01.11.2021

Seminar W5 - 916

Def/th : V k-vector space, $X \subseteq V$

The subspace generated by X is:

$$\langle X \rangle = \bigcap_{\substack{S \leq V \\ S \supseteq X}} S = \left\{ \sum_{i=1}^{n} k_i \cdot a_i \;\middle|\; \begin{array}{l} n \in \mathbb{N},\ k_i \in k \\ a_i \in X \end{array} \right\}$$

2. Consider the following subspaces of the real vector space $\mathbb{R}^3$:
(i) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$;
(ii) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$;
(iii) $C = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$.
Write A, B, C as generated subspaces with a minimal number of generators.

(iv) $D = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{cases} 2x+y=0 \\ y+z=0 \end{cases} \right\}$

(v) $E = \{ (x,y,z,t) \in \mathbb{R}^4 \mid x+2y+z=0 \}$

Sol: **Ex.**: $S = \left\{ (x,y,z,t,u) \in \mathbb{R}^5 \;\middle|\; \begin{cases} x+u+t=0 \\ y+2z=0 \\ u+3t=0 \end{cases} \right\} =$

$$= \left\{ (x,y,z,t,u) \in \mathbb{R}^5 \;\middle|\; \begin{cases} u=-3t \\ y=-2z \\ x-3t+t=0 \end{cases} \right\} =$$

$$= \left\{ (x,y,z,t,u) \in \mathbb{R}^5 \;\middle|\; \begin{cases} x=2t \\ y=-2z \\ u=-3t \end{cases} \right\} =$$

$$= \{ (2t, -2z, z, t, -3t) \mid t,z \in \mathbb{R} \} =$$

$$= \{ (2t, 0, 0, t, -3t) + (0, -2z, z, 0, 0) \mid z, t \in \mathbb{R} \} =$$

$$= \{ t \cdot (2, 0, 0, 1, -3) + z \cdot (0, -2, 1, 0, 0) \mid z, t \in \mathbb{R} \}$$

$$= \langle (2,0,0,1,-3), (0,-2,1,0,0) \rangle$$

This is the minimal number of generators, because

$$(2,0,0,1,-3) \notin \langle (0,-2,1,0,0) \rangle$$

If this were the case, then we'd have $\alpha \in \mathbb{R}$ so that:

$$(2,0,0,1,-3) = \alpha \cdot (0,-2,1,0,0)$$

This would mean that: $\begin{cases} 2 = 0 \\ 0 = -2\alpha \\ 0 = \alpha \\ 1 = 0 \\ -3 = 0 \end{cases}$, absurd, so

$$(2,0,0,1,-3) \notin \langle (0,-2,1,0,0) \rangle$$

(i) $A = \{ (x,y,z) \in \mathbb{R}^3 \mid x = 0 \} = \{ (0,y,z) \mid y,z \in \mathbb{R} \} =$

$= \{ (0,y,0) + (0,0,z) \mid y,z \in \mathbb{R} \} = \{ y \cdot (0,1,0) + z \cdot (0,0,1) \mid y,z \in \mathbb{R} \} =$

$= \langle (0,1,0), (0,0,1) \rangle$

(ii) $B = \{ (x,y,z) \in \mathbb{R}^3 \mid x = -y-z \} =$

$= \{ (-y-z, y, z) \mid y,z \in \mathbb{R} \} =$

$= \{ (-y, y, 0) + (-z, 0, z) \mid y,z \in \mathbb{R} \} =$

$= \{ y \cdot (-1,1,0) + z \cdot (-1,0,1) \mid y,z \in \mathbb{R} \} =$

$= \langle (-1,1,0), (-1,0,1) \rangle$

(iii) $C = \{ (y,y,y) \mid y \in \mathbb{R} \} = \{ y \cdot (1,1,1) \mid y \in \mathbb{R} \} =$

$= \langle (1,1,1) \rangle$

(iv) $D = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{cases} 2x+y=0 \\ y+z=0 \end{cases} \right\} =$

$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{cases} y=-2x \\ z=-y=2x \end{cases} \right\} =$

$= \{ (x, -2x, 2x) \mid x \in \mathbb{R} \} =$

$= \{ x \cdot (1,-2,2) \mid x \in \mathbb{R} \} =$

$= \langle (1,-2,2) \rangle$

(v) $E = \{ (x,y,z,t) \in \mathbb{R}^4 / x+2y+z=0 \} =$

$= \{ (x,y,z,t) \in \mathbb{R}^4 / x = -2y-z \} =$

$= \{ (-2y-z, y, z, t) \mid y,z,t \in \mathbb{R} \} =$

$= \{ (-2y, y, 0, 0) + (-z, 0, z, 0) + (0,0,0,t) \mid y,z,t \in \mathbb{R} \} =$

$= \{ y \cdot (-2,1,0,0) + z \cdot (-1,0,1,0) + t \cdot (0,0,0,1) \mid y,z,t \in \mathbb{R} \} =$

$= \langle (-2,1,0,0), (-1,0,1,0), (0,0,0,1) \rangle$

Def: V, W K-vector spaces, $f: V \to W$ is a homomorphism of vector spaces (or linear map) if:

- $\forall v_1, v_2 \in V$: $f(v_1+v_2) = f(v_1) + f(v_2)$
- $\forall k \in K$, $\forall v \in V$: $f(kv) = k f(v)$

$\to$ $\forall k_1, k_2 \in K$, $\forall v_1, v_2 \in V$: $f(k_1 v_1 + k_2 v_2) = k_1 f(v_1) + k_2 f(v_2)$

8. Let $a \in \mathbb{R}$ and let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x, y) = (x \cos a - y \sin a, x \sin a + y \cos a).$$

Prove that $f \in End_{\mathbb{R}}(\mathbb{R}^2)$.

Sol: Let $k_1, k_2 \in K$, $v_1, v_2 \in \mathbb{R}^2$, $v_1 = (x_1, y_1)$, $v_2 = (x_2, y_2)$

$f(k_1 v_1 + k_2 v_2) = f(k_1(x_1, y_1) + k_2(x_2, y_2)) =$

$= f(k_1 x_1 + k_2 x_2, k_1 y_1 + k_2 y_2) =$

$= ((k_1 x_1 + k_2 x_2) \cdot \cos a - (k_1 y_1 + k_2 y_2) \cdot \sin a, (k_1 x_1 + k_2 x_2) \cdot \sin a +$

$+ (k_1 y_1 + k_2 y_2) \cdot \cos a) = (k_1 x_1 \cos a - k_1 y_1 \sin a, k_1 x_1 \sin a +$

$+ k_1 y_1 \cos a) + (k_2 x_2 \cdot \cos a - k_2 y_2 \cdot \sin a, k_2 x_2 \cdot \sin a + k_2 y_2 \cdot \cos a)$

$= k_1 \cdot (x_1 \cos a - y_1 \sin a, x_1 \sin a + y_1 \cos a) +$

$+ k_2 \cdot (x_2 \cos a - y_2 \sin a, x_2 \sin a + y_2 \cos a) = k_1 \cdot f(v_1) + k_2 \cdot f(v_2)$

Def.: V K-vector space, $S, T \leq_K V$

$V = S + T \iff \forall u \in V : \exists s \in S, \exists t \in T : u = s + t$

$V = S \oplus T \iff \forall u \in V : \exists! s \in S, \exists! t \in T : u = s + t \iff$

$\iff V = S + T, \quad S \cap T = 0$

4. Let
$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\},$$
$$T = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}.$$
Prove that S and T are subspaces of the real vector space $\mathbb{R}^3$ and $\mathbb{R}^3 = S \oplus T$.

already done

Sol.: $S = \langle (-1, 1, 0), (-1, 0, 1) \rangle$
$T = \langle (1, 1, 1) \rangle$

$S = \{ y(-1, 1, 0) + z \cdot (-1, 0, 1) \mid y, z \in \mathbb{R} \}$

$T = \{ x \cdot (1, 1, 1) \mid x \in \mathbb{R} \}$

$S + T = \{ x \cdot (1, 1, 1) + y \cdot (-1, 1, 0) + z \cdot (-1, 0, 1) \mid x, y, z \in \mathbb{R} \}$

$= \{ (x - y - z, x + y, x + z) \mid x, y, z \in \mathbb{R} \}$

Let $u = (a, b, c) \in \mathbb{R}^3$. We have to show that $\exists x, y, z \in \mathbb{R}$:

$(a, b, c) = (x - y - z, x + y, x + z)$

$$\begin{cases} a = x - y - z \\ b = x + y \\ c = x + z \end{cases} \iff \begin{cases} 3x = a + b + c \\ b = x + y \\ c = x + z \end{cases} \iff \begin{cases} x = \frac{a+b+c}{3} \\ y = b - \frac{a+b+c}{3} \\ z = c - \frac{a+b+c}{3} \end{cases}$$

We found these solutions and they are unique

$$\Rightarrow \mathbb{R}^3 = S \oplus T$$

$$(a,b,c) = \underbrace{\left(b - \frac{a+b+c}{3}\right) \cdot (-1,1,0) + \left(c - \frac{a+b+c}{3}\right) \cdot (-1,0,1)}_{\text{the part from } S} +$$

$$+ \underbrace{\frac{a+b+c}{3} \cdot (1,1,1)}_{\text{the part from } T} =$$

$$= \underbrace{\left(a - \frac{a+b+c}{3}, b - \frac{a+b+c}{3}, c - \frac{a+b+c}{3}\right)}_{\in S} + \underbrace{\left(\frac{a+b+c}{3}, \frac{a+b+c}{3}, \frac{a+b+c}{3}\right)}_{\in T}$$

7. Which ones of the following functions are endomorphisms of the real vector space $\mathbb{R}^2$:
(*i*) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (ax + by, cx + dy)$, where $a, b, c, d \in \mathbb{R}$;
(*ii*) $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x, y) = (a + x, b + y)$, where $a, b \in \mathbb{R}$?

<u>Sol</u>: $f(k_1 \overset{(x,y)}{v_1} + k_2 \overset{(z,t)}{v_2}) = f(k_1 x + k_2 z, k_1 y + k_2 t) =$

$$= \left(a \cdot (k_1 x + k_2 z) + b \cdot (k_1 y + k_2 t), c \cdot (k_1 x + k_2 z) + d(k_1 y + k_2 t)\right) =$$

$$= k_1 \cdot (ax + by, cx + dy) + k_2 \cdot (az + bt, cz + dt) =$$

$= k_1 \cdot f(x,y) + k_2 \cdot f(t,t) = k_1 f(u_1) + k_2 f(u_2)$

$\Rightarrow f \in End_{\mathbb{R}}(\mathbb{R}^2)$

(ii) $g : \mathbb{R}^2 \to \mathbb{R}^2$ Let's assume $g \in End_{\mathbb{R}}(\mathbb{R}^2)$

$(x,y) \mapsto (a+x, b+y)$

$g(1,0) = (a+1, b)$

$$\left.\begin{aligned} g(2,0) &= (a+2, b) \\ g(2,0) &= 2 \cdot g(1,0) = (2(a+1), 2b) \end{aligned}\right\} \Rightarrow \begin{cases} a+2 = 2(a+1) \\ b = 2b \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} a+2 = 2a+2 \\ b = 2b \end{cases} \Leftrightarrow \begin{cases} a = 0 \\ b = 0 \end{cases}$$

$\Rightarrow$ The only possibility for $g \in End$ is if $a = b = 0$,

so if $g : \mathbb{R}^2 \to \mathbb{R}^2$

$(x,y) \mapsto (x,y)$

1. Determine the following generated subspaces:
(*i*) $< 1, X, X^2 >$ in the real vector space $\mathbb{R}[X]$.
(*ii*) $\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$ in the real vector space $M_2(\mathbb{R})$.

Sol.:

$$< 1, X, X^2 > \overset{?}{=} \mathbb{R}_2[X] = \{ f \in \mathbb{R}[X] \mid \deg f \leq 2 \}$$

$$< 1, X, X^2 > = \{ a \cdot 1 + b \cdot X + c \cdot X^2 \mid a, b, c \in \mathbb{R} \} = \mathbb{R}_2[X]$$

(ii) $\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle =$

$= \left\{ a \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \right\} =$

$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \right\} = \mathcal{M}_2(\mathbb{R})$

2. **9.** Let $M = \{0, 1, 2, 3\}$ and let $h = (\mathbb{Z}, M, H)$ be a relation, where

$$H = \{(x, y) \in \mathbb{Z} \times M \mid \exists z \in \mathbb{Z} : x = 4z + y\}.$$

"so that"

Is h a function?

Sol: $H = \{ (x, y) \in \mathbb{Z} \times M \mid 4 \mid (x - y) \} =$

$= \{ (x, y) \mid x \in \mathbb{Z}, y \in \{0, 1, 2, 3\} : 4 \mid (x - y) \}$

Because $\forall x \in \mathbb{Z} : \exists!\, y \in \{0, 1, 2, 3\}$ for which $4 \mid (x - y)$,

we have that h is a function.