometry with derivative $\pm d_i$ and $d_i \in \mathbb{Z}_{\geq 2}$ and the $\pm$ sign alternating;

(2) $d = \sum_{i=1}^k d_i$.

Corollary 3.5.5. Let $t \in I = [a, b]$ which is mapped into (a, b). Then $U_{\vec{v}}$ contains no critical ends for all $\vec{v}$ at t not associated to a or b.

Proof. Assume $t \in (a_i, b_i) \subset I$, so the norm of the derivative of F at t is d_i. We first note that the local degree of $D_t F$ at $\vec{v}$ is 1. Otherwise, by counting critical points of $D_t F$, we conclude that the degree of $D_t F$ is $> d_i$. Let $\vec{w}_a, \vec{w}_b \in T_t \mathbb{H}_{Berk}({}^\rho\mathbb{C})$ and $\vec{w}'_a, \vec{w}'_b \in T_{F(t)} \mathbb{H}_{Berk}({}^\rho\mathbb{C})$ correspond to a, b. Then, there exists $\vec{w} \neq \vec{w}_a, \vec{w}_b$ so that $D_t F(\vec{w}) = \vec{w}'_a$. By Lemma 3.3.4., there exists $x \in U_{\vec{w}}$ mapped to a. This is a contradiction as I is backward invariant.

Suppose $U_{\vec{v}}$ contains a critical ends. Since the local degree of $D_t F$ at $\vec{v}$ is 1, proposition 9.41 in [6] gives that $F(U_{\vec{v}}) = \mathbb{H}_{Berk}({}^\rho\mathbb{C})$. Again, this is a contradiction as I is backward invariant. □

We start with the following lemma. The proof uses standard expansion and pullback argument. We refer the readers to [13].

Lemma 3.5.6. Let $A_1, A_0 \subset \mathbb{C}$ be two annuli with $\overline{A_1} \subset A_0$, and let $g: A_1 \to A_0$ be a degree e covering with $e \geq 2$. Then the non-escaping set $J = \cap_{k=0}^{\infty} \overline{g^{-k}(A_0)}$ is a Jordan curve.

Let $[f_n] \in \mathcal{H}$ with markings ϕ_n that gives the bounded escape. Let $F: \mathbb{H}_{Berk}({}^\rho\mathbb{C}) \to \mathbb{H}_{Berk}({}^\rho\mathbb{C})$ be the limiting map. Let $I = [a, b]$ be the geodesic segment as before, and x be a fixed point in (a, b). Then there exists an open set $U^{x-t,x+t} \subset \mathbb{H}_{Berk}({}^\rho\mathbb{C})$ with boundary points $x - t$ and $x + t$ which is mapped to $U^{x-et,x+et}$ for some integer $e \geq 2$, By Corollary 3.5.5., $U^{x-t,x+t}$ contains no critical ends.

Let U_n, V_n be sequences of annuli approximating $U^{x-t,x+t}$, $U^{x-et,x+et}$ as in Lemma 3.3.3.. Then $f_n: U_n \to V_n$ is a degree e covering ω-almost surely. Since $U^{x-t,x+t} \subset U^{x-et,x+et}$, we have $\overline{U_n} \subset V_n$ ω-almost surely.

Let N be in the ω-big set so that the above holds. Define the non-escaping set of $f_N: U_N \to V_N$ by

$$K = \bigcap_{k=0}^{\infty} \overline{f_N^{-k}(V_N)}.$$

Then K is a Jordan curve by Lemma 3.5.6..

Since all f_n comes from a single hyperbolic component, we will abuse notations and regard K as in the topological model $\sigma: J \to J$ of the Julia set. The realization of K in $J_n = J(f_n)$ will be denoted by $\phi_n(K)$.

Let $\mathcal{K}_m := \cup_{i=0}^{m} \sigma^{-i}(K)$ and $\mathcal{K} := \cup_{m=0}^{\infty} \mathcal{K}_m$.

Lemma 3.5.7. $\mathcal{K}$ is a nested set of circles.

Proof. We notice the degree of $[f_n]$ and F in J are d. By Lemma 3.5.3. and use annuli to approximate , we conclude there is a bijiective map π between $\mathcal{K}$ and P. Therefore, distribution of P in a line indicates distribution of circles in $\mathcal{K}$ as a nested set. □.

Theorem 3.5.8. Let $\mathcal{H}$ be a hyperbolic component. If $[f_n] \in \mathcal{H}$ is degenerating with markings ϕ_n such that

$$F : \mathbb{H}_{Berk}({}^{\rho}\mathbb{C}) \to \mathbb{H}_{Berk}({}^{\rho}\mathbb{C})$$

has no repelling periodic ends, then $\mathcal{H}$ has nested Julia sets.

Proof. Let $[f] \in \mathcal{H}$. Abusing the notation, we assume the topological model of the action on the Julia set is given by $f : J \to J$.

First, we will show that the Julia set J is disconnected. To show this, we will show $\sum_{i=1}^{k} \frac{1}{d_i} < 1$, where d_i is defined as in Proposition 3.5.4.. Replacing F by its second iterates and switch the role of a and b if necessary, we may assume a is fixed by F.

Let $p_n, q_n \in P$ with $p_n \to a$, $q_n \to b$, and $C_n = \pi^{-1}(p_n)$, $D_n = \pi^{-1}(q_n)$. We define A_n to be the annulus bounded by C_n and D_n, then

$$A = \cup_{n=1}^{\infty} A_n$$

is again an annulus. Let $p_{i,n}$ and $q_{i,n}$ be the $i-th$ in the linear ordering on $[a, b]$ of the preimages of p_n and q_n, and $C_{i,n} = \pi^{-1}(p_{i,n})$ and $D_{i,n} = \pi^{-1}(q_{i,n})$ respectively. Let $A_{i,n}$ be the annulus bounded by $C_{i,n}$ and $D_{i,n}$, and

$$A_i = \cup_{n=1}^{\infty} A_n.$$

For each n, since $[p_{i,n}, q_{i,n}] \subset (p_m, q_m)$ for sufficiently large m, $A_{i,n} \subset A$. Thus $A_i \subset A$ and the inclusion map is an isomorphism on fundamental groups. Since π is bijiective by Lemma 3.5.7., A_i is mapped to A as a degree d_i covering, so $m(A_i) = \frac{m(A)}{d_i}$. If $\sum_{i=1}^{k} \frac{1}{d_i} = 1$, then by the equality case of the Grotzch inequality, A_i and A_{i+1} shares a Jordan curve boundary. Since $f(A_i) = f(A_{i+1}) = A$, f has a critical point on this boundary, which is a contradiction as this boundary is in the Julia set, and f is hyperbolic.

Since $\sum_{i=1}^{k} \frac{1}{d_i} < 1$, the non-escaping set $\bigcap_{i=0}^{\infty} F^{-i}(I)$ is a Cantor set. Since $P \subset \bigcap_{i=0}^{\infty} F^{-i}(I)$, P is not dense in I, so $J = \overline{\mathcal{K}}$ is not connected.

Now we will prove every components separates two points. Let p_n be close to a, and $C_n = \pi^{-1}(p_n)$. Let U_n be bounded by C_n containing the Julia component associated with a. Then $f: U_n \to f(U_n)$ is a polynomial like map. Since the backward orbits of C_n under the polynomial like restriction are all Jordan curves, the non-escaping set of $f: U_n \to f(U_n)$ is connected.

Thus, the extremal Julia component K_a associated with a is the Julia set associated to this polynomial like restriction. Since f is hyperbolic, there exists a Fatou component U_1 of f whose boundary is contained in K_a. Similarly, we can find a Fatou component U_2 of f whose boundary is contained in K_b.

Let $p_1 \in U_1$, $p_2 \in U_2$ be two points. Then any Jordan curve in $\mathcal{K}$ separates p_1 and p_2. Since $J = \overline{\mathcal{K}}$, every component of J separates p_1, p_2. □

Remark 3.5.9. Until now, we have finish one direction of our main theorem 3.1.5. . For another direction, we need quasi-conformal surgery to construct its bounded escape sequences from its nested Julia sets. We don't plan to introduce quasi-conformal map here and refer the reader to [2] and [4].

第四章 Rescaling limit

In this chapter, we explain a result of Jan Kiwi in 2013. Jan Kiwi proved that most rescaling limits for a degenerated rational map family is postcritically finite and our main theorem 4.2.3. explains this result explicitly. Reviewing the history, rescaling limits already appear in the literature. Rescaling limits appear in Stimson's Ph.D. thesis to describe the asymptotic behavior of certain algebraic curves in quadratic moduli space, in Epstein's proof that hyperbolic components of certain type are precompact in quadratic moduli space, and in De Marco's description of a compactification of quadratic moduli space where the iteration map extends continuously. In section 4.1 and 4.2, we explain degenerated rational map families and give some examples about their rescaling limits. In section 4.3, we construct corresponding Berkovich projective line and translate the complex dynamical problem into a non-archimedean dynamical problem as we do in chapter 3. In section 4.4, we use Berkovich dynamics to get our main theorem.

4.1 Holomorphic families

In our study of rescaling limits, non-achimedean dynamics emerges once degenerate holomorphic families $f_t \subset \mathbb{P}^{2d+1}_{\mathbb{C}}$ are taken into account. Here $\{f_t\}$ is parameterized by a neighborhood of the origin in $\mathbb{C}$ and $f_t \in \partial \mathrm{Rat}_d(\mathbb{C})$ if and only if $t = 0$. Each such family may be regarded as a rational map $\mathbf{f}$ with coefficients in the field of formal Laurent series $\mathbb{C}((t))$. However, we prefer to regard the coefficients as elements of a complete and algebraically closed field containing $\mathbb{C}((t))$, which we denote by $\mathbb{L}$. Then, following ideas is standard: the action of $\mathbf{f} : \mathbb{P}^1_{\mathbb{L}} \to \mathbb{P}^1_{\mathbb{L}}$ becomes easier to understand when extended to the Berkovich projective line $\mathbb{P}^1_{Berk}(\mathbb{L})$ over $\mathbb{L}$. We will explain how the rescalings of $\{f_t\}$ correspond to some of the periodic points in the Julia set of $\mathbf{f} : \mathbb{P}^1_{\mathbb{L}} \to \mathbb{P}^1_{\mathbb{L}}$. Exploiting some basic properties of dynamics on the Berkovich projective line we will obtain our results regarding the rescaling limits of holomorphic families.

Holomorphic families. Given a neighborhood U of the origin in $\mathbb{C}$, a collection $\{f_t\}_{t\in U} \subset \mathbb{P}^{2d+1}_{\mathbb{C}}$ is a 1-dimensional holomorphic family of degree $d \geq 1$ if

$$U \to \mathbb{P}^{2d+1}_{\mathbb{C}},$$

$$t \to f_t$$

is a holomorphic map such that $f_t \in \mathrm{Rat}_d(\mathbb{C})$ for all $t \neq 0$. We say that $\{f_t\}_{t\in U}$ is a degenerate holomorphic family if $f_0 \notin \mathrm{Rat}_d(\mathbb{C})$. A holomorphic family $\{M_t\}_{t\in U}$ of degree 1 will be called a moving frame. We will drop the subscript $t \in U$ from $\{f_t\}_{t\in U}$, since all of our discussion about holomorphic families only depends on the corresponding germs at $t = 0$ and not on the domain U.

4.2 Rescaling limits and examples

Definition 4.2.1. Let $\{f_t\}$ be a holomorphic family of degree at least 2. We say that a moving frame $\{M_t\}$ is a rescaling for $\{f_t\}$ if there exist an integer $q \geq 1$, a degree $d' \geq 2$ rational map $g: \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$, and a finite subset S of $\mathbb{P}^1_{\mathbb{C}}$ such that

$$M_t^{-1} \circ f_t^q \circ M_t(z) \to g(z), \text{ as } t \to 0,$$

uniformly on compact subsets of $\mathbb{P}^1_{\mathbb{C}} \backslash S$. We say that g is rescaling limit for $\{f_t\}$ in $\mathbb{P}^1_{\mathbb{C}} \backslash S$. The minimal $q \geq 1$ such that the above holds is called the period of the rescaling $\{M_t\}$.

We will show that any q for which the above holds for some g of degree at least 2 is a multiple of the period of $\{M_t\}$.

Definition 4.2.2. We say that two moving frames $\{M_t\}$ and $\{L_t\}$ are equivalent if there exists $M \in \mathrm{Rat}_1(\mathbb{C})$ such that $M_t^{-1} \circ L_t \to M$ as $t \to 0$. The equivalence class of M_t will be denoted by $[\{M_t\}]$, and the set formed by the equivalence classes of moving frames will be denoted by $\mathscr{B}$.

It is not difficult to check that the above relation among moving frame is an equivalence relation.

Main Theorem 4.2.3. Let $\{f_t\}$ be a holomorphic family of degree $d \geq 2$ rational maps. Then there are at most $2d - 2$ pairwise dynamically independent rescalings for $\{f_t\}$ such that the corresponding rescaling limits are not postcritically finite.

Example 4.2.4. **Cubic polynomials**. Consider the family of cubic polynomials with one period 3 critical point $w = 0$ which, following Milnor, can be parameterized by

$$F_c(w) = \alpha(c)w^3 + \beta(c)w^2 + 1,$$

where

$$\alpha(c) = -\frac{c^3 + 2c^2 + c + 1}{c(c+1)^2}, \ \beta(c) = c - \alpha(c).$$

The polynomial F_c is well defined and cubic for all $c \in \mathbb{C}$ such that $\alpha(c) \neq 0, \infty$.

Polynomials of this form lead to six different degenerate holomorphic families of degree 3 that correspond to the six points $c \in \mathbb{C} \cup \{\infty\} = \mathbb{P}^1_{\mathbb{C}}$ for which $\alpha(c) = 0$ or ∞. Three of these degenerate holomorphic families will have a rescaling of period 1 while the order three will have period 3.

If $\alpha(c)$ vanishes at $c = c_0$, then the holomorphic family $\{F_{t+c_0}\}$ is a degenerate holomorphic family of degree 3, defined in a neighborhood of $t = 0$. The trivial moving frame $M_t :\ z \to z$ is a rescaling of period 1 for $\{F_{t+c_0}\}$ with rescaling limit $c_0 w^2 + 1$.

Now, for parameters c close to $c = 0$, we have that the moving frame $\{M_t :\ z \to t^2 z\}$ is a period 3 rescaling for $\{F_t\}$ with limit $g(z) = z^2$. Close to $c = -1$, the moving frame $\{z \to t^5 z\}$ is a rescaling of period 3 for the family $\{F_{-1+t}\}$ with rescaling limit $g(z) = 2z^2$. Finally, close to $c = \infty$, the moving frame $\{z \to t^4 z\}$ is a rescaling of period 3 for the family $\{F_{\frac{1}{t}}\}$ with rescaling limit $g(z) = -2z^2$.

4.3　From holomorphic families to Berkovich dynamics

To regard a holomorphic family $\{f_t\}$ as a single dynamical system we consider the field of formal Puiseux series $\mathbb{C}\langle\langle t\rangle\rangle$. This field $\mathbb{C}\langle\langle t\rangle\rangle$ is an algebraic closure of the field of formal Laurent series $\mathbb{C}((t))$. More precisely, $\mathbb{C}\langle\langle t\rangle\rangle$ is the injective limit of $\{\mathbb{C}((t^{\frac{1}{m}}))\}_{m\in\mathbb{N}}$ with the obvious inclusions. The order of vanishing at $t = 0$ induces a non-archimedean absolute value $|\ |$ on $\mathbb{C}\langle\langle t\rangle\rangle$. That is, given an element $z \in \mathbb{C}\langle\langle t\rangle\rangle$ we may consider $m \in \mathbb{N}$, $j_0 \in \mathbb{Z}$, and $c_j \in \mathbb{C}$ such that $z = \sum_{j\geq j_0} c_j t^{\frac{j}{m}}$; then, provided that $z \neq 0$, we have that

$$|z| = \exp(-\min\{\frac{j}{m} | c_j \neq 0\}).$$

Although $\mathbb{C}\langle\langle t\rangle\rangle$ is algebraically closed, it is not complete with respect to $|\ |$. For us, it is more convenient to work with the field $\mathbb{L}$ obtained as the completion of $\mathbb{C}\langle\langle t\rangle\rangle$. It follows that $\mathbb{L}$ is both complete and algebraically closed [14].

Remark 4.3.1. The elements of $\mathbb{L}$ may also be represented by series in t but now of the more general form

$$z = \sum_{j\geq 0} a_j t^{\lambda_j},$$

where $a_j \in \mathbb{C}$, $\lambda_j \in \mathbb{Q}$, and $a_j = 0$ for sufficiently large j or else $\lambda_j \to \infty$ as $j \to \infty$. The absolute value is given by $|z| = \exp(-\min\{\lambda_j \mid a_j \neq 0\})$ provided $z \neq 0$. Hence the value group $|L^*| = \exp(\mathbb{Q})$. Therefore, its Berkovich line contains Type III points. It is also not hard to know the residue field $\tilde{\mathbb{L}}$ is isomorphic to the complex field $\mathbb{C}$.

Definition 4.3.2. Consider a degree $d \geq 1$ holomorphic family $\{f_t\}$. We may write

$$f_t(z) = \frac{a_0(t)z^d + \ldots + a_d(t)}{b_0(t)z^d + \ldots + b_d(t)}.$$

where $a_j(t)$, $b_j(t)$ are holomorphic functions whose domains contain a neighborhood of the origin, for all $j = 0, \ldots, d$. Let $\mathbf{a_j}$, $\mathbf{b_j} \in \mathbb{L}$ be the Taylor series at $t = 0$ of the holomorphic functions $a_j(t), b_j(t)$. Then the degree d rational map $\mathbf{f} : \mathbb{P}^1_{\mathbb{L}} \to \mathbb{P}^1_{\mathbb{L}}$ given by

$$\mathbf{f}(z) = \frac{\mathbf{a_0}z^d + \ldots + \mathbf{a_d}}{\mathbf{b_0}z^d + \ldots + \mathbf{b_d}}$$

is called the rational map associated to $\{f_t\}$.

Lemma 4.3.3. Consider a degree $d \geq 1$ holomorphic family $\{f_t\}$ with associated rational map $\mathbf{f} : \mathbb{P}^1_{\mathbb{L}} \to \mathbb{P}^1_{\mathbb{L}}$. Then, as $t \to 0$,

$$f_t \to \tilde{\mathbf{f}}$$

uniformly on compact subsets of $\mathbb{P}^1_{\mathbb{C}} \backslash \mathscr{H}(\mathbf{f})$.

Proof. Left it to the reader. □

Proposition 4.3.4. Let $\{f_t\}$ be a degree $d \geq 2$ holomorphic family, and let $\{M_t\}$ be a moving frame. Consider the rational map $\mathbf{f} : \mathbb{P}^1_{Berk}(\mathbb{L}) \to \mathbb{P}^1_{Berk}(\mathbb{L})$ associated to $\{f_t\}$, and consider the automorphism $\mathbf{M} : \mathbb{P}^1_{Berk}(\mathbb{L}) \to \mathbb{P}^1_{Berk}(\mathbb{L})$ associates to $\{M_t\}$.

Then, for all $l \geq 1$, the following assertions are equivalent:

(1) There exists a rational map $g : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ of degree at least $d' \geq 1$ such that, as $t \to 0$,

$$M_t^{-1} \circ f_t^l \circ M_t(z) \to g(z)$$

uniformly on compact subsets of $\mathbb{P}^1_{\mathbb{C}}$ with finitely many points removed.

(2) $\mathbf{f}^l(x) = x$ where $x = \mathbf{M}(x_g)$ and $\deg_x \mathbf{f}^l = d' \geq 1$.

In the case in which (1) and (2) hold, $T_x\mathbf{f}^l :\to T_x\mathbb{P}^1_{Berk}(\mathbb{L}) \to \mathbb{P}^1_{Berk}(\mathbb{L})$ is conjugate via a $\mathbb{P}^1_{\mathbb{C}}$ to $g : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$.

Proof. (1) $\Rightarrow$ (2). Note that the rational map associated to $\{M_t^{-1} \circ f_t^l \circ M_t\}$ is $\mathbf{F} = \mathbf{M}^{-1} \circ \mathbf{f}^l \circ \mathbf{M}$. From Lemma 4.3.3., the reduction of $\mathbf{F}$ has degree at least $d' \geq 1$. Thus, by Proposition 2.3.2. $\mathbf{F}(x_g) = x_g$ and $\deg_{x_g}(\mathbf{F}) = d'$. It follows that $\mathbf{f}^l(\mathbf{M}(x_g)) = \mathbf{M}(x_g)$. Moreover, $\deg_x(\mathbf{f}) = \deg_{x_g}(\mathbf{F})$ for $x = \mathbf{M}(x_g)$ since the local degree remains unchanged under pre-composition and post-composition by automorphisms.

(2) $\Rightarrow$ (1). From Lemma 4.3.3., as $t \to 0$, outside a finite set, $M_t^{-1} \circ f_t^l \circ M_t(z) \to \tilde{\mathbf{F}}$

where $\mathbf{F} = \mathbf{M}^{-1} \circ \mathbf{f}^l \circ \mathbf{M}$. By proposition 2.3.2., $\tilde{\mathbf{F}}$ has degree $\mathbf{deg}_x(\mathbf{f}^l) \geq 1$, since $\mathbf{f}^l(x) = x = \mathbf{M}(x_g)$. □

Corollary 4.3.5. Give a degree $d \geq 2$ holomorphic family $\{f_t\}$ and a moving frame $\{M_t\}$, consider the subset S of $\mathbb{N}$ formed by all integers l such that Proposition 4.3.4 holds for some g with $\mathbf{deg}(g) \geq 2$. Then S is empty or $S = q \cdot \mathbb{N}$ for some $q \geq 1$.

Lmma 4.3.6. Let $\{M_t\}$ and $\{L_t\}$ be moving frames. Denote by $\mathbf{M}$ and $\mathbf{L}$ the associated Mobius transformations acting on $\mathbb{P}^1_{Berk}(\mathbb{L})$. The moving frames $\{M_t\}$ and $\{L_t\}$ are equivalent iff $\mathbf{M}(x_g) = \mathbf{L}(x_g)$.

Proof. Left it to the reader. □

Lemma 4.3.7. Let $\{f_t\}$ be a holomorphic family of degree $d \geq 1$. If $\{M_t\}$ is a moving frame, then there exists a moving frame $\{L_t\}$ such that the following equivalent conditions hold:

(1) $\mathbf{f} \circ \mathbf{M}(x_g) = \mathbf{L}(x_g)$.

(2) $L_t^{-1} \circ f_t \circ M_t(z) \to g(z)$ for some non-constant rational map $g: \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ with uniform convergence on compact subsets outside some finite set.

In this case, the reduction of $\mathbf{L}^{-1} \circ \mathbf{f} \circ \mathbf{M}$ coincides with g.

Proof. The equivalence of (1) and (2) can be immediately proved from our discussion before.

Write $g_t = f_t \circ M_t$ as $\frac{P_t}{Q_t}$ where

$$P_t(z) = a_0(t)z^d + \ldots + a_d(t),$$

$$Q_t(z) = b_0(t)z^d + \ldots + b_d(t),$$

for some holomorphic functions $a_j(t), b_j(t)$ defined on a neighborhood of $t = 0$. Taking $A_t(z) = t^m z$ or $\frac{z}{t^m}$ for some $m \geq 0$ and replacing g_t by $A_t \circ g_t$, we may assume that $a_j(0) \neq 0$ for some j and $b_l(0) \neq 0$ for some l. It follows that $\tilde{\mathbf{P}} \neq 0$ and $\tilde{\mathbf{Q}} \neq 0$. If $\tilde{\mathbf{P}} \neq c \cdot \tilde{\mathbf{Q}}$ for all $c \in \mathbb{C}$, then $\tilde{\mathbf{g}}$ has degree at least 1 and, we have $\mathbf{g}(x_g) = x_g$ as required. Otherwise, $\tilde{\mathbf{P}} = c \cdot \tilde{\mathbf{Q}}$ for some $c \in \mathbb{C}$. Hence, all the coefficients of $P_t(z) - cQ_t(z)$ vanish at $t = 0$. Take $c_0 \in \mathbb{C}$ such that $\max|\mathbf{a}_j - c_0\mathbf{b}_j|$ attains its minimum, say, $|t^{l_0}|$, and observe that $\mathbf{N} \circ \mathbf{g} = \frac{\mathbf{P} - c_0\mathbf{Q}}{t^{l_0}\mathbf{Q}}$ has reduction of degree at least 1. Thus, we have $\mathbf{N} \circ \mathbf{g}(x_g) = x_g$ and let $\mathbf{L} = \mathbf{N}^{-1}$. □

We conclude that the set $\mathscr{B}$ of all equivalence classes of moving frames is naturally embedded into the set of type II points of $\mathbb{P}^1_{Berk}(\mathbb{L})$:

$$\tau: \mathscr{B} \to \mathbb{P}^1_{Berk}(\mathbb{L}),$$

$$[\{M_t\}] \to \mathbf{M}(x_g).$$

We also know the moving frame $\{L_t\}$ of the previous lemma is unique modulo equivalence. We write

$$\{f_t\}([\{M_t\}]) = [\{L_t\}].$$

Note that

$$\{f_t\}(\{g_t\}([\{M_t\}])) = \{f_t \circ g_t\}([\{M_t\}]),$$

for all holomorphic families $\{g_t\}$. Also,

$$\mathbf{f} \circ \tau([\{M_t\}]) = \tau \circ \{f_t\}([\{M_t\}])$$

for all $\{M_t\} \in \mathscr{B}$.

Definition 4.3.8. Consider a holomorphic family $\{f_t\}$. Assume that $\{M_t\}$ and $\{L_t\}$ are dynamically dependent rescalings if $\{f_t^l\}([\{M_t\}]) = [\{L_t\}]$, for some $l \geq 0$.

Corollary 4.3.9. Consider a holomorphic family $\{f_t\}$ of degree $d \geq 2$. If $\{M_t'\}$ and M_t are dynamically independent rescalings, then the periodic orbits of $x' = \tau([\{M_t'\}])$ and $x = \tau([\{M_t\}])$, under iterations of $\mathbf{f}$, are distinct.

Lemma 4.3.10. Assume that the moving frames $\{M_t\}$ and $\{L_t\}$ are dynamically dependent rescalings for $\{f_t\}$ with rescaling limits g and h, respectively. Then there exist rational maps S and R such that $g = S \circ R$ and $h = R \circ S$.

Proof. By the definition of dynamical dependence there exist integers $1 < l < q$ such that $\mathbf{f}^l \circ \mathbf{M}(x_g) = \mathbf{L}(x_g)$ and $\mathbf{f}^{q-l} \circ \mathbf{L}(x_g) = \mathbf{M}(x_g)$. Let R and S be the action in the tangent space at $\mathbf{M}(x_g)$ and $\mathbf{L}(x_g)$ of $\mathbf{f}^l$ and $\mathbf{f}^{q-l}$, respectively. Then the action of $\mathbf{f}^q$ in these tangent spaces is $S \circ R$ and $R \circ S$,respectively. Therefore, we have that $g = S \circ R$ and $h = R \circ S$. □

The property of having a postcritically finite is invariant under dynamical dependence.

Lemma 4.3.11. Let R and S be two non-constant complex rational maps. Consider $g = S \circ R$ and $h = R \circ S$. Then the following statements hold:

(1) g is hyperbolic iff h is hyperbolic.

(2) g is postcritically finite iff h is postcritically finite.

Proof. Left it to the reader. □

4.4 Dynamics in the Berkovich projective line and rescalings

Now we are concerned with the critical points in the "basin" of these periodic orbits. More precisely, given a rational map $\phi : \mathbb{P}^1_{Berk}(\mathbb{L}) \to \mathbb{P}^1_{Berk}(\mathbb{L})$ of degree $d \geq 2$ with a periodic orbit $\mathcal{O}$, we say that the basin of $\mathcal{O}$ is the interior of the set of points $x \in \mathbb{P}^1_{Berk}(\mathbb{L})$ so that, for all neighborhoods U of $\mathcal{O}$, the orbits of x is eventually contained in U.

Periodic orbits of rational maps $\phi : \mathbb{P}^1_{Berk}(\mathbb{L}) \to \mathbb{P}^1_{Berk}(\mathbb{L})$ may be classified according to whether they belong to the Julia set $J(\phi)$ or to the Fatou set $F(\phi)$. A point $x \in \mathbb{P}^1_{Berk}(\mathbb{L})$ lies in the Julia set [6] if for all neighborhoods U of x we have that $\bigcup \phi^n(U)$ omits at most two points of $\mathbb{P}^1_{Berk}(\mathbb{L})$. As usual, the complement of the Julia set is the Fatou set. A periodic point $x \in \mathbb{P}^1_{Berk}(\mathbb{L}) \backslash \mathbb{P}^1_{\mathbb{L}}$ of period q lies in the Julia set $J(\phi)$ if either $\deg_x(\phi^q) \geq 2$ or there exists a bad direction of ϕ^q at x with infinite forward orbit under $T_x\phi^q$ [15]. In the first case, x is called a repelling periodic point. In the latter case, when $\deg(T_x\phi) = 1$, we say that x is an indifferent periodic point. Every non-typical periodic point that belongs to the Julia set is of type II [6]. It is worthwhile to emphasize that, despite its name, every repelling periodic point x of type II has nonempty basin. In fact, without loss of generality assume that x is fixed under ϕ. Let $\vec{v} \in T_x\mathbb{P}^1_{Berk}(\mathbb{L})$ be a good direction so that its iterates under $T_x\phi$ are pairwise distinct good directions. Such $\vec{v}$ always exists since $\deg T_x\phi \geq 2$ and the residue field is uncountable. It follows that $\{\phi^n(U_{\vec{v}})\}$ are pairwise disjoint Berkovich open disks with boundary points x. Hence, every point in $U_{\vec{v}}$ belongs to the basin of x because, in the weak topology of $\mathbb{P}^1_{Berk}(\mathbb{L})$, every neighborhood of x contains all but finitely many directions at x.

According to the previous discussion, rescalings give rise to type II repelling periodic orbits. Our next result explores under which conditions the basin of a periodic orbit does not contain classical critical points.

Theorem 4.4.1. Consider a rational map $\phi : \mathbb{P}^1_{Berk}(\mathbb{L}) \to \mathbb{P}^1_{Berk}(\mathbb{L})$ with $\deg(\phi) \geq 2$. Let $\mathcal{O}$ be a type II periodic orbit of period $q \geq 1$ of ϕ. Assume that the basin of $\mathcal{O}$ is free of type I critical points. Then, for all $x \in \mathcal{O}$, every bad direction of ϕ^q at x has finite forward orbit under $T_x\phi^q$. Moreover, if $\deg_x(\phi^q) \geq 2$, then $T_x\phi^q$ is postcritically finite.

The proof relies on the fact that any bad direction contains a rigid critical point and its "images" contains a critical value. This is a particular and simple result about the ramification locus of a rational map which we state and prove in the lemma below. See the more general work by [10] and [11], for a detailed study of the ramification locus.

Lemma 4.4.2. Let $\phi : \mathbb{P}^1_{Berk}(\mathbb{L}) \to \mathbb{P}^1_{Berk}(\mathbb{L})$ be a rational map of degree at least 2.

Consider a type II point $x \in \mathbb{P}^1_{Berk}(\mathbb{L})$. Let $\vec{v} \in T_x\mathbb{P}^1_{Berk}(\mathbb{L})$, and consider $\vec{w} = T_x\phi(\vec{v}) \in T_{\phi(x)}\mathbb{P}^1_{Berk}(\mathbb{L})$.

If $\phi: U_{\vec{v}} \to \phi(U_{\vec{v}})$ is not injective, then there exists a typical critical point $c \in U_{\vec{v}}$ such that the corresponding critical value $\phi(c) \in U_{\vec{w}}$.

To prove the lemma, we will use two basic results about rational maps acting on Berkovich space. The first one is the non-archimedean Rolle's theorem [6].

Lemma 4.4.3. Let $U \subset \mathbb{P}^1_{\mathbb{L}}$ be a disk where a non-constant rational map $\phi \in \mathbb{L}(z)$ has no poles. If ϕ has two or more zeros in U, then the derivative $\phi'(z)$ has at least one zero in U.

An immediate consequence is that if $\phi: U \to \mathbb{L}$ is not one-to-one, then U contains a critical point.

The other result that we will need states that the image of a finite subgraph $\Gamma \subset \mathbb{P}^1_{Berk}(\mathbb{L})$ under a non-constant rational map is again a finite subgraph of $\mathbb{P}^1_{Berk}(\mathbb{L})$ [6].

Proof of Lemma 4.4.2. If $\vec{v}$ is a good direction at x, then the result follows directly from the non-archimedean Rolle's theorem.

Assume that $\vec{v}$ is a bad direction. According to Proposition 2.3.3., there exists at least one preimage of $\phi(x)$ in $U_{\vec{v}}$. Denote by V the connected component $U_{\vec{v}} \backslash \phi^{-1}(\phi(x))$ such that $x \in \partial V$, and observe that V is strictly contained in $U_{\vec{v}}$. Then we have $\phi(V) = U_{\vec{w}}$.

Let $\mathscr{A}_V$ be the convex hull of ∂V. It follows that $\mathscr{A}_V$ is a finite subgraph of $\mathbb{P}^1_{Berk}(\mathbb{L})$. Hence, $\phi(\mathscr{A}_V)$ is a finite subgraph of $\mathbb{P}^1_{Berk}(\mathbb{L})$.

Note that $\mathscr{A}_V$ is formed by type II and type III points. Given $x \in \mathscr{A}_V$, since we know $T_x(\phi)$ is injective at type III points and all the endpoints of $\mathscr{A}_V$ are mapped onto $\phi(x)$, there must exist a type II point $y \in \mathscr{A}_V$ with valence $k \geq 2$ which is mapped to another end point of $\phi(\mathscr{A}_V)$. Therefore, we have $\deg(T_y\phi) \geq k \geq 2$.

All rational maps of degree 2 or more have at least two critical values. Hence, $T_y\phi: T_y\mathbb{P}^1_{Berk}(\mathbb{L}) \to T_{\phi(y)}\mathbb{P}^1_{Berk}(\mathbb{L})$ has at least two critical values. So there exists a critical value $\vec{b}$ of $T_y\phi$ which is a direction at $\phi(y)$ distinct from the one determined by $\phi(\mathscr{A}_V)$. The corresponding critical point $\vec{c}$ of $T_y\phi$ is a direction at y distinct from those determined by the elements of ∂V. Thus, the corresponding disk $U_{\vec{c}}$ is completely contained in V. Therefore, $\phi(U_{\vec{c}}) = U_{\vec{b}}$ and $\phi: U_{\vec{c}} \to U_{\vec{b}}$ is of degree at least 2. By non-archimedean Rolle's theorem, the disk $U_{\vec{c}}$ contains a critical point, $U_{\vec{b}}$ contains a critical value, and the lemma follows. □

Proof of Theorem 4.4.1. If there exists a direction $\vec{w}$ at $x \in O$ which is a bad direction for ϕ^q or which is a critical point of $T_x\phi^q$, then $\phi^q : U_{\vec{w}} \to \phi^q(U_{\vec{w}})$ is not injective and therefore $U_{\vec{w}}$ contains a typical critical point of ϕ^q.

As we notice that bad directions at x are finite, we use Lemma 4.4.2. and we can get a type I critical point in the basin of O if the forward orbit of a bad direction is infinite. Contradiction. □

Corollary 4.4.4. Consider a rational map $\phi : \mathbb{P}^1_{Berk}(\mathbb{L}) \to \mathbb{P}^1_{Berk}(\mathbb{L})$ of degree $d \geq 2$. Then ϕ has at most $2d-2$ repelling type II periodic orbits O such that $T_x\phi^q$ is not postcritically finite for all $x \in O$ where q is the period of O.

Proof of Main Theorem 4.2.3. Suppose that $\{M_t^{(1)}\}$,..., $\{M_t^{(N)}\}$ are pairwise dynamically independent rescalings for $\{f_t\}$ of period $q_1, ..., q_N$, with postcitically infinite rescaling limits. Let $x_j = \mathbf{M}^{(j)}(x_g) \in \mathbb{P}^1_{Berk}(\mathbb{L})$ for $j = 1, ..., N$. From Proposition 4.3.4., we have that $T_{x_j}\mathbf{f}^{q_j} : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ is not postcritically finite, for all $j = 1, ..., N$. In view of Corollary 4.3.9., the points $x_1, ..., x_N$ lie in pairwise distinct periodic orbits of $\mathbf{f}$. It follows that $N \geq 2d-2$. □

参考文献

[1] MILNOR J. Dynamics in One Complex Variable[M]. [S.l.]: Princeton University Press, 2006.

[2] BRANNER B, FAGELLA N. Quasiconformal Surgery in Holomorphic Dynamics[M]. [S.l.]: Cambridge University Press, 2014.

[3] JONSSON M. Dynamics on Berkovich spaces in low dimensions[Z]. 2014. arXiv: 1201.1944 [math.DS].

[4] LUO Y. Trees, length spectra for rational maps via barycentric extensions and Berkovich spaces[Z]. 2021. arXiv: 1907.06220 [math.DS].

[5] KIWI J. Rescaling limits of complex rational maps[J/OL]. Duke Mathematical Journal, 2015, 164(7). https://doi.org/10.1215. DOI: 10.1215/00127094-2916431.

[6] BAKER M, RUMELY R. Potential Theory and Dynamics on the Berkovich Projective Line[M]. [S.l.]: American Mathematical Society, 2010.

[7] Rivera-Letelier, Juan. Geometric Methods in Dynamics (II) Volume in honor of Jacob Palis[M]. [S.l.]: Society of Mathematics in France, 2003.

[8] MCMULLEN C T. Complex Dynamics and Renormalization[M]. Princeton: Princeton University Press, 1995.

[9] KAPOVICH M, LEEB B. On asymptotic cones and quasi-isometry classes of fundamental groups of 3-manifolds[J]. Geometric & Functional Analysis GAFA, 1995.

[10] FABER X. Topology and Geometry of the Berkovich Ramification Locus for Rational Functions[Z]. 2013. arXiv: 1102.1432 [math.NT].

[11] FABER X. Topology and geometry of the Berkovich ramification locus for rational functions, II[J/OL]. Mathematische Annalen, 2012, 356(3): 819-844. https://doi.org/10.1007. DOI: 10.1007/s00208-012-0872-3.

[12] LUO Y. Limits of rational maps, R-trees and barycentric extension[Z]. 2021. arXiv: 1905.00915 [math.GT].

[13] PILGRIM K, TAN L. Rational maps with disconnected Julia set[C]//. [S.l. : s.n.], 2019.

[14] RIBENBOIM P. The Theory of Classical Valuations[M]. [S.l.]: Springer Press, 1999.

[15] KIWI J. Puiseux series dynamics of Quadratic rational maps[Z]. 2011. arXiv: 1106.0059 [math.DS].