SPECIAL CASES OF RATNER'S THEOREMS AND OPPENHEIM'S CONJECTURE

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ABSTRACT. In this article, we study Oppenheim's conjecture and Ratner's theorems on $SL(2,\mathbb{R})$. We translate Oppenheim's conjecture into a homogeneous dynamics problem and prove it using Ratner's theorems. We also give detailed proofs for Ratner's theorems on $SL(2,\mathbb{R})$. At the end of this article, we give a proof of Oppenheim's conjecture without using Ratner's theorems.

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1. INTRODUCTION

Homogeneous dynamics has a long history and is an important branch of modern dynamics. During its development, there emerged many connections with other mathematical fields, especially with number theory. G.A. Margulis proved Oppenheim's conjecture using unipotent flows under the homogeneous space frame, and Lindenstrauss was awarded the Fields Medal for his contribution to Littlewood's conjecture in 2010. Along the way to attacking these two conjectures, people found unipotent flows play an essential role in Oppenheim's conjecture while diagonal flows play an essential role in Littlewood's conjecture.

These two kinds of flows: unipotent flows and diagonal flows, have been at the center of homogeneous dynamics. Many results on their invariant (ergodic) measures and topological properties of orbits have been established. These two kinds of flows have many similar properties but are essentially different. On the one hand, people find measure and topology rigidity properties of unipotent flows-Ratner's theorems, on the other hand, people think the behavior of diagonal flows are wild and chaotic.

Conjecture 1.1. *Weak Oppenheim's Conjecture*[2]*: let Q be a non-degenerate indefinite irrational quadratic form in n variables, $n \geq 3$. Then 0 is contained in the closure of $Q(\mathbb{Z}^n \setminus \mathbf{0})$.*

Date: September 1st, 2022.

Conjecture 1.2. *Strong Oppenheim's Conjecture*[2]*: let Q be a non-degenerate indefinite irrational quadratic form in n variables, $n \geq 3$. Then $Q(\mathbb{Z}^n)$ is dense in $\mathbb{R}$.*

We say a quadratic form is rational, if it is proportional to a form with rational coefficients. Otherwise, we say this form is irrational.

Before M.S. Raghunathan understood Oppenheim's conjecture from the point of view of unipotent flows, mathematicians strove to prove this conjecture with analytic number theory methods, but only partially solved it. After Raghunathan shed new light on this conjecture, Margulis successfully gave a complete proof of it as a result of precisely topological properties of orbits on homogeneous space. Before Margulis settled this conjecture, Raghunathan and Dani had already conjectured a much more general case of orbit closure and measure rigidity of unipotent flow. Eventually, Ratner solved these conjectures on unipotent flows.

Ratner's Theorems

Theorem 1.3. *Classification of Invariant Measures*[7]*: let G be a connected Lie group, and Γ be a discrete subgroup of G. Let H be a connected Lie subgroup of G generated by Ad-unipotent one-parameter groups. Then every ergodic H-invariant Borel probability measure on G/Γ is algebraic.*

"Ad-unipotent" means the image of H under the adjoint representation in $GL(\mathfrak{g})$ consists of unipotent matrices.

"Algebraic " means the following properties hold:

1. there is a connected closed subgroup $L \geq H$ of G, and there exists a point $x_0 \in G/\Gamma$ such that Lx_0 is closed in G/Γ;

2. this H-invariant ergodic measure is also L-invariant and its support is exactly Lx_0 .

Theorem 1.4. *Uniform Distribution*[7]*: let G be a connected Lie group, and Γ be a lattice of G. Let u_t be an Ad-unipotent one-parameter subgroup of G. Then, for any $x \in G/\Gamma$, the orbit $u_t x$ is uniformly distributed with respect to an algebraic measure μ_x on G/Γ.*

"Uniform distribution" means there is a connected closed subgroup $L \geq \{u_t\}$ of G such that

1. Lx is closed and supports an L-invariant measure μ_x;

2. $u_t x$ is dense in Lx and μ_x is an invariant ergodic measure of u_t;

3. x is a generic point of μ_x, i.e. for every compactly-supported continuous function $f : G/\Gamma \to \mathbb{R}$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(u_t x)dt = \int f(y)\mu_x(y)\,.$$

Theorem 1.5. *Topological Rigidity*[7]*: let G be a connected Lie group, and Γ be a lattice of G. Let H be a connected Lie subgroup of G generated by Ad-unipotent one-parameter subgroups. Then for every $x \in G/\Gamma$, there exists $H \leq L \leq G$ and $V \leq H$ some one-parameter Ad-unipotent subgroup of G such that*

1. Lx is closed and supports an L-invariant probability measure μ_L;

2. $\overline{Hx} = \overline{Vx} = Lx$;

3. μ_L is V-invariant and V-ergodic.

From a logical line, Ratner's measure rigidity property can be used to deduce the uniform distribution theorem, and then deduce the topological rigidity property. So, why should we study invariant ergodic measures? Without measure theory, it is difficult to say more words about the topological properties of orbits. Through the classification of invariant measures, we can see the distribution of orbits more clearly!

The proofs of the general Ratner's theorems are difficult to understand though the argument is elementary in Ratner's original articles. Although the author doesn't give a complete proof for general Ratner's theorems, the special case on $SL(2,\mathbb{R})$[5] can also illustrate some original ideas of Ratner.

Littlewood's Conjecture[4]: Let α, β be two arbitrary real numbers. $|| \ ||$ is used to represent the distance to its nearest integer. Then

$$\varliminf_{n\to\infty} n||n\alpha||||n\beta|| = 0\,.$$

The strongest result of this conjecture was due to Einsiedler, Katok and Lindenstrauss. They proved that all pairs (α,β) not satisfying Littlewood's conjecture in $\mathbb{R}^2$ is Hausdorff dimension 0.

2. Oppenheim's Conjecture: A Dynamical Version

In this section, the author puts Oppenheim's conjecture under homogeneous space frame.

Firstly, we carefully analyze Oppenheim conjecture. When $n=2$, Oppenheim's conjecture is not true. For example, we consider

$$Q(x_1,x_2) = x_1x_2 - \sqrt{2}x_2^2\,.$$

We know $\sqrt{2}$ is a badly approximable irrational number. Using continued fractions, it is not hard to know that $|\sqrt{2}-\frac{p}{q}| \geq \frac{C}{q^2}$ for some positive constant C and any rational number $\frac{p}{q}$. Equivalently, we mean the absolute value of $Q(x_1,x_2) = x_1x_2-\sqrt{2}x_2^2$ is bigger than positive constant C unless $x_2=0$. Hence, Oppenheim's conjecture fails when $n=2$.

Secondly, we can't revise "indefinite" as "definite". Because of standard congruence forms of symmetric matrices, we know $Q(\mathbb{Z}^n)$ is discrete on $\mathbb{R}$. And if we consider rational form, it is easy to prove discreteness of $Q(\mathbb{Z}^n)$ on $\mathbb{R}$.

Thirdly, we can only use the word "dense". 0 may or may not be represented non-trivially by a fixed quadratic form. For example, $Q(x_1,x_2,x_3) = x_1x_3-\sqrt{2}x_2^2$ can non-trivially represent 0, while $Q(x_1,x_2,x_3) = x_1^2+x_2^2-\sqrt{2}x_3^2$ can only trivially represent 0.

Fourthly, we can spend some efforts to reduce general Oppenheim conjecture to the case $n=3$. That's because we can restrict this quadratic form on a hyperplane with the same property. We give this reduction at the end of this article.

Now, we translate Oppenheim's conjecture into a dynamical problem.

Notations and Preliminaries

$k = \mathbb{Z},\mathbb{Q},\mathbb{R}$
$G = SL(3,\mathbb{R})$
$\Gamma = SL(3,\mathbb{Z})$
$X_3 = G/\Gamma$
$Q_0 = 2x_1x_3 - x_2^2$
$SO_{Q_0}(k) = \{g \in SL(3,k) \mid g^tQ_0g = Q_0\}$

$SO_Q(k) = \{g \in SL(3,k) \mid g^t Q g = Q\}$

We identify quadratic form with its symmetric matrix form. Some basic facts from Lie theory tell us $SL(n,\mathbb{R})$ is unimodular, i.e. its left Haar measure coincides with right Haar measure. $\Gamma = SL(n,\mathbb{Z})$ is a lattice in $SL(n,\mathbb{R})$, i.e. it is a discrete subgroup and the volume(w.r.t Haar measure) of quotient space $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$ is finite. The author wants to emphasize not all Lie groups are unimodular, and the Haar measure of unimodular Lie groups can induce the Haar measure on quotient spaces G/Γ, even if Γ just is discrete but not a lattice. We can also choose a right invariant Riemannian metric on $SL(n,\mathbb{R})$(it is not a left invariant metric even if it is unimodular) and it induces a metric on quotient spaces(metric topology and quotient topology coincide on the quotient spaces).

We can just admit these facts and do not care about these minor details.

Definition 2.1. A subgroup Λ of $\mathbb{R}^n$ is a (unimodular) lattice if Λ is discrete and $\mathrm{Vol}(\mathbb{R}^n/\Lambda)$ is finite ($\mathrm{Vol}(\mathbb{R}^n/\Lambda)$=1).

All of unimodular lattices Λ of $\mathbb{R}^n$ construct a set, equipped with Chabauty topology, which is homeomorphic with quotient space $X_n = SL(n,\mathbb{R})/SL(n,\mathbb{Z})$.

About Chabauty topology, it suffices for us to know how a sequence of unimodular lattices converges to a specific unimodular lattice. Explicitly, $\Lambda_m \in X_n$ converges to a $\Lambda \in X_n$ if and only if we can find a basis $v_1^m, v_2^m, , , , v_n^m$ of Λ_m so that as $m \to \infty$, v_i^m converges to some $v_i^\infty \in \mathbb{R}^n$ for every i, and (v_i^∞) $\mathbb{Z}$-span Λ.

Definition 2.2. For a discrete subgroup $\Lambda \leq \mathbb{R}^n$ we define

$$sys(\Lambda) = \inf_{0 \neq v \in \Lambda} ||v|| \,.$$

Clearly, we have $C \geq sys(\Lambda) > 0$, if Λ is a unimodular lattice, where C is a positive constant only related with dimension n. It is easy to notice $sys : X_n \to \mathbb{R}_{>0}$ is continuous. However, this map is also proper! We have the following Mahler's criterion.

Proposition 2.3. [7] *A subset $S \subset X_n$ is bounded if and only if there exists $\epsilon > 0$ such that $S \subset X_n^\epsilon = \{\Lambda \in X_n \mid sys(\Lambda \geq \epsilon)\}$.*

Proof. First, if S is bounded, then its closure is bounded hence compact. Because sys is a continuous map, its image is also compact, hence S is contained in some X_n^ϵ.

Second, it is enough to illustrate the compactness of X_n^ϵ. We illustrate that any sequence in X_n^ϵ has a convergent subsequence. We notice a fact: there exist two positive numbers c and C, only related with n and ϵ, for any $\Lambda \in X_n^\epsilon$, there exists a $\mathbb{Z}$-span basis $(v)_i$ such that $c < |v_i| < C$ for every i. If we have this fact, X_n^ϵ is compact apparently.

We prove this fact. First step, we choose a vector v_1 such that $|v_1| = sys(\Lambda)$. We have $c < |v_1| < C$. And v_1 must be primitive. Second step, we consider the orthogonal space V_1 of $\mathrm{span}(v_1)$ and consider the projection onto V_1. The Λ is projected to Λ_1, a lattice in V_1. Clearly, we have $\mathrm{Vol}(V_1/\Lambda_1) \leq \frac{1}{\epsilon}$. We can consider $u_2 \in \Lambda_1$ such that $|u_2| = sys(\Lambda_1)$. It is easily to know $|u_2|$ can not be too large, however we can also know $|u_2| \geq \frac{\sqrt{3}\epsilon}{2}$. That's because we can choose $v_2 \in \Lambda$ as a preimage of u_2, and consider $\mathbb{Z}$-coefficents combination of v_1 and v_2, whose absolute value is larger than v_1. Then we get $|u_2| \geq \frac{\sqrt{3}\epsilon}{2}$. Therefore, we can choose an appropriate v_2 such that $c < |v_2| < C$. And $\mathbb{Z}$-span (v_1, v_2) must be primitive. Still do these steps, we can get a basis have this property.

□

In the following situation, we focus on $n = 3$.

Proposition 2.4. *For a non-degenerate quadratic form Q in 3 variables with real coefficients, the following two are equivalent:*

1. the closure of $Q(\mathbb{Z}^3 \setminus \mathbf{0})$ contains 0;

2. the orbit closure of $SO_Q(\mathbb{R})$ based at the identity coset is unbounded in X_3. Equivalently, $sys(\Lambda)$, Λ in the orbit, can be arbitrary small.

One direction in this proposition is obvious, but how deduce 2 from 1 is not apparent at the first glance. We need the following fact.

Lemma 2.5. *For every $r_{\neq 0} \in \mathbb{R}$, $SO_Q(\mathbb{R})$ acts transitively on the level set*

$$V_r = \{v \in \mathbb{R}^3 \mid Q(v) = r\}.$$

When $r = 0$, $SO_Q(\mathbb{R})$ acts transitively on $V_0 \setminus \mathbf{0}$(if not empty).

Proof. First step. When Q is a positive or negative form, we know orthogonal group acts transitively on unit sphere. Hence, the lemma is true.

Second step. If Q is an indefinite form, we can assume its positive index is l, negative index is s and $l + s = n$. It is not hard to know $SO_Q(\mathbb{R})$ contains(under conjugation) the following matrix:

$$\begin{bmatrix} SO(l, \mathbb{R}) & 0 \\ 0 & SO(s, \mathbb{R}) \end{bmatrix}.$$

Besides, we also notice that $SO_{x_1^2 - x_{l+1}^2}(\mathbb{R})$ can be embedded in $SO_Q(\mathbb{R})$. We calculate $SO_{x_1^2 - x_{l+1}^2}(\mathbb{R})$:

$$\begin{bmatrix} \sqrt{k^2+1} & k \\ k & \sqrt{k^2+1} \end{bmatrix}$$

$k \in \mathbb{R}$. We know $SO_{x^2-y^2}(\mathbb{R})$ acts transitively on $\{(x, y) \mid x^2 - y^2 = r\}$ for fixed $r \neq 0$. Hence we know $SO_Q(\mathbb{R})$ acts transitively on its level set.

As for $r = 0$, notice

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

is contained in $SO_U(\mathbb{R})$, where

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

D can interchange two components of cone $\{x^2 + y^2 = z^2\} \backslash \{\mathbf{0}\}$. Hence, we finish the proof.

□

Proof. Now, we come back proposition 2.4.

First, 2 can easily deduce 1. We just notice $Q(v) = Q(gv)$, where $g \in SO_Q(\mathbb{R})$.

Second, 1 can deduce 2. If $Q(v) = 0, v \in \mathbb{Z}^3$ has only trivial solution, then for every $\epsilon > 0$, we have a non-zero vector $v_\epsilon \in \mathbb{Z}^3$ such that $|Q(v)| < \epsilon$. On the other hand, we can know there are small length vector $u_\epsilon \in \mathbb{R}^3$ such that $Q(u_\epsilon) = Q(v_\epsilon)$. Hence, by transitivity of $SO_Q(\mathbb{R})$, we can know this proposition is right. If $Q(v) = 0, v \in \mathbb{Z}^3$ has a non-trivial solution, then Q is indefinite. We consider small length non-zero vectors v such that $Q(v) = 0, v \in \mathbb{R}^3$. By transitivity of $SO_Q(\mathbb{R})$, we finish the proof.

□

According to the standard congruence forms of symmetric matrices, we know there exist $\lambda \in \mathbb{R}$ and $g \in SL(3,\mathbb{R})$ such that $Q = \lambda g^t Q_0 g$. Therefore, we get

$$SO_Q(\mathbb{R}) = g^{-1} SO_{Q_0}(\mathbb{R}) g\,.$$

From then on, we consider $SO_{Q_0}(\mathbb{R})$ orbit on $gSL(3,\mathbb{Z})$ coset in X_3 rather than $SO_Q(\mathbb{R})$ orbit on identity coset in X_3. Topological properties are equivalent under these two settings.

We firstly analyse $SO_{Q_0}(\mathbb{R})$. Directly calculation shows its Lie algebra:

$$\mathfrak{s} = \begin{bmatrix} a & b & 0 \\ c & 0 & b \\ 0 & c & -a \end{bmatrix},$$

where $a, b, c \in \mathbb{R}$. $SO_{Q_0}(\mathbb{R})$ has two componets, and $SO^\circ_{Q_0}(\mathbb{R})$ represents its component containing I. We denote:

$$A = \{a_t = \begin{bmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} | t \in \mathbb{R}\}$$

$$U = \{u_t = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} | t \in \mathbb{R}\}$$

$$V = \{v_t = \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} | t \in \mathbb{R}\}$$

$$B = AU\,.$$

Easy calculation tells us $A \subset SO^\circ_{Q_0}(\mathbb{R})$, $U \subset SO^\circ_{Q_0}(\mathbb{R})$ and $V \cap SO_{Q_0}(\mathbb{R}) = I$. U and V are commutative, and A normalizes U and V, i.e. $a_t u_s a_{-t} = u_{e^{2t}s}$, $a_t v_s a_{-t} = v_{e^{2t}s}$. Hence, B is a closed subgroup of $SO^\circ_{Q_0}(\mathbb{R})$.

Besides, notice that

$$j = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

belongs to $SO_{Q_0}(\mathbb{R})$. We have $SO_{Q_0}(\mathbb{R}) = SO^\circ_{Q_0}(\mathbb{R}) \cup j SO^\circ_{Q_0}(\mathbb{R})$.

If we carefully check the Lie algebra $\mathfrak{s}$, it seems like $\mathfrak{sl}(2,\mathbb{R})$. In fact, there is a Lie group homomorphism from $SL(2,\mathbb{R})$ to $SO^\circ_{Q_0}(\mathbb{R})$, which is a 2-sheet covering map. To be precise:

Consider a basis of $\mathfrak{sl}(2,\mathbb{R})$, $E_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{-1}{\sqrt{2}} \end{bmatrix}$, $E_3 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$.

A symmetric bilinear form $-Tr : \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R}) \to \mathbb{R}$

$$(X, Y) \to -Tr(XY)\,.$$

We can calculate the matrix of this bilinear form under the basis:

$$(-Tr(E_iE_j))_{i,j} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

We consider the adjoint action of $SL(2,\mathbb{R})$ on its Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. We write $Ad(g)(E_1,E_2,E_3) = (E_1,E_2,E_3)M_g$, where M_g is $Ad(g)$ matrix representation under this basis. Because the adjoint action preserves trace, we have $M_g^t Q_0 M_g = Q_0$, i.e. $M_g \in SO_{Q_0}(\mathbb{R})$.

We get a Lie group homomorphism: $\Phi : SL(2,\mathbb{R}) \to SO^{\circ}_{Q_0}(\mathbb{R})$ as $g \to M_g$. It is not hard to know the preiamge I is $\pm I$. Besides, we know this group homomorphism induces a Lie algebra isomorphism as follows:

$$\phi : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{s}$$
$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix} \to \begin{bmatrix} 2a & -\sqrt{2}b & 0 \\ -\sqrt{2}c & 0 & -\sqrt{2}b \\ 0 & -\sqrt{2}c & -2a \end{bmatrix}.$$

Therefore, it is a 2-sheet covering map. We can deduce that $SO^{\circ}_{Q_0}(\mathbb{R})$ is generated by U and U^t, which are unipotent matrices.

Theorem 2.6. [7]
Let $Q = \lambda Q_0 \circ g$ be a non-degenerate indefinite quadratic form, if $SO_{Q_0}g$ orbit on X_3 is bounded, then it is closed(hence compact).

We now deduce weak Oppenheim's conjecture from theorem 2.6.

Proof. Because Q is an irrational form, we know its orbit is unbounded. Otherwise, by theorem 2.6, we know its orbit is compact. We use proposition 4.1(in section 4), then we know Q must be proportional to a rational form. Contradiction! Therefore, its orbit is unbounded.

Using proposition 2.4, we know weak Oppenheim's conjecture is true. □

We push backward the proof of theorem 2.6.

Theorem 2.7. *Let $Q = \lambda Q_0 \circ g$ be a non-degenerate indefinite irrational quadratic form, then $SO_{Q_0}g$ orbit on X_3 is dense.*

We now deduce strong Oppenheim's conjecture from theorem 2.7.

Proof. Leave it to the reader. □

We push backward the proof of theorem 2.7.

Remark: Margulis firstly proved Oppenheim conjecture with the method before. We want to know: are there more relationships between quadratic forms and orbits in X_3. As a matter of fact, for an indefinite non-degenerate quadratic form $Q = \lambda Q_0 \circ g$ in 3 variables, we have the following facts:

1. Q is a rational form if and only if the orbit $SO_{Q_0}(\mathbb{R})g$ is closed in X_3.
2. If Q is an irrational form, then the orbit $SO_{Q_0}(\mathbb{R})g$ is dense in X_3.
3. Q is a rational form and can only trivially represent 0(not isotropic over $\mathbb{Q}$) if and only if the orbit $SO_{Q_0}(\mathbb{R})g$ is compact in X_3.

If we know Ratner's thoerems, we can get these results not so hard, especially 2. We can even get 2 in a more strong statement: the orbit is not only dense, but is also equidistributed for unipotent flows under the Haar measure on X_3.

As for 1, a rational form having a closed orbit is a result of Borel density theorem in arithmetic groups. On the contrary, a closed orbit deducing a rational form needs a non-divergence theorem for unipotent flows(polynormal growth), which is not so difficult.

As for 3, when $n \geq 5$, a rational indefinite quadratic form is always isotropic over $\mathbb{Q}$.

For theorem 2.8, the author plans to use Margulis's method. But before it, we get a glimpse of Ratner's theorems-the most simple case on $SL(2,\mathbb{R})$. This is an awesome result, if we get it, we can prove Oppenheim's conjecture immediately!

3. Ratner's Theorems for $SL(2,\mathbb{R})$

Firstly, we fix our notations in this section.

$G = SL(2,\mathbb{R})$; Γ a lattice in G

$X_2 = \{$ unimodular lattices in $\mathbb{R}^2$ $\} \cong SL(2,\mathbb{R})/SL(2,\mathbb{Z})$

$\hat{m}$ represents the Haar measure on G/Γ or X_2

$U^+ = \{u_t^+ = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} | t \in \mathbb{R}\}$

$U^- = \{u_t^- = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} | t \in \mathbb{R}\}$

$A = \{a_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} | t \in \mathbb{R}\}$

$B = AU^+$

Before we start the formal discussion, we briefly review some properties about G.

FACT 1: U^+ and U^- generate G.

Notice that: for any $0 \neq a \in \mathbb{R}$, we have:

$$\begin{bmatrix} a & 0 \\ 0 & \frac{1}{a} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{a} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1-a}{a} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -a & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

For any $g \in G$, if $g_{11} \neq 0$, we can use U^+ and U^- to operate elementary row and column transformations to make g look like A. If $g_{11} = 0$, we can use the matrix before to exchange its rows, then $g_{11} \neq 0$.

FACT 2: If we know a little hyperbolic geometry, we can consider $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm I\}$, which can be seen as unit tangent bundle of hyperbolic plane. In this way, G/Γ can be seen as unit tangent bundle of quotient hyperbolic manifold(orbifold) if $\{\pm I\}$ containd in Γ. It can also be thought like that even if $\{\pm I\}$ not contained in Γ in the absence of a covering map. In this setting, $PSL(2,\mathbb{R})$ is orientation-preserving isomorphic group of hyperbolic plane, which faithfully and transitively acts on unit tangent bundle of hyperbolic plane(that's why we can identify two things). Moreover, a_t is a geodesic flows action on unit tangent bundle, while u_t^+ and u_t^- are two kinds of horocycle flows on unit tangent bundle. Of course, these kinds of flows can induce flows on quotient hyperbolic manifolds.

FACT 3: If we know fundamental Dirchilet region, we can judge whether quotient manifold(orbifold) is compact or finite volume. For $SL(2,\mathbb{R})$, we can choose Γ to be cocompact(fundamental region is compact) and also to be lattice(fundamental region is finite volume). For cocompact case, the quotient manifold is compact. For lattice but not compact case, the quotient manifold has some cusps. X_2 has finite volume but is not compact, it has one cusp and its fundamental region can be found in any textbook of hyperbolic geometry[9].

FACT 4: B does not have any lattice, because its left Haar measure is not equal with its right Haar measure.

Now, we recall Ratner's theorems. We firstly notice in this situation Ad-unipotent flows on G can be conjugate to U^+.

Firstly, the author wants to use a simple example to illustrate these theorems[1]. We choose $\mathbb{R}^2$ as G, Γ as $\mathbb{Z}^2$ and choose a unit vector $v_0 = (a, b) \in \mathbb{R}^2$. We take H and u_t as $\mathbb{R}v_0$, and its action on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ as follows:

$$\mathbb{R} \times T^2 \to T^2$$

$$(t, [v]) \to [v + tv_0]$$

where [ ] represents its image in the quotient manifold T^2.

Intuitively, we can think this action as rational or irrational flows on T^2, according to the direction of v_0 is rational or irrational. Choose an(any) initial point $[v] \in T^2$, we know its orbit is periodic in rational flows and dense in irrational flows. In the former case, invariant ergodic measures of a rational flow are numerous, each one is supported on a periodic orbit; in the latter case, invariant ergodic measure of an irrational flow is exactly the Haar measure on T^2(uniquely ergodic). If we use Ratner's theorems, it indeed tells us these measures are algebraic-$[Hv]$ in rational flows and $[Gv] = T^2$ in irrational flows. Rigid proof needs a little Fourier analysis, but it is intuitively right. As for orbits of flows, there is no difficulty to imagine: if the flow is rational, the orbit is periodic(just like a twisted S^1) and uniformly distributed under the Haar measure on S^1, if the flow is irrational, the orbit is dense and uniformly distributed under the Haar meaure on T^2.

Remark: Ratner's thorems can only apply to unipotent groups or groups generated by unipotent elements. If we consider A-action on X, it is well known that we can find $x \in X$ so that the closure of Ax in X is not a manifold. In this way, Ratner's theorems are invalid. Besides, the condition "H is generated by unipotent elements" is not the most general of all possible conditions, as we know that B-invariant ergodic measure on X is the Haar measure $\hat{m}$ and any B-orbit on X is dense!

First step: classify invaraint ergodic measures for U^+ on X.

When we see Ratner's theorems, we know we should have a better understanding of orbits or the closure of orbits.

Lemma 3.1. *If Γ is a cocompact lattice, then there is no closed(periodic) U^+-orbit on $X = G/\Gamma$.*

Proof. Proof by contradiction. Suppose $\Lambda \in X$ has a closed U^+-orbit:

$$U^+ \to X : u_t^+ \to u_t^+\Lambda\,.$$

It is easy to know this map is smooth and immersive. If the image is closed, then the image is compact. Then, this orbit is periodic, i.e. there exists a minimal $t_0 > 0$ such that $u_{t_0}^+\Lambda = \Lambda$. If this orbit is not periodic, then this map is an injective immersion. By the compactness of X, we can find $t_0 \in \mathbb{R}$ and $\{t_i\}_{i=1}^{\infty} \subset \mathbb{R}$ such that $t_i \to \infty$ as $i \to \infty$ and $t_i\Lambda \to t_0\Lambda$. In this way. By homogeneous property of G/Γ , we can translate this phenomenon to every point, including *id* coset. Now we look at *id* coset, and choose a line l pass through *id* coset and transverse the orbit. We know there are at most countably(because there are at most countably many disjoint open intervals in $\mathbb{R}$) many points in the line l which are transverse intersection points with $U^+\Lambda$ orbit. We know every point we construct is not a isolated point because of homogeneous property. But a perfect set is not countable. Hence, this orbit is periodic.

If this orbit is periodic, then we have arbitrary small period U^+-orbit. For $u^+_{t_0}\Lambda = \Lambda$, then we have $u^+_{e^{-2s}t}(a^{-s}\Lambda) = (a^{-s}\Lambda)$.

But it is impossible to have arbitrary small periodic points, because of injective radius is strictly positive for compact X. Injective radius at a point P means the maximal radius r_P such that $B_{r_P}(I) \subset G \to X : g \to gP$ is injective. We know r_P is positive and continuous with P. By compactness of X, they have a common strictly positive lower bound. Therefore, every U^+-orbit is not closed when X is compact. □

Lemma 3.2. [3] *If Γ is a lattice but not cocompact, then there are periodic U^+-orbits. In fact, to every cusp of X there corresponds precisely a one-parameter family of periodic U^+-orbits in X parameterized by the action of the diagonal subgroup A. More precisely, for one point Λ with periodic U^+-orbit we get precisely one element, namely $a_t\Lambda$, of all other periodic U^+-orbits that are associated to the same cusp, by letting $t \in \mathbb{R}$ vary. Moreover, $\Lambda \in X$ is periodic for U^+ if and only if $a_t\Lambda \to \infty$ as $t \to \infty$.*

Proof. The argument is highly relied on our understanding of fundamental Dirichlet region. Γ is a lattice, then its fundamental region is finite-edge convex polygon. The ideal vertexes of fundamental region bijectively correspond to cusps of X. Choosing an appropriate orientation-preserving isometry, we can assume such an ideal vertex is $y = \infty$ ideal vertex under half plane hyperbolic model. In this case, we know the horocycle flow U^+ is just the horizontal line flow, and all periodic orbits are like that. Geodesic flows are just like vertical lines, hence we know these points can be parameterized by the action of diagonal subgroup. The assertion is true. □

Until now, we explicitly express when an U^+-orbit is periodic on X. We can say more about orbits- non-divergence of horocycle flows.

Proposition 3.3. *For every lattice Γ, every compact set $K \subset X$, and every $\epsilon > 0$, there is a compact subset $L = L(K, \epsilon)$ of X such that*

$$\frac{1}{T}\ Leb\{t \in [0, T] \mid u^+_t\Lambda \notin L\} \ \leq\ \epsilon T$$

for all $T > 0$ and all $\Lambda \in K$. Moreover, there is a compact set $L = L(\epsilon) \subset X$(independent of K and Λ) such that for any $\Lambda \in X$ either Λ is periodic or there exists some $T_\Lambda > 0$ such that the above formula holds for all $T \geq T_\Lambda$.

Before, we give proposition 3.3 a proof, we firstly analyse some apparent facts. On the one hand, this claim is automatically right whem Γ is cocompact. On the other hand, when Γ is a lattice but not cocompact, it suffices to only analyse situation $\Gamma = SL(2, \mathbb{Z})$. We know all cusps in X are same like, and U^+ acts on each one in a similar way(conjugation). For general Γ, we can insert each of its cusps into X_2. Basing on above analysis, we just need to handle the X_2 case.

We know X_2 represents unimodular lattices in $\mathbb{R}^2$. Under this circumstance, by a direct calculation we know $U^+\Lambda$ is periodic if and only if Λ contains a horizontal vector(parallels to X-axis). And the orbit period length of Λ is proportional to the length of horizontal vector contained in Λ.

Lemma 3.4. *There exists $C_1 > 0$ and $\alpha_1 > 0$ such that for every interval (a, b) in $\mathbb{R}^2$, every $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$ and every $\rho \in (0, 1)$, we have*

$$\frac{1}{b-a}\ Leb\{s \in (a,b) \mid ||u_s^+ v|| < \rho M_0\}\ \leq C_1 \rho^{\alpha_1}\ ,$$

where $M_0 = \sup_{s\in(a,b)} ||u_s^+ v||$

Proof. The key point is that the growth rate of $u_t^+ v$ is linear, a special case of polynomial growth. If we notice this point, the proof is very elementary and left to the reader. As a matter of fact, this assertion is true for any polynomial growth rate. ☐

Key observation: a rank 2 unimodular lattice $\Lambda \in X_2$ is not allowed to contain two linearly independent vector of length strictly smaller than 1. Otherwise, the volume is less than 1, which is a contradiction.

Proof. For proposition 3.3, we essentially consider the case X_2.

We choose $\delta_0 \in (0, 1)$ such that $K \subset X_2^{\delta_0}$. We shall determine δ later, such that $L = X_2^{\delta}$, depending on δ_0 and ϵ. Consider:

$$I(\Lambda, \delta_0) = \{t \in [0, T] \mid sys(u_t^+\Lambda) < \delta_0\}$$

which decomposes into a disjoint union of open intervals

$$I(\Lambda, \delta_0) = \bigsqcup_{\alpha\in\mathcal{A}} I_\alpha$$

with certain index set $\mathcal{A}$.'

Take one $I_\alpha = (x_\alpha, y_\alpha)$. We know, for each $t \in I_\alpha$ we have a unique primitive vector $v_t \in \Lambda$, satisfying $v_t = sys(u^+\Lambda) < \delta_0$. Because of connectedness of I_α and the key observation, we know this v_t has no relationship with $t \in I_\alpha$, briefly denoted as v_α. By lemma 3.4, we have

$$\frac{1}{I_\alpha}\ Leb\{t \in I_\alpha \mid ||u_t^+ v_\alpha| < \rho\delta_0\} < C_1\rho^{\alpha_1}\ .$$

We take ρ such thar $C_1\rho^{\alpha_1} < \epsilon$. Let $\delta = \rho\delta_0$.

$$\{t \in (a, b) \mid ||u_t v_\alpha|| < \delta\} = \bigsqcup_{\alpha\in\mathcal{A}} \{t \in I_\alpha \mid ||u_t v_\alpha|| < \rho\delta_0\}$$

implying that

$$Leb\{t \in (a, b) \mid ||u_t v_\alpha|| < \delta\} = \Sigma_{\alpha\in\mathcal{A}} Leb\{t \in I_\alpha \mid ||u_t v_\alpha|| < \rho\delta_0\} < \Sigma_{\alpha\in\mathcal{A}} |I_\alpha|\epsilon \leq T\epsilon\, .$$

We can choose $L = X_2^{0.1}$ such that the second sentence is right.

Hence, when $\Gamma = SL(2, \mathbb{Z})$, we finished this case. For general Γ, X may have more than one cusp (but finite), but every cusp is similar to the cusp in X_2 both in geometrical and dynamical viewpoint (just a little hyperbolic geometry). Thus, for general case, the proof is the same as X_2.

☐

Corollary 3.5. *For any lattice* Γ*, if* U^+*-orbit on* $X = G/\Gamma$ *is closed, then it is periodic.*

Remark: This non-divergence property is not only valid for unipotent flows in $SL(2,\mathbb{R})$, but also for unipotent flows on $SL(n,\mathbb{R})$. Of course, when $n \geq 3$, there are more complicated periodic points and we need to discuss more cases. But the essential idea is same. One important result of non-divergence property is no loss of mass for limiting point of a sequence of probability measures.

Theorem 3.6. [3] *The Haar measure on* X *is an invariant ergodic measure for* U^+*.*

Proof. Invariance property is obvious. Recall ergodic property is to say every U^+-invariant square integrable function on X is essentially constant(constant except for a zero measure set). We try to prove this point.

For the Haar meaure $\hat{m}$ on X, G can induce a unitary representation on Hilbert space $L^2(X,\mu)$: $G \times L^2(X,\mu) \to L^2(X,\mu)$

$$(g, f(x)) \to f(g^{-1}x)$$

and this representation is continuous for G under the strong operator topology of unitary operators of $L^2(X,\mu)$.

We know G-action on X is ergodic, and evey G-invariant L^2-integrable function is essentially constant. We claim: every U^+-invariant L^2-integrable function is also G-invariant. Using this, we can prove U^+-action is also ergodic.

We define $\Phi : G \to \mathbb{R}$

$$g \to \langle g.f, f\rangle\ ,$$

where $g.f(x) = f(g^{-1}x)$ and $\langle\ ,\rangle$ is the inner product on $L^2(X,\mu)$. We know that: $\Phi(g) = \langle f, f\rangle$ if and only f is invariant under g.

Arbitrarily fix an U^+-invariant f.

Step 1: Φ is bi-invariant for U^+, i.e. for any $u_1 \in U^+$, $g \in G$ and $u_2 \in U^+$ we have $\Phi(u_1 g u_2) = \Phi(g)$. We can calculate directly: $\Phi(u_1 g u_2) = \langle u_1 g u_2.f, f\rangle = \left\langle g u_2.f, u_1^{-1}.f\right\rangle = \langle g.f, f\rangle = \Phi(g)$.

Step 2: we have the following matrix relationship: let $r, s, \epsilon \in \mathbb{R}$, notice

$$\begin{bmatrix}1 & r\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\\epsilon & 1\end{bmatrix}\begin{bmatrix}1 & s\\0 & 1\end{bmatrix} = \begin{bmatrix}1+r\epsilon & 1+s+rs\epsilon\\\epsilon & 1+s\epsilon\end{bmatrix}.$$

Let $\epsilon \to 0$, but not equals to 0, and $r = \frac{e^t-1}{\epsilon}$, $s = \frac{-r}{1+r\epsilon}$. Then the above matrix simplifies to

$$\begin{bmatrix}e^t & 0\\\epsilon & e^{-t}\end{bmatrix}.$$

Here, it is not hard for us to see f is also invariant under A.

Step 3: similar like step 2, we can also see f is invariant under U^-. Hint:

$$\begin{bmatrix}e^{-t} & 0\\0 & e^t\end{bmatrix}\begin{bmatrix}1 & 0\\\epsilon & 1\end{bmatrix}\begin{bmatrix}e^t & 0\\0 & e^{-t}\end{bmatrix} = \begin{bmatrix}1 & 0\\e^{2t}\epsilon & 1\end{bmatrix}.$$

Let $\epsilon \to 0$ but not equals to 0; choose t to make sure $e^{2t}\epsilon$ not so small.

Therefore, f is G-invariant and is essentially constant. The Haar measure $\hat{m}$ on X is ergodic for U^+. □

In our proof for theorem 3.6, this argument is called Mautner phenomenon.

Proposition 3.7. *Let $\mathcal{H}$ be a Hilbert space carrying a unitary representation of G. Let $g \in G$ be an element that is not conjugate to an element of $SO(2)$. Then any vector $v_0 \in \mathcal{H}$ that is fixed by g is also fixed by all of G.*

Proof. Because g is not conjugate to an element of $SO(2)$, we have the following several situations:

Case 1: g is conjugated to $\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$, where $t \neq 0$.

In this case, the proof is similar like theorem 3.6. We firstly illustrate A fixing v_0, then demonstrate U^+ and U^- fixing v_0.

Case 2: g is conjugated to $\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$, where $t \neq 0$.

We also construct function Φ as before. Then, we can use Step 3 in theorem 3.6. □

Corollary 3.8. *The Haar measure $\hat{m}$ on X is ergodic for any $g^{\mathbb{Z}}$ action, where g is not conjugate to an element of $SO(2)$.*

Theorem 3.9. *Let μ be an U^+-invariant ergodic measure on X, then*
1. *either μ is supported on a closed(periodic) U^+-orbit;*
2. *or μ is B-invariant.*

Theorem 3.10. *If μ is a B-invariant, U^+-ergodic measure on X, then μ is the Haar measure $\hat{m}$.*

Combining theorem 3.9 and theorem 3.10, we can have a clear mind about U^+-invariant ergodic measures. Then we can know all U^+-invariant measures by ergodic decomposition theorem[6].

Theorem 3.9 is not easy to prove. Before we give a detailed proof, we can describe basic ideas behind this proof. When we look at theorem 3.9, it tells us if this measure is not supported on a U^+-closed orbit, then it have more invariance under G-action. Using many kinds of methods to illustrate more invariance of a specific measure is a traditional and useful argument in homogeneous dynamics. In this theorem, we use a method called "H-principle"[1] to fill this argument. This argument is essentially due to the polynomial growth of unipotent flows-polynomial divergence and "shearing property". Precisely, $x, y \in X$ are two nearby points, and they diverge from each other under the action of U^+. If we wait for a period of time, we can find the fastest motion of divergence is along U^+-direction. If we initially choose these two points nearby but not on the same U^+-orbit, they also diverge from each other in other directions, although not as fast as U^+-direction. However, U^+-direction is useless for us. If we use additional U^+-action to ignore this fastest direction, we notice that they are divergent from each other along transverse direction. We can know transverse direction means direction along normalization group direction(A-direction), which is a new direction different from U^+-direction. If we get limit points of the above process, we can add more elements to invariance of μ. With this method, we can demonstrate μ is B-invariant.

As for theorem 3.10, we can use entropy method to illustrate and this method is also widely used in homogeneous dynamics. When measure entropy reach a maximal value or certain value, we can deduce the measure is exactly the Haar measure or other specific form. However, in this article, we use conditional measures method instead of entropy method. Conditional measures method needs less preliminary knowledge and is easier to illustrate.

Now, we give detailed proofs for these two theorems.

Lemma 3.11. *If $x, y \in X$ are both (U, μ)-generic points and $y = gx$ with $g \in G$ normalizing U^+, then μ is g-invariant.*

Proof. Birkhoff ergodic theorem[6] tells us: for generic points $x \in X$, for any compactly-supported continuous function on X, we have $\lim_{T\to\infty} \frac{1}{T}\int_0^T f(u_t x)dt = \int f(y)\mu(y)$. Because g normalizes U^+, we can assume $u_t^+ g = g u_{ct}^+$, where c is a positive real number. We have the following equality:

$$\int f(z)\mu(z) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(u_t y)dt = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(u_t g x)dt = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(g u_{ct} x)dt$$

$$= \lim_{T\to\infty} \frac{1}{cT}\int_0^{cT} f(g u_t x)dt = \int f(gz)\mu(z) = \int f(z) g_*\mu(z)$$

for any compactly-supported continuous function f.

Therefore, μ is g-invariant.

□

Proof. If μ is supported in U^+-orbit, this case is what we want. Otherwise, we fulfill the idea we described before.

We define $E_{U^+,\mu} = \{(U^+, \mu)$ -generic points $\}$ and take E to be a compact subset of $E_{U^+,\mu}$ such that $\mu(E) > 0.99$.

Take T_0 large enough such that the following set

$$F = \{x \in X \mid \frac{1}{T} Leb\{t \in [0, T], u_t^+ x \in E \geq 0.98, \forall T \geq T_0\}\}$$

has $\mu(F) > 0.99$(we can get this F using Birkhoff ergodic theorem).

We claim that there exist pair (x, y) in $F \cap E$ arbitrarily close to each other and yet not on the same local U^+-orbit(μ is not supported on a closed U^+-orbit).

To be precise, two points x, y are said to be on the same local U^+-orbit if $x = u_s^+ y$ for some $s \in (-1, 1)$. If the claim is not true, then there exists $\epsilon > 0$ such that if $x, y \in F \cap E$ and $d(x, y) < \epsilon$ then $x = u_s^+ y$ for some $|s| < 1$. Cover F by countably many measurable sets $\{B_i\}$ of diameter smaller than ϵ. Then $B_i \cap F \cap E \subset u_{(-1,1)} x_i$ for some x_i. So $F \cap E \subset \bigcup_i u(-1, 1)x_i$. Thus for some x_i, we have $\mu(\{u_s^+ x_i \mid |s| < 1\}) > 0$. By homogeneity and ergodicity, $U^+ x_i$ is of full measure, μ is supported on $U^+ x_i$ and hence $U^+ x$ is a closed(periodic) orbit. Contradiction!

Therefore, we can choose nearby pairs $(x_n, y_n) \in F \cap E$ such that $y_n = g_n x_n$, $g_n \to id$ but $g_n \notin U^+$. More precisely, $g_n = \begin{bmatrix} 1 + a_n & b_n \\ c_n & 1 + d_n \end{bmatrix}$, where $a_n d_n + a_n + d_n = b_n c_n$, and $a_n, b_n, c_n, d_n \to 0$ as $n \to \infty$.

Case 1: if there are infinitely many $g_n \in A$, then we are done! By lemma 3.11 and $Stab_\mu$ is a closed subgroup G, we know $A \subset Stab_\mu$.

Case 2: in other situation, we use H-principle to construct case 1. As $y_n = g_n x_n$, we use U^+-action to get $y_n^{'} = u_s^+ y_n$ and $x_n^{'} = u_s^+ x_n$. Then we use U^+-action again to erase the fastest motion U^+-direction and leave the transverse direction to us. Precisely, $y_n^{"} = u_t^+ y_n^{'}$. We have $y_n^{"} = u_{t+s}^+ g_n u_{-s}^+ x_n^{'}$. For simplicity, we write as $y_n^{'} = u_t^+ g_n u_{-s}^+ x_n^{'}$. Here, we calculate:

$$u_t^+ g_n u_{-s}^+ = \begin{bmatrix} 1 + a_n + tc_n & b_n + t - s + td_n - sa_n - stc_n \\ c_n & 1 + d_n - sc_n \end{bmatrix}.$$

Let $b_n + t - s + td_n - sa_n - stc_n = \lambda$, we get $t = \frac{\lambda+s+sa_n-b_n}{1+d_n-sc_n}$. If we make λ vary in a scale $\delta\lambda$, then we make t vary in a scale $\delta t = \frac{\delta\lambda}{1+d_n-sc_n}$. Choose an appropriate s such that $1 + d_n - sc_n$ is neither so small nor so big, i.e. $2\delta_0 > |1 + d_n - sc_n| > \delta_0$, and choose $\lambda < K\delta_0$ and $\delta\lambda = K\delta_0$, where $0 \leq K \leq 1$. From our construction of F, we can choose s and t simultaneously such that $y_n^{'} \in E$ and $x_n^{'} \in E$. By the compactness of E, we can get $y_\infty = gx_\infty$, where $(x_\infty, y_\infty) \in E$ and $g = \begin{bmatrix} \frac{1}{\delta_0} & K\delta_0 \\ 0 & \delta_0 \end{bmatrix}$. Then, for different $k \in (0, 1)$, we consider a sequence in (x_∞, y_∞) as $K \to 0$, and we get a new (x_∞, y_∞) in E satisfying $y_\infty = gx_\infty$(we use the same notation). We have: $g = \begin{bmatrix} \frac{1}{\delta_0} & 0 \\ 0 & \delta_0 \end{bmatrix}$. We can further let δ_0 be arbitrary small, then $A \subset Stab_\mu$.

Therefore, we know this measure μ is B-invariant.

□

In order to give a self-consistent proof for theorem 3.10, we need some knowledge about conditional measure.

Let Y be a nice metric space: locally compact and σ-compact, and $\mathcal{B}_Y$ be its Borel σ-algebra. Let $\mu \in \mathrm{Prob}(Y)$. Let $\mathcal{A}$ be a countably generated sub-σ-algebra of $\mathcal{B}_Y$, i.e. there exists a countable collection of measurable subsets A_i generating $\mathcal{A}$. Assume the complement of every A_i is also contained in $\mathcal{A}$. For $y \in Y$, let the atom containing y be $[y]^{\mathcal{A}} = \bigcap_{y \in A_i} A_i$. It is not hard to realize $[y]^{\mathcal{A}} = \bigcap_{y \in A, A \in \mathcal{A}} A$, hence $[y]^{\mathcal{A}}$ is independent of the choice of a countable generator of $\mathcal{A}$.

Theorem 3.12. *(Conditional measures)*[7] *Let* $(Y, \mathcal{B}_Y, \mu)$ *and* $\mathcal{A}$ *be as above.*

1. Existence of conditional measures.

There exists a measurable map $Y \to$*Prob*(Y) *denoted as* $y \to \mu_y^{\mathcal{A}}$*, i.e. a full measure* $Y^{'} \subset Y$ *such that the above map can be defined. We have:* $\mu_y^{\mathcal{A}}([y]^{\mathcal{A}}) = 1$ *and*

$$\int_A \int f(z)\mu_y^{\mathcal{A}}(z)\mu(y) = \int_A f(y)\mu(y)$$

for every $A \in \mathcal{A}$ *and* $f \in L^1(Y, \mathcal{B}_Y, \mu)$*. Implicitly, we have claimed that* $y \to \int f(z)\mu_y^{\mathcal{A}}(z)$ *is integral on* A*.*

2. Uniqueness of conditional measures.

If $y \to \nu_y^{\mathcal{A}}$ *is another measurable map from a possibly different full measure* $Y^{"} \subset Y$ *to Prob*(Y) *satisfying the above equation for every compactly-supported continuous function* $f \in C_c(X)$ *and* $A = Y^{"}$*, then for some full measure set* $Y^* \subset Y^{'} \cap Y^{"}$ *we have* $\mu_y^{\mathcal{A}} = \nu_y^{\mathcal{A}}$ *for* $y \in Y^*$*.*

We don't give proof for theorem 3.12, but we give some concrete examples to understand this.

Example 1. Let $Y = [0, 1] \times [0, 1]$ and μ be the standard Lebesgue measure defined by $|dx \wedge dy|$. Let $\mathcal{A} = \{A \times [0, 1] \mid A \in \mathcal{B}_{[0,1]}\}$. Then for every $(x, y) \in Y$, $[(x, y)]^{\mathcal{A}}_{(x,y)} = \{x\} \times [0, 1]$ and $\mu^{\mathcal{A}}_{(x,y)}$ is supported on $x \times [0, 1]$ and induced by $|dy|$.

Example 2. Everything is same as in the last example except that we let μ be the standard Lebesgue measure supported on $\Delta = \{(x, x) \mid x \in [0, 1]\}$. Then $[(x, y)]^{\mathcal{A}}_{(x,y)} = \{x\} \times [0, 1]$ and $\mu^{\mathcal{A}}_{(x,y)} = \delta_{(x,y)}$.

Now with these preliminaries, we give a proof for theorem 3.10.

Lemma 3.13. *If* μ *is a* B*-invariant,* U^+*-ergodic measure on* X*, then* μ *is ergodic with respect to* $a^{\mathbb{Z}}$*-action for every* $id \neq a \in A$*.*

This proof is the same as the proof for theorem 3.6. We leave it to the reader.

Proof. Let μ be a B-invariant, a-ergodic probability measure. Here a is a fixed non-trivial element of A. Besides, we know the Haar measure $\hat{m}$ on X also has this property. It suffices to illustrate μ and $\hat{m}$ have a common generic point, and then they are same as we have Birkhoff ergodic theorem.

Fix some o in the support of μ. Choose symmetric neighborhoods of identity $\mathcal{N}_\epsilon^B$(resp.,$\mathcal{N}_\epsilon^V$) in B(resp.,V) that are very small compared to the injectivity radius at o. We say two x, y are on the same local B(resp.,V)-orbit if and only if $x \in \mathcal{N}_\epsilon^B y$(resp.,$x \in \mathcal{N}_\epsilon^V y$). Choose $\delta > 0$ even smaller compared to ϵ.

Let

$$Gene(f, \mu) = \{x \in X \mid \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(a^n x) = \int f(y)\mu(y)\} .$$

Note this set is AU^--invariant and full μ-measure. Let E_f be its intersection with $\mathcal{N}_\delta(o)$.

We define a sub-σ-algebra $\mathcal{A}$ on $\mathcal{N}_\delta(o)$ by specifying its atoms: x and y belong to the same atom if and only if x and y are on the same local B-orbit. Let $E_f^{'} \subset E_f$ be those x such that the conditional measure $\mu_x^{\mathcal{A}}$ is the restriction of some left B-invariant measure when we identify $[x]^{\mathcal{A}}$ as a subset of $\mathcal{N}_\epsilon^B \subset B$ via the orbit map. Let $\hat{E}_f$ consists of elements in $\mathcal{N}_\delta(o)$ that are on the local U^--orbit of some element in $E_f^{'}$. Because μ is B-invariant, we use Fubini theorem to analyse $\hat{E}_f$ via U^--direction and $B = AU^+$-direction under the Haar measure on X. We can get $Gene(f, \mu)$ is full measure in $\mathcal{N}_\delta(o)$ under the Haar measure on X. Hence, it is both conull under $\hat{m}$ and μ. We get the two measure have common generic points. Then $\mu = \hat{m}$.

□

Remark: If we know a little entropy, we can explain entropy method to deduce $\hat{m}$ from B-invariant property. Suppose μ is an A-invariant probability measure on X, then we have $h_\mu(a^s) \leq 2|s|$, with equality if and only if μ is U^+-invariant. This claim is not obvious, and difficult point is how to deduce the U^+-invariance from maximal measure entropy condition. If we admit this claim, then we also have a similar claim for U^--invariant property. That's because there is a measure-preserving transformation on G resulting in $a^s \to a^{-s}$ and $U^+ \to U^-$, which does not change the measure entropy. Therefore, if μ is U^+-invariant, then it satisfied the entropy condition, and then it is also U^--invariant. So $\mu = \hat{m}$.

Second step: orbits of unipotent flows is equally distributed.

Now, we have fully finished classification of invariant ergodic measures for U^+-action on X. In $SL(2, \mathbb{R})/\Gamma$ case, we can easily check whether these U^+-invariant ergodic measures are algebraic. If μ is supported in a periodic U^+-orbit, then we can choose $H = U^+$ and x arbitrarily belonging to this orbit. If μ is $\hat{m}$, then we can choose $H = G$ and x arbitrarily belonging to X.

How should we get information about distribution of U^+-orbits from its invariant ergodic measures?

Before we give a rigid proof, we give some analyses. Choose one point $x \in X$ arbitrarily, what will happen about its U^+-orbit? If this point is periodic, then its orbit is homeomorphic to S^1, and equally distributed along this circle orbit. If this point is not periodic, it shows a property of non-divergence and we can demonstrate this point is a generic point for $\hat{m}$ on X. To be precise, we average its orbit Dirac-measure, and verify any weak-$*$ limit point of this measure sequence is $\hat{m}$ on X. We know any weak-$*$ limit measure is an U^+-invariant probability measure, which is guaranteed by the non-divergence property. With the help of classification of invariant ergodic measures of U^+ and ergodic decomposition theorem, it suffices to illustrate the set consisting of all periodic points is measure 0. Hence, we can prove the limit measure is the Haar measure $\hat{m}$.

Theorem 3.14. *Let $\Lambda_0 \in X$ be such that $U^+\Lambda_0$ is not compact(periodic). Then*

$$\lim_{S\to\infty} \mu_S = \lim_{S\to\infty} \int_0^S (u_s)_* \delta_{\Lambda_0} ds = \hat{m}\,.$$

Proof. For simplicity, we consider X_2.

Step 1. We show any weak-$*$ limit of $\{\mu_S\}_{S\to\infty}$ is an U^+-invariant probability measure. By Banach-Alaoglu theorem, $\{\mu_S\}_{S\to\infty}$ has weak-$*$ convergence subsequences. Assume μ is a weak-$*$ limit point. Recall we have non-divergent property. For $\forall \epsilon > 0$, we have a compact subset $K \subset X_2$ satisfying the non-divergence formula. Then we have

$$\mu(K) \geq \overline{\lim_{S_i\to\infty}}\, \mu_{S_i}(K) \geq 1-\epsilon\,.$$

Therefore, we know μ is a probability measure. Clearly, μ is U^+-invariant.

Step 2. We have classified all U^+-invariant ergodic probability measures. Using ergodic decomposition theorem, we just need to consider its ergodic components. In X_2, we know periodic points equal those lattices having a horizontal vector. Thus, all periodic points are $\mathcal{F} = B.\mathbb{Z}^2$, which is a 2-dimensional submanifold in X_2. We know all kinds of U^+-invariant ergodic measures either are supported on a periodic orbit or just the Haar measure $\hat{m}$. If we can demonstrate $\mu(\mathcal{F}) = 0$, then we get $\mu = \hat{m}$ by ergodic decompostion theorem. Now, we prove $\mu(\mathcal{F}) = 0$.

Fix $t_1 < t_2$, let

$$\mathcal{F}_{[t_1,t_2]} = \{a_t u_s.\mathbb{Z}^2, t \in [t_1,t_2], s \in \mathbb{R}/\mathbb{Z}\}\,.$$

Thus, it suffices to show that $\mu(\mathcal{F}_{[t_1,t_2]}) = 0$ for all $-\infty < t_1 < t_2 < +\infty$. By the definition of weak-$*$ convergence, it suffices to find an open neighborhood $\mathcal{N}_\epsilon$, for every $\epsilon > 0$, of $\mathcal{F}_{[t_1,t_2]}$ such that $\mu(\mathcal{N}_\epsilon) \leq \underline{\lim}_{S_i\to\infty} \mu_{S_i}(\mathcal{N}_\epsilon) \leq \overline{\lim}_{S_i\to\infty} \mu_{S_i}(\mathcal{N}_\epsilon) \leq \epsilon$. Let $\epsilon \to 0$, we finish the proof.

Note that $u_s\Lambda_0$ being close to $\mathcal{F}_{[t_1,t_2]}$ means that, for certain $v \in \mathrm{Prim}(\Lambda_0)$, we have $u_s.v$ is close to

$$A_{[t_1,t_2]} = \{a_t u_s.e_1 \mid t \in [t_1,t_2], s \in \mathbb{R}/\mathbb{Z}\} = [e^{t_1}, e^{t_2}] \times \{0\}\,.$$

For $C, \delta > 0$, consider the box

$$Box_{C,\delta} = [-C,C] \times [-\delta,\delta]\,.$$

Define

$$I(C,\delta) = \{s \geq 0 \mid Prim(u_s\Lambda_0) \cap Box_{C,\delta} \neq \emptyset\}\,.$$

We have

$$I(C,\delta) = \bigcup_{\mathbb{Z}v \in Prim^1(\Lambda_0)} I(C,\delta,\mathbb{Z}v)\,,$$

where $Prim^1(\Lambda_0)$ refers to rank 1 primitive sub-lattices, and

$$I(c,\delta,\mathbb{Z}v) = \{s \geq 0 \mid u_s v \in Box_{C,\delta}\}\,.$$

It is easy to notice $I(c,\delta,\mathbb{Z}v)\cap I(c,\delta,\mathbb{Z}w)=\emptyset$ for two $\mathbb{Z}v\neq\mathbb{Z}w\in Prim^1(\Lambda_0)$ if $\delta C<0.1$.

Because Λ_0 is not periodic, then we have every $I(c,\delta,\mathbb{Z}v)$ is either $\emptyset$ or a bounded closed interval $[a,b]\subset\mathbb{R}$. For fixed t_1,t_2, we can choose a $C>1$ such that $Box_{C,\delta}$ contains $[e^{t_1},e^{t_2}]\times\{0\}$. Now, we find an appropriate δ.

A simple calculation tells us that

$$Leb(I(C,\delta,\mathbb{Z}v)\cap[0,S])\leq 4\epsilon\ Leb(I(\frac{C}{\epsilon},\delta,\mathbb{Z}v)\cap[0,S])$$

if we have $C\delta<0.1\epsilon$ and $C<\frac{0.1}{\epsilon}$.

$$\begin{bmatrix}1 & s\\ 0 & 1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}v_1+sv_2\\ v_2\end{bmatrix}.$$

Observe that this equation is enough to illustrate our simple calculation before. The reader just need to discuss the positional relationship between $[a,b]$ and $[0,S]$ in $\mathbb{R}$.

Therefore, for fixed t_1 and t_2, we choose a sufficient small δ such that $C\delta<0.1\epsilon$, $C<\frac{0.1}{\epsilon}$, we have a sufficiently small neighborhood $Box_{C,\delta}$, and we have the following fact:

$$Leb(I(C,\delta)\cap[0,S])=\sum_{v\in Prim^1(\Lambda_0)}Leb(I(C,\delta,\mathbb{Z}v)\cap[0,S])$$

$$\leq 4\epsilon\sum_{v\in Prim^1(\Lambda_0)}Leb(I(\frac{C}{\epsilon},\delta,\mathbb{Z}v)\cap[0,S])\leq 4\epsilon S\,.$$

Thus, we have a neighborhood $\mathcal{N}_\epsilon$ of $\mathcal{F}_{t_1,t_2}$ such that

$$\overline{\lim_{S_i\to\infty}}\,\mu_S(\mathcal{N}_\epsilon)\leq 4\epsilon\,.$$

Consequently, we finish the proof on X_2.

Step 3. For a general Γ, we can consider X with more cusps. However, this would not essentially affect our dynamics for U^+ on X, rather causes some technical troubles. It is not hard to overcome this.

□

Third step: when we know the equidistribution theorem, it is not hard to know the topological rigidities on $SL(2,\mathbb{R})$.

If H is generated by more than two kinds of unipotent flows, then $H=G$. In this case, we just need to choose an appropriate unipotent flow(conjugate to U^+). If H itself is a unipotent flow, then we do nothing!

4. Oppenheim's Conjecture: Proof using Ratner's Theorems

Our notations in this section is coherent with notations in section 2.

Proposition 4.1. [8] *Let Q be a non-degenerate, indefinite, quadratic form. Then $SO_Q(\mathbb{R})\mathbb{Z}^3$ is closed in X_3 if and only if Q is a rational form.*

Proof. If the orbit is closed, then Q is a rational form(proportional). We notice $SO_Q(\mathbb{R})$ has two components. We write $SO_Q^\circ(\mathbb{R})$ for its component containing I. Evidently, $SO_Q(\mathbb{R})\mathbb{Z}^3$ is closed if and only if $SO_Q^\circ(\mathbb{R})\mathbb{Z}^3$ is closed. We know $SO_Q^\circ(\mathbb{R})$ is generated by two kinds unipotent flows. Let $\Delta=\Gamma\cap SO_Q^\circ(\mathbb{R})$. For any quadratic form Q', we claim: if $\Delta \subset SO_{Q'}(\mathbb{R})$, then $SO^{\circ}_{Q}(\mathbb{R}) \subset SO_{Q'}(\mathbb{R})$ and Q is proportional to Q'.

For any $P \in \mathbb{R}^3$, we consider a continuous function

$$f_P : SO^{\circ}_{Q}(\mathbb{R})/\Delta \to \mathbb{R}$$

$$g \to Q'(gP)\,.$$

When f_P restricted to U'-unipotent flows generating $SO^{\circ}_{Q}(\mathbb{R})$, f_P becomes a polynomial function.

Because the orbit $SO^{\circ}_{Q}(\mathbb{R})\mathbb{Z}^3$ is closed in X_3, $SO^{\circ}_{Q}(\mathbb{R})/\Delta$ is homeomorphic to its image in X_3. By non-divergent property of U'-action on X_3, we can choose a compact set $K \subset X_3$ satisfying non-divergence. Write K' as the preimage of K in $SO^{\circ}_{Q}(\mathbb{R})/\Delta$ and we know $f_P(K')$ is a compact set in $\mathbb{R}$. Our construction tells us f_p is bounded during most of its time when restricted on U'. But a polynomial is bounded during most of time in $\mathbb{R}$ must be a constant, and hence $Q'(gP) = Q'(P), \forall g \in U'$. Clearly, $\{g \in SO^{\circ}_{Q}(\mathbb{R}) \mid Q'(gP) = Q'(P)\}$ is a subgroup, containing every U' generating $SO^{\circ}_{Q}(\mathbb{R})$, hence is $SO^{\circ}_{Q}(\mathbb{R})$. We deduce $SO^{\circ}_{Q}(\mathbb{R}) \subset SO_{Q'}(\mathbb{R})$. For any $g \in SO^{\circ}_{Q}(\mathbb{R})$, notice

$$gQ^{-1}Q'g^{-1} = ((g^{-1})^tQg^{-1})^{-1}((g^{-1})^tQ'g^{-1}) = Q^{-1}Q'\,,$$

meaning $Q^{-1}Q'$ communicates with $SO^{\circ}_{Q}(\mathbb{R})$. Direct calculation tells us $Q^{-1}Q'$ is scalar matrix. Hence Q is proportional to Q'.

We know Q is a solution for equations

$$\gamma^t S\gamma = S,\ \forall \gamma \in \Delta\,,$$

and the coefficients of these equations are integers, then it must have a solution in $\mathbb{Q}$ if it has a solution in $\mathbb{R}$. Using the claim we have proved before, we know Q is proportional to a rational quadratic form.

If Q is a rational form, then the orbit is closed. This is a result of Borel density theorem, we don't plan to give a proof here.

$\square$

Lemma 4.2. $\mathfrak{s}$, *the Lie algebra of* $SO^{\circ}_{Q_0}(\mathbb{R})$, *is a maximal Lie algebra of* $\mathfrak{sl}(3,\mathbb{R})$.

Proof. This proof is elementary but tedious. The reader can give a proof himself.

Suppose $g = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \in \mathfrak{sl}(3,\mathbb{R})$, with $g_{11}+g_{22}+g_{33}=0$, a Lie algebra $\mathfrak{g}$ satisfies $\mathfrak{s} \subset \mathfrak{g} \subset \mathfrak{sl}(3,\mathbb{R})$ and $g \in \mathfrak{g}$.

For simplicity, we denote $a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, and $c = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Calculate $ad(b)^4(g)$ and $ad(c)^4(g)$, we know $\begin{bmatrix} 0 & 0 & g_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{g}$ and

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ g_{31} & 0 & 0 \end{bmatrix} \in \mathfrak{g}$

Calculate $ad(a)(g)$, we know $\begin{bmatrix} 0 & g_{12} & 2g_{13} \\ -g_{21} & 0 & g_{23} \\ -2g_{31} & -g_{32} & 0 \end{bmatrix} \in \mathfrak{g}$.

Therefore, we need to consider $g^{'} = \begin{bmatrix} 0 & g_{12} & 0 \\ -g_{21} & 0 & g_{23} \\ 0 & -g_{32} & 0 \end{bmatrix}$ whether can be more precise.

Direct calculation $ad(b)(g^{'})$, $ad(c)(g^{'})$ tells us the following situations:

Case 1: $g_{12} \neq g_{23}$ and $g_{21} \neq g_{32}$.

Observe $ad(a)^3(g)$, we know $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ in $\mathfrak{g}$. Go on calculating Lie bracket, we know $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$.

Case 2: $g_{12} \neq g_{23}$ but $g_{21} = g_{32}$.

In this case, notice $[x, c] = y$, and $[c, y] = z$, where $x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $y = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$. Then we can consider Case 1.

Case 3: $g_{12} = g_{23}$ but $g_{21} \neq g_{32}$.

Just as Case 2.

Case 4: $g_{12} = g_{23}$ and $g_{21} = g_{32}$.

In this case, if we have $g_{13} \neq 0$ or $g_{31} \neq 0$, we can transform this case to situations above. If $g_{13} = 0$ and $g_{31} = 0$, we just need to analyse diagonal situation. By direct Lie bracket calculation, $\mathfrak{g}$=$\mathfrak{s}$ or $\mathfrak{sl}(3, \mathbb{R})$.

□

Theorem 4.3. *If Q is a non-degenerate, indifferent, quadratic form, then $SO^{\circ}_{Q}(\mathbb{R})\mathbb{Z}^3$ is dense in X_3.*

Proof. Suppose $Q = Q_0 \circ g$.

Recall that $SO^{\circ}_{Q}(\mathbb{R})$ is generated by unipotent flows. We use Ratner's theorems. Topological rigidity tells us there is a closed connected subgroup $SO^{\circ}_{Q_0}(\mathbb{R}) \subset L \subset SL(3, \mathbb{R})$, such that $\overline{SO^{\circ}_{Q_0}(\mathbb{R})g} = Lg$ in X_3.

Because Q is an irrational form, we have $SO^{\circ}_{Q_0}(\mathbb{R}) \neq L$. Becasue $\mathfrak{s}$ is a maximal Lie algebra in $\mathfrak{sl}(3, \mathbb{R})$, we have $L = SL(3, \mathbb{R})$.

Hence, $SO^{\circ}_{Q}(\mathbb{R})\mathbb{Z}^3$ is dense in X_3.

□

Corollary 4.4. *Every $SO_{Q_0}(\mathbb{R})$-orbit in X_3 is either closed or dense. Therefore, theorem 2.7 is true.*

Corollary 4.5. *Strong Oppenheim's conjecture is true.*

Recall that Oppenheim's conjecture can be reduced to $n = 3$ case. Now, we give this process: how transform general Oppenheim's conjecture to $n = 3$ situation.

Proposition 4.6. [8] *If Q is a non-degenerate indefinite irrational quadratic form in n variables, $n \geq 3$, there is a rational hyperplane L such that the restriction of Q to L is non-degenerate, indefinite and irrational.*

Proof. We can find $\mathbb{R}$-linearly independent vectors $b_1, b_2, , , , b_{n-1} \in \mathbb{Q}^n$ such that, denoting by L the $\mathbb{R}$-linear span of $b_1, b_2, , , , b_{n-1}$ and by L' that of $b_1, b_2, , , , b_{n-2}$, the restriction of $Q|_{L'}$ of Q is non-degenerate and $Q|_L$ is indefinite and non-degenerate.

Now take an $x \in \mathbb{Q}^n \backslash L$ and define $L_t = \mathbb{R}$- $span(b_1, b_2, , , , b_{n-2}, b_{n-1} + tx)$ for $t \in \mathbb{Q}$ sufficiently small, so that $Q|_{L_t}$ is non-degenerate and indefinite. We claim that we may choose $t \in \mathbb{Q}$ such that, in addition, $Q|_{L_t}$ is not proportional to a rational quadratic form.

Assume, by contradiction, that this is not true. Then we find, for each t, an $a_t \in \mathbb{R}$ such that

$$b_i^t Q b_j \in a_t\mathbb{Q},\ 1 \leq i, j \leq n-2\ (*)$$

$$b_i^t Q(b_{n-1} + tx) = b_i^t Q b_{n-1} + t b_i^t Q x \in a_t\mathbb{Q},\ 1 \leq i \leq n-2\ (**)$$

$$(b_{n-1} + tx)^t Q(b_{n-1} + tx) = b_{n-1}^t Q b_{n-1} + 2t b_{n-1}^t Q x + t^2 x^t Q x \in a_t\mathbb{Q}\ (***)$$

Since $Q|_{L'}$ is non-zero, $(*)$ shows that $a_t\mathbb{Q} = a_0\mathbb{Q}$ for all t. Hence, $b_i^t Q b_{n-1} \in a_0\mathbb{Q}$ for $1 \leq i \leq n-1$ and $b_i^t Q x \in a_0\mathbb{Q}$ for $1 \leq i \leq n-2$, by taking $t = 0$ in $(**)$ and $(***)$. It follows, by $(***)$, that

$$2t b_{n-1}^t Q x + t^2 x^t Q x \in a_0\mathbb{Q}$$

So, $2b_{n-1}^t Q x + t x^t Q x \in a_0\mathbb{Q}$, and taking $t = 0$ yields

$$b_{n-1}^t Q x \in a_0\mathbb{Q}$$

$$x^t Q x \in a_0\mathbb{Q}$$

.

Since $\{b_1, b_2, , , , b_{n-1}, x\}$ is a rational basis of the whole space, this shows that Q is proportional to a rational quadratic form, a contradiction.

□

5. Oppenheim's Conjecture: Proof without using Ratner's Theorems

Notations in this section is still coherent with Section 2.

In this section, the author wants to give a proof for theorem 2.6. The method we use still is H-principle.

Proposition 5.1. *If $SO_{Q_0}(\mathbb{R})g$ orbit in X_3 is bounded, then either $SO_{Q_0}g$ is closed and hence compact, or the closure of $SO_{Q_0}(\mathbb{R})g$ contains a $\{v_s\}_{s\geq 0}$-orbit or a $\{v_s\}_{s\leq 0}$-orbit.*

We first show how to deduce theorem 2.6 from proposition 5.1.

Proof. We assume there is an element $\Lambda \in X_3$ such that $v_{s\geq 0}\Lambda \subset \overline{SO_{Q_0}g}$ or $v_{s\leq 0}\Lambda \subset \overline{SO_{Q_0}g}$. We know $\overline{Q(P)} \supset Q_0(v_s x) = 2x_1x_3 - x_2^2 + (2x_3^2)s$. Apparently, we can choose $\mathbf{x} = (x_1, x_2, x_3) \in \Lambda$ such that $x_3 \neq 0$ and $Q_0(\mathbf{x}) < 0$ arbitrarily or $Q_0(\mathbf{x}) > 0$ arbitrarily according to $s \geq 0$ or $s \leq 0$. Then we let s varies, we can get the entire real number $\mathbb{R}$. Hence, $\overline{Q(P)} = \mathbb{R}$. Therefore, by proposition 3.6, we know $SO_{Q_0}(\mathbb{R})g$ is unbounded. Contradiction!

Thus, if $SO_{Q_0}(\mathbb{R})$-orbit in X_3 is bounded, then it is compact.

□

For simplicity, we make some conventions. We use $S = SO_{Q_0}(\mathbb{R})$, and $\mathfrak{s}^{\perp}$ represents orthogonal space of $\mathfrak{s}$ in $\mathfrak{sl}(3, \mathbb{R})$, where the Killing form is the inner product. Notice that $\mathfrak{s}^{\perp}$ is just a linear space but not a Lie algebra.

Lemma 5.2. [7]
Assume $\mathfrak{s}^{\perp} \ni w_n \to 0$ but not belongs to Lie(V) when n is large enough. For any $\delta > 0$, any sufficiently large n, there exists $t_{n,\delta}$ such that
1. $||Ad(u_{t_{n,\delta}})w_n|| \in [\frac{\delta}{10^{10}}, 10^{10}\delta]$;
2. every limit point of $Ad(u_{t_{n,\delta}})w_n$ as $n \to \infty$ lies in Lie(V).

Lemma 5.3. [7] *Assume $\mathfrak{s} \ni h_n \to 0$, $\mathfrak{s}^{\perp} \ni w_n \to 0$, as $n \to \infty$,with $h_n + w_n \notin$ Lie(U) Then, for any $\delta > 0$, any n is large enough, there exist $t_{n,\delta}$ and $s_{n,\delta}$ such that*

$$u_{s_{n,\delta}} u_{t_{n,\delta}} exp(h_n) exp(w_n) u_{-t_{n,\delta}} = exp(h_{n,\delta}) exp(w_{n,\delta})$$

for some $h_{n,\delta} \in \mathfrak{s}$ and $w_{n,\delta} \in \mathfrak{s}^{\perp}$ with

$$max\{||h_{n,\delta}||, ||w_{n,\delta}||\} \in [\frac{\delta}{10^{100}}, 10^{100}\delta]$$

and every limit point of $h_{n,\delta} \oplus w_{n,\delta}$ as $n \to \infty$ lies in $Lie(A) \oplus Lie(V)$.

Lemma 5.2 and lemma 5.3 are elementary but tedious. We firstly admit these two lemmas for main proof of proposition 5.1 and then we demonstrate these two lemmas.

Proof. Suppose Sg is bounded but not closed in X_3. Write Y_0 for the closure of Sg.

Consider the following $\mathcal{O} = \{y \in Y_0 \mid Sy \text{ is open in } Y_0\}$.

Thus $\mathcal{O}$ is an S-invariant and open(possible empty) subset of Y_0, in other words, $Y_0 \backslash \mathcal{O}$ is an S-invariant compact set.

Because Sg is not closed, $\mathcal{O} \neq Y_0$. Otherwise, every orbit is open in Y_0, hence is closed in Y_0, hence is closed in X_3. But Sg is not closed, a contradiction!

Let Y_1 be a non-empty U-minimal non-empty set in $Y_0 \backslash \mathcal{O}$. As lemma 3.4, we can use the same method to illustrate Y_1 can not be a U-orbit.

Consider $L = \{s \in S \mid sY_1 = Y_1\}$, and take $x \in Y_1$. Since Ux is not closed, we can find $y_n = exp(h_n)exp(w_n)x \in Y_1$ with $h_n \in \mathfrak{s}$, $w_n \in \mathfrak{s}^{\perp}$, $h_n, w_n \to 0$ and $h_n + w_n \notin Lie(U)$ to approximate x. Using lemma 5.3, by passing to a subsequence, let

$$y_\delta = \lim u_{s_{n,\delta}} u_{t_{n,\delta}} y_n\,;$$
$$x_\delta = \lim u_{t_{n,\delta}} x\,.$$

We get

$$y_\delta = g_\delta x_\delta$$

with $y_\delta \in Y_1$ and $x_\delta \in Y_1$. Here, we write $g_\delta = exp(h_\delta)exp(w_\delta)$, where $h_\delta \oplus w_\delta$ is a limit point in lemma 5.3.

Note g_δ normalizes U. Hence, $g_\delta Y_1 = \overline{g_\delta U x_\delta} = \overline{U y_\delta} = Y_1$. We know L is a closed subgroup, and if we let $\delta \to 0$, there exists some non-zero $v \in Lie(AV)$ such that

$$exp(sv)Y_1 = Y_1, \forall s \in \mathbb{R}\,.$$

If v has non-trivial Lie(V)-component, by erasing Lie(A)-componet, we know Y_0 contains a V-orbit. Otherwise, we can know Y_1 is A-invariant.

If Y_1 is A-invariant, we show that

$$v_{s\geq 0} \text{ or } v_{s\leq 0} \subset \{g \in S \mid gY_1 \subset Y_0\} ,$$

which implies the conclusion of proposition 5.1.

Choose $x \in Y_1$ and $Y_0 \ni y_n \to x$ such that $y_n = exp(w_n)x$, where $\mathfrak{s}^\perp \in w_n \to 0$.

If there are infinitely many $w_n \in Lie(V)$, we finish the proof. Otherwise, by lemma 5.2, we define:

$$x_{n,\delta} = u_{t_{n,\delta}} x$$

and

$$y_{n,\delta} = u_{t_{n,\delta} y_n} ,$$

then

$$y_{n,\delta} = exp(Ad(u_{t_{n,\delta}})w_n)x_{n,\delta} .$$

Using lemma 5.2, we choose a subsequence and get

$$y_\delta = exp(v_\delta)x_\delta ,$$

where $v \in Lie(V)$ and $|v_\delta| \in [\frac{\delta}{10^{10}}, 10^{10}\delta]$.

Thus, we have $Y_0 \supset Uexp(v_\delta)x_\delta = exp(v_\delta)Ux_\delta$, and take the closure, we have $Y_0 \supset exp(v_\delta)Y_1$.

However, we know Y_1 is A-invariant. We use A-action to know: for any $t \in \mathbb{R}$, we have $Y_0 \supset exp(e^{2t}v_\delta)Y_1$. Thus, $\{g \in S \mid gY_1 \subset Y_0\}$ contains a $v_{s\geq 0}$ or $v_{s\leq 0}$-orbit.

To sum up, we get the conclusion of proposition 5.1.

□

Next on, we explain these two lemmas.

Direct calculations tell us:

$$\mathfrak{s}^\perp = \{ \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & -2x_{11} & -x_{12} \\ x_{31} & -x_{21} & x_{11} \end{bmatrix} \} .$$

Take $w = (w_{ij}) \in \mathfrak{s}^\perp$, note that

$$Ad(u_s)w = exp(s\, ad(u_0))w = w + s\, ad(u_0)w + \frac{s^2}{2} ad(u_0)^2 w + \frac{s^3}{3!} ad(u_0)^3 w + \frac{s^4}{4!} ad(u_0)^4 w .$$

We can write as follows: $Ad(u_t)w =$

$$\begin{bmatrix} \frac{t^2}{2}w_{31} + tw_{21} + w_{11} & \frac{t^3}{3!}w_{31} + \frac{t^2}{2}w_{21} + tw_{11} + \frac{-w_{12}}{3} & \frac{t^4}{4!}w_{31} + \frac{t^3}{3!}w_{21} + \frac{t^2}{2}w_{11} + t\frac{-w_{12}}{3} + \frac{w_{13}}{6} \\ tw_{13} + w_{21} & * & * \\ w_{31} & * & * \end{bmatrix}$$

where the terms marked as $*$ are determined by the others, since the matrix is an element in $\mathfrak{s}^\perp$.

We look at this big matrix carefully: no matter how we choose matrix coefficients of $w = (w_{ij})$, the fastest growth of this big matrix belongs to $(1, 3)$-term, which is exactly in Lie(V). Growth rate in this matrix is polynomial growth, and different terms have different growth degree. As every term in w goes to 0, we can choose an appropriate t so that every term other than $(1, 3)$-term goes to 0 and $(1, 3)$-term remains δ-scale. This idea is simple, but we need write many words to say clearly. The author wants to leave it as an exercise to the reader. Hence, we know lemma 5.2 is true.

As for lemma 5.3, recall that we can recognize $S = SO_{Q_0}(\mathbb{R})$ as $SL(2,\mathbb{R})$. We can observe U^+-action on $SL(2,\mathbb{R})$ to understand U-action on S. We can also write concrete matrices to analyse, and it is not hard to get the conclusion.

Acknowledgments

It is a pleasure to thank my supervisors- Prof. Disheng Xu and Prof. Jinpeng An.

Prof. Xu leads me to dynamical systems and Prof. An carefully helps me understand homogeneous dynamics. They are leaders for me in mathematics, and they also give me helpful suggestions on my decisions about my future career.

I also faithfully thank my upperclassmen-Runlin Zhang, Zuo Lin and Yushan Jiang. I ask lots of questions about Ratner's theorems and Oppenheim's conjecture, and Runlin Zhang gives me lots of help. His handout helps me a lot. Zuo Lin and Yushan Jiang also gives me lots of help on Ratner's theorems and they encourages me to go further in mathematics.

I also faithfully thank Prof. Junyi Xie. Prof. Xie helps me a lot in my future direction. He is very warm-hearted and has an insightful eye for mathematics. Maybe he is my ideal mathematician!

I also faithfully thank Prof. Zhiqiang Li. I learn a lot from his seminars. He leads me into complex dynamics and helps me a lot when I encounter difficulties both in life and mathematics.

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