13.12.2021

Seminar W11 - 915

Def: V, V' K-v.s., $\mathcal{B} = (v_1, \dots, v_m)$, $\mathcal{B}' = (v_1', \dots, v_n')$

$f : V \to V'$ linear map.

$$\boxed{[f]_{\mathcal{B},\mathcal{B}'} = \left([f(v_1)]_{\mathcal{B}'} \; \cdots \; [f(v_m)]_{\mathcal{B}'} \right)}$$

Prop.: $\forall v \in V$: $\boxed{[f(v)]_{\mathcal{B}'} = [f]_{\mathcal{B},\mathcal{B}'} \cdot [v]_{\mathcal{B}}}$

Def: V K-v.s., $\mathcal{B}, \mathcal{B}'$ bases of V

$[id_V]_{\mathcal{B},\mathcal{B}'} =: T_{\mathcal{B}',\mathcal{B}}$ = "the base change (transfer) matrix" from $\mathcal{B}'$ to $\mathcal{B}$

$\left(id_V : \begin{array}{c} V \to V \\ v \mapsto v \end{array} \right)$

$\Rightarrow [id_V]_{\mathcal{B},\mathcal{B}'} = \left([v_1]_{\mathcal{B}'} \; [v_2]_{\mathcal{B}'} \; \cdots \; [v_m]_{\mathcal{B}'} \right)$

Cor: For $f = id_V$, then:

$$[v]_{\mathcal{B}'} = [id]_{\mathcal{B},\mathcal{B}'} \cdot [v]_{\mathcal{B}} = T_{\mathcal{B}',\mathcal{B}} \cdot [v]_{\mathcal{B}}$$

Prop: V, V', V'' K-v.s., B, B', B'' basis of V, V', V''

$f: V' \to V''$, $g: V \to V'$ K-linear maps

$f \circ g : V \to V''$

$$\boxed{[f \circ g]_{B, B''} = [f]_{B', B''} \cdot [g]_{B, B'}}$$

Prop: V, V' K-v.s., B_1, B_2 bases of V, B_1', B_2' basis of V', $f: V \to V'$ K-linear maps

$$\begin{tikzcd} V \arrow[r, "f"] & V' \arrow[d, "id_{V'}"] \\ V \arrow[u, "id_V"] \arrow[r, "f"] & V' \end{tikzcd}$$

$$[f]_{B_1', B_2'} = [id_V]_{B_2, B_2'} \cdot [f]_{B_1, B_2} \cdot [id_{V'}]_{B_1', B_1} =$$

$$= T_{B_2', B_2} \cdot [f]_{B_1, B_2} \cdot T_{B_1, B_1'}$$

Prop: V K-v.s., B, B' bases of V

$$\underline{[id]_{B,B'}^{-1} = [id]_{B', B}}$$

This isn't true if instead of id we have a general f

(in other words $T_{B', B}^{-1} = T_{B, B'}$)

2. In the real vector space $\mathbb{R}^2$ consider the bases $B = (v_1, v_2) = ((1,2),(1,3))$ and $B' = (v_1', v_2') = ((1,0),(2,1))$ and let $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ having the matrices $[f]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ and $[g]_{B'} = \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}$. Determine the matrices $[2f]_B$, $[f+g]_B$ and $[f \circ g]_{B'}$. (*Use the matrices of change of basis.*)

$$[(1,2,5,6)]_E = \begin{pmatrix} 1 \\ 2 \\ 5 \\ 6 \end{pmatrix}$$

$$\mathbb{R}^2 = \{(x,y) \mid x, y \in \mathbb{R}\}$$

Sol.: $[2f]_B = 2 \cdot [f]_B = 2 \cdot \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -2 & -2 \end{pmatrix}$

$$[f+g]_B = [f]_B + [g]_B \quad , \quad [f]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

$$[g]_{B'} = \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}$$

$$\underline{[g]_B = [id]_{B',B} \cdot [g]_{B'} \cdot [id]_{B,B'}}$$

$$[id]_{B,B'} = \left([e_1]_{B'} \; [e_2]_{B'} \right) \qquad B = (e_1, e_2) \quad B' = (e_1', e_2')$$

<u>First approach for finding $[id]_{B,B'}$</u>

$$e_1 = k_1 \cdot e_1' + k_2 \cdot e_2'$$

$$\Rightarrow (1,2) = k_1 \cdot (1,0) + k_2 \cdot (2,1)$$

$$\Rightarrow \begin{cases} 1 = k_1 + 2k_2 \\ 2 = k_2 \end{cases} \Rightarrow \begin{cases} k_1 = -3 \\ k_2 = 2 \end{cases} \Rightarrow [e_1]_{B'} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$e_2 = k_1 \cdot e_1' + k_2 \cdot e_2'$$

$$\Rightarrow (1,3) = k_1 \cdot (1,0) + k_2 \cdot (2,1)$$

$$\Rightarrow \begin{cases} 1 = k_1 + 2k_2 \\ 3 = k_2 \end{cases} \Rightarrow \begin{cases} k_1 = -5 \\ k_2 = 3 \end{cases} \Rightarrow [e_2]_{B'} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$$

$$\Rightarrow [id]_{B,B'} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$$

Second approach for finding $[id]_{B,B'}$ (the faster one)

$$[id]_{B,B'} = [id]_{E,B'} \cdot [id]_{B,E} = [id]_{B',E}^{-1} \cdot [id]_{B,E}$$

$$B = ((1,2), (1,3)) \qquad B' = ((1,0), (2,1))$$

$$[id]_{B,E} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \qquad [id]_{B',E} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow [id]_{B,B'} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$$

We found $[id]_{B,B'}$, we still need to find $[id]_{B',B}$

$$[id]_{B',B} = [id]_{B,B'}^{-1} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}$$

$$\underline{[g]_B = [id]_{B',B} \cdot [g]_{B'} \cdot [id]_{B,B'}}$$

$$[g]_B = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} \cdot \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix} \cdot \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 4 & -4 \\ -1 & 5 \end{pmatrix} \cdot \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -20 & -32 \\ 13 & 20 \end{pmatrix}$$

$$[f \circ g]_{B'} = [f]_{B'} \cdot [g]_{B'} \qquad , \quad [f]_{B'} = ?$$

$$= [f]_{B,B'} \cdot [g]_{B',B}$$

$$[f]_{B'} = [id]_{B,B'} \cdot [f]_B \cdot [id]_{B',B} =$$

$$= \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} =$$

$$= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} 8 & 13 \\ -5 & -8 \end{pmatrix}$$

$$[f \circ g]_{B'} = [f]_{B'} \cdot [g]_{B'}$$

$$[f \circ g]_{B'} = \begin{pmatrix} 8 & 13 \\ -5 & -8 \end{pmatrix} \cdot \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 9 & -13 \\ -5 & 9 \end{pmatrix}$$

Eigenvectors, eigenvalues and eigenspaces

Def: V K-v.s., $f : V \to V$.

$v \in V \setminus \{0\}$ eigenvector if $\exists \lambda \in K$ (called an eigenvalue) so that:

$$f(v) = \lambda v$$

$$S(\lambda) := \{ v \in V \mid f(v) = \lambda v \} =$$

$$= \left\{ \begin{matrix} \text{set of eigenvectors} \\ \text{corresponding to } \lambda \end{matrix} \right\} \cup \{0\}$$

$$= \text{the eigenspace of } f \text{ corresponding to } \lambda$$

In practice: B basis of V

$\lambda \in K$ is an eigenvalue of $f \iff \lambda$ is a root of the characteristic polynomial of f, which is:

$$p_f(X) = \det([f]_B - X \cdot I_n)$$

$$\iff p_f(\lambda) = \det([f]_B - \lambda I_n) = 0$$

All these notions translate to matrices:

λ eigenvalue for $A \in M_n(K) \iff \lambda$ eigenvalue for $f : K^n \to K^n$ so that $[f]_E = A$

(E can be replaced by any B)

Compute the eigenvalues and the eigenvectors of the (endomorphisms having) matrices:

$$\textbf{5.}\ \begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ -4 & -8 & -2 \end{pmatrix}. \quad \textbf{6.}\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

<u>Sol</u>: $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

$$P_A(X) = \begin{vmatrix} -X & 0 & 0 & 1 \\ 0 & -X & 1 & 0 \\ 0 & 1 & -X & 0 \\ 1 & 0 & 0 & -X \end{vmatrix} \overset{L_1 \leftarrow L_1 + X L_4}{=} \begin{vmatrix} 0 & 0 & 0 & 1-X^2 \\ 0 & -X & 1 & 0 \\ 0 & 1 & -X & 0 \\ 1 & 0 & 0 & -X \end{vmatrix} =$$

$$= (-1)\cdot(1-X^2)\cdot \begin{vmatrix} 0 & -X & 1 \\ 0 & 1 & -X \\ 1 & 0 & 0 \end{vmatrix} = (-1)\cdot(1-X^2)\cdot \begin{vmatrix} -X & 1 \\ 1 & -X \end{vmatrix} =$$

$$= (-1)\cdot(1-X^2)\cdot(X^2-1) = (X^2-1)^2$$

$\Rightarrow$ the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -1$

We will now find the eigenvectors for $\lambda_1 = 1$.

$v = (x,y,z,t)$ eigenvector for A corresp. to $\lambda_1 = 1 \iff$

$$\iff A\cdot[v]_E = \lambda_1\cdot[v]_E \iff (A - \lambda_1 I_4)\cdot[v]_E = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -x + t = 0 \\ -y + z = 0 \\ y - z = 0 \\ x - t = 0 \end{cases} \Leftrightarrow \begin{cases} t = x \\ z = y \end{cases}$$

$$\Rightarrow S(\lambda_1) = \left\{ (x,y,z,t) \in \mathbb{R}^4 \;\middle|\; \begin{cases} t = x \\ z = y \end{cases} \right\} =$$

$$= \{ (x, y, y, x) \mid x, y \in \mathbb{R} \} =$$

$$= \langle (1,0,0,1), (0,1,1,0) \rangle$$

$$\Rightarrow \dim(S(\lambda_1)) = 2$$