06.12.2021

Seminar W10 – 916

Def: V, W vector spaces, $B = (v_1, v_2, \ldots, v_m)$ basis for V

$B' = (v_1', v_2', \ldots, v_n')$ basis for W

$f : V \to W$ linear map.

$$[f]_{B,B'} = \left([f(v_1)]_{B'} \quad [f(v_2)]_{B'} \quad \cdots \quad [f(v_m)]_{B'} \right)$$

Prop: V, W K-v.s., $f : V \to W$ K-linear map, B, B' bases for V, W

$$\Rightarrow \forall v \in V : \boxed{\underbrace{[f(v)]_{B'}}_{n\times 1} = \underbrace{[f]_{B,B'}}_{n\times m} \cdot \underbrace{[v]_B}_{m\times 1}}$$

2. Let $f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2)$ be defined by

$$f(x, y, z) = (y, -x)$$

and consider the bases $B = (v_1, v_2, v_3) = ((1,1,0), (0,1,1), (1,0,1))$ of $\mathbb{R}^3$, $B' = (v_1', v_2') = ((1,1), (1,-2))$ of $\mathbb{R}^2$ and let $E' = (e_1', e_2')$ be the canonical basis of $\mathbb{R}^2$. Determine the matrices $[f]_{BE'}$ and $[f]_{BB'}$.

$e_1' = (1,0)$, $e_2' = (0,1)$

$e_1 = (1,0,0)$

$e_2 = (0,1,0)$

$e_3 = (0,0,1)$

Sol: To show an example, we will compute $[f]_{E,B'}$

$$[f]_{E,B'} = \left([f(e_1)]_{B'} \quad [f(e_2)]_{B'} \quad [f(e_3)]_{B'} \right)$$

$$f(e_1) = f(1,0,0) = (0,-1) = \alpha_1 \cdot v_1' + \beta_1 \cdot v_2', \qquad [f(e_1)]_{B'} = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$$

$$\Rightarrow (0,-1) = \alpha_1 \cdot (1,1) + \beta_1 \cdot (1,-2)$$

$$\Rightarrow \begin{cases} 0 = \alpha_1 + \beta_1 \\ -1 = \alpha_1 - 2\beta_1 \end{cases} \Leftrightarrow \begin{cases} \beta_1 = -\alpha_1 \\ -1 = \alpha_1 + 2\alpha_1 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = -\frac{1}{3} \\ \beta_1 = \frac{1}{3} \end{cases}$$

$$\Rightarrow [f(e_1)]_{B'} = \begin{pmatrix} -1/3 \\ 1/3 \end{pmatrix}$$

$$f(e_2) = f(0,1,0) = (1,0) = \alpha_2 \cdot e_1' + \beta_2 \cdot e_2'$$

$$\Rightarrow (1,0) = \alpha_2 \cdot (1,1) + \beta_2 \cdot (1,-2)$$

$$\Rightarrow \begin{cases} 1 = \alpha_2 + \beta_2 \\ 0 = \alpha_2 - 2\beta_2 \end{cases} \Leftrightarrow \begin{cases} \alpha_2 = 2\beta_2 \\ 1 = 2\beta_2 + \beta_2 \end{cases} \Leftrightarrow \begin{cases} \beta_2 = \frac{1}{3} \\ \alpha_2 = \frac{2}{3} \end{cases}$$

$$\Rightarrow [f(e_2)]_{B'} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$$

$$f(e_3) = f(0,0,1) = (0,0) = \alpha_3 \cdot (1,1) + \beta_3 \cdot (1,-2)$$

$$\begin{cases} 0 = \alpha_3 + \beta_3 \\ 0 = \alpha_3 - 2\beta_3 \end{cases} \Leftrightarrow \begin{cases} \alpha_3 = 2\beta_3 \\ 0 = 2\beta_3 + \beta_3 \end{cases} \Leftrightarrow \begin{cases} \beta_3 = 0 \\ \alpha_3 = 0 \end{cases}$$

$$\Rightarrow [f(e_3)]_{B'} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} [f(e_1)]_{B'} &= \begin{pmatrix} -1/3 \\ 1/3 \end{pmatrix} \\ [f(e_2)]_{B'} &= \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} \\ [f(e_3)]_{B'} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}\right\} \Rightarrow [f]_{E,B'} = \begin{pmatrix} -1/3 & 2/3 & 0 \\ 1/3 & 1/3 & 0 \end{pmatrix}$$

$$f(u_1) = f(1,1,0) = (1,-1) = \alpha_1 \cdot e_1' + \beta_1 \cdot e_2'$$

$$\Rightarrow \alpha_1(1,0)+\beta_1\cdot(0,1)=(1,-1) \Rightarrow \alpha_1=1 \quad \beta_1=-1$$

$$\Rightarrow [f(v_1)]_{B,E'} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$f(v_2)=f(0,1,1)=(1,0) \Rightarrow [f(v_2)]_E = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$f(v_3)=f(1,0,1)=(0,-1) \Rightarrow [f(v_3)]_E = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\Rightarrow [f]_{B,E'} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$$

$$f(v_1)=(1,-1) = \alpha_1\cdot v_1' + \beta_1\cdot v_2'$$

$$\Rightarrow (1,-1) = \alpha_1\cdot(1,1)+\beta_1\cdot(1,-2)$$

$$\Rightarrow \begin{cases} \alpha_1+\beta_1=1 \\ \alpha_1-2\beta_1=-1 \end{cases} \Rightarrow \begin{cases} \alpha_1=\frac{1}{3} \\ \beta_1=\frac{2}{3} \end{cases} \Rightarrow [f(v_1)]_{B'} = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$$

$$f(v_2)=f(0,1,1)=(1,0)=\alpha_2\cdot(1,1)+\beta_2\cdot(1,-2)$$

$$\Rightarrow \begin{cases} \alpha_2+\beta_2=1 \\ \alpha_2-2\beta_2=0 \end{cases} \Rightarrow \begin{cases} \alpha_2=\frac{2}{3} \\ \beta_2=\frac{1}{3} \end{cases} \Rightarrow [f(v_2)]_{B'} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$$

$$f(v_3)=f(1,0,1)=(0,-1)=\alpha_3\cdot(1,1)+\beta_3\cdot(1,-2)$$

$$\Rightarrow \begin{cases} \alpha_3+\beta_3=0 \\ \alpha_3-2\beta_3=-1 \end{cases} \Rightarrow \begin{cases} \alpha_3=-\frac{1}{3} \\ \beta_3=\frac{1}{3} \end{cases} \Rightarrow [f(v_3)]_{B'} = \begin{pmatrix} -1/3 \\ 1/3 \end{pmatrix}$$

<u>Verification</u>: Make sure that by plugging in the coords. you get the right result

$$[f(v_1)]_{B'} = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$$

$$[f(v_2)]_{B'} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$$

$$[f(v_3)]_{B'} = \begin{pmatrix} -1/3 \\ 1/3 \end{pmatrix}$$

$$\Rightarrow [f]_{B,B'} = \begin{pmatrix} 1/3 & 2/3 & -1/3 \\ 2/3 & 1/3 & 1/3 \end{pmatrix}$$

4. Let $f \in End_{\mathbb{R}}(\mathbb{R}^4)$ with the following matrix in the canonical basis E of $\mathbb{R}^4$:

$$[f]_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix}.$$

(i) Show that $v = (1, 4, 1, -1) \in Ker\, f$ and $v' = (2, -2, 4, 2) \in Im\, f$.
(ii) Determine a basis and the dimension of $Ker\, f$ and $Im\, f$.
(iii) Define f.

Prop: V, W K-v.s., $f: V \to W$ K-linear map, B, B' bases for V, W

$$\Rightarrow \forall v \in V: \boxed{\underbrace{[f(v)]_{B'}}_{n \times 1} = \underbrace{[f]_{B,B'}}_{n \times m} \cdot \underbrace{[v]_B}_{m \times 1}}$$

(i) $v' \in Im\, f \Leftrightarrow \exists \underset{(x,y,z,t)}{v} \in \mathbb{R}^4$: $f(v) = v'$

$$[v']_E = [f]_E \cdot [v]_E$$

$$\Rightarrow \begin{pmatrix} 2 \\ -2 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ -1 & 1 & 1 & 4 & -2 \\ 2 & 1 & -5 & 1 & 4 \\ 1 & 2 & -4 & 5 & 2 \end{array}\right) \underset{\substack{L_3 \leftarrow L_3 - 2L_1 \\ L_4 \leftarrow L_4 - L_1}}{\overset{L_2 \leftarrow L_2 + L_1}{\sim}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ 0 & 2 & -2 & 6 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right) \overset{L_2 \leftarrow \frac{1}{2} L_2}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right) \sim$$

$$\underset{L_4 \leftarrow L_4 - L_2}{\overset{L_3 \leftarrow L_3 + L_2}{\sim}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \text{system is compatible} \Rightarrow$$

$$\Rightarrow \exists u \in \mathbb{R}^4 : f(u) = v' \Rightarrow v' \in \operatorname{Im} f$$

$v = (1, 4, 1, -1)$

$$v \in \operatorname{Ker} f \Leftrightarrow f(v) = 0 \Leftrightarrow [f]_E \cdot [v]_E = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$[f]_E \cdot [v]_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow v \in \operatorname{Ker} f$$

(ii) $$\operatorname{Ker} f = \{ v = (x, y, z, t) \in \mathbb{R}^4 \mid f(v) = 0 \} =$$

$$= \left\{ v = (x, y, z, t) \in \mathbb{R}^4 \;\middle|\; [f]_E \cdot [v]_E = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ v = (x, y, z, t) \in \mathbb{R}^4 \;\middle|\; \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ -1 & 1 & 1 & 4 & 0 \\ 2 & 1 & -5 & 1 & 0 \\ 1 & 2 & -4 & 5 & 0 \end{array}\right) \underset{\substack{L_3 \leftarrow L_3 - 2L_1 \\ L_4 \leftarrow L_4 - L_1}}{\overset{L_2 \leftarrow L_2 + L_1}{\sim}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & 2 & -2 & 6 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right) \sim$$

$$\overset{L_2 \leftarrow \frac{1}{2}L_2}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right) \underset{L_4 \leftarrow L_4 - L_2}{\overset{L_3 \leftarrow L_3 + L_2}{\sim}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim$$

$$\overset{L_1 \leftarrow L_1 - L_2}{\sim} \left(\begin{array}{cccc|c} 1 & 0 & -2 & -1 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \begin{cases} x - 2z - t = 0 \\ y - z + 3t = 0 \end{cases} \Rightarrow \begin{cases} x = 2z + t \\ y = z - 3t \end{cases}$$

$$\operatorname{Ker} f = \left\{ (x,y,z,t) \in \mathbb{R}^4 \;\middle|\; \begin{cases} x = 2z + t \\ y = z - 3t \end{cases} \right\} =$$

$$= \left\{ (2z+t,\ z-3t,\ z,\ t) \mid z,t \in \mathbb{R} \right\} =$$

$$= \left\{ z \cdot (2,1,1,0) + t \cdot (1,-3,0,1) \mid z,t \in \mathbb{R} \right\} =$$

$$= \langle (2,1,1,0), (1,-3,0,1) \rangle$$

$$\begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & -3 & 0 & 1 \end{pmatrix} \overset{L_1 \leftrightarrow L_2}{\sim} \begin{pmatrix} 1 & -3 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix} \overset{L_2 \leftarrow L_2 - 2L_1}{\sim} \begin{pmatrix} 1 & -3 & 0 & 1 \\ 0 & 7 & 1 & -2 \end{pmatrix}$$

$\Rightarrow$ basis for $\operatorname{Ker} f$: $\left((1,-3,0,1), (0,7,1,-2) \right)$

$\Rightarrow \dim \operatorname{Ker} f = 2$

$$\operatorname{Im} f = \{ w = (a,b,c,d) \mid \exists\, u = (x,y,z,t) : f(u) = w \} =$$

$$= \{ (a,b,c,d) \in \mathbb{R}^4 \,/\, \exists (x,y,z,t) : [f]_E \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \}$$

$$\Rightarrow \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ -1 & 1 & 1 & 4 & b \\ 2 & 1 & -5 & 1 & c \\ 1 & 2 & -4 & 5 & d \end{array}\right) \underset{L_4 \leftarrow L_4 - L_1}{\overset{L_2 \leftarrow L_2 + L_1}{\underset{L_3 \leftarrow L_3 - 2L_1}{\sim}}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ 0 & 2 & -2 & 6 & a+b \\ 0 & -1 & 1 & -3 & c-2a \\ 0 & 1 & -1 & 3 & d-a \end{array}\right) \sim$$

$$\underset{\sim}{L_2 \leftrightarrow L_4} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ 0 & 1 & -1 & 3 & d-a \\ 0 & -1 & 1 & -3 & c-2a \\ 0 & 2 & -2 & 6 & a+b \end{array}\right) \underset{L_4 \leftarrow L_4 - 2L_2}{\overset{L_3 \leftarrow L_3 + L_2}{\sim}}$$

$$= \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ 0 & 1 & -1 & 3 & d-a \\ 0 & 0 & 0 & 0 & c+d-3a \\ 0 & 0 & 0 & 0 & a+b-2d+2a \end{array}\right)$$

$(a,b,c,d) \in \operatorname{Im} f \Leftrightarrow$ system is compatible $\Leftrightarrow \begin{cases} c+d-3a = 0 \\ 3a+b-2d = 0 \end{cases}$

$$\Rightarrow \operatorname{Im} f = \left\{ (a,b,c,d) \;\middle|\; \begin{cases} -3a+c+d = 0 \\ 3a+b-2d = 0 \end{cases} \right\}$$

$$= \left\{ (a,b,c,d) \;\middle|\; \begin{array}{l} a = \frac{c+d}{3} \\ b = 2d - (c+d) = d-c \end{array} \right\} =$$

$$= \left\{ \left(\frac{c+d}{3}, d-c, c, d \right) \;\middle|\; c,d \in \mathbb{R} \right\} =$$

$$= \left\langle \left(\frac{1}{3}, -1, 1, 0\right), \left(\frac{2}{3}, 1, 0, 1\right) \right\rangle$$

$$\begin{pmatrix} \frac{1}{3} & -1 & 1 & 0 \\ \frac{2}{3} & 1 & 0 & 1 \end{pmatrix} \underset{L_2 \leftarrow L_2 - L_1}{\sim} \begin{pmatrix} \frac{1}{3} & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{pmatrix} \sim$$

$$\underset{L_1 \leftarrow 3L_1}{\sim} \begin{pmatrix} 1 & -3 & 3 & 0 \\ 0 & 2 & -1 & 1 \end{pmatrix}$$

$\Rightarrow$ basis for $\operatorname{Im} f$: $\left((1,-3,3,0), (0,2,-1,1)\right)$

$\Rightarrow \dim \operatorname{Im} f = 2$

Verification: $\underbrace{\dim \operatorname{Ker} f}_{2} + \underbrace{\dim \operatorname{Im} f}_{2} = \underbrace{\dim (\mathbb{K}^4)}_{4}$

(iii) $\forall x,y,z,t$: $f(x,y,z,t) = ?$

$$[f(x,y,z,t)]_E = [f]_E \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x+y-3z+2t \\ -x+y+z+4t \\ 2x+y-5z+t \\ x+2y-4z+5t \end{pmatrix}$$

$\Rightarrow f(x,y,z,t) = (x+y-3z+2t,\ -x+y+z+4t,$
$2x+y-5z+t,\ x+2y-4z+5t)$

$$\operatorname{rank} [f]_{B,B'} = \dim \operatorname{Im} f$$