Seminar week 9 11

$$r \doteq (\underset{\text{domain}}{A}, \underset{\text{codomain}}{B}, \underset{\text{graph}}{\mathcal{R}}) \qquad \text{relation}$$

A, B sets, $\mathcal{R} \subseteq A \times B$

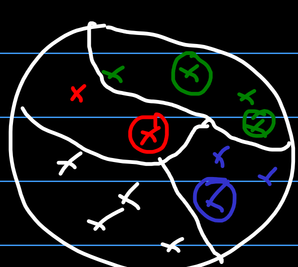
homogeneous relation : $r = (A, A, \mathcal{R})$

<u>Equivalence relation</u> : (binary homogeneous) relation $r = (A, A, \mathcal{R})$ that satisfies the following properties

- reflexivity : $\forall x \in A : \quad x r x$ $\quad (\Leftrightarrow (x, x) \in \mathcal{R})$
- symmetry : $\forall x, y \in A :$ if $x r y$, then $y r x$
- transitivity : $\forall x, y, z \in A$ if $x r y$ and $y r z$, then $x r z$

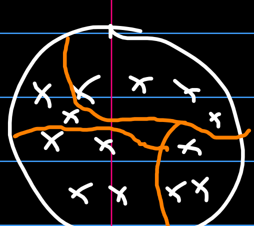
<u>Prop</u> : If A is a set, then we have a bijection :

$$\{\text{equivalence relations on } A\} \xrightarrow{\sim} \{\text{partition of } A\}$$



$r \longmapsto A/r$ quotient set of r

$r_P \longleftarrow P \subseteq \mathcal{P}(A)$



$A/r = \{ \underbrace{r\langle x \rangle}_{=: \hat{x}} \mid x \in A \} = \{$green stuff, red stuff, blue stuff, white stuff$\}$.

$r\langle x \rangle = \{ y \in A \mid x r y \}$

$\mathcal{P}$ partition of A $\Rightarrow$ $\mathcal{P} = \{ A_i \mid i \in I \}$

s.t. $\forall i,j \in I : A_i \cap A_j = \emptyset$

$\bigcup_{i \in I} A_i = A$

$x \, r_{\mathcal{P}} \, y \iff \exists \, S \in \mathcal{P}$ s.t.

$x, y \in S$

2.1.

1. Let r, s, t, v be the homogeneous relations defined on the set $M = \{2, 3, 4, 5, 6\}$ by

$$x \, r \, y \iff x < y$$

$$x \, s \, y \iff x | y \iff y \vdots x$$

$$x \, t \, y \iff g.c.d.(x, y) = 1$$

$$x \, v \, y \iff x \equiv y \pmod 3 . \iff 3 | (x - y) \iff$$

Write the graphs R, S, T, V of the given relations. $\iff x \bmod 3 = y \bmod 3$

Sol.:

$R = \{ (2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6),$
$(4,5), (4,6), (5,6) \}$

$S = \{ (2,2), (2,4), (2,6), (3,3), (3,6), (4,4), (5,5), (6,6) \}$

$T = \{ (2,3), (2,5), (3,4), (3,5), (4,5), (5,6),$
$(3,2), (5,2), (4,3), (5,3), (5,4), (6,5) \}$

$V = \{ (5,2), (6,3), (2,2), (2,5), (3,3), (3,6)$
$(4,4), (5,5), (6,6) \}$

3. Give examples of relations having each one of the properties of reflexivity, transitivity and symmetry, but not the others.

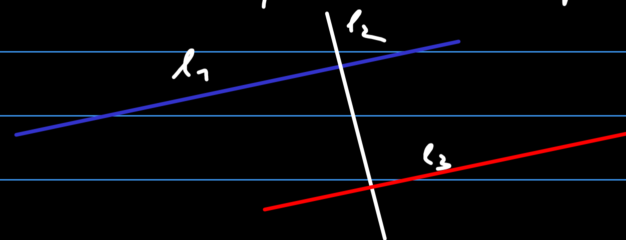
Sol: (r), $\neg(S)$, $\neg(t)$: $M = \{1,2,3\}$

Graph: $\{(1,1), (1,2), (2,2), (2,3), (3,3)\}$

$\neg(r)$, (S), $\neg(t)$: $M =$ set of lines in the plane

$M = \{1,2,3\}$ $\quad l_1 \sim l_2 \Leftrightarrow l_1 \perp l_2$

Graph: $\{(1,2), (2,1), (2,3), (3,2)\}$



$\neg(r)$, $\neg(s)$, (t):

$M = \mathbb{R}$

$x r y \Leftrightarrow x < y$

$M = \{1,2\}$

Graph: $\{(1,2)\}$

5. Let $M = \{1,2,3,4\}$, let r_1, r_2 be homogeneous relations on M and let π_1, π_2, where $R_1 = \Delta_M \cup \{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$, $R_2 = \Delta_M \cup \{(1,2),(1,3)\}$, $\pi_1 = \{\{1\},\{2\},\{3,4\}\}$, $\pi_2 = \{\{1\},\{1,2\},\{3,4\}\}$.

(*i*) Are r_1, r_2 equivalences on M? If yes, write the corresponding partition.

(*ii*) Are π_1, π_2 partitions on M? If yes, write the corresponding equivalence relation.

$\Delta_M = \{(1,1), (2,2), (3,3), (4,4)\}$

(i) r_1 refl., because $\Delta_M \subseteq R_1$

r_1 symmetric and

r_1 transitive

$\Rightarrow M/r_1 = \{\{1,2,3\}, \{4\}\}$

$(1,2) \in R_2$, but $(2,1) \notin R_2 \Rightarrow R_2$ not

symmetric $\Rightarrow R_2$ not an equivalence

Ex: $A = \{1, 2\}$. $A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$

$$P = \{\{(1,1), (2,2)\}, \{(1,2), (2,1)\}\}$$

$r = (A, A, R)$, $\cdot\ R = A \times A$

$$\Rightarrow A/r = \{A\}$$

$\cdot\ R = \Delta_A$

$$\Rightarrow A/r = \{\{x\} \mid x \in A\}$$

(ii)

5. Let $M = \{1,2,3,4\}$, let r_1, r_2 be homogeneous relations on M and let π_1, π_2, where $R_1 = \Delta_M \cup \{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$, $R_2 = \Delta_M \cup \{(1,2),(1,3)\}$, $\pi_1 = \{\{1\},\{2\},\{3,4\}\}$, $\pi_2 = \{\{1\},\{1,2\},\{3,4\}\}$.
(i) Are r_1, r_2 equivalences on M? If yes, write the corresponding partition.
(ii) Are π_1, π_2 partitions on M? If yes, write the corresponding equivalence relation.

π_1 is a partition, because $\forall x \in M \ \exists! A \in \pi_1$

so that $x \in A$

$$R_{\pi_1} = \{(1,1), (2,2), (3,3), (3,4), (4,3), (4,4)\}$$

π_2 is not a partition, because $\{1\} \cap \{1,2\} \neq \emptyset$

8. Determine all equivalence relations and all partitions on the set $M = \{1, 2, 3\}$.

Sol.: Partitions:

$\underbrace{\{\{1,2,3\}\}}_{1 \text{ element}}$

$\underbrace{\{\{1,2\},\{3\}\} \quad \{\{1\},\{2,3\}\} \quad \{\{2\},\{1,3\}\}}_{2 \text{ elements}}$

$\underbrace{\{\{1\},\{2\},\{3\}\}}_{3 \text{ elements}}$

$\Rightarrow$ 5 partitions $\Rightarrow$ 5 equivalences

Ex. 6,7