Physics With Ram, Master Document

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§1 Jamboards

Statics
Forces

§2 Statics

The term "statics" means stationary. In a static system, no objects have any acceleration. This means that two things are true:

- The net force on the system is 0.
- The net torque (spin) on the system is 0.

So when solving a statics problem, draw the free-body diagram and balance forces & torque. There are 4 main forces to understand:

- Tension. Tension is force pulling on a point in a rope. For a massless rope, this force is the same to the left and to the right at any instantaneous spot in the rope, even if it is bent around a pulley, etc. Ropes with mass are trickier to deal with, the general idea is that tension is a function of a point on the rope.
- Normal force is a force opposing a surface, perpendicular to that surface. Think of it as like a spring.
- Friction is force opposing motion. The brief run-down is that kinetic friction satisfies $f_k = \mu_k N$ and static friction satisfies $f_s \leq \mu_s N$.
- Gravity. The gravitational force between two bodies with masses M, m and distance R is
$$F = \frac{GMm}{R^2}$$
Where G is a constant. For the earth, $\frac{GM}{R^2} = g$, hence the force is mg downward in our reference frame.

On to problems.

Example 2.1 (2.1 Red Book)

A rope with length L and mass density per unit length ρ is suspended vertically from one end. Find the tension as a function of height along the rope.

Solution. Consider an instant in the rope. The force is $T(y)$ downward and $T(y + \mathrm{dy})$ upward. The force of gravity is $\rho g\mathrm{dy}$. The system is static, so newton's second law tells us that the net force is zero. Therefore,

$$T(y+\mathrm{dy}) - T(y) - \rho g\mathrm{dy} = 0 \implies T(y+\mathrm{dy}) = T(y) + \rho g\mathrm{dy}$$
$$\frac{T(y+\mathrm{dy}) - T(y)}{\mathrm{dy}} = \rho g$$
$$T'(y) = \rho g$$
$$T(y) = \int \rho g\mathrm{dy} = \rho g y$$

□

Example 2.2 (2.2 Red Book)

A block sits on a plane that is inclined at an angle θ. Assume that the friction force is large enough to keep the block at rest. What are the horizontal components of the friction and normal forces acting on the block? For what θ are these horizontal components maximum?

Solution. Drawing axes parallel to the hypotenuse of the ramp, we get that

$$N = mg\cos(\theta)$$

$$f_s = mg\sin(\theta)$$

The problem asks for the horizontal component with normal axes, which is just $N\sin\theta = f_s\cos\theta = mg\sin\theta\cos\theta = \frac{mg\sin(2\theta)}{2}$. This attains its maximum $\frac{mg}{2}$ at $\theta = 45^{\circ}$. $\square$

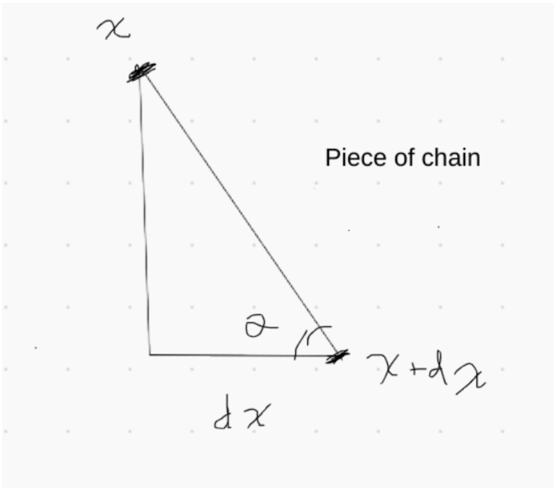
Example 2.3 (2.3 Red Book)

A frictionless tube lies in the vertical plane and is in the shape of a function that has its endpoints at the same height but is otherwise arbitrary. A chain with uniform mass per unit length lies in the tube from end to end. Show that the chain doesn't move.

Solution. Consider an infinitesimal piece of the chain. We want the net force along the direction of the tube to be 0, so the sum of the forces for each piece should be 0. Since the pieces are infinitesimal, we can sum them with an integral. In order words, we want to show that

$$\int a\mathrm{dm} = 0$$

At infinitesimals, stuff is easy to work with, because we can assume the piece of the chain to be a line. Let the function of the tube be $f(x)$. Consider the following diagram.



Notice that $f'(x)$ is defined to be the change in the y direction at an instantaneous moment, i.e. the other leg of the triangle is $f'(x)dx$. The hypotenuse, or length of the chain is then $\sqrt{f'(x)^2+1}dx$. Hence the mass is $\rho\sqrt{f'(x)^2+1}dx$. Since it is a ramp, notice that the force downward is $mg\sin\theta$. Thus by newton's laws

$$ma = -mg\sin\theta \implies a = -g\sin\theta$$

But $\sin\theta = \frac{f'(x)dx}{\sqrt{f'(x)^2+1}dx} = \frac{f'(x)}{\sqrt{f'(x)^2+1}}$. Hence it suffices to show that

$$\int \frac{f'(x)}{\sqrt{f'(x)^2+1}}\rho\sqrt{f'(x)^2+1}dx = 0$$

$$\int \rho f'(x) = 0$$

Which is true due to the fact that $f(x_{\text{initial}}) = f(x_{\text{final}})$. □

Example 2.4 (2.4 Red Book)

A book of mass M is positioned against a vertical wall. The coefficient of friction between the book and the wall is μ. You wish to keep the book from falling by pushing on it with a force F applied to an angle θ with respect to the horizontal.

- For a given θ, what is $\min F$?
- For what θ is $\min F$ minimized? What is the said minimum?
- What is the limiting value of θ such that there doesn't exist an F?

Solution. The upward force applied is $F\sin\theta$. The downward force from the book is Mg. The last force to worry about is friction, which we will denote by f_s. We have $f_s \leq \mu N = \mu F\cos\theta$. Hence we have

$$F\sin\theta + \mu F\cos\theta \geq Mg$$

$$F \geq \frac{Mg}{\sin\theta + \mu\cos\theta}$$

Solving the first part. For the second, we wish to maximize

$$\sin\theta + \mu\cos\theta$$

Taking the derivative w.r.t θ, this is

$$\cos\theta - \mu\sin\theta = 0 \to \theta = \arctan\left(\frac{1}{\mu}\right)$$

Therefore we get

$$\sin\theta = \frac{1}{\sqrt{\mu^2+1}}$$

$$\cos\theta = \frac{\mu}{\sqrt{\mu^2+1}}$$

Hence $\sin\theta + \mu\cos\theta = \sqrt{\mu^2+1}$. Therefore,

$$\min\min F = \frac{Mg}{\sqrt{\mu^2+1}}$$

Onto the last part, there doesn't exist an F if and only if

$$\sin\theta + \mu\cos\theta \leq 0$$

The limiting value is $\sin\theta + \mu\cos\theta = 0$. This implies that said $\theta = \arctan(-\mu)$. □

Example 2.5 (2.5 Red Book)

A rope with length L and mass density per unit length ρ lies on a plane inclined at an angle θ. The top end is nailed to the plane, and the coefficient of friction between the rope and the plane is μ. What are the possible values for the tension at the top of the rope?

Solution. The mass of the rope is ρL. Choose the x and y axes as normally, along the plane. Hence the force in the x direction is $\rho Lg\sin\theta$. The normal force is $N = \rho Lg\cos\theta$, hence the force of static friction is

$$f_s \leq \mu N = \mu\rho Lg\cos\theta$$

Thus

$$T + \mu\rho Lg\cos\theta \geq \rho Lg\sin\theta$$

$$T \geq (\rho Lg)(\sin\theta - \mu\cos\theta)$$

But we also get the upper bound

$$T \leq (\rho Lg)(\sin\theta + \mu\cos\theta)$$

The same way. □

Example 2.6 (2.6 Red Book)

Consider the two problems

- A disk of mass M and radius R is help up by a massless string. The surface of the disk is frictionless. What is the tension in the string? What is the normal force per unit length that the string applies on the disk?
- Let there now be friction between the disk and the string, with coefficient μ. What is the smallest possible tension in the string at its lowest point?

Solution. The solutions are as follows.

- Notice that there is a tension T upward in each end of the string, hence $2T = Mg \to T = \frac{Mg}{2}$. At small instants in the rope, the normal force is essentially just the tension. For an angle θ, the normal force is $N\theta$ and the length is $R\theta$, hence we want
$$\frac{N\theta}{R\theta} = \frac{N}{R} = \frac{Mg}{2R}$$
- Using the rope wrapped around pole example,
$$T(\frac{\pi}{2}) \leq T(0)e^{\mu\frac{\pi}{2}}$$
$$T(0) \geq \frac{Mg}{2}e^{-\mu\frac{\pi}{2}}$$

□

Example 2.7 (2.7 Red Book)

Each of the following planar objects is places (see figure 2.13 in book) between two frictionless circles of radius R. The mass densite per unit area of each object is σ, and the radii to the points of contact make an angle θ with the horizontal. For each case, find the horizontal force that must be applied to the circles to keep them together. For what θ is this force maximum or minimum?

Solution. For the first one, a bit of angle chasing (note the kites) yields that the angle at the apex of the triangle is 2θ. Hence by the sine area formula, it's area is $L^2\sin(2\theta)\frac{1}{2} = L^2\sin\theta\cos\theta$. Hence the mass is $\sigma L^2\sin\theta\cos\theta$. Let F_{CT} be the force from the circle on the triangle. Notice that

$$2F_{CT}\sin\theta = mg = \sigma g L^2\sin\theta\cos\theta$$

$$F_{CT} = \frac{\sigma g L^2\cos\theta}{2}$$

We desire

$$F_{CT}\cos\theta = \frac{L^2\cos^2\theta\sigma g}{2}$$

Similarly for the other cases, the answer is just $\frac{mg\cos\theta}{2\sin\theta}$. And $m = \sigma A$, where A is the area. To find the area of the rectangle, simply draw in the isosceles trapezoid. The unkown side is just $2R(1-\cos\theta)$, hence the area is $2RL(1-\cos\theta)$ and the mass is $2RL\sigma(1-\cos\theta)$. Therefore, the answer is

$$RL\sigma(1-\cos\theta)\cot\theta$$

The last case is a circle. It's a fun geometry puzzle to figure out the area of this. Connect the centers. Let the small circle have radius r. Then $\frac{r+R}{\sin\theta} = \frac{2R}{\sin(180-2\theta)} = \frac{2R}{\sin(2\theta)} = \frac{R}{\sin\theta\cos\theta}$

$$r = R\left(\frac{1}{\cos\theta} - 1\right)$$

And hence the answer is $\pi g R^2\left(\frac{1}{\cos\theta}-1\right)^2\sigma\frac{\cos\theta}{2\sin\theta}$. This is equivalent to

$$\frac{\sigma g\pi R^2(1-\cos\theta)^2}{\sin(2\theta)}$$

Now, we must maximize each one. Fortunately this isn't too difficult because we only must maximize the part that is dependent on θ.

- The first case is to maximize $\cos\theta$, which occurs at $\theta = 0$. The minimum is at $\theta = \frac{\pi}{2}$.
- The second case is to maximize $(1-\cos\theta)\cot\theta$. This is

  $$f(\theta) = \frac{\cos\theta - \cos^2\theta}{\sin\theta}$$

  Using the quotient rule,

  $$f'(\theta) = \cos^3(\theta) - 2\cos(\theta) + 1 = 0$$

  Hence

  $$\theta = \arccos\left(\frac{-1+\sqrt{5}}{2}\right)$$

  The minimimum is at both $\theta = \frac{\pi}{2}, 0$.
- The third case goes to infinity as θ approaches 45 degrees. It's minimized at $\theta = 0$.

□

Example 2.8 (2.8 Red Book)

Consider the two problems:

- A chain with uniform mass density per unit length hangs between two given points on two walls. Find the general shape of the chain. Aside from an arbitrary additive constant, the function describing the shape should contain one unknown constant.
- The unknown constant in your answer depends on the horizontal distance d between the walls, and the vertical distance λ between the support points, and the length ℓ of the chain. Write the constant in terms of these quantities.

Solution. □

Example 2.9 (2.9 Red Book)

A chain with uniform mass density per unit length hands between two supports located at the same height, a distance $2d$ apart. What should the length of the chain be so that the magnitude of the force at the supports is minimized?

Solution. □

§3 Forces

Example 3.1 (3.1)
A massless pulley hangs from a fixed support. A massless string connnecting two masses, m_1 and m_2, hangs over the pulley (see Fig. 3.11). Find the acceleration of the masses and the tension in the string.

Solution. Note that the acceleration is constant, so

$$T - m_1 g = m_1 a$$

$$T - m_2 g = -m_2 a$$

Which, when solved, gives

$$a = \frac{m_2 g - m_1 g}{m_2 + m_1}, T = \frac{2m_1 m_2 g}{m_2 + m_1}$$

□

Example 3.2 (3.2)
A double Atwood's machine is shown in Fig. 3.12, with masses m_1, m_2, m_3. Find the acceleration of each mass.

Solution. The equations are

$$2T - m_1 g = m_1 a_1, T - m_2 g = m_2 a_2, T - m_3 g = m_3 a_3$$

And then from conservation of string,

$$a_1 = \frac{-a_2 - a_3}{2}$$

And the accelerations can be solved for as

$$a_1 = g\frac{4m_2 m_3 - m_1 m_2 - m_1 m_3}{4m_2 m_3 + m_1 m_2 + m_1 m_3}$$

$$a_2 = -g\frac{4m_2 m_3 + m_1 m_2 - 3m_1 m_3}{4m_2 m_3 + m_1 m_2 + m_1 m + 3}$$

$$a_3 = -g\frac{4m_2 m_3 + m_1 m_3 - 3m_2 m_1}{4m_2 m_3 + m_1 m_2 + m_1 m_3}$$

□

Example 3.3 (3.3)
Consider the infinite Atwood's machine shown in Fig. 3.13. A string passes over each pulley, with one end attached to a mass and the other end attached to another pulley. All the masses are equal to m, and all the pulleys and strings are massless. The masses are held fixed and then simultaneously released. What is the acceleration of the top mass? (You may define this infinite system as follows. Consider it to be made of N pulleys, with a nonzero mass replacing what would have been the (N + 1)th pulley. Then take the limit as $N \to \infty$.)

Solution. For a finite system, we get the equations

$$2T - mg = ma_1$$
$$2T - mg = ma_2$$
$$\dots$$
$$T - mg = ma_n$$
$$T - mg = ma_{n+1}$$

Conservation of string equations aren't needed yet. We get that

$$g + a_1 = g + a_2 = \cdots = g + a_{n-1} = 2g + 2a_n = 2g + 2a_{n+1}$$

So $a_1 = a_2 = a_3... = a_{n-1}$. But by conservation of string, we get that

$$a_1 = a_{n-1} = -\frac{a_n + a_{n+1}}{2} = -a_n$$
$$a_1 = 2a_n + g$$
$$a_n = -\frac{g}{3}$$
$$a_1 = \frac{g}{3}$$

□

Example 3.4 (3.4)
$N + 2$ equal masses hand from a system of pulleys, as shown in Fig. 3.14. What are the accelerations of all the masses?

Solution. We have that

$$T - mg = ma$$
$$2T - mg = ma_1$$

But $2a_1N = -2a$ by conservation of string. Hence

$$a = -\frac{Ng}{2N+1}, a_1 = \frac{g}{2N+1}$$

□

Example 3.5 (3.5)
Consider the system of pulleys shown in FIg. 3.15. The string (which is a loop with no ends) hangs over N fixed pulleys that circle around the underside of a ring. N masses, $m_1, m_2, \cdots, m_N$ are attached to N pulleys that hang on the string. What are the accelerations of all the masses.

Solution. We get

$$2T - m_k g = m_k a_k$$

For each k. Since the masses are placed around a ring,

$$a_1 + a_2 + ... + a_N = 0$$

But $a_k = \frac{2T - m_k g}{m_k}$. Hence

$$\sum a_k = \sum \frac{2T}{m_k} - g = 2T \sum \frac{1}{m_k} - Ng = 0$$

$$T = \frac{Ng}{2 \sum \frac{1}{m_k}}$$

And hence

$$a_k = \frac{\frac{Ng}{\sum \frac{1}{m_k}} - m_k g}{m_k}$$

□

Example 3.6 (3.6)

Problems are as follows:

- A block starts at rest and slides down a frictionless plane inclined at an angle θ. What should θ be so that the block travels a given horizontal distance in the minimum amount of time.
- Same question, now there is a coefficient of friction μ.

Solution. Consider the following solutions.

- Suppose the distance is d and the time is t. The force downward has magnitude mg, so the force down the plane has magnitude $mg \sin\theta$. Hence the acceleration is $g \sin\theta$. The initial velocity is zero, so we can use

  $$d = \frac{1}{2}at^2$$

  $$g \sin\theta = \frac{2d}{t^2}$$

  $$\theta = \arcsin\left(\frac{2d}{t^2 g}\right)$$

  We want theta such that t is minimized, so $\theta = 45$.

- Same thing, except the acceleration is now $g\sin\theta - \mu g \cos\theta$. Hence

  $$g \sin\theta - \mu g \cos\theta = \frac{2d}{t^2}$$

  $$\sin\theta - \mu \cos\theta = \frac{2d}{t^2 g}$$

  For now, define $y := \frac{2d}{t^2 g}$. Let $x = \sin\theta$. We want to solve

  $$x - y = \mu\sqrt{1 - x^2}$$

$$(x-y)^2 = \mu^2(1-x^2)$$
$$x^2 - 2xy + y^2 = \mu^2 - x^2\mu^2$$
$$x^2(1+\mu^2) - 2xy + y^2 - \mu^2 = 0$$
$$x = \frac{2y \pm 2\sqrt{y^2 + (1+\mu^2)\mu^2}}{2+2\mu^2}$$
$$x = \frac{y \pm \sqrt{y^2+\mu^2+\mu^4}}{1+\mu^2}$$

Taking only the positive value,

$$\theta = \arcsin\left(\frac{\frac{2d}{t^2 g} + \sqrt{\frac{4d^2}{t^4 g^2} + \mu^2 + \mu^4}}{1+\mu^2}\right)$$

To minimize t, we want the value in the parentheses to be fixed as a small value, specifically $\theta = \frac{\arctan(-\frac{1}{\mu})}{2}$ (which can be obtained with nasty calculations after taking the derivative).

□

Example 3.7 (3.8)

A block of mass m is held motionless on a frictionless plane of mass M and angle of inclination θ (see Fig. 3.16). The plane rests on a frictionless horizontal surface. The block is released. What is the horizontal acceleration of the plane?

Solution. Note that we have

$$mg - N\cos\theta = ma_y$$
$$N\sin\theta = ma_x = Ma'_x$$

Hence $N = \frac{mg - ma_y}{\cos\theta} = \frac{ma_x}{\sin\theta}$. Thus $a_x = (g - a_y)\tan\theta$. In addition, the ratio of the y accleration to the x acceleration must remain $\tan\theta$, so

$$a_y = (a_x + a'_x)\tan\theta$$
$$mg - N\cos\theta = N\sin\theta + \frac{m}{M'}N\sin\theta$$
$$mg - N\cos\theta = N\sin\theta(\frac{M'+m}{M'})$$

A bunch of algebra gives

$$a'_x = \frac{mg\sin\theta\cos\theta}{M + m\sin^2\theta}$$

□

Example 3.8 (3.9)

A particle of mass m is subject to a force $F(t) = ma_0 e^{-bt}$. The initial position and speed are zero. Find $x(t)$.

Solution. The magnitude of the acceleration is hence $a = a_0 e^{-bt}$. Notice

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = a_0 e^{-bt}$$

$$x = \int\int a_0 e^{-bt}\mathrm{dtdt}$$

Now,

$$\int a_0 e^{-bt}\mathrm{dt} = a_0 \int e^{-bt}\mathrm{dt} = \frac{a_0 e^{-bt}}{-b} + C_1$$

The initial speed is zero so

$$C_1 = \frac{a_0}{b}$$

Then

$$x = \frac{a_0 e^{-bt}}{b^2} + \frac{a_0}{b}t + C_2$$

Now, the initial position is zero, so likewise we find

$$C_2 = -\frac{a_0}{b^2}$$

Thus

$$x(t) = \frac{a_0 e^{-bt}}{b^2} + \frac{a_0}{b}t - \frac{a_0}{b^2}$$

□

Example 3.9 (3.10)

Same question just initial position is x_0 and $F(x) = -kx$.

Solution. We have that

$$mv\frac{\mathrm{dv}}{\mathrm{dx}} = -kx$$

$$mv\mathrm{dx} = -kx\mathrm{dx}$$

$$\int_{x_0}^{x} -kx\mathrm{dx} = \int_0^v mv\mathrm{dv}$$

$$kx_0^2 - kx^2 = mv^2$$

Employing $\omega = \sqrt{\frac{k}{m}}$, this is

$$v = \omega\sqrt{x_0^2 - x^2}$$

$$\int_{x_0}^{x} \frac{1}{\sqrt{x_0^2 - x^2}}\mathrm{dx} = \int_0^t \omega\mathrm{dt}$$

$$\int_{x_0}^{x} \frac{1}{\sqrt{x_0^2 - x^2}}\mathrm{dx} = \omega t$$

Substitute $x = x_0\cos\theta$, then

$$\int_0^\theta \frac{1}{x_0\sin\theta}\mathrm{dx} = \int_0^\theta -\mathrm{d}\theta$$

Hence $\theta = -\omega t$, and $x = x_0\cos(\omega t)$ □

Example 3.10 (3.11)

A chain with length ℓ is held stretched out on a frictionless horizontal table, with a length y_0 hanging down through a hole in the table. The chain is released. As a function of time, find the length that hangs down through the hole (don't bother with t after the chain loses contact with the table). Also, find the speed of the chain right when it loses contact with the table.

Solution. □

Example 3.11 (3.14)

A ball is thrown at speed v from zero height on level ground. At what angle should it be thrown so that the area under the trajectory is maximum?

Solution. □

Example 3.12 (3.15)

A ball is thrown straight upward so that it reaches a height h. It falls down and bounces repeatedly. After each bounce, it returns to a certain fraction f of its previous height. Find the total distance traveled, and also the total time, before it comes to rest. What is its average speed?

Solution. □

Example 3.13 (3.17)

A ball is thrown with speed v from the edge of a cliff of height h. At what inclination angle should it be trhwon so that it travels the maximum horizontal distance? What is this maximum distance? Assume that the ground below the cliff is horizontal.

Solution. □