22.11.2021

Seminar W8 - 914

$$(S): \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn} = b_m \end{cases}$$

. In order to decide if a system is compatible or not:

$$M = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \qquad \overline{M} = \left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right)$$

<u>Th.</u> (Kronecker - Capelli):

(S) is compatible $\Leftrightarrow$ rank M = rank $\overline{M}$

<u>Th.</u> (Rouché):

Let Δ_p be a principal minor (minor inside $\Delta = \det M$ that is of maximal size, rank M), $(\Delta_i)_{i=\overline{1,s}}$ characteristic minors

(e.g. $\Delta_i = \left(\begin{array}{c|c} \Delta_p & \text{column of free terms} \\ \hline \text{row from the matrix} & \end{array}\right)$

(S) compatible $\Leftrightarrow$ $\forall i \in \{1, \dots, s\}$: $\Delta_i = 0$

To find the solutions:

- Find the principal minor Δ_p
- The rows in Δ_p correspond to the principal equations
  The columns in Δ_p correspond to the principal unknowns
- Discard the secondary equations
- Treat the secondary unknowns as parameters (rename them) and move them to the column of free terms
- We are left with a square system, which we solve by using Cramer's rule.

$$(S): \begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{n1}x_1 + \dots + a_{nn}x_n = b_n \end{cases}$$

$$(S) \text{ compatible} \Leftrightarrow \Delta = \det M \neq 0$$

$$\text{If } (S) \text{ compatible} \Rightarrow \forall i \in \{1, \dots, n\} : x_i = \frac{\Delta_{x_i}}{\Delta}$$

$$\Delta_{x_i} = \begin{vmatrix} a_{11} & \dots & a_{1\,i-1} & b_1 & a_{1\,i+1} & \dots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & & a_{n\,i-1} & b_n & a_{n\,i+1} & \dots & a_{nn} \end{vmatrix}$$

8.2, 8.3.

2. Using the Kronecker-Capelli theorem, decide if the following linear systems are compatible and then solve the compatible ones:

$$(i)\begin{cases} x_1+x_2+x_3-2x_4=5 \\ 2x_1+x_2-2x_3+x_4=1 \\ 2x_1-3x_2+x_3+2x_4=3 \end{cases} \qquad (ii)\begin{cases} x_1-2x_2+x_3+x_4=1 \\ x_1-2x_2+x_3-x_4=-1 \\ x_1-2x_2+x_3+5x_4=5 \end{cases}$$

$$(iii)\begin{cases} x+y+z=3 \\ x-y+z=1 \\ 2x-y+2z=3 \\ x+z=4 \end{cases}$$

3. Using the Rouché theorem, decide if the systems from **2.** are compatible and then solve the compatible ones.

Sol.

$$(i)\begin{cases} x_1+x_2+x_3-2x_4=5 \\ 2x_1+x_2-2x_3+x_4=1 \\ 2x_1-3x_2+x_3+2x_4=3 \end{cases}$$

$$M=\begin{pmatrix} 1 & 1 & 1 & -2 \\ 2 & 1 & -2 & 1 \\ 2 & -3 & 1 & 2 \end{pmatrix} \qquad \bar{M}=\left(\begin{array}{cccc|c} 1 & 1 & 1 & -2 & 5 \\ 2 & 1 & -2 & 1 & 1 \\ 2 & -3 & 1 & 2 & 3 \end{array}\right)$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -2 \\ 2 & -3 & 1 \end{vmatrix} = -14 \neq 0 \Rightarrow \text{rank } M = 3 \Rightarrow \text{rank } \bar{M} = 3$$

$$\Rightarrow \text{rank } M = \text{rank } \bar{M} \overset{K-C}{\Rightarrow} \text{System is compatible}$$

Principal equations : all of them

Principal unknowns : x_1, x_2, x_3

Secondary unknowns : x_4

Let $x_4=\alpha$, so the system is:

$$\begin{cases} x_1 + x_2 + x_3 = 5 + 2\alpha \\ 2x_1 + x_2 - 2x_3 = 1 - \alpha \\ 2x_1 - 3x_2 + x_3 = 3 - 2\alpha \end{cases}$$

$$\Delta = -14, \qquad \Delta_{x_1} = \begin{vmatrix} 5+2\alpha & 1 & 1 \\ 1-\alpha & 1 & -2 \\ 3-2\alpha & -3 & 1 \end{vmatrix} =$$

$$= 5+2\alpha - 2(3-2\alpha) - 3(1-\alpha) - (3-2\alpha) - (1-\alpha) - 6(5+2\alpha) =$$

$$= -38$$

$$\Rightarrow x_1 = -\frac{38}{-14} = \frac{19}{7}.$$

Same thing for x_2 and x_3

Solve the following linear systems by the Gauss and Gauss-Jordan methods:

5. (i) $\begin{cases} 2x + 2y + 3z = 3 \\ x - y \quad = 1 \\ -x + 2y + z = 2 \end{cases}$ (ii) $\begin{cases} 2x + 5y + z = 7 \\ x + 2y - z = 3 \\ x + y - 4z = 2 \end{cases}$ (iii) $\begin{cases} x + y + z = 3 \\ x - y + z = 1 \\ 2x - y + 2z = 3 \\ x + z = 4 \end{cases}$

6. $\begin{cases} 2x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 + 2x_2 - x_3 + 4x_4 = 2 \\ x_1 + 5x_2 - 4x_3 + 11x_4 = \lambda \end{cases}$ $(\lambda \in \mathbb{R})$

7. $\begin{cases} ax + y + z = 1 \\ x + ay + z = a \\ x + y + az = a^2 \end{cases}$ $(a \in \mathbb{R})$

row echelon form = "# of zeros before the first nonzero strictly increases from row to row"

Sol.: 5(i) $\left(\begin{array}{ccc|c} 2 & 2 & 3 & 3 \\ 1 & -1 & 0 & 1 \\ -1 & 2 & 1 & 2 \end{array}\right)$

We will use this line to make zeros under the pivot

pivot

$$\left(\begin{array}{ccc|c} 2 & 2 & 3 & 3 \\ 1 & -1 & 0 & 1 \\ -1 & 2 & 1 & 2 \end{array}\right) \underset{L_1 \leftrightarrow L_2}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 2 & 2 & 3 & 3 \\ -1 & 2 & 1 & 2 \end{array}\right) \begin{array}{l} \text{pivot line} \\ \sim \end{array}$$

$$\begin{array}{c} L_2 \leftarrow L_2 - 2L_1 \\ \sim \\ L_3 \leftarrow L_3 + L_1 \end{array} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 4 & 3 & 1 \\ 0 & 1 & 1 & 3 \end{array}\right) \underset{L_2 \leftrightarrow L_3}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & \overset{\text{pivot}}{1} & 1 & 3 \\ 0 & 4 & 3 & 1 \end{array}\right) \begin{array}{l} \text{pivot line} \sim \end{array}$$

$$\underset{\sim}{L_3 \leftarrow L_3 - 4L_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -11 \end{array}\right) \underset{L_3 \leftarrow (-1) L_3}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 11 \end{array}\right)$$

This is a row echelon form

If we use the Gauss method:

→ We revert to the system and we solve it

$$\begin{cases} x - y = 1 \\ y + z = 3 \\ z = 11 \end{cases} \Rightarrow \begin{cases} z = 11 \\ y = 3 - z = -8 \\ x = 1 + y = -7 \end{cases}$$

If we use the Gauss-Jordan method:

→ We continue the process by picking pivots in reverse and making zeros above them:

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 11 \end{array}\right) \underset{\sim}{L_2 \leftarrow L_2 - L_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 11 \end{array}\right) \sim$$

pivot line

pivot line

pivot

$$\underset{\sim}{L_1 \leftarrow L_1 + L_2} \begin{pmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 11 \end{pmatrix}$$

We revert to the system and collect our solutions

$$\Rightarrow \begin{cases} x = -7 \\ y = -8 \\ z = 11 \end{cases}$$

Remark: If the system is incompatible, then in the process of applying Gaussian elimination, we will a line that looks like this:

$$\left(\begin{array}{cccccc|c} 0 & 0 & \cdots & 0 & 0 & 0 & \alpha \end{array}\right) \quad \alpha \neq 0$$

If we get a zero line:

$$\left(\begin{array}{ccccc|c} 0 & 0 & \cdots & 0 & 0 & 0 \end{array}\right)$$

then it just means that the corresponding equation was redundant

6. $\begin{cases} 2x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 + 2x_2 - x_3 + 4x_4 = 2 \\ x_1 + 5x_2 - 4x_3 + 11x_4 = \lambda \end{cases} \qquad (\lambda \in \mathbb{R})$

<u>Sol.</u>: $\left(\begin{array}{cccc|c} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 4 & 2 \\ 1 & 5 & -4 & 11 & \lambda \end{array}\right) \overset{L_1 \leftrightarrow L_2}{\sim} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 5 & -4 & 11 & \lambda \end{array}\right) \sim$

$\overset{L_2 \leftarrow L_2 - 2L_1}{\sim} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 0 & -3 & 3 & -7 & -3 \\ 1 & 5 & -4 & 11 & \lambda \end{array}\right) \overset{L_3 \leftarrow L_3 - L_1}{\sim}$

$\sim \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 0 & -3 & 3 & -7 & -3 \\ 0 & 3 & -3 & 7 & \lambda - 2 \end{array}\right) \overset{L_3 \leftarrow L_3 + L_2}{\sim} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 0 & -3 & 3 & -7 & -3 \\ 0 & 0 & 0 & 0 & \lambda - 5 \end{array}\right)$

If $\lambda - 5 \neq 0 \Rightarrow$ system is incompatible

If $\lambda = 5 \Rightarrow$ the extended matrix becomes:

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 0 & -3 & 3 & -7 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow \begin{cases} x_1 + 2x_2 - x_3 + 4x_4 = 2 \\ -3x_2 + 3x_3 - 7x_4 = -3 \end{cases}$

$\Rightarrow \quad x_3 = \alpha \qquad x_4 = \beta$

$\begin{cases} x_1 + 2x_2 = \alpha - 4\beta + 2 \\ -3x_2 = -3\alpha + 7\beta - 3 \end{cases}$

$$\Rightarrow \quad x_2 = \frac{-3\alpha + 7\beta - 3}{-3}$$

$$x_1 = \alpha - 4\beta + 2 - 2x_2 = \alpha - 4\beta + 2 - \frac{-6\alpha + 14\beta - 6}{-3} =$$

$$= -\alpha + \frac{2\beta}{3}$$

$$\Rightarrow \quad \text{solution}: \quad \begin{cases} x_1 = -\alpha + \frac{2\beta}{3} \\ x_2 = \alpha - \frac{7}{3}\beta + 1 \\ x_3 = \alpha \\ x_4 = \beta \end{cases}$$