1.11.2024

Seminar W5 - 914

Def/Th: V K-vector space, $X \subseteq V$

The subspace of V generated by X is:

$$\langle X \rangle = \bigcap_{\substack{S \leq V \\ S \supseteq X}} S = \left\{ \sum_{i=1}^{n} \alpha_i \cdot a_i \;\middle|\; n \in \mathbb{N},\ \alpha_i \in K,\ a_i \in X \right\}$$

2. Consider the following subspaces of the real vector space $\mathbb{R}^3$:
(i) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$;
(ii) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$;
(iii) $C = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$.
Write A, B, C as generated subspaces with a minimal number of generators.

(iv) $D = \left\{ (x,y,z,t) \in \mathbb{R}^4 \;\middle|\; \begin{cases} y = 0 \\ x + t = 0 \end{cases} \right\}$

(v) $E = \{ (x,y,z) \in \mathbb{R}^3 \mid x + 4y = 0 \}$

Sol: **Ex:** $S = \left\{ (x,y,z,t) \in \mathbb{R}^4 \;\middle|\; \begin{cases} x + y = 0 \\ z + 3t = 0 \end{cases} \right\} =$

$= \left\{ (x,y,z,t) \in \mathbb{R}^4 \;\middle|\; \begin{cases} x = -y \\ z = -3t \end{cases} \right\} =$

$= \{ (-y, y, -3t, t) \mid y, t \in \mathbb{R} \}$

$= \{ (-y, y, 0, 0) + (0, 0, -3t, t) \mid y, t \in \mathbb{R} \} =$

$= \{ y \cdot (-1, 1, 0, 0) + t \cdot (0, 0, -3, 1) \mid y, t \in \mathbb{R} \} =$

$= \langle \{ (-1,1,0,0), (0,0,-3,1) \} \rangle = \langle (-1,1,0,0), (0,0,-3,1) \rangle$

Because $(-1,1,0,0) \notin \langle (0,0,-3,1) \rangle$, the number of generators is minimal.

(i) $A = \{(x,y,z) \in \mathbb{R}^3 \mid x=0\} = \{(0,y,z) \mid y,z \in \mathbb{R}\} =$

$= \{(0,y,0)+(0,0,z) \mid y,z \in \mathbb{R}\} = \{y\cdot(0,1,0)+z\cdot(0,0,1) \mid y,z\in\mathbb{R}\} =$

$= \langle (0,1,0), (0,0,1) \rangle$

$(0,1,0) \notin \langle (0,0,1) \rangle$, so the number of generators is minimal

(ii) $B = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0\} =$

$= \{(x,y,z) \in \mathbb{R}^3 \mid x = -y-z\} =$

$= \{(-y-z, y, z) \mid y,z\in\mathbb{R}\} =$

$= \{(-y,y,0)+(-z,0,z) \mid y,z\in\mathbb{R}\}$

$= \{y\cdot(-1,1,0)+z\cdot(-1,0,1) \mid y,z\in\mathbb{R}\}$

$= \langle (-1,1,0), (-1,0,1) \rangle$

The number of generators is minimal, since $(-1,1,0) \notin \langle(-1,0,1)\rangle$

If $(-1,1,0) \in \langle(-1,0,1)\rangle \Rightarrow \exists\, \alpha \in \mathbb{R} : (-1,1,0) = \alpha(-1,0,1)$

$\Rightarrow \begin{cases} -1 = -\alpha \\ 1 = 0 \\ 0 = 1 \end{cases}$ absurd $\Rightarrow (-1,1,0) \notin \langle(-1,0,1)\rangle$

(iii) $C = \{(x,y,z) \in \mathbb{R}^3 / x = y = z\} =$

$= \{(x,x,x) \mid x \in \mathbb{R}\} = \{x \cdot (1,1,1) \mid x \in \mathbb{R}\} =$

$= \langle (1,1,1) \rangle$

(iv) $D = \{(x,y,z,t) \in \mathbb{R}^4 \mid \begin{cases} y = 0 \\ x+t = 0 \end{cases} \} =$

$= \{(x,0,z,-x) \mid x, z \in \mathbb{R}\} =$

$= \{(x,0,0,-x) + (0,0,z,0) \mid x, z \in \mathbb{R}\} =$

$= \{x \cdot (1,0,0,-1) + z \cdot (0,0,1,0) \mid x, z \in \mathbb{R}\} =$

$= \langle (1,0,0,-1), (0,0,1,0) \rangle$

(v) $E = \{(x,y,z) \in \mathbb{R}^3 / x + 4y = 0\}$

$= \{(-4y, y, z) \mid y, z \in \mathbb{R}\} =$

$= \{(-4y, y, 0) + (0,0,z) \mid y, z \in \mathbb{R}\} =$

$= \{y \cdot (-4,1,0) + z \cdot (0,0,1) \mid y, z \in \mathbb{R}\} =$

$= \langle (-4,1,0), (0,0,1) \rangle$

Def: V, W k-vector spaces, $f: V \to W$ is a <u>(homo)morphism of vector spaces</u> (or <u>a linear map</u>) if:

- $\forall u_1, u_2 \in V$: $f(u_1 + u_2) = f(u_1) + f(u_2)$
- $\forall u \in V, \forall k \in k$: $f(k \cdot u) = k \cdot f(u)$

$\forall k_1, k_2 \in K$ $\forall u_1, u_2 \in V$: $f(k_1 u_1 + k_2 u_2) = k_1 f(u_1) + k_2 f(u_2)$

6. Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ and $h : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$f(x, y) = (x + y, x - y),$$
$$g(x, y) = (2x - y, 4x - 2y),$$
$$h(x, y, z) = (x - y, y - z, z - x).$$

Show that $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ and $h \in End_{\mathbb{R}}(\mathbb{R}^3)$.

$f(\underbrace{(x,y)}_{u}) = \underline{(x+y, x-y)}$

Sol: Let $u_1 = (x_1, y_1)$, $u_2 = (x_2, y_2)$. We will show that

$$f(u_1 + u_2) \overset{?}{=} f(u_1) + f(u_2)$$

$f(u_1 + u_2) = f(x_1 + x_2, y_1 + y_2) = (x_1 + x_2 + y_1 + y_2, x_1 + x_2 - y_1 - y_2)$

$f(u_1) + f(u_2) = f(x_1, y_1) + f(x_2, y_2) = (x_1 + y_1, x_1 - y_1) + (x_2 + y_2, x_2 - y_2) =$

$= (x_1 + x_2 + y_1 + y_2, x_1 + x_2 - y_1 - y_2)$

$f(ku) \overset{?}{=} k \cdot f(u)$

$u_1 = (x_1, y_1)$ $\qquad f(ku_1) = f(kx_1, ky_1) = (kx_1 + ky_1, kx_1 - ky_1)$

$$k f(v_1) = k \cdot (x_1+y_1, \ x_1-y_1) = (kx_1+ky_1, \ kx_1-ky_1)$$

$$\Rightarrow \forall v_1 \in \mathbb{R}^2, \ \forall k \in \mathbb{R}: \ f(kv_1) = k f(v_1)$$

$$\Rightarrow f \in \mathrm{End}_{\mathbb{R}}(\mathbb{R}^2)$$

$$g: \mathbb{R}^2 \to \mathbb{R}^2$$
$$(x,y) \mapsto (2x-y, \ 4x-2y)$$

Let $k_1, k_2 \in \mathbb{R}$, $v_1 = (x_1, y_1)$, $v_2 = (x_2, y_2) \in \mathbb{R}^2$

$$g(k_1 v_1 + k_2 v_2) \overset{?}{=} k_1 g(v_1) + k_2 \cdot g(v_2)$$

$$g(k_1 v_1 + k_2 v_2) = g(k_1 x_1 + k_2 x_2, \ k_1 y_1 + k_2 y_2) =$$

$$= (2k_1x_1 + 2k_2x_2 - k_1y_1 - k_2y_2, \ 4k_1x_1 + 4k_2x_2 - 2k_1y_1 - 2k_2y_2)$$

$$k_1 g(v_1) + k_2 g(v_2) = k_1 (2x_1 - y_1, \ 4x_1 - 2y_1) +$$

$$+ k_2 \cdot (2x_2 - y_2, \ 4x_2 - 2y_2) = (2k_1x_1 - k_1y_1, \ 4k_1x_1 - 2k_1y_1) +$$

$$+ (2k_2x_2 - k_2y_2, \ 4k_2x_2 - 2k_2y_2) =$$

$$= (2k_1x_1 + 2k_2x_2 - k_1y_1 - k_2y_2, \ 4k_1x_1 + 4k_2x_2 - 2k_1y_1 - 2k_2y_2)$$

$$\Rightarrow g(k_1v_1 + k_2v_2) = k_1 g(v_1) + k_2 g(v_2) \Rightarrow g \in \mathrm{End}_{\mathbb{R}}(\mathbb{R}^2)$$

Def: V k-vector space; $S, T \leq_k V$

$V = S + T \iff \forall v \in V: \exists s \in S, \exists t \in T: v = s + t$

$V = S \oplus T \iff \forall v \in V: \exists! s \in S, \exists! t \in T: v = s + t \iff$

$\iff V = S + T$ and $S \cap T = \{0\}$

4. Let

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\},$$
$$T = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}.$$

Prove that S and T are subspaces of the real vector space $\mathbb{R}^3$ and $\mathbb{R}^3 = S \oplus T$.

already done

Sol: $T = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$

Let $(x, y, z) \in S \cap T \Rightarrow \begin{cases} x + y + z = 0 \\ x = y = z \end{cases} \Rightarrow \begin{cases} 3x = 0 \\ x = y = z \end{cases} \Rightarrow x = y = z = 0$

$\Rightarrow S \cap T \subseteq \{0\} \Rightarrow S \cap T = \{0\}$

We will now show that $\mathbb{R}^3 = S + T$, so we have to show

that $\forall v \in \mathbb{R}^3\ \exists s \in S\ \exists t \in T: \quad v = s + t$

We assume that this is true (to get a handle on what s and t might be)

Let $v = (x, y, z) \in \mathbb{R}^3$, $v = s + t$, $s = (a, b, c)$, $a + b + c = 0$
$t = (\alpha, \alpha, \alpha)$

$$\Rightarrow \begin{cases} x = a+\alpha \\ y = b+\alpha \\ z = c+\alpha \end{cases} \Rightarrow x+y+z = a+b+c+3\alpha$$

$$\Rightarrow \alpha = \frac{x+y+z}{3} \Rightarrow a = x - \frac{x+y+z}{3}$$

$$b = y - \frac{x+y+z}{3}$$

$$c = z - \frac{x+y+z}{3}$$

Let now $v = (x,y,z) \in \mathbb{R}^3$. Then we have:

$$(x,y,z) = \underbrace{\left(\frac{x+y+z}{3}, \frac{x+y+z}{3}, \frac{x+y+z}{3}\right)}_{\in T} +$$

$$+ \underbrace{\left(x - \frac{x+y+z}{3}, y - \frac{x+y+z}{3}, z - \frac{x+y+z}{3}\right)}_{\in S}$$

$$\Rightarrow \mathbb{R}^3 = S + T$$

$$\Rightarrow \mathbb{R}^3 = S \oplus T$$