03.11.2021

Seminar W6 - 917

Def: V K-vector space, $v_1, v_2, \dots, v_n \in V$

$v_1, v_2, \dots, v_n$ are <u>linearly independent</u> if:

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0 \Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$$

$v_1, \dots, v_n$ are linearly dependent if:

$\exists\, \alpha_1, \dots, \alpha_n$ not all zero so that $\underbrace{\alpha_1 v_1 + \dots + \alpha_n v_n = 0}$

"dependence relationship between $v_1, \dots, v_n$"

1. Let $v_1 = (1, -1, 0)$, $v_2 = (2, 1, 1)$, $v_3 = (1, 5, 2)$ be vectors in the canonical real vector space $\mathbb{R}^3$. Prove that:

(*i*) v_1, v_2, v_3 are linearly dependent and determine a dependence relationship.

(*ii*) v_1, v_2 are linearly independent.

Sol: (i) $2v_2 - 3v_1 = v_3$

Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ so that $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

$$\alpha_1 \cdot (1, -1, 0) + \alpha_2 \cdot (2, 1, 1) + \alpha_3 \cdot (1, 5, 2) = (0, 0, 0)$$

$$\Rightarrow \begin{cases} \alpha_1 + 2\alpha_2 + \alpha_3 = 0 \\ -\alpha_1 + \alpha_2 + 5\alpha_3 = 0 \\ \alpha_2 + 2\alpha_3 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_2 = -2\alpha_3 \\ \alpha_1 - 4\alpha_3 + \alpha_3 = 0 \\ -\alpha_1 - 2\alpha_3 + 5\alpha_3 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_2 = -2\alpha_3 \\ \alpha_1 = 3\alpha_3 \\ \alpha_1 = 3\alpha_3 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = 3\alpha_3 \\ \alpha_2 = -2\alpha_3 \end{cases}$$

For $\alpha_3 = 1$ we get: $3u_1 - 2u_2 + u_3 = 0$

(ii) Let $\alpha_1, \alpha_2 \in \mathbb{R}$: $\alpha_1 v_1 + \alpha_2 v_2 = 0$

$\Rightarrow \alpha_1 \cdot (1,-1,0) + \alpha_2 \cdot (2,1,1) = 0$

$$\Rightarrow \begin{cases} \alpha_1 + 2\alpha_2 = 0 \\ -\alpha_1 + \alpha_2 = 0 \\ \alpha_2 = 0 \end{cases} \Rightarrow \alpha_1 = \alpha_2 = 0$$

$\Rightarrow u_1, u_2$ linearly independent

3. Let $v_1 = (1, a, 0)$, $v_2 = (a, 1, 1)$, $v_3 = (1, 0, a)$ be vectors in ${}_{\mathbb{R}}\mathbb{R}^3$. Determine $a \in \mathbb{R}$ such that the vectors v_1, v_2, v_3 are linearly independent.

Sol: Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$: $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 u_3 = 0$

$$\begin{cases} \alpha_1 + a\alpha_2 + \alpha_3 = 0 \\ a\alpha_1 + \alpha_2 = 0 \\ \alpha_2 + a\alpha_3 = 0 \end{cases}$$

The system is compatible determined if and only if $\Delta = \begin{vmatrix} 1 & a & 1 \\ a & 1 & 0 \\ 0 & 1 & a \end{vmatrix} \neq 0$

$\Updownarrow$

v_1, v_2, v_3 are linearly independent

$$\Delta = \begin{vmatrix} 1 & a & 1 \\ a & 1 & 0 \\ 0 & 1 & a \end{vmatrix} = a + a + 0 - 0 - a^3 - 0 = 2a - a^3 = a(2 - a^2)$$

$\Rightarrow v_1, v_2, v_3$ lin. indep $\Leftrightarrow a \in \mathbb{R} \setminus \{0, -\sqrt{2}, \sqrt{2}\}$

Def: V K-vector space, $v_1, \dots, v_n \in V$

$v_1, \dots, v_n$ form a basis of V $\Leftrightarrow$ • $V = \langle v_1, \dots, v_n \rangle$ $\Leftrightarrow$ ...
• $v_1, \dots, v_n$ are linearly independent

$\Leftrightarrow$ $\forall v \in V$: $\exists!\, \alpha_1, \alpha_2, \dots, \alpha_n$: $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$

($\alpha_1, \dots, \alpha_n$ are the coordinates of v in the basis $(v_1, \dots, v_n)$)

[$\dim_K V$:= # of elements in any basis]

7. Let $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $A_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Prove that the lists (E_1, E_2, E_3, E_4) and (A_1, A_2, A_3, A_4) are bases of the real vector space $M_2(\mathbb{R})$ and determine the coordinates of $B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ in each of the two bases.

Sol: We will prove first that $M_2(\mathbb{R}) = \langle E_1, E_2, E_3, E_4 \rangle$

Let $M \in M_2(\mathbb{R})$, $M = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$

$$M = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix} =$$

$$= x \cdot E_1 + y \cdot E_2 + z \cdot E_3 + t \cdot E_4$$

$$\Rightarrow M_2(\mathbb{R}) = \langle E_1, E_2, E_3, E_4 \rangle$$

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$ so that $\alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 E_4 = O_2$

$$\Rightarrow \begin{pmatrix} \alpha_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \alpha_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \alpha_3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \alpha_4 \end{pmatrix} = O_2$$

$$\Rightarrow \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$$

$\Rightarrow$ E_1, E_2, E_3, E_4 lin indep

$\Rightarrow$ (E_1, E_2, E_3, E_4) basis for ${}_{\mathbb{R}}M_2(\mathbb{R})$

$$[B]_{(E_1, E_2, E_3, E_4)} = ?$$

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = 2 \cdot E_1 + 1 \cdot E_2 + 1 \cdot E_3 + 0 \cdot E_4$$

$$\Rightarrow [B]_{(E_1, E_2, E_3, E_4)} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Let $\beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{R}$ so that:

$$\beta_1 A_1 + \beta_2 A_2 + \beta_3 A_3 + \beta_4 A_4 = O_2$$

$$\Rightarrow \beta_1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta_2 \cdot \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \beta_3 \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \beta_4 \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{matrix} (L_1) \\ (L_2) \\ (L_3) \\ (L_4) \end{matrix} \begin{cases} \beta_1 + \beta_2 + \beta_3 + \beta_4 = 0 \\ \beta_2 + \beta_3 + \beta_4 = 0 \\ \beta_3 + \beta_4 = 0 \\ \beta_1 + \beta_4 = 0 \end{cases} \Leftrightarrow \begin{cases} \beta_1 = 0 \\ \beta_2 + \beta_3 + \beta_4 = 0 \\ \beta_3 + \beta_4 = 0 \\ \beta_1 + \beta_4 = 0 \end{cases} \Leftrightarrow \begin{cases} \beta_1 = 0 \\ \beta_2 = 0 \\ \beta_3 + \beta_4 = 0 \\ \beta_1 + \beta_4 = 0 \end{cases} \Leftrightarrow$$

$\Leftrightarrow$ $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$ $\Rightarrow$ A_1, A_2, A_3, A_4 lin. indep

$$A_3 - A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = E_3$$

$$A_4 - A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = E_4$$

We will now show that $M_2(\mathbb{R}) = \langle A_1, A_2, A_3, A_4 \rangle$

Let $M = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$. We want to find $\beta_1, \beta_2, \beta_3, \beta_4$ so that

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \beta_1 A_1 + \beta_2 A_2 + \beta_3 A_3 + \beta_4 A_4$$

$$\Rightarrow \begin{cases} x = \beta_1 + \beta_2 + \beta_3 + \beta_4 \\ y = \beta_2 + \beta_3 + \beta_4 \\ z = \beta_3 + \beta_4 \\ t = \beta_1 + \beta_4 \end{cases} \Leftrightarrow \begin{cases} \beta_1 = t - \beta_4 \\ x = \beta_1 + \beta_2 + \beta_3 + \beta_4 \\ y = \beta_2 + \beta_3 + \beta_4 \\ z = \beta_3 + \beta_4 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \beta_1 = t - \beta_4 \\ x = t + \beta_2 + \beta_3 \\ y = \beta_2 + \beta_3 + \beta_4 \\ z = \beta_3 + \beta_4 \end{cases} \Leftrightarrow \begin{cases} \beta_1 = t - \beta_4 \\ \beta_3 = z - \beta_4 \\ x = t + \beta_2 + \beta_3 \\ y = \beta_2 + \beta_3 + \beta_4 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \beta_1 = t - \beta_4 \\ \beta_3 = z - \beta_4 \\ x = t + \beta_2 + z - \beta_4 \\ y = \beta_2 + z \end{cases} \Leftrightarrow \begin{cases} \beta_1 = t - \beta_4 \\ \beta_2 = y - z \\ \beta_3 = z - \beta_4 \\ x = t + y - z + z - \beta_4 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \beta_4 = t + y - x \\ \beta_1 = x - y \\ \beta_2 = y - z \\ \beta_3 = x + z - t - y \end{cases}$$

$$\Rightarrow \forall M \quad \exists \beta_1, \beta_2, \beta_3, \beta_4 : M = \sum_{i=1}^{4} \beta_i A_i \Rightarrow M_2(\mathbb{R}) = \langle A_1, A_2, A_3, A_4 \rangle$$

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \qquad [B]_{(A_1,A_2,A_3,A_4)} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}$$

$$\begin{cases} \beta_4 = t + y - x \\ \beta_1 = x - y \\ \beta_2 = y - z \\ \beta_3 = x + z - t - y \end{cases} \Rightarrow \begin{cases} \beta_4 = 0 + 1 - 2 = -1 \\ \beta_1 = 2 - 1 = 1 \\ \beta_2 = 1 - 1 = 0 \\ \beta_3 = 2 + 1 - 1 - 0 = 2 \end{cases}$$

$$\Rightarrow [B]_{(A_1,A_2,A_3,A_4)} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix}$$