09.11.2021

Seminar W7 – 911

Th.: V K-vector space, $\dim_K V = n$. $B = (v_1, \dots, v_n)$ a list of vectors

Then:

B basis $\Leftrightarrow$ B linearly independent $\Leftrightarrow$ B system of generators for V

1.* Determine a basis and the dimension of the following subspaces of the real vector space $\mathbb{R}^3$:

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$$
$$B = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y + z = 0\}$$
$$C = \{(x, y, z) \in \mathbb{R}^3 \mid \cancel{x = y = z}\}.$$
$$\begin{cases} x + 2y = 0 \\ z + y = 0 \end{cases}$$

Sol: $B = \{(x,y,z) \in \mathbb{R}^3 \mid z = -x - 2y\} = \{(x, y, -x-2y) \mid x, y \in \mathbb{R}\} =$

$= \{x \cdot (1, 0, -1) + y \cdot (0, 1, -2) \mid x, y \in \mathbb{R}\} =$

$= \langle (1,0,-1), (0,1,-2) \rangle$

$b = ((1,0,-1), (0,1,-2))$ basis, because B linearly independent

$\dim B = 2$

"f is a linear map between V and W"

Def: V, W K-v.s., $\underbrace{f \in \mathrm{Hom}_K(V, W)}$, Then:

$$\underset{(\text{"kernel"})}{\mathrm{Ker} f} = \{ v \in V \mid f(v) = 0_W \}$$

$$\mathrm{Im} f = \{ f(v) \mid v \in V \} = \{ w \in W \mid \exists v \in V : f(v) = w \}$$

Example: $f: \mathbb{R}^3 \to \mathbb{R}^3$ Find a basis for $\mathrm{Ker} f$ and $\mathrm{Im} f$.
$(x,y,z) \mapsto (x+2y, y+x, z)$

$$\mathrm{Ker} f = \{ (x,y,z) \in \mathbb{R}^3 \mid (x+2y, y+x, z) = (0,0,0) \} =$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{cases} x+2y = 0 \\ y+x = 0 \\ z = 0 \end{cases} \right\} =$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{cases} z = 0 \\ y = -x \\ y = -\frac{x}{2} \end{cases} \right\} = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{cases} z = 0 \\ y = 0 \\ x = 0 \end{cases} \right\} =$$

$$= \{ (0,0,0) \} \Rightarrow \mathrm{Ker} f = 0$$

$$\mathrm{Im} f = \{ (x+2y, y+x, z) \mid x,y,z \in \mathbb{R} \} =$$

$$= \{ x \cdot (1,1,0) + y \cdot (2,1,0) + z \cdot (0,0,1) \mid x,y,z \in \mathbb{R} \} =$$

$$= \langle (1,1,0), (2,1,0), (0,0,1) \rangle$$

Sneak peek: $v_1, v_2, \ldots, v_n \in_K V$, $\mathrm{rank}(v_1, \ldots, v_n) :=$ max # of lin. indep. vectors among $v_1, v_2, \ldots, v_n$

If $V = K^n$, then $\mathrm{rank}(v_1, \ldots, v_n) = \mathrm{rank} \begin{pmatrix} v_1 \\ \hline v_2 \\ \hline \vdots \\ \hline v_n \end{pmatrix} = \mathrm{rank}(v_1 | v_2 | \ldots | v_n)$

To decide if $(1,1,0)$, $(2,1,0)$, $(0,0,1)$ are lin. indep, we can either use the definition of linear independence, or just calculate the rank of the matrix formed by the vectors.

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$|M| = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1 \neq 0 \Rightarrow \text{rank } M = 3 \Rightarrow$$

$\Rightarrow (1,1,0), (2,1,0), (0,0,1)$ are linearly independent $\Rightarrow$ they form a basis of $\operatorname{Im} f$

4. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $f(x, y, z) = (y, -x)$. Prove that f is an $\mathbb{R}$-linear map and determine a basis and the dimension of $Ker\, f$ and $Im\, f$.

Sol.: $u_1, u_2 \in \mathbb{R}^3$, $u_1 = (x_1, y_1, z_1)$, $u_2 = (x_2, y_2, z_2)$

$$\underline{f(u_1 + u_2) \overset{?}{=} f(u_1) + f(u_2)}$$

$$f(u_1 + u_2) = f(x_1 + x_2, y_1 + y_2, z_1 + z_2) = (y_1 + y_2, -(x_1 + x_2))$$

$$f(u_1) + f(u_2) = f(x_1, y_1, z_1) + f(x_2, y_2, z_2) = (y_1, -x_1) + (y_2, -x_2)$$

$$\Rightarrow f(u_1 + u_2) = f(u_1) + f(u_2)$$

$$\underline{f(k u_1) \overset{?}{=} k f(u_1)}$$

$$f(k u_1) = f(k(x_1, y_1, z_1)) = f(kx_1, ky_1, kz_1) = (ky_1, -kx_1) =$$
$$= k \cdot (y_1, -x_1) = k f(v_1)$$
$$\Rightarrow f \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2)$$
$$\operatorname{Ker} f = \{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = (0,0)\} =$$
$$= \{(x,y,z) \in \mathbb{R}^3 \mid (y, -x) = (0,0)\} =$$
$$= \{(x,y,z) \in \mathbb{R}^3 \mid y = x = 0\} =$$
$$= \{(0,0,z) \mid z \in \mathbb{R}\} = \{z \cdot (0,0,1) \mid z \in \mathbb{R}\}$$
$$= \langle (0,0,1) \rangle, \quad B = ((0,0,1)) \Rightarrow \dim \operatorname{Ker} f = 1$$
$$\operatorname{Im} f = \{(y, -x) \mid x, y \in \mathbb{R}\} = \{y \cdot (1,0) + x \cdot (0,-1) \mid x, y \in \mathbb{R}\} =$$
$$= \langle (1,0), (0,-1) \rangle$$
$$\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1 \neq 0 \Rightarrow (1,0), (0,-1) \text{ are linearly independent}$$
$$\Rightarrow B = ((1,0), (0,1)) \text{ basis for } \operatorname{Im} f \Rightarrow \dim \operatorname{Im} f = 2$$

Th (Steinitz) (rephrased): V K - vector space, $\dim V = n$

If $u_1, u_2, \dots, u_m \in V$, $m \leq n$ are linearly independent,

then $\exists\, u_{m+1}, u_{m+2}, \dots, u_n \in V$ so that

$\mathcal{B} = (u_1, u_2, \dots, u_m, u_{m+1}, \dots, u_n)$ basis for V

In order to complete a linearly independent family $u_1, \dots, u_m$ to a basis:

- We choose $u_{m+1} \in V \setminus \langle u_1, \dots, u_m \rangle$
- Do we have enough vectors (n) ?

Yes — No

Yes, basis! — We choose $u_{m+2} \in V \setminus \langle u_1, \dots, u_m, u_{m+1} \rangle$

Yes — No

6. Complete the bases of the subspaces from Exercise **1.** to some bases of the real vector space $\mathbb{R}^3$ over $\mathbb{R}$.

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$$
$$B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$$
$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}.$$

...

Sol: $A = \langle (1,0,0), (0,1,0) \rangle$

$(0,0,1) \notin A \Rightarrow (1,0,0), (0,1,0), (0,0,1)$ lin. indep $\Rightarrow$

$\Rightarrow \big((1,0,0),\ (0,1,0),\ (0,0,1)\big)$ basis for $\mathbb{R}^3$

$B = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0\}$

$= \langle (1,0,-1),\ (0,1,-1) \rangle$

We choose to add $(0,0,1)$, because $\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$, hence

$(1,0,-1), (0,1,-1), (0,0,1)$ lin. indep $\overset{\dim \mathbb{R}^3 = 3}{\Longrightarrow}$ this is a basis for $\mathbb{R}^3$

$C = \{(x,y,z) \in \mathbb{R}^3 \mid x=y=z\} =$

$= \langle (1,1,1) \rangle$

We first add $(0,1,0) \in \mathbb{R}^3 \setminus C$. So $(1,1,1)$ and $(0,1,0)$

are linearly independent. To complete this to a basis of $\mathbb{R}^3$,

we need another vector, which must not belong to $\langle (1,1,1), (0,1,0) \rangle$

$\langle (1,1,1), (0,1,0) \rangle = \{ a(1,1,1) + b\cdot(0,1,0) \mid a,b \in \mathbb{R} \} =$

$= \{ (a, a+b, a) \mid a,b \in \mathbb{R} \} =$ we could just stop here and choose a vector not of the form $(a, a+b, a)$

$= \left\{ (x,y,z) \,\middle|\, \begin{cases} x=a \\ y=a+b \\ z=a \end{cases}, a,b \in \mathbb{R} \right\} =$

$= \left\{ (x,y,z) \,\middle|\, \begin{matrix} a=x \\ b=y-a \\ z=x \end{matrix} \right\} =$

$= \{ (x,y,z) \mid z = x \}$

We can just choose, for instance

$(1,0,2) \notin \mathbb{R}^3 \setminus < (1,1,1), (0,1,0) >$

$\Rightarrow (1,0,2), (1,1,1), (0,1,0)$ lin. indep $\Rightarrow$ they form a basis of $\mathbb{R}^3$

<u>Th. (1^{st} dim. theorem)</u>: $f: V \to W$ K-linear map, then:

$$\dim V = \dim(\ker f) + \dim(\operatorname{Im} f)$$

<u>Th. (2^{nd} dim theorem)</u> V K-v.s., $S, T \leq_K V$:

$$\dim(S+T) = \dim(S) + \dim(T) - \dim(S \cap T)$$

9. Consider the subspaces

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\},$$

$$T =< (0, 1, 1), (1, 1, 0) >$$

of the real vector space $\mathbb{R}^3$. Determine $S \cap T$ and show that $S + T = \mathbb{R}^3$.

<u>Sol.</u>: $T = \{ a \cdot (0,1,1) + b \cdot (1,1,0) \mid a, b \in \mathbb{R} \} =$

$= \{ (b, a+b, a) \mid a, b \in \mathbb{R} \} =$

$= \{ (x,y,z) \in \mathbb{R}^3 \mid \begin{cases} x = b \\ y = a+b \\ z = a \end{cases} \} =$

$= \{ (x,y,z) \in \mathbb{R}^3 \mid y = x + z \}$

$$S \cap T = \{ (x,y,z) \in \mathbb{R}^3 \mid y = x+z \text{ and } x = 0 \} =$$

$$= \{ (0,y,y) \in \mathbb{R}^3 \mid y \in \mathbb{R} \} =$$

$$= \langle (0,1,1) \rangle \Rightarrow \dim(S \cap T) = 1$$

In order to show that $S+T = \mathbb{R}^3$, since $S+T \leq \mathbb{R}^3$,

it suffices to show that $\dim(S+T) = \dim(\mathbb{R}^3) = 3$

$$S = \{ (x,y,z) \in \mathbb{R}^3 \mid x = 0 \}$$

because $(0,1,1)$ and $(1,1,0)$ are lin. indep.

$$T = \langle (0,1,1), (1,1,0) \rangle \Rightarrow \dim T = 2$$

because $(0,1,0)$ and $(0,0,1)$ are lin. indep.

$$S = \langle (0,1,0), (0,0,1) \rangle \Rightarrow \dim S = 2$$

$$\dim(S+T) = \dim S + \dim T - \dim(S \cap T) = 2+2-1 = 3$$

$$\Rightarrow S+T = \mathbb{R}^3$$