03.12.2021

Seminar W9 – 915

Definition 3.3.1 Let V be a vector space over K, $B = (v_1, \dots, v_n)$ a basis of V and $X = (u_1, \dots, u_m)$ a list of vectors in V. Let

$$\begin{cases} u_1 = a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ u_2 = a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \dots\dots\dots\dots\dots \\ u_m = a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{cases}$$

be the unique writings of the vectors in X as linear combinations of vectors of the basis B, for some $a_{ij} \in K$. The *matrix of the list of vectors* X in the basis B is the matrix having as its rows the coordinates of the vectors in X in the basis B, that is,

$$[X]_B = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

I don't care about this

I care about this (a lot!)

$f : V \to W$ linear map. $B = (u_1, \dots, u_m)$ basis of V, $B' = (u_1', \dots, u_n')$ basis of V'

$$[f]_{B,B'} = \left([f(u_1)]_{B'} \ [f(u_2)]_{B'} \ \dots \ [f(u_m)]_{B'} \right) \in M_{n,m}(K)$$

Compute by applying elementary operations the ranks of the matrices:

1. $\begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix}$. **2.** $\begin{pmatrix} 1 & -1 & 3 & 2 \\ -2 & 0 & 3 & -1 \\ -1 & 2 & 0 & -1 \end{pmatrix}$. **3.** $\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix}$ $(\alpha, \beta \in \mathbb{R})$.

<u>Sol.</u> : 1.

$$M = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix} \overset{L_1 \leftrightarrow L_3}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 3 \\ 0 & 2 & 3 \\ 2 & 2 & 4 \end{pmatrix} \underset{L_4 \leftarrow L_4 - 2L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix} \sim$$

$$\overset{L_3 \leftarrow L_3 - L_2}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} \overset{L_4 \leftarrow L_4 - L_3}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

rank M = rank (echelon form) = # of nonzero rows in the echelon form

3. $\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix} \overset{L_1 \leftrightarrow L_2}{\sim} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ \beta & 1 & 3 & 4 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix} \underset{L_3 \leftarrow L_3 - 2L_1}{\overset{L_2 \leftarrow L_2 - \beta L_1}{\sim}}$

$\sim \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & 1-\alpha\beta & 3-3\beta & 4-3\beta \\ 0 & \alpha & -2 & 1 \end{pmatrix} \overset{L_2 \leftrightarrow L_3}{\sim} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & \alpha & -2 & 1 \\ 0 & 1-\alpha\beta & 3-3\beta & 4-3\beta \end{pmatrix}$

I) $\alpha = 0$: $\begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 0 & -2 & 1 \\ 0 & 1 & 3-3\beta & 4-3\beta \end{pmatrix} \overset{L_2 \leftrightarrow L_3}{\sim} \begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 3-3\beta & 4-3\beta \\ 0 & 0 & -2 & 1 \end{pmatrix}$

$\Rightarrow \text{rank } M = 3$

II) $\alpha \neq 0$ $\begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & \alpha & -2 & 1 \\ 0 & 1-\alpha\beta & 3-3\beta & 4-3\beta \end{pmatrix} \overset{L_3 \leftarrow L_3 + \frac{\alpha\beta-1}{\alpha} L_2}{\sim}$

$\sim \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & \alpha & -2 & 1 \\ 0 & 0 & 3-3\beta-2\cdot\frac{\alpha\beta-1}{\alpha} & 4-3\beta+\frac{\alpha\beta-1}{\alpha} \end{pmatrix}$

$\Rightarrow \text{rank } M \in \{2, 3\}$

$\text{rank } M = 2 \iff \begin{cases} 3-3\beta-2\cdot\frac{\alpha\beta-1}{\alpha} = 0 \\ 4-3\beta+\frac{\alpha\beta-1}{\alpha} = 0 \end{cases} \iff$

$\iff \begin{cases} 3\alpha-3\alpha\beta-2\alpha\beta+2 = 0 \\ 4\alpha-3\alpha\beta+\alpha\beta-1 = 0 \end{cases} \iff \begin{cases} 3\alpha-5\alpha\beta+2 = 0 \\ 4\alpha-2\alpha\beta-1 = 0 \end{cases} \iff$

$$\Leftrightarrow \begin{cases} 6\alpha - 10\alpha\beta + 4 = 0 \\ 20\alpha - 10\alpha\beta - 5 = 0 \end{cases} \Leftrightarrow \begin{cases} 6\alpha - 10\alpha\beta + 4 = 0 \\ 14\alpha - 9 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha = \frac{9}{14} \\ 6 \cdot \frac{9}{14} - 10 \cdot \frac{9}{14} \cdot \beta + 4 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha = \frac{9}{14} \\ \beta = \dfrac{4 + \frac{54}{14}}{\frac{90}{14}} = \frac{110}{90} = \frac{11}{9} \end{cases}$$

We have:

$$\operatorname{rank} M = 2 \Leftrightarrow \begin{cases} \alpha = \frac{9}{14} \\ \beta = \frac{11}{9} \end{cases}$$

Compute by applying elementary operations the inverses of the matrices:

4. $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$. **5.** $\begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}$.

Sol.:

$$\left(A \mid I_n\right) \sim \overset{\text{Gauss-Jordan}}{\cdots} \sim \left(I_n \mid A^{-1}\right)$$

5.

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \underset{L_3 \leftarrow L_3 - 3L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right) \sim$$

$$\overset{L_2 \leftarrow -\frac{1}{5} L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right) \overset{L_3 \leftarrow L_3 + 12L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} & \frac{9}{5} & -\frac{12}{5} & 1 \end{array}\right)$$

$$\overset{L_3 \leftarrow 5L_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right) \underset{L_1 \leftarrow L_1 - 2L_3}{\overset{L_2 \leftarrow L_2 - \frac{3}{5} L_3}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 4 & 0 & -17 & 24 & -10 \\ 0 & 1 & 0 & -5 & 7 & -3 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right) \sim$$

$$\overset{L_1 \leftarrow L_1 - 4L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 2 \\ 0 & 1 & 0 & -5 & 7 & -3 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right)$$

$$\Rightarrow \begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & -4 & 2 \\ -5 & 7 & -3 \\ 9 & -12 & 5 \end{pmatrix}$$

Always check the validity of your calculations by multiplying the matrices:

$$\begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 3 & -4 & 2 \\ -5 & 7 & -3 \\ 9 & -12 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Ex: If we apply this method to a non-invertible matrix, then in the process we will obtain a zero row, thus halting the algorithm

↳ on the left side!

$\begin{pmatrix} 1 & 2 & 0 \\ 4 & 3 & 1 \\ 6 & 7 & 1 \end{pmatrix}$ is non-invertible, let's see what the algorithm gives:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 4 & 3 & 1 & 0 & 1 & 0 \\ 6 & 7 & 1 & 0 & 0 & 1 \end{array}\right) \underset{L_3 \leftarrow L_3 - 6L_1}{\overset{L_2 \leftarrow L_2 - 4L_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -5 & 1 & -4 & 1 & 0 \\ 0 & -5 & 1 & -6 & 0 & 1 \end{array}\right) \sim$$

$$\overset{L_3 \leftarrow L_3 - L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -5 & 1 & -4 & 1 & 0 \\ 0 & 0 & 0 & -2 & -1 & 1 \end{array}\right)$$

$\Rightarrow$ the matrix is not invertible

4.
$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & -2 & 0 & 1 & 0 \\ 2 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \underset{L_3 \leftarrow L_3 - 2L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right) \sim$$

$$\overset{L_2 \leftarrow -\frac{1}{3} L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right) \overset{L_3 \leftarrow L_3 + 6L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 9 & 2 & -2 & 1 \end{array}\right) \sim$$

$$\overset{L_3 \leftarrow \frac{1}{9} L_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{array}\right) \underset{L_1 \leftarrow L_1 - 2L_3}{\overset{L_2 \leftarrow L_2 - 2L_3}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{5}{9} & \frac{4}{9} & -\frac{2}{9} \\ 0 & 1 & 0 & \frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{array}\right) \sim$$

$$\overset{L_1 \leftarrow L_1 - 2L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ 0 & 1 & 0 & \frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{array}\right)$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}^{-1} = \frac{1}{9} \cdot \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

7. In the real vector space $\mathbb{R}^3$ consider the list $X = (v_1, v_2, v_3, v_4)$, where $v_1 = (1, 0, 4)$, $v_2 = (2, 1, 0)$, $v_3 = (1, 5, -36)$ and $v_4 = (2, 10, -72)$. Determine $dim < X >$ and a basis of $< X >$.

Sol.:

7. In the real vector space $\mathbb{R}^3$ consider the list $X = (v_1, v_2, v_3, v_4)$, where $v_1 = (1,0,4)$ $v_2 = (2,1,0)$, $v_3 = (1,5,-36)$ and $v_4 = (2,10,-72)$. Determine $dim < X >$ and a basis o $< X >$.

Sol.
$$\begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 5 & -36 \\ 2 & 10 & -72 \end{pmatrix} \underset{\substack{L_3 \leftarrow L_3 - L_1 \\ L_4 \leftarrow L_4 - 2L_1}}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 5 & -40 \\ 0 & 10 & -80 \end{pmatrix} \underset{L_4 \leftarrow L_4 - 10L_2}{\overset{L_3 \leftarrow L_3 - 5L_2}{\sim}} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Row echelon form $\Rightarrow \dim \langle X \rangle = 2$, $((1,0,4),(0,1,-8))$ basis for $\langle X \rangle$

9. Determine the dimension of the subspaces S, T, $S+T$ and $S \cap T$ of the real vector space $\mathbb{R}^3$ and a basis for the first three of them, where

$$S = < (1,0,4), (2,1,0), (1,1,-4) >,$$

$$T = < (-3,-2,4), (5,2,4), (-2,0,-8) > .$$

Sol. :
$$\begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{pmatrix} \underset{L_3 \leftarrow L_3 - L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 1 & -8 \end{pmatrix} \overset{L_3 \leftarrow L_3 - L_2}{\sim} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \dim S = 2$, basis for S : $((1,0,4), (0,1,-8))$

$$\begin{pmatrix} -3 & -2 & 4 \\ 5 & 2 & 4 \\ -2 & 0 & -8 \end{pmatrix} \overset{L_1 \leftarrow -\frac{1}{3}L_1}{\sim} \begin{pmatrix} 1 & \frac{2}{3} & -\frac{4}{3} \\ 5 & 2 & 4 \\ -2 & 0 & -8 \end{pmatrix} \underset{L_3 \leftarrow L_3 + 2L_1}{\overset{L_2 \leftarrow L_2 - 5L_1}{\sim}}$$

$$\sim \begin{pmatrix} 1 & \frac{2}{3} & -\frac{4}{3} \\ 0 & -\frac{4}{3} & \frac{32}{3} \\ 0 & \frac{4}{3} & -\frac{32}{3} \end{pmatrix} \overset{L_2 \leftarrow \frac{L_2}{-4/3}}{\sim} \begin{pmatrix} 1 & \frac{2}{3} & -\frac{4}{3} \\ 0 & 1 & -8 \\ 0 & \frac{4}{3} & -\frac{32}{3} \end{pmatrix} \sim$$

$$\overset{L_3 \leftarrow L_3 - \frac{4}{3}L_2}{\sim} \begin{pmatrix} 1 & \frac{2}{3} & -\frac{4}{3} \\ 0 & 1 & -8 \\ 0 & 0 & 0 \end{pmatrix} \overset{L_1 \leftarrow 3L_1}{\sim} \begin{pmatrix} 3 & 2 & -4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \dim T = 2$, basis for T: $\left((3,2,-4), (0,1,-8) \right)$

For $S+T$:

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 3 & 2 & -4 \\ 0 & 1 & -8 \end{pmatrix} \underset{L_3 \leftarrow L_3 - 3L_1}{\sim} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 2 & -16 \\ 0 & 1 & -8 \end{pmatrix} \underset{\substack{L_3 \leftarrow L_3 - 2L_2 \\ L_4 \leftarrow L_4 - L_2}}{\sim}$$

$$\sim \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \dim(S+T) = 2$$

basis for $S+T$: $\left((1,0,4), (0,1,-8) \right)$

$$\dim(S \cap T) = \dim S + \dim T - \dim(S+T) = 2+2-2 = 2$$

(In particular, since $\dim S = \dim T = 2$, $S \cap T \leq S$, $S \cap T \leq T$

$\Rightarrow S \cap T = S = T$)