01.11.2021

Seminar W5 - 915

Def. : V K-vector space $X \subseteq V$

The subspace of V generated by X is:

$$\langle X \rangle = \bigcap_{\substack{S \leq_K V \\ S \supseteq X}} S = \left\{ \sum_{i=1}^{n} k_i \cdot u_i \;\middle|\; n \in \mathbb{N},\ k_i \in K,\ u_i \in X \right\}$$

2. Consider the following subspaces of the real vector space $\mathbb{R}^3$:
(i) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$;
(ii) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$;
(iii) $C = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$.
Write A, B, C as generated subspaces with a minimal number of generators.

(iv) $D = \left\{(x,y,z,t) \in \mathbb{R}^4 \;\middle|\; \begin{cases} x + 2y = 0 \\ t = 0 \end{cases} \right\}$

(v) $E = \{(x,y,z) \in \mathbb{R}^3 \mid 4y + 5z = 0\}$

Sol. : $\underline{\text{Ex.}}$: $S = \left\{ (x,y,z,t,u) \in \mathbb{R}^5 \;\middle|\; \begin{cases} x + y = 0 \\ y + 2t + z = 0 \\ u + x = 0 \end{cases} \right\} =$

$= \left\{ (x,y,z,t,u) \in \mathbb{R}^5 \;\middle|\; \begin{cases} y = -x \\ u = -x \\ -x + 2t + z = 0 \end{cases} \right\} =$

$= \left\{ (x,y,z,t,u) \in \mathbb{R}^5 \;\middle|\; \begin{cases} y = -x \\ u = -x \\ z = x - 2t \end{cases} \right\} =$

$= \{ (x, -x, x - 2t, t, -x) \mid x, t \in \mathbb{R} \} =$

$= \{ (x, -x, x, 0, -x) + (0, 0, -2t, t, 0) \mid x, t \in \mathbb{R} \} =$

$= \{ x \cdot (1, -1, 1, 0, -1) + t \cdot (0, 0, -2, 1, 0) \mid x, t \in \mathbb{R} \} =$

$=$

$= \langle (1,-1,1,0,-1) \; , \; (0,0,-2,1,0) \rangle$

The number of generators is minimal, because $(1,-1,1,0,-1) \notin \langle(0,0,-2,1,0)\rangle$

(h) $E = \{ (x,y,z) \in \mathbb{R}^3 \mid 4y+5z = 0 \} =$

$= \{(x,y,z) \in \mathbb{R}^3 \mid z = -\frac{4}{5} y\} = \{ (x, y, -\frac{4}{5}y) \mid x,y \in \mathbb{R} \} =$

$= \{ (x,0,0) + (0,y,-\frac{4}{5}y) \mid x,y \in \mathbb{R} \} = \{ x\cdot(1,0,0) + y\cdot(0,1,-\frac{4}{5}) \mid x,y \in \mathbb{R}\}$

$= \langle (1,0,0) \; , \; (0,1,-\frac{4}{5}) \rangle = \langle (1,0,0) \; , \; (0,5,-4) \rangle$

(i) $B = \{ (x,y,z) \in \mathbb{R}^3 \mid x = -y-z \} =$

$= \{ (-y-z, y, z) \mid y,z \in \mathbb{R} \} =$

$= \{ (-y,y,0) + (-z,0,z) \mid y,z \in \mathbb{R} \} =$

$= \{ y\cdot(-1,1,0) + z\cdot(-1,0,1) \mid y,z \in \mathbb{R} \} =$

$= \langle (-1,1,0) \; , \; (-1,0,1) \rangle$

$(-1,1,0) \notin \langle (-1,0,1) \rangle$

If $(-1,1,0) \in \langle (-1,0,1) \rangle \Rightarrow \exists \alpha \in \mathbb{R} : (-1,1,0) = \alpha\cdot(-1,0,1)$

$\Rightarrow \begin{cases} -1 = -\alpha \\ 1 = 0 \\ 0 = 1 \end{cases}$ absurd $\Rightarrow (-1,1,0) \notin \langle(-1,0,1)\rangle$

(iii) $C = \{ (x,y,z) \in \mathbb{R}^3 \mid x = y = z \} = \{ (x,x,x) \mid x \in \mathbb{R} \} =$

$= \{ x \cdot (1,1,1) \mid x \in \mathbb{R} \} = \langle (1,1,1) \rangle$

(i) $A = \{ (x,y,z) \in \mathbb{R}^3 \mid x = 0 \} =$

$= \{ (0,y,z) \mid y,z \in \mathbb{R} \} =$

$= \{ (0,y,0) + (0,0,z) \mid y,z \in \mathbb{R} \} =$

$= \{ y (0,1,0) + z \cdot (0,0,1) \mid y, z \in \mathbb{R} \} =$

$= \langle (0,1,0), (0,0,1) \rangle$

(ii) $D = \{ (x,y,z,t) \in \mathbb{R}^4 \mid \begin{cases} x+2y = 0 \\ t = 0 \end{cases} \} =$

$= \{ (x,y,z,t) \in \mathbb{R}^4 \mid \begin{cases} x = -2y \\ t = 0 \end{cases} \} =$

$= \{ (-2y, y, z, 0) \mid y, z \in \mathbb{R} \} =$

$= \{ (-2y, y, 0, 0) + (0,0,z,0) \mid y,z \in \mathbb{R} \} =$

$= \{ y \cdot (-2,1,0,0) + z \cdot (0,0,1,0) \mid y,z \in \mathbb{R} \} =$

$= \langle (-2,1,0,0), (0,0,1,0) \rangle =$

$= \langle (-2,1,0,0), (0,0,1,0), (-2,1,1,0) \rangle$

Def: V, W k-vector spaces, $f: V \to W$

(homo)morphism of vector spaces (or a linear map) if:

$\left\{ \begin{array}{l} \cdot \ \forall v_1, v_2 \in V: f(v_1+v_2) = f(v_1)+f(v_2) \\ \cdot \ \forall k \in k, \forall v \in V: f(kv) = k \cdot f(v) \end{array} \right.$

$\to \forall k_1, k_2 \in k, \forall v_1, v_2 \in V: f(k_1v_1+k_2v_2) = k_1 f(v_1) + k_2 f(v_2)$

6. Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ and $h : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$f(x,y) = (x+y, x-y),$$
$$g(x,y) = (2x-y, 4x-2y),$$
$$h(x,y,z) = (x-y, y-z, z-x).$$

Show that $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ and $h \in End_{\mathbb{R}}(\mathbb{R}^3)$.

$Fix(f) = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = (x,y)\} =$

$= \left\{(x,y) \in \mathbb{R}^2 \mid \left\{ \begin{array}{l} x+y = x \\ x-y = y \end{array} \right. \right\} =$

$= \{(0,0)\}$

Sol.: Let $k_1, k_2 \in \mathbb{R}$, $v_1, v_2 \in \mathbb{R}^2$, $v_1 = (x_1, y_1)$, $v_2 = (x_2, y_2)$

$f(k_1v_1 + k_2v_2) = f(k_1(x_1,y_1) + k_2 \cdot (x_2,y_2)) = f((k_1x_1, k_1y_1) + (k_2x_2, k_2y_2))$

$= f(k_1x_1 + k_2x_2, k_1y_1 + k_2y_2) = (k_1x_1 + k_2x_2 + k_1y_1 + k_2y_2,$

$k_1x_1 + k_2x_2 - k_1y_1 - k_2y_2)$

$k_1 f(v_1) + k_2 f(v_2) = k_1 \cdot (x_1+y_1, x_1-y_1) + k_2 \cdot (x_2+y_2, x_2-y_2) =$

$= (k_1x_1 + k_1y_1, k_1x_1 - k_1y_1) + (k_2x_2 + k_2y_2, k_2x_2 - k_2y_2) =$

$= (k_1x_1 + k_1y_1 + k_2x_2 + k_2y_2, k_1x_1 - k_1y_1 + k_2x_2 - k_2y_2)$

$$\Rightarrow \quad f(k_1 e_1 + k_2 e_2) = k_1 \, f(e_1) + k_2 f(e_2)$$

$$\Rightarrow \quad f \in End_{\mathbb{R}}(\mathbb{R}^2)$$

$$g: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x,y) \longmapsto (2x-y, 4x-2y)$$

$k_1, k_2 \in \mathbb{R}$, $e_1, e_2 \in \mathbb{R}^2$, $e_1 = (x_1, y_1)$, $e_2 = (x_2, y_2)$

$$g(k_1 e_1 + k_2 e_2) = g\left((k_1 x_1, k_1 y_1) + (k_2 x_2, k_2 y_2)\right) =$$

$$= g(k_1 x_1 + k_2 x_2, k_1 y_1 + k_2 y_2) =$$

$$= \left(2(k_1 x_1 + k_2 x_2) - (k_1 y_1 + k_2 y_2), 4\cdot(k_1 x_1 + k_2 x_2) - 2(k_1 y_1 + k_2 y_1)\right)$$

$$= (2k_1 x_1 - k_1 y_1, 4k_1 x_1 - 2k_1 y_1) +$$

$$+ (2k_2 x_2 - k_2 y_2, 4k_2 x_2 - 2k_2 y_1) =$$

$$= k_1 \cdot (2x_1 - y_1, 4x_1 - 2y_1) + k_2 \cdot (2x_2 - y_2, 4x_2 - 2y_1) =$$

$$= k_1 \cdot g(x_1, y_1) + k_2 \cdot g(x_2, y_2)$$

Def: V k-vector space, $S, T \leq_k V$

$V = S + T \iff \forall v \in V\ \exists s \in S, \exists t \in T : v = s + t$

$V = S \oplus T \iff \forall v \in V\ \exists! s \in S, \exists! t \in T : v = s + t \iff$

$\iff$ $\underline{V = S + T \text{ and } S \cap T = 0 = \{0\}}$

5. Let S and T be the set of all even functions and of all odd functions in $\mathbb{R}^{\mathbb{R}}$ respectively. Prove that S and T are subspaces of the real vector space $\mathbb{R}^{\mathbb{R}}$ and $\mathbb{R}^{\mathbb{R}} = S \oplus T$.

(homework)

Sol.: $f : A \to B$ is an odd function if $f(-x) = -f(x)$

even $f(-x) = f(x)$

We will prove that $S \cap T = 0$

Let $f \in S \cap T \Rightarrow f$ is both odd and even

So $\forall x \in \mathbb{R}$: $\left.\begin{aligned} f(-x) &\overset{f \text{ odd}}{=} -f(x) \\ f(-x) &\underset{f \text{ even}}{=} f(x) \end{aligned}\right\} \Rightarrow \forall x \in \mathbb{R} : f(x) = -f(x)$, so $f(x) = 0$

$\Rightarrow f = 0_{\mathbb{R}^{\mathbb{R}}} \Rightarrow S \cap T \subseteq 0 \Rightarrow S \cap T = 0$

It remains to show that $\forall f : \mathbb{R} \to \mathbb{R}$ there exists

an f_S and an f_T so that $f = f_S + f_T$, $f_S \in S$, $f_T \in T$

If we assume that $f = f_S + f_T$, let's see what we can deduce.

Let $x \in \mathbb{R}$: $\Rightarrow$ $f(-x) = \underbrace{f_S(-x)}_{=f_S(x)} + \underbrace{f_T(-x)}_{-f_T(x)}$

$$\Rightarrow \quad f(-x) = f_S(x) - f_T(x)$$

$$f(x) = f_S(x) + f_T(x)$$

$$\Rightarrow 2f_S(x) = f(x) + f(-x)$$

$$\Rightarrow f_S(x) = \frac{f(x)+f(-x)}{2}$$

Let $f \in \mathbb{R}^{\mathbb{R}}$, we define $f_S \in \mathbb{R}^{\mathbb{R}}$ by:

$$f_S(x) = \frac{f(x)+f(-x)}{2}$$

We define $f_T(x) = f(x) - f_S(x) = \frac{f(x)-f(-x)}{2}$, where $f(x) - \frac{f(x)+f(-x)}{2} = \frac{f(x)-f(-x)}{2}$

Now for any $x \in \mathbb{R}$:

$$f_S(-x) = \frac{f(-x)+f(x)}{2} = \frac{f(x)+f(-x)}{2} = f_S(x)$$

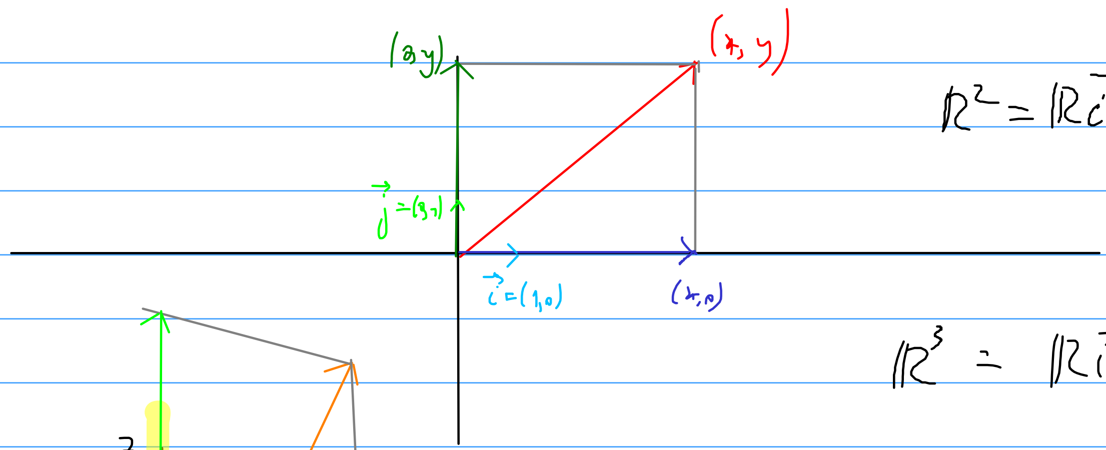
$\Rightarrow f_S \in S$

$f_T(-x) = \dfrac{f(-x) - f(x)}{2} = -\dfrac{f(x) - f(-x)}{2} = -f_T(x)$

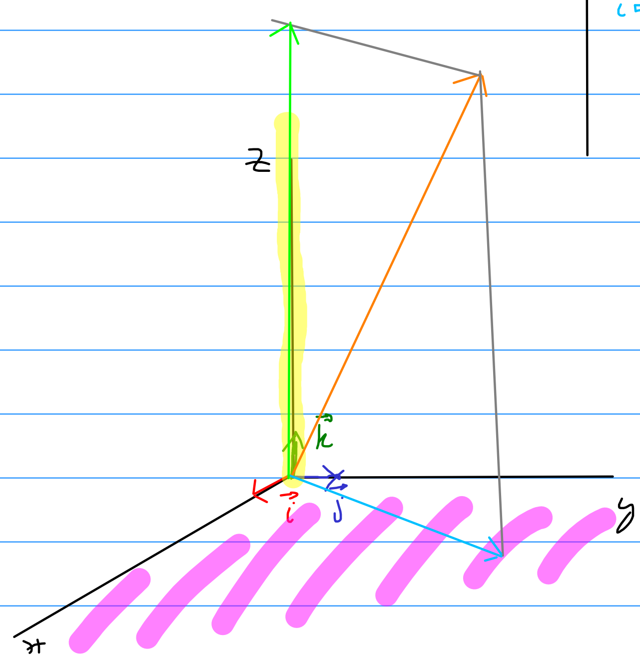
$\Rightarrow f_T \in T$

So for any $f \in \mathbb{R}^{\mathbb{R}}$ $\exists f_S \in S$, $\exists f_T \in T$ so that $f = f_S + f_T \Rightarrow \mathbb{R}^{\mathbb{R}} = S + T$

Given that $S \cap T = 0 \Rightarrow \mathbb{R}^{\mathbb{R}} = S \oplus T$



$\mathbb{R}^2 = \mathbb{R}\vec{i} \oplus \mathbb{R}\vec{j}$



$\mathbb{R}^3 = \mathbb{R}\vec{i} \oplus \mathbb{R}\vec{j} \oplus \mathbb{R}\vec{k}$

$= \underbrace{\langle i, j \rangle}_{\substack{\parallel \\ \mathbb{R}\vec{i} \oplus \mathbb{R}\vec{j}}} \oplus \underbrace{\langle k \rangle}_{\substack{\parallel \\ \mathbb{R}\vec{k}}}$

10. Let V be a vector space over K and $f \in End_K(V)$. Show that the set

$$S = \{x \in V \mid f(x) = x\}$$

of fixed points of f is a subspace of V.

Sol: We have to show that $\forall\, v_1, v_2 \in S$, $\forall\, k_1, k_2 \in K$

we have $k_1 v_1 + k_2 v_2 \in S$

$v_1, v_2 \in S \Rightarrow f(v_1) = v_1,\quad f(v_2) = v_2$

$f \in End_K(V)$

$$f(k_1 v_1 + k_2 v_2) = k_1 \underbrace{f(v_1)}_{= v_1} + k_2 \underbrace{f(v_2)}_{= v_2} = k_1 v_1 + k_2 v_2$$

$\Rightarrow k_1 v_1 + k_2 v_2 \in S$

$\Rightarrow S \leq_K V$