Nice Inequality from IMO

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Example 0.1 (IMO 1984)

Prove that $0 \leq yz + zx + xy - 2xyz \leq \frac{7}{27}$, where x, y and z are non-negative real numbers satisfying $x + y + z = 1$.

Solution. For the lower bound, $xy + yz + xz - 2xyz = (xy + yz + xz)(x + y + z) - 2xyz \geq 0$ upon expansion. For the upper bound,

$$2\left(\frac{1}{2} - x\right)\left(\frac{1}{2} - y\right)\left(\frac{1}{2} - z\right) = \frac{1}{4} - \frac{1}{2}(x + y + z) + xy + yz + xz - 2xyz$$

$$xy + yz + xz - 2xyz = \frac{1}{4} + 2\prod_{cyc}\left(\frac{1}{2} - x\right)$$

Simple AM-GM on the $\frac{1}{2} - x$ terms gives

$$\frac{1}{6} \geq \sqrt[3]{\prod_{cyc}\left(\frac{1}{2} - x\right)}$$

$$\prod_{cyc}\left(\frac{1}{2} - x\right) \leq \frac{1}{216}$$

$$xy + yz + xz - 2xyz \leq \frac{7}{27}$$

As desired. AM-GM is allowed, unless suppose that say $x > \frac{1}{2}$. In this case, $xy+yz+xz-2xyz \leq \frac{1}{4}$, which we don't care about. $\square$