Seminar 1

1. Which ones of the usual symbols of addition, subtraction, multiplication and division define an operation (composition law) on the numerical sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$?

2. Let $A = \{a_1, a_2, a_3\}$. Determine the number of:
(i) operations on A;
(ii) commutative operations on A;
(iii) operations on A with identity element.
Generalization for a set A with n elements ($n \in \mathbb{N}^*$).

3. Decide which ones of the numerical sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are groups together with the usual addition or multiplication.

4. Let "$*$" be the operation defined on $\mathbb{R}$ by $x * y = x + y + xy$. Prove that:
(i) $(\mathbb{R}, *)$ is a commutative monoid.
(ii) The interval $[-1, \infty)$ is a stable subset of $(\mathbb{R}, *)$.

5. Let "$*$" be the operation defined on $\mathbb{N}$ by $x * y = \text{g.c.d.}(x, y)$.
(i) Prove that $(\mathbb{N}, *)$ is a commutative monoid.
(ii) Show that $D_n = \{x \in \mathbb{N} \mid x/n\}$ ($n \in \mathbb{N}^*$) is a stable subset of $(\mathbb{N}, *)$ and $(D_n, *)$ is a commutative monoid.
(iii) Fill in the table of the operation "$*$" on D_6.

6. Determine the finite stable subsets of $(\mathbb{Z}, \cdot)$.

7. Let $(G, \cdot)$ be a group. Show that:
(i) G is abelian $\Longleftrightarrow \forall x, y \in G$, $(xy)^2 = x^2y^2$.
(ii) If $x^2 = 1$ for every $x \in G$, then G is abelian.

8. Let "$\cdot$" be an operation on a set A and let $X, Y \subseteq A$. Define an operation "$*$" on the power set $\mathcal{P}(A)$ by

$$X * Y = \{x \cdot y \mid x \in X, y \in Y\}.$$

Prove that:
(i) If $(A, \cdot)$ is a monoid, then $(\mathcal{P}(A), *)$ is a monoid.
(ii) If $(A, \cdot)$ is a group, then in general $(\mathcal{P}(A), *)$ is not a group.

Seminar 2

1. Let r, s, t, v be the homogeneous relations defined on the set $M = \{2, 3, 4, 5, 6\}$ by

$$x\,r\,y \Longleftrightarrow x < y$$

$$x\,s\,y \Longleftrightarrow x|y$$

$$x\,t\,y \Longleftrightarrow g.c.d.(x, y) = 1$$

$$x\,v\,y \Longleftrightarrow x \equiv y \pmod 3\,.$$

Write the graphs R, S, T, V of the given relations.

2. Let A and B be sets with n and m elements respectively ($m, n \in \mathbb{N}^*$). Determine the number of:
(i) relations having the domain A and the codomain B;
(ii) homogeneous relations on A.

3. Give examples of relations having each one of the properties of reflexivity, transitivity and symmetry, but not the others.

4. Which ones of the properties of reflexivity, transitivity and symmetry hold for the following homogeneous relations: the strict inequality relations on $\mathbb{R}$, the divisibility relation on $\mathbb{N}$ and on $\mathbb{Z}$, the perpendicularity relation of lines in space, the parallelism relation of lines in space, the congruence of triangles in a plane, the similarity of triangles in a plane?

5. Let $M = \{1, 2, 3, 4\}$, let r_1, r_2 be homogeneous relations on M and let π_1, π_2, where $R_1 = \Delta_M \cup \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}$, $R_2 = \Delta_M \cup \{(1, 2), (1, 3)\}$, $\pi_1 = \{\{1\}, \{2\}, \{3, 4\}\}$, $\pi_2 = \{\{1\}, \{1, 2\}, \{3, 4\}\}$.
(i) Are r_1, r_2 equivalences on M? If yes, write the corresponding partition.
(ii) Are π_1, π_2 partitions on M? If yes, write the corresponding equivalence relation.

6. Define on $\mathbb{C}$ the relations r and s by:

$$z_1\,r\,z_2 \Longleftrightarrow |z_1| = |z_2|\,; \qquad z_1\,s\,z_2 \Longleftrightarrow arg\,z_1 = arg\,z_2 \text{ or } z_1 = z_2 = 0\,.$$

Prove that r and s are equivalence relations on $\mathbb{C}$ and determine the quotient sets (partitions) $\mathbb{C}/r$ and $\mathbb{C}/s$ (geometric interpretation).

7. Let $n \in \mathbb{N}$. Consider the relation ρ_n on $\mathbb{Z}$, called the *congruence modulo* n, defined by:

$$x\,\rho_n\,y \Longleftrightarrow n|(x - y)\,.$$

Prove that ρ_n is an equivalence relation on $\mathbb{Z}$ and determine the quotient set (partition) $\mathbb{Z}/\rho_n$. Discuss the cases $n = 0$ and $n = 1$.

8. Determine all equivalence relations and all partitions on the set $M = \{1, 2, 3\}$.

9. Let $M = \{0, 1, 2, 3\}$ and let $h = (\mathbb{Z}, M, H)$ be a relation, where

$$H = \{(x, y) \in \mathbb{Z} \times M \mid \exists z \in \mathbb{Z} : x = 4z + y\}\,.$$

Is h a function?

10. Consider the following homogeneous relations on $\mathbb{N}$, defined by:

$$m\,r\,n \Longleftrightarrow \exists a \in \mathbb{N} : m = 2^a n\,,$$

$$m\,s\,n \Longleftrightarrow (m = n \text{ or } m = n^2 \text{ or } n = m^2)\,.$$

Are r and s equivalence relations?

Seminar 3

1. Let M be a non-empty set and let $S_M = \{f : M \to M \mid f \text{ is bijective}\}$. Show that $(S_M, \circ)$ is a group, called the *symmetric group* of M.

2. Let M be a non-empty set and let $(R, +, \cdot)$ be a ring. Define on $R^M = \{f \mid f : M \to R\}$ two operations by: $\forall f, g \in R^M$,

$$f + g : M \to R\,, \quad (f+g)(x) = f(x) + g(x)\,, \quad \forall x \in M\,,$$

$$f \cdot g : M \to R\,, \quad (f \cdot g)(x) = f(x) \cdot g(x)\,, \quad \forall x \in M\,.$$

Show that $(R^M, +, \cdot)$ is a ring. If R is commutative or has identity, does R^M have the same property?

3. Prove that $H = \{z \in \mathbb{C} \mid |z| = 1\}$ is a subgroup of $(\mathbb{C}^*, \cdot)$, but not of $(\mathbb{C}, +)$.

4. Let $U_n = \{z \in \mathbb{C} \mid z^n = 1\} \quad (n \in \mathbb{N}^*)$ be the *set of n-th roots of unity*. Prove that U_n is a subgroup of $(\mathbb{C}^*, \cdot)$.

5. Let $n \in \mathbb{N}$, $n \geq 2$. Prove that:
(i) $GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid det(A) \neq 0\}$ is a stable subset of the monoid $(M_n(\mathbb{C}), \cdot)$;
(ii) $(GL_n(\mathbb{C}), \cdot)$ is a group, called the *general linear group of rank n*;
(iii) $SL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid det(A) = 1\}$ is a subgroup of the group $(GL_n(\mathbb{C}), \cdot)$.

6. Show that the following sets are subrings of the corresponding rings:
(i) $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ in $(\mathbb{C}, +, \cdot)$.
(ii) $\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\}$ in $(M_2(\mathbb{R}), +, \cdot)$.

7. (i) Let $f : \mathbb{C}^* \to \mathbb{R}^*$ be defined by $f(z) = |z|$. Show that f is a group homomorphism between $(\mathbb{C}^*, \cdot)$ and $(\mathbb{R}^*, \cdot)$.

(ii) Let $g : \mathbb{C}^* \to GL_2(\mathbb{R})$ be defined by $g(a + bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Show that g is a group homomorphism between $(\mathbb{C}^*, \cdot)$ and $(GL_2(\mathbb{R}), \cdot)$.

8. Let $n \in \mathbb{N}$, $n \geq 2$. Prove that the groups $(\mathbb{Z}_n, +)$ of residue classes modulo n and $(U_n, \cdot)$ of n-th roots of unity are isomorphic.

9. Let $n \in \mathbb{N}$, $n \geq 2$. Consider the ring $(\mathbb{Z}_n, +, \cdot)$ and let $\widehat{a} \in \mathbb{Z}_n^*$.
(i) Prove that $\widehat{a}$ is invertible $\iff (a, n) = 1$.
(ii) Deduce that $(\mathbb{Z}_n, +, \cdot)$ is a field $\iff n$ is prime.

10. Let $\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| a, b \in \mathbb{R} \right\} \subseteq M_2(\mathbb{R})$. Show that $(\mathcal{M}, +, \cdot)$ is a field isomorphic to $(\mathbb{C}, +, \cdot)$.