07.12.2021

Seminar W10 - 911

Def: V, V' K-vector spaces, $B = (v_1, \dots, v_m)$ basis of V ("source basis")
$B' = (v_1', \dots, v_n')$ basis of V' ("target basis")

$f : V \to V'$ K-linear map.

$$[f]_{B,B'} = \left([f(v_1)]_{B'} \quad [f(v_2)]_{B'} \quad \cdots \quad [f(v_m)]_{B'} \right) \in M_{n,m}(K)$$

Prop: V, V' K-v.s., B, B' bases of V, V', $f : V \to V'$ linear map,

$$\forall v \in V : \quad [f(v)]_{B'} = [f]_{B,B'} \cdot [v]_B$$

2. Let $f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2)$ be defined by

$$f(x, y, z) = (y, -x)$$

and consider the bases $B = (v_1, v_2, v_3) = ((1,1,0),(0,1,1),(1,0,1))$ of $\mathbb{R}^3$, $B' = (v_1', v_2') = ((1,1),(1,-2))$ of $\mathbb{R}^2$ and let $E' = (\underset{(1,0)}{\underset{\shortparallel}{e_1'}}, \underset{(0,1)}{\underset{\shortparallel}{e_2'}})$ be the canonical basis of $\mathbb{R}^2$. Determine the matrices $[f]_{BE'}$ and $[f]_{BB'}$.

Sol. Ex: $[f]_{E,B'}$, $E = (\underset{(1,0,0)}{\underset{\shortparallel}{e_1}}, \underset{(0,1,0)}{\underset{\shortparallel}{e_2}}, \underset{(0,0,1)}{\underset{\shortparallel}{e_3}})$

$$[f]_{E,B'} = \left([f(e_1)]_{B'} \quad [f(e_2)]_{B'} \quad [f(e_3)]_{B'} \right)$$

$$f(e_1) = f(1,0,0) = (0,-1) = \alpha_1 \cdot v_1' + \beta_1 \cdot v_2'$$

$$\Rightarrow (0,-1) = \alpha_1 \cdot (1,1) + \beta_1 \cdot (1,-2) \Rightarrow \begin{cases} \alpha_1 + \beta_1 = 0 \\ \alpha_1 - 2\beta_1 = -1 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = 2\beta_1 - 1 \\ 2\beta_1 - 1 + \beta_1 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \beta_1 = \frac{1}{3} \\ \alpha_1 = -\frac{1}{3} \end{cases} \Rightarrow [f(e_1)]_{B'} = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 1/3 \end{pmatrix};$$

$$f(e_2) = f(0,1,0) = (1,0) = \alpha_2 \cdot e_1' + \beta_2 \cdot e_2'$$

$$\Rightarrow (1,0) = \alpha_2 \cdot (1,1) + \beta_2 \cdot (1,-2) \Rightarrow \begin{cases} \alpha_2 + \beta_2 = 1 \\ \alpha_2 - 2\beta_2 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_2 = \frac{2}{3} \\ \beta_2 = \frac{1}{3} \end{cases}$$

$$\Rightarrow [f(e_2)]_{B'} = \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$$

$$f(e_3) = f(0,0,1) = (0,0) = \alpha_3 \cdot (1,1) + \beta_3 \cdot (1,-2)$$

$$\Rightarrow \begin{cases} 0 = \alpha_3 + \beta_3 \\ 0 = \alpha_3 - 2\beta_2 \end{cases} \Rightarrow \begin{cases} \alpha_3 = 0 \\ \beta_3 = 0 \end{cases}$$

$$\Rightarrow [f(e_3)]_{B'} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{l} [f(e_1)]_{B'} = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 1/3 \end{pmatrix} \\ [f(e_2)]_{B'} = \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} \\ [f(e_3)]_{B'} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{array}\right\} \Rightarrow [f]_{E,B'} = \begin{pmatrix} -1/3 & 2/3 & 0 \\ 1/3 & 1/3 & 0 \end{pmatrix}$$

$[f]_{B,B'} = ?$

$f(u_1) = f(1,1,0) = (1,-1) = \alpha_1 \cdot u_1' + \beta_1 \cdot u_2'$

$\Rightarrow (1,-1) = \alpha_1 \cdot (1,1) + \beta_1 \cdot (1,-2)$

$\Rightarrow \begin{cases} 1 = \alpha_1 + \beta_1 \\ -1 = \alpha_1 - 2\beta_1 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = 1 - \beta_1 \\ -1 = 1 - 3\beta_1 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = \frac{1}{3} \\ \beta_1 = \frac{2}{3} \end{cases}$

$\Rightarrow [f(u_1)]_{B'} = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$

$f(u_2) = f(0,1,1) = (1,0) = \alpha_2 \cdot u_1' + \beta_2 \cdot u_2' = \alpha_2 (1,1) + \beta_2 \cdot (1,-2)$

$\Rightarrow \begin{cases} 1 = \alpha_2 + \beta_2 \\ 0 = \alpha_2 - 2\beta_2 \end{cases} \Leftrightarrow \begin{cases} \alpha_2 = 1 - \beta_2 \\ 0 = 1 - 3\beta_2 \end{cases} \Leftrightarrow \begin{cases} \beta_2 = \frac{1}{3} \\ \alpha_2 = \frac{2}{3} \end{cases}$

$\Rightarrow [f(u_2)]_{B'} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$

$f(u_3) = f(1,0,1) = (0,-1) = \alpha_3 \cdot (1,1) + \beta_3 \cdot (1,-2)$

$\Rightarrow \begin{cases} 0 = \alpha_3 + \beta_3 \\ -1 = \alpha_3 - 2\beta_3 \end{cases} \Leftrightarrow \begin{cases} \alpha_3 = -\beta_3 \\ -1 = -3\beta_3 \end{cases} \Leftrightarrow \begin{cases} \beta_3 = \frac{1}{3} \\ \alpha_3 = -\frac{1}{3} \end{cases}$

$\Rightarrow [f(u_3)]_{B'} = \begin{pmatrix} -1/3 \\ 1/3 \end{pmatrix}$

$\Rightarrow [f]_{B,B'} = \begin{pmatrix} 1/3 & 2/3 & -1/3 \\ 2/3 & 1/3 & 1/3 \end{pmatrix}$

$$f(v_1) = (1,-1) = \alpha_1 \cdot (1,0) + \beta_1 \cdot (0,1)$$

$$\Rightarrow \alpha_1 = 1 \, , \, \beta_1 = -1$$

$$\Rightarrow [f(v_1)]_E = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$[f(v_2)]_E = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$[f(v_3)]_E = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\Rightarrow [f]_E = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$$

4. Let $f \in End_{\mathbb{R}}(\mathbb{R}^4)$ with the following matrix in the canonical basis E of $\mathbb{R}^4$:

$$[f]_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix}.$$

(i) Show that $v = (1, 4, 1, -1) \in Ker\, f$ and $v' = (2, -2, 4, 2) \in Im\, f$.
(ii) Determine a basis and the dimension of $Ker\, f$ and $Im\, f$.
(iii) Define f.

Prop: V, V' K-v.s., B, B' bases of V, V', $f: V \to V'$ linear map,

$$\forall v \in V: \quad [f(v)]_{B'} = [f]_{B,B'} \cdot [v]_B$$

(i) $$[f(v)]_E = [f]_E \cdot [v]_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow f(v) = 0 \Rightarrow v \in Ker f.$

We have to find $v'' = (x, y, z, t)$ so that $f(v'') = v'$

$$[f(v'')] = [f]_E \cdot [v'']_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

We have to show that $\exists x, y, z, t$ so that

$$\begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 4 \\ 2 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ -1 & 1 & 1 & 4 & -2 \\ 2 & 1 & -5 & 1 & 4 \\ 1 & 2 & -4 & 5 & 2 \end{array}\right) \underset{\substack{L_3 \leftarrow L_3 - 2L_1 \\ L_4 \leftarrow L_4 - L_1}}{\overset{L_2 \leftarrow L_2 + L_1}{\sim}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ 0 & 2 & -2 & 6 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right) \sim$$

$$\overset{L_2 \leftrightarrow L_4}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 2 & -2 & 6 & 0 \end{array}\right) \begin{array}{l} L_3 \leftarrow L_3 + L_2 \\ \sim \\ L_4 \leftarrow L_4 - 2L_2 \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow$ the system is compatible $\Rightarrow \exists x, y, z, t \Rightarrow v' \in \operatorname{Im} f$.

(ii) $\operatorname{Ker} f = \{ v = (x, y, z, t) \mid f(v) = 0 \} =$

$$= \left\{ v = (x, y, z, t) \mid [f]_E \cdot [v]_E = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} =$$

$$= \left\{ v = (x, y, z, t) \in \mathbb{R}^4 \mid \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ -1 & 1 & 1 & 4 & 0 \\ 2 & 1 & -5 & 1 & 0 \\ 1 & 2 & -4 & 5 & 0 \end{array}\right) \begin{array}{l} L_2 \leftarrow L_2 + L_1 \\ \sim \\ L_3 \leftarrow L_3 - 2L_1 \\ L_4 \leftarrow L_4 - L_1 \end{array} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & 2 & -2 & 6 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right) \sim$$

$$\begin{array}{l} L_3 \leftarrow L_3 + \frac{1}{2} L_2 \\ \sim \\ L_4 \leftarrow L_4 - \frac{1}{2} L_2 \end{array} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & 2 & -2 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \overset{L_2 \leftarrow \frac{1}{2} L_2}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \begin{cases} x + y - 3z + 2t = 0 \\ y - z + 3t = 0 \end{cases}$$

$$\operatorname{Ker} f = \left\{ (x,y,z,t) \in \mathbb{R}^4 \;\middle|\; \begin{cases} x + y - 3z + 2t = 0 \\ y - z + 3t = 0 \end{cases} \right\} =$$

$$= \left\{ (x,y,z,t) \;\middle|\; \begin{cases} y = z - 3t \\ x = 3z - 2t - z + 3t = 2z + t \end{cases} \right\} =$$

$$= \left\{ (2z+t,\ z-3t,\ z,\ t) \;\middle|\; z,t \in \mathbb{R} \right\} =$$

$$= \left\{ (2z,\ z,\ z,\ 0) + (t,\ -3t,\ 0,\ t) \;\middle|\; z,t \in \mathbb{R} \right\} =$$

$$= \left\{ z \cdot (2,1,1,0) + t \cdot (1,-3,0,1) \;\middle|\; z,t \in \mathbb{R} \right\} =$$

$$= \langle (2,1,1,0),\ (1,-3,0,1) \rangle$$

$$\begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & -3 & 0 & 1 \end{pmatrix} \overset{L_1 \leftrightarrow L_2}{\sim} \begin{pmatrix} 1 & -3 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix} \overset{L_2 \leftarrow L_2 - 2L_1}{\sim} \begin{pmatrix} 1 & -3 & 0 & 1 \\ 0 & 7 & 1 & -2 \end{pmatrix}$$

$\Rightarrow$ basis for $\operatorname{Ker} f$: $\left((1,-3,0,1),\ (0,7,1,-2) \right)$

$\Rightarrow \dim(\operatorname{Ker} f) = 2$

$$\operatorname{Im} f = \left\{ f(v) \;\middle|\; v \in \mathbb{R}^4 \right\} = \left\{ f(x,y,z,t) \;\middle|\; x,y,z,t \in \mathbb{R} \right\} =$$

$$= \left\{ w = (a,b,c,d) \;\middle|\; \exists\, v = (x,y,z,t) : f(v) = w \right\} =$$

$$= \left\{ w = (a,b,c,d) \;\middle|\; \exists\, v = (x,y,z,t) : [f]_E \cdot [v]_E = [w]_E \right\}$$

$$= \left\{ w = (a,b,c,d) \;\middle|\; \exists\, v = (x,y,z,t) : \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \right\}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ -1 & 1 & 1 & 4 & b \\ 2 & 1 & -5 & 1 & c \\ 1 & 2 & -4 & 5 & d \end{array}\right) \underset{\substack{L_3 \leftarrow L_3 - 2L_1 \\ L_4 \leftarrow L_4 - L_1}}{\overset{L_2 \leftarrow L_2 + L_1}{\sim}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ 0 & 2 & -2 & 6 & a+b \\ 0 & -1 & 1 & -3 & c-2a \\ 0 & 1 & -1 & 3 & d-a \end{array}\right) \sim$$

$$\overset{L_2 \leftrightarrow L_4}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ 0 & 1 & -1 & 3 & d-a \\ 0 & -1 & 1 & -3 & c-2a \\ 0 & 2 & -2 & 6 & a+b \end{array}\right) \underset{L_4 \leftarrow L_4 - 2L_2}{\overset{L_3 \leftarrow L_3 + L_2}{\sim}}$$

$$= \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ 0 & 1 & -1 & 3 & d-a \\ 0 & 0 & 0 & 0 & c+d-3a \\ 0 & 0 & 0 & 0 & a+b-2d+2a \end{array}\right)$$

The system is compatible $\Leftrightarrow \begin{cases} c+d-3a=0 \\ 3a+b-2d=0 \end{cases}$

$$\operatorname{Im} f = \left\{ (a,b,c,d) \in \mathbb{R}^4 \,\middle|\, \begin{cases} c+d-3a=0 \\ 3a+b-2d=0 \end{cases} \right\} =$$

$$= \left\{ (a,b,c,d) \,\middle|\, \begin{cases} c = 3a-d \\ b = 2d-3a \end{cases} \right\} =$$

$$= \{ (a, 2d-3a, 3a-d, d) \mid a,d \in \mathbb{R} \}$$

$$= \langle (1,-3,3,1), (0,2,-1,1) \rangle$$

$\Rightarrow$ basis for $\operatorname{Im} f$: $((1,-3,3,1), (0,2,-1,1))$

$\Rightarrow \dim \operatorname{Im} f = 2$

Verification: $\underbrace{\dim \ker f}_{2} + \underbrace{\dim \operatorname{Im} f}_{2} = \underbrace{\dim(\mathbb{R}^4)}_{4}$ ✓

(iii)

$f(x,y,z,t) = ?$

$$[f(x,y,z,t)]_E = [f]_E \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x+y-3z+2t \\ -x+y+z+4t \\ 2x+y-5z+t \\ x+2y-4z+5t \end{pmatrix}$$

$$\Rightarrow f(x,y,z,t) = (x+y-3z+2t,\ -x+y+z+4t,\ 2x+y-5z+t,\ x+2y-4z+5t)$$