25.10.2021

Seminar W4 – 916

Def: $(V,+)$ abelian group, $(K,+,\cdot)$ field, $\cdot : K \times V \to V$
$(k,v) \mapsto kv$
external operation

V is a <u>K-vector space</u> if:

- $\forall \alpha \in K, \forall v_1, v_2 \in V$: $\alpha \cdot (v_1+v_2) = \alpha v_1 + \alpha v_2$
- $\forall \alpha, \beta \in K, \forall v \in V$: $(\alpha+\beta)\cdot v = \alpha v + \beta v$
- $\forall \alpha, \beta \in K, \forall v \in V$: $(\alpha\beta) v = \alpha \cdot (\beta v)$
- $\forall v \in V$: $1 \cdot v = v$

Ex.: $\mathbb{R}^2$, $\mathbb{R}^n$, K^n, $K^A = \{f: A \to K\}$ with $A \neq \emptyset$, $\mathcal{M}_{mn}(K)$, $K[X]$, $\mathcal{C}([a,b])$, $\mathcal{C}^n([a,b])$

Def./Th.: V K-vector space, $S \subseteq V$

$S \leq_K V \iff$
(i) $S \neq \emptyset$
(ii) $\forall v_1, v_2 \in S : v_1 + v_2 \in S$
(iii) $\forall k \in K, \forall v \in S : kv \in S$

("S is a K-subspace of V")

(ii) and (iii) $\iff$ $\forall k_1, k_2 \in K$, $\forall v_1, v_2 \in S$: $k_1 v_1 + k_2 v_2 \in S$

4. Let $V = \{x \in \mathbb{R} \mid x > 0\}$ and define the operations: $x \perp y = xy$ and $k \top x = x^k$, $\forall k \in \mathbb{R}$ and $\forall x, y \in V$. Prove that V is a vector space over $\mathbb{R}$.

$\perp$ = "eet"
$\top$ = "tee"

Sol.: Let $k_1, k_2 \in \mathbb{R}$, $x_1, x_2 \in V$

- $k \top (x_1 \perp x_2) \overset{?}{=} (k \top x_1) \perp (k \top x_2)$

$k \top (x_1 \perp x_2) = k \top (x_1 x_2) = (x_1 x_2)^k$

$(k \top x_1) \perp (k \top x_2) = x_1^k \perp x_2^k = x_1^k \cdot x_2^k = (x_1 x_2)^k =$

$= k \top (x_1 \perp x_2)$

- $(k_1 + k_2) \top x = x^{k_1 + k_2}$

$(k_1 \top x) \perp (k_2 \top x) = x^{k_1} \perp x^{k_2} = x^{k_1} \cdot x^{k_2} = x^{k_1 + k_2} = (k_1 + k_2) \top x$

- $(k_1 k_2) \top x = x^{k_1 k_2}$

$k_1 \top (k_2 \top x) = k_1 \top (x^{k_2}) = (x^{k_2})^{k_1} = x^{k_1 k_2} = (k_1 k_2) \top x$

- $1 \top x = x^1 = x$

$(V, \perp)$ abelian group (homework)

8. Which ones of the following sets are subspaces:

(*i*) $[-1,1]$ of the real vector space $\mathbb{R}$;

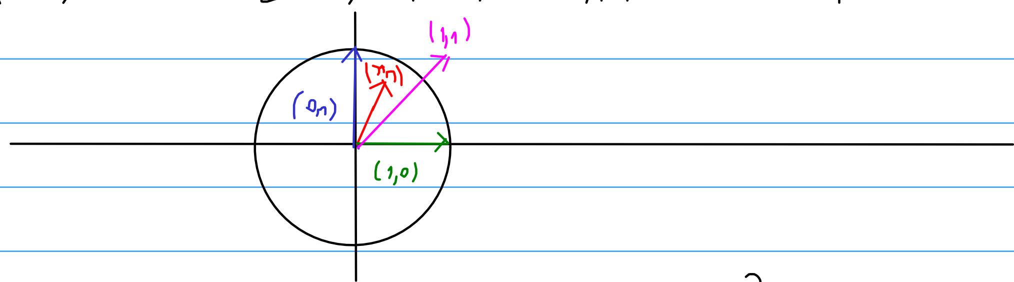
(*ii*) $\underbrace{\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 1\}}_{S_2}$ of the real vector space $\mathbb{R}^2$;

(*iii*) $\left\{\begin{pmatrix} a & b \\ 0 & c\end{pmatrix} \middle| a,b,c\in\mathbb{Q}\right\} = S_3$ of ${}_{\mathbb{Q}}M_2(\mathbb{Q})$ or of ${}_{\mathbb{R}}M_2(\mathbb{R})$;

(*iv*) $\underbrace{\{f:\mathbb{R}\to\mathbb{R} \mid f \text{ continuous}\}}_{S_4}$ of the real vector space $\mathbb{R}^{\mathbb{R}}$?

Sol: (i) $k = 100$, $x = 1 \in [-1,1]$, $kx = 100 \notin [-1,1] \Rightarrow [-1,1] \not\leq_{\mathbb{R}} \mathbb{R}$

(ii) $(0,1),(1,0) \in S_2$, $(0,1)+(1,0) = (1,1) \notin S_2 \Rightarrow S_2 \not\leq_{\mathbb{R}} \mathbb{R}^2$



(iii) $S_3 = \left\{\begin{pmatrix} a & b \\ 0 & c\end{pmatrix} \middle| a,b,c\in\mathbb{Q}\right\} \overset{?}{\leq_{\mathbb{Q}}} M_2(\mathbb{Q})$

$A_1 = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1\end{pmatrix} \in S_3$, $A_2 = \begin{pmatrix} a_2 & b_2 \\ 0 & c_2\end{pmatrix} \in S_3$

$$A_1 + A_2 = \begin{pmatrix} a_1+a_2 & b_1+b_2 \\ 0 & c_1+c_2\end{pmatrix} \in S_3$$

Let $k \in \mathbb{Q}$: $k\cdot A_1 = \begin{pmatrix} ka_1 & kb_1 \\ 0 & kc_1\end{pmatrix} \in S_3$

$\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} \in S_3 \Rightarrow S_3 \neq \emptyset$

$$\Rightarrow S_3 \leq_{\mathbb{Q}} M_2(\mathbb{Q})$$

$$S_3 = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \,\middle|\, a,b,c \in \mathbb{Q} \right\} \overset{?}{\leq_{\mathbb{R}}} M_2(\mathbb{R})$$

$$\sqrt{2} \in \mathbb{R}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in S_3, \quad \sqrt{2} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \notin S_3$$

$$\Rightarrow S_3 \not\leq_{\mathbb{R}} M_2(\mathbb{R})$$

(iv) $S_4 = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous} \}$

$$\mathrm{id}_{\mathbb{R}} \in S_4 \Rightarrow S_4 \neq \emptyset$$

$\forall \alpha, \beta \in \mathbb{R}, \ \forall f, g \in S_4 : \quad \alpha f + \beta g$ continuous

$$\forall x_0 \in \mathbb{R} : \quad \lim_{x \to x_0} (\alpha f + \beta g) \overset{f,g \text{ continuous}}{=} \alpha f(x_0) + \beta g(x_0) = (\alpha f + \beta g)(x_0)$$

$\Rightarrow \alpha f + \beta g$ continuous

9. Which ones of the following sets are subspaces of the K-vector space $K[X]$:
(*i*) $K_n[X] = \{f \in K[X] \mid \text{degree}(f) \leq n\}$ $(n \in \mathbb{N})$;
(*ii*) $K'_n[X] = \{f \in K[X] \mid \text{degree}(f) = n\}$ $(n \in \mathbb{N})$.

Sol.: $f = a_n X^n + \dots + a_1 X + a_0, \quad a_n \neq 0, \quad \deg f := n$

$f, g \in K[X]$: $\deg(fg) = \deg f + \deg g$

$f \in K[X], \alpha \in K$: $\deg(\alpha f) = \begin{cases} \deg f, & \alpha \neq 0 \\ -\infty, & 0 \end{cases}$ $\qquad \deg(0) = -\infty$

$\forall \alpha \in K \setminus \{0\}$: $\deg(\alpha) = 0$

$f, g \in K[X]$ $\deg(f+g) \leq \max(\deg f, \deg g)$

(i) Let $k_1, k_2 \in K$, $f, g \in K_n[X]$

If $k_1 = k_2 = 0$: $k_1 f + k_2 g = 0 \in K_n[X]$, because
$\deg(0) = -\infty$

If $k_1 = 0$, $k_2 \neq 0$: $k_1 f + k_2 g = k_2 g$

$\deg(k_2 g) = \deg g \leq n$ ($g \in K_n[X]$)

$\Rightarrow k_1 f + k_2 g \in K_n[X]$

If $k_1 \neq 0$, $k_2 = 0$, same as before

If $k_1 \neq 0$, $k_2 \neq 0$: $\deg(k_1 f + k_2 g) \leq \max(\deg(k_1 f), \deg(k_2 g))$

$= \max(\underbrace{\deg f}_{\leq n}, \underbrace{\deg g}_{\leq n}) \leq n$

$\Rightarrow \forall k_1, k_2 \in K, \forall f, g \in K_n[X] : \alpha f + \beta g \in K_n[X]$

$0 \in K_n[X] \Rightarrow K_n[X] \neq \emptyset$

$\Rightarrow K_n[X] \leq_K K[X]$

(ii) $K_n'[X] = \{ f \in K[X] \mid \deg f = n \}$

$\forall f, g \in K_n'[X]$, $\alpha = \beta = 0$: $\alpha f + \beta g = 0 \notin K_n'[X]$

7. Which ones of the following sets are subspaces of the real vector space $\mathbb{R}^3$:
(i) $A = \{(x,y,z) \in \mathbb{R}^3 \mid x = 0\}$;
(ii) $B = \{(x,y,z) \in \mathbb{R}^3 \mid x = 0 \text{ or } z = 0\}$;
(iii) $C = \{(x,y,z) \in \mathbb{R}^3 \mid x \in \mathbb{Z}\}$; ✓
(iv) $D = \{(x,y,z) \in \mathbb{R}^3 \mid x + y + z = 0\}$;
(v) $E = \{(x,y,z) \in \mathbb{R}^3 \mid x + y + z = 1\}$; ✓
(vi) $F = \{(x,y,z) \in \mathbb{R}^3 \mid x = y = z\}$? ✓

Sol: (iii) $\frac{1}{2} \in \mathbb{R}, \quad (1,0,0) \in C$

$\frac{1}{2} \cdot (1,0,0) = (\frac{1}{2}, 0, 0) \notin C$

$\Rightarrow C \not\leq_{\mathbb{R}} \mathbb{R}^3$

(v) $0 \in \mathbb{R}, \quad (1,0,0) \in E$

$0 \cdot (1,0,0) = (0,0,0) \notin E$

(vi) $F = \{(x,y,z) \in \mathbb{R}^3 \mid x = y = z\}$

$(0,0,0) \in F \Rightarrow F \neq \emptyset$

$v = (a,a,a) \in F, \quad w = (b,b,b) \in F$

$v + w = (a+b, a+b, a+b) \in F$

$\forall k \in \mathbb{R}: \quad k \cdot v = (ka, ka, ka) \in F$

$\Rightarrow F \leq_{\mathbb{R}} \mathbb{R}^3$