27.10.2021

Seminar W5 - 917

4. **10.** Show that the set of all solutions of a homogeneous linear system of two equations and two unknowns with real coefficients is a subspace of the real vector space $\mathbb{R}^2$.

Sol.: $\mathbb{R}^2 = \{ (x,y) \mid x, y \in \mathbb{R} \}$

$$S = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{cases} a_1 x + b_1 y = 0 \\ a_2 x + b_2 y = 0 \end{cases} \right\}$$

We have to show that $S \leq_{\mathbb{R}} \mathbb{R}^2$

$(0,0) \in S \Rightarrow S \neq \emptyset$

Let $v_1, v_2 \in S$, $v_1 = (x_1, y_1)$, $v_2 = (x_2, y_2)$, $\alpha, \beta \in \mathbb{R}$

We will show that $\alpha v_1 + \beta v_2 \in S$

$$v_1, v_2 \in S \Rightarrow \begin{cases} a_1 x_1 + b_1 y_1 = 0 \\ a_2 x_1 + b_2 y_1 = 0 \end{cases}, \quad \begin{cases} a_1 x_2 + b_1 y_2 = 0 \\ a_2 x_2 + b_2 y_2 = 0 \end{cases}$$

$$\alpha v_1 + \beta v_2 = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2)$$

$$a_1 \cdot (\alpha x_1 + \beta x_2) + b_1 \cdot (\alpha y_1 + \beta y_2) =$$

$$= \alpha \underbrace{(a_1 x_1 + b_1 y_1)}_{=0} + \beta \underbrace{(a_1 x_2 + b_1 y_2)}_{=0} = 0$$

$$a_2 \cdot (\alpha x_1 + \beta x_2) + b_2 \cdot (\alpha y_1 + \beta y_2) =$$

$$= \alpha \overbrace{(a_2 x_1 + b_2 y_1)}^{=0} + \beta \overbrace{(a_2 x_2 + b_2 y_2)}^{=0} = 0$$

$\Rightarrow \forall v_1, v_2 \in S, \forall \alpha, \beta \in \mathbb{R} : \alpha v_1 + \beta v_2 \in S$

$\Rightarrow S \leq_{\mathbb{R}} \mathbb{R}^2$

Generated subspaces

Def/Th.: ${}_K V$, $X \subseteq V$

We have the subspace generated by X:

$$\langle X \rangle = \bigcap_{\substack{S \leq_K V \\ S \supseteq X}} S = \left\{ \sum_{i=1}^{n} k_i \cdot v_i \mid n \in \mathbb{N}, k_i \in K, v_i \in X \right\}$$

= "the set of all linear combinations of elements in X"

2. Consider the following subspaces of the real vector space $\mathbb{R}^3$:
(*i*) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$;
(*ii*) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$;
(*iii*) $C = \{(x, y, z) \in \mathbb{R}^3 \mid x = y = z\}$.
Write A, B, C as generated subspaces with a minimal number of generators.

(iv) $D = \left\{ (x,y,z) \in \mathbb{R}^3 \mid \begin{cases} x - z = 0 \\ y + x = 0 \end{cases} \right\}$

Ex: $E = \left\{ (x,y,z,t) \in \mathbb{R}^4 \mid \begin{cases} x + z = 0 \\ y + 2z + t = 0 \end{cases} \right\} =$

$= \left\{ (x,y,z,t) \in \mathbb{R}^4 \mid \begin{cases} x = -z \\ y = -2z - t \end{cases} \right\} =$

$= \{ (-z, -2z - t, z, t) \mid z, t \in \mathbb{R} \} =$

$= \{ (-z, -2z, z, 0) + (0, -t, 0, t) \mid z, t \in \mathbb{R} \} =$

$= \{ z \cdot (-1, -2, 1, 0) + t \cdot (0, -1, 0, 1) \mid z, t \in \mathbb{R} \} =$

$= \langle \{(-1,-2,1,0), (0,-1,0,1)\} \rangle = \langle (-1,-2,1,0), (0,-1,0,1) \rangle$

This is the minimal number of generators because none of them can be generated using the others, i.e.

$$\nexists \alpha \in \mathbb{R} : (-1,-2,1,0) = \alpha \cdot (0,-1,0,1)$$

If there were such an α, then $\begin{cases} -1 = 0 \\ -2 = -\alpha \\ 1 = 0 \\ 0 = 1 \end{cases}$, absurd

(i) $A = \{(x,y,z) \in \mathbb{R}^3 \mid x = 0\} = \{(0,y,z) \mid y,z \in \mathbb{R}\} =$

$= \{(0,y,0) + (0,0,z) \mid y,z \in \mathbb{R}\} = \{y \cdot (0,1,0) + z \cdot (0,0,1) \mid y,z \in \mathbb{R}\}$

$= \langle (0,1,0), (0,0,1) \rangle$

This is the minimal number of generators

(ii) $B = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z = 0\} = \{(x,y,z) \in \mathbb{R}^3 \mid x = -y-z\} =$

$= \{(-y-z, y, z) \mid y,z \in \mathbb{R}\} = \{y \cdot (-1,1,0) + z \cdot (-1,0,1) \mid y,z \in \mathbb{R}\}$

$= \langle (-1,1,0), (-1,0,1) \rangle$

(iii) $C = \{(x,y,z) \in \mathbb{R}^3 \mid x = y = z\} = \{(x,x,x) \mid x \in \mathbb{R}\} =$

$= \{x \cdot (1,1,1) \mid x \in \mathbb{R}\} = \langle (1,1,1) \rangle$ This is the minimal number of generators

(iv) $D = \left\{ (x,y,z) \in \mathbb{R}^3 \;/\; \begin{cases} x = z \\ y = -x \end{cases} \right\} =$

$= \{ (x, -x, x) \mid x \in \mathbb{R} \} = \{ x \cdot (1,-1,1) \mid x \in \mathbb{R} \} =$

$= \langle (1,-1,1) \rangle$

<u>Def</u>: V, W K-vector spaces, $f : V \to W$

is a <u>homomorphism of vector spaces</u> (or <u>linear map</u>)

if: • $\forall v_1, v_2 \in V$: $f(v_1 + v_2) = f(v_1) + f(v_2)$

• $\forall k \in K, \forall v \in V$: $f(kv) = k f(v)$

$\forall k_1, k_2 \in K, \forall v_1, v_2 \in V$:

$$f(k_1 v_1 + k_2 v_2) = k_1 f(v_1) + k_2 f(v_2)$$

6. Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ and $h : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$f(x,y) = (x+y, x-y),$$

$$g(x,y) = (2x-y, 4x-2y),$$

$$h(x,y,z) = (x-y, y-z, z-x).$$

Show that $f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ and $h \in End_{\mathbb{R}}(\mathbb{R}^3)$.

<u>Sol</u>: Let $k_1, k_2 \in \mathbb{R}$, $v_1, v_2 \in \mathbb{R}^2$

$$f(k_1 v_1 + k_2 v_2) \overset{?}{=} k_1 f(v_1) + k_2 f(v_2)$$

$v_1 = (a,b)$, $v_2 = (c,d)$

$$f(k_1 w_1 + k_2 w_2) = f(k_1 (a,b) + k_2 (c,d)) = f(k_1 a + k_2 c, k_1 b + k_2 d) =$$
$$= (k_1 a + k_2 c + k_1 b + k_2 d, k_1 a + k_2 c - k_1 b - k_2 d) =$$
$$= (k_1 \cdot (a+b) + k_2 \cdot (c+d), k_1 (a-b) + k_2 (c-d)) =$$
$$= k_1 (a+b, a-b) + k_2 \cdot (c+d, c-d) =$$
$$= k_1 f(w_1) + k_2 f(w_2)$$

$$h(x,y,z) = (x-y, y-z, z-x)$$

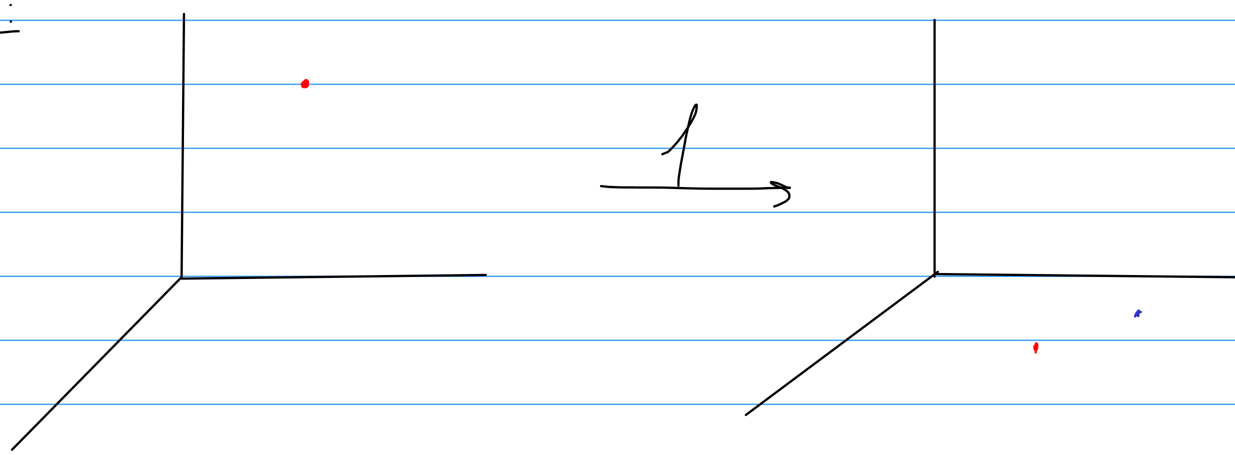
Let $k_1, k_2, k_3 \in \mathbb{R}$, $\forall w_1, w_2, w_3 \in \mathbb{R}^3$, $w_i = (x_i, y_i, z_i)$, $i = \overline{1,3}$

$$h(k_1 w_1 + k_2 w_2 + k_3 w_3) = h(k_1(x_1, y_1, z_1) + k_2 \cdot (x_2, y_2, z_2) + k_3 (x_3, y_3, z_3))$$
$$= h(k_1 x_1 + k_2 x_2 + k_3 x_3, k_1 y_1 + k_2 y_2, k_3 y_3, k_1 z_1 + k_2 z_2 + k_3 z_3)$$
$$= (k_1 x_1 + k_2 x_2 + k_3 x_3 - k_1 y_1 - k_2 y_2 - k_3 y_3,$$
$$k_1 y_1 + k_2 y_2 + k_3 y_3 - k_1 z_1 - k_2 z_2 - k_3 z_3,$$
$$k_1 z_1 + k_2 z_2 + k_3 z_3 - k_1 x_1 - k_2 x_2 - k_3 x_3) =$$
$$= (k_1 x_1 - k_1 y_1, k_1 y_1 - k_1 z_1, k_1 z_1 - k_1 x_1) +$$
$$+ (k_2 x_2 - k_2 y_2, k_2 y_2 - k_2 z_2, k_2 z_2 - k_2 x_2) +$$

$$+ (k_2 x_3 - k_3 y_3 , k_3 y_3 - k_2 z_3 , k_3 z_3 - k_3 x_3) =$$

$$= k_1 \cdot h(w_1) + k_2 \cdot h(w_2) + k_3 \cdot h(w_3)$$

<u>Ex:</u>



$$f : \mathbb{R}^3 \to \mathbb{R}^3$$
$$(x, y, z) \mapsto (0, y, z)$$