My Favorite Olympiad Problems!

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§1 Introduction

These are my favorite olympiad problems! I have only done a few, and they are all "beginner" problems.

§2 Algebra

Algebra is the best.

Example 2.1 (IMO SL, 1967)
If x, y, z are real numbers satisfying relations

$$x + y + z = 1 \quad \text{and} \quad \arctan x + \arctan y + \arctan z = \frac{\pi}{4},$$

prove that $x^{2n+1} + y^{2n+1} + z^{2n+1} = 1$ holds for all positive integers n.

Solution. Summing the arctans with the formula,

$$\arctan\left(\frac{x + y + z - xyz}{1 - (xy + yz + xz)}\right) = \frac{\pi}{4}$$

$$x + y + z - xyz = 1 - (xy + yz + xz)$$

$$xyz = xy + yz + xz$$

So x, y, z are roots of $t^3 - t^2 + kt - k = 0$, where $k = xyz = xy + yz + xz$. This factors as $(t^2 + k)(t - 1) = 0$. So the roots are $1, \sqrt{-k}, -\sqrt{-k}$, from which the result comes immediately.

$\square$

Example 2.2 (Iran 2007, Round 3)
Let a, b be two complex numbers. Prove that roots of $z^4 + az^2 + b$ form a rhombus with origin as center, if and only if $\frac{a^2}{b}$ is a non-positive real number.

Solution. First of all, the vertices are of the form $t, -t, p, -p$. It's a rhombus if and only if $p = rit$ for some real r. Notice that there must exist some complex numbers t, r such that

$$z^4 + az^2 + b = (z^2 - t^2)(z^2 + r^2t^2)$$

And then $a = t^2(r^2-1), b = -r^2t^4$. Hence $\frac{a^2}{b} = -\left(\frac{r^2-1}{r}\right)^2$, which is non-positive real if and only if $r \in \mathbb{R}$, i.e. $p = rit$, done.

$\square$

Example 2.3 (Putnam 1971)
Find all polynomials $f(x)$ such that $f(0) \equiv 0$ and $f(x^2+1) = [f(x)]^2 + 1$

Solution. The answer is only $f \equiv \text{id}$. Let S be the set of numbers x such that $f(x) = x$. It suffices to show that the cardinality of S is infinite.

Suppose for the sake of contradiction that S has finite cardinality. Due to the fact that $f(0) = 0$, $S \neq \emptyset$. Then it is valid to let k be the maximum element found in S. Notice that $f(k^2+1) = f(k)^2 + 1 = k^2 + 1$, thus $k^2 + 1 \in S$, contradiction. $\square$

Example 2.4 (USAMO 1975)
If $P(x)$ denotes a polynomial of degree n such that $P(k) = \frac{k}{k+1}$ for $k = 0, 1, 2, \ldots, n$, determine $P(n+1)$.

Solution. Let $Q(k) = (k+1)P(k) - k$. Hence for $k = 0, 1, 2, \cdots, n$, $Q(k) = 0$. Therefore, we can let

$$Q(x) = c\prod_{i=0}^{n}(x-i)$$

For a constant c. Notice that $Q(-1) = 1$, hence

$$-1 = c\prod_{i=0}^{n}(-i-1) = c(-1)^{n+1}(n+1)!$$

$$c = \frac{1}{(-1)^{n+1}(n+1)!}$$

And hence $Q(n+1) = \frac{1}{(-1)^{n+1}} = (n+2)P(n+1) - (n+1)$ thus

$$\boxed{P(n+1) = \frac{n+1+(-1)^{1-n}}{n+2}},$$

or if you desire a piecewise representation:

$$\boxed{P(n+1) = \begin{cases} 1 & n \equiv 1 \pmod 2 \\ \frac{n}{n+2} & n \equiv 0 \pmod 2 \end{cases}}$$

$\square$

Example 2.5 (IMO 1963)
Prove that $\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}$.

Solution. Let $\omega = \text{cis}\left(\frac{\pi}{14}\right)$. Thus it suffices to show that $\omega+\omega^{-1}-\omega^2-\omega^{-2}+\omega^3+\omega^{-3} = 1$. Now using the fact that $\omega^k = \omega^{14+k}$ and $-\omega^2 = \omega^9$, this is equivalent to

$$\omega + \omega^3 + \omega^5 + \omega^7 + \omega^9 + \omega^{11} + \omega^{13} - \omega^7$$

$$\omega\left(\frac{\omega^{14}-1}{\omega^2-1}\right) - \omega^7$$

But since ω is a 14th root of unity, $\omega^{14} = 1$. The answer is then $-\omega^7 = 1$, as desired. $\square$

Example 2.6 (JMMO)

Let x, y and z be positive real numbers such that $x+y+z = 1$. Prove the inequality:

$$\frac{x^2}{1+y} + \frac{y^2}{1+z} + \frac{z^2}{1+x} \leq 1$$

Solution. Notice that $x < 1-y < 1+y$, hence $\frac{x}{1+y} \leq 1 \rightarrow \frac{x^2}{1+y} \leq x$. Summing cyclically yields the desired result. $\square$

Example 2.7 (IMO 1984)

Prove that $0 \leq yz + zx + xy - 2xyz \leq \frac{7}{27}$, where x, y and z are non-negative real numbers satisfying $x+y+z = 1$.

Solution. For the lower bound, $xy+yz+xz-2xyz = (xy+yz+xz)(x+y+z)-2xyz \geq 0$ upon expansion. For the upper bound,

$$2\left(\frac{1}{2}-x\right)\left(\frac{1}{2}-y\right)\left(\frac{1}{2}-z\right) = \frac{1}{4} - \frac{1}{2}(x+y+z) + xy + yz + xz - 2xyz$$

$$xy + yz + xz - 2xyz = \frac{1}{4} + 2\prod_{cyc}\left(\frac{1}{2}-x\right)$$

Simple AM-GM on the $\frac{1}{2} - x$ terms gives

$$\frac{1}{6} \geq \sqrt[3]{\prod_{cyc}\left(\frac{1}{2}-x\right)}$$

$$\prod_{cyc}\left(\frac{1}{2}-x\right) \leq \frac{1}{216}$$

$$xy + yz + xz - 2xyz \leq \frac{7}{27}$$

As desired. AM-GM is allowed, unless suppose that say $x > \frac{1}{2}$. In this case, $xy + yz + xz - 2xyz \leq \frac{1}{4}$, which we don't care about. $\square$

Example 2.8 (Kyrgyzstan)

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x)^2 + f(y)) = xf(x) + y$$

Solution. Asserting $P(0, 0)$, there must be some a such that $f(a) = 0$. Hence assert $P(a, 0)$ to get

$$f(f(y)) = y$$

Thus f is an involution and thus a bijection (well-known, easy to prove). Assert $P(f(b), y)$. We then get

$$f(b^2 + f(y)) = bf(b) + y$$

Assert $P(b, y)$. We then get

$$f(f(b)^2 + f(y)) = bf(b) + y = f(b^2 + f(y))$$

Remembering that f is a bijection, $[f(b)]^2 = b^2$. Hence $f(b) = \pm b$. Unfortunately, we now run into the pointwise trap. Suppose that $f(x) = x$ and $f(y) = -y$. Thus

$$f(x^2 - y) = x^2 + y$$

Since $f(b) = \pm b$, then $y = 0 \to f(y) = 0 = y$, or $x = 0 \to f(x) = 0 = -x$. The alternative case is isomorphic. □

§3 Combinatorics

Combinatorics is both the life and death of me.

Example 3.1 (Canada)

Let there be a fixed positive integer n. Find the sum of all integers such that, when represented in base 2, has $2n$ digits, consisting of n ones, and n zeroes.

Solution. If $n = 1$ we can easily get that the sum is 2. For $n \geq 2$, the first digit is one, so there are $\binom{2n-1}{n-1}$ ways to put the 1's in the empty slots. Then $\binom{2n-2}{n-2}$, etc. So $\binom{2n-1}{n-1} + \binom{2n-2}{n-2} + ... = \binom{2n-1}{n} + \binom{2n-2}{n} + ... = \binom{2n-1}{n}2^{2n-1}$. Then there is $\binom{2n-2}{n}$ of other powers of 2. The requested result is

$$\binom{2n-2}{n}(1+2+2^2+...+2^{2n-2}) + \binom{2n-1}{n}2^{2n-1} = \boxed{\binom{2n-2}{n}(2^{2n-1} - 1) + \binom{2n-1}{n}2^{2n-1}}$$

□

Example 3.2 (USAJMO 2010)

Two permutations $a_1, a_2, \ldots, a_{2010}$ and $b_1, b_2, \ldots, b_{2010}$ of the numbers $1, 2, \ldots, 2010$ are said to intersect if $a_k = b_k$ for some value of k in the range $1 \leq k \leq 2010$. Show that there exist 1006 permutations of the numbers $1, 2, \ldots, 2010$ such that any other such permutation is guaranteed to intersect at least one of these 1006 permutations.

Solution. **Construction:** Pick $S \subseteq \{1, 2, ..., 2010\}$ with $|S| = 1004$. Let Q be the set of elements in $\{1, 2, ..., 2010\} \notin S$. Clearly $|Q| = 1006$.

Suppose the said permutations are $b_1, b_2, ..., b_{1006}$. For each set $\mathcal{A}_i$, pick $\mathcal{A}_{i_n}$ to be the nth element in $\mathcal{A}_i$. Pick $b_{i_j} = S_{j-1006} \forall 1007 \leq j \leq 2010$. Define the kth *loop* of a permutation of $\{1, ..., n\}$ to be some $\{k, ..., n, 1, ..., k-1\}$ with $n \geq k \geq 1$. Set the first 1006 elements of b_i to be the ith *loop* of Q.

Proof that the construction is valid: From pigeonhole, there exists an element ϵ of Q such that ϵ is in one of b_{i_j} with $1 \leq j \leq 1006$.

But with our construction, in each of the first 1006 columns of some b_i, each of the numbers in Q exists. Since ϵ is also an element of Q, there must be an intersection, so we're done. □

§4 Number Theory

Number theory is not my favorite due to the casework, but there are some nice ones here and there.

Example 4.1 (IMO 2006)

Determine all pairs (x, y) of integers such that

$$1 + 2^x + 2^{2x+1} = y^2.$$

Solution. Factoring,

$$2^x(1 + 2^{x+1}) = (y+1)(y-1)$$

This implies that one of $y+1, y-1$ has ν_2 less than or equal to 1. Hence $y = 2^{x-1}a + b$ for odd a and $b^2 = 1$. Hence

$$2^{x-2}(a^2 - 8) = 1 - ab$$

Clearly $b = -1$, so

$$2^{x-2}(a^2 - 8) = a + 1$$

This is a finite check since the LHS grows much faster in a than the RHS. The solutions then are $(0, \pm 2)$ and $(4, \pm 23)$. □

Example 4.2 (Putnam 1969)

Let n be a positive integer such that $24|(n+1)$. Prove that the sum of all divisors of n is also divisible by 24

Solution. Note that $n \equiv -1 \pmod{24}$, n can't be a square. So any $d|n$ satisfies $d \equiv 1, 2 \pmod 3$ and $d \equiv 1, 3, 5, 7 \pmod 8$. In $d, \frac{n}{d}$ one is 1 and the other is 2 mod 3. Hence the possibilities are

$$d \equiv 1, \frac{n}{d} \equiv 2 \pmod 3$$

$$d \equiv 1, \frac{n}{d} \equiv 7 \pmod 8$$

$$d \equiv 3, \frac{n}{d} \equiv 5 \pmod 8$$

Hence the sum is always $0 \pmod 3$ and $0 \pmod 8$, i.e. $0 \pmod{24}$. □

Example 4.3 (IMO 1989)

Prove that for each positive integer n there exist n consecutive positive integers none of which is an integral power of a prime number.

Solution. I present three solutions. The first is the beautiful one. The second is the more obvious one after doing high-level math. The third is a construction, suggested by the user dblues on AoPS, and proven by the user ComplexPhi

- Take some x such that
$$x \equiv -i \pmod{p_i q_i}$$
For $1 \leq i \leq n$ and distinct primes p_i, q_i. There exists a solution by CRT. So take $x+1, x+2, \cdots, x+n$. Each is divisible by two distinct primes, so it can't be a perfect power of a prime.

- It suffices to prove that the density of p^k for prime p in $\mathbb{Z}$ is 0. This is a very weak statement, since by the prime density theorem, the density of *the primes themselves* is 0 in $\mathbb{Z}$, done.

- Consider the set of n integers $\{[(n+1)!]^2+2, [(n+1)!]^2+3, \ldots, [(n+1)!]^2+(n+1)\}$. Let's assume that one of the numbers is the power of a prime. Let it be $[(n+1)!]^2+i = p^k$ with p prime and $2 \leq i \leq n+1$. From this we get that i divides p^k so $i = p^l$ with $l \geq 1$. So $p \leq i \leq n+1$. Obviously $k = v_p([(n+1)!]^2+i) = \min(v_p([(n+1)!]^2), v_p(i)) = v_p(i) = l$ $i = p^l = p^k = [(n+1)!]^2+i$ a contradiction.

□

Example 4.4 (USAMO 2003)

Prove that for every positive integer n there exists an n-digit number divisible by 5^n all of whose digits are odd.

Solution. I claim that the possible m for $n+1$ is just the m for n with a new odd digit at the beginning. This sufficiently solves the problem (obviously 5^n is $\leq$ n digits otherwise 5 is larger than 9). To prove this, we use induction. The base case is easy, for $n=1$, we can use $m=5$. Then the new number is $(2k+1)\cdot 10^{n-1} + a\cdot 5^{n-1} = 5^{n-1}\cdot((2k+1)2^{n-1}+a)$, so we just need to show that there exists some k such that $(2k+1)2^{n-1}+a \equiv 0 \pmod 5$ for some fixed a. Or rephrased, $(2k+1)2^r \equiv t \pmod 5$ has solutions for k for any t, r. This is true since the odd digits are complete $\pmod 5$, so we're done □

Example 4.5 (IMO 1979)

If p and q are natural numbers so that

$$\frac{p}{q} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{1318} + \frac{1}{1319},$$

prove that p is divisible with 1979.

Solution. Notice that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{1318} + \frac{1}{1319}$$

$$= \left(\sum_{k=1}^{1319} \frac{1}{k}\right) - 2\left(\sum_{k=1}^{659} \frac{1}{2k}\right)$$

$$= \sum_{k=660}^{1319} \frac{1}{k}$$

Grouping all terms $\frac{1}{k} + \frac{1}{1979-k} = \frac{1979}{k(1979-k)}$, we see that the numerator must be divisible by 1979 due to the fact that 1979 is prime.

Stronger version (Titu Andreescu):

For prime $p \equiv 1 \pmod 3$ and $q = \lfloor \frac{2p}{3} \rfloor$, with

$$\frac{m}{n} = \frac{1}{1*2} + \frac{1}{3*4} + ... + \frac{1}{(q-1)q}$$

then $p|m$.

The proof is identical, with partial fraction decomposition required at the very beginning. □

Example 4.6 (USAMO 1972)

Prove that $\forall a, b, c \in \mathbb{Z}^+$,

$$\frac{\gcd(a,b,c)^2}{\gcd(a,b)\gcd(b,c)\gcd(a,c)} = \frac{\text{lcm}(a,b,c)^2}{\text{lcm}(a,b)\text{lcm}(a,c)\text{lcm}(b,c)}$$

Solution. We use p-adics. It's clear that if $\nu_p(LHS) = \nu_p(RHS)$ for all primes p, $LHS = RHS$. Pick an arbitrary prime p. Suppose that $\nu_p(a) \le \nu_p(b) \le \nu_p(c)$. Hence we get

$$\nu_p(LHS) = \frac{\nu_p(a)^2}{\nu_p(a)\nu_p(b)\nu_p(a)} = \frac{1}{\nu_p(b)}$$

$$\nu_p(RHS) = \frac{\nu_p(c)^2}{\nu_p(b)\nu_p(c)\nu_p(c)} = \frac{1}{\nu_p(b)}$$

As desired. □

§5 Geometry

I'm not great at geometry, but here we go.

Example 5.1 (Baltic Way 2000)

Prove that for all positive real numbers a, b, c we have

$$\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} \ge \sqrt{a^2 + ac + c^2}$$

Solution. Let $ABCD$ be a convex quadrilateral. Construct $ABCD$ such that $\angle ADB = 60, \angle BDC = 60, AD = a, BD = b, CD = c$. By the Law of Cosines:

$$\triangle ADC \to AC = \sqrt{a^2 + ac + c^2}$$

$$\triangle BDC \to BC = \sqrt{b^2 - bc + c^2}$$

$$\triangle ADB \to AB = \sqrt{a^2 - ab + c^2}$$

And by the triangle inequality in $\triangle ABC$,

$$\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} \geq \sqrt{a^2 + ac + c^2}$$

We are done because the quadrilateral is clearly always constructible for any $a, b, c > 0$. $\square$

Example 5.2 (JBMO 2019)

Triangle ABC is such that $AB < AC$. The perpendicular bisector of side BC intersects lines AB and AC at points P and Q, respectively. Let H be the orthocentre of triangle ABC, and let M and N be the midpoints of segments BC and PQ, respectively. Prove that lines HM and AN meet on the circumcircle of ABC.

Solution. There is a spiral similarity sending $\triangle BMH$ to $\triangle QNA$, so $\angle BMH = \angle QNA$ and $\angle HMC = 90 - \angle BMH = 90 - \angle QNA$, thus $HM \perp NA$ and so AN is a tangent to the circumcircle of ABC at which it meets MA, as desired. $\square$