15.12.2021

Seminar W 11 - 917

Def.: V, V' K-v.s., $B = (v_1, v_2, \dots, v_m)$ basis for V
$B' = (v_1', v_2', \dots, v_n')$ basis of V'

$f : V \to V'$ K-linear map.

$$\boxed{[f]_{B,B'} = \left([f(v_1)]_{B'} \;\; [f(v_2)]_{B'} \;\; \dots \;\; [f(v_m)]_{B'} \right)}$$

Prop.: $\forall v \in V$: $\boxed{[f(v)]_{B'} = [f]_{B,B'} \cdot [v]_B}$

Def.: $[id]_{B',B} = T_{B,B'}$ = the base-change matrix from B to B'

$\hookrightarrow \left([v_1']_B \;\; [v_2']_B \;\; \dots \;\; [v_n']_B \right)$

Cor.: $f = id_V \Rightarrow [v]_{B'} = [id]_{B,B'} \cdot [v]_B = T_{B',B} \cdot [v]_B$

Prop.: V, V', V'' K-v.s., B, B', B'' bases of V, V', V'' respectively

$f : V' \to V''$, $g : V \to V'$ K-linear maps

$\Rightarrow f \circ g : V \to V''$ linear map

$$\boxed{[f \circ g]_{B,B''} = [f]_{B',B''} \cdot [g]_{B,B'}}$$

$\rightarrow$ Cor.: V, V' K-v.s. $\quad B_1, B_2$ bases of V

B_1', B_2' bases of V'

$f: V \to V'$ linear map

$$[f]_{B_2, B_2'} = [id_{V'}]_{B_1', B_2'} \cdot [f]_{B_1, B_1'} \cdot [id_V]_{B_2, B_1} =$$

$$= T_{B_2', B_1'} \cdot [f]_{B_1, B_1'} \cdot T_{B_1, B_2}$$

Prop.: $[id]_{B,B'}^{-1} = [id]_{B',B}$ $\rightarrow$ This property does NOT work for any linear map.

$$\left(T_{B',B}^{-1} = T_{B,B'} \right)$$

$[f]_{B,B'}^{-1} \neq [f]_{B',B}$

2. In the real vector space $\mathbb{R}^2$ consider the bases $B = (v_1, v_2) = ((1,2),(1,3))$ and $B' = (v_1', v_2') = ((1,0),(2,1))$ and let $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ having the matrices $[f]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ and $[g]_{B'} = \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}$. Determine the matrices $[2f]_B$, $[f+g]_B$ and $[f \circ g]_{B'}$. (*Use the matrices of change of basis.*)

Sol.: $$[2f]_B = 2 \cdot [f]_B = 2 \cdot \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -2 & -2 \end{pmatrix}$$

$$[f+g]_B = [f]_B + [g]_B$$

$$[g]_B = [id]_{B',B} \cdot [g]_{B'} \cdot [id]_{B,B'}$$

We will now calculate $[id]_{B,B'}$

1st approach (tedious, but easy):

$$[id]_{B,B'} = \left([v_1]_{B'} \;\; [v_2]_{B'} \right)$$

$$[e_1]_{B'} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Rightarrow e_1 = \alpha e_1' + \beta e_2'$$

$$\Rightarrow (1,2) = \alpha \cdot (1,0) + \beta \cdot (2,1)$$

$$\Rightarrow \begin{cases} 1 = \alpha + 2\beta \\ 2 = \beta \end{cases} \Rightarrow \begin{cases} \beta = 2 \\ \alpha = -3 \end{cases} \Rightarrow [e_1]_{B'} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$[e_2]_{B'} = \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} \Rightarrow e_2 = \alpha' \cdot e_1' + \beta' \cdot e_2' \Rightarrow (1,3) = \alpha' \cdot (1,0) + \beta' \cdot (2,1)$$

$$\Rightarrow \begin{cases} 1 = \alpha' + 2\beta' \\ 3 = \beta' \end{cases} \Rightarrow \begin{cases} \beta' = 3 \\ \alpha' = -5 \end{cases} \Rightarrow [e_2]_{B'} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$$

$$\Rightarrow [id]_{B,B'} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$$

2nd approach (slightly fancier) $\quad E = ((1,0), (0,1))$

$$[id]_{B,B'} = [id]_{E,B'} \cdot [id]_{B,E} = [id]_{B',E}^{-1} \cdot [id]_{B,E}$$

$$B = ((1,2), (1,3)), \qquad B' = ((1,0), (2,1))$$

$$[id]_{B,E} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \qquad [id]_{B',E} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow [id]_{B,B'} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$$

$$[id]_{B,B'} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$$

$$[g]_B = [id]_{B',B} \cdot \underbrace{[g]_{B'}}_{\checkmark} \cdot \underbrace{[id]_{B,B'}}_{\checkmark}$$

$$[id]_{B',B} = [id]_{B,B'}^{-1} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}$$

$$\Rightarrow [g]_B = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} \cdot \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix} \cdot \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 4 & -4 \\ -1 & 5 \end{pmatrix} \cdot \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -20 & -32 \\ 13 & 20 \end{pmatrix}$$

$$[f+g]_B = [f]_B + [g]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} + \begin{pmatrix} -20 & -32 \\ 13 & 20 \end{pmatrix} = \begin{pmatrix} -19 & -30 \\ 12 & 19 \end{pmatrix}$$

$$[f \circ g]_{B'} = [f]_{B'} \cdot [g]_{B'}$$

We have to find $[f]_{B'}$.

$$[f]_{B'} = [id]_{B,B'} \cdot [f]_B \cdot [id]_{B',B} =$$

$$= \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} =$$

$$= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} 8 & 13 \\ -5 & -8 \end{pmatrix}$$

$$[f \circ g]_{B'} = \begin{pmatrix} 8 & 13 \\ -5 & -8 \end{pmatrix} \cdot \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 9 & -13 \\ -5 & 9 \end{pmatrix}$$

Eigenvalues and eigenvectors

Def.: V K-v.s., $f: V \to V$ linear map.

$v \in V \setminus \{0\}$ eigenvector for $f \iff \exists \lambda \in K$ (called an eigenvalue):

$$f(v) = \lambda v$$

$$V(\lambda) = \{ v \in V \mid f(v) = \lambda v \}$$

$\hookrightarrow$ eigenspace of f corresp. to λ

Prop: λ eigenvalue for $f \iff \lambda$ is a root of the characteristic polynomial

$$p_f(X) = \det([f]_B - X \cdot I_n)$$

$(\forall B \text{ basis})$

eigenvalues / eigenvectors for a $f \in \mathrm{End}_K(V)$ $\iff$ eigenvalues / eigenvectors for $[f]_B$, $\forall$ basis B

Compute the eigenvalues and the eigenvectors of the (endomorphisms having) matrices:

6. $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Sol.: $$P_A(X) = \begin{vmatrix} -X & 0 & 0 & 1 \\ 0 & -X & 1 & 0 \\ 0 & 1 & -X & 0 \\ 1 & 0 & 0 & -X \end{vmatrix} \overset{L_1 \leftarrow L_1 + X L_4}{=} \begin{vmatrix} 0 & 0 & 0 & 1-X^2 \\ 0 & -X & 1 & 0 \\ 0 & 1 & -X & 0 \\ 1 & 0 & 0 & -X \end{vmatrix} =$$

$$= (-1) \cdot (1-x^2) \cdot \begin{vmatrix} 0 & -x & 1 \\ 0 & 1 & -x \\ 1 & 0 & 0 \end{vmatrix} = (-1) \cdot (1-x^2) \cdot \begin{vmatrix} -x & 1 \\ 1 & -x \end{vmatrix} =$$

$$= (-1) \cdot (1-x^2) \cdot (x^2-1) = (x-1)^2 \cdot (x+1)^2$$

$$\Rightarrow \lambda_1 = 1 \ , \ \lambda_2 = -1$$

Let's find the eigenvectors corresponding to $\lambda_1 = 1$

$$V(\lambda_1) = \{ u = (x,y,z,t) \in \mathbb{R}^4 \mid A \cdot [u]_E = \lambda_1 \cdot [u]_E \} =$$

$$= \left\{ u = (x,y,z,t) \mid (A - \lambda_1 I_4) \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} =$$

$$= \left\{ u = (x,y,z,t) \mid \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} =$$

$$= \left\{ u = (x,y,z,t) \mid \begin{cases} -x + t = 0 \\ -y + z = 0 \\ y - z = 0 \\ x - t = 0 \end{cases} \right\} =$$

$$= \{ u = (x,y,z,t) \mid t = x \ , \ y = z \} =$$

$$= \{ u = (x,y,y,x) \mid x,y \in \mathbb{R} \} =$$

$$= \langle (1,0,0,1), (0,1,1,0) \rangle$$

$$\Rightarrow \dim V(\lambda_1) = 2$$