Seminar W1 – 9.14

Algebra → study of algebraic structures
$(S, *, \cdot, \dots)$ (operations; S set)

Ex.: monoids, groups, rings, fields, (+ vector space)
↳ internal law

→ $\cdot$ operation on a set S if:

$$\cdot : S \times S \to S$$
$$(x,y) \mapsto x \cdot y$$

- $(G, *)$ group

Semigroup
→ $*$ is an operation: $\forall x, y \in G: x * y \in G$
→ associativity:
$\forall x, y, z \in G: (x * y) * z = x * (y * z)$

monoid
→ neutral element:
$\exists e \in G \ \forall x \in G: x * e = e * x = x$

Semigroup (monoid)

→ invertibility element:
$\forall x \in G, \exists x' \in G: x * x' = x' * x = e$

→ commutativity:
$\forall x, y \in G: x * y = y * x$

3. Decide which ones of the numerical sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are groups together with the usual addition or multiplication.

Sol.: $\underset{F}{(\mathbb{N},+)}$, $\underset{T}{(\mathbb{Z},+)}$, $\underset{T}{(\mathbb{Q},+)}$, $\underset{T}{(\mathbb{R},+)}$, $\underset{T}{(\mathbb{C},+)}$

$\underset{F}{(\mathbb{N},\cdot)}$, $\underset{F}{(\mathbb{Z},\cdot)}$, $\underset{F}{(\mathbb{Q},\cdot)}$, $\underset{F}{(\mathbb{R},\cdot)}$, $\underset{F}{(\mathbb{C},\cdot)}$

5. Let "$*$" be the operation defined on $\mathbb{N}$ by $x * y = \text{g.c.d.}(x, y)$.

(i) Prove that $(\mathbb{N}, *)$ is a commutative monoid.

(ii) Show that $D_n = \{x \in \mathbb{N} \mid x/n\}$ ($n \in \mathbb{N}^*$) is a stable subset of $(\mathbb{N}, *)$ and $(D_n, *)$ is a commutative monoid. $x | n \Leftrightarrow n \vdots x$

(iii) Fill in the table of the operation "$*$" on D_6.

You can take without a proof the following result:

$$\gcd(\gcd(x,y),z) = \gcd(x, \gcd(y,z)) \quad (\star)$$

Sol: (i) $\forall x,y \in \mathbb{N}$: $\gcd(x,y) \in \mathbb{N}$

Associativity is relation $(\star)$

Neutral element:

~~$\forall x \in \mathbb{N}$: $x * 1 = \gcd(x,1) = 1$~~

$\forall x \in \mathbb{N}$: $x * 0 = \gcd(x, 0) = x$

Commutativity: $\forall x,y$: $x * y = \gcd(x,y) = \gcd(y,x) = y * x$

Def.: S set, $*$ operation on S

$A \subseteq S$ is a stable part of S with regards to $*$

if: $\forall x,y \in A$: $x * y \in A$

(ii) $x,y \in D_n \Rightarrow x | n$, $y | n$

$\gcd(x,y)$

(Ex.: $n = 180$ $x = 15$ $y = 10$ $(15,10) = 5$)

$$\left(\begin{array}{l} a \mid b,\ b \mid c \Rightarrow a \mid c \\ \left.\begin{array}{l} a \mid b \Rightarrow b = ak \\ b \mid c \Rightarrow c = b \cdot m \end{array}\right\} \Rightarrow \begin{array}{l} c = a \cdot km \\ \Rightarrow a \mid c \end{array} \end{array} \right)$$

$\gcd(x,y) \mid x,\ x \mid n \Rightarrow \gcd(x,y) \mid n \Rightarrow \gcd(x,y) \in D_n$

$\Rightarrow D_n$ stable part of $(\mathbb{N}, *)$ ($\Rightarrow *$ is an op on D_n)

The commutativity and associativity are inherited from $(\mathbb{N}, *)$.

$0 \notin D_n \Rightarrow$ we will look for a different neutral element.

$n \in D_n$ and $\forall x \in D_n$: $x * n = \gcd(x,n) = x$

$\Rightarrow n$ is the neutral element of $(D_n, *)$

(iii) $D_6 = \{1, 2, 3, 6\}$

*	1	2	3	6
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
6	1	2	3	6

6. Determine the <u>finite</u> stable subsets of $(\mathbb{Z}, \cdot)$.

<u>Sol.</u>: Let $S \subseteq \mathbb{Z}$ be such a subset

If $x \in S$ with $x \notin \{-1, 0, 1\}$: $\{x^k \mid k \in \mathbb{N}\} \subseteq S$

$\{x^k \mid k \in \mathbb{N}\}$ infinite $\Rightarrow$ S infinite $\Rightarrow$ contradiction

$\Rightarrow$ $S \subseteq \{-1, 0, 1\}$

If $|S| = 1$ $\Rightarrow$ $S = \{0\}$ or $S = \{1\}$

If $|S| = 2$ $\Rightarrow$ $S = \{-1, 1\}$ or $S = \{0, 1\}$

If $|S| = 3$ $\Rightarrow$ $S = \{-1, 0, 1\}$