3D-PCP

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Least square fit in 3D:

$$z = a_1 x + a_2 y + a_3$$

$$e_i = z_i - (a_1 x_i + a_2 y_i + a_3) \text{ for points } (x_i, y_i, z_i)$$

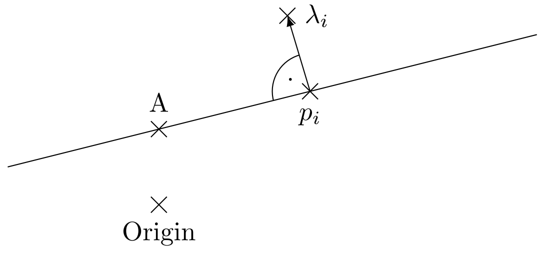
$$E(a_1, a_2, a_3) = \sum_i \|e_i\|^2 = \sum_i \|z_i - (a_1 x_i + a_2 y_i + a_3)\|^2$$

$$\underbrace{\begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix}}_{Z} = \underbrace{\begin{pmatrix} x_1 & y_1 & 1 \\ & \vdots & \\ x_n & y_n & 1 \end{pmatrix}}_{M} \cdot \underbrace{\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}}_{A}$$

2 15.05.2015

in 3D for 3D planes:
$\vec{A} = \vec{A}_0 + t \cdot \vec{D} + s \cdot \vec{E}$ or $N(x - A) = 0$, where N is the unit-length normal to plane and A is a point on the Plane.

Let x_i be the sample-points:
$x_i = A\lambda_i - N + p_i N_i^{\perp}$ where $\underbrace{\lambda_i = N(x_i - A)}_{\text{dist to plane}}$ and $N_i^{\perp}$ is some unit length vector perpendicular to N with appropriate coefficient p_i.



Define $y_i = x_i - A$. The vecotr from x_i to its projection onto the plane is $\lambda_i N$. Then

$$\lambda_i^2 = (N \cdot y_i)^2$$

is the squared distance to the plane. We define a Error-function

$$\begin{aligned} E(A,N) &= \sum_{i=1}^{m} \lambda_i^2 \\ &= \sum_i y_i^T (NN^T) y_i \\ &= N^T (\sum_i y_i y_i^T) N \\ &= N^T M(A) N \end{aligned}$$

Using the first form of E:

$$\frac{dE}{dA} = -2[NN^T] \sum_i y_i$$

This partial derivative becomes 0 whenever $\sum_i y_i = 0$ in which case $A = \frac{1}{m} \sum_i x_i$ (the average of all the sample points).
Given A, the matrix $M(A)$ is determined in the second form of E

$$N^T M(A) N$$

is a quadratic frm, whose minimum is the smallest eigenvalue of $M(A)$. The corresponding unit length vector (eigenvector) N completes our construction of the least square plane.

$$M(A) = \begin{pmatrix} \sum(x_i - a)^2 & \sum(x_i - a)(y_i - b) & \sum(x_i - a)(z_i - c) \\ \sum(x_i - a(y_i - b) & \sum(y_i - b)^2 & \sum(y_i - b)(z_i - c) \\ \sum(x_i - a)(z_i - c) & \sum(z_i - c)(y_i - b) & \sum(z_i - c)^2 \end{pmatrix}$$

$$f(N) = N^T M(A) N$$
$$\min f(N) \text{ subject to } N^T N = 1$$

Solve this with constraint-minimization – Lagrange-multipliers:

$$\mathcal{L}(n, \lambda) = f(N) - \lambda(N^T N - 1)$$
$$\frac{d\mathcal{L}}{dN} = 0 \Leftrightarrow M(A) = \lambda N$$
$$\frac{d\mathcal{L}}{d\lambda} = 0 \Leftrightarrow N^T N = 1$$

N is the eigenvector of $M(A)$ with the smallest eigenvalue.

3 18.05.2015

$$\begin{aligned} E(R,t) &= \sum_{i=1}^{\sharp m} \sum_{j=1}^{\sharp D} w_{ij} \|\hat{m}_i - (R\hat{d}_i + t)\|^2 \\ &= \frac{1}{N} \sum_{i=1}^{N} \|m_i - (Rd_i + t)\|^2 \end{aligned}$$

where R needs to be orthonormal.

Complete the centroids:

$$c_m = \frac{1}{N}\sum_{i=1}^{N} m_i \qquad c_d = \frac{1}{N}\sum_{i=1}^{n} d_i$$

$$M' = \{m'_i = m_i - c_m\} \qquad D' = \{d'_i = d_i - c_d\}$$

$$\begin{aligned} E(R,t) &= \sum_{i=1}^{N} \|m'_i - Rd'_i - (t - c_m - Rc_d)\|^2 \\ &= \frac{1}{N}\sum \|m'_i - Rd'_i\| - \underbrace{\frac{2}{N} t \sum (m'_i - Rd'_i)}_{=0} + \underbrace{\frac{1}{N}\sum \|\tilde{t}\|^2}_{\tilde{t}=0 \to t = c_m - Rc_d} \end{aligned}$$

Lemma 1.

$$tr\left(AA^T\right) \geq tr\left(BAA^T\right)$$

Proof. without proof. □

Theorem 1. *The optimal rotation is $R = VU^T$. V and U are derived by singular value decomposition $H = U\Lambda V^T$ of a correlation matrix H*

$$H = \sum_{i=1}^{N} m_i'^T d'_i = \begin{pmatrix} S_x x & S_x y & S_x z \\ S_y x & S_y y & S_y z \\ S_z x & S_z y & Szz \end{pmatrix}$$

where $S_x x = \sum_{i=1}^{N} m'_{x_i} d'_{x_i} \ldots$

Proof.

$$\begin{aligned} E &= \sum \|m'_i - Rd'_i\|^2 \\ &= \sum_{i_1}^{N} \|m'_i\|^2 - 2\sum_{i=1}^{N} m'_i R d'_i + \underbrace{\sum_{i=1}^{N} \|Rd_i\|^2}_{\to \sum_{i=1}^{N} \|d'_i\|^2} \end{aligned}$$

$$\begin{aligned} \min E \Rightarrow \max &\sum_{i=1}^{N} m'_i R d'_i \\ &= \sum_{i=1}^{N} m_i'^T R d'_i \end{aligned}$$

Using the trace:

$$\operatorname{tr}\left(\sum_{i=1}^{N} R d'_i m_i'^T\right) = \operatorname{tr}(RH)$$

find the R that maximizes the trace $\operatorname{tr}(RH)$.
Assume that the SVD of H is

$$H = U\Lambda V^T$$

U and V are orthonormal and Λ is a 3×3 diagonal matrix without negative elements.

$$R := VU^T$$

R is orthonormal and

$$\begin{aligned} RH &= VU^T U\Lambda V^T \\ &= V\Lambda V^T \end{aligned}$$

using Lemma 1 $\operatorname{tr}(RH) \geq \operatorname{tr}(BRH)$ for any orthonormal matrix B. $\square$