6. Let p be a prime number and let V be a vector space over the field $\mathbb{Z}_p$.
(i) Prove that $\underbrace{x + \cdots + x}_{p \text{ times}} = 0$, $\forall x \in V$.

(ii) Is there a scalar multiplication endowing $(\mathbb{Z}, +)$ with a structure of a vector space over $\mathbb{Z}_p$?

$\hookrightarrow$ to endow = a înzestra

Sol. : $\mathbb{Z}_p = \{ \bar{0}, \bar{1}, \ldots, \overline{p-1} \}$

(i) $x + \ldots + x = (1_{\mathbb{Z}_p} + \ldots + 1_{\mathbb{Z}_p}) x = (\underbrace{\bar{1} + \ldots + \bar{1}}_{p \text{ times}}) \cdot x = \bar{0} \cdot x = 0_V$

(ii) Suppose that we have such a structure:

${}_{\mathbb{Z}_p}\mathbb{Z}$ vector space.

$\Rightarrow$ using (i) we have that $\forall x \in \mathbb{Z}$:

$$\underbrace{x + x + \ldots + x}_{p \text{ times}} = 0$$

For example, $1 + 1 + \ldots + 1 = 0$, so $p = 0$, absurd

$\Rightarrow$ there is no such scalar multiplication.

2. Let M be a non-empty set and let $(R, +, \cdot)$ be a ring. Define on $R^M = \{f \mid f : M \to R\}$ two operations by: $\forall f, g \in R^M$,

$$f + g : M \to R, \quad (f+g)(x) = f(x) + g(x), \quad \forall x \in M,$$

$$f \cdot g : M \to R, \quad (f \cdot g)(x) = f(x) \cdot g(x), \quad \forall x \in M.$$

Show that $(R^M, +, \cdot)$ is a ring. If R is commutative or has identity, does R^M have the same property?

$\exists\, 0 \in R^M$, $0 : M \to R$
$x \mapsto 0_R$

$\forall f \in \mathbb{R}^M, \quad \exists \; -f : M \to \mathbb{R}$

$x \longmapsto -f(x)$

4. Let $U_n = \{z \in \mathbb{C} \mid z^n = 1\}$ $(n \in \mathbb{N}^*)$ be the *set of n-th roots of unity*. Prove that U_n is a subgroup of $(\mathbb{C}^*, \cdot)$.

Sol.: $U_n = \{z \in \mathbb{C} \mid z^n = 1\} = \{\cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n} \mid k = 0, \dots, n-1\}$

$1 \in U_n \Rightarrow U_n \neq \emptyset$

Let $z_1, z_2 \in U_n \Rightarrow z_1^n = z_2^n = 1$. We have to show that:

$z_1 \cdot z_2^{-1} \in U_n$

$(z_1 \cdot z_2^{-1})^n = z_1^n \cdot (z_2^{-1})^n = z_1^n \cdot (z_2^n)^{-1} = 1 \cdot 1 = 1$

9. Consider the subspaces

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\},$$

$$T = < (0, 1, 1), (1, 1, 0) >$$

of the real vector space $\mathbb{R}^3$. Determine $S \cap T$ and show that $S + T = \mathbb{R}^3$.

Sol.: $S = \{(0, y, t) \mid y, t \in \mathbb{R}\} = < (0,1,0), (0,0,1) >$

$\Rightarrow \dim S = 2$, because $(0,1,0)$ and $(0,0,1)$ lin. indep.

$(0,1,1)$ and $(1,1,0)$ lin. indep. $\Rightarrow \dim T = 2$

$S+T = \langle (0,1,0), (0,0,1), (0,1,1), (1,1,0) \rangle$

$$\operatorname{rnk} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} = 3, \text{ because } \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 1 \neq 0$$

$\Rightarrow ((0,1,0), (0,0,1), (1,1,0))$ basis for $S+T$

$\Rightarrow \dim(S+T) = 3$

From the 2^{nd} dim thm: $\dim(S\cap T) = \dim S + \dim T - \dim(S+T) =$
$= 2+2-3 = 1$

Another option:

$T = \langle (0,1,1), (1,1,0) \rangle = \{ a\cdot(0,1,1) + b\cdot(1,1,0) \mid a,b \in \mathbb{R} \}$
$= \{ (b, a+b, a) \mid a,b \in \mathbb{R} \}$

$S\cap T = \{ (b, a+b, a) \mid a,b \in \mathbb{R}, b=0 \} =$
$= \{ (0, a, a) \mid a \in \mathbb{R} \} = \langle (0,1,1) \rangle$

$\Rightarrow \dim(S\cap T) = 1$

$\Rightarrow \dim(S+T) = \dim S + \dim T - \dim(S\cap T) = 2+2-1 = 3$

Conclusion: Because $\left.\begin{array}{l} \dim(S+T) = 3 \\ \dim(\mathbb{R}^3) = 3 \\ S+T \leq \mathbb{R}^3 \end{array}\right\} \Rightarrow S+T = \mathbb{R}^3.$

6. Let $n \in \mathbb{N}^*$. Show that the vectors

$$v_1 = (1, \ldots, 1, 1), v_2 = (1, \ldots, 1, 2), v_3 = (1, \ldots, 1, 2, 3), \ldots, v_n = (1, 2, \ldots, n-1, n)$$

form a basis of the real vector space $\mathbb{R}^n$ and write the coordinates of a vector $(x_1, \ldots, x_n)$ in this basis.

Sol: $v_1, v_2, \ldots, v_n$ basis for $\mathbb{R}^n \Leftrightarrow v_1, \ldots, v_n$ lin. indep $\Leftrightarrow$

$$\Leftrightarrow \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 2 \\ 1 & 1 & \cdots & 1 & 2 & 3 \\ \vdots & & & \vdots & \vdots & \vdots \\ 1 & 1 & & & & \\ 1 & 2 & \cdots & n-2 & n-1 & n \end{vmatrix} \neq 0 \overset{L_i \leftarrow L_i - L_1,\ \forall i \neq 1}{\Leftrightarrow} \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 2 \\ \vdots & & & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & n-3 & n-2 & n-1 \end{vmatrix} \neq 0 \Leftrightarrow$$

$$\Leftrightarrow \begin{vmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 2 \\ \vdots & & & & \\ 0 & \cdots & n-4 & n-3 & n-2 \\ 1 & \cdots & n-3 & n-2 & n-1 \end{vmatrix} \neq 0 \Leftrightarrow \begin{vmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 2 \\ \vdots & & & & & \\ 1 & 0 & \cdots & n-4 & n-3 & n-2 \end{vmatrix} \neq 0 \Leftrightarrow$$

$$\Leftrightarrow \cdots \Leftrightarrow \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} \neq 0 \Leftrightarrow \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -1 \neq 0$$

Developing a determinant according to a line/column

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1\,n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2\,n-1} & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{m\,n-1} & a_{mn} \end{vmatrix} =$$

$$= a_{21} \cdot (-1)^{2+1} \cdot \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1\,n-1} & a_{1n} \\ a_{32} & a_{33} & \cdots & a_{3\,n-1} & a_{3n} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m2} & a_{m3} & \cdots & a_{m\,n-1} & a_{mn} \end{vmatrix} +$$

$$+ a_{22} \cdot (-1)^{2+2} \cdot \begin{vmatrix} a_{11} & a_{13} & \cdots & a_{1\,n-1} & a_{1n} \\ a_{31} & a_{33} & \cdots & a_{3\,n-1} & a_{3n} \\ \vdots & & & & \\ a_{m1} & a_{m3} & \cdots & a_{m\,n-1} & a_{mn} \end{vmatrix} +$$

$$+ \cdots + a_{2n} \cdot (-1)^{2+n} \cdot \Delta_{2,n}$$

10. Determine the number of elements of the general linear group $(GL_3(\mathbb{Z}_2), \cdot)$ of invertible 3×3-matrices over $\mathbb{Z}_2$.

<u>Sol</u>: $A \in GL_3(\mathbb{Z}_2) \iff \det A = 1$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in GL_3(\mathbb{Z}_2) \iff \underbrace{v_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}, v_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}, v_3 = \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}}_{\text{lin. indep.}}$$

$GL_3(\mathbb{Z}_2) = \{ M \in M_3(\mathbb{Z}_2) \mid \det M \neq 0 \}$

We have to find the number of triples of vectors in $\mathbb{Z}_2^3$ that are lin. indep.

We have to choose $v_1, v_2, v_3 \in \mathbb{Z}_2^3$ s.t. v_1, v_2, v_3 lin. indep.

To choose v_1 : $2^3 - 1 = 7$ choices

To choose v_2 : $2^3 - 1 - 1 = 6$ choices
(once v_1 was chosen)

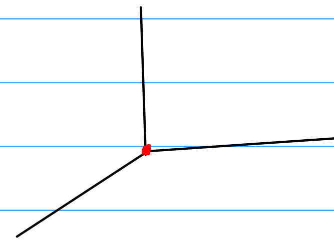
To choose v_3 : $2^3 - 1 - 1 - 1 - 1 = 4$ choices
(once v_1 and v_2 were chosen)

$\longrightarrow \quad 7 \times 6 \times 4 = 168$

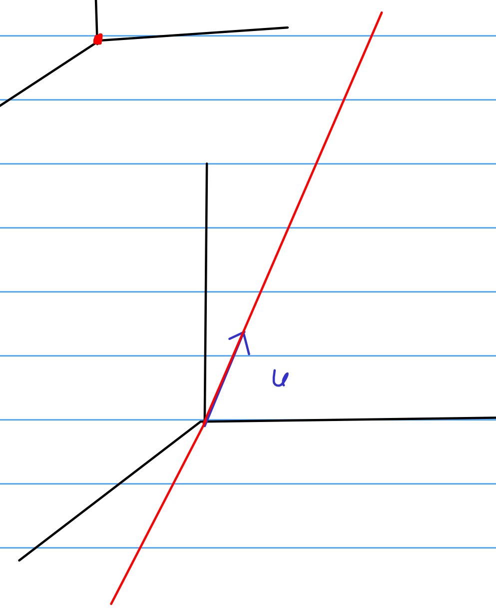
So we have 168 triples of vectors in $\mathbb{Z}_2^3$ that are lin. indep. $\Rightarrow$ we have 168 matrices in $GL_3(\mathbb{Z}_2)$

Subspaces of $\mathbb{R}^3$: $S \leq \mathbb{R}^3$

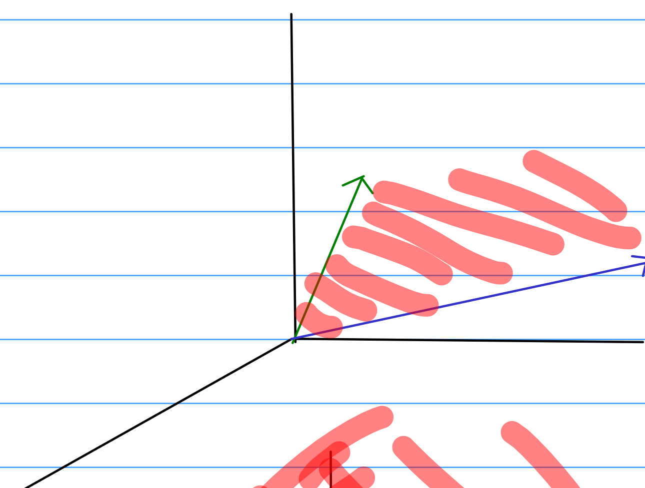
- $\dim S = 0 \Rightarrow S = 0$



- $\dim S = 1 \Rightarrow S = \langle u \rangle$



- $\dim S = 2 \Rightarrow S = \langle v, w \rangle$



- $\dim S = 3 \Rightarrow S = \langle u, v, w \rangle$

$V = 0 = \{0\}$