23.11.2021

Seminar W 9 – 911

Compute by applying elementary operations the ranks of the matrices:

1. $\begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix}$. **2.** $\begin{pmatrix} 1 & -1 & 3 & 2 \\ -2 & 0 & 3 & -1 \\ -1 & 2 & 0 & -1 \end{pmatrix}$. **3.** $\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix}$ $(\alpha, \beta \in \mathbb{R})$.

Sol.: 1.

$$\begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix} \overset{L_1 \leftrightarrow L_3}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 3 \\ 0 & 2 & 3 \\ 2 & 2 & 4 \end{pmatrix} \underset{L_4 \leftarrow L_4 - 2L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix} \sim$$

$$\overset{L_3 \leftarrow L_3 - L_2}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} \overset{L_4 \leftarrow L_4 - L_3}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ rank M = # of nonzero rows in the row echelon form $= 3$

Compute by applying elementary operations the ranks of the matrices:

1. $\begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix}$. **2.** $\begin{pmatrix} 1 & -1 & 3 & 2 \\ -2 & 0 & 3 & -1 \\ -1 & 2 & 0 & -1 \end{pmatrix}$. **3.** $\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix}$ $(\alpha, \beta \in \mathbb{R})$.

Sol.: 3.

$$\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix} \overset{L_1 \leftrightarrow L_2}{\sim} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ \beta & 1 & 3 & 4 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix} \sim$$

$$L_2 - \beta L_1 \atop L_3 - 2L_1 \sim \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & 1-\beta\alpha & 3-3\beta & 4-3\beta \\ 0 & \alpha & -2 & 1 \end{pmatrix}$$

If $\alpha = 0$: $\begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 3-3\beta & 4-3\beta \\ 0 & 0 & -2 & 1 \end{pmatrix}$ row echelon form

$\Rightarrow$ rank $M = 3$

If $\alpha \neq 0$: $\begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & \alpha & -2 & 1 \\ 0 & 1-\beta\alpha & 3-3\beta & 4-3\beta \end{pmatrix} \sim$

$$L_3 \leftarrow L_3 - \frac{1-\beta\alpha}{\alpha} \cdot L_2 \sim \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & \alpha & -2 & 1 \\ 0 & 0 & 3-3\beta + 2\cdot\frac{1-\beta\alpha}{\alpha} & 4-3\beta - \frac{1-\beta\alpha}{\alpha} \end{pmatrix}$$

$\Rightarrow$ rank $M \in \{2, 3\}$

$$\text{rank } M = 2 \iff \begin{cases} 3-3\beta + \frac{2\cdot(1-\beta\alpha)}{\alpha} = 0 \\ 4-3\beta - \frac{1-\beta\alpha}{\alpha} = 0 \end{cases} \iff$$

$$\iff \begin{cases} 3\alpha - 5\beta\alpha + 1 = 0 \\ 4\alpha - 2\beta\alpha - 1 = 0 \end{cases} \Rightarrow \begin{cases} 7\alpha - 7\beta\alpha = 0 \\ 3\alpha - 5\beta\alpha + 1 = 0 \end{cases} \iff$$

$$\iff \begin{cases} 7\alpha(1-\beta) = 0 \\ 3\alpha - 5\beta\alpha + 1 = 0 \end{cases} \overset{\alpha\neq 0}{\iff} \begin{cases} 1-\beta = 0 \\ 3\alpha - 5\beta\alpha + 1 = 0 \end{cases} \Rightarrow \begin{cases} \beta = 1 \\ -2\alpha + 1 = 0 \end{cases}$$

$\Rightarrow$ rank $M = 2 \iff \alpha = \frac{1}{2}$ $\beta = 1$

rank $M = 3 \iff \alpha \neq \frac{1}{2}$ or $\beta \neq 1$

Compute by applying elementary operations the inverses of the matrices:

4. $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$.   **5.** $\begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}$.

$$\left(A \;\middle|\; I_n\right) \overset{\text{Gauss-Jordan}}{\sim} \;---\; \sim \left(I_n \;\middle|\; A^{-1}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & -2 & 0 & 1 & 0 \\ 2 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \underset{L_3 \leftarrow L_3 - 2L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right) \sim$$

$$\overset{L_2 \leftarrow L_2 \cdot \frac{1}{-3}}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right) \overset{L_3 \leftarrow L_3 + 6L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 9 & 2 & -2 & 1 \end{array}\right) \sim$$

$$\overset{L_3 \leftarrow \frac{1}{9} \cdot L_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{array}\right) \underset{L_1 \leftarrow L_1 - 2L_3}{\overset{L_2 \leftarrow L_2 - 2L_3}{=}} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{5}{9} & \frac{4}{9} & -\frac{2}{9} \\ 0 & 1 & 0 & \frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{array}\right)$$

$$\overset{L_1 \leftarrow L_1 - 2L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ 0 & 1 & 0 & \frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{array}\right)$$

$$\Rightarrow \quad A^{-1} = \begin{pmatrix} 1/9 & 2/9 & 2/9 \\ 2/9 & 1/9 & -2/9 \\ 2/9 & -2/9 & 1/9 \end{pmatrix}$$

Compute by applying elementary operations the inverses of the matrices:

4. $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$. **5.** $\begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}$.

Sol.:

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \overset{L_2 \leftarrow L_2 - 2L_1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 3 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\overset{L_3 \leftarrow L_3 - 3L_1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right) \overset{L_2 \leftarrow \frac{1}{-5} L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right) \sim$$

$$\overset{L_3 \leftarrow L_3 + 12L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} & \frac{9}{5} & \frac{-12}{5} & 1 \end{array}\right) \overset{L_3 \leftarrow 5L_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & \frac{-1}{5} & 0 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right) \sim$$

$$\overset{L_2 \leftarrow L_2 - \frac{3}{5} L_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -5 & 7 & -3 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right) \overset{L_1 \leftarrow L_1 - 2L_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 0 & -17 & 24 & -10 \\ 0 & 1 & 0 & -5 & 7 & -3 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right)$$

$$\overset{L_1 \leftarrow L_1 - 4L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 2 \\ 0 & 1 & 0 & -5 & 7 & -3 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right)$$

$= A^{-1}$

Ex.: $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 5 \\ 1 & -1 & 2 \end{pmatrix}$ is not invertible

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 1 & 5 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \underset{L_3 \leftarrow L_3 - L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -1 & -2 & 1 & 0 \\ 0 & -3 & -1 & -1 & 0 & 1 \end{array}\right) \sim$$

$$\overset{L_3 \leftarrow L_3 - L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right)$$

$\Rightarrow$ the matrix is not invertible, because we have a zero row on the left.

7. In the real vector space $\mathbb{R}^3$ consider the list $X = (v_1, v_2, v_3, v_4)$, where $v_1 = (1, 0, 4)$, $v_2 = (2, 1, 0)$, $v_3 = (1, 5, -36)$ and $v_4 = (2, 10, -72)$. Determine $dim < X >$ and a basis of $< X >$.

<u>Sol.</u>: To find a basis of $\langle X \rangle$, we just need to bring the matrix whose rows are the elements of X to a row echelon form.

$$\begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 5 & -36 \\ 2 & 10 & -72 \end{pmatrix} \underset{\substack{L_4 \leftarrow L_4 - 2L_1 \\ L_3 \leftarrow L_3 - L_1}}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 5 & -40 \\ 0 & 10 & -80 \end{pmatrix} \underset{L_4 \leftarrow L_4 - 10L_2}{\overset{L_3 \leftarrow L_3 - 5L_2}{\sim}}$$

$$\sim \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Rightarrow \dim \langle X \rangle = 2$$

A basis $\langle X \rangle$ is:

$$\big((1,0,4),\ (0,1,-8) \big)$$

9. Determine the dimension of the subspaces S, T, $S+T$ and $S \cap T$ of the real vector space $\mathbb{R}^3$ and a basis for the first three of them, where

$$S = < (1,0,4), (2,1,0), (1,1,-4) > ,$$

$$T = < (-3,-2,4), (5,2,4), (-2,0,-8) > .$$

Sol. :

$$\begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{pmatrix} \underset{L_3 \leftarrow L_3 - L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 1 & -8 \end{pmatrix} \overset{L_3 \leftarrow L_3 - L_2}{\sim} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ basis for S : $\big((1,0,4), (0,1,-8) \big) \Rightarrow \dim S = 2$

$$\begin{pmatrix} -3 & -2 & 4 \\ 5 & 2 & 4 \\ -2 & 0 & -8 \end{pmatrix} \overset{L_1 \leftrightarrow L_3}{\sim} \begin{pmatrix} -2 & 0 & -8 \\ 5 & 2 & 4 \\ -3 & -2 & 4 \end{pmatrix} \overset{L_1 \leftarrow L_1 - L_3}{\sim}$$

$$\sim \begin{pmatrix} 1 & 2 & -12 \\ 5 & 2 & 4 \\ -3 & -2 & 4 \end{pmatrix} \underset{L_3 \leftarrow L_3 + 3L_1}{\overset{L_2 \leftarrow L_2 - 5L_1}{\sim}} \begin{pmatrix} 1 & 2 & -12 \\ 0 & -8 & 64 \\ 0 & 4 & -32 \end{pmatrix} \sim$$

$$\overset{L_3 \leftarrow L_3 + \frac{1}{2}L_2}{\sim} \begin{pmatrix} 1 & 2 & -12 \\ 0 & -8 & 64 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ basis for T : $\big((1,2,-12), (0,-8,64) \big) \Rightarrow \dim T = 2$

$S+T = < S \cup T >$

$$\begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \\ -3 & -2 & 4 \\ 5 & 2 & 4 \\ -2 & 0 & -8 \end{pmatrix} \begin{matrix} L_2 \leftarrow L_2 - 2L_1 \\ L_3 \leftarrow L_3 - L_1 \\ \sim \\ L_4 \leftarrow L_4 + 3L_1 \\ L_5 \leftarrow L_5 - 5L_1 \\ L_6 \leftarrow L_6 + 2L_1 \end{matrix} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 1 & -8 \\ 0 & -2 & 16 \\ 0 & 2 & -16 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} L_3 \leftarrow L_3 - L_2 \\ \sim \\ L_4 \leftarrow L_4 + 2L_2 \\ L_5 \leftarrow L_5 - 2L_2 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{basis for } S+T: \; \big((1,0,4),\ (0,1,-8)\big) \Rightarrow \dim(S+T) = 2$$

$$\dim(S\cap T) = \dim S + \dim T - \dim(S+T) =$$

$$= 2 + 2 - 2 = 2$$

6. Let K be a field, let $B = (e_1, e_2, e_3, e_4)$ be a basis and let $X = (v_1, v_2, v_3)$ be a list in the canonical K-vector space K^4, where

$$v_1 = 3e_1 + 2e_2 - 5e_3 + 4e_4\,,$$

$$v_2 = 3e_1 - e_2 + 3e_3 - 3e_4\,,$$

$$v_3 = 3e_1 + 5e_2 - 13e_3 + 11e_4\,.$$

Write the matrix of the list X in the basis B, determine an echelon form for it and deduce that X is linearly dependent.

Sol: $X = (v_1, v_2, \dots, v_m)$ list of vectors

$B = (b_1, b_2, \dots, b_n)$ basis

$$v_1 = a_{11} b_1 + a_{12} b_2 + \dots + a_{1n} b_n$$

$$\vdots$$

$$v_m = a_{m1} b_1 + a_{m2} b_2 + \dots + a_{mn} b_n$$

$$[X]_B = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$$[x]_B = \begin{pmatrix} 3 & 2 & -5 & 4 \\ 3 & -1 & 3 & -3 \\ 3 & 5 & -13 & 11 \end{pmatrix} \sim$$

$$\underset{L_3 \leftarrow L_3 - L_1}{\overset{L_2 \leftarrow L_2 - L_1}{\sim}} \begin{pmatrix} 3 & 2 & -5 & 4 \\ 0 & -3 & 8 & -7 \\ 0 & 3 & -8 & 7 \end{pmatrix} \overset{L_3 \leftarrow L_3 + L_2}{\sim} \begin{pmatrix} 3 & 2 & -5 & 4 \\ 0 & -3 & 8 & -7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ the initial vectors were linearly dependent