My Favorite Olympiad Problems!

Anay Aggarwal

July 22, 2021

§1 Introduction

These are my favorite olympiad problems! I have only done a few, and they are all "beginner" problems.

§2 Algebra

Algebra is the best.

Example 2.1 (IMO SL, 1967)
If x, y, z are real numbers satisfying relations

$$x + y + z = 1 \quad \text{and} \quad \arctan x + \arctan y + \arctan z = \frac{\pi}{4},$$

prove that $x^{2n+1} + y^{2n+1} + z^{2n+1} = 1$ holds for all positive integers n.

Solution. Summing the arctans with the formula,

$$\arctan\left(\frac{x+y+z-xyz}{1-(xy+yz+xz)}\right) = \frac{\pi}{4}$$

$$x+y+z-xyz = 1-(xy+yz+xz)$$

$$xyz = xy+yz+xz$$

So x, y, z are roots of $t^3 - t^2 + kt - k = 0$, where $k = xyz = xy + yz + xz$. This factors as $(t^2 + k)(t - 1) = 0$. So the roots are $1, \sqrt{-k}, -\sqrt{-k}$, from which the result comes immediately.

□

Example 2.2 (Iran 2007, Round 3)
Let a, b be two complex numbers. Prove that roots of $z^4 + az^2 + b$ form a rhombus with origin as center, if and only if $\frac{a^2}{b}$ is a non-positive real number.

Solution. First of all, the vertices are of the form $t, -t, p, -p$. It's a rhombus if and only if $p = rit$ for some real r. Notice that there must exist some complex numbers t, r such that

$$z^4 + az^2 + b = (z^2 - t^2)(z^2 + r^2t^2)$$

And then $a = t^2(r^2 - 1), b = -r^2t^4$. Hence $\frac{a^2}{b} = -\left(\frac{r^2-1}{r}\right)^2$, which is non-positive real if and only if $r \in \mathbb{R}$, i.e. $p = rit$, done.

□

Example 2.3 (Putnam 1971)

Find all polynomials $f(x)$ such that $f(0) \equiv 0$ and $f(x^2 + 1) = [f(x)]^2 + 1$

Solution. The answer is only $f \equiv \text{id}$. Let S be the set of numbers x such that $f(x) = x$. It suffices to show that the cardinality of S is infinite.

Suppose for the sake of contradiction that S has finite cardinality. Due to the fact that $f(0) = 0$, $S \neq \emptyset$. Then it is valid to let k be the maximum element found in S. Notice that $f(k^2 + 1) = f(k)^2 + 1 = k^2 + 1$, thus $k^2 + 1 \in S$, contradiction. □

Example 2.4 (USAMO 1975)

If $P(x)$ denotes a polynomial of degree n such that $P(k) = \frac{k}{k+1}$ for $k = 0, 1, 2, \ldots, n$, determine $P(n + 1)$.

Solution. Let $Q(k) = (k + 1)P(k) - k$. Hence for $k = 0, 1, 2, \cdots, n$, $Q(k) = 0$. Therefore, we can let

$$Q(x) = c\prod_{i=0}^{n}(x - i)$$

For a constant c. Notice that $Q(-1) = 1$, hence

$$-1 = c\prod_{i=0}^{n}(-i - 1) = c(-1)^{n+1}(n + 1)!$$

$$c = \frac{1}{(-1)^{n+1}(n + 1)!}$$

And hence $Q(n + 1) = \frac{1}{(-1)^{n+1}} = (n + 2)P(n + 1) - (n + 1)$ thus

$$\boxed{P(n + 1) = \frac{n + 1 + (-1)^{1-n}}{n + 2}},$$

or if you desire a piecewise representation:

$$\boxed{P(n + 1) = \begin{cases} 1 & n \equiv 1 \pmod 2 \\ \frac{n}{n+2} & n \equiv 0 \pmod 2 \end{cases}}$$

□

Example 2.5 (IMO 1963)

Prove that $\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}$.

Solution. Let $\omega = \text{cis}\left(\frac{\pi}{14}\right)$. Thus it suffices to show that $\omega+\omega^{-1}-\omega^2-\omega^{-2}+\omega^3+\omega^{-3} = 1$. Now using the fact that $\omega^k = \omega^{14+k}$ and $-\omega^2 = \omega^9$, this is equivalent to

$$\omega + \omega^3 + \omega^5 + \omega^7 + \omega^9 + \omega^{11} + \omega^{13} - \omega^7$$

$$\omega\left(\frac{\omega^{14}-1}{\omega^2-1}\right) - \omega^7$$

But since ω is a 14th root of unity, $\omega^{14} = 1$. The answer is then $-\omega^7 = 1$, as desired. $\square$

Example 2.6 (JMMO)

Let x, y and z be positive real numbers such that $x+y+z = 1$. Prove the inequality:

$$\frac{x^2}{1+y} + \frac{y^2}{1+z} + \frac{z^2}{1+x} \leq 1$$

Solution. Notice that $x < 1-y < 1+y$, hence $\frac{x}{1+y} \leq 1 \to \frac{x^2}{1+y} \leq x$. Summing cyclically yields the desired result. $\square$

Example 2.7 (IMO 1984)

Prove that $0 \leq yz + zx + xy - 2xyz \leq \frac{7}{27}$, where x, y and z are non-negative real numbers satisfying $x + y + z = 1$.

Solution. For the lower bound, $xy+yz+xz-2xyz = (xy+yz+xz)(x+y+z)-2xyz \geq 0$ upon expansion. For the upper bound,

$$2\left(\frac{1}{2}-x\right)\left(\frac{1}{2}-y\right)\left(\frac{1}{2}-z\right) = \frac{1}{4} - \frac{1}{2}(x+y+z) + xy + yz + xz - 2xyz$$

$$xy + yz + xz - 2xyz = \frac{1}{4} + 2\prod_{cyc}\left(\frac{1}{2}-x\right)$$

Simple AM-GM on the $\frac{1}{2} - x$ terms gives

$$\frac{1}{6} \geq \sqrt[3]{\prod_{cyc}\left(\frac{1}{2}-x\right)}$$

$$\prod_{cyc}\left(\frac{1}{2}-x\right) \leq \frac{1}{216}$$

$$xy + yz + xz - 2xyz \leq \frac{7}{27}$$

As desired. AM-GM is allowed, unless suppose that say $x > \frac{1}{2}$. In this case, $xy + yz + xz - 2xyz \leq \frac{1}{4}$, which we don't care about. $\square$

Example 2.8 (Kyrgyzstan)

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x)^2 + f(y)) = xf(x) + y$$

Solution. Asserting $P(0,0)$, there must be some a such that $f(a) = 0$. Hence assert $P(a,0)$ to get

$$f(f(y)) = y$$

Thus f is an involution and thus a bijection (well-known, easy to prove). Assert $P(f(b), y)$. We then get

$$f(b^2 + f(y)) = bf(b) + y$$

Assert $P(b,y)$. We then get

$$f(f(b)^2 + f(y)) = bf(b) + y = f(b^2 + f(y))$$

Remembering that f is a bijection, $[f(b)]^2 = b^2$. Hence $f(b) = \pm b$. Unfortunately, we now run into the pointwise trap. Suppose that $f(x) = x$ and $f(y) = -y$. Thus

$$f(x^2 - y) = x^2 + y$$

Since $f(b) = \pm b$, then $y = 0 \to f(y) = 0 = y$, or $x = 0 \to f(x) = 0 = -x$. The alternative case is isomorphic. □

§3 Combinatorics

Combinatorics is both the life and death of me.

Example 3.1 (Canada)

Let there be a fixed positive integer n. Find the sum of all integers such that, when represented in base 2, has $2n$ digits, consisting of n ones, and n zeroes.

Solution. If $n = 1$ we can easily get that the sum is 2. For $n \geq 2$, the first digit is one, so there are $\binom{2n-1}{n-1}$ ways to put the 1's in the empty slots. Then $\binom{2n-2}{n-2}$, etc. So $\binom{2n-1}{n-1} + \binom{2n-2}{n-2} + ... = \binom{2n-1}{n} + \binom{2n-2}{n} + ... = \binom{2n-1}{n}2^{2n-1}$. Then there is $\binom{2n-2}{n}$ of other powers of 2. The requested result is

$$\binom{2n-2}{n}(1+2+2^2+...+2^{2n-2})+\binom{2n-1}{n}2^{2n-1} = \boxed{\binom{2n-2}{n}(2^{2n-1}-1)+\binom{2n-1}{n}2^{2n-1}}$$

□

Example 3.2 (USAJMO 2010)

Two permutations $a_1, a_2, \ldots, a_{2010}$ and $b_1, b_2, \ldots, b_{2010}$ of the numbers $1, 2, \ldots, 2010$ are said to intersect if $a_k = b_k$ for some value of k in the range $1 \leq k \leq 2010$. Show that there exist 1006 permutations of the numbers $1, 2, \ldots, 2010$ such that any other such permutation is guaranteed to intersect at least one of these 1006 permutations.

Solution. **Construction:** Pick $S \subseteq \{1, 2, ..., 2010\}$ with $|S| = 1004$. Let Q be the set of elements in $\{1, 2, ..., 2010\} \notin S$. Clearly $|Q| = 1006$.

Suppose the said permutations are $b_1, b_2, ..., b_{1006}$. For each set $\mathcal{A}_i$, pick $\mathcal{A}_{i_n}$ to be the nth element in $\mathcal{A}_i$. Pick $b_{i_j} = S_{j-1006} \forall 1007 \leq j \leq 2010$. Define the kth *loop* of a permutation of $\{1, ..., n\}$ to be some $\{k, ..., n, 1, ..., k-1\}$ with $n \geq k \geq 1$. Set the first 1006 elements of b_i to be the ith *loop* of Q.

Proof that the construction is valid: From pigeonhole, there exists an element ϵ of Q such that ϵ is in one of b_{i_j} with $1 \leq j \leq 1006$.

But with our construction, in each of the first 1006 columns of some b_i, each of the numbers in Q exists. Since ϵ is also an element of Q, there must be an intersection, so we're done. □

Example 3.3 (2020 ISL)

Let n be a positive integer. Find the number of permutations $a_1, a_2, \ldots a_n$ of the sequence $1, 2, \ldots, n$ satisfying

$$a_1 \leq 2a_2 \leq 3a_3 \leq \cdots \leq na_n$$

.

Solution. The answer is the $n+1$th Fibonacci number. I prove this with strong induction. Let there be $P(n)$ ways for a sequence of n numbers. The base cases are easy since

$$P(1) = |\{1\}| = 1 = F_2$$

$$P(2) = |\{1, 2\}, \{2, 1\}| = 2 = F_3$$

We do casework on the index of n. If n is at the last position, there are $P(n-1)$ ways. If n is at the second-to-last position, $n-1$ must be at the last position, hence there are $P(n-2)$ ways. I now prove that n cannot be at any other position, effectively solving the problem.

Suppose the contrary. Clearly, n cannot be at the first position. Hence there is a valid sequence of the first k numbers, then n, then a jumble of the other numbers. Consider the number $k+1$. Let's say its at index r. Then

$$n(k+1) \leq (k+1)r$$

$$n \leq r$$

$$r = n$$

Hence $k+1$ is the last number. Suppose that the number at index $n-1$ is r. Thus

$$(n-1)r \leq (k+1)n$$

But $k+2 \leq r$, hence

$$(k+2)(n-1) \leq (k+1)n$$

$$nk + 2n - k - 2 \leq nk + n$$

$$n \leq k+2$$

Meaning that there are at least $n-2$ numbers before the number n, i.e. n is at one of the last two spots, as desired. □

Example 3.4 (IMO 2021 P5)

Two squirrels, Bushy and Jumpy, have collected 2021 walnuts for the winter. Jumpy numbers the walnuts from 1 through 2021, and digs 2021 little holes in a circular pattern in the ground around their favourite tree. The next morning Jumpy notices that Bushy had placed one walnut into each hole, but had paid no attention to the numbering. Unhappy, Jumpy decides to reorder the walnuts by performing a sequence of 2021 moves. In the k-th move, Jumpy swaps the positions of the two walnuts adjacent to walnut k. Prove that there exists a value of k such that, on the k-th move, Jumpy swaps some walnuts a and b such that $a < k < b$.

Solution. Let's suppose that Jumpy is hungry and eats every walnut they swap the neighbors of before they swap them. Notice that we want to show that Jumpy swaps some uneaten walnut with some eaten walnut at some point in time (assume an eaten walnut is just the crumbs). Assume the contrary, i.e. they always swaps some two walnuts of the same status.

Lemma 3.5

Swapping two walnuts of the same status preserves the number of pairs of adjacent uneaten walnuts.

This is obvious since swapping two walnuts of the same status doesn't change anything about the statuses at each index at all.

Now, the number of pairs of adjacent uneaten walnuts can't remain the same throughout, so we must have swapped an eaten one with an uneaten one. □

§4 Number Theory

Number theory is not my favorite due to the casework, but there are some nice ones here and there.

Example 4.1 (IMO 2006)

Determine all pairs (x, y) of integers such that

$$1 + 2^x + 2^{2x+1} = y^2.$$

Solution. Factoring,

$$2^x(1 + 2^{x+1}) = (y + 1)(y - 1)$$

This implies that one of $y + 1, y - 1$ has ν_2 less than or equal to 1. Hence $y = 2^{x-1}a + b$ for odd a and $b^2 = 1$. Hence

$$2^{x-2}(a^2 - 8) = 1 - ab$$

Clearly $b = -1$, so

$$2^{x-2}(a^2 - 8) = a + 1$$

This is a finite check since the LHS grows much faster in a than the RHS. The solutions then are $(0, \pm 2)$ and $(4, \pm 23)$. □

Example 4.2 (Putnam 1969)

Let n be a positive integer such that $24|(n+1)$. Prove that the sum of all divisors of n is also divisible by 24

Solution. Note that $n \equiv -1 \pmod{24}$, n can't be a square. So any $d|n$ satisfies $d \equiv 1, 2 \pmod 3$ and $d \equiv 1, 3, 5, 7 \pmod 8$. In $d, \frac{n}{d}$ one is 1 and the other is 2 mod 3. Hence the possibilities are

$$d \equiv 1, \frac{n}{d} \equiv 2 \pmod 3$$
$$d \equiv 1, \frac{n}{d} \equiv 7 \pmod 8$$
$$d \equiv 3, \frac{n}{d} \equiv 5 \pmod 8$$

Hence the sum is always $0 \pmod 3$ and $0 \pmod 8$, i.e. $0 \pmod{24}$. □

Example 4.3 (IMO 1989)

Prove that for each positive integer n there exist n consecutive positive integers none of which is an integral power of a prime number.

Solution. I present three solutions. The first is the beautiful one. The second is the more obvious one after doing high-level math. The third is a construction, suggested by the user dblues on AoPS, and proven by the user ComplexPhi

- Take some x such that
$$x \equiv -i \pmod{p_i q_i}$$
For $1 \le i \le n$ and distinct primes p_i, q_i. There exists a solution by CRT. So take $x+1, x+2, \cdots, x+n$. Each is divisible by two distinct primes, so it can't be a perfect power of a prime.

- It suffices to prove that the density of p^k for prime p in $\mathbb{Z}$ is 0. This is a very weak statement, since by the prime density theorem, the density of *the primes themselves* is 0 in $\mathbb{Z}$, done.

- Consider the set of n integers $\{[(n+1)!]^2+2, [(n+1)!]^2+3, \ldots, [(n+1)!]^2+(n+1)\}$. Let's assume that one of the numbers is the power of a prime. Let it be $[(n+1)!]^2+i = p^k$ with p prime and $2 \le i \le n+1$. From this we get that i divides p^k so $i = p^l$ with $l \ge 1$. So $p \le i \le n+1$. Obviously $k = v_p([(n+1)!]^2+i) = \min(v_p([(n+1)!]^2), v_p(i)) = v_p(i) = l$ $i = p^l = p^k = [(n+1)!]^2 + i$ a contradiction.

□

Example 4.4 (USAMO 2003)

Prove that for every positive integer n there exists an n-digit number divisible by 5^n all of whose digits are odd.

Solution. I claim that the possible m for $n+1$ is just the m for n with a new odd digit at the beginning. This sufficiently solves the problem (obviously 5^n is $\leq$ n digits otherwise 5 is larger than 9). To prove this, we use induction. The base case is easy, for $n=1$, we can use $m=5$. Then the new number is $(2k+1)\cdot 10^{n-1}+a\cdot 5^{n-1}=5^{n-1}\cdot((2k+1)2^{n-1}+a)$, so we just need to show that there exists some k such that $(2k+1)2^{n-1}+a\equiv 0 \pmod 5$ for some fixed a. Or rephrased, $(2k+1)2^r\equiv t \pmod 5$ has solutions for k for any t, r. This is true since the odd digits are complete $\pmod 5$, so we're done $\square$

Example 4.5 (IMO 1979)

If p and q are natural numbers so that

$$\frac{p}{q}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots-\frac{1}{1318}+\frac{1}{1319},$$

prove that p is divisible with 1979.

Solution. Notice that

$$\begin{aligned}1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots-\frac{1}{1318}+\frac{1}{1319}\\ =\left(\sum_{k=1}^{1319}\frac{1}{k}\right)-2\left(\sum_{k=1}^{659}\frac{1}{2k}\right)\\ =\sum_{k=660}^{1319}\frac{1}{k}\end{aligned}$$

Grouping all terms $\frac{1}{k}+\frac{1}{1979-k}=\frac{1979}{k(1979-k)}$, we see that the numerator must be divisible by 1979 due to the fact that 1979 is prime.

Stronger version (Titu Andreescu):

For prime $p\equiv 1 \pmod 3$ and $q=\lfloor\frac{2p}{3}\rfloor$, with

$$\frac{m}{n}=\frac{1}{1*2}+\frac{1}{3*4}+\ldots+\frac{1}{(q-1)q}$$

then $p|m$.

The proof is identical, with partial fraction decomposition required at the very beginning. $\square$

Example 4.6 (USAMO 1972)

Prove that $\forall a,b,c\in\mathbb{Z}^+$,

$$\frac{\gcd(a,b,c)^2}{\gcd(a,b)\gcd(b,c)\gcd(a,c)}=\frac{\text{lcm}(a,b,c)^2}{\text{lcm}(a,b)\text{lcm}(a,c)\text{lcm}(b,c)}$$

Solution. We use p-adics. It's clear that if $\nu_p(LHS)=\nu_p(RHS)$ for all primes p, $LHS=RHS$. Pick an arbitrary prime p. Suppose that $\nu_p(a)\leq\nu_p(b)\leq\nu_p(c)$. Hence we get

$$\nu_p(LHS)=\frac{\nu_p(a)^2}{\nu_p(a)\nu_p(b)\nu_p(a)}=\frac{1}{\nu_p(b)}$$

$$\nu_p(RHS) = \frac{\nu_p(c)^2}{\nu_p(b)\nu_p(c)\nu_p(c)} = \frac{1}{\nu_p(b)}$$

As desired. □

Example 4.7 (Classic)
Prove that
$$\mu^2(n) = \sum_{d|n} \mu(d) 2^{\omega\left(\frac{n}{d}\right)}$$
holds $\forall n \in \mathbb{Z}^+$.

Lemma 4.8
$\sum_{d|n} |\mu(d)| = 2^{\omega(n)}$

Let $n = \prod p_i^{a_i}$. In order for the mobius function to be non-zero, if $d = \prod p_i^{d_i}$, all $d_i < 2$. Hence there are two options, 0 and 1, to choose each d_i. There are $\omega(n)$ such d_i, completing the proof.

Back to the main problem. Note that $\mu^2(n) = |\mu(n)|$. By symmetry, the summation on the RHS is
$$\sum_{d|n} \mu\left(\frac{n}{d}\right) 2^{\omega(d)}$$

Definition 4.9. For two arithmetic functions f and g, $(f * g)(n)$ is the convolution of f and g, i.e.
$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right)$$

Using this definition, we want to prove that
$$|\mu(n)| = (\mu * 2^{\omega})(n)$$

Using the lemma, this is
$$|\mu(n)| = (\mu * (|\mu| * \mathbb{1}))(n)$$

Since the convolution function is commutative and associative,
$$|\mu(n)| = ((\mu * \mathbb{1}) * |\mu(n)|)(n)$$

But the function
$$\epsilon_0(n) = (\mu * \mathbb{1})(n) = \begin{cases} 1 & n = 1 \\ 0 & \text{else} \end{cases}$$

Hence
$$(\epsilon_0(n) * |\mu(n)|)(n) = \sum_{d|n} \epsilon_0(d) |\mu\left(\frac{n}{d}\right)|$$

Using the piecewise definition of ϵ_0, all terms cancel except the $d = 1$ term. This is simply $|\mu(n)|$, as desired. □

Example 4.10 (Henry John Stephen Smith)
Define the matrix
$$A =: [a_{i,j}]_{n\times n} | a_{i,j} = \gcd(i,j)$$
Find a closed-form for $\det(A)$.

This is quite a difficult problem. The idea is to reduce the matrix using elementary row operations.

Lemma 4.11 (The Algorithm)
Consider the algorithm as follows: For each $1 \le i \le n$, subtract each row R_{ik} by R_i with $1 < k$ and $ik \le n$. When this algorithm is run on the matrix A to become A', A' is an upper triangular matrix.

To prove this lemma, we simply need to show that $a_{ij} = 0$ whenever $j < i$ in the new matrix. Notice that for each $d|j, d < i$ we are subtracting a_{id} from a_{ij}. Using this, it's easy to see that the lemma is true.

Lemma 4.12
Each $a_{k,k} = \varphi(k)$ after the algorithm has been run.

Notice that $\gcd(k,k) = k$, so the element was originally just k. It's easy to see that the algorithm spits out what is essentially the definition of the totient function.

Using the two lemmas, the determinant is simply $\varphi(1)\varphi(2)\cdots\varphi(n)$.

§5 Geometry

I'm not great at geometry, but here we go.

Example 5.1 (Baltic Way 2000)
Prove that for all positive real numbers a, b, c we have
$$\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} \ge \sqrt{a^2 + ac + c^2}$$

Solution. Solution. Let $ABCD$ be a convex quadrilateral. Construct $ABCD$ such that $\angle ADB = 60, \angle BDC = 60, AD = a, BD = b, CD = c$. By the Law of Cosines:
$$\triangle ADC \to AC = \sqrt{a^2 + ac + c^2}$$
$$\triangle BDC \to BC = \sqrt{b^2 - bc + c^2}$$
$$\triangle ADB \to AB = \sqrt{a^2 - ab + c^2}$$
And by the triangle inequality in $\triangle ABC$,
$$\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} \ge \sqrt{a^2 + ac + c^2}$$
We are done because the quadrilateral is clearly always constructible for any $a, b, c > 0$. $\square$

Example 5.2 (JBMO 2019)

Triangle ABC is such that $AB < AC$. The perpendicular bisector of side BC intersects lines AB and AC at points P and Q, respectively. Let H be the orthocentre of triangle ABC, and let M and N be the midpoints of segments BC and PQ, respectively. Prove that lines HM and AN meet on the circumcircle of ABC.

Solution. There is a spiral similarity sending $\triangle BMH$ to $\triangle QNA$, so $\angle BMH = \angle QNA$ and $\angle HMC = 90 - \angle BMH = 90 - \angle QNA$, thus $HM \perp NA$ and so AN is a tangent to the circumcircle of ABC at which it meets MA, as desired. □

Example 5.3 (USA TST 2010)

Let h_a, h_b, h_c be the lengths of the altitudes of a triangle ABC from A, B, C respectively. Let P be any point inside the triangle. Show that

$$\sum_{cyc} \frac{PA}{h_b + h_c} \geq 1$$

Solution. Notice that $h_b = \frac{2[ABC]}{b}$, etc. Hence the inequality is

$$\sum_{cyc} \frac{PA}{\frac{1}{b} + \frac{1}{c}} \geq 2[ABC]$$

$$\sum_{cyc} PA\frac{bc}{b+c} \geq 2[ABC]$$

Let $x = d(P, AC), y = d(P, AB), z = d(P, BC)$. This implies that

$$2[ABC] = xb + yc + za$$

By the key lemma in the proof of Erdos-Mordell (I think the name is Mordell lemma),

$$PAa \geq xb + yc, xc + yb$$

Adding yields

$$PA \geq \frac{(x+y)(b+c)}{2a}$$

$$\sum_{cyc} PA\frac{bc}{b+c} \geq \sum_{cyc} \frac{(x+y)bc}{2a}$$

But notice that

$$\sum_{cyc} \frac{xbc + ybc}{2a} = \frac{1}{2}\sum_{cyc} xb\frac{c}{a} + yc\frac{b}{a}$$

But

$$\sum_{cyc} xb\left(\frac{c}{a} + \frac{a}{c}\right) \geq 2\sum_{cyc} xb$$

by AM-GM. This directly implies the result. □