06.12.2021

Seminar Wg – 914

Compute by applying elementary operations the ranks of the matrices:

1. $\begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix}$. **2.** $\begin{pmatrix} 1 & -1 & 3 & 2 \\ -2 & 0 & 3 & -1 \\ -1 & 2 & 0 & -1 \end{pmatrix}$. **3.** $\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix}$ $(\alpha, \beta \in \mathbb{R})$.

Sol.: 1.

$$\begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix} \overset{L_1 \leftrightarrow L_3}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 3 \\ 0 & 2 & 3 \\ 2 & 2 & 4 \end{pmatrix} \underset{L_4 \leftarrow L_4 - 2L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix} \sim$$

$$\overset{L_3 \leftarrow L_3 - L_2}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} \overset{L_4 \leftarrow L_4 - L_3}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

nonzero row

zero row

This is a row echelon form

$\Rightarrow$ rank M = rank (row echelon form) = # of nonzero rows

3. $\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix}$ $(\alpha, \beta \in \mathbb{R})$.

Sol.:

$$\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix} \underset{L_2 \leftrightarrow L_3}{\overset{L_1 \leftrightarrow L_2}{\sim}} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \\ \beta & 1 & 3 & 4 \end{pmatrix} \underset{L_3 \leftarrow L_3 - \beta L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & \alpha & -2 & 1 \\ 0 & 1-\alpha\beta & 3-3\beta & 4-3\beta \end{pmatrix}$$

If $\alpha = 0$:

$$\begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 0 & -2 & 1 \\ 0 & 1 & 3-3\beta & 4-3\beta \end{pmatrix} \overset{L_2 \leftrightarrow L_3}{\sim} \begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 3-3\beta & 4-3\beta \\ 0 & 0 & -2 & 1 \end{pmatrix}$$

$\Rightarrow$ rank $= 3$

2) $\alpha \neq 0$ $\begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & \alpha & -2 & 1 \\ 0 & 1-\alpha\beta & 3-3\beta & 4-3\beta \end{pmatrix} \underset{L_3 \leftarrow L_3 + \beta L_2}{\overset{L_1 \leftarrow L_1 - L_2}{\sim}} \begin{pmatrix} 1 & 0 & 5 & 2 \\ 0 & \alpha & -2 & 1 \\ 0 & 1 & 3-5\beta & 4-2\beta \end{pmatrix} \sim$

$\overset{L_2 \leftrightarrow L_3}{\sim} \begin{pmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 3-3\beta & 4-2\beta \\ 0 & \alpha & -2 & 1 \end{pmatrix} \overset{L_3 \leftarrow L_3 - \alpha L_2}{\sim} \begin{pmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 3-3\beta & 4-2\beta \\ 0 & 0 & -2-3\alpha+3\alpha\beta & 1-4\alpha+2\alpha\beta \end{pmatrix}$

$\Rightarrow \operatorname{rank} M \in \{2, 3\}$

$\operatorname{rank} M = 2 \iff \begin{cases} -2-3\alpha+3\alpha\beta = 0 \\ 1-4\alpha+2\alpha\beta = 0 \end{cases} \iff \begin{cases} -4-6\alpha+6\alpha\beta = 0 \\ 3-12\alpha+6\alpha\beta = 0 \end{cases} \iff$

$\iff \begin{cases} -2-3\alpha+3\alpha\beta = 0 \\ 6\alpha - 7 = 0 \end{cases} \iff \begin{cases} \alpha = \frac{7}{6} \\ -2-\frac{7}{2}+\frac{7}{2}\beta = 0 \end{cases} \iff \begin{cases} \alpha = \frac{7}{6} \\ \beta = \frac{11}{7} \end{cases}$

Compute by applying elementary operations the inverses of the matrices:

4. $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$. **5.** $\begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}$.

Sol.: $\left(A \mid I_n \right) \sim \dots \overset{\text{Gauss-Jordan}}{\sim} \left(I_n \mid A^{-1} \right)$

5. $\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \underset{L_3 \leftarrow L_3 - 3L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right) \sim$

$\overset{L_2 \leftarrow \frac{1}{-5} L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right) \overset{L_3 \leftarrow L_3 + 12L_2}{\sim} \begin{pmatrix} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & \frac{-1}{5} & 0 \\ 0 & 0 & \frac{1}{5} & \frac{9}{5} & \frac{-12}{5} & 1 \end{pmatrix} \sim$

$$\underset{\sim}{L_3 \leftarrow 5L_3} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right) \underset{L_1 \leftarrow L_1 - 2L_3}{\overset{L_2 \leftarrow L_2 - \frac{3}{5}L_3}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 4 & 0 & -17 & 24 & -10 \\ 0 & 1 & 0 & -5 & 7 & -3 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right) \sim$$

$$\underset{\sim}{L_1 \leftarrow L_1 - 4L_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 2 \\ 0 & 1 & 0 & -5 & 7 & -3 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right)$$

$$\Rightarrow \begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & -4 & 2 \\ -5 & 7 & -3 \\ 9 & -12 & 5 \end{pmatrix}$$

<u>Verification</u>: $\begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 3 & -4 & 2 \\ -5 & 7 & -3 \\ 9 & -12 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

<u>Remark</u>: If the matrix is not invertible, then at some point in the algorithm above we will get a zero row on the left side (thereby halting our Gauss-Jordan elimination)

<u>Ex.</u>: $\begin{pmatrix} 1 & 2 & -1 \\ 3 & 6 & 7 \\ 5 & 11 & 5 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 6 & 7 & 0 & 1 & 0 \\ 5 & 10 & 5 & 0 & 0 & 1 \end{array}\right) \underset{L_3 \leftarrow L_3 - 5L_1}{\overset{L_2 \leftarrow L_2 - 3L_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 10 & -3 & 1 & 0 \\ 0 & 0 & 10 & -5 & 0 & 1 \end{array}\right) \sim$$

$$\underset{\sim}{L_3 \leftarrow L_3 - L_2} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 10 & -3 & 1 & 0 \\ 0 & 0 & 0 & -2 & -1 & 1 \end{array}\right)$$

This row doesn't let us continue the argument

$\Rightarrow$ the initial matrix was not invertible.

4. $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$.

Sol :

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & -2 & 0 & 1 & 0 \\ 2 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \underset{L_3 \leftarrow L_3 - 2L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right) \sim$$

$$\overset{L_2 \leftarrow -\frac{1}{3} L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & \frac{-1}{3} & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right) \overset{L_3 \leftarrow L_3 + 6L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & \frac{-1}{3} & 0 \\ 0 & 0 & 9 & 2 & -2 & 1 \end{array}\right) \sim$$

$$\overset{L_3 \leftarrow \frac{1}{9} L_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & \frac{-1}{3} & 0 \\ 0 & 0 & 1 & \frac{2}{9} & \frac{-2}{9} & \frac{1}{9} \end{array}\right) \underset{L_2 \leftarrow L_2 - 2L_3}{\overset{L_1 \leftarrow L_1 - 2L_3}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{5}{9} & \frac{4}{9} & \frac{-2}{9} \\ 0 & 1 & 0 & \frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ 0 & 0 & 1 & \frac{2}{9} & \frac{-2}{9} & \frac{1}{9} \end{array}\right) \sim$$

$$\overset{L_1 \leftarrow L_1 - 2L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ 0 & 1 & 0 & \frac{2}{9} & \frac{1}{9} & \frac{-2}{9} \\ 0 & 0 & 1 & \frac{2}{9} & \frac{-2}{9} & \frac{1}{9} \end{array}\right)$$

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{1}{9} & \frac{-2}{9} \\ \frac{2}{9} & \frac{-2}{9} & \frac{1}{9} \end{pmatrix} = \frac{1}{9} \cdot \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

7. In the real vector space $\mathbb{R}^3$ consider the list $X = (v_1, v_2, v_3, v_4)$, where $v_1 = (1, 0, 4)$, $v_2 = (2, 1, 0)$, $v_3 = (1, 5, -36)$ and $v_4 = (2, 10, -72)$. Determine $dim < X >$ and a basis of $< X >$.

Sol.:

$$\begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 5 & -36 \\ 2 & 10 & -72 \end{pmatrix} \underset{\substack{L_3 \leftarrow L_3 - L_1 \\ L_4 \leftarrow L_4 - 2L_1}}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 5 & -40 \\ 0 & 10 & -80 \end{pmatrix} \underset{L_4 \leftarrow L_4 - 10L_2}{\overset{L_3 \leftarrow L_3 - 5L_2}{\sim}}$$

$$\sim \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow ((1,0,4), (0,1,-8)) \text{ basis for } < X >$$

and $\dim < X > = 2$

9. Determine the dimension of the subspaces S, T, $S + T$ and $S \cap T$ of the real vector space $\mathbb{R}^3$ and a basis for the first three of them, where

$$S = < (1, 0, 4), (2, 1, 0), (1, 1, -4) >,$$

$$T = < (-3, -2, 4), (5, 2, 4), (-2, 0, -8) > .$$

$S + T = < S \cup T >$

$$\begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{pmatrix} \underset{L_3 \leftarrow L_3 - L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 1 & -8 \end{pmatrix} \overset{L_3 \leftarrow L_3 - L_2}{\sim} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ basis for S : $((1,0,4), (0,1,-8))$, $\dim S = 2$

$$\begin{pmatrix} -3 & -2 & 4 \\ 5 & 2 & 4 \\ -2 & 0 & -8 \end{pmatrix} \overset{L_1 \leftarrow -\frac{1}{3} L_1}{\sim} \begin{pmatrix} 1 & \frac{2}{3} & -\frac{4}{3} \\ 5 & 2 & 4 \\ -2 & 0 & -8 \end{pmatrix} \sim$$

$$\underset{L_3 \leftarrow L_3 + 2L_1}{\overset{L_2 \leftarrow L_2 - 5L_1}{\sim}} \begin{pmatrix} 1 & \frac{2}{3} & -\frac{4}{3} \\ 0 & -\frac{4}{3} & \frac{32}{3} \\ 0 & \frac{4}{3} & -\frac{32}{3} \end{pmatrix} \overset{L_3 \leftarrow L_3 + L_2}{\sim} \begin{pmatrix} 1 & \frac{2}{3} & -\frac{4}{3} \\ 0 & -\frac{4}{3} & \frac{32}{3} \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ basis for T: $\left(\left(1, \frac{2}{3}, -\frac{4}{3}\right), \left(0, -\frac{4}{3}, \frac{32}{3}\right) \right)$

$\Rightarrow \dim T = 2$

For $S+T$:

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 3 & 2 & -4 \\ 0 & -4 & 32 \end{pmatrix} \overset{L_3 \leftarrow L_3 - 3L_1}{\sim} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & -16 \\ 0 & -4 & 32 \end{pmatrix} \sim$$

$$\overset{L_4 \leftarrow L_4 + 4L_2}{\sim} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & -16 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{basis for } S+T\text{:}$$

$\left((1,0,4), (0,1,-8), (0,0,-16) \right)$

$\Rightarrow \dim(S+T) = 3$

(Calculations are probably wrong)

$$\dim(S \cap T) = \dim(S) + \dim(T) - \dim(S+T) = 2+2-3 = 1$$