26.10.2021

Seminar W5 – 911

Prop: V K-vector space, $X \subseteq V$

The subspace generated by X is:

$$\langle X \rangle = \bigcap_{\substack{S \leq_K V \\ S \supseteq X}} S = \left\{ \sum_{i=1}^{n} k_i \cdot v_i \;\middle|\; n \in \mathbb{N},\ k_i \in K,\ v_i \in X \right\} =$$

= "the set of all finite linear combinations of elements from X"

2. Consider the following subspaces of the real vector space $\mathbb{R}^3$:
(i) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$;
(ii) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$;
(iii) $C = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$.
Write A, B, C as generated subspaces with a minimal number of generators.

(iv) $D = \left\{ (x,y,z,t) \in \mathbb{R}^4 \;\middle|\; \begin{cases} x + 2y = 0 \\ y + t = 0 \end{cases} \right\}$

(v) $E = \{ (x,y,z) \in \mathbb{R}^3 \mid y + 2z = 0 \}$

Sol: (i) $A = \{(x,y,z) \in \mathbb{R}^3 \mid x = 0\} = \{(0,y,z) \mid y,z \in \mathbb{R}\} =$

$= \{(0,y,0) + (0,0,z) \mid y,z \in \mathbb{R}\} = \{y \cdot (0,1,0) + z \cdot (0,0,1) \mid y,z \in \mathbb{R}\}$

$= \langle \{(0,1,0), (0,0,1)\} \rangle = \langle (0,1,0), (0,0,1) \rangle$

(ii) $B = \{(x,y,z) \in \mathbb{R}^3 \mid x = -y - z\} = \{(-y-z, y, z) \mid y,z \in \mathbb{R}\} =$

$= \{(-y,y,0) + (-z,0,z) \mid y,z \in \mathbb{R}\} = \{y \cdot (-1,1,0) + z \cdot (-1,0,1) \mid y,z \in \mathbb{R}\}$

$= \langle (-1,1,0), (-1,0,1) \rangle$

It is the minimal number of generators, since $(-1,0,1) \notin \langle (-1,1,0) \rangle$

(iii) $C = \{(x,y,z) \mid x=y=z\} = \{(x,x,x) \mid x \in \mathbb{R}\} =$

$= \{x\cdot(1,1,1) \mid x \in \mathbb{R}\} = \langle (1,1,1) \rangle$

(iv) $D = \left\{ (x,y,z,t) \in \mathbb{R}^4 \;\middle|\; \begin{cases} x+2y=0 \\ y+t=0 \end{cases} \right\} =$

$= \left\{ (x,y,z,t) \in \mathbb{R}^4 \;\middle|\; \begin{cases} x=-2y \\ t=-y \end{cases} \right\} =$

$= \{(-2y, y, z, -y) \mid y,z \in \mathbb{R}\} =$

$= \{(-2y, y, 0, -y) + (0,0,z,0) \mid y,z \in \mathbb{R}\} =$

$= \{y(-2,1,0,-1) + z\cdot(0,0,1,0) \mid y,z \in \mathbb{R}\} =$

$= \langle (-2,1,0,-1), (0,0,1,0) \rangle$

This the minimal number of generators, because $(0,0,1,0) \notin \langle (-2,1,0,-1) \rangle$

Proof: if $(0,0,1,0) \in \langle (-2,1,0,-1) \rangle$, then $\exists \alpha \in \mathbb{R}$:

$$(0,0,1,0) = \alpha\cdot(-2,1,0,-1)$$

$$\Rightarrow \begin{cases} 0=0 \\ \alpha=0 \\ 0=1 \\ -\alpha=0 \end{cases}, \text{ which has no solutions}$$

$\Rightarrow (0,0,1,0) \notin \langle (-2,1,0,-1) \rangle$

(v) $E = \{(x,y,z) \in \mathbb{R}^3 \mid y+2z=0\}$

$= \{(x,y,z) \in \mathbb{R}^3 \mid y=-2z\} = \{(x,-2z,z) \mid x,z \in \mathbb{R}\}$

$$= \{ (x,0,0) + (0,-2z,z) \mid x,z \in \mathbb{R} \} = \{ x\cdot(1,0,0) + z\cdot(0,-2,1) \mid x,z \in \mathbb{R} \}$$

$$= \langle (1,0,0), (0,-2,1) \rangle$$

This is the minimal number of generators, because none of them can be obtained as a linear combination of the others

<u>Def</u>: V, W K-vector spaces, $f : V \to W$ is a

<u>(hom)-orphism of vector spaces</u> (<u>linear map</u>) if:

$$\left.\begin{array}{l} \cdot\ \forall v_1, v_2 : \ f(v_1+v_2) = f(v_1) + f(v_2) \\ \cdot\ \forall k \in K, \forall v \in V : \ f(kv) = k f(v) \end{array}\right\}$$

$$\Downarrow$$

$$\forall k_1, k_2 \in K, \forall v_1, v_2 \in V : \ f(k_1 v_1 + k_2 v_2) = k_1 f(v_1) + k_2 f(v_2)$$

6. Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ and $h : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$f(x,y) = (x+y, x-y),$$
$$g(x,y) = (2x-y, 4x-2y),$$
$$h(x,y,z) = (x-y, y-z, z-x).$$

Show that $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ and $h \in End_{\mathbb{R}}(\mathbb{R}^3)$.

<u>Sol</u>: Let $v_1, v_2 \in \mathbb{R}^2$, $v_1 = (x_1, y_1)$, $v_2 = (x_2, y_2)$

Let $k_1, k_2 \in \mathbb{R}$:

$$f(k_1 v_1 + k_2 v_2) = f(k_1(x_1,y_1) + k_2(x_2,y_2)) =$$

$$= f\left((k_1x_1, k_1y_1) + (k_2x_2, k_2y_2) \right) = f(k_1x_1 + k_2x_2, k_1y_1 + k_2y_2) =$$

$$= (k_1x_1 + k_2x_2 + k_1y_1 + k_2y_2, k_1x_1 + k_2x_2 - k_1y_1 - k_2y_2) =$$

$$= (k_1x_1 + k_1y_1, k_1x_1 - k_1y_1) + (k_2x_2 + k_2y_2, k_2x_2 - k_2y_2) =$$

$$= k_1(x_1+y_1, x_1-y_1) + k_2(x_2+y_2, x_2-y_2) =$$

$$= k_1 \cdot f(x_1,y_1) + k_2 \cdot f(x_2,y_2) = k_1 f(u_1) + k_2 f(u_2)$$

$$g(x,y) = (2x-y, 4x-2y)$$

$$\forall k_1,k_2 \in \mathbb{R}, \quad \forall u_1 = (x_1,y_1), \quad u_2 = (x_2,y_2) \in \mathbb{R}^2$$

$$k_1 g(u_1) + k_2 g(u_2) = k_1 \cdot (2x_1 - y_1, 4x_1 - 2y_1) +$$

$$+ k_2 \cdot (2x_2 - y_2, 4x_2 - 2y_2) = (2k_1x_1 - k_1y_1, 4k_1x_1 - 2k_1y_1) +$$

$$+ (2k_2x_2 - k_2y_2, 4k_2x_2 - 2k_2y_2) = (2k_1x_1 - k_1y_1 + 2k_2x_2 - k_2y_2,$$

$$4k_1x_1 - 2k_1y_1 + 4k_2x_2 - 2k_2y_2) = (\underbrace{2k_1x_1 + 2k_2x_2}_{2(k_1x_1 + k_2x_2)} - (k_1y_1 + k_2y_2),$$

$$4(k_1x_1 + k_2x_2) - 2(k_1y_1 + k_2y_2)) =$$

$$= g(k_1x_1 + k_2x_2, k_1y_1 + k_2y_2) = g(k_1u_1 + k_2u_2)$$

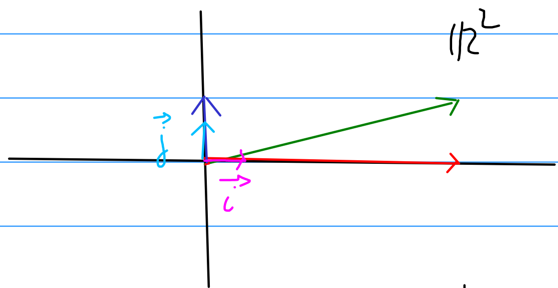
$$\Rightarrow g \in End_{\mathbb{R}}(\mathbb{R}^2)$$

Def: $_k V$, $S, T \le_k V$

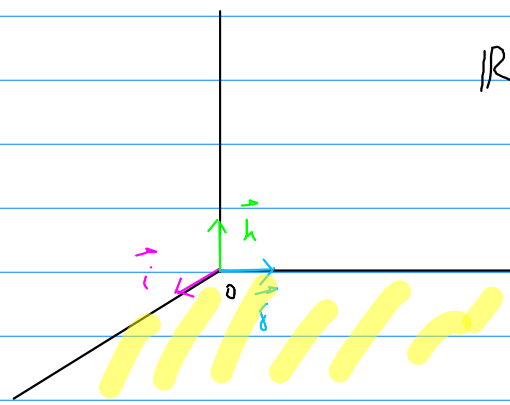
$$V = S + T \iff \forall v \in V\ \exists s \in S,\ \exists t \in T: \ v = s + t$$

$$V = S \oplus T \iff \begin{matrix} V = S + T \\ S \cap T = \{0\} \end{matrix} \iff \forall v \in V\ \exists!\, s \in S, t \in T: \ v = s + t$$

("direct sum")



$$\mathbb{R}^2 = \underset{\langle \vec{i} \rangle}{\underset{\shortparallel}{\mathbb{R}\vec{i}}} \oplus \underset{\langle \vec{j} \rangle}{\underset{\shortparallel}{\mathbb{R}\vec{j}}}$$



$$\mathbb{R}^3 = \mathbb{R}\vec{i} \oplus \mathbb{R}\vec{j} \oplus \mathbb{R}\vec{k} =$$
$$= \langle \vec{k} \rangle \oplus \underbrace{\langle \vec{i}, \vec{j} \rangle}_{\approx \mathbb{R}^2}$$

4. Let

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\},$$
$$T = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}.$$

Prove that S and T are subspaces of the real vector space $\mathbb{R}^3$ and $\mathbb{R}^3 = S \oplus T$.

Sol: $S = \{(-y-z, y, z) \mid y, z \in \mathbb{R}\} = \{(-y, y, 0) + (-z, 0, z) \mid y, z \in \mathbb{R}\}$

$= \{y \cdot (-1, 1, 0) + z \cdot (-1, 0, 1) \mid y, z \in \mathbb{R}\}$

Let $(x, y, z) \in S \cap T \Rightarrow \begin{cases} x + y + z = 0 \\ x = y = z \end{cases} \Rightarrow x = y = z = 0$

$\Rightarrow S \cap T \subseteq 0 \Rightarrow S \cap T = 0 = \{0\}$

$\mathbb{R}^3 = S + T$

We have to show that $\forall v = (x,y,z) \in \mathbb{R}^3 \ \exists s \in S, t \in T$:

$$v = s + t$$

Assume we have found such a decomposition.
Say $t = (a,a,a) \Rightarrow s = (x-a, y-a, z-a) \in S$

$$\Rightarrow x - a + y - a + z - a = 0 \Rightarrow a = \frac{x+y+z}{3}$$

Now, for any $v = (x,y,z)$, we can decompose:

$$(x,y,z) = \underbrace{\left(\frac{x+y+z}{3}, \frac{x+y+z}{3}, \frac{x+y+z}{3}\right)}^{=:t} +$$

$$+ \underbrace{\left(x - \frac{x+y+z}{3}, y - \frac{x+y+z}{3}, z - \frac{x+y+z}{3}\right)}_{=:s}$$

We can now clearly see that $s \in S$ and $t \in T$

$\Rightarrow$ we have, thus, shown, that $\forall v \in \mathbb{R}^3$: $\exists s \in S, \exists t \in T$:

$$v = s + t$$

Hence we have shown that $\mathbb{R}^3 = S + T$

Because $S \cap T = 0 \Rightarrow \mathbb{R}^3 = S \oplus T$