Chapter 5/8 Homework

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Example 0.1 (Context Rich Problem)
Pulleys

Solution. To solve this problem, we need to know how to balance forces and we need the basics of torque and moments of inertia. Specifically,

$$\sum \tau = I\alpha$$

$$a = r\alpha$$

$$I = \int r^2 \mathrm{dm}$$

where τ is the torque on an object, I is the moment of inertia, α is the angular acceleration, a is the tangential acceleration, r is the radius, m is the mass.

Our solution will look like this:

- Balance forces on m_1
- Balance forces on m_2
- Balance torques on the pulley
- Combine all previous results to solve the problem

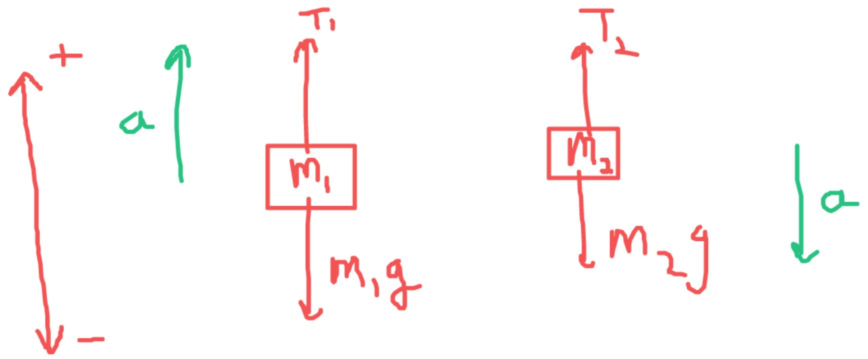
Balancing forces will be straightforward, as we will see soon. To balance torques on the pulley, we need to:

- Find all the torques on the pulley
- Compute I
- Use the previous information to compute α
- Use that information to compute a

Onto the process.

Key piece of information: The acceleration of each object is the same, a. This is because they are all connected by a rope.

To balance forces on m_1 and m_2, we can use the following free-body diagrams:



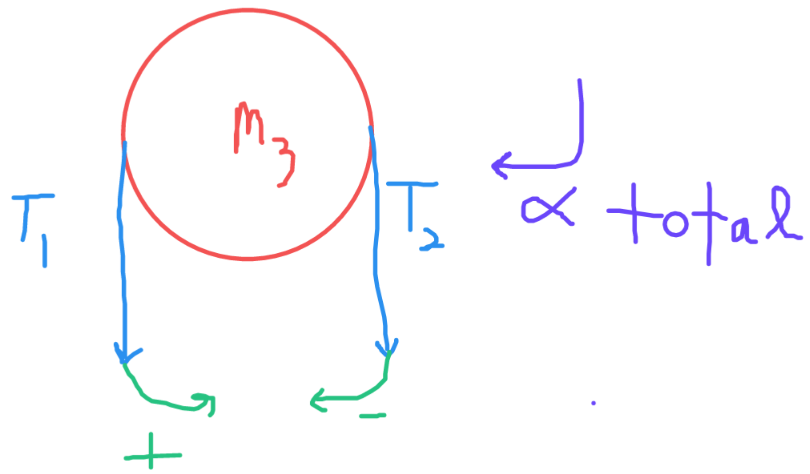
Balancing the forces in the y-direction on m_1, we have by Newton's second law that

$$T_1 - m_1 g = m_1 a \quad (1)$$

Balancing for m_2, we have that

$$m_2 g - T_2 = m_2 a \quad (2)$$

To balance torques on the pulley, we can use the following diagram:



Notice that there are two torques on m_3. The torque due to T_1, and the torque due to T_2. There is no torque due to gravity since the axis of rotation is the center of mass of the pulley, which is where gravity acts on the pulley. Let r be the radius of the pulley. By Newton's second law for torques, we have that

$$\sum \tau = I \cdot -\alpha$$

$$T_1 r - T_2 r = I \cdot -\alpha \quad (3)$$

In order to compute the moment of inertia of the pulley, we can use

$$I = \int r^2 \text{dm}$$

Let the mass density of the pulley be λ and A be the area of the disk. Then

$$\lambda A = m_3$$

$$\lambda \mathrm{dA} = \mathrm{d}m_3$$

Let r be the radius of the pulley. Then $A = \pi r^2$ so

$$dA = 2\pi r \mathrm{dr}$$

Hence,

$$I = \int_0^R r^2 \mathrm{d}m_3 = \int_0^R r^2 \lambda \mathrm{dA}$$

$$I = \int_0^R r^2 \lambda 2\pi r \mathrm{dr}$$

$$I = 2\pi\lambda \int_0^R r^3 \mathrm{dr}$$

$$I = 2\pi\lambda \left(\frac{R^4}{4} - \frac{0^4}{4}\right)$$

$$I = 2\pi\lambda \frac{R^4}{4} = \frac{R^4 \pi\lambda}{2}$$

Where R is the radius of the pulley (just so we don't mix it up with the variable being integrated). Substituting $\lambda = \frac{m_3}{A} = \frac{m_3}{\pi R^2}$, we have

$$I = \frac{m_3 R^2}{2} \qquad (4)$$

Combining equations (3) and (4), we have that

$$T_1 r - T_2 r = m_3 r^2 \frac{-\alpha}{2}$$

(recall that we use r and R interchangeably). Cancelling a factor of r,

$$T_1 - T_2 = m_3 r \frac{-\alpha}{2}$$

Now, remember that the tangential acceleration of the pulley, which is the same a, is equal to $r\alpha$. Therefore,

$$T_1 - T_2 = \frac{m_3(-a)}{2}$$

$$T_2 - T_1 = \frac{m_3 a}{2} \qquad (5)$$

We can finally finish the problem off now. Using equations $(1), (2), (5)$, we have

$$T_1 - m_1 g = m_1 a$$

$$m_2 g - T_2 = m_2 a$$

$$T_2 - T_1 = \frac{m_3 a}{2}$$

We want to solve for a. We have three equations, and three unknowns (T_1, T_2, a). Adding the three equations, we get

$$m_2g - m_1g = m_1a + m_2a + \frac{m_3a}{2}$$

$$a = \frac{m_2g - m_1g}{m_1 + m_2 + \frac{m_3}{2}}$$

Which is the answer!

Answer check: Firstly, the units are clearly correct since the units of the numerator is N and the denominator is kg. Second, in the limit that $m_3 \to 0$ (the pulley is massless), we get

$$a = \frac{m_2g - m_1g}{m_1 + m_2}$$

Which agrees with the result that we had when we did massless pulleys. When m_2 is giant compared to everything else, we get $a \sim g$, which also makes sense since the everything will essentially be freefall. This information suggest that our answer is reasonable. □

Example 0.2 (Explanation Task)
Tracks

Solution. To solve this problem, we need to know the basic principles of circular motion. Specifically, we will use that for an object in circular motion,

$$F = \frac{mv^2}{r}$$

where F is the inward force on the object, v is the tangential speed of the object, m is the mass of the object, and r is the radius of the circle.

In the case of this problem, the only inward force is the frictional force. Hence

$$F_{friction} = \frac{mv^2}{r}$$

I am assuming the cars have the same mass. This was not specified in the problem, but the problem would be unsolvable without this (Since $F_{friction} \propto m$).

We can just plug into the formula. For the red car, we have

$$F_{red} = \frac{mv^2}{2} = 0.5mv^2$$

For the white one,

$$F_{white} = \frac{m(1.5v)^2}{3} = 0.75mv^2$$

For the green one,

$$F_{green} = \frac{m(2v)^2}{4} = mv^2$$

For the orange one,

$$F_{orange} = \frac{m(4v)^2}{5} = 3.2mv^2$$

Hence, since mv^2 is a positive quantity,

$$\boxed{F_{red} < F_{white} < F_{green} < F_{orange}}$$

$\square$