19.10.2021

Seminar W4 – 911

Def.: $(V,+)$ abelian group, $(K,+,\cdot)$ field

$$\cdot : K \times V \to V \quad \text{external operation}$$
$$(k, v) \mapsto k \cdot v$$

$\to$ V <u>K-vector space</u> if:

- $\forall \alpha, \beta \in K, \forall v \in V$: $(\alpha+\beta)\cdot v = \alpha v + \beta v$
- $\forall \alpha \in K, \forall v_1, v_2 \in V$: $\alpha\cdot(v_1+v_2) = \alpha v_1 + \alpha v_2$
- $\forall \alpha, \beta \in K, \forall v \in V$: $(\alpha \beta) v = \alpha \cdot (\beta v)$
- $\forall v \in V$: $1 \cdot v = v$

4. Let $V = \{x \in \mathbb{R} \mid x > 0\}$ and define the operations: $x \perp y = xy$ and $k \top x = x^k$, $\forall k \in \mathbb{R}$ and $\forall x, y \in V$. Prove that V is a vector space over $\mathbb{R}$.

Sol.: $(V, \perp)$ abelian group, external operation: $\top$

- $\perp$ is an op: $\forall v_1, v_2 \in V \overset{?}{\Rightarrow} v_1 \perp v_2 \in V$

$$v_1 \perp v_2 = \underbrace{v_1}_{>0} \cdot \underbrace{v_2}_{>0} > 0 \Rightarrow v_1 \perp v_2 \in V$$

- assoc. of $\perp$: $\forall v_1, v_2, v_3 \in V \overset{?}{\Rightarrow} (v_1 \perp v_2) \perp v_3 = v_1 \perp (v_2 \perp v_3)$

$$(v_1 \perp v_2) \perp v_3 = (v_1 v_2) \perp v_3 = v_1 v_2 v_3$$
$$v_1 \perp (v_2 \perp v_3) = v_1 \perp (v_2 v_3) = v_1 v_2 v_3 \quad \checkmark$$

$\Rightarrow$ assoc.

- commutativity of $\perp$: $\forall v_1, v_2 \in V$:

$$v_1 \perp v_2 = v_1 v_2 = v_2 v_1 = v_2 \perp v_1$$

- neutral element of $\perp$: $1 \in V$, so $\forall v \in V: 1 \perp v = v = v \perp 1$

- invertibility of $\perp$: $\forall v \in V \Rightarrow v > 0 \Rightarrow \frac{1}{v} \in V \Rightarrow$

$\Rightarrow \forall v$ has an inverse

$\Rightarrow (V, \perp)$ abelian group

We will now prove the axioms :

$\underline{\forall \alpha, \beta \in \mathbb{R}, \forall v \in V: \quad (\alpha + \beta) \top v \overset{?}{=} (\alpha \top v) \perp (\beta \top v)}$

$$(\alpha + \beta) \top v = v^{\alpha + \beta}$$

$$(\alpha \top v) \perp (\beta \top v) = v^{\alpha} \perp v^{\beta} = v^{\alpha} \cdot v^{\beta} = v^{\alpha + \beta}$$

$\underline{\forall \alpha \in \mathbb{R}, \quad \forall v_1, v_2 \in V: \quad \alpha \top (v_1 \perp v_2) = (\alpha \top v_1) \perp (\alpha \top v_2)}$

$$\left.\begin{aligned} LHS &= \alpha \top (v_1 v_2) = (v_1 v_2)^{\alpha} \\ RHS &= v_1^{\alpha} \perp v_2^{\alpha} = v_1^{\alpha} \cdot v_2^{\alpha} = (v_1 v_2)^{\alpha} \end{aligned}\right\} \Rightarrow \checkmark$$

$\underline{\forall \alpha, \beta \in \mathbb{R}, \quad \forall v \in V \quad : \quad (\alpha \beta) \top v \overset{?}{=} \alpha \top (\beta \top v)}$

$LHS = v^{\alpha \beta} \qquad RHS = \alpha \top (v^{\beta}) = v^{\beta \alpha} = v^{\alpha \beta} \Rightarrow \checkmark$

$$\underline{\forall v \in V : \quad 1 \top v \overset{?}{=} v}$$

$$1 \top v = v^1 = v$$

$\Rightarrow$ V is an $\mathbb{R}$-vector space

5. Let K be a field and let $V = K \times K$. Decide whether V is a K-vector space with respect to the following addition and scalar multiplication:

(*i*) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + 2y_2)$ and $k \cdot (x_1, y_1) = (kx_1, ky_1)$, $\forall (x_1, y_1), (x_2, y_2) \in V$ and $\forall k \in K$.

(*ii*) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $k \cdot (x_1, y_1) = (kx_1, y_1)$, $\forall (x_1, y_1), (x_2, y_2) \in V$ and $\forall k \in K$.

<u>Sol.</u> : (i) If we assume that V is a vector space

$$(1+1) \cdot (1,1) = (2,2)$$

$2 := 1+1$, $3 := 1+1+1$

$$(1+1) \cdot (1,1) = 1 \cdot (1,1) + 1 \cdot (1,1) = (1+1, 1+2 \cdot 1) = (2,3)$$

$\Rightarrow$ $2 = 3$ $\Rightarrow$ $1 = 0$ $\Rightarrow$ K is not a field $\Rightarrow$ contradiction $\Rightarrow$

$\Rightarrow$ V is not a vector space.

Another approach: $\forall \alpha, \beta \in K$, $\forall v \in V$:, $v = (v_1, v_2)$

$$(\alpha + \beta) \cdot v = \alpha v + \beta v$$

$$\Rightarrow \left((\alpha+\beta) v_1, (\alpha+\beta) v_2\right) = (\alpha v_1, \alpha v_2) + (\beta v_1, \beta v_2)$$

$$\Rightarrow \beta v_2 = 2\beta v_2 \quad \Rightarrow \beta v_2 = 0$$

$$\forall \beta, v_2 \in K$$

(ii) We assume that V is a vector space.

$$\forall \alpha, \beta \in K, \forall v = (x,y) \in V \Rightarrow (\alpha+\beta)\cdot(x,y) = \alpha(x,y) + \beta\cdot(x,y)$$

$$(\alpha+\beta)\cdot(x,y) = ((\alpha+\beta)x, y)$$

$$\alpha(x,y) + \beta(x,y) = (\alpha x, y) + (\beta x, y) = ((\alpha+\beta)x, 2y)$$

$$\Rightarrow 2y = y, \forall y \in K \Rightarrow y = 0, \forall y \in K \quad \text{contradiction}$$

<u>Def</u>: V K-vector space, $S \subseteq V$

$S \leq_K V$ (K-subspace) $\Leftrightarrow$
(i) $S \neq \emptyset$
(ii) $(S,+) \leq (V,+)$ (subgroup) : $\forall x, y \in S : x - y \in S$
(iii) S is compatible with scalar multiplication: $\forall x \in S, \forall k \in K : kx \in S$

$\forall \alpha, \beta \in K, \forall x, y \in S : \alpha x + \beta y \in S$

7. Which ones of the following sets are subspaces of the real vector space $\mathbb{R}^3$:
(i) $A = \{(x,y,z) \in \mathbb{R}^3 \mid x = 0\}$;
(ii) $B = \{(x,y,z) \in \mathbb{R}^3 \mid x = 0 \text{ or } z = 0\}$;
(iii) $C = \{(x,y,z) \in \mathbb{R}^3 \mid x \in \mathbb{Z}\}$;
(iv) $D = \{(x,y,z) \in \mathbb{R}^3 \mid x + y + z = 0\}$;
(v) $E = \{(x,y,z) \in \mathbb{R}^3 \mid x + y + z = 1\}$;
(vi) $F = \{(x,y,z) \in \mathbb{R}^3 \mid x = y = z\}$?

<u>Sol</u>: (i) $A \neq \emptyset$, because $(0,0,0) \in A$

Let $v_1 = (\overset{0}{\overset{\parallel}{x_1}}, y_1, z_1)$, $v_2 = (\overset{0}{\overset{\parallel}{x_2}}, y_2, z_2)$

$$v_1 - v_2 = (\overbrace{x_1 - x_2}^{=0}, y_1 - y_2, z_1 - z_2) \in A$$

Let $\alpha \in \mathbb{R}$: $\alpha v_1 = \alpha \cdot (x_1, y_1, z_1) = (\underbrace{\alpha x_1}_{=0}, \alpha y_1, \alpha z_1) \in \mathbb{R}^3 \Rightarrow$

$\Rightarrow \alpha v_1 \in A \Rightarrow A \leq_{\mathbb{R}} \mathbb{R}^3$

(ii) $\underset{B}{\underset{\Uparrow}{(0,1,1)}} + \underset{B}{\underset{\Uparrow}{(1,1,0)}} = (1,2,1) \notin B \Rightarrow B \not\leq \mathbb{R}^3$

(iii) $\underbrace{\frac{1}{2}}_{\in \mathbb{R}} \cdot \underset{C}{\underset{\Uparrow}{(1,2,3)}} = (\frac{1}{2}, 1, \frac{3}{2}) \notin C \Rightarrow C \not\leq \mathbb{R}^3$

(iv) $D \neq \emptyset$, because $(0,0,0) \in D$

Let $v = (v_1, v_2, v_3)$, $u = (u_1, u_2, u_3) \in D$

$\Rightarrow v_1 + v_2 + v_3 = u_1 + u_2 + u_3 = 0$

$v - u = (v_1 - u_1, v_2 - u_2, v_3 - u_3)$

$(v_1 - u_1) + (v_2 - u_2) + (v_3 - u_3) = (v_1 + v_2 + v_3) - (u_1 + u_2 + u_3) = 0$

$\Rightarrow v - u \in D$

Let $v = (v_1, v_2, v_3)$, $a \in \mathbb{R}$

$av = (av_1, av_2, av_3)$

$av_1 + av_2 + av_3 = a(v_1 + v_2 + v_3) = 0$

$\Rightarrow av \in D \Rightarrow D \leq_{\mathbb{R}} \mathbb{R}^3$