08.11.2021

Seminar W6 - 914

Def.: V K-vector space, $v_1, \dots, v_n \in V$ are linearly independent if:

if $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$, then $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$

($v_1, \dots, v_n$ are linearly dependent if $\exists\, \alpha_1, \dots, \alpha_n \in \mathbb{R}$, not all zero:

$$\underbrace{\alpha_1 v_1 + \dots + \alpha_n v_n = 0}_{\text{depending relation between the vectors } v_1, \dots, v_n}$$

2. Prove that the following vectors are linearly independent:
(i) $v_1 = (1, 0, 2)$, $v_2 = (-1, 2, 1)$, $v_3 = (3, 1, 1)$ in ${}_{\mathbb{R}}\mathbb{R}^3$.
(ii) $v_1 = (1, 2, 3, 4)$, $v_2 = (2, 3, 4, 1)$, $v_3 = (3, 4, 1, 2)$, $v_4 = (4, 1, 2, 3)$ in ${}_{\mathbb{R}}\mathbb{R}^4$.

Sol.: (i) Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ so that $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

$$\alpha_1 \cdot (1,0,2) + \alpha_2 \cdot (-1, 2, 1) + \alpha_3 \cdot (3,1,1) = (0,0,0)$$

$$\Rightarrow \begin{cases} \alpha_1 - \alpha_2 + 3\alpha_3 = 0 \\ 2\alpha_2 + \alpha_3 = 0 \\ 2\alpha_1 + \alpha_2 + \alpha_3 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = \alpha_2 - 3\alpha_3 \\ 2\alpha_2 + \alpha_3 = 0 \\ 2\alpha_1 + \alpha_2 + \alpha_3 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = \alpha_2 - 3\alpha_3 \\ 2\alpha_2 + \alpha_3 = 0 \\ 2\alpha_2 - 6\alpha_3 + \alpha_2 + \alpha_3 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_1 = \alpha_2 - 3\alpha_3 \\ \alpha_3 = -2\alpha_2 \\ 3\alpha_2 - 5\alpha_3 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = 7\alpha_2 \\ \alpha_3 = -2\alpha_2 \\ 3\alpha_2 + 10\alpha_2 = 0 \end{cases} \Leftrightarrow \begin{cases} 13\alpha_2 = 0 \\ \alpha_1 = 7\alpha_2 \\ \alpha_3 = -2\alpha_2 \end{cases} \Rightarrow \alpha_2 = \alpha_1 = \alpha_3 = 0$$

(ii) Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$: $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = 0$

$$\alpha_1\cdot(1,2,3,4)+\alpha_2\cdot(2,3,4,1)+\alpha_3\cdot(3,4,1,2)+\alpha_4\cdot(4,1,2,3)=0$$

$$\Rightarrow \begin{cases}\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4=0\\ 2\alpha_1+3\alpha_2+4\alpha_3+\alpha_4=0\\ 3\alpha_1+4\alpha_2+\alpha_3+2\alpha_4=0\\ 4\alpha_1+\alpha_2+2\alpha_3+3\alpha_4=0\end{cases} \Leftrightarrow \begin{cases}\alpha_1=-2\alpha_2-3\alpha_3-4\alpha_4\\ 2\alpha_1+3\alpha_2+4\alpha_3+\alpha_4=0\\ 3\alpha_1+4\alpha_2+\alpha_3+2\alpha_4=0\\ 4\alpha_1+\alpha_2+2\alpha_3+3\alpha_4=0\end{cases}$$

$$\Leftrightarrow \begin{cases}\alpha_1=-2\alpha_2-3\alpha_3-4\alpha_4\\ -4\alpha_2-6\alpha_3-8\alpha_4+3\alpha_2+4\alpha_3+\alpha_4=0\\ -6\alpha_2-9\alpha_3-12\alpha_4+4\alpha_2+\alpha_3+2\alpha_4=0\\ -8\alpha_2-12\alpha_3-16\alpha_4+\alpha_2+2\alpha_3+3\alpha_4=0\end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases}\alpha_1=-2\alpha_2-3\alpha_3-4\alpha_4\\ -\alpha_2-2\alpha_3-7\alpha_4=0\\ -2\alpha_2-8\alpha_3-10\alpha_4=0\\ -7\alpha_2-10\alpha_3-13\alpha_4=0\end{cases} \Leftrightarrow \begin{cases}\alpha_1=-2\alpha_2-3\alpha_3-4\alpha_4\\ \alpha_2=-2\alpha_3-7\alpha_4\\ -2\alpha_2-8\alpha_3-10\alpha_4=0\\ -7\alpha_2-10\alpha_3-13\alpha_4=0\end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases}\alpha_1=-2\alpha_2-3\alpha_3-4\alpha_4\\ \alpha_2=-2\alpha_3-7\alpha_4\\ 4\alpha_3+14\alpha_4-8\alpha_3-10\alpha_4=0\\ 14\alpha_3+49\alpha_4-10\alpha_3-13\alpha_4=0\end{cases} \Leftrightarrow \begin{cases}\alpha_1=-2\alpha_2-3\alpha_3-4\alpha_4\\ \alpha_2=-2\alpha_3-7\alpha_4\\ -4\alpha_3+4\alpha_4=0\\ 4\alpha_3+36\alpha_4=0\end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases}\alpha_1=-2\alpha_2-3\alpha_3-4\alpha_4\\ \alpha_2=-2\alpha_3-7\alpha_4\\ \alpha_3=\alpha_4\\ \alpha_3+9\alpha_4=0\end{cases} \Leftrightarrow \begin{cases}\alpha_1=-2\alpha_2-3\alpha_3-4\alpha_4\\ \alpha_2=-2\alpha_3-7\alpha_4\\ \alpha_3=\alpha_4\\ \alpha_4=0\end{cases} \Rightarrow$$

$\Rightarrow \alpha_4=\alpha_3=\alpha_2=\alpha_1=0 \Rightarrow v_1, v_2, v_3, v_4$ linearly independent

Def: V k-vector space, $u_1, \dots, u_n \in V$

$(u_1, u_2, \dots, u_n)$ basis for$_k$ V $\Leftrightarrow$ $\begin{cases} \cdot\ u_1, \dots, u_n \text{ linearly independent } (\Leftrightarrow \\ \cdot\ V = \langle u_1, \dots, u_n \rangle \end{cases}$

$\Leftrightarrow \forall u \in V\ \exists!\ \underbrace{\alpha_1, \dots, \alpha_n}_{} \in \mathbb{R} : \quad u = \alpha_1 u_1 + \dots + \alpha_n u_n$

$\hookrightarrow$ the coordinates of u in the basis $(u_1, \dots, u_n)$ $\quad [u]_{(u_1,\dots,u_n)} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$

$\dim_k V$ = # of elements in any one of its bases

8. Let $\mathbb{R}_2[X] = \{f \in \mathbb{R}[X] \mid degree(f) \leq 2\}$. Show that the lists $E = (1, X, X^2)$, $B = (1, X-a, (X-a)^2)$ $(a \in \mathbb{R})$ are bases of the real vector space $\mathbb{R}_2[X]$ and determine the coordinates of a polynomial $\cancel{f = a_0 + a_1 X + a_2 X^2} \in \mathbb{R}_2[X]$ in each basis.

$f = 5 + 7X^2$

Sol: Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$: $\quad \alpha_1 \cdot 1 + \alpha_2 \cdot X + \alpha_3 \cdot X^2 = 0 \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$

$\Rightarrow 1, X, X^2$ linearly independent.

We will show that $1, X, X^2$ generate $\mathbb{R}_2[X]$.

$\left(V = \langle u_1, \dots, u_n \rangle \Leftrightarrow \forall u \in V\ \exists\ \alpha_1, \alpha_2, \dots, \alpha_n \in K : u = \alpha_1 u_1 + \dots + \alpha_n u_n \right)$

$V = \mathbb{R}_2[X]$. Let $f \in \mathbb{R}_2[X] \Rightarrow f = a_0 \cdot 1 + a_1 \cdot X + a_2 \cdot X^2 \Rightarrow$

$\Rightarrow \mathbb{R}_2[X] = \langle 1, X, X^2 \rangle$

$\Rightarrow (1, X, X^2)$ basis for $\mathbb{R}_2[X]$ $\quad (\Rightarrow \dim_{\mathbb{R}} \mathbb{R}_2[X] = 3)$

$B = (1,\ X-a,\ (X-a)^2)$

Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$: $\alpha_1 + \alpha_2 \cdot (X-a) + \alpha_3 \cdot (X-a)^2 = 0 \Rightarrow$

$\Rightarrow \alpha_1 + \alpha_2 X - \alpha_2 a + \alpha_3 X^2 - 2a\alpha_3 X + \alpha_3 a^2 = 0 \Rightarrow$

$\Rightarrow (\alpha_1 - \alpha_2 a + \alpha_3 a^2) + (\alpha_2 - 2a\alpha_3) \cdot X + \alpha_3 X^2 = 0 \Rightarrow$

$$\Rightarrow \begin{cases} \alpha_1 - \alpha_2 a + \alpha_3 a^2 = 0 \\ \alpha_2 - 2a\alpha_3 = 0 \\ \alpha_3 = 0 \end{cases} \Rightarrow \begin{cases} \alpha_3 = 0 \\ \alpha_2 = 2a\alpha_3 \\ \alpha_1 - \alpha_2 a + \alpha_3 a^2 = 0 \end{cases} \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$$

$\Rightarrow$ $1,\ X-a,\ (X-a)^2$ linearly independent

Let $f \in \mathbb{R}_2[X]$, $f = a_0 + a_1 X + a_2 X^2$

We want to find $b_0, b_1, b_2 \in \mathbb{R}$: $a_0 + a_1 X + a_2 X^2 = b_0 + b_1 (X-a) + b_2 \cdot (X-a)^2$

$\Rightarrow a_0 + a_1 X + a_2 X^2 = b_0 + b_1 X - b_1 a + b_2 X^2 - 2ab_2 X + a^2 b_2$

$\Rightarrow a_0 + a_1 X + a_2 X^2 = (b_0 - b_1 a + b_2 a^2) + (b_1 - 2ab_2) X + b_2 X^2$

$$\Rightarrow \begin{cases} a_0 = b_0 - b_1 a + b_2 a^2 \\ a_1 = b_1 - 2ab_2 \\ a_2 = b_2 \end{cases} \Leftrightarrow \begin{cases} b_2 = a_2 \\ b_1 = a_1 + 2ab_2 \\ a_0 = b_0 - b_1 a + b_2 a^2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} b_2 = a_2 \\ b_1 = a_1 + 2aa_2 \\ a_0 = b_0 - b_1 a + a_2 a^2 \end{cases} \Leftrightarrow \begin{cases} b_2 = a_2 \\ b_1 = a_1 + 2aa_2 \\ a_0 = b_0 - aa_1 + 2a^2 a_2 + a_2 a^2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} b_2 = a_2 \\ b_1 = a_1 + 2a\, a_2 \\ b_0 = a_0 + a_1 a - a_2 \cdot 3a^2 \end{cases} \Rightarrow 1, (x-a), (x-a)^2 \text{ generating set for } \mathbb{R}_2[x]$$

$$\Rightarrow B = \{1, x-a, (x-a)^2\} \text{ basis for } \mathbb{R}_2[x]$$

We will determine the coordinates of $f = 5 + 7x^2$ in the basis $B = \{1, x-2, (x-2)^2\}$

$$5 + 7x^2 = b_0 + b_1(x-2) + b_2 \cdot (x-2)^2$$

$$\begin{cases} b_2 = a_2 \\ b_1 = a_1 + 2a\, a_2 \\ b_0 = a_0 + a_1 a - a_2 \cdot 3a^2 \end{cases} \quad \begin{cases} b_2 = 7 \\ b_1 = 0 + 2 \cdot 2 \cdot 7 \\ b_0 = 5 + 0 \cdot 2 - 7 \cdot 3 \cdot 2^2 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} b_2 = 7 \\ b_1 = 28 \\ b_0 = -79 \end{cases} \Rightarrow [5 + 7x^2]_{(1, x-2, (x-2)^2)} = \begin{pmatrix} -79 \\ 28 \\ 7 \end{pmatrix}$$