08.12.2021

Seminar W10 – 917

Def: V, V' K-vector spaces, $f: V \to V'$ K-linear map
$B = (v_1, v_2, \dots, v_m)$ basis of V ("source basis")
$B' = (v_1', v_2', \dots, v_n')$ basis of V' ("target basis")

$$[f]_{B,B'} = \left([f(v_1)]_{B'} \; [f(v_2)]_{B'} \; \dots \; [f(v_m)]_{B'} \right)$$

Prop: In the same conditions as above,

$$\forall v \in V: \quad \boxed{[f(v)]_{B'} = [f]_{B,B'} \cdot [v]_B}$$

2. Let $f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2)$ be defined by

$$f(x, y, z) = (y, -x)$$

and consider the bases $B = (v_1, v_2, v_3) = ((1,1,0), (0,1,1), (1,0,1))$ of $\mathbb{R}^3$, $B' = (v_1', v_2') = ((1,1), (1,-2))$ of $\mathbb{R}^2$ and let $E' = (e_1', e_2')$ be the canonical basis of $\mathbb{R}^2$. Determine the matrices $[f]_{BE'}$ and $[f]_{BB'}$.

$e_1' = (1,0)$, $e_2' = (0,1)$

$E = (e_1, e_2, e_3)$, $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

Sol: example for determining $[f]_{E,B'}$

$$[f]_{E,B'} = \left([f(e_1)]_{B'} \; [f(e_2)]_{B'} \; [f(e_3)]_{B'} \right)$$

$$f(e_1) = f(1,0,0) = (0,-1) = \alpha_1 \cdot v_1' + \beta_1 \cdot v_2'$$

$$\Rightarrow (0,-1) = \alpha_1 \cdot (1,1) + \beta_1 \cdot (1,-2)$$

$$\Rightarrow \begin{cases} \alpha_1 + \beta_1 = 0 \\ \alpha_1 - 2\beta_1 = -1 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = 2\beta_1 - 1 \\ 2\beta_1 - 1 + \beta_1 = 0 \end{cases} \Leftrightarrow \begin{cases} \beta_1 = \frac{1}{3} \\ \alpha_1 = -\frac{1}{3} \end{cases}$$

$$\Rightarrow [f(e_1)]_{B'} = \begin{pmatrix} -1/3 \\ 1/3 \end{pmatrix}$$

$$f(e_2) = f(0,1,0) = (1,0) = \alpha_2 \cdot \underbrace{e_1'}_{(1,1)} + \beta_2 \cdot \underbrace{e_2'}_{(1,-2)}$$

$$\Rightarrow \begin{cases} 1 = \alpha_2 + \beta_2 \\ 0 = \alpha_2 - 2\beta_2 \end{cases} \Leftrightarrow \begin{cases} \alpha_2 = 2\beta_2 \\ 1 = 2\beta_2 + \beta_2 \end{cases} \Leftrightarrow \begin{cases} \beta_2 = \frac{1}{3} \\ \alpha_2 = \frac{2}{3} \end{cases}$$

$$\Rightarrow [f(e_2)]_{B'} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$$

$$f(e_3) = f(0,0,1) = (0,0) = \alpha_3 \cdot e_1' + \beta_3 \cdot e_2'$$

$$\begin{cases} 0 = \alpha_3 + \beta_3 \\ 0 = \alpha_3 - 2\beta_3 \end{cases} \Leftrightarrow \begin{cases} \alpha_3 = 2\beta_3 \\ 2\beta_3 + \beta_3 = 0 \end{cases} \Leftrightarrow \begin{cases} \beta_3 = 0 \\ \alpha_3 = 0 \end{cases}$$

$$\Rightarrow [f(e_3)]_{B'} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} [f(e_1)]_{B'} &= \begin{pmatrix} -1/3 \\ 1/3 \end{pmatrix} \\ [f(e_2)]_{B'} &= \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} \\ [f(e_3)]_{B'} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}\right\} \Rightarrow [f]_{E,B'} = \begin{pmatrix} -1/3 & 2/3 & 0 \\ 1/3 & 1/3 & 0 \end{pmatrix}$$

$$[f]_{B,E'} = \left([f(v_1)]_{E'} \quad [f(v_2)]_{E'} \quad [f(v_3)]_{E'} \right)$$

$$f(v_1) = f(1,1,0) = (1,-1) = \alpha_1 e_1' + \beta_1 \cdot e_2' = \alpha_1 \cdot (1,0) + \beta_1 \cdot (0,1)$$

$$\Rightarrow \begin{cases} \alpha_1 = 1 \\ \beta_1 = -1 \end{cases} \Rightarrow [f(v_1)]_{E'} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$f(e_2) = f(0,1,1) = (1,0) = \alpha_2 e_1' + \beta_2 \cdot e_2' = \alpha_2 (1,0) + \beta_2 \cdot (0,1)$$

$$\Rightarrow \begin{cases} \alpha_2 = 1 \\ \beta_2 = 0 \end{cases} \Rightarrow [f(e_2)]_{E'} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$f(e_3) = f(1,0,1) = (0,-1) \Rightarrow [f(e_3)]_{E'} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\Rightarrow [f]_{B,E'} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$$

$$[f]_{B,B'} = \left([f(e_1)]_{B'} \quad [f(e_2)]_{B'} \quad [f(e_3)]_{B'} \right)$$

$$f(e_1) = (1,-1) = \alpha_1 \cdot e_1' + \beta_1 \cdot e_2' = \alpha_1 \cdot (1,1) + \beta_1 \cdot (1,-2)$$

$$\Leftrightarrow \begin{cases} \alpha_1 + \beta_1 = 1 \\ \alpha_1 - 2\beta_1 = -1 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = 1 - \beta_1 \\ 1 - \beta_1 - 2\beta_1 = -1 \end{cases} \Leftrightarrow \begin{cases} -3\beta_1 = -2 \\ \alpha_1 = 1 - \beta_1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \beta_1 = \frac{2}{3} \\ \alpha_1 = \frac{1}{3} \end{cases} \Rightarrow [f(e_1)]_{B'} = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$$

$$f(e_2) = (1,0) = \alpha_2 \cdot e_1' + \beta_2 \cdot e_2' = \alpha_2 (1,1) + \beta_2 \cdot (1,-2)$$

$$\Leftrightarrow \begin{cases} \alpha_2 + \beta_2 = 1 \\ \alpha_2 - 2\beta_2 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_2 = 2\beta_2 \\ 2\beta_2 + \beta_2 = 1 \end{cases} \Leftrightarrow \begin{cases} \beta_2 = \frac{1}{3} \\ \alpha_2 = \frac{2}{3} \end{cases}$$

$$\Rightarrow [f(e_2)]_{B'} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$$

$$f(e_3) = (0,-1) = \alpha_3 \cdot e_1' + \beta_3 \cdot e_2' = \alpha_3 \cdot (1,1) + \beta_3 \cdot (1,-2)$$

$$\begin{cases} \alpha_3+\beta_3=0 \\ \alpha_3-2\beta_3=-1 \end{cases} \Rightarrow \begin{cases} \alpha_3=-\beta_3 \\ -\beta_3-2\beta_3=-1 \end{cases} \Leftrightarrow \begin{cases} \beta_3=\frac{1}{3} \\ \alpha_3=-\frac{1}{3} \end{cases}$$

$$\Rightarrow [f(v_3)]_{B'} = \begin{pmatrix} -1/3 \\ 1/3 \end{pmatrix}$$

$$\Rightarrow [f]_{B,B'} = \begin{pmatrix} 1/3 & 2/3 & -1/3 \\ 2/3 & 1/3 & 1/3 \end{pmatrix}$$

Not.: V K-v.s. $B=(v_1,\dots,v_n)$ basis for V

Let $v\in V \Rightarrow \exists!\,\alpha_1,\alpha_2,\dots,\alpha_n\in K$ so that

$$v=\alpha_1v_1+\alpha_2v_2+\dots+\alpha_nv_n$$

Not.: $[v]_B = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$

4. Let $f\in End_{\mathbb{R}}(\mathbb{R}^4)$ with the following matrix in the canonical basis E of $\mathbb{R}^4$:

$$[f]_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix}.$$

(i) Show that $v=(1,4,1,-1)\in Ker\,f$ and $v'=(2,-2,4,2)\in Im\,f$.
(ii) Determine a basis and the dimension of $Ker\,f$ and $Im\,f$.
(iii) Define f.

Prop.: In the same conditions as above.

$\forall v\in V$: $\boxed{[f(v)]_{B'} = [f]_{B,B'}\cdot[v]_B}$

(i) $v\in Ker f \Leftrightarrow f(v)=0 \Leftrightarrow [f(v)]_E = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \Leftrightarrow [f]_E\cdot[v]_E = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$

$$[f]_E\cdot[v]_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix}\cdot\begin{pmatrix}1\\4\\1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$v' \in \mathrm{Im} f \iff \exists u \in \mathbb{R}^4 : f(u) = v' \iff \exists u \in \mathbb{R}^4 : [f]_E \cdot [u]_E = [v']_E$

$$\iff \exists u = (x,y,z,t) : \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 4 \\ 2 \end{pmatrix} \iff$$

$$\iff \begin{cases} x + y - 3z + 2t = 2 \\ -x + y + z + 4t = -2 \\ 2x + y - 5z + t = 4 \\ x + 2y - 4z + 5t = 2 \end{cases}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ -1 & 1 & 1 & 4 & -2 \\ 2 & 1 & -5 & 1 & 4 \\ 1 & 2 & -4 & 5 & 2 \end{array}\right) \underset{L_4 \leftarrow L_4 - L_1}{\overset{\substack{L_2 \leftarrow L_2 + L_1 \\ L_3 \leftarrow L_3 - 2L_1}}{\sim}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ 0 & 2 & -2 & 6 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right) \sim$$

$$\overset{L_4 \leftrightarrow L_2}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 2 & -2 & 6 & 0 \end{array}\right) \underset{L_4 \leftarrow L_4 - 2L_2}{\overset{L_3 \leftarrow L_3 + L_2}{\sim}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow$ system is compatible $\Rightarrow \exists u = (x,y,z,t) : f(u) = v'$

$\Rightarrow v' \in \mathrm{Im} f$.

(ii) $\mathrm{Ker} f = \{ v = (x,y,z,t) \mid f(v) = 0 \} =$

$$= \left\{ v = (x,y,z,t) \;\middle|\; [f(v)]_E = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ v = (x,y,z,t) \;\middle|\; [f]_E \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} =$$

$$= \left\{ u = (x,y,z,t) \;\middle|\; \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ -1 & 1 & 1 & 4 & 0 \\ 2 & 1 & -5 & 1 & 0 \\ 1 & 2 & -4 & 5 & 0 \end{array}\right) \underset{\substack{L_3 \leftarrow L_3 - 2L_1 \\ L_4 \leftarrow L_4 - L_1}}{\overset{L_2 \leftarrow L_2 + L_1}{\sim}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & 2 & -2 & 6 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right) \sim$$

$$\overset{L_2 \leftrightarrow L_4}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 2 & -2 & 6 & 0 \end{array}\right) \underset{L_4 \leftarrow L_4 - 2L_2}{\overset{L_3 \leftarrow L_3 + L_2}{\sim}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\overset{L_1 \leftarrow L_1 - L_2}{\sim} \left(\begin{array}{cccc|c} 1 & 0 & -2 & -1 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\operatorname{Ker} f = \left\{ u = (x,y,z,t) \;\middle|\; \begin{cases} x - 2z - t = 0 \\ y - z + 3t = 0 \end{cases} \right\} =$$

$$= \left\{ u = (x,y,z,t) \;\middle|\; \begin{cases} x = 2z + t \\ y = z + 3t \end{cases} \right\} =$$

$$= \{ v = (2z + t, z + 3t, z, t) \mid z, t \in \mathbb{R} \} =$$

$$= \{ z \cdot (2,1,1,0) + t \cdot (1,3,0,1) \mid z, t \in \mathbb{R} \} =$$

$$= \langle (2,1,1,0), (1,3,0,1) \rangle$$

$$\begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{pmatrix} \overset{L_2 \leftarrow L_2 - \frac{1}{2} L_1}{\sim} \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & \frac{5}{2} & -\frac{1}{2} & 1 \end{pmatrix} \overset{L_2 \leftarrow 2L_2}{\sim} \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 5 & -1 & 2 \end{pmatrix}$$

$\Rightarrow$ basis for $\ker f$: $\left((2,1,1,0), (0,5,-1,2) \right)$

$\dim \ker f = 2$

$\operatorname{Im} f = \{ w = (a,b,c,d) \mid \exists u = (x,y,z,t) : f(u) = w \}$

$= \{ w = (a,b,c,d) \mid \exists u = (x,y,z,t) : [w]_E = [f]_E \cdot [u]_E \}$

$= \left\{ (a,b,c,d) \in \mathbb{R}^4 \;\middle|\; \exists x,y,z,t : \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \right\}$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ -1 & 1 & 1 & 4 & b \\ 2 & 1 & -5 & 1 & c \\ 1 & 2 & -4 & 5 & d \end{array}\right) \underset{\substack{L_3 \leftarrow L_3 - 2L_1 \\ L_4 \leftarrow L_4 - L_1}}{\overset{L_2 \leftarrow L_2 + L_1}{\sim}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ 0 & 2 & -2 & 6 & a+b \\ 0 & -1 & 1 & -3 & c-2a \\ 0 & 1 & -1 & 3 & d-a \end{array}\right) \sim$$

$$\overset{L_2 \leftrightarrow L_4}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ 0 & 1 & -1 & 3 & d-a \\ 0 & -1 & 1 & -3 & c-2a \\ 0 & 2 & -2 & 6 & a+b \end{array}\right) \underset{L_4 \leftarrow L_4 - 2L_2}{\overset{L_3 \leftarrow L_3 + L_2}{\sim}}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ 0 & 1 & -1 & 3 & d-a \\ 0 & 0 & 0 & 0 & c+d-3a \\ 0 & 0 & 0 & 0 & a+b-2d+2a \end{array}\right)$$

$\Rightarrow$ system is compatible $\Leftrightarrow \begin{cases} c+d-3a = 0 \\ 3a+b-2d = 0 \end{cases}$

$\Rightarrow \operatorname{Im} f = \left\{ (a,b,c,d) \;\middle|\; \begin{cases} c+d-3a = 0 \\ 3a+b-2d = 0 \end{cases} \right\}$

$$\Rightarrow \operatorname{Im} f = \left\{ (a,b,c,d) \;\middle|\; \begin{matrix} c = 3a - d \\ b = 2d - 3a \end{matrix} \right\} =$$

$$= \{ (a, 2d-3a, 3a-d, d) \mid a,d \in \mathbb{R} \} =$$

$$= \{ a(1,-3,3,0) + d\cdot(0,2,-1,1) \mid a,d \in \mathbb{R} \}$$

$$= \langle (1,-3,3,0), (0,2,-1,1) \rangle$$

$(1,-3,3,0)$ and $(0,2,-1,1)$ lin. indep.

$\Rightarrow ((1,-3,3,0),(0,2,-1,1))$ basis for $\operatorname{Im} f$

$\Rightarrow \dim \operatorname{Im} f = 2$

(iii)