11.10.2021

Seminar W2 — 915

- r relation : $r = (A, B, R)$

  A — domain, B — codomain

  $R \subseteq A \times B$ — graph

  $(x,y) \in R \Leftrightarrow x\,r\,y$

  - homogeneous relation : $A = B$

    - equivalence relations : they satisfy:
      - reflexivity : $\forall x \in A$: $x\,r\,x$
      - symmetry : $\forall x, y \in A$: if $x\,r\,y$, then $y\,r\,x$
      - transitivity : $\forall x, y, z \in A$: if $x\,r\,y$ and $y\,r\,z$, then $x\,r\,z$

1. Let r, s, t, v be the homogeneous relations defined on the set $M = \{2,3,4,5,6\}$ by

$$x\,r\,y \iff x < y$$

$$x\,s\,y \iff x|y \Leftrightarrow y \,\vdots\, x$$

$$x\,t\,y \iff g.c.d.(x,y) = 1$$

$$x\,v\,y \iff x \equiv y \pmod 3 . \Leftrightarrow 3 \mid (x-y) \Leftrightarrow x \bmod 3 = y \bmod 3$$

Write the graphs R, S, T, V of the given relations.

Sol. : $R = \{(2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6)\}$

$S = \{(2,4), (2,6), (3,6), (2,2), (3,3), (4,4), (5,5), (6,6)\}$

$T = \{(2,3), (2,5), (3,4), (3,5), (4,5), (5,6), (3,2), (5,2), (4,3), (5,3), (5,4), (6,5)\}$

$V = \{(3,6), (2,5), (2,2), (3,3), (4,4), (5,5), (6,3), (5,2), (6,6)\}$

3. Give examples of relations having each one of the properties of reflexivity, transitivity and symmetry, but not the others.

Sol.:

refl., $\neg$ sym, $\neg$ trans	$\neg$ refl., sym., $\neg$ trans	$\neg$ refl., $\neg$ sym., trans
~~$xry \Leftrightarrow x=y$~~ $A = \{1,2,3\}$	$A = \{1,2,3\}$	$A = \{1,2,3\}$
$R = \{(1,1),(2,2),(3,3),(3,1),(1,2)\}$	$R = \{(1,2),(2,1),(1,3),(3,1)\}$	$R = \{(1,2),(2,3),(1,3)\}$

Def.: A set, $P = \{A_i \mid i \in I\} \subseteq \mathcal{P}(A)$ (i.e. $A_i \subseteq A$) is called a partition of A iff:

- $\forall i,j \in I$, $i \neq j$: $A_i \cap A_j = \emptyset$
- $\bigcup_{i \in I} A_i = A$

Th.: A set

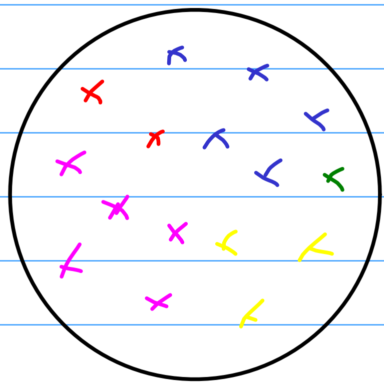
$$\left\{\begin{matrix}\text{equivalence relations}\\ \text{on } A\end{matrix}\right\} \xleftrightarrow{\sim} \left\{\begin{matrix}\text{partitions}\\ \text{of } A\end{matrix}\right\}$$

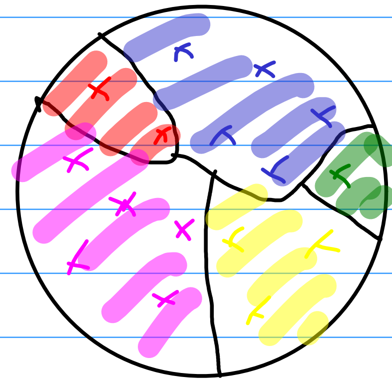
$$r \longmapsto A/r \quad (\text{quotient set})$$

$$r_P \longleftarrow P$$

$$A/r = \{\underbrace{r\langle x\rangle}_{=:\,\hat{x}\, =\, \{y \in A \mid x r y\}} \mid x \in A\}$$

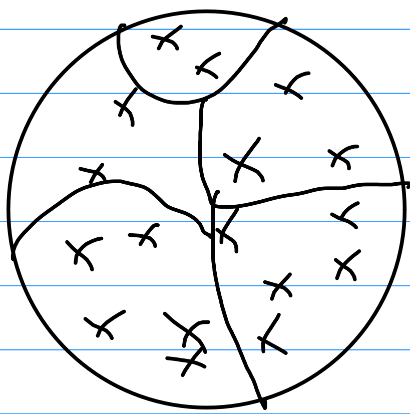
$x \; r_{\mathcal{P}} \; y \iff \exists\, B \in \mathcal{P} : \; x, y \in B$



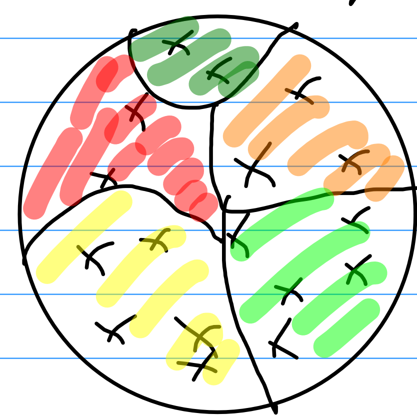


$r =$ "they have the same colour"

$A/r =$ red $\cup$ blue $\cup$ green $\cup$ yellow $\cup$ magenta



$\longrightarrow$



$\mathcal{P}$

$x \; r_{\mathcal{P}} \; y \iff$ "x and y have the same colour"

5. Let $M = \{1,2,3,4\}$, let r_1, r_2 be homogeneous relations on M and let π_1, π_2, where $R_1 = \Delta_M \cup \{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$, $R_2 = \Delta_M \cup \{(1,2),(1,3)\}$, $\pi_1 = \{\{1\},\{2\},\{3,4\}\}$, $\pi_2 = \{\{1\},\{1,2\},\{3,4\}\}$.

(i) Are r_1, r_2 equivalences on M? If yes, write the corresponding partition.

(ii) Are π_1, π_2 partitions on M? If yes, write the corresponding equivalence relation.

$\Delta_M = \{(1,1), (2,2), (3,3), (4,4)\}$

5. Let $M = \{1,2,3,4\}$, let r_1, r_2 be homogeneous relations on M and let π_1, π_2, where $R_1 = \Delta_M \cup \{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$, $R_2 = \Delta_M \cup \{(1,2),(1,3)\}$, $\pi_1 = \{\{1\},\{2\},\{3,4\}\}$, $\pi_2 = \{\{1\},\{1,2\},\{3,4\}\}$.

(i) Are r_1, r_2 equivalences on M? If yes, write the corresponding partition.

(ii) Are π_1, π_2 partitions on M? If yes, write the corresponding equivalence relation.

(i) r_1 is an equivalence

$M/r_1 = \{\{1,2,3\},\{4\}\}$

r_2 is not an equivalence, because it is not symmetrical,

since $(1,2) \in R_2$, but $(2,1) \notin R_2$

(ii) π_1 is a partition

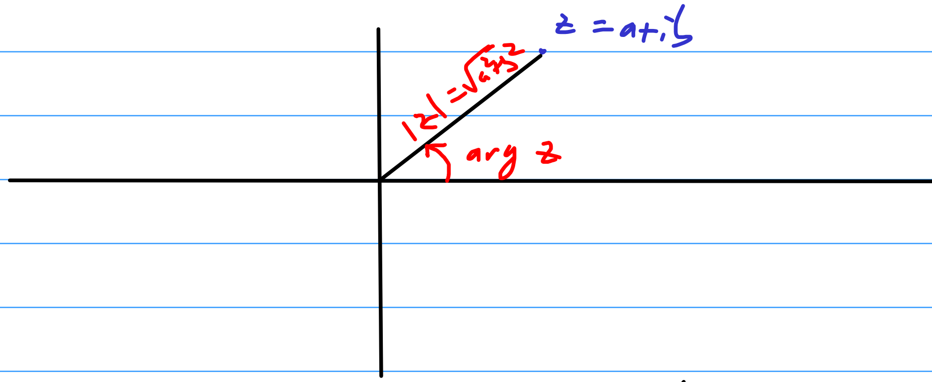
$R_{\pi_1} = \{(1,1),(2,2),(3,3),(4,4),(3,4),(4,3)\}$

π_2 is not a partition, because $1 \in \{1\}$ and $1 \in \{1,2\}$

6. Define on $\mathbb{C}$ the relations r and s by:

$$z_1\, r\, z_2 \iff |z_1| = |z_2|\,; \qquad z_1\, s\, z_2 \iff arg\, z_1 = arg\, z_2 \text{ or } z_1 = z_2 = 0\,.$$

Prove that r and s are equivalence relations on $\mathbb{C}$ and determine the quotient sets (partitions) $\mathbb{C}/r$ and $\mathbb{C}/s$ (geometric interpretation).



$\arg z \in [0, 2\pi)$

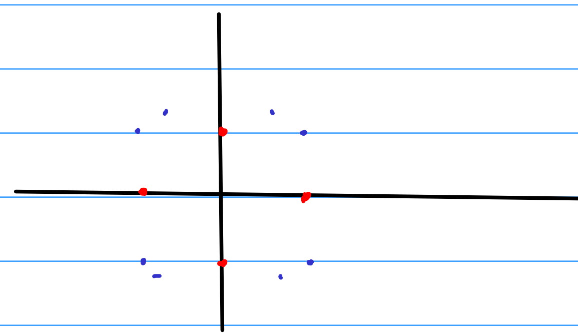
$x, y \in \mathbb{C}$, $x = a+bi$, $y = c+di$

$|x| = \sqrt{a^2+b^2}$ $|y| = \sqrt{c^2+d^2}$

- if $x \, r \, y$, then $|x| = |y| \Rightarrow |y| = |x| \Rightarrow y \, r \, x$
- if $x \in \mathbb{C}$, then $|x| = |x| \Rightarrow x \, r \, x$
- if $x, y, z \in \mathbb{C}$, $x \, r \, y$, $y \, r \, z \Rightarrow |x| = |y|$, $|y| = |z| \Rightarrow |x| = |z|$

$\Rightarrow x \, r \, z$

$\mathbb{C}/r = \{ r\langle x \rangle \mid x \in \mathbb{C} \}$

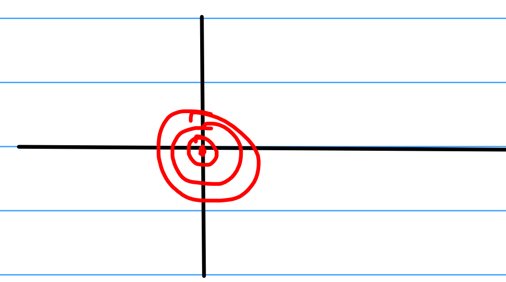


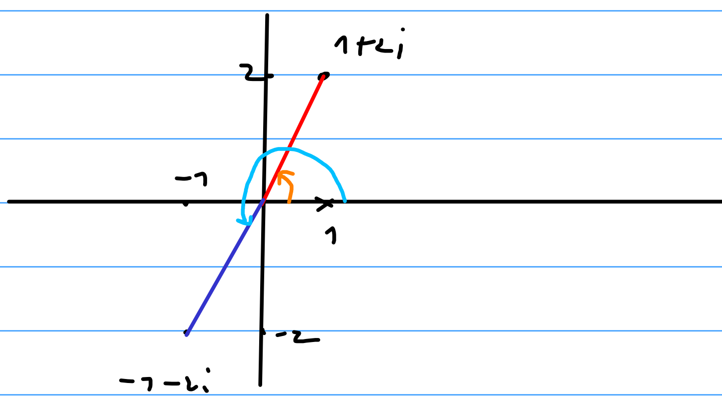
$= \{ \hat{x} \mid x \in \mathbb{C} \}$

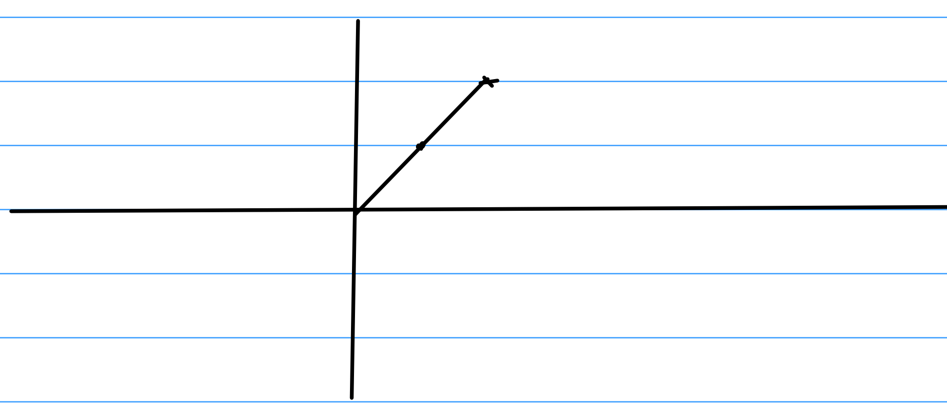
$r\langle x \rangle = \{ y \in \mathbb{C} \mid x \, r \, y \} = \{ y \in \mathbb{C} \mid \underbrace{|x|}_{= k \in \mathbb{R}_{\geq 0}} = |y| \} =$

$= \mathcal{C}(0, k)$

$\Rightarrow \quad \mathbb{C}/r \simeq \bigcup_{k \in \mathbb{R}_{\geq 0}} \mathcal{C}(0, k)$







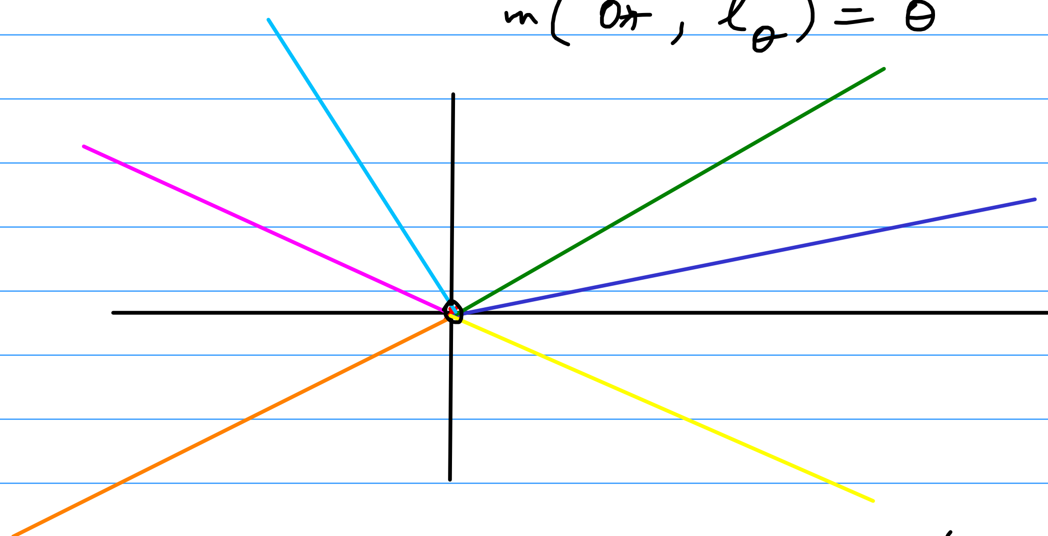
- $\forall z \in \mathbb{C} \Rightarrow \arg z = \arg z \Rightarrow z \sim z$
- $\forall z, w \in \mathbb{C}$, if $\arg z = \arg w \Rightarrow \arg w = \arg z$
- $\forall z_1, z_2, z_3 \in \mathbb{C}$, if $\arg z_1 = \arg z_2$ and $\arg z_2 = \arg z_3 \Rightarrow \arg z_1 = \arg z_3$

$$\mathbb{C}/s = \{ s\langle z\rangle \mid z \in \mathbb{C} \} = \{0\} \cup \{ s\langle z\rangle \mid z \in \mathbb{C}\setminus\{0\} \}$$

$$s\langle z\rangle = \{ w \in \mathbb{C} \mid \arg z = \arg w \text{ or } z = w = 0 \}$$

Let ℓ_θ be the semiline containing the origin, so that

$$m(\widehat{Ox, \ell_\theta}) = \theta$$



$$\Rightarrow \mathbb{C}/s = \{0\} \cup \bigcup_{\theta \in [0, 2\pi)} \ell_\theta$$