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Seminar W6 – 911

Def: V K-vector space.

We say that $v_1, v_2, \dots, v_n \in V$ are <u>linearly independent</u> if:

$\forall \alpha_1, \alpha_2, \dots, \alpha_n \in K$, if $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$, then $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$

Conversely, $v_1, \dots, v_n \in V$ are <u>linearly dependent</u> if:

$\exists \alpha_1, \dots, \alpha_n \in K$, not all zero, so that $\underbrace{\alpha_1 v_1 + \dots + \alpha_n v_n = 0}_{\text{"dependence relationship"}}$

1. Let $v_1 = (1, -1, 0)$, $v_2 = (2, 1, 1)$, $v_3 = (1, 5, 2)$ be vectors in the canonical real vector space $\mathbb{R}^3$. Prove that:
(*i*) v_1, v_2, v_3 are linearly dependent and determine a dependence relationship.
(*ii*) v_1, v_2 are linearly independent.

Sol: We want to find $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ so that:

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

$$\alpha_1 \cdot (1, -1, 0) + \alpha_2 \cdot (2, 1, 1) + \alpha_3 \cdot (1, 5, 2) = (0, 0, 0)$$

$$\Rightarrow \begin{cases} \alpha_1 + 2\alpha_2 + \alpha_3 = 0 \\ -\alpha_1 + \alpha_2 + 5\alpha_3 = 0 \\ \alpha_2 + 2\alpha_3 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_2 = -2\alpha_3 \\ \alpha_1 - 4\alpha_3 + \alpha_3 = 0 \\ -\alpha_1 - 2\alpha_3 + 5\alpha_3 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_2 = -2\alpha_3 \\ \alpha_1 = 3\alpha_3 \\ \alpha_1 = 3\alpha_3 \end{cases}$$

$\Rightarrow \forall \alpha_1, \alpha_2, \alpha_3$ that satisfy these conditions, we can get a dependence relationship:

$\Rightarrow 3v_1 - 2v_2 + v_3 = 0$

(ii) Let $\alpha_1, \alpha_2 \in \mathbb{R}$ so that

$$\alpha_1 v_1 + \alpha_2 v_2 = 0$$

$$\alpha_1 \cdot (1,-1,0) + \alpha_2 \cdot (2,1,1) = 0$$

$$\Rightarrow \begin{cases} \alpha_1 + 2\alpha_2 = 0 \\ -\alpha_1 + \alpha_2 = 0 \\ \alpha_2 = 0 \end{cases} \Rightarrow \alpha_1 = \alpha_2 = 0$$

$\Rightarrow v_1, v_2$ are linearly independent

3. Let $v_1 = (1, a, 0)$, $v_2 = (a, 1, 1)$, $v_3 = (1, 0, a)$ be vectors in ${}_{\mathbb{R}}\mathbb{R}^3$. Determine $a \in \mathbb{R}$ such that the vectors v_1, v_2, v_3 are linearly independent.

Sol: Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ so that:

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

$$\Rightarrow \begin{cases} \alpha_1 + a\alpha_2 + \alpha_3 = 0 \\ a\alpha_1 + \alpha_2 = 0 \\ \alpha_2 + a\alpha_3 = 0 \end{cases} \Leftrightarrow \begin{cases} a(\alpha_1 - \alpha_3) = 0 \\ \alpha_1 + a\alpha_2 + \alpha_3 = 0 \\ a\alpha_1 + \alpha_2 = 0 \end{cases}$$

If $a \neq 0 \Rightarrow \alpha_1 = \alpha_3$: $\begin{cases} 2\alpha_1 + a\alpha_2 = 0 \\ a\alpha_1 + \alpha_2 = 0 \end{cases} \Rightarrow \begin{cases} \alpha_2 = -a\alpha_1 \\ 2\alpha_1 - a^2\alpha_1 = 0 \end{cases} \Rightarrow$

$$\Rightarrow \begin{cases} \alpha_2 = -a\alpha_1 \\ \alpha_1(2 - a^2) = 0 \\ \alpha_3 = \alpha_1 \end{cases}$$

If $a \neq \pm\sqrt{2} \Rightarrow \alpha_1 = 0 \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$ so v_1, v_2, v_3 lin. indep.

If $a = \pm\sqrt{2} \Rightarrow \begin{cases} \alpha_1 = \alpha_1 \\ \alpha_2 = -a\alpha_1 \\ \alpha_3 = \alpha_1 \end{cases}$ in this case v_1, v_2, v_3 lin. dependent

If $a=0$: $\begin{cases} \alpha_1+\alpha_3=0 \\ \alpha_2=0 \end{cases}$ This system is compatible indetermined, therefore v_1, v_2, v_3 are linearly dependent.

Def: V K-vector space. $X \subseteq V$ basis for V if:

- X linearly independent
- X system of generators for V $(V=\langle X\rangle)$

($\dim V$ = # of elements in every basis)

7. Let $E_1=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $E_2=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $E_3=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, $E_4=\begin{pmatrix}0&0\\0&1\end{pmatrix}$, $A_1=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $A_2=\begin{pmatrix}1&1\\0&0\end{pmatrix}$, $A_3=\begin{pmatrix}1&1\\1&0\end{pmatrix}$, $A_4=\begin{pmatrix}1&1\\1&1\end{pmatrix}$. Prove that the lists (E_1,E_2,E_3,E_4) and (A_1,A_2,A_3,A_4) are bases of the real vector space $M_2(\mathbb{R})$ and determine the coordinates of $B=\begin{pmatrix}2&1\\1&0\end{pmatrix}$ in each of the two bases.

Sol: We prove first that (E_1,E_2,E_3,E_4) is lin. indep.

Let $\alpha_1,\alpha_2,\alpha_3,\alpha_4 \in \mathbb{R}$ so that $\alpha_1E_1+\alpha_2E_2+\alpha_3E_3+\alpha_4E_4=0$

$\Rightarrow \alpha_1\cdot\begin{pmatrix}1&0\\0&0\end{pmatrix}+\alpha_2\cdot\begin{pmatrix}0&1\\0&0\end{pmatrix}+\alpha_3\cdot\begin{pmatrix}0&0\\1&0\end{pmatrix}+\alpha_4\cdot\begin{pmatrix}0&0\\0&1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$

$\Rightarrow \begin{pmatrix}\alpha_1&\alpha_2\\\alpha_3&\alpha_4\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix} \Rightarrow \alpha_1=\alpha_2=\alpha_3=\alpha_4=0$

So E_1,E_2,E_3,E_4 lin. indep.

We prove that $M_2(\mathbb{R})=\langle E_1,E_2,E_3,E_4\rangle$

Let $M = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$. We have:

$$M = \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 E_4$$

$$\Rightarrow \mathcal{M}_2(\mathbb{R}) = \langle E_1, E_2, E_3, E_4 \rangle$$

$$\Rightarrow (E_1, E_2, E_3, E_4) \text{ basis for } {}_{\mathbb{R}}\mathcal{M}_2(\mathbb{R})$$

$$\dim \mathcal{M}_2(\mathbb{R}) = 4$$

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$ so that

$$\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \alpha_4 A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \alpha_1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \alpha_3 \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \alpha_4 \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0 \\ \alpha_2 + \alpha_3 + \alpha_4 = 0 \\ \alpha_3 + \alpha_4 = 0 \\ \alpha_1 + \alpha_4 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = -\alpha_4 \\ \alpha_3 = -\alpha_4 \\ \alpha_2 - \alpha_4 + \alpha_4 = 0 \\ -\alpha_4 + \alpha_2 - \alpha_4 + \alpha_4 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = -\alpha_4 \\ \alpha_3 = -\alpha_4 \\ \alpha_2 = 0 \\ \alpha_4 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$$

$$\Rightarrow A_1, A_2, A_3, A_4 \text{ are linearly independent}$$

We will show now that $\mathcal{M}_2(\mathbb{R}) = \langle A_1, A_2, A_3, A_4 \rangle$

Let $M = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$.

$$M = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \alpha_4 A_4$$

$$\Rightarrow \begin{cases} x = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \\ y = \alpha_2 + \alpha_3 + \alpha_4 \\ z = \alpha_3 + \alpha_4 \\ t = \alpha_1 + \alpha_4 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = t - \alpha_4 \\ \alpha_3 = z - \alpha_4 \\ x = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \\ y = \alpha_2 + \alpha_3 + \alpha_4 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_1 = t - \alpha_4 \\ \alpha_3 = z - \alpha_4 \\ x = t - \alpha_4 + \alpha_2 + z - \alpha_4 + \alpha_4 \\ y = \alpha_2 + z - \alpha_4 + \alpha_4 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = t - \alpha_4 \\ \alpha_3 = z - \alpha_4 \\ x = z + t + \alpha_2 - \alpha_4 \\ y = \alpha_2 + z \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_1 = t - \alpha_4 \\ \alpha_3 = z - \alpha_4 \\ \alpha_2 = y - z \\ x = z + t + y - z - \alpha_4 \end{cases} \Leftrightarrow \begin{cases} \alpha_4 = y + t - x \\ \alpha_2 = y - z \\ \alpha_3 = z + x - y - t \\ \alpha_1 = x - y \end{cases}$$

$$\Rightarrow \quad M_2(\mathbb{R}) = \langle A_1, A_2, A_3, A_4 \rangle$$

<u>Not.</u>: If V K-v.s., $B = (v_1, v_2, \dots, v_n)$ basis for V

$\forall v \in V$: the coordinates of v in the basis B are:

$\alpha_1, \alpha_2, \dots, \alpha_n$ (denoted $[v]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$)

if $\boxed{v = \alpha_1 v_1 + \dots + \alpha_n v_n}$

We have to determine $\left[\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\right]_{(E_1, E_2, E_3, E_4)}$, $\left[\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\right]_{(A_1, A_2, A_3, A_4)}$

$$\left[\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\right]_{(E_1, E_2, E_3, E_4)} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \text{ because}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = 2 \cdot E_1 + 1 \cdot E_2 + 1 \cdot E_3 + 0 \cdot E_4$$

$$\left[\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\right]_{(A_1, A_2, A_3, A_4)} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}$$

so that $\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \alpha_4 A_4$

$$\begin{cases} \alpha_4 = y + t - x \\ \alpha_2 = y - z \\ \alpha_3 = z + x - y - t \\ \alpha_1 = x - y \end{cases} \Rightarrow \begin{cases} \alpha_4 = 1 + 0 - 2 = -1 \\ \alpha_2 = 1 - 1 = 0 \\ \alpha_3 = 1 + 2 - 1 - 0 = 2 \\ \alpha_1 = 2 - 1 = 1 \end{cases}$$

$$\Rightarrow \left[\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\right]_{(A_1, A_2, A_3, A_4)} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix}$$