Seminar W1 – 9.15

Algebra

$\hookrightarrow$ study of algebraic structures

$(\underbrace{S}_{\text{set}}, \underbrace{*, \cdot, \otimes, \dots}_{\text{operations}})$

→ (binary) operation: (internal law)

$\cdot : S \times S \to S$

$(x, y) \mapsto x \cdot y$

Ex.: of algebraic structures: monoids, groups, rings, fields, (+ vector spaces)

Def.: $(G, *)$ group:

semigroup
→ $*$ is an operation: $\forall x, y \in G: x * y \in G$
→ associativity:
$\forall x, y, z \in G: x*(y*z) = (x*y)*z$

monoid
→ neutral element:
$\exists e \in G \ \forall x \in G: x * e = e * x = x$

→ invertibility (symmetrical element, inverse element)
$\forall x \in G \ \exists x' \in G: x * x' = x' * x = e$

(+) → commutativity:
$\forall x, y \in G: x * y = y * x$

abelian group

semigroup (monoid)

1. Which ones of the usual symbols of addition, subtraction, multiplication and division define an operation (composition law) on the numerical sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$?

Sol. :

	$+$	$-$	$\cdot$	$/$
$\mathbb{N}$	Yes	No	Yes	No
$\mathbb{Z}$	Yes	Yes	Yes	No
$\mathbb{Q}$	Yes	Yes	Yes	No
$\mathbb{R}$	Yes	Yes	Yes	No
$\mathbb{C}$	Yes	Yes	Yes	No

4. Let "$*$" be the operation defined on $\mathbb{R}$ by $x * y = x + y + xy$. Prove that:
(i) $(\mathbb{R}, *)$ is a commutative monoid.
(ii) The interval $[-1, \infty)$ is a stable subset of $(\mathbb{R}, *)$.

Sol. : (i) $\forall x, y \in \mathbb{R} \Rightarrow x+y+xy \in \mathbb{R} \Rightarrow x*y \in \mathbb{R} \Rightarrow *$ operation

$\forall x,y,z \in \mathbb{R}$: $(x*y)*z \overset{?}{=} x*(y*z)$

$(x*y)*z = (x+y+xy)*z = x+y+xy+z+(x+y+xy)\cdot z =$

$= x+y+xy+z+xz+yz+xyz$

$x*(y*z) = x*(y+z+yz) = x+y+z+yz+xy+xz+xyz$

$\Rightarrow (x*y)*z = x*(y*z)$

If e is a neutral element, then: $x*e = x$

$\Rightarrow x+e+xe = x \Rightarrow e+xe = 0 \Rightarrow e(1+x) = 0 \Rightarrow e = 0$

We show that 0 is, indeed, a neutral element.

$\forall x \in \mathbb{R}:$ $x * 0 = x + 0 + x \cdot 0 = x$

$0 * x = 0 + x + 0 \cdot x = x$

$\Rightarrow (\mathbb{R}, *)$ is a monoid

Let $x, y \in \mathbb{R}$: $x * y = x + y + xy = y + x + yx = y * x$

$\Rightarrow (\mathbb{R}, *)$ is a commutative monoid

Def.: $*$ is an operation on S

$A \subseteq S$ is a stable subset of S with regards to the operation $*$ if $\forall x, y \in A$: $x * y \in A$

(ii) $\forall x, y \in [-1, \infty)$: $x * y \overset{?}{\in} [-1, \infty)$

$x * y = x + y + xy + 1 - 1 = x(1+y) + (1+y) - 1 =$

$= (1+x) \cdot (1+y) - 1$

$\forall x, y \geq -1 \Rightarrow \underbrace{(1+x)}_{\geq 0} \underbrace{(1+y)}_{\geq 0} \geq 0$

$\Rightarrow (1+x) \cdot (1+y) - 1 \geq -1 \Rightarrow x * y \in [-1, \infty)$

8. Let "$\cdot$" be an operation on a set A and let $X, Y \subseteq A$. Define an operation "$*$" on the power set $\mathcal{P}(A)$ by

$$X * Y = \{x \cdot y \mid x \in X, y \in Y\}.$$

Prove that:

(i) If $(A, \cdot)$ is a monoid, then $(\mathcal{P}(A), *)$ is a monoid.

(ii) If $(A, \cdot)$ is a group, then in general $(\mathcal{P}(A), *)$ is not a group.

Sol.: $\mathcal{P}(A) = \{S \subseteq A\} = \{\emptyset, \dots, S\}$

(i) To show that $*$ is an operation it suffices to show that $X * Y \in \mathcal{P}(A)$, i.e. $X * Y \subseteq A$

Let $z \in X * Y \Rightarrow \exists x \in X, \exists y \in Y : z = xy$

$(A, \cdot)$ monoid $\Rightarrow$ $\cdot$ operation on A $\overset{x, y \in A}{\Rightarrow}$ $x \cdot y \in A \Rightarrow X * Y \subseteq A$

$\Rightarrow$ $*$ is an operation on $\mathcal{P}(A)$

Associativity: $(X * Y) * Z = X * (Y * Z)$

Let $\alpha \in (X * Y) * Z \Rightarrow \exists x \in X, y \in Y, z \in Z$

$$\alpha = (x \cdot y) \cdot z$$

$\cdot$ assoc. on A
$\Rightarrow$

$$\alpha = (x \cdot y) \cdot z = \overbrace{x \cdot \underbrace{(y \cdot z)}_{\in Y * Z}}^{\in X * (Y * Z)}$$

$$\Rightarrow (X * Y) * Z \subseteq X * (Y * Z)$$

Likewise we show that $X*(Y*Z) \supseteq (X*Y)*Z$

$\Rightarrow (\mathcal{P}(A), *)$ semigroup

Let e be the neutral element in the monoid $(A, \cdot)$

Let $E = \{e\}$

Let $X \in \mathcal{P}(A) \Rightarrow$ $X * E = \{ \underbrace{x \cdot e}_{=x} \mid x \in X \} = X$

$E * X = \{ \underbrace{e \cdot x}_{=x} \mid x \in X \} = X$

(ii) $\forall X \in \mathcal{P}(A)$: $X * \emptyset = \emptyset$

$\Rightarrow \forall X \in \mathcal{P}(A)$: $X * \emptyset \neq E$

$\Rightarrow$ $\emptyset$ cannot be invertible