10.11.2020

Seminar WZ - 917

Th.: V k-v.s., $\dim_k V = n$, $B = (v_1, v_2, \dots, v_n)$ family of vectors.

B basis $\Leftrightarrow$ B lin. indep $\Leftrightarrow$ B system of generators

Def: V k-v.s., $v_1, v_2, \dots, v_m \in V$

$\operatorname{rank}(v_1, \dots, v_m) :=$ max number of linearly independent vectors among $v_1, \dots, v_m$ $= \dim \langle v_1, \dots, v_m \rangle$

Prop: $V = k^n$, $v_1, \dots, v_m \in V$

$$\Rightarrow \operatorname{rank}(v_1, \dots, v_m) = \operatorname{rank}\begin{pmatrix} v_1 \\ \hline v_2 \\ \hline \vdots \\ \hline v_m \end{pmatrix} = \operatorname{rank}\left(v_1 \,|\, v_2 \,|\, \cdots \,|\, v_m\right)$$

In the lecture: $V = k^n$, $v_1, \dots, v_n \in V$

$$v_1, \dots, v_n \text{ lin. indep.} \Leftrightarrow \det\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \neq 0 \Leftrightarrow \det(v_1 | \cdots | v_n) \neq 0$$

Th. (Steinitz): V k-v.s., $v_1, \dots, v_m$ linearly independent in V
(a simplification)

$\dim_k V = n \geq m \Rightarrow \exists\, w_{m+1}, \dots, w_n \in V$ so that

$(v_1, \dots, v_m, w_{m+1}, \dots, w_n)$ is a basis

How to complete a lin. indep. family to a basis:

$u_1, \ldots, u_m$ lin. indep.

- Choose a vector $w_{m+1} \in V \setminus \langle u_1, \ldots, u_m \rangle$

Do we have enough vectors now?

yes — Yay, we have a basis

no — Choose a vector $w_{m+2} \in V \setminus \langle u_1, \ldots, u_m, u_{m+1} \rangle$

Enough vectors?

1. Determine a basis and the dimension of the following subspaces of the real vector space $\mathbb{R}^3$:

$$A = \{(x,y,z) \in \mathbb{R}^3 \mid z = 0\}$$
$$B = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z = 0\}$$
$$C = \{(x,y,z) \in \mathbb{R}^3 \mid x = y = z\}.$$

6. Complete the bases of the subspaces from Exercise **1.** to some bases of the real vector space $\mathbb{R}^3$ over $\mathbb{R}$.

Sol.: $A = \{ (x,y,0) \mid x,y \in \mathbb{R} \} = \{ (x,0,0) + (0,y,0) \mid x,y \in \mathbb{R} \} =$

$= \langle (1,0,0), (0,1,0) \rangle$

$\operatorname{rank}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = 2 \Rightarrow (1,0,0), (0,1,0)$ lin. indep. $\Rightarrow$ they form a basis

To complete this to a basis of $\mathbb{R}^3$, we add a vector that is not in $\langle (1,0,0), (0,1,0) \rangle = A$. Let's add $(0,0,1)$.

$\Rightarrow \big((1,0,0), (0,1,0), (0,0,1) \big)$ basis of $\mathbb{R}^3$

$B = \langle (-1,1,0), (-1,0,1) \rangle$

$\text{rank} \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = 2 \Rightarrow (-1,1,0), (-1,0,1)$ lin. indep $\Rightarrow$ they form a basis

To complete this to a basis of $\mathbb{R}^3$, we need add one more vector.

We try adding $(1,0,0)$. We check if what we get is lin. indep:

$\begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 1 \neq 0 \Rightarrow (1,0,0), (-1,1,0), (-1,0,1)$ lin. indep $\Rightarrow$

$\Rightarrow$ they form a basis of $\mathbb{R}^3$

$C = \{ (x,x,x) \mid x \in \mathbb{R} \} = \langle (1,1,1) \rangle$

$\Rightarrow \dim C = 1$.

To complete this basis to a basis of $\mathbb{R}^3$ we need to add two vectors. Let the first one be $(1,0,0)$.

$(1,0,0)$ and $(1,1,1)$ are linearly independent.

We add $(0,1,0)$

- we determine $\langle (1,0,0), (1,1,1) \rangle$ and we see that $(0,1,0) \notin \langle (1,0,0), (1,1,1) \rangle$
- we check for linear indep of the three vectors by using the rank
- use the definition

$$\operatorname{rank}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = 3, \quad \text{because} \quad \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1 \neq 0$$

$$\Rightarrow (1,0,0), (0,1,0), (1,1,1) \text{ lin. indep } \overset{\dim_{\mathbb{R}} \mathbb{R}^3 = 3}{\Rightarrow} \text{ they form a basis}$$

<u>Consequence</u>: V K-v.s., $S \leq V$

If we find a basis $u_1, \dots, u_m$ for S, then if we complete this basis by $w_{m+1}, w_{m+2}, \dots, w_n$ to a basis V, then:

$$T := \langle w_{m+1}, w_{m+2}, \dots, w_n \rangle$$

$$\Rightarrow \quad V = S \oplus T$$

<u>Def</u>: V, W K-v.s., $\underbrace{f \in \operatorname{Hom}_K(V, W)}_{\text{"} f: V \to W \text{ is a } K\text{-linear map"}}$

$$\operatorname{Ker} f = \{ v \in V \mid f(v) = 0_W \} \leq_K V$$
("kernel")

$$\operatorname{Im} f = \{ f(v) \mid v \in V \} = \{ w \in W \mid \exists v \in V : f(v) = w \}$$

4. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $f(x, y, z) = (y, -x)$. Prove that f is an $\mathbb{R}$-linear map and determine a basis and the dimension of $Ker\, f$ and $Im\, f$.

Sol.: $\operatorname{Ker} f = \{ (x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = (0,0) \} =$

$= \{ (x,y,z) \in \mathbb{R}^3 \mid (y,-x) = (0,0) \} =$

$= \{ (x,y,z) \in \mathbb{R}^3 \mid y = 0 \quad x = 0 \} =$

$= \{ (0,0,z) \mid z \in \mathbb{R} \} = \langle (0,0,1) \rangle$

$((0,0,1))$ is a basis for $\operatorname{Ker} f \Rightarrow \dim \operatorname{Ker} f = 1$

$\operatorname{Im} f = \{ f(x,y,z) \mid x,y,z \in \mathbb{R} \} = \{ (y,-x) \mid x,y,z \in \mathbb{R} \} =$

$= \{ (y,-x) \mid x,y \in \mathbb{R} \} = \{ y \cdot (1,0) + x \cdot (0,-1) \mid x,y \} =$

$= \langle (1,0), (0,-1) \rangle$

$\operatorname{rang} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2 \Rightarrow (1,0), (0,-1)$ lin. indep. $\Rightarrow ((1,0),(0,-1))$ basis for $\operatorname{Im} f$

$\Rightarrow \dim \operatorname{Im} f = 2$

Th. (1^{st} dim thm.) V, W K-v.s., $f \in \operatorname{Hom}_K(V,W)$:

$$\dim V = \dim \operatorname{Ker} f + \dim \operatorname{Im} f$$

Th. (2^{nd} dim. thm.) $S, T \leq_K V$, then: $\dim(S+T) = \dim S + \dim T - \dim(S \cap T)$

Remark : 7) $S = \langle v_1, \ldots, v_n \rangle$, $T = \langle w_1, \ldots, w_m \rangle$

$\Rightarrow S+T = \langle v_1, v_2, \ldots, v_n, w_1, \ldots, w_m \rangle$

(do ex. 9)