20.10.2021

Seminar W4 – 917

Def: $(V,+)$ abelian group, $(K,+,\cdot)$ field

$\cdot : K \times V \to V$ external operation

$(k,v) \longmapsto k\cdot v$

- $\forall \alpha,\beta \in K, \forall v \in V : (\alpha+\beta)\cdot v = \alpha v + \beta v$
- $\forall \alpha \in K, \forall v_1,v_2 \in V : \alpha\cdot(v_1+v_2) = \alpha v_1 + \alpha v_2$
- $\forall \alpha,\beta \in K, \forall v \in V : (\alpha\beta)\cdot v = \alpha\cdot(\beta v)$

$\left(v\cdot(\alpha\beta) = (v\alpha)\beta \right)$

- $\forall v \in V : 1\cdot v = v$

V K-vector space

$\Downarrow$ not

${}_K V$

Ex.: ${}_K K[X]$, ${}_K \mathcal{M}_{m,n}(K)$, ${}_K K^A$, ${}_K K^n$

3. Let K be a field, $A \neq \emptyset$ and denote $K^A = \{f \mid f : A \to K\}$. Show that K^A is a K-vector space, where the addition and the scalar multiplication are defined as follows: $\forall f,g \in K^A$, $\forall k \in K$, $f+g \in K^A$, $kf \in K^A$,

$$(f+g)(x) = f(x)+g(x)\,, \quad (k\cdot f)(x) = k\cdot f(x)\,, \forall x \in A\,.$$

Sol.: [To prove that for $f,g : A \to B$ we have $f=g$, we must show that $\forall x \in A : f(x) = g(x)$]

$(K^A,+)$ abelian group.

$\forall f,g \in K^A : f+g \in K^A$

- assoc. of $+$: $\forall f, g, h \in K^A$: $(f+g)+h = f+(g+h)$

Let $x \in A \Rightarrow ((f+g)+h)(x) = (f+g)(x) + h(x) =$

$+$ is associative in K

$= (f(x)+g(x)) + h(x) = f(x) + (g(x)+h(x)) =$

$= f(x) + (g+h)(x) = (f+(g+h))(x)$

$\Rightarrow (f+g)+h = f+(g+h)$

- neutral element for $+$: $0_{K^A} : A \to K$, $x \mapsto 0_K$

$\forall f \in K^A, \forall x \in A$: $(f+0_{K^A})(x) = f(x) + 0_{K^A}(x) = f(x) =$

$= 0_{K^A}(x) + f(x) = (0_{K^A}+f)(x)$

- invertibility of $+$: Let $f \in K^A$, we define $-f : A \to K$, $x \mapsto -f(x)$

$\Rightarrow f+(-f) = (-f)+f = 0_{K^A}$

- commutativity of $+$: $\forall f, g \in K^A, \forall x \in A$:

$+$ is commutative in K

$(f+g)(x) = f(x)+g(x) = g(x)+f(x) = (g+f)(x)$

$\Rightarrow f+g = g+f$

– $\forall \alpha, \beta \in K, \forall f \in K^A$: $(\alpha+\beta) f \overset{?}{=} \alpha f + \beta f$

distributivity in K

Let $x \in A$: $((\alpha+\beta) f)(x) = (\alpha+\beta) \cdot f(x) = \alpha f(x) + \beta f(x) = (\alpha f)(x) + (\beta f)(x)$

$$\Rightarrow \quad (\alpha+\beta) f = \alpha f + \beta f$$

$- \ \forall \alpha \in K, \ \forall f_1, f_2 \in K^A : \quad \alpha (f_1 + f_2) \overset{?}{=} \alpha f_1 + \alpha f_2$

Let $x \in A$: $\quad (\alpha \cdot (f_1+f_2))(x) = \alpha \cdot (f_1+f_2)(x) = \alpha \cdot (f_1(x) + f_2(x)) =$

$$\overset{\text{distributivity in } K}{=} \alpha f_1(x) + \alpha f_2(x) = (\alpha f_1)(x) + (\alpha f_2)(x)$$

$$\Rightarrow \quad \alpha (f_1+f_2) = \alpha f_1 + \alpha f_2$$

$- \ \forall \alpha, \beta \in K, \ \forall f \in K^A : \quad (\alpha\beta) f = \alpha \cdot (\beta \cdot f)$

Let $x \in A$: $\quad ((\alpha\beta) f)(x) = (\alpha\beta) \cdot f(x) \overset{\cdot \text{ associative in } K}{=} \alpha \cdot (\beta \cdot f(x)) = \alpha \cdot ((\beta f)(x)) =$

$$= (\alpha \cdot (\beta f))(x) \quad \Rightarrow \quad (\alpha\beta) f = \alpha \cdot (\beta f)$$

$- \ \forall f \in K^A : \quad 1 f \overset{?}{=} f$

Let $x \in A$: $\quad (1 \cdot f)(x) = 1 \cdot f(x) \overset{1 \text{ is the identity in } K}{=} f(x)$

$$\Rightarrow \quad 1 \cdot f = f$$

Remark: $K^A = \{ f : A \to K \}$

- if $|A| = n$, then $K^A \simeq K^n$

$$A = \{ a_1, a_2, \dots, a_n \}, \quad f \in K^A$$

$$\begin{aligned} f : A &\to K \\ a_1 &\mapsto x_1 \\ a_2 &\mapsto x_2 \\ &\vdots \\ a_n &\mapsto x_n \end{aligned}$$

Notation: $f = (x_1, x_2, \dots, x_n)$

- if A is countably infinite, $A = \mathbb{N}$

$\Rightarrow$ $f \in K^A$ can be identified with a sequence

$$f = f(1), f(2), f(3), \dots$$

Def/Th. V K-vector space, $S \subseteq V$

$S \leq_K V$ (S is a K-subspace of V) $\Leftrightarrow$ (i) $S \neq \emptyset$

(ii) $(S,+) \leq (V,+)$: $\forall x, y \in S$: $x - y \in S$

(iii) the external op is well defined on S : $\forall k \in K$, $\forall v \in S$: $k \cdot v \in S$

$\to$ $\forall \alpha, \beta \in K$, $\forall x, y \in S$: $\alpha x + \beta y \in S$

7. Which ones of the following sets are subspaces of the $\underbrace{\text{real vector space } \mathbb{R}^3}_{\mathbb{R}^3_{\mathbb{R}}}$:

(i) $A = \{(x,y,z) \in \mathbb{R}^3 \mid x = 0\}$;

(ii) $B = \{(x,y,z) \in \mathbb{R}^3 \mid x = 0 \text{ or } z = 0\}$;

(iii) $C = \{(x,y,z) \in \mathbb{R}^3 \mid x \in \mathbb{Z}\}$;

(iv) $D = \{(x,y,z) \in \mathbb{R}^3 \mid x + y + z = 0\}$;

(v) $E = \{(x,y,z) \in \mathbb{R}^3 \mid x + y + z = 1\}$;

(vi) $F = \{(x,y,z) \in \mathbb{R}^3 \mid x = y = z\}$?

Sol: (i) $A \neq \emptyset$, because $(0,1,2) \in A$

Let $v_1 = (x_1, y_1, z_1)$, $v_2 = (x_2, y_2, z_2) \in A \Rightarrow x_1 = 0$, $x_2 = 0$

$\Rightarrow v_1 - v_2 = (\underbrace{x_1 - x_2}_{=0}, y_1 - y_2, z_1 - z_2) = (0, y_1 - y_2, z_1 - z_2) \in A$

$\forall k \in \mathbb{R}$, $\forall v = (x,y,z) \in A \overset{?}{:} kv \in A$

$k \cdot v = (\underbrace{kx}_{=0}, ky, kz) = (0, ky, kz) \in A$

$\Rightarrow A \leq_{\mathbb{R}} \mathbb{R}^3$

(ii) $B = \{(x,y,z) \in \mathbb{R}^3 \mid x = 0 \text{ or } z = 0\}$

$$\left.\begin{aligned}&(0,1,3)\in B,\quad (5,6,0)\in B\\&(0,1,3)-(5,6,0)=(-5,-5,3)\notin B\end{aligned}\right\}\Rightarrow B\not\leq \mathbb{R}^3$$

(iii) $C=\{(x,y,z)\in\mathbb{R}^3 \mid x\in\mathbb{Z}\}$

$(1,2,3)\in C$, but $(1.02)\cdot(1,2,3)=(1.02,\ 2.04,\ 3.06)\notin C$

$\Rightarrow C\not\leq \mathbb{R}^3$

(iv) $D=\{(x,y,z)\in\mathbb{R}^3 \mid x+y+z=0\}$

$(-1,1,0)\in D\Rightarrow D\neq\emptyset$

Let $u_1=(x_1,y_1,z_1)$, $u_2=(x_2,y_2,z_2)\in D\Rightarrow x_1+y_1+z_1=x_2+y_2+z_2=0$

$u_1-u_2=(x_1-x_2,\ y_1-y_2,\ z_1-z_2)$

$(x_1-x_2)+(y_1-y_2)+(z_1-z_2)=(x_1+y_1+z_1)-(x_2+y_2+z_2)=0$

$\Rightarrow u_1-u_2\in D$

Let $k\in\mathbb{R}$, $u=(x,y,z)\in D\Rightarrow x+y+z=0$

$k\cdot u=(kx,ky,kz)$

$kx+ky+kz=k(x+y+z)=0$

$\Rightarrow ku\in D\Rightarrow D\leq\mathbb{R}^3$

(v) $E = \{ (x,y,z) \in \mathbb{R}^3 \mid x+y+z=1 \}$

$(-1, 2, 0) \in E$, $(-2, 4, -1) \in E$

$(-1, 2, 0) - (-2, 4, -1) = (1, -2, 1) \notin E$

$\Rightarrow E \leq \mathbb{R}^3$

(vi) $F = \{ (x,y,z) \in \mathbb{R}^3 / x = y = z \}$

$(0,0,0) \in F$, so $F \neq \emptyset$

Let $u_1 = (x_1, y_1, z_1)$, $u_2 = (x_2, y_2, z_2) \in F$ and $\alpha_1, \alpha_2 \in \mathbb{R}$

$\alpha_1 u_1 + \alpha_2 u_2 \overset{?}{\in} F$

$\alpha_1 u_1 + \alpha_2 u_2 = (\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 y_1 + \alpha_2 y_2, \alpha_1 z_1 + \alpha_2 z_2)$

$u_1, u_2 \in F \Rightarrow x_1 = y_1 = z_1$, $x_2 = y_2 = z_2$

$\Rightarrow \alpha_1 x_1 + \alpha_2 x_2 = \alpha_1 y_1 + \alpha_2 y_2 = \alpha_1 z_1 + \alpha_2 z_2$

$\Rightarrow \alpha u_1 + \beta u_2 \in F \Rightarrow F \leq \mathbb{R}^3$