25.10.2021

Seminar W4 – 914

the external operation (the scalar multiplication)

Def.: $(V,+)$ abelian group, $(K,+,\cdot)$ field, $\cdot : K\times V \to V$, $(k,v)\mapsto kv$

We say that V is a <u>K-vector space</u> (denote ${}_K V$) if:

- $\forall \alpha,\beta\in K,\ \forall v\in V:\ (\alpha+\beta)\cdot v = \alpha v + \beta v$
- $\forall \alpha\in K,\ \forall v_1,v_2\in V:\ \alpha\cdot(v_1+v_2) = \alpha\cdot v_1 + \alpha\cdot v_2$
- $\forall \alpha,\beta\in K,\ \forall v\in V:\ (\alpha\beta)\cdot v = \alpha\cdot(\beta v)$
- $\forall v\in V:\ 1\cdot v = v$

Ex.: K^n, $K[X]$, $M_{m,n}(K)$, $K^A = \{f : A\to K\}$ with A a set,

$\mathcal{C}([a,b])$, $\mathcal{C}^n([a,b])$

Def/Th.: V K-vector space, $S\subseteq V$

$S \leq_K V$ ("S is a K-subspace of V") $\Leftrightarrow$
(i) $S\neq\emptyset$
(ii) $(S,+)\leq(V,+)$: $\forall v_1,v_2\in S:\ v_1-v_2\in S$
(iii) $\cdot$ is well defined on S: $\forall\alpha\in K,\ \forall v\in S:\ \alpha\cdot v\in S$

(ii)' $\forall v_1,v_2\in S:\ v_1+v_2\in S$

$\forall \alpha,\beta\in K,\ \forall v_1,v_2\in S:\ \alpha v_1+\beta v_2\in S$

4. Let $V = \{x \in \mathbb{R} \mid x > 0\}$ and define the operations: $x \perp y = xy$ and $k \top x = x^k$, $\forall k \in \mathbb{R}$ and $\forall x, y \in V$. Prove that V is a vector space over $\mathbb{R}$.

<u>Sol</u>: $(V, \perp)$ abelian group

- $\forall x, y \in V$: $x \perp y = \underbrace{x}_{>0}\underbrace{y}_{>0} \in \mathbb{R}$

- assoc. : $\forall x, y, z \in V$: $(x \perp y) \perp z = (xy) \perp z = (xy)z = x \cdot (yz) =$

$$= x \perp (y \perp z)$$

- neutral element : $\forall x \in V$: $x \perp 1 = x \cdot 1 = x = 1 \perp x$

- invertibility : $\forall x \in V$ $\exists x' \in V$: $x \perp x' = 1 = x' \perp x$

$$x' = \frac{1}{x}, \text{ because } x > 0$$

- commutativity : $\forall x, y \in V$: $x \perp y = xy = yx = y \perp x$

We will now prove the axioms:

- Let $k \in \mathbb{R}$, $v_1, v_2 \in V$: $k \top (v_1 \perp v_2) \overset{?}{=} (k \top v_1) \perp (k \top v_2)$

$k \top v_1 = v_1^k$

$k \top v_2 = v_2^k$

$(k \top v_1) \perp (k \top v_2) = v_1^k \perp v_2^k = v_1^k \cdot v_2^k = (v_1 v_2)^k = (v_1 \perp v_2)^k =$

$= k \top (v_1 \perp v_2)$

- Let $k_1, k_2 \in \mathbb{R}$, $v \in V$: $(k_1+k_2)Tv \stackrel{?}{=} (k_1Tv)\perp(k_2Tv)$

$$(k_1Tv)\perp(k_2Tv) = v^{k_1}\perp v^{k_2} = v^{k_1}\cdot v^{k_2} = v^{k_1+k_2} = (k_1+k_2)Tv$$

- Let $k_1, k_2 \in \mathbb{R}$, $v \in V$: $(k_1\cdot k_2)Tv \stackrel{?}{=} k_1T(k_2Tv)$

$$k_1T(k_2Tv) = k_1T(v^{k_2}) = (v^{k_2})^{k_1} = v^{k_1k_2} = (k_1\cdot k_2)Tv$$

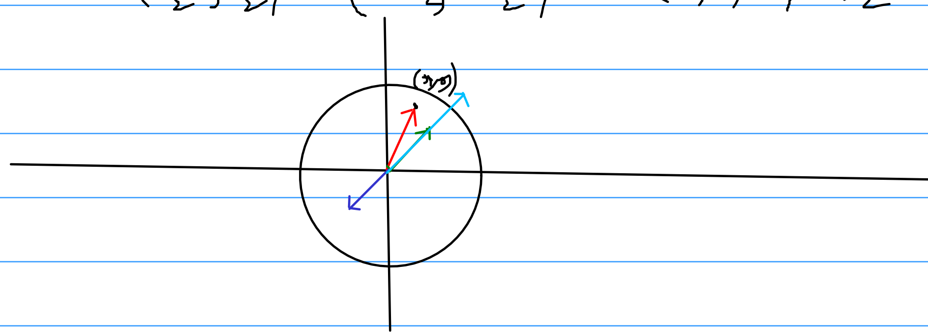
- Let $v \in V$: $1Tv = v^1 = v$

8. Which ones of the following sets are subspaces:
(*i*) $[-1,1]$ of the real vector space $\mathbb{R}$;
(*ii*) $\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\leq 1\} = S_2$ of the real vector space $\mathbb{R}^2$;
(*iii*) $\left\{\begin{pmatrix}a & b\\ 0 & c\end{pmatrix} \middle| a,b,c\in\mathbb{Q}\right\}$ of ${}_{\mathbb{Q}}M_2(\mathbb{Q})$ or of ${}_{\mathbb{R}}M_2(\mathbb{R})$;
(*iv*) $\{f:\mathbb{R}\to\mathbb{R} \mid f \text{ continuous}\}$ of the real vector space $\mathbb{R}^{\mathbb{R}}$?

Sol: (i) $1,-1\in[-1,1]$, $1-(-1)=2\notin[-1,1] \Rightarrow [-1,1]\not\leq_{\mathbb{R}}\mathbb{R}$

(ii) $\left(\frac{1}{2},\frac{1}{2}\right)\in S_2$, $\left(-\frac{1}{2},-\frac{1}{2}\right)\in S_2$

$$\left(\frac{1}{2},\frac{1}{2}\right)-\left(-\frac{1}{2},-\frac{1}{2}\right)=(1,1)\notin S_2 \Rightarrow S_2\not\leq_{\mathbb{R}}\mathbb{R}^2$$



(iii) $S_3=\left\{\begin{pmatrix}a & b\\ 0 & c\end{pmatrix} \middle| a,b,c\in\mathbb{Q}\right\} \stackrel{?}{\leq_{\mathbb{Q}}} M_2(\mathbb{Q})$

$\begin{pmatrix}1 & 2\\ 0 & 3\end{pmatrix}\in S_3 \Rightarrow S_3\neq\emptyset$

Let $A_1 = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}$, $A_2 = \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix} \in S_3$

$$A_1 + A_2 = \begin{pmatrix} a_1+a_2 & b_1+b_2 \\ 0 & c_1+c_2 \end{pmatrix} \in S_3$$

Let $A_3 = \begin{pmatrix} a_3 & b_3 \\ 0 & c_3 \end{pmatrix} \in S_3$, $\alpha \in \mathbb{Q}$

$$\alpha A_3 = \begin{pmatrix} \alpha a_3 & \alpha b_3 \\ 0 & \alpha c_3 \end{pmatrix} \in S_3$$

$$\Rightarrow \quad S_3 \leq_{\mathbb{Q}} M_2(\mathbb{Q})$$

$$S_3 = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \,\middle|\, a, b, c \in \mathbb{Q} \right\} \overset{?}{\leq}_{\mathbb{R}} M_2(\mathbb{R})$$

$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \in S_3$, $\sqrt{2} \in \mathbb{R}$, but

$$\sqrt{2} \cdot \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 3\sqrt{2} \end{pmatrix} \notin S_3$$

$$\Rightarrow \quad S_3 \not\leq_{\mathbb{R}} M_2(\mathbb{R})$$

(iv) $S_4 = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous} \}$

$S_4 \neq \emptyset$, because $id_{\mathbb{R}} \in S_4$

$\forall \alpha, \beta \in \mathbb{R}$, $\forall f, g \in S_4$ $\overset{?}{:}$ $\alpha f + \beta g$ continuous

Let $x_0 \in \mathbb{R}$ then

$$\lim_{x \to x_0} (\alpha f + \beta g)(x) = \lim_{x \to x_0} (\alpha \cdot f(x) + \beta g(x)) \overset{f, g \text{ continuous}}{=} \alpha f(x_0) + \beta g(x_0) =$$
$$= (\alpha f + \beta g)(x_0)$$

$$\Rightarrow \alpha f + \beta g \in S_4 \quad \Rightarrow \quad S_4 \leq_{\mathbb{R}} \mathbb{R}^{\mathbb{R}}$$

7. Which ones of the following sets are subspaces of the real vector space $\mathbb{R}^3$:
(*i*) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$;
(*ii*) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0 \text{ or } z = 0\}$; ✓
(*iii*) $C = \{(x, y, z) \in \mathbb{R}^3 \mid x \in \mathbb{Z}\}$;
(*iv*) $D = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$; ✓
(*v*) $E = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1\}$;
(*vi*) $F = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$?

Sol: (ii) $(0, 10, 20) \in B$, $(15, 7, 0) \in B$

$$(0, 10, 20) + (15, 7, 0) = (15, 17, 20) \notin B$$

$$\Rightarrow B \not\leq_{\mathbb{R}} \mathbb{R}^3$$

(iv) $(0,0,0) \in D \Rightarrow D \neq \emptyset$

Let $(a,b,c), (d,e,f) \in D$

$$(a,b,c) + (d,e,f) = (a+d, b+e, c+f)$$

$$(a+d) + (b+e) + (c+f) = \underbrace{(a+b+c)}_{=0} + \underbrace{(d+e+f)}_{=0} = 0$$

Let $\alpha \in \mathbb{R}$:

$$\alpha \cdot (a,b,c) = (\alpha a, \alpha b, \alpha c)$$

$$\alpha a + \alpha b + \alpha c = \alpha(a+b+c) = \alpha \cdot 0 = 0$$

$$\Rightarrow D \leq_{\mathbb{R}} \mathbb{R}^3$$