22-11.2021

Seminar W8 - 915

$$(S): \begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{cases}$$

$$M = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

$$\overline{M} = \left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right)$$

Th. (Kronecker - Capelli) : (S) compatible $\Leftrightarrow$ rank M = rank $\overline{M}$

Th. (Rouché) : Δ_p principal minor in M

We build the characteristic minors : $(\Delta_i)_{i=\overline{1,s}}$

$$\Delta_i = \left(\begin{array}{c|c} \Delta_p & \begin{matrix}\text{column} \\ \text{of free} \\ \text{terms}\end{matrix} \\ \hline \text{one row from } M & \end{array}\right)$$

(S) compatible $\Leftrightarrow$ $\forall i \in \{1, \dots, s\}$; $\Delta_i = 0$

To solve a system:

- We choose a principal minor $\leftarrow$ minor in M that is nonzero of maximal size
- The rows in this minor correspond to the principal equations (the rows not in this minor correspond to the secondary equations)
- The columns in this minor correspond to the principal unknowns.
- The columns not in this minor correspond to the secondary unknowns.
- Discard the secondary equations.
- Regard the secondary unknowns as parameters (rename them) and move them to the other side as free terms.
- We now have a square system.
- Solve by using Cramer's rule.

$$(S): \begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{n1}x_1 + \dots + a_{nn}x_n = b_n \end{cases}$$

$$(S) \text{ compatible} \iff \Delta = \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} \neq 0$$

$\exists!$ (S) compatible: $x_i = \dfrac{\Delta_{x_i}}{\Delta}$

$$\Delta_{x_i} = \begin{vmatrix} a_{11} & \dots & a_{1\,i-1} & b_1 & a_{1\,i+1} & \dots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \dots & a_{n\,i-1} & b_n & a_{n\,i+1} & \dots & a_{nn} \end{vmatrix}$$

2. Using the Kronecker-Capelli theorem, decide if the following linear systems are compatible and then solve the compatible ones:

$$(i)\begin{cases} x_1+x_2+x_3-2x_4=5 \\ 2x_1+x_2-2x_3+x_4=1 \\ 2x_1-3x_2+x_3+2x_4=3 \end{cases} \qquad (ii)\begin{cases} x_1-2x_2+x_3+x_4=1 \\ x_1-2x_2+x_3-x_4=-1 \\ x_1-2x_2+x_3+5x_4=5 \end{cases}$$

$$(iii)\begin{cases} x+y+z=3 \\ x-y+z=1 \\ 2x-y+2z=3 \\ x+z=4 \end{cases}$$

3. Using the Rouché theorem, decide if the systems from **2.** are compatible and then solve the compatible ones.

Sol.:
$$(i)\begin{cases} x_1+x_2+x_3-2x_4=5 \\ 2x_1+x_2-2x_3+x_4=1 \\ 2x_1-3x_2+x_3+2x_4=3 \end{cases} \qquad M=\begin{pmatrix} 1 & 1 & 1 & -2 \\ 2 & 1 & -2 & 1 \\ 2 & -3 & 1 & 2 \end{pmatrix}$$

$$\Delta_p=\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -2 \\ 2 & -3 & 1 \end{vmatrix}=-7$$

x_1, x_2, x_3 principal unknowns

x_4 secondary unknown, $x_4=:\alpha$

$$\Rightarrow \begin{cases} x_1+x_2+x_3=5+2\alpha \\ 2x_1+x_2-2x_3=1-\alpha \\ 2x_1-3x_2+x_3=3-2\alpha \end{cases}$$

$$\Rightarrow x_1=\frac{\Delta_{x_1}}{\Delta}=\frac{\begin{vmatrix} 5+2\alpha & 1 & 1 \\ 1-\alpha & 1 & -2 \\ 3-2\alpha & -3 & 1 \end{vmatrix}}{-7}=$$

$$=\frac{5+2\alpha-3(1-\alpha)-2(3-2\alpha)-(3-2\alpha)-6\cdot(5+2\alpha)-(1-\alpha)}{-7}$$

$$=\frac{-38}{-7}$$

$$x_2 = \frac{\Delta_{x_2}}{\Delta}, \quad x_3 = \frac{\Delta_{x_3}}{\Delta}$$

Solve the following linear systems by the Gauss and Gauss-Jordan methods:

5. (i) $\begin{cases} 2x+2y+3z=3 \\ x-y \quad =1 \\ -x+2y+z=2 \end{cases}$ (ii) $\begin{cases} 2x+5y+z=7 \\ x+2y-z=3 \\ x+y-4z=2 \end{cases}$ (iii) $\begin{cases} x+y+z=3 \\ x-y+z=1 \\ 2x-y+2z=3 \\ x+z=4 \end{cases}$

6. $\begin{cases} 2x_1+x_2+x_3+x_4=1 \\ x_1+2x_2-x_3+4x_4=2 \\ x_1+5x_2-4x_3+11x_4=\lambda \end{cases}$ $(\lambda \in \mathbb{R})$

7. $\begin{cases} ax+y+z=1 \\ x+ay+z=a \\ x+y+az=a^2 \end{cases}$ $(a \in \mathbb{R})$

5.(ii) $\left(\begin{array}{ccc|c} 2 & 5 & 1 & 7 \\ 1 & 2 & -1 & 3 \\ 1 & 1 & -4 & 2 \end{array}\right)$

row echelon form $\to$ the number of zeros until the first nonzero increases strictly with every row

pivot

$$\left(\begin{array}{ccc|c} 2 & 5 & 1 & 7 \\ 1 & 2 & -1 & 3 \\ 1 & 1 & -4 & 2 \end{array}\right) \underset{L_1 \leftrightarrow L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 5 & 1 & 7 \\ 1 & 1 & -4 & 2 \end{array}\right) \text{ pivot line } \sim$$

$$\underset{L_3 \leftarrow L_3 - L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & -1 & -3 & -1 \end{array}\right) \overset{L_3 \leftarrow L_3 + L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\boxed{1}$ If we apply <u>the Gauss method:</u>

- we revert to the system, which we solve

$$\begin{cases} x + 2y - z = 3 \\ \quad\; y + 3z = 1 \end{cases}$$

$$\Rightarrow \begin{cases} y = 1 - 3z \\ x = 3 + z - 2y = 3 + z - 2 + 6z = 7z + 1 \end{cases}$$

$$\Rightarrow \begin{cases} x = 7\alpha + 1 \\ y = 1 - 3\alpha \\ z = \alpha \end{cases}$$

Ex:

$$\begin{array}{c}
\begin{array}{cccc|ccc} \multicolumn{4}{c|}{\text{principal unknowns}} & \multicolumn{3}{c}{\text{secondary unknowns}} \end{array} \\
\left(\begin{array}{cccc|ccc|c}
6 & 7 & 1 & 3 & 0 & 5 & 7 & 1 \\
0 & 0 & 3 & 5 & 8 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 2 & 1 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 5 & 1 & 3 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 &
\end{array}\right)
\end{array}$$

$\boxed{2}$ If we apply the Gauss-Jordan:

- we choose pivots in the reverse direction and make zeros above them

- we will end up with the solution

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \overset{L_1 \leftarrow L_1 - 2L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & -7 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x \quad - 7z = 1 \\ y + 3z = 1 \end{cases} \Rightarrow \begin{cases} x = 7z + 1 \\ y = 1 - 3z \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x = 7\alpha + 1 \\ y = 1 - 3\alpha \\ z = \alpha \end{cases}$$

<u>Remark</u>: The system (S) is incompatible $\Leftrightarrow$ in the process of applying Gaussian elimination we obtain a row that looks like this:

$$\left(\begin{array}{ccccc|c} 0 & 0 & 0 & \cdots & 0 & \alpha \end{array}\right), \ \alpha \neq 0$$

6. $\begin{cases} 2x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 + 2x_2 - x_3 + 4x_4 = 2 \\ x_1 + 5x_2 - 4x_3 + 11x_4 = \lambda \end{cases} \qquad (\lambda \in \mathbb{R})$

Sol.:

$$\left(\begin{array}{cccc|c} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 4 & 2 \\ 1 & 5 & -4 & 11 & \lambda \end{array}\right) \overset{L_1 \leftrightarrow L_2}{\sim} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 5 & -4 & 11 & \lambda \end{array}\right) \sim$$

$$\underset{L_3 \leftarrow L_3 - L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 0 & -3 & 3 & -7 & -3 \\ 0 & 3 & -3 & 7 & \lambda - 2 \end{array}\right) \overset{L_3 \leftarrow L_3 + L_2}{\sim} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 0 & -3 & 3 & -7 & -3 \\ 0 & 0 & 0 & 0 & \lambda - 5 \end{array}\right)$$

If $\lambda \neq 5 \Rightarrow$ system is incompatible

If $\lambda = 5 \Rightarrow$ the extended matrix becomes

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 0 & -3 & 3 & -7 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We revert to the system:

$$\begin{cases} x_1 + 2x_2 - \underset{\alpha}{x_3} + 4\underset{\beta}{x_4} = 2 \\ \quad -3x_2 + 3x_3 - 7x_4 = -3 \end{cases}$$

$$\Rightarrow \begin{cases} x_2 = \dfrac{3\alpha - 7\beta + 3}{3} \\ x_1 = \alpha - 4\beta + 2 - 2 \cdot \dfrac{3\alpha - 7\beta + 3}{3} \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = -\alpha + \frac{2\beta}{3} \\ x_2 = \alpha - \frac{7}{3}\beta + 3 \\ x_3 = \alpha \\ x_4 = \beta \end{cases}$$