06.12.2021

Seminar W10 - 915

Def: V, W K-vector spaces, $\underset{\text{source basis}}{B} = (v_1, \dots, v_m)$ basis for V

$\underset{\text{target basis}}{B'} = (v'_1, v'_2, \dots, v'_n)$ basis for V'

$f : V \to W$ K-linear map. We have:

$$[f]_{B,B'} = \left([f(v_1)]_{B'} \,\middle|\, [f(v_2)]_{B'} \,\middle|\, \dots \,\middle|\, [f(v_m)]_{B'} \right)$$

2. Let $f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2)$ be defined by

$$f(x, y, z) = (y, -x)$$

and consider the bases $B = (v_1, v_2, v_3) = ((1,1,0), (0,1,1), (1,0,1))$ of $\mathbb{R}^3$, $B' = (v'_1, v'_2) = ((1,1), (1,-2))$ of $\mathbb{R}^2$ and let $E' = (e'_1, e'_2)$ be the canonical basis of $\mathbb{R}^2$. Determine the matrices $[f]_{BE'}$ and $[f]_{BB'}$.

Sol: $f(v_1) = f(1,1,0) = (1,-1)$

$f(v_2) = f(0,1,1) = (1,0)$

$f(v_3) = f(1,0,1) = (0,-1)$

$\Rightarrow [f]_{BE'} = \left([f(v_1)]_{E'} \; [f(v_2)]_{E'} \; [f(v_3)]_{E'} \right)$

To find $[f(v_1)]_{E'}$, we must find k_1, k_2 so that

$$f(v_1) = k_1 \cdot \underbrace{e'_1}_{(1,0)} + k \cdot \underbrace{e'_2}_{(0,1)}$$

$\Rightarrow (1,-1) = k_1 \cdot (1,0) + k_2 \cdot (0,1) \Rightarrow (1,-1) = (k_1, k_2) \Rightarrow$

$\Rightarrow k_1 = 1, \quad k_2 = -1 \quad \Rightarrow [f(v_1)]_{E'} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$f(u_2) = (1,0) \Rightarrow [f(u_2)]_{E'} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$f(u_3) = (0,-1) \Rightarrow [f(u_3)]_{E'} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\Rightarrow [f]_{B,E'} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$$

$$[f]_{B,B'} = ?$$

$$f(u_1) = (1,-1), \quad [f(u_1)]_{B'} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

$$(1,-1) = k_1 \cdot u_1' + k_2 \cdot u_2' = k_1 \cdot (1,1) + k_2 \cdot (1,-2)$$

$$\Rightarrow (1,-1) = (k_1 + k_2, k_1 - 2k_2)$$

$$\Rightarrow \begin{cases} k_1 + k_2 = 1 \\ k_1 - 2k_2 = -1 \end{cases}$$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -2 & -1 \end{array}\right) \overset{L_2 \leftarrow L_2 - L_1}{\sim} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -3 & -2 \end{array}\right) \overset{L_1 \leftarrow L_1 + \frac{1}{3} L_3}{\sim} \left(\begin{array}{cc|c} 1 & 0 & \frac{1}{3} \\ 0 & -3 & -2 \end{array}\right)$$

$$\Rightarrow \begin{cases} k_1 = \frac{1}{3} \\ k_2 = \frac{2}{3} \end{cases} \Rightarrow [f(u_1)]_{B'} = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$$

$$f(u_2) = (1,0) \qquad [f(u_2)]_{B'} = \begin{pmatrix} k_1' \\ k_2' \end{pmatrix}$$

$$\Rightarrow (1,0) = k_1' \cdot u_1' + k_2' \cdot u_2'$$

$$\Rightarrow (1,0) = k_1' \cdot (1,1) + k_2' \cdot (1,-2)$$

$$\Rightarrow \begin{cases} 1 = k_1' + k_2' \\ 0 = k_1' - 2k_2' \end{cases} \Rightarrow \begin{cases} k_1' = 2k_2' \\ 1 = 2k_2' + k_2' \end{cases} \Rightarrow \begin{cases} k_2' = \frac{1}{3} \\ k_1' = \frac{2}{3} \end{cases}$$

$$\Rightarrow [f(u_2)]_{B'} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$$

$$f(u_3) = (0,-1) \quad \Rightarrow \quad (0,-1) = k_1'' \cdot u_1' + k_2'' \cdot u_2'$$

$$\Rightarrow (0,-1) = k_1'' \cdot (1,1) + k_2'' \cdot (1,-2)$$

$$\Rightarrow \begin{cases} 0 = k_1'' + k_2'' \\ -1 = k_1'' - 2k_2'' \end{cases} \Rightarrow \begin{cases} k_2'' = -k_1'' \\ -1 = k_1'' + 2k_1'' \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} k_1'' = -\frac{1}{3} \\ k_2'' = \frac{1}{3} \end{cases} \Rightarrow [f(u_3)]_{B'} = \begin{pmatrix} -1/3 \\ 1/3 \end{pmatrix}$$

$$\left.\begin{aligned} [f(u_1)]_{B'} &= \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix} \\ [f(u_2)]_{B'} &= \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} \\ [f(u_3)]_{B'} &= \begin{pmatrix} -1/3 \\ 1/3 \end{pmatrix} \end{aligned}\right\} \Rightarrow [f]_{B,B'} = \begin{pmatrix} 1/3 & 2/3 & -1/3 \\ 2/3 & 1/3 & 1/3 \end{pmatrix}$$

4. Let $f \in End_{\mathbb{R}}(\mathbb{R}^4)$ with the following matrix in the canonical basis E of $\mathbb{R}^4$:

$$\underset{[f]_{E,E}}{[f]_E} = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix}.$$

$$E = \big(\underbrace{(1,0,0,0)}_{e_1}, \underbrace{(0,1,0,0)}_{e_2}, \underbrace{(0,0,1,0)}_{e_3}, \underbrace{(0,0,0,1)}_{e_4}\big)$$

(i) Show that $v = (1,4,1,-1) \in Ker\, f$ and $v' = (2,-2,4,2) \in Im\, f$.
(ii) Determine a basis and the dimension of $Ker\, f$ and $Im\, f$.
(iii) Define f.

Prop.: $f : V \to V'$, B, B' basis for V, V'

$\forall\ u \in V$: $\boxed{[f(u)]_{B'} = [f]_{B,B'} \cdot [u]_B}$

4. Let $f \in End_{\mathbb{R}}(\mathbb{R}^4)$ with the following matrix in the canonical basis E of $\mathbb{R}^4$:

$$[f]_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix}.$$

(i) Show that $v = (1, 4, 1, -1) \in Ker\, f$ and $v' = (2, -2, 4, 2) \in Im\, f$.
(ii) Determine a basis and the dimension of $Ker\, f$ and $Im\, f$.
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(i) $u \in Ker f \iff f(u) = 0$

$$[f(u)]_E = [f]_E \cdot [u]_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ 1 \\ -1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow f(u) = 0 \Rightarrow u \in Ker f$$

$$Im f = \{ (a, b, c, d) \mid \exists\, u = (x, y, z, t) : f(x, y, z, t) = (a, b, c, d) \}$$

$u' \in Im f \iff \exists\, x, y, z, t : f(x, y, z, t) = u' \iff \exists\, x, y, z, t : [u']_E = [f]_E \cdot \underbrace{[u]_E}_{\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}}$

$$\Rightarrow \begin{pmatrix} 2 \\ -2 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ -1 & 1 & 1 & 4 & -2 \\ 2 & 1 & -5 & 1 & 4 \\ 1 & 2 & -4 & 5 & 2 \end{array}\right) \underset{\substack{L_3 \leftarrow L_3 - 2L_1 \\ L_4 \leftarrow L_4 - L_1}}{\overset{L_2 \leftarrow L_2 + L_1}{\sim}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ 0 & 2 & -2 & 6 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right) \sim$$

$$\underset{L_4 \leftarrow L_4 + \frac{1}{2}L_2}{\overset{L_3 \leftarrow L_3 + \frac{1}{2}L_2}{\sim}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ 0 & 2 & -2 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow$ the system is compatible $\Rightarrow$

$\Leftrightarrow \exists u = (x,y,z,t) : f(u) = u'$

$\Rightarrow u' \in \operatorname{Im} f$

(ii) $\operatorname{Ker} f = \{ u \in \mathbb{R}^4 \mid f(u) = 0 \} =$

$= \left\{ u = (x,y,z,t) \;\middle|\; [f(u)]_E = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} =$

$= \left\{ u = (x,y,z,t) \;\middle|\; [f]_E \cdot [u]_E = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} =$

$= \left\{ u = (x,y,z,t) \;\middle|\; \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} =$

$= \left\{ u = (x,y,z,t) \;\middle|\; \begin{cases} x + y - 3z + 2t = 0 \\ -x + y + z + 4t = 0 \\ 2x + y - 5z + t = 0 \\ x + 2y - 4z + 5t = 0 \end{cases} \right\}$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ -1 & 1 & 1 & 4 & 0 \\ 2 & 1 & -5 & 1 & 0 \\ 1 & 2 & -4 & 5 & 0 \end{array}\right) \underset{\substack{L_3 \leftarrow L_3 - 2L_1 \\ L_4 \leftarrow L_4 - L_1}}{\overset{L_2 \leftarrow L_2 + L_1}{\sim}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & 2 & -2 & 6 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right) \overset{L_2 \leftarrow \frac{1}{2}L_2}{\sim}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right) \underset{L_4 \leftarrow L_4 - L_2}{\overset{L_3 \leftarrow L_3 + L_2}{\sim}} \left(\begin{array}{cccc|c} 1 & -1 & -3 & 2 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \begin{cases} x - y - 3z + 2t = 0 \\ y - z + 3t = 0 \end{cases} \Leftrightarrow \begin{cases} y = z - 3t \\ x = z - 3t + 3z - 2t = 4z - 5t \end{cases}$$

$$\Rightarrow \operatorname{Ker} f = \{ (x,y,z,t) \in \mathbb{R}^4 \mid x = 4z - 5t,\ y = z - 3t \} =$$

$$= \{ (4z - 5t,\ z - 3t,\ z,\ t) \mid z, t \in \mathbb{R} \} =$$

$$= \{ z \cdot (4,1,1,0) + t \cdot (-5,-3,0,1) \mid z,t \in \mathbb{R} \} =$$

$$= \langle (4,1,1,0),\ (-5,-3,0,1) \rangle$$

$$\begin{pmatrix} 4 & 1 & 1 & 0 \\ -5 & -3 & 0 & 1 \end{pmatrix} \overset{L_2 \leftarrow L_2 + \frac{5}{4} L_1}{\sim} \begin{pmatrix} 4 & 1 & 1 & 0 \\ 0 & -\frac{7}{4} & \frac{5}{4} & 1 \end{pmatrix} \sim$$

$$\overset{L_2 \leftarrow 4L_2}{\sim} \begin{pmatrix} 4 & 1 & 1 & 0 \\ 0 & -7 & 5 & 1 \end{pmatrix} \Rightarrow \text{basis for } \operatorname{Ker} f:$$

$$\big((4,1,1,0),\ (0,-7,5,1) \big)$$

$$\dim \operatorname{Ker} f = 2$$

$$\operatorname{Im} f = \{ w = f(u) \mid u \in \mathbb{R}^4 \} =$$

$$= \{ w = (a,b,c,d) \mid \exists\, u = (x,y,z,t) \in \mathbb{R}^4 : f(u) = w \} =$$

$$= \{ w = (a,b,c,d) \mid \exists\, x,y,z,t \in \mathbb{R} : [w]_E = [f]_E \cdot [u]_E \} =$$

$$= \left\{ w = (a,b,c,d) \;\middle|\; \exists x,y,z,t : \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \right\}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ -1 & 1 & 1 & 4 & b \\ 2 & 1 & -5 & 1 & c \\ 1 & 2 & -4 & 5 & d \end{array}\right) \underset{\substack{L_3 \leftarrow L_3 - 2L_1 \\ L_4 \leftarrow L_4 - L_1}}{\overset{L_2 \leftarrow L_2 + L_1}{\sim}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ 0 & 2 & -2 & 6 & a+b \\ 0 & -1 & 1 & -3 & c-2a \\ 0 & 1 & -1 & 3 & d-a \end{array}\right) \sim$$

$$\overset{L_2 \leftrightarrow L_4}{\sim} \left(\begin{array}{ccccc} 1 & 1 & -3 & 2 & a \\ 0 & 1 & -1 & 3 & d-a \\ 0 & -1 & 1 & -3 & c-2a \\ 0 & 2 & -2 & 6 & a+b \end{array}\right) \underset{L_4 \leftarrow L_4 - 2L_2}{\overset{L_3 \leftarrow L_3 + L_2}{\sim}}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ 0 & 1 & -1 & 3 & d-a \\ 0 & 0 & 0 & 0 & c+d-3a \\ 0 & 0 & 0 & 0 & a+b-2d+2a \end{array}\right)$$

$\Rightarrow$ the system is compatible (i.e. $\exists x,y,z,t$) iff

$$\begin{cases} c+d-3a = 0 \\ 3a+b-2d = 0 \end{cases}$$

$$\Rightarrow \operatorname{Im} f = \left\{ w = (a,b,c,d) \in \mathbb{R}^4 \;\middle|\; \begin{cases} c+d-3a=0 \\ 3a+b-2d=0 \end{cases} \right\}$$

$$\left(\begin{array}{cccc|c} -3 & 0 & 1 & 1 & 0 \\ 3 & 1 & 0 & -2 & 0 \end{array}\right) \overset{L_2 \leftarrow L_2 + L_1}{\sim} \left(\begin{array}{cccc|c} -3 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{array}\right)$$

$$\Rightarrow \operatorname{Im} f = \left\{ (a,b,c,d) \in \mathbb{R}^4 \;\middle|\; \begin{cases} -3a+c+d=0 \\ b+c-d=0 \end{cases} \right\}$$

$$\Rightarrow \operatorname{Im} f = \left\{ (a,b,c,d) \in \mathbb{R}^4 \;\middle|\; \begin{cases} a = \dfrac{c+d}{3} \\ b = d - c \end{cases} \right\} =$$

$$= \left\{ \left(\frac{c+d}{3}, d-c, c, d \right) \;\middle|\; c, d \in \mathbb{R} \right\} =$$

$$= \left\langle \left(\frac{1}{3}, -1, 1, 0 \right), \left(\frac{1}{3}, 1, 0, 1 \right) \right\rangle =$$

$$= \langle (1, -3, 3, 0), (1, 3, 0, 3) \rangle$$

$$\begin{pmatrix} 1 & -3 & 3 & 0 \\ 1 & 3 & 0 & 3 \end{pmatrix} \overset{L_2 \leftarrow L_2 - L_1}{\sim} \begin{pmatrix} 1 & -3 & 3 & 0 \\ 0 & 6 & -3 & 3 \end{pmatrix} \sim$$

$$\overset{L_2 \leftarrow \frac{1}{3} L_2}{\sim} \begin{pmatrix} 1 & -3 & 3 & 0 \\ 0 & 2 & -1 & 1 \end{pmatrix}$$

$\Rightarrow$ basis for $\operatorname{Im} f$: $\big((1, -3, 3, 0), (0, 2, -1, 1) \big)$

$$\Rightarrow \dim(\operatorname{Im} f) = 2$$

Verification : $\underbrace{\dim(\operatorname{Ker} f)}_{2} + \underbrace{\dim \operatorname{Im}(f)}_{2} = \underbrace{\dim(\mathbb{R}^4)}_{4}$

(ii) In order to define f, we need to find $f(x,y,z,t)$.

$$[f(x,y,z,t)]_E = [f]_E \cdot [(x,y,z,t)]_E =$$

$$= \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} =$$

$$= \begin{pmatrix} x+y-3z+2t \\ -x+y+z+4t \\ 2x+y-5z+t \\ x+2y-4z+5t \end{pmatrix}$$

$$\Rightarrow f(x,y,z,t) = (x+y-3z+2t)\cdot \overset{(1,0,0,0)}{\overset{\parallel}{e_1}} + (-x+y+z+4t)\cdot \overset{(0,1,0,0)}{\overset{\parallel}{e_2}} +$$

$$+ (2x+y-5z+t)\cdot \underset{(0,0,1,0)}{\underset{\parallel}{e_3}} + (x+2y-4z+5t)\cdot \underset{(0,0,0,1)}{\underset{\parallel}{e_4}}$$

$$\Rightarrow f(x,y,z,t) = (x+y-3z+2t,\ -x+y+z+4t,\ 2x+y-5z+t,\ x+2y-4z+5t)$$