13. 12. 2021

Seminar W11-911

Def: V, V' K-v.s., $f: V \to V'$ linear map
$B = (v_1, \dots, v_m)$, $B' = (v_1', \dots, v_n')$

$$[f]_{B,B'} = \left([f(v_1)]_{B'} \ \dots \ [f(v_m)]_{B'} \right)$$

Prop: $\forall v \in V$: $[f(v)]_{B'} = [f]_{B,B'} \cdot [v]_B$

Def: $[id]_{B,B'} =: T_{B',B}$ = the base change (or transfer) matrix from the basis B' to the basis B.

Corollary: If we put $f = id$:

$$[v]_{B'} = [id]_{B,B'} \cdot [v]_B = T_{B',B} \cdot [v]_B$$

Prop: $\forall B, B'$ bases of V: $T_{B,B'}^{-1} = T_{B',B}$

$$\left(\underline{[id]_{B',B}^{-1} = [id]_{B,B'}} \right)$$

This is only true for id!!!

Prop.: V, V', V'' K-v.s., $g: V \to V'$, $f: V' \to V''$ linear maps.
B, B', B'' bases for V, V', V'' respectively

$$\boxed{[f \circ g]_{B,B''} = [f]_{B',B''} \cdot [g]_{B,B'}}$$

$id_V : V \to V$
$x \mapsto x$
$id_{V'} : V' \to V'$
$x \mapsto x$

Cor.: V, V', B_1, B_2 bases of V
B_1', B_2' bases of V'

$$[f]_{B_1', B_2'} = \underbrace{[id_{V'}]_{B_2, B_2'}}_{T_{B_2', B_2}} \cdot [f]_{B_1, B_2} \cdot \underbrace{[id_V]_{B_1', B_1}}_{T_{B_1, B_1'}}$$

2. In the real vector space $\mathbb{R}^2$ consider the bases $B = (v_1, v_2) = ((1,2),(1,3))$ and $B' = (v_1', v_2') = ((1,0),(2,1))$ and let $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ having the matrices $[f]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ and $[g]_{B'} = \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}$. Determine the matrices $[2f]_B$, $[f+g]_B$ and $[f \circ g]_{B'}$. (*Use the matrices of change of basis.*)

Hint: Convert everything to the canonical basis

Sol: $[2f]_B = 2 \cdot [f]_B = 2 \cdot \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -2 & -2 \end{pmatrix}$

$[f+g]_B = [f]_B + [g]_B$

$$[g]_B = [id]_{B',B} \cdot [g]_{B'} \cdot \underline{[id]_{B,B'}}$$

$$[id]_{B,B'} = \left([v_1]_{B'} \quad [v_2]_{B'} \right)$$

One approach: (we'll see another one later) $\quad [v_1]_{B'} = ?$

$\Rightarrow v_1 = \alpha \cdot v_1' + \beta \cdot v_2'$

$\Rightarrow (1,2) = \alpha \cdot (1,0) + \beta \cdot (2,1)$

$\Rightarrow \begin{cases} \alpha + 2\beta = 1 \\ \beta = 2 \end{cases} \Rightarrow \begin{cases} \alpha = -3 \\ \beta = 2 \end{cases} \Leftrightarrow [v_1]_{B'} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

$[v_2]_{B'} = ?$

$\Rightarrow (1,3) = \alpha \cdot (1,0) + \beta \cdot (2,1)$

$\Rightarrow \begin{cases} \alpha + 2\beta = 1 \\ \beta = 3 \end{cases} \Rightarrow \begin{cases} \alpha = -5 \\ \beta = 3 \end{cases} \Rightarrow [v_2]_{B'} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$

$\Rightarrow [id]_{B,B'} = T_{B',B} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$

To find $[id]_{B',B}$, we can either do the same thing, or use the fact that $[id]_{B',B} = [id]_{B,B'}^{-1}$

$$\left(\begin{array}{cc|cc} -3 & -5 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right) \underset{L_1 \leftarrow \frac{1}{-3} L_1}{\sim} \left(\begin{array}{cc|cc} 1 & \frac{5}{3} & -\frac{1}{3} & 0 \\ 2 & 3 & 0 & 1 \end{array}\right) \sim$$

$$\underset{L_2 \leftarrow L_2 - 2L_1}{\sim} \left(\begin{array}{cc|cc} 1 & \frac{5}{3} & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & \frac{2}{3} & 1 \end{array}\right) \underset{L_2 \leftarrow -3L_2}{\sim} \left(\begin{array}{cc|cc} 1 & \frac{5}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & -2 & -3 \end{array}\right) \sim$$

$$\underset{L_1 \leftarrow L_1 - \frac{5}{3} L_2}{\sim} \left(\begin{array}{cc|cc} 1 & 0 & 3 & 5 \\ 0 & 1 & -2 & -3 \end{array}\right) \Rightarrow [id]_{B',B} = T_{B,B'} = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}$$

$$[g]_B = [id]_{B',B} \cdot [g]_{B'} \cdot [id]_{B,B'} =$$

$$= \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} \cdot \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix} \cdot \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} \cdot \begin{pmatrix} -5 & -4 \\ -1 & -4 \end{pmatrix} = \begin{pmatrix} -20 & -32 \\ 13 & 20 \end{pmatrix}$$

$$[f \circ g]_{B'} = ?$$

$$[f \circ g]_{B'} = [f]_{B'} \cdot \underbrace{[g]_{B'}}_{\begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}}$$

$$[f]_{B'} = ?$$

$$[f]_{B'} = [id]_{B,B'} \cdot [f]_B \cdot [id]_{B',B} =$$

$$= \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} =$$

$$= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} 8 & 13 \\ -5 & -8 \end{pmatrix}$$

$$\Rightarrow [f \circ g]_{B'} = \begin{pmatrix} 8 & 13 \\ -5 & -8 \end{pmatrix} \cdot \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 9 & -13 \\ -5 & 9 \end{pmatrix}$$

If we didn't already have the base change matrices, then we could also have used this approach to find them:

$[id]_{B,B'} = ?$

$$[id]_{B,B'} = [id]_{E,B'} \cdot [id]_{B,E} = [id]_{B',E}^{-1} \cdot [id]_{B,E}$$

$$[id]_{B,E} = \left([v_1]_E \quad [v_2]_E \right) = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$$

$$[id]_{B',E} = \left([v_1']_E \quad [v_2']_E \right) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow [id]_{B,B'} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$$

Eigenvectors & eigenvalues

V K-vector space; $f: V \to V$

$v \in V \setminus \{0\}$ eigenvector of f if $\exists \lambda \in K$ (eigenvalue of f):

$$f(v) = \lambda \cdot v$$

$V(\lambda) := \{ v \in V \mid f(v) = \lambda v \} = \{ \text{set of eigenvectors corresp. to } \lambda \} \cup \{0\}$

$\hookrightarrow$ the eigenspace of f corresponding to λ

B basis of V, then:

λ eigenvalue for f $\iff$ λ root of the characteristic polynomial: $\iff$

$$P_f(X) = \det([f]_B - X \cdot I_n)$$

$\iff \det([f]_B - \lambda I_n) = 0$

eigenvalues and eigenvectors of a matrix A $\iff$ e-values and e-vectors of a linear map f so that $[f]_E = A$

Compute the eigenvalues and the eigenvectors of the (endomorphisms having) matrices:

5. $\begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ -4 & -8 & -2 \end{pmatrix}$. **6.** $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Sol.: 5. $p_A(X) = \begin{vmatrix} 3-X & 1 & 0 \\ -4 & -1-X & 0 \\ -4 & -8 & -2-X \end{vmatrix} = (-2-X)\cdot \begin{vmatrix} 3-X & 1 \\ -4 & -1-X \end{vmatrix} =$

$= -(X+2)\cdot\left((3-X)(-1-X)+4\right) = -(X+2)\cdot\left(-3-3X+X+X^2+4\right)$

$= -(X+2)\cdot\left(X^2-2X+1\right) = -(X+2)\cdot(X-1)^2$

$\Rightarrow$ the eigenvalues are $\lambda_1 = -2$, $\lambda_2 = 1$

The eigenvectors corresponding to $\lambda_1 = -2$:

$A\cdot[u]_E = \lambda\cdot[u]_E \iff (A-\lambda I_3)\cdot[u]_E = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$u = (x, y, z)$:

$$\begin{pmatrix} 3-(-2) & 1 & 0 \\ -4 & -1-(-2) & 0 \\ -4 & -8 & -2-(-2) \end{pmatrix}\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 5 & 1 & 0 \\ -4 & 1 & 0 \\ -4 & -8 & 0 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 5 & 1 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ -4 & -8 & 0 & 0 \end{array}\right) \underset{L_3 \leftarrow L_3 + \frac{4}{5} L_1}{\overset{L_2 \leftarrow L_2 + \frac{4}{5} L_1}{\sim}} \left(\begin{array}{ccc|c} 5 & 1 & 0 & 0 \\ 0 & \frac{9}{5} & 0 & 0 \\ 0 & -\frac{36}{5} & 0 & 0 \end{array}\right) \sim$$

$$\overset{L_3 \leftarrow L_3 + 4 L_2}{\sim} \left(\begin{array}{ccc|c} 5 & 1 & 0 & 0 \\ 0 & \frac{9}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \begin{cases} 5x + y = 0 \\ \frac{9}{5} y = 0 \end{cases} \Rightarrow \begin{cases} y = 0 \\ x = 0 \end{cases}$$

$\Rightarrow$ the eigenvectors corresponding to $\lambda_1 = -2$ are the vectors of the form:

$$(0, 0, z)$$

$$\Rightarrow S(\lambda_1) = \{(0, 0, z) \mid z \in \mathbb{R}\} = \langle (0, 0, 1) \rangle$$