08. 12. 2021

Seminar W10 - 914

Def: V, V' K-v.s., $f: V \to V'$ K-linear map

$B = (v_1, \dots, v_m)$ basis of V "source basis"

$B' = (v_1', \dots, v_n')$ basis of V' "target basis"

$$[f]_{B,B'} = \left([f(v_1)]_{B'} \; [f(v_2)]_{B'} \; \dots \; [f(v_m)]_{B'} \right)$$

Prop: V, V' K-v.s., $f: V \to V'$ K-linear map

B basis for V, B' basis for V'

$$\Rightarrow \forall v \in V: \quad \boxed{[f(v)]_{B'} = [f]_{B,B'} \cdot [v]_B}$$

2. Let $f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2)$ be defined by

$$f(x, y, z) = (y, -x)$$

and consider the bases $B = (v_1, v_2, v_3) = ((1,1,0), (0,1,1), (1,0,1))$ of $\mathbb{R}^3$, $B' = (v_1', v_2') = ((1,1), (1,-2))$ of $\mathbb{R}^2$ and let $E' = (e_1', e_2')$ be the canonical basis of $\mathbb{R}^2$. Determine the matrices $[f]_{BE'}$ and $[f]_{BB'}$.

$e_1' = (1,0)$, $e_2' = (0,1)$

$E = (e_1, e_2, e_3)$, $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

Sol.: $[f]_{E,B'} = \left([f(e_1)]_{B'} \; [f(e_2)]_{B'} \; [f(e_3)]_{B'} \right)$

$$f(e_1) = f(1,0,0) = (0,-1) = \alpha_1 \cdot v_1' + \beta_1 \cdot v_2'$$

$$(0,-1) = \alpha_1 \cdot (1,1) + \beta_1 \cdot (1,-2) \Rightarrow \begin{cases} \alpha_1 + \beta_1 = 0 \\ \alpha_1 - 2\beta_1 = -1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_1 = 2\beta_1 - 1 \\ 2\beta_1 - 1 + \beta_1 = 0 \end{cases} \Leftrightarrow \begin{cases} \beta_1 = \frac{1}{3} \\ \alpha_1 = -\frac{1}{3} \end{cases} \Rightarrow [f(e_1)]_{B'} = \begin{pmatrix} -1/3 \\ 1/3 \end{pmatrix}$$

$$f(e_2) = f(0,1,0) = (1,0) = \alpha_2 \cdot (1,1) + \beta_2 \cdot (1,-2)$$

$$\Rightarrow \begin{cases} 1 = \alpha_2 + \beta_2 \\ 0 = \alpha_2 - 2\beta_2 \end{cases} \Leftrightarrow \begin{cases} \alpha_2 = 2\beta_2 \\ 2\beta_2 + \beta_2 = 1 \end{cases} \Leftrightarrow \begin{cases} \beta_2 = \frac{1}{3} \\ \alpha_2 = \frac{2}{3} \end{cases}$$

$$\Rightarrow [f(e_2)]_{B'} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$$

$$f(e_3) = f(0,0,1) = (0,0) = \alpha_3 \cdot e_1' + \beta_3 \cdot e_2'$$

$$\begin{cases} 0 = \alpha_3 + \beta_3 \\ 0 = \alpha_3 - 2\beta_3 \end{cases} \Rightarrow \begin{cases} \alpha_3 = 2\beta_3 \\ 2\beta_3 + \beta_3 = 0 \end{cases} \Rightarrow \begin{cases} \beta_3 = 0 \\ \alpha_3 = 0 \end{cases}$$

$$\Rightarrow [f(e_3)]_{B'} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} [f(e_1)]_{B'} &= \begin{pmatrix} -1/3 \\ 1/3 \end{pmatrix} \\ [f(e_2)]_{B'} &= \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} \\ [f(e_3)]_{B'} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}\right\} \Rightarrow [f]_{E,B'} = \begin{pmatrix} -1/3 & 2/3 & 0 \\ 1/3 & 1/3 & 0 \end{pmatrix}$$

$$[f]_{B,E'} = \left([f(u_1)]_{E'} \quad [f(u_2)]_{E'} \quad [f(u_3)]_{E'} \right)$$

$$f(u_1) = f(1,1,0) = (1,-1) = \alpha \cdot e_1' + \beta \cdot e_2'$$

$$\Rightarrow (1,-1) = \alpha \cdot (1,0) + \beta \cdot (0,1)$$

$$\Rightarrow \begin{cases} \alpha = 1 \\ \beta = -1 \end{cases} \Rightarrow [f(u_1)]_{E'} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$f(u_2) = f(0,1,1) = (1,0) \Rightarrow [f(u_2)]_{E'} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$f(u_3) = f(1,0,1) = (0,-1) \Rightarrow [f(u_3)]_{E'} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\Rightarrow [f]_{B,E'} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$$

$$[f]_{B,B'} = \left([f(u_1)]_{B'} \quad [f(u_2)]_{B'} \quad [f(u_3)]_{B'} \right)$$

$$f(u_1) = (1,-1) = \alpha u_1' + \beta u_2' \Rightarrow (1,-1) = \alpha \cdot (1,1) + \beta(1,-2)$$

$$\Rightarrow \begin{cases} 1 = \alpha + \beta \\ -1 = \alpha - 2\beta \end{cases} \Rightarrow \begin{cases} \alpha = 1 - \beta \\ -1 = 1 - \beta - 2\beta \end{cases} \Leftrightarrow \begin{cases} \beta = \frac{2}{3} \\ \alpha = \frac{1}{3} \end{cases}$$

$$\Rightarrow [f(u_1)]_{B'} = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$$

$$f(u_2) = (1,0) = \alpha \cdot u_1' + \beta \cdot u_2'$$

$$\Rightarrow \begin{cases} \alpha + \beta = 1 \\ \alpha - 2\beta = 0 \end{cases} \Rightarrow \begin{cases} \alpha = 2\beta \\ 2\beta + \beta = 1 \end{cases} \Rightarrow \begin{cases} \beta = \frac{1}{3} \\ \alpha = \frac{2}{3} \end{cases}$$

$$\Rightarrow [f(u_2)]_{B'} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$$

$$f(u_3) = (0,-1) = \alpha u_1' + \beta u_2'$$

$$\Rightarrow \begin{cases} \alpha + \beta = 0 \\ \alpha - 2\beta = -1 \end{cases} \Rightarrow \begin{cases} \alpha = -\beta \\ -3\beta = -1 \end{cases} \Rightarrow \begin{cases} \beta = \frac{1}{3} \\ \alpha = -\frac{1}{3} \end{cases}$$

$$[f(u_3)]_{B'} = \begin{pmatrix} -1/3 \\ 1/3 \end{pmatrix}$$

$$\Rightarrow \quad [f]_{B,B'} = \begin{pmatrix} 1/3 & 2/3 & -1/3 \\ 2/3 & 1/3 & 1/3 \end{pmatrix}$$

4. Let $f \in End_{\mathbb{R}}(\mathbb{R}^4)$ with the following matrix in the canonical basis E of $\mathbb{R}^4$:

$$[f]_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix}.$$

(i) Show that $v = (1, 4, 1, -1) \in Ker\, f$ and $v' = (2, -2, 4, 2) \in Im\, f$.
(ii) Determine a basis and the dimension of $Ker\, f$ and $Im\, f$.
(iii) Define f.

Prop: V, V' K-v.s., $f: V \to V'$ K-linear map
B basis for V, B' basis for V'

$$\Rightarrow \forall v \in V: \quad \boxed{[f(v)]_{B'} = [f]_{B,B'} \cdot [v]_B}$$

(i) $v \in Ker f \Leftrightarrow f(v) = 0 \Leftrightarrow [f(v)]_E = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow$

$$\Leftrightarrow [f]_E \cdot [v]_E = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$[f]_E \cdot [v]_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$v' \in Im f \Leftrightarrow \exists u \in \mathbb{R}^4 : f(u) = v' \Leftrightarrow \exists u = (x, y, z, t) : [f(u)]_E = [v']_E \Leftrightarrow$

$\Leftrightarrow \exists u = (x, y, z, t) : [v']_E = [f]_E \cdot [u]_E \Leftrightarrow$

$\Leftrightarrow \exists\, x, y, z, t \in \mathbb{R} : \begin{pmatrix} 2 \\ -2 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ -1 & 1 & 1 & 4 & -2 \\ 2 & 1 & -5 & 1 & 4 \\ 1 & 2 & -4 & 5 & 2 \end{array}\right) \underset{\substack{L_3 \leftarrow L_3 - 2L_1 \\ L_4 \leftarrow L_4 - L_1}}{\overset{L_2 \leftarrow L_2 + L_1}{\sim}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ 0 & 2 & -2 & 6 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right) \sim$$

$$\underset{L_4 \leftarrow L_4 - \frac{1}{2}L_2}{\overset{L_3 \leftarrow L_3 + \frac{1}{2}L_2}{\sim}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ 0 & 2 & -2 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

row echelon form, no incompatibility row $\Rightarrow$ System is compatible

$\Rightarrow \exists\, x, y, z, t : f(x,y,z,t) = (2,-2,4,2) \Rightarrow u' = (2,-2,4,2) \in \operatorname{Im} f$

(ii) $\operatorname{Ker} f = \{ u \in \mathbb{R}^4 \mid f(u) = 0 \} = \{ u = (x,y,z,t) \in \mathbb{R}^4 \mid [f(u)]_E = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \}$

$= \{ u = (x,y,z,t) \in \mathbb{R}^4 \mid [f]_E \cdot [u]_E = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \} =$

$= \{ u = (x,y,z,t) \in \mathbb{R}^4 \mid \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \}$

$$\begin{cases} x + y - 3z + 2t = 0 \\ -x + y + z + 4t = 0 \\ 2x + y - 5z + t = 0 \\ x + 2y - 4z + 5t = 0 \end{cases}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ -1 & 1 & 1 & 4 & 0 \\ 2 & 1 & -5 & 1 & 0 \\ 1 & 2 & -4 & 5 & 0 \end{array}\right) \underset{\substack{L_3 \leftarrow L_3 - 2L_2 \\ L_4 \leftarrow L_4 - L_1}}{\overset{L_2 \leftarrow L_2 + L_1}{\sim}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & 2 & -2 & 6 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right) \sim$$

$$\overset{L_2 \leftarrow \frac{1}{2}L_2}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right) \underset{L_4 \leftarrow L_4 - L_2}{\overset{L_3 \leftarrow L_3 + L_2}{\sim}}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim$$

$$\overset{L_1 \leftarrow L_1 - L_2}{\sim} \left(\begin{array}{cccc|c} 1 & 0 & -2 & -1 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \begin{cases} x - 2z - t = 0 \\ y - z + 3t = 0 \end{cases}$$

$$\Rightarrow \operatorname{Ker} f = \left\{ (x,y,z,t) \;\middle|\; \begin{cases} x - 2z - t = 0 \\ y - z + 3t = 0 \end{cases} \right\} =$$

$$= \left\{ (x,y,z,t) \;\middle|\; \begin{array}{l} x = 2z+t \\ y = z-3t \end{array} \right\} =$$

$$= \{ (2z+t,\ z-3t,\ z,\ t) \mid z,t \in \mathbb{R} \} =$$

$$= \{ (2z, z, z, 0) + (t, -3t, 0, t) \mid z, t \in \mathbb{R} \}$$

$$= \{ z \cdot (2,1,1,0) + t \cdot (1,-3,0,1) \mid z,t \in \mathbb{R} \}$$

$$= \langle (2,1,1,0), (1,-3,0,1) \rangle$$

$$\begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & -3 & 0 & 1 \end{pmatrix} \underset{\sim}{L_1 \leftrightarrow L_2} \begin{pmatrix} 1 & -3 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix} \sim$$

$$\underset{\sim}{L_2 \leftarrow L_2 - 2L_1} \begin{pmatrix} 1 & -3 & 0 & 1 \\ 0 & 7 & 1 & -2 \end{pmatrix}$$

$\Rightarrow$ basis of $\operatorname{Ker} f$: $\left((1,-3,0,1), (0,7,1,-2) \right)$

$\Rightarrow \dim \operatorname{Ker} f = 2$

$$\operatorname{Im} f = \left\{ w = (a,b,c,d) \in \mathbb{R}^4 \mid \exists\, u = (x,y,z,t) \in \mathbb{R}^4 : f(u) = w \right\} =$$

$$= \left\{ w = (a,b,c,d) \in \mathbb{R}^4 \mid \exists\, u = (x,y,z,t) : [w]_E = [f]_E \cdot [u]_E \right\} =$$

$$= \left\{ (a,b,c,d) \in \mathbb{R}^4 \mid \exists\, (x,y,z,t) : \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \right\}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ -1 & 1 & 1 & 4 & b \\ 2 & 1 & -5 & 1 & c \\ 1 & 2 & -4 & 5 & d \end{array}\right) \underset{\substack{L_3 \leftarrow L_3 - 2L_1 \\ L_4 \leftarrow L_4 - L_1}}{\overset{L_2 \leftarrow L_2 + L_1}{\sim}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ 0 & 2 & -2 & 6 & a+b \\ 0 & -1 & 1 & -3 & c-2a \\ 0 & 1 & -1 & 3 & d-a \end{array}\right) \sim$$

$$\overset{L_2 \leftrightarrow L_4}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ 0 & 1 & -1 & 3 & d-a \\ 0 & -1 & 1 & -3 & c-2a \\ 0 & 2 & -2 & 6 & a+b \end{array}\right) \underset{L_4 \leftarrow L_4 - 2L_2}{\overset{L_3 \leftarrow L_3 + L_2}{\sim}}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ 0 & 1 & -1 & 3 & d-a \\ 0 & 0 & 0 & 0 & c+d-3a \\ 0 & 0 & 0 & 0 & a+b-2d+2a \end{array}\right)$$

System is compatible $\iff \begin{cases} c+d-3a=0 \\ 3a+b-2d=0 \end{cases}$

$$\operatorname{Im} f = \left\{ (a,b,c,d) \;\middle|\; \begin{cases} c+d-3a=0 \\ 3a+b-2d=0 \end{cases} \right\} =$$

$$= \left\{ (a,b,c,d) \;\middle|\; \begin{cases} c = 3a-d \\ b = 2d-3a \end{cases} \right\}$$

$$= \{ (a, 2d-3a, 3a-d, d) \mid a,d \in \mathbb{R} \} =$$

$$= \langle (1,-3,3,0), (0,2,-1,1) \rangle$$

These gens are lin indep $\Rightarrow$ basis for $\operatorname{Im} f$:

$$((1,-3,3,0), (0,2,-1,1))$$

$$\Rightarrow \dim \operatorname{Im} f = 2$$

<u>Verification:</u> $\underbrace{\dim \operatorname{Ker} f}_{2} + \underbrace{\dim \operatorname{Im} f}_{2} = \underbrace{\dim(\mathbb{R}^4)}_{4}$ ✓

(iii)