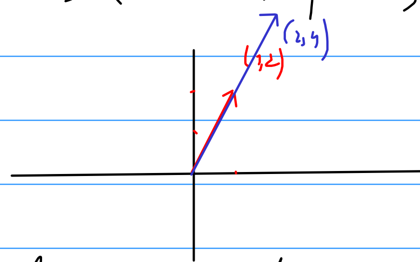
25.10.2021

Seminar W4 – 915

Def: $(V,+)$ abelian group, $(K,+,\cdot)$ field, $\cdot : K\times V \to V$, $(k,v)\mapsto kv$



V is a <u>K-vector space</u> if the following statements are true:

- $\forall \alpha\in K, \forall v_1,v_2\in V: \ \alpha\cdot(v_1+v_2) = \alpha\cdot v_1 + \alpha\cdot v_2$
- $\forall \alpha,\beta\in K, \forall v\in V: \ (\alpha+\beta)\cdot v = \alpha v + \beta v$
- $\forall \alpha,\beta \in K, \ \forall v\in V: \ (\alpha\beta)\cdot v = \alpha\cdot(\beta v)$
- $\forall v\in V: \ 1\cdot v = v$

Ex.: ${}_K K^n$, ${}_{\mathbb{R}}\mathbb{R}^n$, ${}_{\mathbb{R}}\mathbb{R}^2$, ${}_K K^A = \{f : A\to K\}, A\neq\emptyset$, ${}_K \mathcal{M}_{mn}(K)$,

${}_K K[X]$, ${}_{\mathbb{R}}\mathcal{C}([a,b])$, ${}_{\mathbb{R}}\mathcal{C}^n([a,b])$

Def/Th.: V K-vector space, $S\subseteq V$

$$S \leq_K V \iff \left.\begin{array}{l}(i)\ S\neq\emptyset \\ (ii)\ \forall v_1,v_2\in S: v_1+v_2\in S \\ (iii)\ \forall k\in K, \forall v\in S: kv\in S\end{array}\right\} \to \begin{array}{l}\forall k_1,k_2\in K \\ \forall v_1,v_2\in S: \\ k_1v_1+k_2v_2\in S\end{array}$$

("S is a K-subspace of V")

3. Let K be a field, $A \neq \emptyset$ and denote $K^A = \{f \mid f : A \to K\}$. Show that K^A is a K-vector space, where the addition and the scalar multiplication are defined as follows: $\forall f, g \in K^A$, $\forall k \in K$, $f + g \in K^A$, $kf \in K^A$,

$$(f+g)(x) = f(x) + g(x)\,, \quad (k \cdot f)(x) = k \cdot f(x)\,, \forall x \in A\,.$$

Sol: $(K^A, +)$ abelian group

$+$ operation: $\forall f, g \in K^A$: $f+g \in K^A$

$+$ commutative: $\forall f, g \in K^A$: $f+g \overset{?}{=} g+f$

Let $x \in A$: $(f+g)(x) = (g+f)(x)$

$$(f+g)(x) = f(x)+g(x) \overset{+\text{ is commutative in } K}{=} g(x)+f(x) = (g+f)(x)$$

$+$ associative: $\forall f, g, h \in K^A$: $f+(g+h) = (f+g)+h$

Let $x \in A$: $\big(f+(g+h)\big)(x) = f(x) + (g+h)(x) = f(x) + \big(g(x)+h(x)\big) =$

$$\overset{+\text{ is associative in } K}{=} \big(f(x)+g(x)\big)+h(x) = (f+g)(x)+h(x) = \big((f+g)+h\big)(x)$$

$$\Rightarrow f+(g+h) = (f+g)+h$$

$+$ has an identity: $\exists\, e \in K^A$: $f+e = e+f = f$

Take $e: A \to K$

$x \longmapsto 0_K$

$$\forall x \in A: \ (f+e)(x) = f(x)+e(x) = f(x) + 0 \overset{0 \text{ is a neutral element for } + \text{ in } K}{=} f(x)$$

$$\Rightarrow f+e = e+f = f$$

$+$ has invertibility: $\forall f \in K^A$, $\exists -f$, defined by:

$$\begin{aligned} -f : A &\longrightarrow K \\ x &\longmapsto -f(x) \end{aligned}$$

$$\forall x \in A: \Rightarrow \big(f+(-f)\big)(x) = f(x) + (-f)(x) = f(x) + \big(-f(x)\big) \overset{\text{invertibility of } + \text{ in } K}{=} 0$$

Let $\alpha \in K$, $f_1, f_2 \in K^A$

$$\alpha \cdot (f_1 + f_2) \overset{?}{=} \alpha f_1 + \alpha f_2$$

Let $x \in A$: $\big(\alpha \cdot (f_1+f_2)\big)(x) = \alpha \cdot (f_1+f_2)(x) = \alpha \cdot \big(f_1(x) + f_2(x)\big) =$

distributivity in K

$= \alpha \cdot f_1(x) + \alpha \cdot f_2(x) = (\alpha f_1 + \alpha f_2)(x)$

$$\Rightarrow \alpha \cdot (f_1 + f_2) = \alpha f_1 + \alpha f_2$$

Let $\alpha, \beta \in K$, $f \in K^A$: $(\alpha+\beta) \cdot f = \alpha f + \beta f$

Let $x \in A$: $\big((\alpha+\beta) \cdot f\big)(x) = (\alpha+\beta) \cdot f(x) \overset{\text{distributivity from } K}{=} \alpha \cdot f(x) + \beta \cdot f(x) =$

$= (\alpha f + \beta f)(x) \Rightarrow (\alpha+\beta) f = \alpha f + \beta f$

Let $\alpha, \beta \in K$, $f \in K^A$: $(\alpha\beta) \cdot f = \alpha \cdot (\beta f)$

Let $x \in A$: $\big((\alpha\beta) \cdot f\big)(x) = (\alpha\beta) \cdot f(x) \overset{\text{associativity of } \cdot \text{ in } K}{=} \alpha \cdot \big(\beta f(x)\big) =$

$= \alpha \cdot \big((\beta f)(x)\big) = \big(\alpha \cdot (\beta f)\big)(x)$

Let $f \in K^A$: $1 \cdot f = f$

Let $x \in A$: $(1 \cdot f)(x) = 1 \cdot f(x) \underset{\text{1 identity in } K}{=} f(x)$

Remark : $\cdot$ $A = \{1, 2, 3, \dots, n\}$

$K^A = \{ f : \{1, 2, \dots, n\} \to K \} \approx K^n = \{ (x_1, \dots, x_n) \mid x_i \in K \}$

$\cdot$ $A = \mathbb{N}$

$K^{\mathbb{N}} = \{ f : \mathbb{N} \to K \} \approx \{ (x_n)_{n \in \mathbb{N}} \mid x_n \in K, \forall n \}$

7. Which ones of the following sets are subspaces of the real vector space $\mathbb{R}^3$:
(*i*) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$;
(*ii*) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0 \text{ or } z = 0\}$;
(*iii*) $C = \{(x, y, z) \in \mathbb{R}^3 \mid x \in \mathbb{Z}\}$;
(*iv*) $D = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$;
(*v*) $E = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1\}$;
(*vi*) $F = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$?

Sol: (i) $(0, 1, 1) \in A \Rightarrow A \neq \emptyset$

Let $u_1, u_2 \in A$: $u_1 + u_2 \overset{?}{\in} A$

$u_1 = (\underbrace{x_1}_{=0}, y_1, z_1)$, $u_2 = (\underbrace{x_2}_{=0}, y_2, z_2)$

$u_1 + u_2 = (\underbrace{x_1 + x_2}_{=0}, y_1 + y_2, z_1 + z_2) \in A$

Let $k \in \mathbb{R}$, $u = (x, y, z) \in A$: $ku \overset{?}{\in} A$

$u \in A \Rightarrow x = 0$, $ku = k(x, y, z) = (\underbrace{kx}_{=0}, ky, kz) \in A$

$\Rightarrow A \leq_{\mathbb{R}} \mathbb{R}^3$

(ii) $B = \{ (x,y,z) \in \mathbb{R}^3 \mid x = 0 \text{ or } z = 0 \}$

$(0,1,1) \in B$, $(1,0,0) \in B$

$(0,1,1) + (1,0,0) = (1,1,1) \notin B \Rightarrow B \not\leq_{\mathbb{R}} \mathbb{R}^3$

(iii) $C = \{ (x,y,z) \in \mathbb{R}^3 \mid x \in \mathbb{Z} \}$

$(-1,1,1) \in C \Rightarrow C \neq \emptyset$

Let $v_1, v_2 \in C \overset{?}{\Rightarrow} v_1 + v_2 \in C$

$v_1 = (x_1, y_1, z_1)$, $v_2 = (x_2, y_2, z_2)$

$$\left.\begin{array}{l} v_1 + v_2 = (x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2) \\ x_1, x_2 \in \mathbb{Z} \Rightarrow x_1 + x_2 \in \mathbb{Z} \end{array}\right\} \Rightarrow v_1 + v_2 \in C$$

Let $k \in \mathbb{R}$, $v \in C \overset{?}{\Rightarrow} kv \in C$

$\sqrt{3} \in \mathbb{R}$, $(-1,2,3) \in C$, but $\sqrt{3} \cdot (-1,2,3) = (-\sqrt{3}, 2\sqrt{3}, 3\sqrt{3}) \notin C$

$\Rightarrow C \not\leq_{\mathbb{R}} \mathbb{R}^3$

(iv) $D = \{ (x,y,z) \in \mathbb{R}^3 \mid x + y + z = 0 \}$

$(-1,0,1) \in D \Rightarrow D \neq \emptyset$

Let $v_1, v_2 \in D$, $\alpha_1, \alpha_2 \in \mathbb{R}$ $\overset{?}{\Rightarrow}$ $\alpha_1 v_1 + \alpha_2 v_2 \in D$

$v_1 = (x_1, y_1, z_1)$ $v_2 = (x_2, y_2, z_2) \in D \Rightarrow x_1 + y_1 + z_1 = x_2 + y_2 + z_2 = 0$

$\alpha_1 v_1 + \alpha_2 v_2 = \alpha_1 (x_1, y_1, z_1) + \alpha_2 \cdot (x_2, y_2, z_2) =$

$= (\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 y_1 + \alpha_2 y_2, \alpha_1 z_1 + \alpha_2 z_2)$

$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_1 y_1 + \alpha_2 y_2 + \alpha_1 z_1 + \alpha_2 z_2 =$

$= \alpha_1 \cdot \underbrace{(x_1 + y_1 + z_1)}_{=0} + \alpha_2 \underbrace{(x_2 + y_2 + z_2)}_{=0} = 0$

$\Rightarrow D \leq_{\mathbb{R}} \mathbb{R}^3$

(v), (vi) homework

9. Which ones of the following sets are subspaces of the K-vector space $K[X]$:
(*i*) $K_n[X] = \{f \in K[X] \mid \text{degree}(f) \leq n\}$ $(n \in \mathbb{N})$;
(*ii*) $K'_n[X] = \{f \in K[X] \mid \text{degree}(f) = n\}$ $(n \in \mathbb{N})$.

Sol: $f = a_n X^n + \dots + a_1 X + a_0$, $a_n \neq 0$, $\deg f = n$

$\deg(0) = -\infty$

$\alpha \in K \setminus \{0\}$ $\deg(\alpha) = 0$

(i) $K_n[X] \neq \emptyset$, because $X \in K_n[X]$

Let $f, g \in K[X]$, $\alpha, \beta \in K$

$$\deg(fg) = \deg f + \deg g$$

$$\deg(\alpha f) = \begin{cases} \deg f, & \alpha \neq 0 \\ -\infty, & \alpha = 0 \end{cases}$$

$$\deg(f+g) \leq \max(\deg f, \deg g)$$

$$\deg(\alpha f + \beta g) \leq \max(\underbrace{\deg(\alpha f)}_{\leq \deg f}, \underbrace{\deg(\beta g)}_{\leq \deg g}) \leq \max(\underbrace{\deg f}_{\leq n}, \underbrace{\deg g}_{\leq n}) \leq n$$

$$\Rightarrow \alpha f + \beta g \in K_n[x]$$

(ii) $x^n, -x^n \in K_n'[x]$, but $x^n + (-x^n) = 0 \notin K_n'[x]$