15.11.2021

Seminar W7 – 915

Corollary (Steinitz): V K-vector space, $S \leq V$.

We can complete any basis of S to a basis of V.

In practice: V K-v.s., $S \leq V$, $(u_1, \dots, u_n)$ basis of S

- Do we have enough vectors?
  - yes: $(u_1, \dots, u_n)$ basis for V
  - no: Choose $w_{n+1} \in V \setminus \langle u_1, \dots, u_n \rangle$. Do we have enough vectors?
    - yes: $(u_1, \dots, u_n, w_{n+1})$ basis for V
    - no: Choose $w_{n+2} \in V \setminus \langle u_1, \dots, u_n, w_{n+1} \rangle$. Do we have enough vectors?

...

- How do we extract a basis from a system of generators? (how do we decide which vectors are linearly independent?)

$u_1, u_2, \dots, u_n \in {}_K V$, $\text{rank}(u_1, \dots, u_n) = \dim \langle u_1, \dots, u_n \rangle =$

= "maximal number of linearly independent vectors in V"

$\rightarrow$ if $V = K^m \Rightarrow \text{rank}(u_1, \dots, u_n) = \text{rank}\begin{pmatrix} u_1 \\ \hline u_2 \\ \hline \vdots \\ \hline u_n \end{pmatrix} = \text{rank}(u_1 | \dots | u_n)$

1. Determine a basis and the dimension of the following subspaces of the real vector space $\mathbb{R}^3$:

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$$
$$B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$$
$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}.$$

6. Complete the bases of the subspaces from Exercise **1.** to some bases of the real vector space $\mathbb{R}^3$ over $\mathbb{R}$.

Sol: $A = \{(x,y,0) \mid x,y \in \mathbb{R}\} = \{(x,0,0) + (0,y,0) \mid x,y \in \mathbb{R}\} =$

$= \{x \cdot (1,0,0) + y \cdot (0,1,0) \mid x,y \in \mathbb{R}\} = \langle (1,0,0), (0,1,0) \rangle$

$\text{rank}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = 2 \Rightarrow \dim A = 2$, $((1,0,0), (0,1,0))$ basis for A

Two explanations:

— $(0,0,1) \in \mathbb{R}^3 \setminus A \Rightarrow (0,0,1) \notin \langle (1,0,0), (0,1,0) \rangle \Rightarrow (0,0,1), (1,0,0), (0,1,0)$ basis

or

— $\text{rank}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 3 \Rightarrow (1,0,0), (0,1,0), (0,0,1)$ lin. indep.

$B = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z = 0\} = \{(x, y, -x-y) \mid x,y \in \mathbb{R}\} =$

$= \{(x, 0, -x) + (0, y, -y) \mid x,y \in \mathbb{R}\} = \langle (1,0,-1), (0,1,-1) \rangle$

$\text{rank}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} = 2 \Rightarrow$ lin. indep. $\Rightarrow \dim B = 2$

$((1,0,-1), (0,1,-1))$ basis for B

$(1,1,0) \notin B \Rightarrow (1,1,0) \in \mathbb{R}^3 \setminus \langle (1,0,-1), (0,1,-1) \rangle$

$\Rightarrow (1,0,-1), (0,1,-1), (1,1,0)$ basis for $\mathbb{R}^3$

$C = \{ (x,y,z) \in \mathbb{R}^3 \mid x = y = z \} =$

$= \langle (1,1,1) \rangle \Rightarrow \dim C = 1,\ ((1,1,1))$ basis for C

We need to add two vectors.

Let's add $(1,0,0)$ first. $(1,0,0) \notin C \Rightarrow (1,1,1), (1,0,0)$ lin. indep.

(we can check this using $\operatorname{rank}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} = 2$)

To add another vector, we have to first describe

$\langle (1,1,1), (1,0,0) \rangle = \{ a \cdot (1,1,1) + b \cdot (1,0,0) \mid a, b \in \mathbb{R} \} =$

$= \{ (a+b, a, a) \mid a, b \in \mathbb{R} \} = \left\{ (x,y,z) \,\middle|\, \begin{cases} x = a+b \\ y = a \\ z = a \end{cases} \right\} = \left\{ (x,y,z) \,\middle|\, \begin{matrix} a = y = z \\ b = x - a = x - y \end{matrix} \right.$

We can now choose $(1,0,1) \in \mathbb{R}^3 \setminus \langle (1,1,1), (1,0,0) \rangle \Rightarrow$

$\Rightarrow (1,0,1), (1,1,1), (1,0,0)$ lin. indep. $\overset{\dim \mathbb{R}^3 = 3}{\Rightarrow}$ they form a basis

Rem: $\langle (2,3,1), (1,-1,2) \rangle = \{ a \cdot (2,3,1) + b \cdot (1,-1,2) \mid a, b \in \mathbb{R} \} =$

$= \{ (2a+b, 3a-b, a+2b) \mid a, b \in \mathbb{R} \} =$

$= \left\{ (x,y,z) \,\middle|\, \begin{cases} 2a+b = x \\ 3a-b = y \\ a+2b = z \end{cases} \right\} = \left\{ (x,y,z) \,\middle|\, \begin{cases} b = 3a-y \\ 2a+b = x \\ a+2b = z \end{cases} \right\} =$

$= \left\{ (x,y,z) \,\middle|\, \begin{matrix} b = 3a - y \\ 2a + 3a - y = x \\ a + 6a - 2y = z \end{matrix} \right\} = \left\{ (x,y,z) \,\middle|\, \begin{matrix} b = 3a - y \\ 5a = x + y \\ 7a - 2y = z \end{matrix} \right\} =$

$$= \left\{ (x,y,z) \;\middle|\; \begin{cases} b = 3a - y \\ a = \frac{x+y}{5} \\ \frac{7(x+y)}{5} - 2y = z \end{cases} \right\} = \left\{ (x,y,z) \;\middle|\; \frac{7x+7y}{5} - 2y - z = 0 \right\}$$

$$= \left\{ (x,y,z) \in \mathbb{R} \;\middle|\; \frac{7}{5}x - \frac{3}{5}y - z = 0 \right\}$$

<u>Def.</u>: V, W K - vector spaces, $f : V \to W$ linear map

$Ker\, f = \{ v \in V \mid f(v) = 0_W \} \leq_K V$
("kernel")

$Im\, f = \{ f(v) \in W \mid v \in V \} = \{ w \in W \mid \exists v \in V : f(v) = w \}$

5. Let $f \in End_{\mathbb{R}}(\mathbb{R}^3)$ be defined by $f(x,y,z) = (-y+5z, x, y-5z)$. Determine a basis and the dimension of $Ker\, f$ and $Im\, f$.

<u>Sol.</u>: $Ker\, f = \{ (x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = (0,0,0) \} =$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{cases} -y+5z = 0 \\ x = 0 \\ y - 5z = 0 \end{cases} \right\} =$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{cases} y = 5z \\ x = 0 \\ 5z - 5z = 0 \end{cases} \right\} \Leftrightarrow \{ (x,y,z) \mid y = 5z, x = 0 \} =$$

$$= \{ (0, 5z, z) \mid z \in \mathbb{R} \} = \{ z \cdot (0,5,1) \mid z \in \mathbb{R} \} =$$

$$= \langle (0,5,1) \rangle \Rightarrow \dim Ker\, f = 1$$

$\operatorname{Im} f = \{ f(u) \mid u \in \mathbb{R}^3 \} = \{ (-y+5z, x, y-5z) \mid x,y,z \in \mathbb{R} \} =$

$= \{ (0,x,0) + (-y, 0, y) + (5z, 0, -5z) \mid x,y,z \in \mathbb{R} \} =$

$= \{ x\cdot(0,1,0) + y\cdot(-1,0,1) + z\cdot(5,0,-5) \mid x,y,z \in \mathbb{R} \} =$

$= \langle (0,1,0), (-1,0,1), (5,0,-5) \rangle$

$$\operatorname{rank}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 5 & 0 & -5 \end{pmatrix} = ? \qquad \begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 5 & 0 & -5 \end{vmatrix} = 0$$

$\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \neq 0 \Rightarrow (0,1,0)$ and $(-1,0,1)$ form a basis of $\operatorname{Im} f$

$\Rightarrow \dim \operatorname{Im} f = 2$

<u>Th.</u> (1st dim th., rank-nullity th.) $\quad f: V \to W$ linear map

$\Rightarrow \dim V = \dim \operatorname{Ker} f + \dim \operatorname{Im} f$

(2nd dim th.) $\quad V$ K-v.s., $\quad S, T \leq V$

$\dim(S+T) = \dim S + \dim T - \dim(S \cap T)$

<u>Rem</u>: $S = \langle v_1, v_2, \dots, v_n \rangle$, $T = \langle w_1, w_2, \dots, w_m \rangle$

$\Rightarrow S+T = \langle v_1, v_2, \dots, v_n, w_1, \dots, w_m \rangle$

"pool" ↳ useful for finding $\dim(S \cap T)$ without computing $S \cap T$