15.11.2021

Seminar W7 - 916

Corollary (Steinitz) V K-vector space, $S \leq V$

We can extend any basis of S to a basis of V.

In practice: $u_1, \dots, u_n$ basis for S.

Do we have enough vectors?

- yes: $u_1, \dots, u_n$ basis for V
- no: We choose a vector $w_{n+1} \in V \setminus \langle u_1, \dots, u_n \rangle$

  Do we have enough vectors?

  - yes: $u_1, u_2, \dots, u_n, w_{n+1}$ basis for V
  - no: We choose a vector $w_{n+2} \in V \setminus \langle u_1, u_2, \dots, u_n, w_{n+1} \rangle$

    ...

Remark: V K-vector space, $u_1, u_2, \dots, u_n \in V$

$$\operatorname{rank}(u_1, \dots, u_n) = \dim \langle u_1, u_2, \dots, u_n \rangle =$$

= max number of linearly independent vectors among the $u_1, u_2, \dots, u_n$

If $V = K^n$: $\operatorname{rank}(u_1, u_2, \dots, u_n) = \operatorname{rank}\begin{pmatrix} u_1 \\ \hline u_2 \\ \hline \vdots \\ \hline u_n \end{pmatrix} = \operatorname{rank}(u_1 | u_2 | \dots | u_n)$

1. Determine a basis and the dimension of the following subspaces of the real vector space $\mathbb{R}^3$:

$$A = \{(x,y,z) \in \mathbb{R}^3 \mid z = 0\}$$
$$B = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z = 0\}$$
$$C = \{(x,y,z) \in \mathbb{R}^3 \mid x = y = z\}.$$

6. Complete the bases of the subspaces from Exercise **1.** to some bases of the real vector space $\mathbb{R}^3$ over $\mathbb{R}$.

Sol: $A = \{(x,y,0) \mid x,y \in \mathbb{R}\} = \{(x,0,0) + (0,y,0) \mid x,y \in \mathbb{R}\}$

$= \{x \cdot (1,0,0) + y \cdot (0,1,0) \mid x,y \in \mathbb{R}\} =$

$= \langle (1,0,0), (0,1,0) \rangle$

$(1,0,0), (0,1,0)$ are lin indep (components are not proportional)

$\Rightarrow ((1,0,0), (0,1,0))$ basis of $A \Rightarrow \dim A = 2$

We will choose a vector in $\mathbb{R}^3 \setminus A$, in order to complete the basis.

$(0,0,1)$ is such a vector $\Rightarrow (1,0,0), (0,1,0), (0,0,1)$ lin indep $\Rightarrow$

$\Rightarrow$ basis of $\mathbb{R}^3$

$B = \{(x,y,z) \in \mathbb{R}^3 / x+y+z = 0\} =$

$= \{(x,y,-x-y) \mid x,y \in \mathbb{R}\} =$

$= \{(x,0,-x) + (0,y,-y) \mid x,y \in \mathbb{R}\} =$

$= \{ x \cdot (1,0,-1) + y \cdot (0,1,-1) \mid x, y \in \mathbb{R} \} =$

$= \langle (1,0,-1), (0,1,-1) \rangle$

$\operatorname{rank} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} = 2 \Rightarrow (1,0,-1), (0,1,-1)$ lin. indep. $\Rightarrow$

$\Rightarrow (1,0,-1)$ and $(0,1,-1)$ form a basis of B

We need to add a vector that is not in $B = \langle (1,0,-1), (0,1,-1) \rangle$

$(1,1,1) \notin B \Rightarrow ((1,0,-1), (0,1,-1), (1,1,1))$ lin. indep. $\Rightarrow$

$\Rightarrow$ they form a basis.

$C = \{ (x,y,z) \in \mathbb{R}^3 \mid x = y = z \}$

$C = \{ (x,x,x) \mid x \in \mathbb{R} \} = \langle (1,1,1) \rangle \Rightarrow \dim C = 1$

We have to add 2 more vectors to complete this to a basis of $\mathbb{R}^3$.

We first add a vector that does not belong to C.

Let's add $(1,0,0)$. Now we have a lin. indep. family $(1,0,0)$ and $(1,1,1)$.

We will add a third vector, that needs to not be in

$<(1,0,0),(1,1,1)>.$

$$<(1,0,0),(1,1,1)> = \{a(1,0,0)+b\cdot(1,1,1) \mid a,b\in\mathbb{R}\} =$$

$$= \{(a+b,b,b) \mid a,b\in\mathbb{R}\} = \left\{(x,y,z) \;\middle|\; \begin{cases} x=a+b \\ y=b \\ z=b \end{cases}\right\} =$$

$$= \left\{(x,y,z) \;\middle|\; \begin{matrix} b=y=z \\ x=a+y \end{matrix}\right\} = \{(x,y,z)\in\mathbb{R}^3 \mid y=z\}$$

We choose $(0,1,0)\in\mathbb{R}^3 \setminus <(1,0,0),(1,1,1)> \Rightarrow$

$\Rightarrow (0,1,0),(1,0,0),(1,1,1)$ lin. indp. $\Rightarrow$ they form a basis.

(We could have also found the last vector by making sure that $\operatorname{rank}\begin{pmatrix}1&0&0\\1&1&1\\a&b&c\end{pmatrix}=3$)

<u>Def.</u>: V, W vector spaces, $f: V\to W$ linear map

$$\operatorname{Ker} f = \{v\in V \mid f(v)=0_W\} \leq V$$
("kernel")

$$\operatorname{Im} f = \{f(v)\in W \mid v\in V\} = \{w\in W \mid \exists v\in V: f(v)=w\}$$

4. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $f(x,y,z) = (y,-x)$. Prove that f is an $\mathbb{R}$-linear map and determine a basis and the dimension of $Ker\, f$ and $Im\, f$.

Sol.: Let $v_1 = (x_1, y_1, z_1)$, $v_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$

$$f(v_1 + v_2) = f(x_1 + x_2, y_1 + y_2, z_1 + z_2) = (y_1 + y_2, -x_1 - x_2) =$$

$$= (y_1, -x_1) + (y_2, -x_2) = f(v_1) + f(v_2)$$

$v = (x, y, z)$, $k \in \mathbb{R}$

$$f(kv) = f(kx, ky, kz) = (ky, -kx) = k \cdot (y, -x) = k \cdot f(v)$$

$\Rightarrow f$ is an $\mathbb{R}$-linear map.

$$Ker\, f = \{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = (0,0)\} =$$

$$= \{(x,y,z) \in \mathbb{R}^3 \mid (y, -x) = (0,0)\} = \{(x,y,z) \in \mathbb{R}^3 \mid x = y = 0\} =$$

$$= \{(0,0,z) \mid z \in \mathbb{R}\} = \langle (0,0,1) \rangle$$

$((0,0,1))$ basis of the kernel $\Rightarrow \dim Ker\, f = 1$

$$Im\, f = \{f(x,y,z) \mid x,y,z \in \mathbb{R}^3\} = \{(y, -x) \mid x,y,z \in \mathbb{R}\} =$$

$$= \{y \cdot (1,0) + x \cdot (0,-1) \mid x, y \in \mathbb{R}\} = \langle (1,0), (0,-1) \rangle$$

$((1,0), (0,-1))$ basis $\Rightarrow \dim Im\, f = 2$

Th. (1^{st} dimension theorem ; the rank-nullity theorem)

V, W k-v.s., $f: V \to W$ linear map, then:

$$\dim V = \underbrace{\dim \operatorname{Ker} f}_{\text{nullity}(f)} + \underbrace{\dim \operatorname{Im} f}_{\text{rank}(f)}$$

Th. (2^{nd} dimension theorem) : V k-v.s., $S, T \leq_k V$

$$\dim (S+T) = \dim S + \dim T - \dim (S \cap T)$$

Remark: $S = \langle u_1, \dots, u_n \rangle$, $T = \langle w_1, w_2, \dots, w_m \rangle$

$$\Rightarrow \quad S+T = \langle u_1, u_2, \dots, u_n, w_1, w_2, \dots, w_m \rangle$$

9. Consider the subspaces

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\},$$

$$T = \langle (0, 1, 1), (1, 1, 0) \rangle$$

of the real vector space $\mathbb{R}^3$. Determine $S \cap T$ and show that $S + T = \mathbb{R}^3$.

Sol. : $S = \langle (0,1,0), (0,0,1) \rangle$, $((0,1,0), (0,0,1))$ basis for S

$\Rightarrow \dim S = 2$

$(0,1,1)$ and $(1,1,0)$ lin. indep. $\Rightarrow \big((0,1,1), (1,1,0)\big)$ basis for $T \Rightarrow$

$\Rightarrow \dim T = 2$

$S+T = \langle (0,0,1), (0,1,0), (0,1,1), (1,1,0) \rangle$

$$\operatorname{rank}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} = ?$$

$\begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 1 \neq 0 \Rightarrow \dim(S+T) = 3$ and a basis of $S+T$:

is $\left((0,1,0), (0,1,1), (1,1,0) \right)$

$$\left.\begin{array}{l} \dim \mathbb{R}^3 = 3 \\ \dim(S+T) = 3 \\ S+T \leq \mathbb{R}^3 \end{array}\right\} \Rightarrow S+T = \mathbb{R}^3$$

$\dim(S \cap T) = \dim S + \dim T - \dim(S+T) = 2+2-3 = 1$

$T = \langle (0,1,1), (1,1,0) \rangle = \{ a \cdot (0,1,1) + b \cdot (1,1,0) \mid a,b \in \mathbb{R} \} =$

$= \{ (b, a+b, a) \mid a,b \in \mathbb{R} \}$

$S = \{ (x,y,z) \mid x = 0 \}$

$S \cap T = \{ (b, a+b, a) \mid b = 0, a \in \mathbb{R} \} = \{ (0,a,a) \mid a \in \mathbb{R} \} =$

$= \langle (0,1,1) \rangle$

8. Let V be a vector space over K and let S, T and U be subspaces of V such that $dim(S \cap U) = dim(T \cap U)$ and $dim(S+U) = dim(T+U)$. Prove that if $S \subseteq T$, then $S = T$.

$\underline{\text{Sol}}$: $\dim(S \cap U) = \dim(T \cap U)$

$\Rightarrow \dim S + \underline{\dim U} - \underline{\dim(S+U)} = \dim T + \underline{\dim U} - \underline{\dim(T+U)}$

$$\left.\begin{array}{l} \Rightarrow \dim S = \dim T \\ \\ S \subseteq T \end{array}\right\} \Rightarrow S = T$$