11.10.2021

Seminar W2 — 916

- relation: $r = (A, B, R)$, where A = domain, B = codomain, $R \subseteq A \times B$ = graph

$x r y \overset{\text{not}}{\iff} (x,y) \in R$

→ homogeneous relation: $A = B$

→ equivalence relation $r = (A, A, R)$

→ must satisfy:

- reflexivity: $\forall x \in A$: $x r x$
- symmetry: $\forall x, y \in A$: if $x r y$, then $y r x$
- transitivity: $\forall x, y, z \in A$: if $x r y$ and $y r z$, then $x r z$

1. Let r, s, t, v be the homogeneous relations defined on the set $M = \{2, 3, 4, 5, 6\}$ by

$$x\, r\, y \iff x < y$$

$$x\, s\, y \iff x | y \iff y \overset{\cdot}{\vdots}\, x$$

$$x\, t\, y \iff g.c.d.(x, y) = 1$$

$$x\, v\, y \iff x \equiv y \pmod 3 \iff 3 \mid (x-y) \iff x \bmod 3 = y \bmod 3$$

Write the graphs R, S, T, V of the given relations.

Sol: $R = \{(2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6)\}$

$S = \{ (2,2), (2,4), (2,6), (3,3), (3,6), (4,4), (5,5), (6,6) \}$

$T = \{ (2,3), (3,2), (2,5), (5,2), (3,4), (4,3), (3,5), (5,3), (5,6), (6,5) \}$

$V = \{ (2,5), (5,2), (3,6), (6,3), (4,4), (2,2), (3,3), (5,5), (6,6) \}$

3. Give examples of relations having each one of the properties of reflexivity, transitivity and symmetry, but not the others.

Sol:

refl., $\neg$sym, $\neg$trans	$\neg$refl, sym, $\neg$trans	$\neg$refl, $\neg$sym, trans
$A = \mathbb{N}$	$A = \{1,2\}$	$A = \mathbb{N}$
$x r y \iff x \mid y$	$R = \{(1,2),(2,1)\}$	$x R y \iff x < y$
$\therefore x \mid y \Rightarrow y = xk,\ k \in \mathbb{N}$	✓	✓
$y \mid z \Rightarrow z = ym,\ m \in \mathbb{N}$		$A = \{1,2,3\}$
$z = ym = xkm \Rightarrow x \mid z$		$R = \{(1,2),(2,3),(1,3)\}$

This does not work because it's transitive

~~$A = \{2,3,5\}$~~

$A = \{1,2,3\}$

$R = \{(1,2),(2,3),(1,1),(2,2),(3,3)\}$

✓

Def: A set, $\mathcal{P} = \{A_i \mid i \in I\} \subseteq \mathcal{P}(A)$

is a partition of A if:

- $\forall i,j \in I,\ i \neq j : A_i \cap A_j = \emptyset$
- $\bigcup_{i \in I} A_i = A$

Thm: A set. We have the bijection:

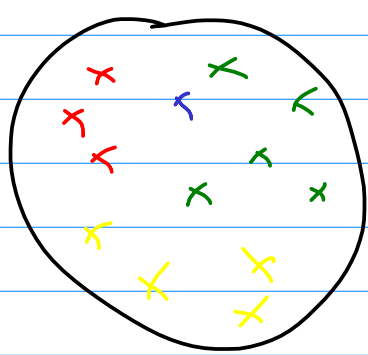
$$\{\text{equivalence relations on } A\} \xleftrightarrow{\sim} \{\text{partitions of } A\}$$

$$r \longmapsto A/r$$

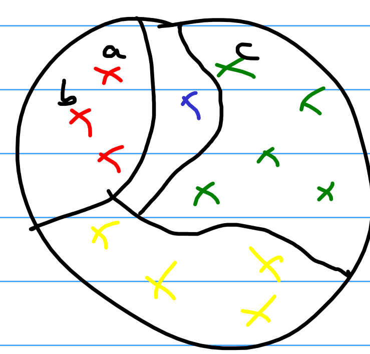
$$r_{\mathcal{P}} \longleftarrow\!\shortmid\ \mathcal{P}$$

$$A/r = \{ r\langle x \rangle \mid x \in A \}$$

$$\hat{x} := r\langle x \rangle = \{ y \in A \mid x r y \}$$



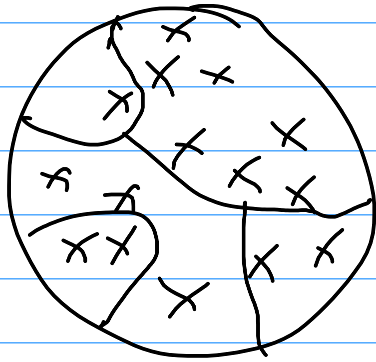
$r =$ "have the same colour"

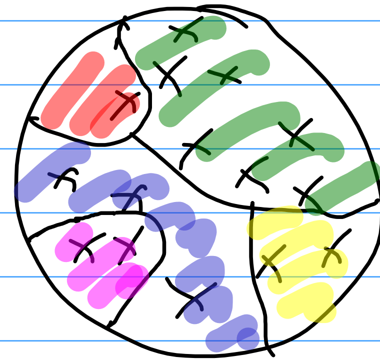


$A/r =$ red $\cup$ blue $\cup$ green $\cup$ yellow

↓

the set of elements having the same color as a (or b)





$\mathcal{P}$

$x \; r_{\mathcal{P}} \; y \iff$ "there is a set in $\mathcal{P}$ that both x and y are part of"

$\Delta_M = \{(1,1), (2,2), (3,3), (4,4)\}$

5. Let $M = \{1,2,3,4\}$, let r_1, r_2 be homogeneous relations on M and let π_1, π_2, where $R_1 = \Delta_M \cup \{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$, $R_2 = \Delta_M \cup \{(1,2),(1,3)\}$, $\pi_1 = \{\{1\},\{2\},\{3,4\}\}$, $\pi_2 = \{\{1\},\{1,2\},\{3,4\}\}$.

(i) Are r_1, r_2 equivalences on M? If yes, write the corresponding partition.

(ii) Are π_1, π_2 partitions on M? If yes, write the corresponding equivalence relation.

<u>Sol.</u> : (i) r_1 is an equivalence relation

$$M/r_1 = \{\{1,2,3\}, \{4\}\}$$

r_2 is not an equivalence relation, because it is not symmetrical, since $(1,2) \in R_2$, but $(2,1) \notin R_2$

(ii) π_1 partition, because $\forall x \in M : \exists!\, B \in \pi_1 : x \in B$

$$R_{\pi_1} = \{(1,1), (2,2), (3,3), (3,4), (4,3), (4,4)\}$$

π_2 is not a partition, because $\{1\} \cap \{1,2\} \neq \emptyset$

Ex: $M = \{1, 2, 3, 4, 5, 6\}$

$r = (M, M, R)$

$R = \Delta_M \cup \{(1,2), (2,6), (6,1), (1,6), (6,2), (2,1),$

$(4,5), (5,4)\}$

$M/r = \{\{1,2,6\}, \{3\}, \{4,5\}\}$

7. Let $n \in \mathbb{N}$. Consider the relation ρ_n on $\mathbb{Z}$, called the *congruence modulo* n, defined by:

$$x \,\rho_n\, y \iff n|(x-y).$$

Prove that ρ_n is an equivalence relation on $\mathbb{Z}$ and determine the quotient set (partition) $\mathbb{Z}/\rho_n$. Discuss the cases $n = 0$ and $n = 1$.

Sol: $\forall x \in \mathbb{Z}$: $n \mid 0 \Rightarrow n \mid (x-x) \Rightarrow x \rho_n x$

$\forall x, y \in \mathbb{Z}$, if $n \mid x-y \Rightarrow n \mid y-x$, so if $x \rho_n y$, then $y \rho_n x$

$\forall x, y, z \in \mathbb{Z}$ if $x \rho_n y$ and $y \rho_n z \Rightarrow n \mid (x-y)$, $n \mid (y-z) \Rightarrow$

$\Rightarrow n \mid (x-y)+(y-z) \Rightarrow n \mid x-z \Rightarrow x \rho_n z$

$\Rightarrow \rho_n$ is an equivalence relation.

- For $n = 0 \Rightarrow \forall x, y : x \rho_0 y \iff x = y$

$\Rightarrow \mathbb{Z}/\rho_0 = \{\{x\} \mid x \in \mathbb{Z}\}$

- For $n = 1 \Rightarrow \forall x, y : x \rho_1 y \Rightarrow \mathbb{Z}/\rho_1 = \{\mathbb{Z}\}$

For $n \in \mathbb{N} \setminus \{0,1\}$:

$$\mathbb{Z}/\rho_n = \{ \rho_n\langle x\rangle \mid x \in \mathbb{Z} \}$$

$$\rho_n\langle x\rangle = \{ y \in \mathbb{Z} \mid x \rho_n y \} =$$

$$= \{ y \in \mathbb{Z} \mid n \mid (x-y) \} =$$

$$= \{ y \in \mathbb{Z} \mid x \equiv y \pmod{n} \}$$

$$\mathbb{Z}/\rho_n = \{ \underbrace{\{nk+r \mid k \in \mathbb{Z}\}}_{=\hat{r}} \mid r \in \{0,\dots,n-1\} \}$$

$$= \mathbb{Z}_n$$

8. Determine all equivalence relations and all partitions on the set $M = \{1, 2, 3\}$.

<u>Sol.</u> : Partitions

$\pi_1 = \{\{1\},\{2\},\{3\}\}$ $\quad R_{\pi_1} = \{(1,1),(2,2),(3,3)\} = \Delta_M$

$\pi_2 = \{\{1\},\{2,3\}\}$ $\quad R_{\pi_2} = \{(1,1),(2,2),(2,3),(3,2),(3,3)\}$

$\pi_3 = \{\{2\},\{1,3\}\}$

$\pi_4 = \{\{3\},\{1,2\}\}$

$\pi_5 = \{\{1,2,3\}\}$ $\quad R_{\pi_5} = \{(1,1),(1,2),(1,3),$
$(2,1),(2,2),(2,3),$
$(3,1),(3,2),(3,3)\}$