13.12.2021

Seminar W11 – 916

Def: V, V' K-v.s., $\quad B = (v_1, \dots, v_m)$ basis of V

$B' = (v_1', \dots, v_n')$ basis of V'

$f: V \to V'$ linear map

$$[f]_{B B'} = \left([f(v_1)]_{B'} \; [f(v_2)]_{B'} \; \dots \; [f(v_m)]_{B'} \right)$$

Prop: $\forall v \in V$: $\quad [f(v)]_{B'} = [f]_{B,B'} \cdot [v]_B$

Def: V K-v.s., B, B' bases of V, $\quad id_V : V \to V$, $v \mapsto v$

$[id_V]_{B,B'} =: T_{B',B}$ = base-change (or transfer) matrix from B' to B

$$\| $$

$$\left([v_1]_{B'} \; [v_2]_{B'} \; \dots \; [v_m]_{B'} \right)$$

Cor: $\exists f$ $\; f = id_V$:

$$\boxed{[v]_{B'} = [id]_{B,B'} \cdot [v]_B = T_{B',B} \cdot [v]_B}$$

Prop: $\boxed{[id]^{-1}_{B',B} = [id]_{B,B'}}$ only true because we have id as our linear map; NOT TRUE for any f!!!

$$\left(T^{-1}_{B,B'} = T_{B',B} \right)$$

Prop. : V, V', V'' K-v.s., B, B', B'' bases for V, V', V''

$f : V' \to V''$, $g : V \to V'$ linear maps

$f \circ g : V \to V''$

$$\boxed{[f \circ g]_{B, B''} = [f]_{B', B''} \cdot [g]_{B, B'}}$$

Cor. : V, V' K-v.s., B_1, B_2 bases of V, B_1', B_2' bases of V', $f : V \to V'$ linear map

$$[f]_{B_2, B_2'} = [id_{V'}]_{B_1', B_2'} \cdot [f]_{B_1, B_1'} \cdot [id_V]_{B_2, B_1} =$$

$$= T_{B_2', B_1'} \cdot [f]_{B_1, B_1'} \cdot T_{B_1, B_2}$$

2. In the real vector space $\mathbb{R}^2$ consider the bases $B = (v_1, v_2) = ((1,2),(1,3))$ and $B' = (v_1', v_2') = ((1,0),(2,1))$ and let $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ having the matrices $[f]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$ and $[g]_{B'} = \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}$. Determine the matrices $[2f]_B$, $[f+g]_B$ and $[f \circ g]_{B'}$. (*Use the matrices of change of basis.*)

Sol. : $[2f]_B = 2 \cdot [f]_B = 2 \cdot \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -2 & -2 \end{pmatrix}$

$[f+g]_B = [f]_B + [g]_B$, $[g]_B = ?$

$$[g]_B = [id]_{B', B} \cdot [g]_{B'} \cdot [id]_{B, B'}$$

$$[id]_{B,B'} = \begin{pmatrix} [u_1]_{B'} & [u_2]_{B'} \end{pmatrix}$$

<u>The first approach</u> (the tedious but simple one)

$$[u_1]_{B'} = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \Rightarrow u_1 = \alpha_1 u_1' + \beta_1 u_2'$$

$$\Rightarrow (1,2) = \alpha_1 \cdot (1,0) + \beta_1 \cdot (2,1)$$

$$\Rightarrow \begin{cases} 1 = \alpha_1 + 2\beta_1 \\ 2 = \beta_1 \end{cases} \Rightarrow \begin{cases} \alpha_1 = -3 \\ \beta_1 = 2 \end{cases} \Rightarrow [u_1]_{B'} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$[u_2]_{B'} = \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \Rightarrow u_2 = \alpha_2 u_1' + \beta_2 u_2'$$

$$\Rightarrow (1,3) = \alpha_2 \cdot (1,0) + \beta_2 \cdot (2,1)$$

$$\Rightarrow \begin{cases} 1 = \alpha_2 + 2\beta_2 \\ 3 = \beta_2 \end{cases} \Rightarrow \begin{cases} \alpha_2 = -5 \\ \beta_2 = 3 \end{cases} \Rightarrow [u_2]_{B'} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$$

$$\Rightarrow [id]_{B,B'} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$$

<u>The second approach</u> (the fancier one) $\quad E = ((1,0), (0,1))$

$$[id]_{B,B'} = [id]_{E,B'} \cdot [id]_{B,E} = [id]^{-1}_{B',E} \cdot [id]_{B,E}$$

$$B = ((1,2), (1,3)) \qquad B' = ((1,0), (2,1))$$

$$[id]_{B,E} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, \qquad [id]_{B',E} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad [id]_{B,B'} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$$

$$[g]_B = \underbrace{[id]_{B',B}}_{?} \cdot \underbrace{[g]_{B'}}_{\checkmark} \cdot \underbrace{[id]_{B,B'}}_{\checkmark}$$

$$[id]_{B',B} = [id]_{B,B'}^{-1} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}$$

$$\Rightarrow \quad [g]_B = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} \cdot \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix} \cdot \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 4 & -4 \\ -1 & 5 \end{pmatrix} \cdot \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -20 & -32 \\ 13 & 20 \end{pmatrix}$$

$$[f \circ g]_{B'} = [f]_{B'} \cdot [g]_{B'} \qquad\qquad [f]_{B'} = ?$$

$$= [f]_{B,B'} \, [g]_{B',B}$$

$$[f]_{B'} = [id]_{B,B'} \cdot [f]_B \cdot [id]_{B',B} =$$

$$= \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} =$$

$$= \begin{pmatrix} 8 & 13 \\ -5 & -8 \end{pmatrix}$$

$$[f \circ g]_{B'} = [f]_{B'} \cdot [g]_{B'} = \begin{pmatrix} 8 & 13 \\ -5 & -8 \end{pmatrix} \cdot \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix} =$$

$$= \begin{pmatrix} 9 & -13 \\ -5 & 9 \end{pmatrix}$$

Def: V K-vector space, $f: V \to V$ linear map

$v \in V \setminus \{0\}$ eigenvector for f $\Leftrightarrow$ $\exists \lambda \in K$ (called an eigenvalue):

$$f(v) = \lambda \cdot v$$

$$V(\lambda) = \{v \in V \mid f(v) = \lambda v\} = \{\text{set of eigenvectors corresp to } \lambda\} \cup \{0\}$$

$= $ eigenspace of λ w.r.t. f

Prop: $\mathcal{B}$ basis for V : λ eigenvalue for f $\Leftrightarrow$ λ is a root of the characteristic polynomial $P_f(X)$ of f

$$P_f(X) = \det([f]_{\mathcal{B}} - X I_n)$$

$$\Updownarrow$$

$$\det([f]_{\mathcal{B}} - \lambda I_n) = 0$$

$A \in M_n(K)$

eigenvalues / eigenvectors of A $\Leftrightarrow$ eigenvalues / eigenvectors of f

where $[f]_E = A$

Compute the eigenvalues and the eigenvectors of the (endomorphisms having) matrices:

5. $\begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ -4 & -8 & -2 \end{pmatrix}$. **6.** $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Sol.: 5.

$$A = \begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ -4 & -8 & -2 \end{pmatrix}$$

$$p_A(X) = \det(A - X I_3) = \begin{vmatrix} 3-X & 1 & 0 \\ -4 & -1-X & 0 \\ -4 & -8 & -2-X \end{vmatrix} =$$

$$= \begin{vmatrix} 3-X & 1 & 0 \\ 4 & 1+X & 0 \\ 4 & 8 & X+2 \end{vmatrix} = (X+2) \cdot \begin{vmatrix} 3-X & 1 \\ 4 & 1+X \end{vmatrix} =$$

$$= (X+2) \cdot (-X^2 - X + 3X + 3 - 4) = (X+2) \cdot (-X^2 + 2X - 1) =$$

$$= -(X+2) \cdot (X-1)^2$$

$\Rightarrow$ eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 1$

$$S(\lambda_1) = \{ u = (x,y,z) \in \mathbb{R}^3 \mid A \cdot [u]_E = \lambda_1 \cdot [u]_E \} =$$

$$= \left\{ u = (x,y,z) \;\middle|\; (A - \lambda_1 I_3) \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} =$$

$$= \left\{ u = (x,y,z) \;\middle|\; \begin{pmatrix} 5 & 1 & 0 \\ -4 & 1 & 0 \\ -4 & -8 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ u = (x,y,z) \;\middle|\; \begin{cases} 5x + y = 0 \\ -4x + y = 0 \\ -4x - 8y = 0 \end{cases} \right\} =$$

$$= \left\{ u = (x,y,z) \;\middle|\; \begin{cases} 9x = 0 \\ -4x + y = 0 \\ -4x - 8y = 0 \end{cases} \right\}$$

$= \{ u = (x, y, z) \mid x = y = 0 \} =$

$= \{ u = (0, 0, z) \mid z \in \mathbb{R} \} = \langle (0, 0, 1) \rangle$

$\Rightarrow \dim S(\lambda_1) = 1$