24. 11. 2021

Seminar W9 – 917

Compute by applying elementary operations the ranks of the matrices:

1. $\begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix}$. **2.** $\begin{pmatrix} 1 & -1 & 3 & 2 \\ -2 & 0 & 3 & -1 \\ -1 & 2 & 0 & -1 \end{pmatrix}$. **3.** $\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix}$ $(\alpha, \beta \in \mathbb{R})$.

9.2.

$$M = \begin{pmatrix} 1 & -1 & 3 & 2 \\ -2 & 0 & 3 & -1 \\ -1 & 2 & 0 & -1 \end{pmatrix} \underset{L_3 \leftarrow L_3 + L_1}{\overset{L_2 \leftarrow L_2 + 2L_1}{\sim}} \begin{pmatrix} 1 & -1 & 3 & 2 \\ 0 & -2 & 9 & 3 \\ 0 & 1 & 3 & 1 \end{pmatrix} \sim$$

$$\overset{L_3 \leftarrow L_3 + \frac{1}{2}L_2}{\sim} \begin{pmatrix} 1 & -1 & 3 & 2 \\ 0 & -2 & 9 & 3 \\ 0 & 0 & \frac{15}{2} & \frac{5}{2} \end{pmatrix} \Rightarrow \text{rank } M = \text{\# of nonzero rows in the echelon form} = 3$$

9.3.

$$\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix} \underset{L_2 \leftrightarrow L_3}{\overset{L_1 \leftrightarrow L_2}{\sim}} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \\ \beta & 1 & 3 & 4 \end{pmatrix} \underset{L_3 \leftarrow L_3 - \beta L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}}$$

$$\sim \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & \alpha & -2 & 1 \\ 0 & 1-\alpha\beta & 3-3\beta & 4-3\beta \end{pmatrix}$$

I) $\alpha = 0 \Rightarrow \begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 0 & -2 & 1 \\ 0 & 1 & 3-3\beta & 4-3\beta \end{pmatrix} \overset{L_2 \leftrightarrow L_3}{\sim} \begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 3-3\beta & 4-3\beta \\ 0 & 0 & -2 & 1 \end{pmatrix}$

$\Rightarrow \text{rank } M = 3$

3) $\alpha \neq 0 \Rightarrow \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & \alpha & -2 & 1 \\ 0 & 1-\alpha\beta & 3-3\beta & 4-3\beta \end{pmatrix} \overset{L_3 \leftarrow \left(-\frac{1}{\alpha}+\beta\right)L_2 + L_3}{\sim}$

$$\sim \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & \alpha & -2 & 1 \\ 0 & 0 & 3-3\beta+\frac{2}{\alpha}-2\beta & 4-3\beta-\frac{1}{\alpha}+\beta \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & \alpha & -2 & 1 \\ 0 & 0 & 3+\frac{2}{\alpha}-5\beta & 4-\frac{1}{\alpha}-2\beta \end{pmatrix}$$

$\Rightarrow \operatorname{rank} M \in \{2, 3\}$

$$\operatorname{rank} M = 2 \iff \begin{cases} 3+\frac{2}{\alpha}-5\beta = 0 \\ 4-\frac{1}{\alpha}-2\beta = 0 \end{cases} \iff$$

$$\iff \begin{cases} \frac{1}{\alpha} = 4-2\beta \\ 3+\frac{2}{\alpha}-5\beta = 0 \end{cases} \iff \begin{cases} \frac{1}{\alpha} = 4-2\beta \\ 3+8-4\beta-5\beta = 0 \end{cases} \iff$$

$$\iff \begin{cases} \frac{1}{\alpha} = 4-2\beta \\ 11-9\beta = 0 \end{cases} \iff \begin{cases} \beta = \frac{11}{9} \\ \alpha = \frac{1}{4-2\cdot\frac{11}{9}} = \frac{9}{14} \end{cases}$$

To invert a matrix A

$$\left(A \,\middle|\, I_n\right) \sim \underset{\text{Gauss-Jordan}}{\cdots} \sim \left(I_n \,\middle|\, A^{-1}\right)$$

Compute by applying elementary operations the inverses of the matrices:

4. $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$. **5.** $\begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}$.

Sol.: 4.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & -2 & 0 & 1 & 0 \\ 2 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \underset{L_3 \leftarrow L_3 - 2L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right) \sim$$

$$\overset{L_2 \leftarrow \frac{1}{-3} \cdot L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right) \overset{L_3 \leftarrow L_3 + 6L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 9 & 2 & -2 & 1 \end{array}\right) \sim$$

$$\overset{L_3 \leftarrow \frac{1}{9} L_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & \frac{-1}{3} & 0 \\ 0 & 0 & 1 & \frac{2}{9} & \frac{-2}{9} & \frac{1}{9} \end{array}\right) \underset{L_1 \leftarrow L_1 - 2L_3}{\overset{L_2 \leftarrow L_2 - 2L_3}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{5}{9} & \frac{4}{9} & \frac{-2}{9} \\ 0 & 1 & 0 & \frac{2}{9} & \frac{1}{9} & \frac{-2}{9} \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{array}\right) \sim$$

$$\overset{L_1 \leftarrow L_1 - 2L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ 0 & 1 & 0 & \frac{2}{9} & \frac{2}{9} & -\frac{2}{9} \\ 0 & 0 & 1 & \frac{2}{9} & \frac{-2}{9} & \frac{1}{9} \end{array}\right)$$

$$\Rightarrow A^{-1} = \begin{pmatrix} \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ \frac{2}{9} & \frac{-2}{9} & \frac{1}{9} \end{pmatrix}$$

Ex :
$$\begin{pmatrix} 1 & -2 & 3 \\ 1 & 2 & 5 \\ 3 & -2 & 11 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 1 & 2 & 5 & 0 & 1 & 0 \\ 3 & -2 & 11 & 0 & 0 & 1 \end{array}\right) \underset{L_3 \leftarrow L_3 - 3L_1}{\overset{L_2 \leftarrow L_2 - L_1}{\sim}}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 4 & 2 & -1 & 1 & 0 \\ 0 & 4 & 2 & -3 & 0 & 1 \end{array}\right) \overset{L_3 \leftarrow L_3 - L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 4 & 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 & -1 & 1 \end{array}\right)$$

We got a zero row on the left, therefore the algorithm cannot continue, so the initial matrix was not invertible

5. Compute by applying elementary operations the inverses of the matrices:

4. $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$. **5.** $\begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}$.

Sol. :
$$\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \underset{L_3 \leftarrow L_3 - 3L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right) \sim$$

$$\overset{L_2 \leftarrow -\frac{1}{5}L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right) \overset{L_3 \leftarrow L_3 + 12L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} & \frac{9}{5} & -\frac{12}{5} & 1 \end{array}\right) \sim$$

$\underset{\sim}{L_3 \leftarrow 5L_3}$ $\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right) \underset{L_1 \leftarrow L_1 - 2L_3}{\overset{L_2 \leftarrow L_2 - \frac{3}{5}L_3}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 4 & 0 & -17 & 24 & -10 \\ 0 & 1 & 0 & -5 & 7 & -3 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right) \sim$

$\underset{\sim}{L_1 \leftarrow L_1 - 4L_2}$ $\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 2 \\ 0 & 1 & 0 & -5 & 7 & -3 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right)$

$$\Rightarrow A^{-1} = \begin{pmatrix} 3 & -4 & 2 \\ -5 & 7 & -3 \\ 9 & -12 & 5 \end{pmatrix}$$

To check: $\begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 3 & -4 & 2 \\ -5 & 7 & -3 \\ 9 & -12 & 5 \end{pmatrix} =$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

7. In the real vector space $\mathbb{R}^3$ consider the list $X = (v_1, v_2, v_3, v_4)$, where $v_1 = (1, 0, 4)$, $v_2 = (2, 1, 0)$, $v_3 = (1, 5, -36)$ and $v_4 = (2, 10, -72)$. Determine $dim < X >$ and a basis of $< X >$.

Sol.: $\begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 5 & -36 \\ 2 & 10 & -72 \end{pmatrix} \underset{\substack{L_3 \leftarrow L_3 - L_1 \\ L_4 \leftarrow L_4 - 2L_1}}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 5 & -40 \\ 0 & 10 & -80 \end{pmatrix} \sim$

$\underset{L_4 - 10L_2}{\overset{L_3 - 5L_2}{\sim}} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\Rightarrow \dim < X > = 2$

Basis for $< X >$: $((1, 0, 4), (0, 1, -8))$

10. Determine the dimension of the subspaces S, T, $S+T$ and $S \cap T$ of the real vector space $\mathbb{R}^4$ and a basis for the first three of them, where

$$S = < (1,2,-1,-2), (3,1,1,1), (-1,0,1,-1) > ,$$

$$T = < (2,5,-6,-5), (-1,2,-7,-3) > .$$

Sol.:
$$\begin{pmatrix} 1 & 2 & -1 & -2 \\ 3 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 \end{pmatrix} \underset{L_3 \leftarrow L_3 + L_1}{\overset{L_2 \leftarrow L_2 - 3L_1}{\sim}} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -5 & 4 & 7 \\ 0 & 2 & 0 & -3 \end{pmatrix} \sim$$

$$\overset{L_3 \leftarrow L_3 + \frac{2}{5}L_2}{\sim} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -5 & 4 & 7 \\ 0 & 0 & \frac{8}{5} & -\frac{1}{5} \end{pmatrix}$$

$\Rightarrow$ $\dim S = 3$ and $\left((1,2,-1,-2), (0,-5,4,7), (0,0,\frac{8}{5},-\frac{1}{5})\right)$ basis for S

For T:

$$\begin{pmatrix} 2 & 5 & -6 & -5 \\ -1 & 2 & -7 & -3 \end{pmatrix} \overset{L_1 \leftrightarrow L_2}{\sim} \begin{pmatrix} -1 & 2 & -7 & -3 \\ 2 & 5 & -6 & -5 \end{pmatrix} \sim$$

$$\overset{L_2 \leftarrow L_2 + 2L_1}{\sim} \begin{pmatrix} -1 & 2 & -7 & -3 \\ 0 & 9 & -20 & -11 \end{pmatrix}$$

$\Rightarrow \dim T = 2$, basis for T: $\left((-1,2,7,-3), (0,9,-20,-11)\right)$

To find a basis for $S+T$:

$$\begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -5 & 4 & 7 \\ 0 & 0 & \frac{8}{5} & -\frac{2}{5} \\ -1 & 2 & -7 & -3 \\ 0 & 9 & -20 & -11 \end{pmatrix} \overset{L_4 \leftarrow L_4 + L_1}{\sim} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -5 & 4 & 7 \\ 0 & 0 & \frac{8}{5} & -\frac{2}{5} \\ 0 & 4 & -8 & -5 \\ 0 & 9 & -20 & -11 \end{pmatrix} \sim$$

$$\begin{matrix} L_4 \leftarrow L_4 + \frac{4}{5} L_2 \\ \sim \\ L_5 \leftarrow L_5 + \frac{9}{5} L_2 \end{matrix} \quad \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -5 & 4 & 7 \\ 0 & 0 & \frac{8}{5} & -\frac{2}{5} \\ 0 & 0 & -\frac{24}{5} & \frac{3}{5} \\ 0 & 0 & -\frac{64}{5} & \frac{8}{5} \end{pmatrix} \sim$$

$$\begin{matrix} L_4 \leftarrow L_4 + 3L_3 \\ \sim \\ L_5 \leftarrow L_5 + 8L_3 \end{matrix} \quad \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -5 & 4 & 7 \\ 0 & 0 & \frac{8}{5} & -\frac{2}{5} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \dim(S+T) = 3$ basis for $S+T$:

$$\left((1,2,-1,-2), (0,-5,4,7), \left(0,0,\tfrac{8}{5},-\tfrac{2}{5}\right) \right)$$

$$\dim(S \cap T) = \dim S + \dim T - \dim(S+T) =$$

$$= 3 + 2 - 3 = 2$$

Ex: $S = \langle (2,1), (4,3) \rangle$, $T = \langle (0,5), (6,7) \rangle$

$$S \cap T = \{ \alpha_1 \cdot (2,1) + \alpha_2 \cdot (4,3) \mid \exists \beta_1, \beta_2 : \alpha_1 (2,1) + \alpha_2 (4,3) = \beta_1 (0,5) + \beta_2 \cdot (6,7) \}$$