A Nice Baltic Way Problem

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Example 0.1 (Baltic Way 2000)

Prove that for all positive real numbers a, b, c we have

$$\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} \geq \sqrt{a^2 + ac + c^2}$$

Let $ABCD$ be a convex quadrilateral. Construct $ABCD$ such that $\angle ADB = 60, \angle BDC = 60, AD = a, BD = b, CD = c$. By the Law of Cosines:

$$\triangle ADC \rightarrow AC = \sqrt{a^2 + ac + c^2}$$

$$\triangle BDC \rightarrow BC = \sqrt{b^2 - bc + c^2}$$

$$\triangle ADB \rightarrow AB = \sqrt{a^2 - ab + c^2}$$

And by the triangle inequality in $\triangle ABC$,

$$\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} \geq \sqrt{a^2 + ac + c^2}$$

We are done because the quadrilateral is clearly always constructible for any $a, b, c > 0$.