The condition is
$0 \leq \tan(x-y) \leq \frac{1}{\sqrt{3}}$
If $0 \leq x-y \leq 2\pi$, we must have $0 \leq x \leq \frac{\pi}{6}$.
Hence $2\pi n \leq x-y \leq 2\pi n + \frac{\pi}{6}$ for integer n.
Take all numbers modulo π. We want to show that there must be two such that their positive difference is less than or equal to $\frac{\pi}{6}$.
Consider the number line from 0 to π partitioned into 6 segments of length $\frac{\pi}{6}$
The Pigeonhole Principle tells us that at least 2 points must be on the same segment i.e. have the desired difference, done.