11. 10. 2021

Seminar W2 - 914

- r relation : $r = (A, B, \mathcal{R})$
  - (binary)
  - A domain, B codomain, $\mathcal{R} \subseteq A \times B$ graph

$(a,b) \in \mathcal{R} \Leftrightarrow a\,r\,b \Leftrightarrow$ "a is in relation with b w.r. to the relation r"

→ homogeneous relations : $A = B$

→ equivalence relations must satisfy:
- reflexivity : $\forall x \in A$: $x\,r\,x$
- symmetry : $\forall x,y \in A$: if $x\,r\,y$, then $y\,r\,x$
- transitivity : $\forall x,y,z \in A$: if $x\,r\,y$ and $y\,r\,z$, then $x\,r\,z$

1. Let r, s, t, v be the homogeneous relations defined on the set $M = \{2,3,4,5,6\}$ by

$$x\,r\,y \Longleftrightarrow x < y$$

$$x\,s\,y \Longleftrightarrow x|y \Leftrightarrow y \,\vdots\, x$$

$$x\,t\,y \Longleftrightarrow g.c.d.(x,y) = 1$$

$$x\,v\,y \Longleftrightarrow x \equiv y \pmod 3. \Leftrightarrow 3 \mid (x-y) \Leftrightarrow x \bmod 3 = y \bmod 3$$

Write the graphs R, S, T, V of the given relations.

Sol. :

$R = \{(2,3), (3,4), (4,5), (5,6), (2,4), (2,5), (2,6), (3,5), (3,6), (4,6)\}$

$S = \{(2,4), (2,6), (3,6), \underbrace{(2,2), (3,3), (4,4), (5,5), (6,6)}_{\Delta_M}\}$

$T = \{(2,3), (2,5), (3,4), (3,2), (3,5), (4,3), (4,5), (5,2), (5,3), (5,4), (5,6), (6,5)\}$

$V = \{(2,5), (3,6), (5,2), (6,3)\} \cup \Delta_M$

3. Give examples of relations having each one of the properties of reflexivity, transitivity and symmetry, but not the others.

Sol.:

refl, $\neg$ sym, $\neg$ trans

~~$x r x \iff x = x$~~

$A = \{1,2,3\}$

$R = \{(1,2),(2,3)\} \cup \Delta_A$

$\neg$ refl., sym., $\neg$ trans

$A = \mathbb{Z}$

$x r y \iff xy > 0$

$\to$ sym ✓

$\to \neg$ refl: $0 \not> 0$

$\to \neg$ trans: ✗ doesn't work

$A = \{1,2,3\}$

$R = \{(1,1),(3,2),$
$(1,2),(2,3),(2,1),(3,2)\}$

$A = \mathbb{Z}$

$x r y \iff xy < 0$

$\to$ sym ✓

$\to \neg$ refl. ✓

$\to \neg$ trans.

if $xy < 0$, $yz < 0$

$\Rightarrow xy^2z < 0 \Rightarrow xz > 0$ ✓

$A = \{1,2,3,4,5\}$

$x r y \iff x + y = 5$ ✓

$\neg$ refl., $\neg$ sym., trans

$A = \mathbb{N}$

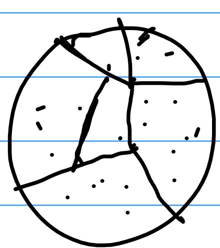
$x r y \iff x < y$

$1 \not< 1$

$x < y,\ y < z \Rightarrow x < z$

$1 < 2$, but $2 \not< 1$

Def.: A set. $\mathcal{P} \subseteq \mathcal{P}(A)$ is a partition of A if:



$\to \forall A_i, A_j \in \mathcal{P}: \quad A_i \cap A_j = \emptyset$

$\to \bigcup_{A_i \subseteq A} A_i = A$

Thm:

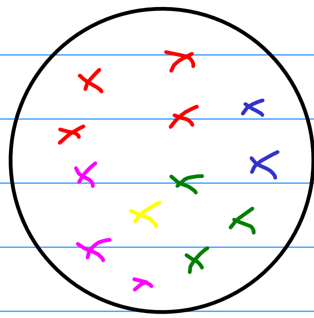
$$\left\{\begin{array}{c}\text{equivalence relations}\\ \text{on } A\end{array}\right\} \xleftrightarrow{\sim} \left\{\begin{array}{c}\text{partitions}\\ \text{of } A\end{array}\right\}$$

$$r \longmapsto A/r \quad \left(\begin{array}{c}\text{quotient set of } A\\ \text{w.r. to } r\end{array}\right)$$

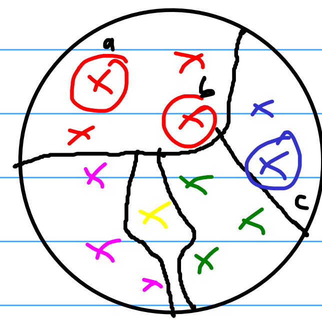
$$r_{\mathcal{P}} \longleftarrow\!\!| \ \mathcal{P}$$

$$A/r = \{ \overbrace{\underbrace{r\langle x \rangle}_{\{y \in A \,|\, xry\}}}^{\hat{x}} \mid x \in A \}$$

r = "they have the same colour"



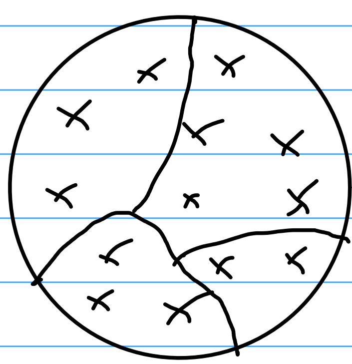
A



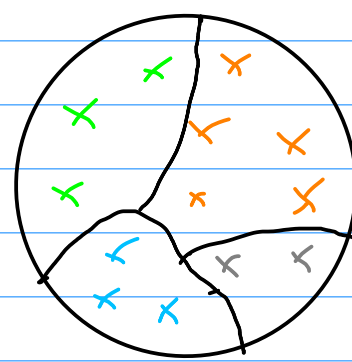
$A/r = \text{red} \cup \text{blue} \cup \text{green} \cup \text{yellow magenta}$

red: "a and all the elements having the same colour as him"

blue: "c and all the elements having the same colour as him"



$\mathcal{P}$



to get $r_{\mathcal{P}}$ we color the elements according to the set they come from

$\mathcal{P}$ partition $\Rightarrow$ $x \, r_{\mathcal{P}} \, y \iff \exists B \in \mathcal{P} : x, y \in B$

5. Let $M = \{1,2,3,4\}$, let r_1, r_2 be homogeneous relations on M and let π_1, π_2, where $R_1 = \Delta_M \cup \{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$, $R_2 = \Delta_M \cup \{(1,2),(1,3)\}$, $\pi_1 = \{\{1\},\{2\},\{3,4\}\}$, $\pi_2 = \{\{1\},\{1,2\},\{3,4\}\}$.
(*i*) Are r_1, r_2 equivalences on M? If yes, write the corresponding partition.
(*ii*) Are π_1, π_2 partitions on M? If yes, write the corresponding equivalence relation.

(i) r_1 is an equivalence relation on M

$M/r_1 = \{ \{1,2,3\}, \{4\} \}$

$(1,2) \in R_2$, but $(2,1) \notin R_2 \Rightarrow$ r_2 is not symmetrical $\Rightarrow r_2$ is not an equivalence

(ii) π_1 is a partition

$\Rightarrow$ $R_{\pi_1} = \Delta_M \cup \{ (3,4), (4,3) \}$

π_2 is not a partition, because $\{1\} \cap \{1,2\} \neq \emptyset$

8. Determine all equivalence relations and all partitions on the set $M = \{1,2,3\}$.

Sol.: Partitions:

$\pi_1 = \{ \{1,2,3\} \} \longleftrightarrow R_{\pi_1} = \{ (1,1),(2,2),(3,3), (1,2),(1,3),(2,3), (2,1),(3,1),(3,2) \}$

$\pi_2 = \{ \{1,2\},\{3\} \}$, $\pi_3 = \{ \{1,3\}, \{2\} \}$, $\pi_4 = \{ \{2,3\}, \{1\} \}$

$\pi_5 = \{ \{1\}, \{2\}, \{3\} \}$

$R_{\pi_2} = \{ (1,1), (2,2), (3,3), (1,2), (2,1) \}$

$R_{\pi_5} = \{ (1,1), (2,2), (3,3) \}$