22.11.2021

Seminar W/8 - 916

To check for compatibility: $(S): \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn} = b_m \end{cases}$

<u>Th.</u> (Kronecker - Capelli)

(S) compatible $\iff$ rank M = rank $\bar{M}$

$$M = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \qquad \bar{M} = \begin{pmatrix} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{pmatrix}$$

<u>Th.</u> (Rouché): Δ_p principal minor of M

(S) compatible $\iff$ $\forall i \in \{1, \dots, \underset{m-r,\ r = \text{rank } M}{s}\}$: $\Delta_i = 0$

(all the characteristic minors are 0)

$$\left(\Delta_i = \left(\begin{array}{c|c} \Delta_p & \text{column of free terms} \\ \hline \text{row from } M & \end{array} \right) \right)$$

To find the solutions of a system:

- Find a principal minor Δ_p
- The rows in Δ_p correspond to the principal equations.
- The rows not in Δ_p correspond to the secondary equations.

- The columns in Δ_p correspond to the principal unknowns
- The columns not in Δ_p correspond to the secondary unknowns
- We discard the secondary equations.
- We regard the secondary unknowns as parameters (rename them) and move them to the column of free terms
- We are left with a square system, which we solve by using Cramer's rule.

If we have a square system

$$(S'):\begin{cases} c_{11}x_1 + \dots + c_{1n}x_n = d_1 \\ \vdots \\ c_{n1}x_1 + \dots + c_{nn}x_n = d_n \end{cases}$$

$$\Rightarrow (S') \text{ compatible} \Leftrightarrow \Delta = \begin{vmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{vmatrix} \neq 0$$

If S' compatible, then the solutions are:

$$\forall i \in \{1, \dots, n\} : x_i = \frac{\Delta_{x_i}}{\Delta}$$

$$\Delta_{x_i} = \begin{vmatrix} c_{11} & \cdots & c_{1\,i-1} & d_1 & c_{1\,i+1} & \cdots & c_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ c_{n1} & & c_{n\,i-1} & d_n & c_{n\,i+1} & \cdots & c_{nn} \end{vmatrix}$$

2. Using the Kronecker-Capelli theorem, decide if the following linear systems are compatible and then solve the compatible ones:

$$(i)\begin{cases} x_1 + x_2 + x_3 - 2x_4 = 5 \\ 2x_1 + x_2 - 2x_3 + x_4 = 1 \\ 2x_1 - 3x_2 + x_3 + 2x_4 = 3 \end{cases} \qquad (ii)\begin{cases} x_1 - 2x_2 + x_3 + x_4 = 1 \\ x_1 - 2x_2 + x_3 - x_4 = -1 \\ x_1 - 2x_2 + x_3 + 5x_4 = 5 \end{cases}$$

$$(iii)\begin{cases} x + y + z = 3 \\ x - y + z = 1 \\ 2x - y + 2z = 3 \\ x + z = 4 \end{cases}$$

3. Using the Rouché theorem, decide if the systems from **2.** are compatible and then solve the compatible ones.

Sol.: (i) $\bar{M} = \left(\begin{array}{cccc|c} 1 & 1 & 1 & -2 & 5 \\ 2 & 1 & -2 & 1 & 1 \\ 2 & -3 & 1 & 2 & 3 \end{array}\right)$

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -2 \\ 2 & -3 & 1 \end{vmatrix} = -19 \neq 0$$

$\Rightarrow$ rank $\bar{M} = 3$ rank $M = 3$

$\overset{K-C}{\Rightarrow}$ the system is compatible

All the equations are principal.

x_1, x_2, x_3 principal unknowns

x_4 secondary unknown, $x_4 =: \alpha$

The system is:

$$\begin{cases} x_1 + x_2 + x_3 = 5+2\alpha \\ 2x_1 + x_2 - 2x_3 = 1-\alpha \\ 2x_1 - 3x_2 + x_3 = 3-2\alpha \end{cases}$$

$$\Rightarrow x_1 = \frac{\Delta_{x_1}}{\Delta} = \frac{\begin{vmatrix} 5+2\alpha & 1 & 1 \\ 1-\alpha & 1 & -2 \\ 3-2\alpha & -3 & 1 \end{vmatrix}}{-19} = \frac{-38}{-19} = 2$$

$$x_2 = \frac{\Delta_{x_2}}{\Delta} \qquad x_3 = \frac{\Delta_{x_3}}{\Delta}$$

5. $(i)\begin{cases} 2x+2y+3z=3 \\ x-y \quad =1 \\ -x+2y+z=2 \end{cases}$ $\boxed{(ii)\begin{cases} 2x+5y+z=7 \\ x+2y-z=3 \\ x+y-4z=2 \end{cases}}$ $(iii)\begin{cases} x+y+z=3 \\ x-y+z=1 \\ 2x-y+2z=3 \\ x+z=4 \end{cases}$

6. $\begin{cases} 2x_1+x_2+x_3+x_4=1 \\ x_1+2x_2-x_3+4x_4=2 \\ x_1+5x_2-4x_3+11x_4=\lambda \end{cases} \quad (\lambda \in \mathbb{R})$

7. $\begin{cases} ax+y+z=1 \\ x+ay+z=a \\ x+y+az=a^2 \end{cases} \quad (a \in \mathbb{R})$

Sol.: 5(ii) $\left(\begin{array}{ccc|c} 2 & 5 & 1 & 7 \\ 1 & 2 & -1 & 3 \\ 1 & 1 & -4 & 2 \end{array}\right)$

row echelon form : – form where the number of zero elements to the left of the first nonzero element in a row increases strictly with the rows

$$\left(\begin{array}{ccc|c} 2 & 5 & 1 & 7 \\ 1 & 2 & -1 & 3 \\ 1 & 1 & -4 & 2 \end{array}\right) \underset{\sim}{L_1 \leftrightarrow L_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 5 & 1 & 7 \\ 1 & 1 & -4 & 2 \end{array}\right) \sim$$

pivot

pivot line

$$\underset{L_3 \leftarrow L_3 - L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & -1 & -3 & -1 \end{array}\right) \overset{L_3 \leftarrow L_3 + L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

From now on:
- Gauss method
- Gauss - Jordan method

<u>Gauss method</u>: We revert to the initial system and solve it

$$\begin{cases} x + 2y - z = 3 \\ y + 3z = 1 \end{cases} \Rightarrow \begin{cases} y = 1 - 3z \\ x = z + 3 - 2y = z + 3 - 2 + 6z = \\ \quad = 7z + 1 \end{cases}$$

$$\Rightarrow \begin{cases} x = 7\alpha + 1 \\ y = 1 - 3\alpha \\ z = \alpha \end{cases}$$

<u>Gauss - Jordan</u>: Choose pivots in reverse, make zeros above them, find the solutions directly

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \overset{L_1 \leftarrow L_1 - 2L_2}{\sim} \begin{pmatrix} 1 & 0 & -7 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x - 7z = 1 \\ y + 3z = 1 \end{cases} \Rightarrow \begin{cases} x = 1 + 7z \\ y = 1 - 3z \end{cases} \Rightarrow \begin{cases} x = 1 + 7\alpha \\ y = 1 - 3\alpha \\ z = \alpha \end{cases}$$

Remark: (S) is incompatible $\Leftrightarrow$ in the process of applying Gaussian elimination, we obtain a line of the form:

$$\left(0 \quad 0 \quad 0 \quad 0 \quad \ldots \quad 0 \,\middle|\, \alpha\right), \quad \alpha \neq 0$$

7. $\begin{cases} ax + y + z = 1 \\ x + ay + z = a \\ x + y + az = a^2 \end{cases} \quad (a \in \mathbb{R})$

Sol:

$$\left(\begin{array}{ccc|c} a & 1 & 1 & 1 \\ 1 & a & 1 & a \\ 1 & 1 & a & a^2 \end{array}\right) \underset{L_2 \leftrightarrow L_3}{\overset{L_1 \leftrightarrow L_2}{\sim}} \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 1 & 1 & a & a^2 \\ a & 1 & 1 & 1 \end{array}\right) \underset{L_3 \leftarrow L_3 - aL_1}{\overset{L_2 \leftarrow L_2 - L_1}{\sim}}$$

$$\sim \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 0 & 1-a & a-1 & a^2-a \\ 0 & 1-a^2 & 1-a & 1-a^2 \end{array}\right) \overset{L_3 \leftarrow L_3 - (1+a)\cdot L_2}{\sim} \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 0 & 1-a & a-1 & a^2-a \\ 0 & 0 & \underbrace{2-a-a^2}_{(1-a)(2+a)} & (1-a)\cdot(a+1)^2 \end{array}\right)$$

$$1-a^2 + \left(-(1+a)\cdot a(a-1)\right) =$$
$$= (1-a)(a+1)\,(1+a) = (1-a)\,(a+1)^2$$

$$\sim \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 0 & 1-a & a-1 & a^2-a \\ 0 & 0 & (1-a)(2+a) & (1-a)\,(a+1)^2 \end{array}\right)$$

I) $a = 1$:
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \quad x+y+z=1 \quad \Rightarrow \quad \begin{cases} x = -\alpha-\beta \\ y = \alpha \\ z = \beta \end{cases}$$

2) $a=-2 \Rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & -2 \\ 0 & 3 & -3 & 6 \\ 0 & 0 & 0 & 3 \end{array}\right)$

$\Rightarrow$ the system is incompatible

3) $a=-1 \Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right)$

$$\Rightarrow \begin{cases} x-y+z=-1 \\ 2y-2z=0 \\ 2z=0 \end{cases} \Rightarrow \begin{cases} z=0 \\ y=0 \\ x=-1 \end{cases}$$

4) $a \neq 1, -1, -2$:

$$\left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 0 & 1-a & a-1 & a^2-a \\ 0 & 0 & (1-a)(2+a) & (1-a)(a+1)^2 \end{array}\right) \quad \begin{array}{l} L_3 \leftarrow \frac{1}{1-a} L_3 \\ L_2 \leftarrow \frac{1}{1-a} \cdot L_2 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 0 & 1 & -1 & -a \\ 0 & 0 & 2+a & (a+1)^2 \end{array}\right)$$

$$\Rightarrow \begin{cases} x+ay+z=a \\ y-z=-a \\ (2+a)z=(a+1)^2 \end{cases} \Rightarrow \begin{cases} z = \frac{(a+1)^2}{2+a} \\ y = \frac{(a+1)^2}{2+a} - a = \frac{1}{2+a} \\ x = -\frac{(a+1)^2}{2+a} - \frac{a}{a+2} = a - \frac{a^2+3a+1}{a+2} \end{cases}$$

$$\Rightarrow \begin{cases} z = \dfrac{(a+1)^2}{2+a} \\ y = \dfrac{1}{a+2} \\ x = -\dfrac{a+1}{a+2} \end{cases}$$

5 (iii) $\begin{cases} x+y+z=3 \\ x-y+z=1 \\ 2x-y+2z=3 \\ x+z=4 \end{cases}$

Sol.:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & 1 & 1 \\ 2 & -1 & 2 & 3 \\ 1 & 0 & 1 & 4 \end{array}\right) \underset{L_2 \leftarrow L_2 - L_1}{\sim} \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -2 & 0 & -2 \\ 2 & -1 & 2 & 3 \\ 1 & 0 & 1 & 4 \end{pmatrix} \sim$$

$$\underset{L_3 \leftarrow L_3 - 2L_1}{\sim} \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -2 & 0 & -2 \\ 0 & -3 & 0 & -3 \\ 1 & 0 & 1 & 4 \end{pmatrix} \underset{L_4 \leftarrow L_4 - L_1}{\sim} \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -2 & 0 & -2 \\ 0 & -3 & 0 & -3 \\ 0 & -1 & 0 & 1 \end{pmatrix} \sim$$

$$\underset{L_3 \leftarrow L_3 - \frac{3}{2}L_2}{\sim} \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \underset{L_4 \leftarrow L_4 - \frac{1}{2}L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right)$$

$\Rightarrow$ system is incompatible