17.11.2021

Seminar W8 – 917

$$(S)\ \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$$M = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \qquad \overline{M} = \left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right)$$

Cramer's rule: If $m = n$:

(S) is compatible determined $\Leftrightarrow \Delta = \det M \neq 0$

If $\Delta \neq 0 \Rightarrow$ the unique solution of (S) is given by:

$$x_1 = \frac{\Delta_{x_1}}{\Delta}, \dots, x_n = \frac{\Delta_{x_n}}{\Delta}$$

$$\forall i \in \{1, \dots, n\}: \Delta_{x_i} = \begin{vmatrix} a_{11} & \dots & a_{1\,i-1} & b_1 & a_{1\,i+1} & \dots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & & a_{m\,i-1} & b_m & a_{m\,i+1} & \dots & a_{mn} \end{vmatrix}$$

Procedure for solving a general system using Cramer's rule

- find a principal minor in Δ
  $\hookrightarrow$ minor of size rank M

- the rows in this minor correspond to principal equations (the ones that we use)

- the columns in this minor correspond to principal unknowns (the ones that will be kept as unknowns)

- move the secondary unknowns to the column of free terms and treat them as parameter.

- we know have a Cramer system of size rank M

- Apply Cramer's rule to get the unknowns in terms of the parameters.

<u>Th.</u> (Kronecker–Capelli):

(S) is compatible $\Leftrightarrow$ $\operatorname{rank}(M) = \operatorname{rank}(\bar{M})$

<u>Th.</u> (Rouché) Δ_p is a principal minor

(S) is compatible $\Leftrightarrow$ all the characteristic minors are 0

Characteristic minor: $\left|\begin{array}{c|c} \Delta_p & \text{column of free terms} \\ \hline \text{row from } \Delta \text{ not in } \Delta_p & \end{array}\right|$

2. Using the Kronecker-Capelli theorem, decide if the following linear systems are compatible and then solve the compatible ones:

$$(i)\begin{cases} x_1+x_2+x_3-2x_4=5 \\ 2x_1+x_2-2x_3+x_4=1 \\ 2x_1-3x_2+x_3+2x_4=3 \end{cases} \qquad (ii)\begin{cases} x_1-2x_2+x_3+x_4=1 \\ x_1-2x_2+x_3-x_4=-1 \\ x_1-2x_2+x_3+5x_4=5 \end{cases}$$

$$(iii)\begin{cases} x+y+z=3 \\ x-y+z=1 \\ 2x-y+2z=3 \\ x+z=4 \end{cases}$$

3. Using the Rouché theorem, decide if the systems from **2.** are compatible and then solve the compatible ones.

Sol.:

(ii) $\begin{cases} x_1-2x_2+x_3+x_4=1 \\ x_1-2x_2+x_3-x_4=-1 \\ x_1-2x_2+x_3+5x_4=5 \end{cases}$

$$M=\begin{pmatrix} 1 & -2 & 1 & 1 \\ 1 & -2 & 1 & -1 \\ 1 & -2 & 1 & 5 \end{pmatrix} \qquad \overline{M}=\left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & -1 & -1 \\ 1 & -2 & 1 & 5 & 5 \end{array}\right)$$

$$\begin{vmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{vmatrix}=0 \qquad \begin{vmatrix} -2 & 1 & 1 \\ -2 & 1 & -1 \\ -2 & 1 & 5 \end{vmatrix}=0$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 5 \end{vmatrix}=0$$

We can see that $\operatorname{rank}\overline{M}=2$,

$$\Delta_{(2,3)\times(1,4)}=\begin{vmatrix} 1 & -1 \\ 1 & 5 \end{vmatrix}\neq 0$$

(We actually need to check that $\begin{vmatrix} 1 & -2 & 1 \\ 1 & -2 & -1 \\ 1 & -2 & 5 \end{vmatrix}=\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 5 \end{vmatrix}=\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 5 & 5 \end{vmatrix}=0$)

$$\operatorname{rank} M\leq\operatorname{rank}\overline{M}=2$$

$\begin{vmatrix} 1 & -1 \\ 1 & 5 \end{vmatrix}\neq 0 \Rightarrow \operatorname{rank} M=2 \Rightarrow$ system is compatible

$$M = \begin{pmatrix} 1 & -2 & 1 & 1 \\ 1 & -2 & 1 & -1 \\ 1 & -2 & 1 & 5 \end{pmatrix} \qquad \bar{M} = \left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & -1 & -1 \\ 1 & -2 & 1 & 5 & 5 \end{array}\right)$$

$\Rightarrow$ principal equations : second and third
principal unknowns : x_1 and x_4

We denote $x_2 = \alpha$ and $x_3 = \beta$. Our system becomes:

$$\begin{cases} x_1 - 2\alpha + \beta - x_4 = -1 \\ x_1 - 2\alpha + \beta + 5x_4 = 5 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 - x_4 = 2\alpha - \beta - 1 \\ x_1 + 5x_4 = 2\alpha - \beta + 5 \end{cases}$$

$$\Rightarrow x_1 = \frac{\Delta_{x_1}}{\Delta} = \frac{\begin{vmatrix} 2\alpha-\beta-1 & -1 \\ 2\alpha-\beta+5 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 1 & 5 \end{vmatrix}} =$$

$$= \frac{10\alpha - 5\beta - 5 + 2\alpha - \beta + 5}{6} = \frac{12\alpha - 6\beta}{6} = 2\alpha - \beta$$

$$x_4 = \frac{\Delta_{x_4}}{\Delta} = \frac{\begin{vmatrix} 1 & 2\alpha-\beta-1 \\ 1 & 2\alpha-\beta+5 \end{vmatrix}}{6} = \frac{2\alpha - \beta + 5 - 2\alpha + \beta + 1}{6} = 1$$

$$\Rightarrow \begin{cases} x_1 = 2\alpha - \beta \\ x_2 = \alpha \\ x_3 = \beta \\ x_4 = 1 \end{cases}$$

Solve the following linear systems by the Gauss and Gauss-Jordan methods:

5. $(i)\begin{cases}2x+2y+3z=3\\x-y\quad=1\\-x+2y+z=2\end{cases}$ $(ii)\begin{cases}2x+5y+z=7\\x+2y-z=3\\x+y-4z=2\end{cases}$ $(iii)\begin{cases}x+y+z=3\\x-y+z=1\\2x-y+2z=3\\x+z=4\end{cases}$

6. $\begin{cases}2x_1+x_2+x_3+x_4=1\\x_1+2x_2-x_3+4x_4=2\\x_1+5x_2-4x_3+11x_4=\lambda\end{cases}$ $(\lambda\in\mathbb{R})$

7. $\begin{cases}ax+y+z=1\\x+ay+z=a\\x+y+az=a^2\end{cases}$ $(a\in\mathbb{R})$

Sol. : 5 (i) $\begin{cases}2x+2y+3z=3\\x-y\quad=1\\-x+2y+z=2\end{cases}$

$$\left(\begin{array}{ccc|c}2&2&3&3\\1&-1&0&1\\-1&2&1&2\end{array}\right)\underset{L_1\leftrightarrow L_2}{\sim}\begin{pmatrix}1&-1&0&1\\2&2&3&3\\-1&2&1&2\end{pmatrix}\text{ pivot line}$$

pivot

$$\underset{\substack{L_2\leftarrow L_2-2L_1\\L_3\leftarrow L_3+L_1}}{\sim}\begin{pmatrix}1&-1&0&1\\0&4&3&1\\0&1&1&3\end{pmatrix}\underset{L_2\leftrightarrow L_3}{\sim}\begin{pmatrix}1&-1&0&1\\0&1&1&3\\0&4&3&1\end{pmatrix}\text{ pivot line}$$

pivot

$$\underset{L_3\leftarrow L_3-4L_2}{\sim}\begin{pmatrix}1&-1&0&1\\0&1&1&3\\0&0&-1&-11\end{pmatrix}$$

row echelon form

From now on, we have two options :

Gauss method : we revert to the system and solve it.

$$\begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -11 \end{pmatrix} \qquad \begin{cases} x - y = 1 \\ y + z = 3 \\ -z = -11 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} z = 11 \\ y = 3 - z = -8 \\ x = y + 1 = -7 \end{cases}$$

<u>Gauss-Jordan:</u> We keep performing transformations on the rows of the matrix, until we get zeros over the diagonal:

pivot line

$$\begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -11 \end{pmatrix} \overset{L_2 \leftarrow L_2 + L_3}{\sim} \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & -1 & -11 \end{pmatrix} \sim$$

pivot

pivot line

pivot

$$\overset{L_1 \leftarrow L_1 + L_2}{\sim} \begin{pmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & -1 & -11 \end{pmatrix} \Rightarrow \begin{cases} x = -7 \\ y = -8 \\ z = 11 \end{cases}$$

Here's what can happen:

1) in the process of Gaussian elimination we get a row like this:

$(0 \ 0 \ \cdots \ 0 \mid \alpha)$, $\alpha \neq 0$ $\rightarrow$ system is incompatible

$(0 \ 0 \ \cdots \ 0 \mid 0)$ $\rightarrow$ redundant equation

6. $\begin{cases} 2x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 + 2x_2 - x_3 + 4x_4 = 2 \\ x_1 + 5x_2 - 4x_3 + 11x_4 = \lambda \end{cases} \quad (\lambda \in \mathbb{R})$

Sol. :

$$\left(\begin{array}{cccc|c} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 4 & 2 \\ 1 & 5 & -4 & 11 & \lambda \end{array}\right) \underset{L_1 \leftrightarrow L_2}{\sim} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 5 & -4 & 11 & \lambda \end{array}\right) \sim$$

$$\underset{L_3 \leftarrow L_3 - L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 0 & -3 & 3 & -7 & -3 \\ 0 & 3 & -3 & 7 & \lambda-2 \end{array}\right) \overset{L_3 \leftarrow L_3 + L_2}{\sim}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 0 & -3 & 3 & -7 & -3 \\ 0 & 0 & 0 & 0 & \lambda-5 \end{array}\right)$$

↳ row echelon form

If $\lambda - 5 \neq 0 \Rightarrow$ system is incompatible

If $\lambda = 5 \Rightarrow$ the system is

$$\begin{cases} x_1 + 2x_2 - x_3 + 4x_4 = 2 \\ -3x_2 + 3\underbrace{x_3}_{=:\alpha} - 7\underbrace{x_4}_{=:\beta} = -3 \end{cases}$$

$$\Rightarrow \begin{cases} x_2 = \frac{1}{3}(3 + 3\alpha - 7\beta) \\ x_1 = 2 - 2x_2 + \alpha - 4\beta = 2 - 2 - 2\alpha + \frac{14}{3}\beta + \end{cases}$$

$$+ \alpha - 4\beta = -\alpha + \frac{2}{3}\beta$$

$$\Rightarrow \begin{cases} x_1 = -\alpha + \frac{2}{3}\beta \\ x_2 = \alpha - \frac{7}{3}\beta + 1 \\ x_3 = \alpha \\ x_4 = \beta \end{cases}$$