3.12.2021

Seminar W9 – 916

Compute by applying elementary operations the ranks of the matrices:

1. $\begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix}$. **2.** $\begin{pmatrix} 1 & -1 & 3 & 2 \\ -2 & 0 & 3 & -1 \\ -1 & 2 & 0 & -1 \end{pmatrix}$. **3.** $\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix}$ $(\alpha, \beta \in \mathbb{R})$.

Sol.: 1.

$$\begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix} \overset{L_1 \leftrightarrow L_3}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 3 \\ 0 & 2 & 3 \\ 2 & 2 & 4 \end{pmatrix} \underset{L_4 \leftarrow L_4 - 2L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix} \sim$$

$$\overset{L_3 \leftarrow L_3 - L_2}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} \overset{L_4 \leftarrow L_4 - L_3}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

This is an echelon form. So $\operatorname{rank} M = \#$ nonzero rows in the echelon form $= 3$

3.

$$\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix} \overset{L_1 \leftrightarrow L_2}{\sim} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ \beta & 1 & 3 & 4 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix} \underset{L_3 \leftarrow L_3 - 2L_1}{\overset{L_2 \leftarrow L_2 - \beta L_1}{\sim}}$$

$$\sim \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & 1-\alpha\beta & 3-3\beta & 4-3\beta \\ 0 & \alpha & -2 & 1 \end{pmatrix} \overset{L_2 \leftrightarrow L_3}{\sim} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & \alpha & -2 & 1 \\ 0 & 1-\alpha\beta & 3-3\beta & 4-3\beta \end{pmatrix}$$

If $\alpha = 0 \Rightarrow \begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 3-3\beta & 4-3\beta \\ 0 & 0 & -2 & 1 \end{pmatrix} \Rightarrow \operatorname{rank} M = 3$

II) If $\alpha \neq 0 \Rightarrow \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & \alpha & -2 & 1 \\ 0 & 1-\alpha\beta & 3-3\beta & 4-3\beta \end{pmatrix} \quad L_3 \leftarrow L_3 - \frac{1-\alpha\beta}{\alpha} L_2 \quad \sim$

$(1-\alpha\beta) + k\cdot\alpha = 0$

$k = \frac{\alpha\beta - 1}{\alpha}$

$$\sim \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & \alpha & -2 & 1 \\ 0 & 0 & 3-3\beta + \frac{2(1-\alpha\beta)}{\alpha} & 4-3\beta - \frac{1-\alpha\beta}{\alpha} \end{pmatrix}$$

rank $M \in \{2, 3\}$ and

rank $M = 2 \iff \begin{cases} 3-3\beta + \frac{2(1-\alpha\beta)}{\alpha} = 0 \\ 4-3\beta - \frac{1-\alpha\beta}{\alpha} = 0 \end{cases} \iff$

$$\iff \begin{cases} 3\alpha - 3\alpha\beta + 2 - 2\alpha\beta = 0 \\ 4\alpha - 3\alpha\beta - 1 + \alpha\beta = 0 \end{cases} \iff \begin{cases} 3\alpha - 5\alpha\beta + 2 = 0 \\ 4\alpha - 2\alpha\beta - 1 = 0 \end{cases} \iff$$

$$\iff \begin{cases} 6\alpha - 10\alpha\beta + 4 = 0 \\ 20\alpha - 10\alpha\beta - 5 = 0 \end{cases} \iff \begin{cases} 6\alpha - 10\alpha\beta + 4 = 0 \\ 14\alpha - 9 = 0 \end{cases} \iff$$

$$\iff \begin{cases} \alpha = \frac{9}{14} \\ \frac{54}{14} - 10\alpha\beta + 4 = 0 \end{cases} \iff \begin{cases} \alpha = \frac{9}{14} \\ \frac{55}{4} = 10\alpha\beta \end{cases} \iff \begin{cases} \alpha = \frac{9}{14} \\ \beta = \frac{55}{40\cdot\frac{9}{14}} = \end{cases}$$

$$= \frac{55\cdot 14}{40\cdot 9} = \frac{77}{36}$$

Compute by applying elementary operations the inverses of the matrices:

4. $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$. **5.** $\begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}$.

Sol:

$$\left(A \mid I_n\right) \sim \overset{\text{Gauss-Jordan}}{\cdots} \sim \left(I_n \mid A^{-1}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & -2 & 0 & 1 & 0 \\ 2 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \underset{L_3 \leftarrow L_3 - 2L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right) \sim$$

$$\overset{L_2 \leftarrow \frac{1}{-3}L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right) \overset{L_3 \leftarrow L_3 + 6L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 9 & 2 & -2 & 1 \end{array}\right) \sim$$

$$\overset{L_3 \leftarrow \frac{L_3}{9}}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{array}\right) \underset{L_1 \leftarrow L_1 - 2L_3}{\overset{L_2 \leftarrow L_2 - 2L_3}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{5}{9} & \frac{4}{9} & -\frac{2}{9} \\ 0 & 1 & 0 & \frac{2}{9} & \frac{1}{9} & \frac{-2}{9} \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{array}\right) \sim$$

$$\overset{L_1 \leftarrow L_1 - 2L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ 0 & 1 & 0 & \frac{2}{9} & \frac{1}{9} & \frac{-2}{9} \\ 0 & 0 & 1 & \frac{2}{9} & \frac{-2}{9} & \frac{1}{9} \end{array}\right)$$

$$\Rightarrow \quad A^{-1} = \begin{pmatrix} \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{1}{9} & \frac{-2}{9} \\ \frac{2}{9} & \frac{-2}{9} & \frac{1}{9} \end{pmatrix}$$

Verification:

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{1}{9} & \frac{-2}{9} \\ \frac{2}{9} & \frac{-2}{9} & \frac{1}{9} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Remark: if A not invertible, then in the process of Gaussian elimination we will get a zero row, which halts the process.

Ex.: $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 6 \\ 2 & 3 & 12 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & 6 & 0 & 1 & 0 \\ 2 & 3 & 12 & 0 & 0 & 1 \end{array}\right) \overset{L_3 \leftarrow L_3 - 2L_1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -6 & 0 & 1 & 0 \\ 0 & -1 & 6 & -2 & 0 & 1 \end{array}\right) \sim$$

$$\overset{L_3 \leftarrow L_3 + L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -6 & 0 & 1 & 0 \\ \underbrace{0 \quad 0 \quad 0} & & & -2 & 1 & 1 \end{array}\right)$$

↳ we cannot obtain I_3 here, so A not invertible

5. $$\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \underset{L_3 \leftarrow L_3 - 3L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right) \sim$$

$$\overset{L_2 \leftarrow -\frac{1}{5} L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right) \overset{L_3 \leftarrow L_3 + 12 L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} & \frac{9}{5} & \frac{-12}{5} & 1 \end{array}\right)$$

$$\overset{L_3 \leftarrow 5L_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right) \overset{L_2 \leftarrow L_2 - \frac{3}{5} L_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -5 & 7 & -3 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right) \sim$$

$\sim$

$$\overset{L_1 \leftarrow L_1 - 2L_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 0 & -17 & 24 & -10 \\ 0 & 1 & 0 & -5 & 7 & -3 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right) \overset{L_1 \leftarrow L_1 - 4L_2}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 2 \\ 0 & 1 & 0 & -5 & 7 & -3 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right)$$

$$\Rightarrow \begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & -4 & 2 \\ -5 & 7 & -3 \\ 9 & -12 & 5 \end{pmatrix}$$

8. In the real vector space $\mathbb{R}^4$ consider the list $X = (v_1, v_2, v_3)$, where $v_1 = (1, 0, 4, 3)$, $v_2 = (0, 2, 3, 1)$ and $v_3 = (0, 4, 6, 2)$. Determine $dim < X >$ and a basis of $< X >$.

Sol.:

$$\begin{pmatrix} 1 & 0 & 4 & 3 \\ 0 & 2 & 3 & 1 \\ 0 & 4 & 6 & 2 \end{pmatrix} \overset{L_3 \leftarrow L_3 - 2L_2}{\sim} \begin{pmatrix} 1 & 0 & 4 & 3 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \dim \langle X \rangle = \operatorname{rank} M = 2$$

$$\Rightarrow \text{basis}: \left((1,0,4,3) ; (0,2,3,1) \right)$$

9. Determine the dimension of the subspaces S, T, $S + T$ and $S \cap T$ of the real vector space $\mathbb{R}^3$ and a basis for the first three of them, where

$$S =< (1, 0, 4), (2, 1, 0), (1, 1, -4) >,$$

$$T =< (-3, -2, 4), (5, 2, 4), (-2, 0, -8) > .$$

$$S+T = \langle S \cup T \rangle$$

$$\begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{pmatrix} \underset{L_3 \leftarrow L_3 - L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 1 & -8 \end{pmatrix} \overset{L_3 \leftarrow L_3 - L_2}{\sim} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \dim S = 2$ basis for S : $\big((1,0,4), (0,1,-8)\big)$

$$\begin{pmatrix} -3 & -2 & 4 \\ 5 & 2 & 4 \\ -2 & 0 & -8 \end{pmatrix} \overset{L_1 \leftarrow \frac{1}{-3} L_1}{\sim} \begin{pmatrix} 1 & \frac{2}{3} & -\frac{4}{3} \\ 5 & 2 & 4 \\ -2 & 0 & -8 \end{pmatrix} \underset{L_3 \leftarrow L_3 + 2L_1}{\overset{L_2 \leftarrow L_2 - 5L_1}{\sim}}$$

$$\sim \begin{pmatrix} 1 & \frac{2}{3} & -\frac{4}{3} \\ 0 & -\frac{4}{3} & \frac{32}{3} \\ 0 & \frac{4}{3} & -\frac{32}{3} \end{pmatrix} \overset{L_3 \leftarrow L_3 + L_2}{\sim} \begin{pmatrix} 1 & \frac{2}{3} & -\frac{4}{3} \\ 0 & -\frac{4}{3} & \frac{32}{3} \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \dim T = 2$, basis of T: $\left(\left(1, \frac{2}{3}, -\frac{4}{3}\right), \left(0, -\frac{4}{3}, \frac{32}{3}\right)\right)$

For $S+T$:

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 3 & 2 & -4 \\ 0 & -4 & 32 \end{pmatrix} \overset{L_3 \leftarrow L_3 - 3L_1}{\sim} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 2 & -16 \\ 0 & -4 & 32 \end{pmatrix} \underset{L_4 \leftarrow L_4 + 4L_2}{\overset{L_3 \leftarrow L_3 - 2L_2}{\sim}}$$

$$\sim \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \dim(S+T) = 2$

basis: $\big((1,0,4), (0,1,-8)\big)$

$$\dim(S \cap T) = \dim(S) + \dim(T) - \dim(S+T) = 2+2-2 = 2$$