08.11.2021

Seminar W6 - 916

Def.: V K-vector space, $v_1, v_2, \ldots, v_n \in V$

We say that $v_1, v_2, \ldots, v_n$ are linearly independent if $\forall \alpha_1, \ldots, \alpha_n \in K$:

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0 \Rightarrow \alpha_1 = \cdots = \alpha_n = 0$$

We say that $v_1, v_2, \ldots, v_n$ are linearly dependent if

$\exists \alpha_1, \ldots, \alpha_n \in K$, not all zero, so that $\underbrace{\alpha_1 v_1 + \ldots + \alpha_n v_n = 0}_{\text{dependence relationship}}$

2. Prove that the following vectors are linearly independent:
(*i*) $v_1 = (1,0,2)$, $v_2 = (-1,2,1)$, $v_3 = (3,1,1)$ in ${}_{\mathbb{R}}\mathbb{R}^3$.
(*ii*) $v_1 = (1,2,3,4)$, $v_2 = (2,3,4,1)$, $v_3 = (3,4,1,2)$, $v_4 = (4,1,2,3)$ in ${}_{\mathbb{R}}\mathbb{R}^4$.

Sol.: (i) Let $k_1, k_2, k_3 \in \mathbb{R}$ so that $k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$

$$k_1 \cdot (1,0,2) + k_2 \cdot (-1,2,1) + k_3 \cdot (3,1,1) = (0,0,0)$$

$$\Rightarrow \begin{cases} k_1 + 3k_3 - k_2 = 0 \\ 2k_2 + k_3 = 0 \\ 2k_1 + k_2 + k_3 = 0 \end{cases} \Leftrightarrow \begin{cases} k_3 = -2k_2 \\ k_1 - k_2 - 6k_2 = 0 \\ 2k_1 + k_2 - 2k_2 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} k_3 = -2k_2 \\ k_1 = 7k_2 \\ 14k_2 - k_2 = 0 \end{cases} \Leftrightarrow k_2 = k_1 = k_3 = 0$$

$\Rightarrow v_1, v_2, v_3$ linearly independent

(ii) Let $k_1, k_2, k_3, k_4 \in \mathbb{R}$ so that $k_1 v_1 + k_2 v_2 + k_3 v_3 + k_4 v_4 = 0$

$\Rightarrow k_1 \cdot (1,2,3,4) + k_2 \cdot (2,3,4,1) + k_3 \cdot (3,4,1,2) + k_4 \cdot (4,1,2,3) = (0,0,0,0)$

$$\Rightarrow \begin{cases} k_1 + 2k_2 + 3k_3 + 4k_4 = 0 \\ 2k_1 + 3k_2 + 4k_3 + k_4 = 0 \\ 3k_1 + 4k_2 + k_3 + 2k_4 = 0 \\ 4k_1 + k_2 + 2k_3 + 3k_4 = 0 \end{cases}$$

The matrix of the system is $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix}$

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix} \overset{L_1 \leftarrow L_1 + L_2 + L_3 + L_4}{=} \begin{vmatrix} 10 & 10 & 10 & 10 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix} =$$

$$= 10 \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix} \underset{C_4 \leftarrow C_4 - C_1}{\overset{\substack{C_2 \leftarrow C_2 - C_1 \\ C_3 \leftarrow C_3 - C_1}}{=}} 10 \cdot \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 2 & -1 \\ 3 & 1 & -2 & -1 \\ 4 & -3 & -2 & -1 \end{vmatrix} =$$

$$= 10 \cdot \begin{vmatrix} 1 & 2 & -1 \\ 1 & -2 & -1 \\ -3 & -2 & -1 \end{vmatrix} = 10 \cdot (2 + 2 - 2 + 6 + 2 - 2) = 80 \neq 0$$

$\Rightarrow$ by Cramer's rule, the system is compatible determined

$\Rightarrow$ the only solution is the trivial one: $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$

$\Rightarrow$ v_1, v_2, v_3, v_4 lin. independent.

Def: V K-vector space, $v_1, v_2, \dots, v_n \in V$

$B = (v_1, \dots, v_n)$ is a **basis** for V $\iff \begin{cases} \cdot\ v_1, \dots, v_n \text{ lin. indep} \\ \cdot\ V = \langle v_1, v_2, \dots, v_n \rangle \end{cases} \iff$

$\iff \forall v \in V \ \exists! \underbrace{\alpha_1, \alpha_2, \dots, \alpha_n}_{= \text{"the coordinates of } v \text{ in the basis } V\text{"}} \in K : v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$, $\quad [v]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$

($\dim_K V$:= # of vectors in each basis of V)

8. Let $\mathbb{R}_2[X] = \{f \in \mathbb{R}[X] \mid degree(f) \leq 2\}$. Show that the lists $E = (1, X, X^2)$, $B = (1, X - a, (X - a)^2)$ $(a \in \mathbb{R})$ are bases of the real vector space $\mathbb{R}_2[X]$ and determine the coordinates of a polynomial ~~$f = a_0 + a_1X + a_2X^2 \in \mathbb{R}_2[X]$ in each basis.~~

$f = 3 + 8x^2$ in the basis $(1, X+2, (X+2)^2)$

Sol. We will show that: $\forall f \in \mathbb{R}_2[X] \ \exists! \ \alpha_0, \alpha_1, \alpha_2 \in \mathbb{R} : f = \alpha_0 + \alpha_1 \cdot X + \alpha_2 \cdot X^2$

For any $f \in \mathbb{R}_2[X]$: $f = a_0 + a_1 X + a_2 X^2$, $a_0, a_1, a_2 \in \mathbb{R}$

$\Rightarrow \begin{cases} \alpha_0 = a_0 \\ \alpha_1 = a_1 \\ \alpha_2 = a_2 \end{cases}$ and they are unique from the uniqueness of the form of a polynomial

$\Rightarrow E$ basis

We will show that $\forall f \in \mathbb{R}_2[X] \ \exists! \ \alpha_0, \alpha_1, \alpha_2 \in \mathbb{R} : f = \alpha_0 + \alpha_1(X - a) + \alpha_2(X - a)^2$

$f = \beta_0 + \beta_1 X + \beta_2 X^2 = \alpha_0 + \alpha_1(X - a) + \alpha_2(X - a)^2 =$

$= \alpha_0 + \alpha_1 X - a\alpha_1 + \alpha_2 X^2 - 2a\alpha_2 X + \alpha_2 a^2 =$

$$= (\alpha_0 - \alpha_1 a + \alpha_2 a^2) + (\alpha_1 - 2\alpha_2 a)\cdot X + \alpha_2 X^2$$

$$\Rightarrow \begin{cases} \beta_0 = \alpha_0 - \alpha_1 a + \alpha_2 a^2 \\ \beta_1 = \alpha_1 - 2\alpha_2 a \\ \beta_2 = \alpha_2 \end{cases} \Leftrightarrow \begin{cases} \alpha_2 = \beta_2 \\ \beta_0 = \alpha_0 - \alpha_1 a + \beta_2 a^2 \\ \beta_1 = \alpha_1 - 2\beta_2 a \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_2 = \beta_2 \\ \alpha_1 = \beta_1 + 2\beta_2 a \\ \beta_0 = \alpha_0 - \alpha_1 a + \beta_2 a^2 \end{cases} \Leftrightarrow \begin{cases} \alpha_2 = \beta_2 \\ \alpha_1 = \beta_1 + 2\beta_2 a \\ \alpha_0 = \beta_0 + a\beta_1 + 2\beta_2 a^2 + \beta_2 a^2 \end{cases}$$

The system has a unique solution $\Rightarrow (1, X-a, (X-a)^2)$ basis for $\mathbb{R}_2[X]$

$$\left(\dim \mathbb{R}_2[X] = 3 \quad , \quad \mathbb{R}_2[X] \simeq \mathbb{R}^3 \right)$$

$$f = 3 + 8X^2 = \alpha_0 + \alpha_1 \cdot (X+2) + \alpha_2 \cdot (X+2)^2$$

$$\begin{cases} \alpha_2 = \beta_2 \\ \alpha_1 = \beta_1 + 2\beta_2 a \\ \alpha_0 = \beta_0 + a\beta_1 + 2\beta_2 a^2 + \beta_2 a^2 \end{cases} \Rightarrow \begin{cases} \alpha_2 = 8 \\ \alpha_1 = 0 + 2\cdot 8 \cdot (-2) \\ \alpha_0 = 3 + (-2)\cdot 0 + 2\cdot 8\cdot 4 + 8\cdot 4 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \alpha_2 = 8 \\ \alpha_1 = -32 \\ \alpha_0 = 99 \end{cases} \Rightarrow [3+8X^2]_{(1,\, X+2,\, (X+2)^2)} = \begin{pmatrix} 99 \\ -32 \\ 8 \end{pmatrix}$$

4. Let $v_1 = (1, -2, 0, -1)$, $v_2 = (2, 1, 1, 0)$, $v_3 = (0, a, 1, 2)$ be vectors in ${}_{\mathbb{R}}\mathbb{R}^4$. Determine $a \in \mathbb{R}$ such that the vectors v_1, v_2, v_3 are linearly dependent.

Sol: $k_1, k_2, k_3 \in \mathbb{R}$ so that $k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$

$\Rightarrow k_1 \cdot (1, -2, 0, -1) + k_2 \cdot (2, 1, 1, 0) + k_3 \cdot (0, a, 1, 2) = (0, 0, 0, 0)$

$$\Rightarrow \begin{cases} k_1 + 2k_2 = 0 \\ -2k_1 + k_2 + ak_3 = 0 \\ k_2 + k_3 = 0 \\ -k_1 + 2k_3 = 0 \end{cases} \Leftrightarrow \begin{cases} k_1 = -2k_2 \\ k_3 = -k_2 \\ 4k_2 + k_2 - ak_2 = 0 \\ 2k_2 - 2k_2 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} k_1 = -2k_2 \\ k_3 = -k_2 \\ (5-a)\, k_2 = 0 \end{cases}$$

If $a \neq 5 \Rightarrow \begin{cases} k_1 = -2k_2 \\ k_3 = -k_2 \\ k_2 = 0 \end{cases} \Rightarrow k_1 = k_2 = k_3 = 0$

$\Rightarrow v_1, v_2, v_3$ lin. indep.

(NOT WHAT WE WANT!)

If $a = 5 \Rightarrow \begin{cases} k_1 = -2k_2 \\ k_3 = -k_2 \end{cases} \Rightarrow v_1, v_2, v_3$ lin. dep.

9. Determine the number of bases of the vector space $\mathbb{Z}_2^3$ over $\mathbb{Z}_2$.

Sol: $K = \mathbb{Z}_2 = \{0, 1\}$

$\mathbb{Z}_2^3 = \{(a, b, c) \mid a, b, c \in \mathbb{Z}_2\}$

$\dim \mathbb{Z}_2^3 = 3$

$X \subseteq \mathbb{Z}_2^3$ is a basis $\Rightarrow |X| = 3$

$\Rightarrow X = \{v_1, v_2, v_3\}$

# of bases in $\mathbb{Z}_2^3$ = # of lists of 3 linearly independent vectors in $\mathbb{Z}_2^3$

We must choose 3 vectors:

I	II	III
7 choices	6 choices	4 choices
$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$	$\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$	$\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}$

$\Rightarrow 42 \cdot 4 = 168$ choices of bases