08.11.2021

Seminar W6 - 915

Def: V K-vector space, $v_1, \dots, v_n \in V$

We say that $v_1, \dots, v_n$ are <u>linearly independent</u> if: $\forall \alpha_1, \dots, \alpha_n \in K$

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0 \Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$$

$v_1, v_2, \dots, v_n$ linearly dependent if $\exists \alpha_1, \dots, \alpha_n \in K$, not all zero, so that:

$$\underbrace{\alpha_1 v_1 + \dots + \alpha_n v_n = 0}_{\text{depending relation}}$$

1. Let $v_1 = (1, -1, 0)$, $v_2 = (2, 1, 1)$, $v_3 = (1, 5, 2)$ be vectors in the canonical real vector space $\mathbb{R}^3$. Prove that:
(*i*) v_1, v_2, v_3 are linearly dependent and determine a dependence relationship.
(*ii*) v_1, v_2 are linearly independent.

Sol: Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ so that $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

$$\Rightarrow \alpha_1 \cdot (1, -1, 0) + \alpha_2 \cdot (2, 1, 1) + \alpha_3 \cdot (1, 5, 2) = (0, 0, 0)$$

$$\Rightarrow \begin{cases} \alpha_1 + 2\alpha_2 + \alpha_3 = 0 \\ -\alpha_1 + \alpha_2 + 5\alpha_3 = 0 \\ \alpha_2 + 2\alpha_3 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_2 = -2\alpha_3 \\ \alpha_1 + 2\alpha_2 + \alpha_3 = 0 \\ -\alpha_1 + \alpha_2 + 5\alpha_3 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_2 = -2\alpha_3 \\ \alpha_1 - 4\alpha_3 + \alpha_3 = 0 \\ -\alpha_1 - 2\alpha_3 + 5\alpha_3 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_2 = -2\alpha_3 \\ \alpha_1 = 3\alpha_3 \\ -3\alpha_3 + 3\alpha_3 = 0 \end{cases}$$

$\Rightarrow v_1, v_2, v_3$ linearly dependent:

$$3v_1 - 2v_2 + v_3 = 0$$

(ii) Let $\alpha_1, \alpha_2 \in \mathbb{R}$: $\alpha_1 v_1 + \alpha_2 v_2 = 0 \Rightarrow \alpha_1 \cdot (1,-1,0) + \alpha_2 \cdot (2,1,1) = (0,0,0)$

$$\Rightarrow \begin{cases} \alpha_1 + 2\alpha_2 = 0 \\ -\alpha_1 + \alpha_2 = 0 \\ \alpha_2 = 0 \end{cases} \Rightarrow \alpha_1 = \alpha_2 = 0$$

$\Rightarrow v_1, v_2$ linearly independent

4. Let $v_1 = (1,-2,0,-1)$, $v_2 = (2,1,1,0)$, $v_3 = (0,a,1,2)$ be vectors in ${}_{\mathbb{R}}\mathbb{R}^4$. Determine $a \in \mathbb{R}$ such that the vectors v_1, v_2, v_3 are linearly dependent.

Sol.: Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$: $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

$$\Rightarrow \alpha_1 \cdot (1,-2,0,-1) + \alpha_2 \cdot (2,1,1,0) + \alpha_3 \cdot (0,a,1,2) = (0,0,0,0)$$

$$\Rightarrow \begin{cases} \alpha_1 + 2\alpha_2 = 0 \\ -2\alpha_1 + \alpha_2 + \alpha_3 \cdot a = 0 \\ \alpha_2 + \alpha_3 = 0 \\ -\alpha_1 + 2\alpha_3 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_2 = -\alpha_3 \\ \alpha_1 = 2\alpha_3 \\ \alpha_1 + 2\alpha_2 = 0 \\ -2\alpha_1 + \alpha_2 + \alpha_3 \cdot a = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_2 = -\alpha_3 \\ \alpha_1 = 2\alpha_3 \\ 2\alpha_3 - 2\alpha_3 = 0 \\ -4\alpha_3 - \alpha_3 + \alpha_3 \cdot a = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_2 = -\alpha_3 \\ \alpha_1 = 2\alpha_3 \\ (a-5) \cdot \alpha_3 = 0 \end{cases}$$

If $a = 5 \Rightarrow \begin{cases} \alpha_2 = -\alpha_3 \\ \alpha_1 = 2\alpha_3 \end{cases}$ compatible indetermined $\Rightarrow v_1, v_2, v_3$ linearly dependent

2) $a \neq 5 \Rightarrow \begin{cases} \alpha_2 = -\alpha_3 \\ \alpha_1 = 2\alpha_3 \\ \alpha_3 = 0 \end{cases} \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$

$\Rightarrow v_1, v_2, v_3$ linearly independent

<u>Def</u>: V K-vector space, $v_1, \dots, v_n \in V$

$\{v_1, \dots, v_n\}$ is a basis for V $\Leftrightarrow \begin{cases} \cdot\ v_1, v_2, \dots, v_n \text{ linearly independent} \Leftrightarrow \\ \cdot\ V = \langle v_1, \dots, v_n \rangle \end{cases}$

$\Leftrightarrow \forall v \in V : \exists!\ \underbrace{\alpha_1, \alpha_2, \dots, \alpha_n} : \alpha_1 v_1 + \dots + \alpha_n v_n = v$

"the coordinates of v in the basis $B = \{v_1, \dots, v_n\}$"

$$[v]_B = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

8. Let $\mathbb{R}_2[X] = \{f \in \mathbb{R}[X] \mid degree(f) \leq 2\}$. Show that the lists $E = (1, X, X^2)$, $B = (1, X-a, (X-a)^2)$ $(a \in \mathbb{R})$ are bases of the real vector space $\mathbb{R}_2[X]$ and determine the coordinates of a polynomial ~~$f = a_0 + a_1X + a_2X^2 \in \mathbb{R}_2[X]$ in each basis.~~

$E = \{1, x, x^2\}$

$f = 3 - 7x^2$ in the basis $(1, x-6, (x-6)^2)$

<u>Sol</u>: We show, first, that $1, x, x^2$ linearly independent.

Let $a, b, c \in \mathbb{R}$ so that $a \cdot 1 + b \cdot x + c \cdot x^2 = 0$

$\Rightarrow a + bx + cx^2 = 0 \Rightarrow a = b = c = 0$

We show that $\mathbb{R}_2[X] = \langle 1, x, x^2 \rangle$

Let $f \in \mathbb{R}_2[X] \Rightarrow \deg f \leq 2 \Rightarrow f = a_0 + a_1 x + a_2 \cdot x^2 \Rightarrow$

$f = a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2 \Rightarrow f \in \langle 1, x, x^2 \rangle$

$\Rightarrow \mathbb{R}_2[X] = \langle 1, X, X^2 \rangle \Rightarrow 1, X, X^2$ generating set for $\mathbb{R}_2[X]$

$\Rightarrow E = \{1, X, X^2\}$ basis for $\mathbb{R}_2[X]$

$B = \{1, X-a, (X-a)^2\}$

Let $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}$ so that

$$\alpha_0 \cdot 1 + \alpha_1 (X-a) + \alpha_2 \cdot (X-a)^2 = 0$$

$$\Rightarrow \alpha_0 + \alpha_1 X - a\alpha_1 + \alpha_2 X^2 - 2a\alpha_2 X + \alpha_2 a^2 = 0$$

$$\Rightarrow (\alpha_0 + \alpha_2 a^2 - a\alpha_1) + (\alpha_1 - 2a\alpha_2) \cdot X + \alpha_2 X^2 = 0$$

$$\Rightarrow \begin{cases} \alpha_0 - a\alpha_1 + \alpha_2 a^2 = 0 \\ \alpha_1 - 2\alpha_2 = 0 \\ \alpha_2 = 0 \end{cases} \Rightarrow \alpha_2 = \alpha_1 = \alpha_0 = 0$$

$\Rightarrow 1, X-a, (X-a)^2$ linearly independent

Let $f \in \mathbb{R}_2[X] \Rightarrow \deg f \leq 2 \Rightarrow f = \beta_0 + \beta_1 X + \beta_2 X^2$

We want to find $\gamma_0, \gamma_1, \gamma_2 \in \mathbb{R}$, so that:

$$f = \gamma_0 \cdot 1 + \gamma_1 \cdot (X-a) + \gamma_2 \cdot (X-a)^2$$

$$\Rightarrow \beta_0 + \beta_1 X + \beta_2 X^2 = (\gamma_0 - a\gamma_1 + \gamma_2 \cdot a^2) + (\gamma_1 - 2a\gamma_2) \cdot X +$$

$$+ \gamma_2 X^2$$

$$\Rightarrow \begin{cases} \beta_0 = \gamma_0 - a\gamma_1 + \gamma_2 \cdot a^2 \\ \beta_1 = \gamma_1 - 2a\gamma_2 \\ \beta_2 = \gamma_2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \gamma_2 = \beta_2 \\ \beta_1 = \gamma_1 - 2a\beta_2 \\ \beta_0 = \gamma_0 - a\gamma_1 + \beta_2 \cdot a^2 \end{cases} \Leftrightarrow \begin{cases} \gamma_2 = \beta_2 \\ \gamma_1 = \beta_1 + 2a\beta_2 \\ \gamma_0 = \beta_0 + a \cdot (\beta_1 + a\beta_2) - \beta_2 a^2 \end{cases}$$

$\Rightarrow \forall f = \beta_0 + \beta_1 X + \beta_2 X^2 \quad \exists\ \gamma_0, \gamma_1, \gamma_2 \in \mathbb{R}$ so that:

$$f = \gamma_0 + \gamma_1 (X-a) + \gamma_2 (X-a)^2$$

$$\Rightarrow \mathbb{R}_2[X] = \langle 1, X-a, (X-a)^2 \rangle$$

We want to write:

$$3 - 7X^2 = \gamma_0 + \gamma_1 (X-6) + \gamma_2 (X-6)^2$$

$$\begin{cases} \gamma_2 = \beta_2 \\ \gamma_1 = \beta_1 + 2a\beta_2 \\ \gamma_0 = \beta_0 + a \cdot (\beta_1 + a\beta_2) - \beta_2 a^2 \end{cases} \Rightarrow \begin{cases} \gamma_2 = -7 \\ \gamma_1 = 0 + 2 \cdot 6 \cdot (-7) \\ \gamma_0 = 3 + 6 \cdot (0 + 6(-7)) - (-7) \cdot 6^2 \end{cases}$$

$$\Rightarrow \begin{cases} \gamma_2 = -7 \\ \gamma_1 = -84 \\ \gamma_0 = 3 - 42 \cdot 6 + 42 \cdot 6 = 3 \end{cases}$$

$$\Rightarrow [3 - 7X^2]_{1, X-6, (X-6)^2} = \begin{pmatrix} 3 \\ -84 \\ -7 \end{pmatrix}$$

Recap:

V is a K-vector space with respect to the

$(K, +, \cdot)$

$(V, +)$, an abelian group

external operation $\cdot : K \times V \to V$