14.12.2021

Seminar W11 – 911

Def: V, V' K-vector space, $B = (v_1, v_2, \dots, v_n)$ basis for V
$B' = (v_1', v_2', \dots, v_n)$ basis for V'
$f : V \to V'$ K-linear map

$$\boxed{[f]_{B,B'} = \left([f(v_1)]_{B'} \quad [f(v_2)]_{B'} \quad \cdots \quad [f(v_n)]_{B'} \right)}$$

Prop: $\forall v \in V$: $\boxed{[f(v)]_{B'} = [f]_{B,B'} \cdot [v]_B}$

Def: $[id_V]_{B,B'} =: T_{B',B}$ = "the base-change matrix from B' to B"

Corollary: $f = id_V \Rightarrow [v]_{B'} = [id]_{B,B'} \cdot [v]_B$

$$\left([v]_{B'} = T_{B',B} \cdot [v]_B \right)$$

Prop: $[id]_{B,B'}^{-1} = [id]_{B',B}$

$$\left(T_{B',B}^{-1} = T_{B,B'} \right)$$

This doesn't work for any linear map f, just for id!
$[f]_{B,B'}^{-1} \neq [f]_{B',B}$

Prop : V, V', V'' K-v.s. , B, B', B'' bases

$f: V' \to V''$, $g: V \to V'$ K-linear maps

$f \circ g : V \to V''$

$$\boxed{[f \circ g]_{B, B''} = [f]_{B', B''} \cdot [g]_{B, B'}}$$

Corollary : V, V' K-v.s. , B_1, B_2 bases for V, B_1', B_2' bases for V' , $f: V \to V'$ linear map

$$[f]_{B_2, B_2'} = [id_{V'}]_{B_1', B_2'} \cdot [f]_{B_1, B_1'} \cdot [id_V]_{B_2, B_1}$$

$$= T_{B_2', B_1'} \cdot [f]_{B_1, B_1'} \cdot T_{B_1, B_2}$$

2. In the real vector space $\mathbb{R}^2$ consider the bases $B = (v_1, v_2) = ((1,2),(1,3))$ and $B' = (v_1', v_2') = ((1,0),(2,1))$ and let $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ having the matrices $[f]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ and $[g]_{B'} = \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}$. Determine the matrices $[2f]_B$, $[f+g]_B$ and $[f \circ g]_{B'}$. (*Use the matrices of change of basis.*)

Sol. : $[2f]_B = 2 \cdot [f]_B = 2 \cdot \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -2 & -2 \end{pmatrix}$

$[f+g]_B = [f]_B + [g]_B$

$[g]_B = [id]_{B', B} \cdot [g]_{B'} \cdot [id]_{B, B'}$

In order to find $[id]_{B',B}$.

<u>1st approach (tedious, but simple one)</u>:

$$[id]_{B',B} = \left([u_1']_B \quad [u_2']_B \right)$$

$$[u_1']_B = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Rightarrow u_1' = \alpha u_1 + \beta u_2$$

$$\Rightarrow (1,0) = \alpha \cdot (1,2) + \beta \cdot (1,3)$$

$$\Rightarrow \begin{cases} 1 = \alpha + \beta \\ 0 = 2\alpha + 3\beta \end{cases} \Rightarrow \begin{cases} \beta = 1 - \alpha \\ 0 = 2\alpha + 3 - 3\alpha \end{cases} \Rightarrow \begin{cases} \beta = 1 - \alpha \\ 0 = -\alpha + 3 \end{cases} \Rightarrow \begin{cases} \alpha = 3 \\ \beta = -2 \end{cases}$$

$$\Rightarrow [u_1']_B = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$[u_2']_B = \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \Rightarrow u_2' = \alpha_2 u_1 + \beta_2 u_2$$

$$\Rightarrow (2,1) = \alpha_2 \cdot (1,2) + \beta_2 \cdot (1,3)$$

$$\Rightarrow \begin{cases} \alpha_2 + \beta_2 = 2 \\ 2\alpha_2 + 3\beta_2 = 1 \end{cases} \Rightarrow \begin{cases} \alpha_2 = 5 \\ \beta_2 = -3 \end{cases} \Rightarrow [u_2']_B = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$$

$$\Rightarrow [id]_{B',B} = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}$$

2nd approach (the fancier one):

$$[id]_{B',B} = [id]_{E,B} \cdot [id]_{B',E} = [id]_{B,E}^{-1} \cdot [id]_{B',E}$$

$$B = ((1,2),(1,3)), \quad B' = ((1,0),(2,1))$$

$$[id]_{B,E} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \qquad [id]_{B',E} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow [id]_{B',B} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}$$

$$[g]_B = [id]_{B',B} \cdot [g]_{B'} \cdot [id]_{B,B'}$$

$$[id]_{B,B'} = [id]_{B',B}^{-1} = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}^{-1} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$$

$$\Rightarrow [g]_B = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} \cdot \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix} \cdot \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 4 & -4 \\ -1 & 5 \end{pmatrix} \cdot \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -20 & -32 \\ 13 & 20 \end{pmatrix}$$

$$\Rightarrow [f+g]_B = [f]_B + [g]_B$$

$$[f \circ g]_{B'} = [f]_{B'} \cdot [g]_{B'}$$

$$[g]_{B'} = \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}$$

$$[f]_{B'} = [id]_{B,B'} \cdot [f]_B \cdot [id]_{B',B} =$$

$$= \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} =$$

$$= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} 8 & 13 \\ -5 & -8 \end{pmatrix}$$

$$\Rightarrow [f \circ g]_{B'} = \begin{pmatrix} 8 & 13 \\ -5 & -8 \end{pmatrix} \cdot \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 9 & -13 \\ -5 & 9 \end{pmatrix}$$

Eigenvalues and eigenvectors

Def: V K-v.s., $f: V \to V$ linear map.

$v \in V \setminus \{0\}$ <u>eigenvector</u> for f if $\exists\, \lambda \in K$ (called an <u>eigenvalue</u>):

$$f(v) = \lambda \cdot v$$

$V(\lambda) = \{ v \in V \mid f(v) = \lambda v \} = \{\text{the set of eigenvectors corresponding to } \lambda\} \cup \{0\}$

$\hookrightarrow$ the <u>eigenspace</u> of f corresponding to λ.

Prop: λ eigenvalue of $f \iff \lambda$ is a root of the characteristic polynomial $p_f(X) = \det([f]_B - X I_n)$

(B basis of V, $n = \dim V$)

Rem.: eigenvalues & eigenvectors for $f \in \mathrm{End}_K(V)$ $\Rightarrow$ eigenvalues and eigenvectors for $A \in \mathcal{M}_n(K)$, $A = [f]_B$

Compute the eigenvalues and the eigenvectors of the (endomorphisms having) matrices:

7. $\begin{pmatrix} a & 0 & b \\ 0 & a & 0 \\ b & 0 & a \end{pmatrix}$ $(a, b \in \mathbb{R}^*)$.

Sol.: $p_A(X) = \det(A - X I_3) = \begin{vmatrix} a-X & 0 & b \\ 0 & a-X & 0 \\ b & 0 & a-X \end{vmatrix} = (a-X)^3 - b^2(a-X)$

$$= (a-x)\left((a-x)^2 - b^2\right) = (a-x)(a-x-b)(a-x+b)$$

$\Rightarrow$ the eigenvalues are:

$$\lambda_1 = a, \quad \lambda_2 = a-b, \quad \lambda_3 = a+b$$

To find $V(\lambda)$, we just need to solve the equation:

$$A \cdot [v]_E = \lambda \cdot [v]_E$$

$$(A - \lambda \cdot I_3) \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$V(\lambda_1) = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ b & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} =$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{cases} bz = 0 \\ bx = 0 \end{cases} \right\} =$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{cases} x = 0 \\ z = 0 \end{cases} \right\} =$$

$$= \{ (0, y, 0) \mid y \in \mathbb{R} \} = \langle (0,1,0) \rangle$$

$$V(\lambda_2) = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} b & 0 & b \\ 0 & b & 0 \\ b & 0 & b \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} =$$

$$= \left\{ (x,y,z) \;\middle|\; \begin{cases} bx + bz = 0 \\ by = 0 \end{cases} \right\} =$$

$$\overset{b \neq 0}{=} \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{cases} y = 0 \\ x = -z \end{cases} \right\} =$$

$$= \{(-z, 0, z) \mid z \in \mathbb{R}\} = \langle (-1, 0, 1) \rangle$$

$$V(\lambda_3) = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} -b & 0 & b \\ 0 & -b & 0 \\ b & 0 & -b \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{cases} -b(x-z) = 0 \\ -by = 0 \end{cases} \right\} =$$

$$= \{(z, 0, z) \mid z \in \mathbb{R}\} = \langle (1, 0, 1) \rangle$$

$\dim V(\lambda_1) = 1 = \dim V(\lambda_2) = \dim V(\lambda_3)$

$\lambda_1, \lambda_2, \lambda_3$ have multiplicity 1.

$\Rightarrow$ we can make a basis of eigenvectors:

$$B = ((0,1,1), (-1,0,1), (1,0,1))$$

Let f be a linear map so that $[f]_E = A = \begin{pmatrix} a & 0 & b \\ 0 & a & 0 \\ b & 0 & a \end{pmatrix}$

$$[f]_B = [id]_{E,B} \cdot [f]_E \cdot [id]_{B,E}$$

$$[id]_{B,E} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

We will get $[T]_B = \begin{pmatrix} a & 0 & 0 \\ 0 & a-b & 0 \\ 0 & 0 & a+b \end{pmatrix}$

Which is a diagonal matrix.