16.11.2021

Seminar W8-911

Prop: S linear system. M its matrix, $\overline{M}$ its extended matrix

$$S \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$$M = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \qquad \overline{M} = \left(M \,\middle|\, \begin{matrix} b_1 \\ \vdots \\ b_m \end{matrix} \right)$$

Kronecker-Capelli thm: (S) is compatible $\Leftrightarrow$ rank (M) = rank $\overline{M}$

Rouché's theorem: We find a principal minor in M Δ

[minor = determinant formed by selecting k rows and k columns from M; principal minor = minor of maximal size (minor of size rank M)]

We form all the characteristic minors

$$\begin{vmatrix} \text{principal minor} & \text{elements from column of free terms} \\ \text{row from } M & \end{vmatrix}$$

(S) is compatible $\Leftrightarrow$ all the char. minors are zero

Cramer's rule: If we have a square system S (n unknowns, n equations)

S compatible determined $\Leftrightarrow \Delta = \det(M) \neq 0$

$\rightarrow$ solution is unique

If S compatible $\Rightarrow$ $x_1 = \dfrac{\Delta_{x_1}}{\Delta}$, ..., $x_n = \dfrac{\Delta_{x_n}}{\Delta}$

2. Using the Kronecker-Capelli theorem, decide if the following linear systems are compatible and then solve the compatible ones:

$$(i)\begin{cases} x_1 + x_2 + x_3 - 2x_4 = 5 \\ 2x_1 + x_2 - 2x_3 + x_4 = 1 \\ 2x_1 - 3x_2 + x_3 + 2x_4 = 3 \end{cases} \qquad (ii)\begin{cases} x_1 - 2x_2 + x_3 + x_4 = 1 \\ x_1 - 2x_2 + x_3 - x_4 = -1 \\ x_1 - 2x_2 + x_3 + 5x_4 = 5 \end{cases}$$

$$(iii)\begin{cases} x + y + z = 3 \\ x - y + z = 1 \\ 2x - y + 2z = 3 \\ x + z = 4 \end{cases}$$

3. Using the Rouché theorem, decide if the systems from **2.** are compatible and then solve the compatible ones.

Sol.: 3(ii)
$$\begin{cases} x_1 - 2x_2 + x_3 + x_4 = 1 \\ x_1 - 2x_2 + x_3 - x_4 = -1 \\ x_1 - 2x_2 + x_3 + 5x_4 = 5 \end{cases}$$

$$M = \begin{pmatrix} 1 & -2 & 1 & 1 \\ 1 & -2 & 1 & -1 \\ 1 & -2 & 1 & 5 \end{pmatrix} \Rightarrow \text{rank } M \leq 3$$

$$\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} \neq 0 \Rightarrow \text{rank } M \geq 2$$

Any 3 columns we choose, two of them will be proportional, so all minors of order 3 are 0.

$\Rightarrow$ rank $M = 2$

Principal unknowns: x_1, x_4

Principal equations: first, third

Now we apply Rouché's theorem to see if the system is compatible

$$\Delta = \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} \qquad M = \begin{pmatrix} 1 & -2 & 1 & 1 \\ 1 & -2 & 1 & -1 \\ 1 & -2 & 1 & 5 \end{pmatrix}$$

$$\overline{M} = \left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & -1 & -1 \\ 1 & -2 & 1 & 5 & 5 \end{array}\right)$$

$$\Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 5 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 4 & 4 \end{vmatrix} = -8-(-8) = 0 \overset{\text{Rouché}}{\Longrightarrow} \text{the system is compatible}$$

We take the secondary unknowns as parameters:

$$x_2 = \alpha \qquad x_3 = \beta.$$

So our system is :

$$\begin{cases} x_1 + x_4 = 1 + 2\alpha - \beta \\ x_1 + 5x_4 = 5 + 2\alpha - \beta \end{cases}$$

$$\Rightarrow \begin{cases} x_1 + x_4 = 1 + 2\alpha - \beta \\ 4x_4 = 4 \end{cases} \Rightarrow \begin{cases} x_4 = 1 \\ x_1 = 2\alpha - \beta \\ x_2 = \alpha \\ x_3 = \beta \end{cases}$$

Solve the following linear systems by the Gauss and Gauss-Jordan methods:

5. $(i)\begin{cases}2x+2y+3z=3\\ x-y\quad=1\\ -x+2y+z=2\end{cases}$ $\quad(ii)\begin{cases}2x+5y+z=7\\ x+2y-z=3\\ x+y-4z=2\end{cases}$ $\quad(iii)\begin{cases}x+y+z=3\\ x-y+z=1\\ 2x-y+2z=3\\ x+z=4\end{cases}$

6. $\begin{cases}2x_1+x_2+x_3+x_4=1\\ x_1+2x_2-x_3+4x_4=2\\ x_1+5x_2-4x_3+11x_4=\lambda\end{cases}\quad(\lambda\in\mathbb{R})$

7. $\begin{cases}ax+y+z=1\\ x+ay+z=a\\ x+y+az=a^2\end{cases}\quad(a\in\mathbb{R})$

Sol. : (i) $\begin{cases}2x+2y+3z=3\\ x-y=1\\ -x+2y+z=2\end{cases}$

row echelon form

$$\left(\begin{array}{ccc|c}2&2&3&3\\1&-1&0&1\\-1&2&1&2\end{array}\right)\underset{L_1\leftrightarrow L_2}{\sim}\left(\begin{array}{ccc|c}1&-1&0&1\\2&2&3&3\\-1&2&1&2\end{array}\right)\sim$$

$$\begin{array}{c}L_2\leftarrow L_2-2L_1\\ \sim\\ L_3\leftarrow L_3+L_1\end{array}\left(\begin{array}{ccc|c}1&-1&0&1\\0&4&3&1\\0&1&1&3\end{array}\right)\underset{L_2\leftrightarrow L_3}{\sim}\left(\begin{array}{ccc|c}1&-1&0&1\\0&1&1&3\\0&4&3&1\end{array}\right)\sim$$

$$\underset{\sim}{L_3\leftarrow L_3-4L_2}\left(\begin{array}{ccc|c}1&-1&0&1\\0&1&1&3\\0&0&-1&-11\end{array}\right)\to$$ we have obtained a row echelon form

Gauss — from the row echelon form we rewrite the system and solve it.

Gauss-Jordan — keep eliminating zeros above the diagonal and you get solutions

Gauss:

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -11 \end{array}\right)$$

$$\begin{cases} x - y = 1 \\ y + z = 3 \\ -z = -11 \end{cases} \Rightarrow \begin{cases} z = 11 \\ y = 3 - 11 = -8 \\ x = -8 + 1 = -7 \end{cases}$$

Gauss-Jordan

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -11 \end{array}\right) \overset{L_2 \leftarrow L_2 + L_3}{\sim}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & -1 & -11 \end{array}\right) \underset{L_3 \leftarrow -L_3}{\overset{L_1 \leftarrow L_1 + L_2}{\sim}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 11 \end{array}\right)$$

$$\Rightarrow \begin{cases} x = -7 \\ y = -8 \\ z = 11 \end{cases}$$

When we're using Gaussian elimination to solve a system, there are some remarkable situations:

$$\left(\begin{array}{ccccc|c} 0 & 0 & 0 & 0 \cdots & 0 & a \end{array}\right), a \neq 0 \Rightarrow \text{incompatible system}$$

$$\left(\begin{array}{cccccc|c} 0 & 0 \cdots & 0 & 0 & \cdots & 0 & 0 \end{array}\right) \Rightarrow \text{redundancy (redundant equation)}$$

5. $(i)\begin{cases}2x+2y+3z=3\\ x-y\quad=1\\ -x+2y+z=2\end{cases}$ $(ii)\begin{cases}2x+5y+z=7\\ x+2y-z=3\\ x+y-4z=2\end{cases}$ $(iii)\begin{cases}x+y+z=3\\ x-y+z=1\\ 2x-y+2z=3\\ x+z=4\end{cases}$

$$\textbf{6.}\quad \begin{cases}2x_1+x_2+x_3+x_4=1\\ x_1+2x_2-x_3+4x_4=2\\ x_1+5x_2-4x_3+11x_4=\lambda\end{cases}\qquad (\lambda\in\mathbb{R})$$

$$\textbf{7.}\quad \begin{cases}ax+y+z=1\\ x+ay+z=a\\ x+y+az=a^2\end{cases}\qquad (a\in\mathbb{R})$$

Sol :

$$\left(\begin{array}{ccc|c}1&1&1&3\\1&-1&1&1\\2&-1&2&3\\1&0&1&4\end{array}\right)\underset{L_2\leftarrow L_2-L_1}{\sim}\left(\begin{array}{ccc|c}1&1&1&3\\0&-2&0&-2\\2&-1&2&3\\1&0&1&4\end{array}\right)\underset{L_3\leftarrow L_3-2L_1}{\sim}\left(\begin{array}{ccc|c}1&1&1&3\\0&-2&0&-2\\0&-3&0&-3\\1&0&1&4\end{array}\right)\sim$$

$$\underset{L_4\leftarrow L_4-L_1}{\sim}\left(\begin{array}{ccc|c}1&1&1&3\\0&-2&0&-2\\0&-3&0&-3\\0&-1&0&-1\end{array}\right)\underset{\substack{L_2\leftarrow \frac{L_2}{-2}\\ L_3\leftarrow\frac{L_3}{-3}\\ L_4\leftarrow\frac{L_4}{-1}}}{\sim}\left(\begin{array}{ccc|c}1&1&1&3\\0&1&0&1\\0&1&0&1\\0&1&0&1\end{array}\right)\sim$$

$$\underset{L_3\leftarrow L_3-L_2}{\sim}\left(\begin{array}{ccc|c}1&1&1&3\\0&1&0&1\\0&0&0&0\\0&1&0&1\end{array}\right)\underset{L_4\leftarrow L_4-L_2}{\sim}\left(\begin{array}{ccc|c}1&1&1&3\\0&1&0&1\\0&0&0&0\\0&0&0&0\end{array}\right)$$

$$\Rightarrow\begin{cases}x+y+z=3\\ y=1\end{cases}\Rightarrow\begin{cases}x=2-z\\ y=1\end{cases}$$

(ii)

$$\left(\begin{array}{ccc|c}2&5&1&7\\1&2&-1&3\\1&1&-4&2\end{array}\right)\underset{\substack{L_2\leftarrow 2L_2-L_1\\ L_3=2L_3-L_1}}{\sim}\left(\begin{array}{ccc|c}2&5&1&7\\0&-1&-3&-1\\0&-3&-9&-3\end{array}\right)\sim$$

Gauss-Jordan
$\Longrightarrow$

$$\underset{\sim}{L_3 \leftarrow L_3 - 3L_2} \begin{pmatrix} 2 & 5 & 1 & \vert & 7 \\ 0 & -1 & -3 & \vert & -1 \\ 0 & 0 & 0 & \vert & 0 \end{pmatrix} \underset{\sim}{L_1 \leftarrow L_1 + 5L_2} \begin{pmatrix} 2 & 0 & -14 & \vert & 2 \\ 0 & -1 & -3 & \vert & -1 \\ 0 & 0 & 0 & \vert & 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 2x - 14z = 2 \\ -y - 3z = -1 \end{cases} \Rightarrow \begin{cases} x = 7z + 1 \\ y = -3z + 1 \end{cases}$$