04.10.2023

Seminar W2 – 917

(binary) relation : $r = (\underset{\text{set}}{\overset{\text{domain}}{A}}, \underset{\text{set}}{\overset{\text{codomain}}{B}}, \overset{\text{graph}}{\underline{R}})$, $R \subseteq A \times B$

homogeneous relation : $A = B$

equivalence relation $r = (A, A, R)$

has the properties

- reflexivity : $\forall x \in A$: $x \, r \, x$ $\big(\Leftrightarrow (x,x) \in R \big)$
- symmetry : $\forall x, y \in A$: if $x \, r \, y$, then $y \, r \, x$
- transitivity : $\forall x, y, z \in A$: if $x \, r \, y$ and $y \, r \, z$, then $x \, r \, z$

1. Let r, s, t, v be the homogeneous relations defined on the set $M = \{2, 3, 4, 5, 6\}$ by

$$x \, r \, y \iff x < y$$
$$x \, s \, y \iff x | y \Leftrightarrow y \, \vdots \, x$$
$$x \, t \, y \iff g.c.d.(x, y) = 1$$
$$x \, v \, y \iff x \equiv y \pmod 3 \Leftrightarrow 3 \mid (x - y) \Leftrightarrow x \bmod 3 = y \bmod 3$$

Write the graphs R, S, T, V of the given relations.

Sol. : $R = \{(2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6)\}$

$S = \{(2,2), (2,4), (2,6), (3,3), (3,6), (4,4), (5,5), (6,6)\}$

$T = \{(2,3), (2,5), (3,2), (3,5), (5,2), (3,4), (5,3), (5,4), (5,6), (6,5), (4,5), (4,3)\}$

$V = \{(2,5), (5,2), (3,6), (6,3), (2,2), (3,3), (4,4), (5,5), (6,6)\}$

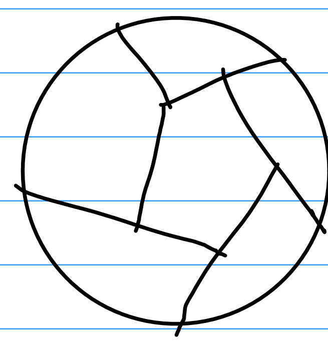
3. Give examples of relations having each one of the properties of reflexivity, transitivity and symmetry, but not the others.

Sol :

refl., $\neg$sym, $\neg$trans	$\neg$refl., sym, $\neg$trans	$\neg$refl, $\neg$sym, trans
$A=\{1,2,3\}$ $\mathcal{R}=\{(1,1),(2,2),(3,3),(1,2),(2,3)\}$	$A=\{1,2\}$ $\mathcal{R}=\{(1,2),(2,1)\}$	$x\,r\,y \Leftrightarrow x>y$ $A=\mathbb{R}$
		$A=\{1,2\}$ $\mathcal{R}=\{(1,1),(1,2)\}$

Def : S set , $P=\{A_i \mid i\in I\}$ partition of S if:

- $\forall i,j \in I: \; A_i \cap A_j = \emptyset$
- $\bigcup_{i\in I} A_i = S$



Prop : S set. We have a bijection:

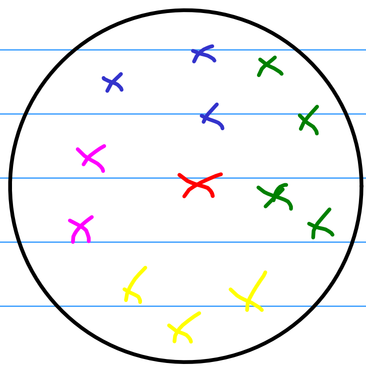
$$\{\text{equivalence relations on } S\} \longleftrightarrow \{\text{partitions of } S\}$$

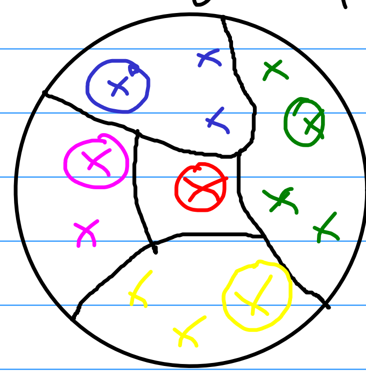
$$r \longmapsto S/r$$

quotient set ("S modulo r")

$$r_P \longmapsfrom P$$

$$S/r = \{ \underbrace{r\langle x\rangle}_{= \{ y \in S \mid x r y \} =: \hat{x}} \mid x \in S \}$$

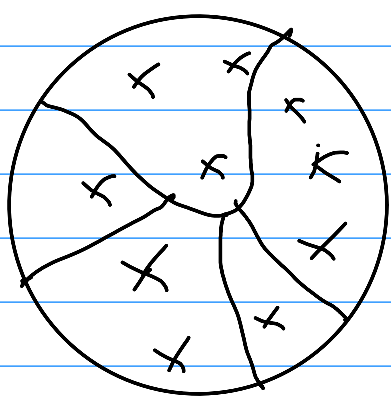




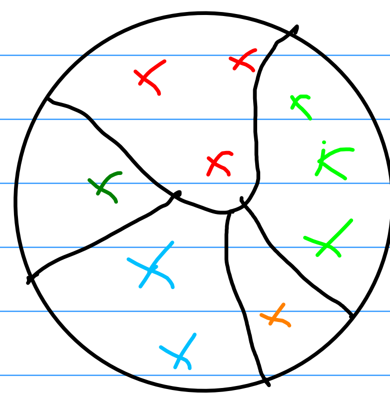
r shows that some elements have the same color

S/r = the blue slice $\cup$ the green slice $\cup$ the magenta slice $\cup$ the yellow slice $\cup$ the red slice

$$x \, r_{\mathcal{P}} \, y \iff \exists A \in \mathcal{P} : x, y \in A$$



$\mathcal{P}$



$r_{\mathcal{P}}$

5. Let $M = \{1, 2, 3, 4\}$, let r_1, r_2 be homogeneous relations on M and let π_1, π_2, where $R_1 = \Delta_M \cup \{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$, $R_2 = \Delta_M \cup \{(1,2),(1,3)\}$, $\pi_1 = \{\{1\},\{2\},\{3,4\}\}$, $\pi_2 = \{\{1\},\{1,2\},\{3,4\}\}$.

$\Delta_M = \{(1,1),(2,2),(3,3),(4,4)\}$

(i) Are r_1, r_2 equivalences on M? If yes, write the corresponding partition.

(ii) Are π_1, π_2 partitions on M? If yes, write the corresponding equivalence relation.

(i) $\Delta_M \subseteq R_1 \Rightarrow r_1$ refl.
r_1 sym, r_1 trans $\Big] \Rightarrow r_1$ equivalence

$$M/r_1 = \{ \{1,2,3\}, \{4\} \}$$

$(1,3) \in R_2$, but $(3,1) \notin R_2 \Rightarrow r_2$ not symmetric
$\Rightarrow r_2$ is not an equivalence relation

(ii) Π_1 is a partition, because $\forall x \in M\ \exists! A \in \Pi_1 : x \in A$

$$\Rightarrow R_{\Pi_1} = \{ (1,1), (2,2), (3,3), (4,4), (3,4), (4,3) \}$$

Π_2 is not a partition $\{1\} \cap \{1,2\} \neq \emptyset$

8. Determine all equivalence relations and all partitions on the set $M = \{1, 2, 3\}$.

<u>Sol.</u>: The partitions:

$\Pi_1 = \{ \{1\}, \{2\}, \{3\} \}$ $\quad R_{\Pi_1} = \{(1,1),(2,2),(3,3)\}$

$\Pi_2 = \{ \{1,2,3\} \}$ $\quad R_{\Pi_2} = \{1,2,3\} \times \{1,2,3\}$

$\Pi_3 = \{ \{1,2\}, \{3\} \}$ $\quad R_{\Pi_3} = \{(1,1),(2,2),(1,2),(2,1),(3,3)\}$

$\Pi_4 = \{ \{1\}, \{2,3\} \}$ $\quad R_{\Pi_4} = \{(1,1),(2,2),(3,3),(2,3),(3,2)\}$

$\Pi_5 = \{ \{2\}, \{1,3\} \}$ $\quad R_{\Pi_5} = \{(2,2),(1,1),(1,3),(3,1),(3,3)\}$

1.6. Finite stable subsets of $(\mathbb{R}, \cdot)$

$\{-1, 0, 1\}$ ✓

$\{-1, 0\}$ ✗

$\{0, 1\}$ ✗

$\{-1, 1\}$ ✓

$\{-1\}$ ✗

$\{0\}$ ✓

$\{1\}$ ✓