Ram Math

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§1 Jamboards

Polynomials Sequences

§2 Polynomials

Example 2.1 (AMC 12 2017)
For certain reals a, b, c, the polynomial

$$g(x) = x^3 + ax^2 + x + 10$$

Has three distinct roots, each of which is also a root of

$$f(x) = x^4 + x^3 + bx^2 + 100x + c$$

Compute $f(1)$

Solution. Since the sum of the three roots is $-a$, and the sum of the four roots is -1, we know that the last root is $a-1$. We want $102 + b + c$. Notice that the product of the three roots is -10, hence $c = (a-1)\cdot -10$. If x_1, x_2, x_3 are the roots, and x_4 is the fourth, then

$$b = \sum x_1x_2 = x_1x_2+x_2x_3+x_1x_3+x_1x_4+x_2x_4+x_3x_4 = 1+x_4(x_1+x_2+x_3) = 1+(a-1)(-a)$$

Note that

$$-100 = x_1x_2x_3+x_1x_2x_4+x_2x_3x_4+x_1x_3x_4 = -10+x_4(x_1x_2+x_2x_3+x_1x_3) = -10+(a-1)$$

Thus $a - 1 = -90 \to a = -89$. Putting everything together, the answer is 7007. $\square$

Example 2.2 (AMC 12 2010)
The graph of $y = x^6 - 10x^5 + 29x^4 - 4x^3 + ax^2$ lies above the line $y = bx + c$ except at 3 values of x, where they intersect. What's the largest of the three values?

Solution. We must have $f(x) = x^6 - 10x^5 + 29x^4 - 4x^3 + ax^2 - bx - c \geq 0$ for all real x. We want the equality cases. Note that all 3 roots must be double roots. Let

$p(x) = x^3 - ux^2 + vx - w$ be the polynomial with the single roots instead of the double roots. In other words, $(p(x))^2 = f(x)$. Comparing coefficients in the expansion of $(p(x))^2$,

$$u^2 + 2v = 29$$

$$2uv + 2w = 4$$

$$2u = 10 \to u = 5 \to v = 2, w = -8$$

Hence $p(x) = x^3 - 5x^2 + 2x + 8 = (x-4)(x-2)(x+1)$. So equality of the original inequality occurs at $4, 2, -1$, of which the maximum is 4. □

Example 2.3 (AIME 1996)

Suppose that the roots of $x^3 + 3x^2 + 4x - 11 = 0$ are a, b, and c, and that the roots of $x^3 + rx^2 + sx + t = 0$ are $a+b, b+c$, and $a+c$. Find t.

Solution. Note that

$$-t = (a+b)(b+c)(a+c) = (ab+ac+bc+b^2)(a+c) = a^2b+a^2c+abc+ab^2+abc+ac^2+bc^2+b^2c$$

$$= (ab+ac+bc)(a+b+c) - abc = 4(-3) - 11 = -23$$

Hence $t = 23$. □

Example 2.4 (CMIMC 2018)

Let $P(x) = x^2 + 4x + 1$. What is the product of the real roots of

$$P(P(x)) = 0?$$

Solution. If $P(P(x)) = 0$, then $P(x)$ is a root of $x^2 + 4x + 1$. Hence

$$P(x) = -2 \pm \sqrt{3}$$

If $x^2 + 4x + t$ has real roots, then

$$16 - 4t \geq 0 \to t \leq 4$$

Hence we either have $x^2 + 4x + 3 + \sqrt{3} = 0$ or $x^2 + 4x + 3 - \sqrt{3} = 0$. The former is impossible, and the latter has both real roots. Hence the product is $3 - \sqrt{3}$. □

Example 2.5 (PuMAC 2016)

Let $f(x) = 15x - 2016$. If $f(f(f(f(f(x))))) = f(x)$, find the sum of all possible values of x.

Solution. The given equation is linear, hence there is 1 solution. Notice that $f(x) = x$ suffices, hence $x = 144$. □

Example 2.6 (HMMT 2014)

Find the sum of the real roots of

$$5x^4 - 10x^3 + 10x^2 - 5x - 11 = 0$$

Solution. Note that

$$(x-1)^5 = x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$$

$$-(x-1)^5 = -x^5 + 5x^4 - 10x^3 + 10x^2 - 5x + 1$$

$$x^5 - (x-1)^5 = 5x^4 - 10x^3 + 10x^2 - 5x + 1$$

$$5x^4 - 10x^3 + 10x^2 - 5x - 11 = x^5 - (x-1)^5 - 12$$

Hence

$$x^5 - (x-1)^5 = 12$$

There is a positive solution and a negative solution. The graph has symmetry at $x = 0.5$, hence the answer is $2(0.5) = 1$. $\square$

Example 2.7 (HMMT 2014)

Let b and c be real numbers and define the polynomial $P(x) = x^2 + bx + c$. Suppose that $P(P(1)) = P(P(2)) = 0$ and that $P(1) \neq P(2)$. Find $P(0)$.

Solution. We get

$$P(1)P(2) = c$$

$$P(1) + P(2) = -b$$

But $P(1) = b + c + 1, P(2) = 2b + c + 4$. Hence

$$4b + 2c + 5 = 0$$

$$(b + c + 1)(2b + c + 4) = c$$

Solving, we want $c = -\frac{3}{2}$. $\square$

Example 2.8 (AIME 2010)

Let $P(x)$ be a quadratic with real coefficients satisfying

$$x^2 - 2x + 2 \leq P(x) \leq 2x^2 - 4x + 3$$

for all real numbers x, and suppose $P(11) = 181$. Find $P(16)$.

Solution. Plugging in $x = 1$, we get $1 \leq P(1) \leq 1$, hence $P(1) = 1$. In addition, all three parabolas share the same vertex. Thus $P(x) = c(x-1)^2 + 1$ for some c. Plugging $x = 11$, $100c + 1 = 181 \to c = \frac{9}{5}$ hence the answer is 406. $\square$

Example 2.9 (AIME 2007)

The polynomial $P(x)$ is cubic. What is the largest value of k for which the polynomials $Q_1(x) = x^2 + (k-29)x - k$ and $Q_2(x) = 2x^2 + (2k-43)x + k$ are both factors of $P(x)$?

Solution. Q_1, Q_2 must have a common root, say a. If $Q_1(a) = 0, Q_2(a) = 0$, then

$$Q_2(a) - 2Q_1(a) = 15a + 3k = 0$$

Hence $a = -\frac{k}{5}$. Plugging this into Q_1, we get the answer is 30. □

Example 2.10 (AIME 2015)

Let $f(x) \in R[x]$ be a cubic such that

$$|f(1)| = |f(2)| = |f(3)| = |f(5)| = |f(6)| = |f(7)| = 12$$

Solution. Cubics have two bumps, so $f(1) = f(5) = f(6), f(2) = f(3) = f(7), f(4) = 0$. Let $f(1) = 2$ hence the others are -12. The third difference is 12, hence $f(0) = 12 + 24 + 24 + 12 = 72$. □

§3 Sequences

Example 3.1 (AIME 1988)

For any positive integer k, let $f_1(k)$ be the sum of the digits of k. For $n \geq 2$, let $f_n(k) = f_1(f_{n-1}(k))$. Find $f_{1988}(11)$.

Solution. □

Example 3.2 (Pumac 2016)

Let $a_1 = 20, a_2 = 16$, and for $k \geq 3$, let $a_k = \sqrt[3]{k - a_{k-1}^3 - a_{k-2}^3}$. Compute $a_1^3 + a_2^3 + \cdots + a_{10}^3$.

Solution. □

Example 3.3 (HMMT 2013)

Let $\{a_n\}_{n\geq 1}$ be an arithmetic sequence and $\{g_n\}_{n\geq 1}$ be a geometric sequence such that the first four terms of $\{a_n + g_n\}$ are $0, 0, 1, 0$. What is the 10th term of $\{a_n + g_n\}$?

Solution. □

Example 3.4 (HMMT 2016)

An infinite sequence of reals $a_1, a_2, \cdots$ satisfies

$$a_{n+3} = a_{n+2} - 2a_{n+1} + a_n$$

Given that $a_1 = a_3 = 1$ and $a_{98} = a_{99}$, compute $a_1 + a_2 + \cdots + a_{100}$.

Solution. □

Example 3.5 (AIME 2009)

The sequence (a_n) satisfies $a_1 = 1$ and $5^{a_{n+1}-a_n} - 1 = \frac{1}{n+\frac{2}{3}}$. Find the least $k > 1$ such that $a_k \in \mathbb{Z}$.

Solution. □

Example 3.6 (PUMAC)

Solution. □

Example 3.7

Solution. □

Example 3.8

Solution. □

Example 3.9

Solution. □

Example 3.10

Solution. □