15.11.2021

Seminar W7-914

Th. (Steinitz): (rephrased) If V K-v.s., $S \leq_K V$ $\Rightarrow$ any basis of S can be completed to a basis a V

How to:
- we have $u_1, \dots, u_n$ basis of S
- we pick a vector $w_{n+1} \in V \setminus \underbrace{\langle u_1, \dots, u_n \rangle}_{= S}$
- Do we have enough vectors?

yes — we have our basis

no — We pick a vector $w_{n+2} \in V \setminus \langle u_1, \dots, u_n, w_{n+1} \rangle$

yes — no

1. Determine a basis and the dimension of the following subspaces of the real vector space $\mathbb{R}^3$:

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$$
$$B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$$
$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}.$$

6. Complete the bases of the subspaces from Exercise **1.** to some bases of the real vector space $\mathbb{R}^3$ over $\mathbb{R}$.

$u_1, \dots, u_n \in_K V$, $\operatorname{rank}(u_1, \dots, u_n) := \dim \langle u_1, \dots, u_n \rangle =$ the maximal number of lin. indep. vectors in $u_1, \dots, u_n$

If $V = K^m$: $\operatorname{rank}(u_1, \dots, u_n) = \operatorname{rank} \begin{pmatrix} u_1 \\ \hline u_2 \\ \hline \vdots \\ \hline u_n \end{pmatrix} = \operatorname{rank} (u_1 \mid u_2 \mid \dots \mid u_n)$

$A = \{ (x,y,z) \in \mathbb{R}^3 \mid z = 0 \} = \{ (x,y,0) \mid x, y \in \mathbb{R} \} =$

$= \langle (1,0,0), (0,1,0) \rangle$

$((1,0,0), (0,1,0))$ basis for A, because $(1,0,0)$ and $(0,1,0)$ lin. indep

We add $(0,0,1) \in \mathbb{R}^3 \setminus A$, so $(0,0,1)$, $(1,0,0)$, $(0,0,1)$ lin. indep

We have 3 lin. indep. vectors $\Rightarrow$ $(1,0,0)$, $(0,0,1)$, $(0,1,0)$ basis.

$B = \{ (x,y,z) \in \mathbb{R}^3 \mid x+y+z = 0 \} = \{ (x,y,z) \mid x = -y-z \} =$

$= \{ (-y-z, y, z) \mid y,z \in \mathbb{R} \} = \{ y \cdot (-1,1,0) + z \cdot (-1,0,1) \mid y,z \in \mathbb{R} \}$

$= \langle (-1,1,0), (-1,0,1) \rangle$

$\text{rank} \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = 2 \Rightarrow (-1,1,0), (-1,0,1)$ lin indep $\Rightarrow$ they form a basis

$(0,1,0) \notin B \Rightarrow (0,1,0) \notin \langle (-1,1,0), (-1,0,1) \rangle \Rightarrow$

$\Rightarrow (0,1,0), (-1,1,0), (-1,0,1)$ lin. indep $\Rightarrow$ they form a basis of $\mathbb{R}^3$

$C = \{ (x,y,z) \in \mathbb{R}^3 \mid x = y = z \} =$

$= \{ (x,x,x) \mid x \in \mathbb{R} \} = \langle (1,1,1) \rangle$

$((1,1,1))$ is a basis for C, which need to complete to a basis of $\mathbb{R}^3$.

We add $(1,0,0)$ and $(0,1,1)$. We check if $(1,1,1)$, $(1,0,0)$ and $(0,1,1)$ form a basis. It suffices to check if they are linearly independent.

$$\operatorname{rank}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = ? \qquad \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

, therefore our choice doesn't work

$\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \neq 0 \Rightarrow$ we can keep $(1,1,1)$ and $(1,0,0)$, because they're linearly independent

Let's replace $(0,1,1)$ by $(0,1,2)$ and check for linear independence:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{vmatrix} = -1 \neq 0 \Rightarrow \operatorname{rank}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix} = 3 \Rightarrow$$

$\Rightarrow (1,1,1), (1,0,0), (0,1,2)$ are lin. indep. $\Rightarrow$ they form a basis

Def: V, W k-vector spaces, $\underbrace{f \in \operatorname{Hom}_k(V, W)}_{\text{"}f \text{ is a linear map"}}$

$$\operatorname{Ker} f = \{ v \in V \mid f(v) = 0_W \} \leq_k V$$
("kernel")

$$\operatorname{Im} f = \{ f(v) \mid v \in V \} = \{ w \in W \mid \exists\, v \in V : f(v) = w \} \leq W$$

5. Let $f \in End_{\mathbb{R}}(\mathbb{R}^3)$ be defined by $f(x, y, z) = (-y + 5z, x, y - 5z)$. Determine a basis and the dimension of $Ker\, f$ and $Im\, f$.

Sol.: $Ker\, f = \{ (x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = (0,0,0) \} =$

$= \{ (x,y,z) \mid (-y+5z, x, y-5z) = (0,0,0) \} =$

$= \left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, \begin{cases} -y+5z=0 \\ x=0 \\ y-5z=0 \end{cases} \right\} =$

$= \left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, \begin{cases} x=0 \\ y=5z \\ y=5z \end{cases} \right\} =$

$= \{ (0, 5z, z) \mid z \in \mathbb{R} \} = \langle (0,5,1) \rangle$

$\Rightarrow \dim Ker\, f = 1$

$Im\, f = \{ f(u) \mid u \in \mathbb{R}^3 \} = \{ f(x,y,z) \mid x,y,z \in \mathbb{R} \} =$

$= \{ (-y+5z, x, y-5z) \mid x,y,z \in \mathbb{R} \} =$

$= \{ x \cdot (0,1,0) + y \cdot (-1,0,1) + z \cdot (5,0,-5) \mid x,y,z \in \mathbb{R} \} =$

$= \langle (0,1,0), (-1,0,1), (5,0,-5) \rangle$

$\operatorname{rank} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 5 & 0 & -5 \end{pmatrix} = ?$ $\quad \begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 5 & 0 & -5 \end{vmatrix} = 0$

$\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \neq 0 \Rightarrow$ we pick $(0,1,0), (-1,0,1)$ as our basis vectors

$\Rightarrow \left((0,1,0), (-1,0,1) \right)$ basis for $\operatorname{Im} f \Rightarrow \dim \operatorname{Im} f = 2$

Th. (1st dim thm) $\quad f: V \to W$ linear map, then:

$$\dim V = \underbrace{\dim \ker f}_{\text{nullity}(f)} + \underbrace{\dim \operatorname{Im} f}_{\text{rank}(f)}$$

Th. (2nd dim thm)

V K-v.s., $\quad S, T \leq_K V$, then:

$$\dim (S+T) = \dim S + \dim T - \dim (S \cap T)$$

Rem: If $S = \langle u_1, \dots, u_n \rangle$, $T = \langle w_1, \dots, w_m \rangle$

$$\Rightarrow S+T = \langle u_1, \dots, u_n, w_1, \dots, w_m \rangle$$

10. Determine the dimensions of the subspaces S, T, $S+T$ and $S \cap T$ of the real vector space $M_2(\mathbb{R})$, where

$$S = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle, \qquad T = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle.$$

Sol: $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ lin. indep $\Rightarrow \dim S = 2$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ lin. indep $\Rightarrow \dim T = 2$

$$S+T = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle$$

Let's check if $\begin{pmatrix}1&1\\0&0\end{pmatrix}$, $\begin{pmatrix}1&0\\1&1\end{pmatrix}$, $\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\begin{pmatrix}0&0\\1&1\end{pmatrix}$ are lin indep

$a,b,c,d \in \mathbb{R}$: $a\cdot\begin{pmatrix}1&1\\0&0\end{pmatrix} + b\cdot\begin{pmatrix}1&0\\1&1\end{pmatrix} + c\cdot\begin{pmatrix}0&1\\1&0\end{pmatrix} + d\cdot\begin{pmatrix}0&0\\1&1\end{pmatrix} = 0$

$$\Rightarrow \begin{cases} a+b=0 \\ a+c=0 \\ b+c+d=0 \\ b+d=0 \end{cases} \Rightarrow \begin{cases} b=-a \\ c=-a \\ d=-b=a \\ a-a-a=0 \end{cases} \Rightarrow a=b=c=d=0$$

$\Rightarrow$ the 4 vectors are lin indep $\Rightarrow \dim(S+T)=4$

$$\dim(S\cap T) = -\dim(S+T) + \dim S + \dim T = -4+2+2 = 0$$

$$\Rightarrow S\cap T = 0 \Rightarrow S+T = S\oplus T$$