Math Notes: Kurtosis Categorification for Stock Price and Volume Changes: CALA RDUS and XBIT

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The Mathematical Learning Space Research Portfolio: Math Note 2

Kurtosis is the fourth standardized moment, defined as [1]

$$\mathrm{K}[X]=\mathrm{E}\left[\left(\frac{X-\mu}{\sigma}\right)^4\right]=\frac{\mathrm{E}\left[(X-\mu)^4\right]}{\left(\mathrm{E}\left[(X-\mu)^2\right]\right)^2}=\frac{\mu_4}{\sigma^4}, \tag{1}$$

$$\mathrm{K}[Y]-3=\frac{1}{n^2}\sum_{i=1}^{n}\left(\mathrm{K}\left[X_i\right]-3\right). \tag{2}$$

Consider excess kurtosis KE with α and β shape parameters for the Beta distribution given by

$$\text{KE}(Y)= \tag{3}$$
$$=\frac{6[(\alpha-\beta)^2(\alpha+\beta+1)-\alpha\beta(\alpha+\beta+2)]}{\alpha\beta(\alpha+\beta+2)(\alpha+\beta+3)} \tag{4}$$

where the beta distribution with $\Gamma(x)$ is the gamma function is given by

$$f(x;\alpha,\beta)=\mathrm{constant}\cdot x^{\alpha-1}(1-x)^{\beta-1} \tag{5}$$
$$=\frac{x^{\alpha-1}(1-x)^{\beta-1}}{\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\,du} \tag{6}$$
$$=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\,x^{\alpha-1}(1-x)^{\beta-1} \tag{7}$$
$$=\frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1} \tag{8}$$

Kurtosis is bounded below by the squared skewness plus 1

$$\frac{\mu_4}{\sigma^4}\geq\left(\frac{\mu_3}{\sigma^3}\right)^2+1 \tag{9}$$

where μ_3 is the third central moment. The lower bound is realized by the Bernoulli distribution where there is no upper limit to the kurtosis of a general probability distribution as shown in Table 2. Consider the kurtosis of X and Y along with their cokurtosis given by

$$\mathrm{K}[X+Y]=\frac{1}{\sigma_{X+Y}^4}\big(\sigma_X^4\,\mathrm{K}[X]+4\sigma_X^3\sigma_Y\,\mathrm{K}[X,X,X,Y] \tag{10}$$
$$+6\sigma_X^2\sigma_Y^2\,\mathrm{K}[X,X,Y,Y] \tag{11}$$
$$+4\sigma_X\sigma_Y^3\,\mathrm{K}[X,Y,Y,Y]+ \tag{12}$$
$$\sigma_Y^4\,\mathrm{K}[Y]\big). \tag{13}$$

where $K(X,X,X,X)=\frac{\mathrm{E}\left[(X-\mathrm{E}[X])^4\right]}{\sigma_X^4}=\mathrm{kurtosis}\left[X\right]$ and

$$K(X,X,X,Y)=\frac{\mathrm{E}\left[(X-\mathrm{E}[X])^3(Y-\mathrm{E}[Y])\right]}{\sigma_X^3\sigma_Y} \tag{14}$$
$$K(X,X,Y,Y)=\frac{\mathrm{E}\left[(X-\mathrm{E}[X])^2(Y-\mathrm{E}[Y])^2\right]}{\sigma_X^2\sigma_Y^2} \tag{15}$$
$$K(X,Y,Y,Y)=\frac{\mathrm{E}\left[(X-\mathrm{E}[X])(Y-\mathrm{E}[Y])^3\right]}{\sigma_X\sigma_Y^3}, \tag{16}$$

Cokurtosis is a measure the magnitude of change fo two random variables together. A high level of cokurtosis requires extreme positive and negative deviations at the same time. [2]

The kurtosis goes to infinity for high and low values of p but for p = 1/2 the two-point distributions including the Bernoulli distribution have a lower excess kurtosis than any other probability distribution,-2. Let

$$E(X^n)=\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}=\frac{1}{n+1}\sum_{k=0}^{n}a^kb^{n-k}. \tag{17}$$

The Bernoulli distribution is a special case of the binomial distribution with n = 1. The Bernoulli distributions for $0\leq p\leq 1$ form an exponential family where for n¿ 1 the excess kurtosis given by (1-6pq)/npq.

Consider data from yahoo finance for both daily closing price and volume for three sysmbols CALA RDUS and XBIT over the period august 24 2020 to august 24 2021. Table 1 has the moments for the daily closing prices august 24 2020 to august 24 2021 that demonstrates negative kurtosis in the interval [-1,2,-0.44].[1001] A platykurtic distribution has thinner tails such as continuous and discrete uniform distributions, and the raised cosine distribution. The Hurst (H) exponent given by $\mathbb{E}\left[\frac{R(n)}{S(n)}\right]=Cn^H$ as $n\to\infty$, H is directly related to fractal dimension, D, where $1<D<2$, such that D = 2 - H. The values of the Hurst exponent vary between 0 and 1, with higher values indicating a smoother trend, less volatility, and less roughness [900] In other words, H in [0 – 0.5) has long-term switching between high and low values in adjacent pairs, a single high value followed by a low value then a high value instead of the opposite of high value followed by high value like H in (0.5,1]. H=0.5 is neither A or not A in category and therefore uncorrelated.

	Mean	StDEV	Skew	Kurtosis	SKR	hurst
CALA	3.06	0.98	0.76	-0.44	0.30	0.26
RDUS	17.42	3.56	-0.13	-1.12	-0.07	0.26
XBIT	17.90	1.36	0.12	-1.20	0.07	0.26

The Skewness Kurtosis Ratio (SKR) can be examined with a Taylor Series by the ratio of X and Y random variables given by

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$$
\mathrm{E}\left[\frac{X}{Y}\right] \approx \frac{\mathrm{E}[X]}{\mathrm{E}[Y]} - \frac{\operatorname{cov}[X,Y]}{\mathrm{E}[Y]^2} + \frac{\mathrm{E}[X]}{\mathrm{E}[Y]^3}\operatorname{var}[Y] \tag{18}
$$

$$
\operatorname{var}\left[\frac{X}{Y}\right] \approx \frac{\operatorname{var}[X]}{\mathrm{E}[Y]^2} - \frac{2\,\mathrm{E}[X]}{\mathrm{E}[Y]^3}\operatorname{cov}[X,Y] + \frac{\mathrm{E}[X]^2}{\mathrm{E}[Y]^4}\operatorname{var}[Y] \,.\, \operatorname{var}[f(X)] \approx \left(f'(\mathrm{E}[X])\right)^2 \operatorname{var}[X] + \frac{\left(f''(\mathrm{E}[X])\right)^2}{2}\left(\operatorname{var}[X]\right)^2 = \left(f'(\mu_X)\right)^2 \sigma_X^2 + \frac{1}{2}\left(f''(\mu_X)\right)^2 \sigma_X^4 + \left(f'(\mu_X)\right)\left(f'''(\mu_X)\right)\sigma_X^4 \tag{19}
$$

The cokurtosis for the closing prices of CALA and RDUS -24.6 and CALA and XBIT is 1.51 and for RDUS and XBIT is 0.048.

Table 2 for the daily volumes with positive kurtosis from the interval [7.21,79.01] Leptokurtic distribution has fatter tails such as Student's t-distribution, Rayleigh distribution, Laplace distribution, exponential distribution, Poisson distribution and the logistic distribution and are super-Gaussian.[1001]

	Mean	StDEV	Skew	Kurtosis	SKR	hurst
CALA	988598.41	1286410.36	7.75	79.01	0.09	0.45
RDUS	426508.33	232447.09	2.22	7.21	0.22	0.36
XBIT	100107.14	87652.85	3.60	20.33	0.15	0.40

The cokurtosis for the volumes of CALA and RDUS 2.166156e+22 and CALA and XBIT is -7.885063e+20 and for RDUS and XBIT is 3.222973e+19.

Figure 1 has charts for Daily Closing Prices For CALA August 24 2020-August 24 2021 with the following functions addBBands(n = 20, sd = 2, maType = "SMA", draw = 'bands', on = -1) addSMA() addZLEMA() addWPR(n=24) addRSI(n = 14, maType = "EMA", wilder = TRUE) addWPR(n = 2) addCCI(n = 20, maType="SMA", c=0.015) for Figures 1-6.[1001]

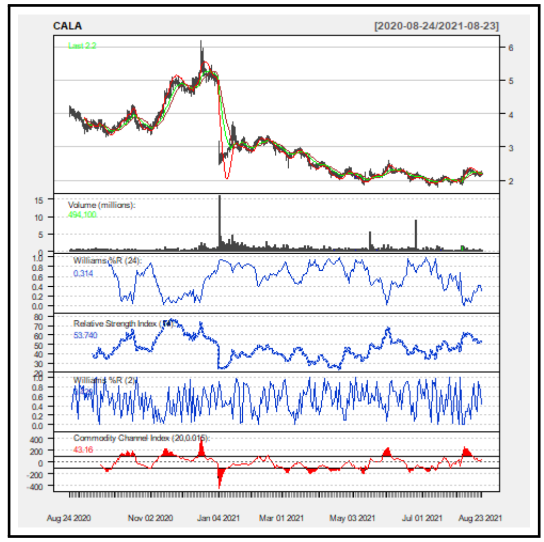

Figure 1: Daily Closing Prices For CALA August 24 2020-August 24 2021.[1001]

Figure 2 has the charts for Daily Closing Prices for RDUS August 24 2020-August 24 2021.

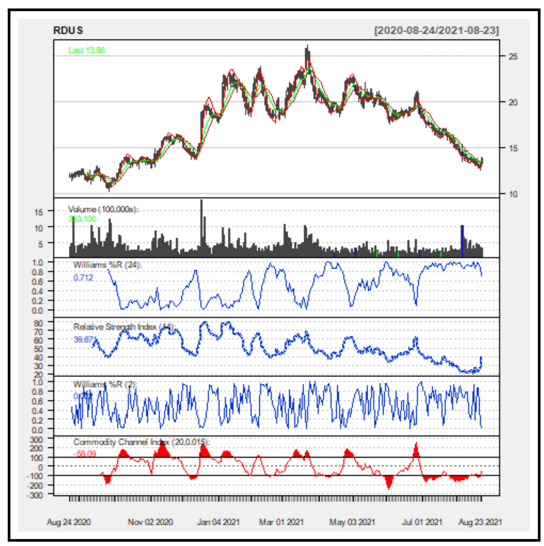

Figure 2: Daily Closing Prices for RDUS August 24 2020-August 24 2021.[1001]

Figure 3 has the charts for Daily Closing Prices for XBIT August 24 2020-August 24 2021.

Figure 3: Daily Closing Prices for XBIT August 24 2020-August 24 2021. [1001]

Figure 4 has the charts for Daily Volume For CALA August 24 2020-August 24 2021.

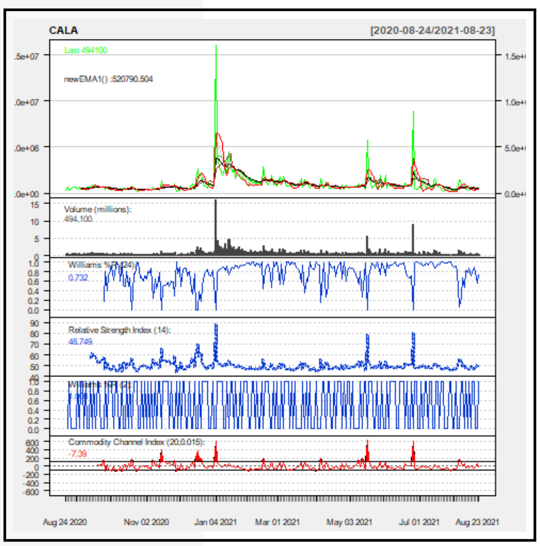

Figure 4: Daily Volume For CALA August 24 2020-August 24 2021. [1001]

Figure 5 has the charts for Daily Volume for RDUS August 24 2020-August 24 2021.

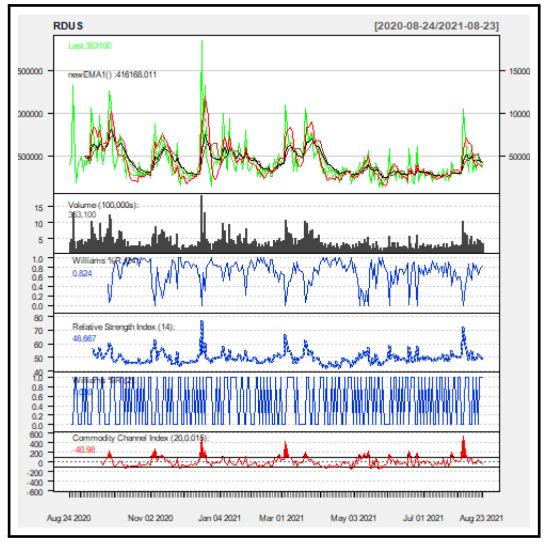

Figure 5: Daily Volume for RDUS August 24 2020-August 24 2021. [1001]

Figure 6 has charts for Daily Volume for XBIT August 24 2020-August 24 2021.

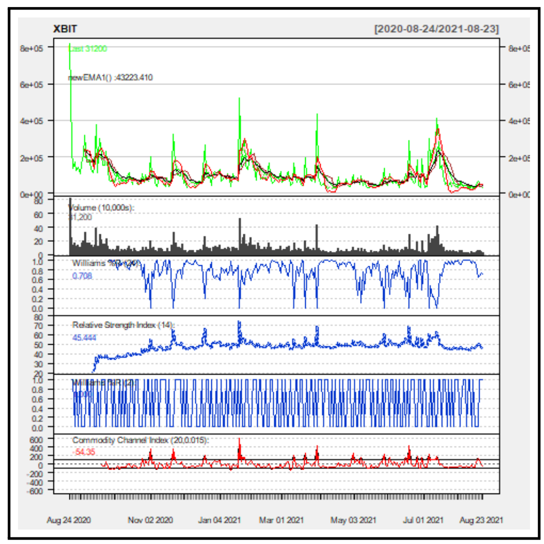

Figure 6: Daily Volume for XBIT August 24 2020-August 24 2021. [1001]

All are type 1 Pearsons (i.e. generalized Beta) with maximum likelihood estimation except for RDUS and XBIT Volume which is Type 6 amd 5 where F-distribution (type VI), Inverse-chi-squared distribution (type V) and Inverse-gamma distribution (type V) are similar distributions.[1000]

For closing prices, Kendall's rank correlation tau for CALA and XBIT z = 8.8148, p-value < 2.2e-16 alternative hypothesis: true tau is greater than 0 sample estimates: tau 0.3740798 . For Kendall's rank correlation tau data: CALA and RDUS z = 3.0821, p-value = 0.001028 alternative hypothesis: true tau is greater than 0 sample estimates: tau 0.1302078 . [1000]

For volumes, Kendall's rank correlation tau data: CALA and RDUS z = 3.0821, p-value = 0.001028 alternative hypothesis: true tau is greater than 0 sample estimates: tau 0.1303605 . Kendall's rank correlation tau data: CALA and XBIT z = 3.0777, p-value = 0.001043 alternative hypothesis: true tau is greater than 0 sample estimates:

tau 0.1302078. For RDUS and XBIT z = 3.5039, p-value = 0.0002293 and tau 0.1482489.

In the combination of closing prices and volumes for Kenda11's rank correlation CALA close price and RDUS volume has z = 4.3849, p-value = 5.803e-06 and tau 0.1858. CALA close price and XBIT volume z = 4.9083, p-value = 4.593e-07 with tau 0.208.

Figure 7 has the distribution for Daily Closing Prices August 24 2020-August 24 2021.

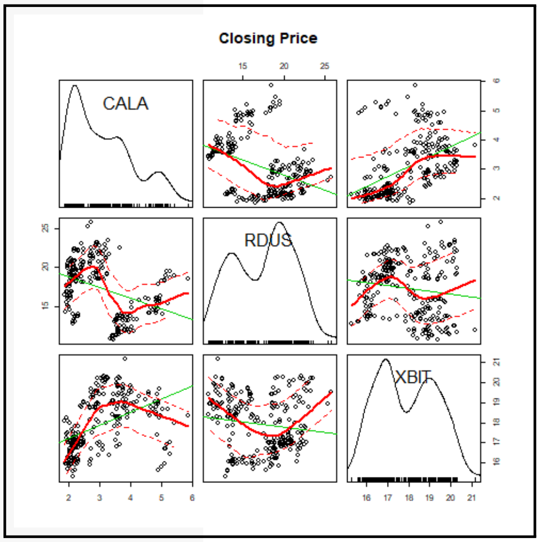

Figure 7: Daily Closing Prices August 24 2020-August 24 2021. [1001]

Figure 8 has the distributions for Daily Volumes.

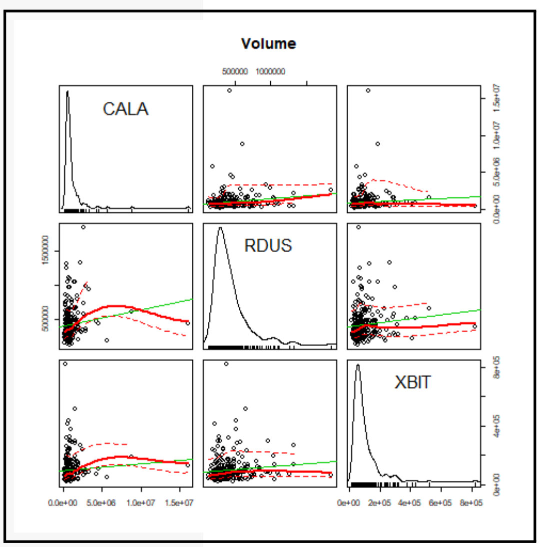

Figure 8: Daily Volumes August 24 2020-August 24 2021 [1001]

Figure 9 has the XBIT 30 Day Forecasts based on Nonlinear Models August 24 2020-August 24 2021.[1004]

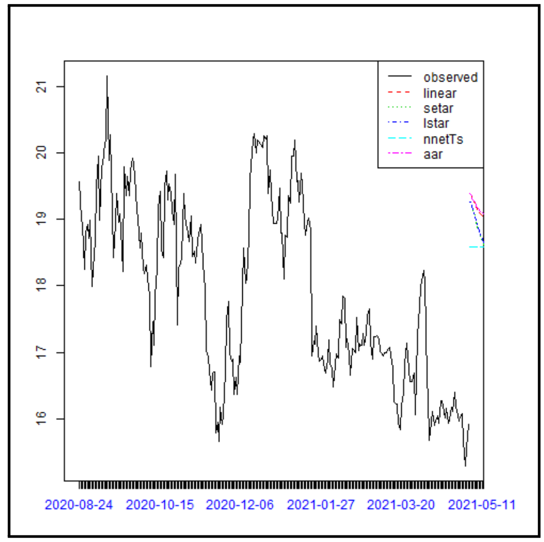

Figure 9: XBIT 30 Day Forecasts based on Nonlinear Models August 24 2020-August 24 2021 [1001]

Table 3 has the forecasts from each of the five models. [1004]

OBS	Date	linear	setar	lstar	nnets	aar
1	2021-08-25	19.39392	19.25635	19.25734	18.59424	19.39157
2	2021-08-26	19.35554	19.24037	19.24417	18.59424	19.35528
3	2021-08-27	19.29759	19.12950	19.13578	18.59424	19.30158
4	2021-08-28	19.24791	19.05504	19.06474	18.59424	19.25806
5	2021-08-29	19.20117	18.97283	18.98618	18.59424	19.21888
6	2021-08-30	19.15799	18.89749	18.91497	18.59424	19.18431
7	2021-08-31	19.11796	18.82395	18.84587	18.59424	19.15363
8	2021-09-01	19.08088	18.75374	18.78043	18.59424	19.12641
9	2021-09-02	19.04653	18.68619	18.71793	18.59424	19.10221
10	2021-09-03	19.01470	18.62138	18.65841	18.59424	19.08069
11	2021-09-04	18.98521	18.55913	18.60168	18.59424	19.06152
12	2021-09-05	18.95789	18.49937	18.54762	18.59424	19.04444
13	2021-09-06	18.93257	18.44199	18.49886	18.59424	19.02920
14	2021-09-07	18.90912	18.63483	18.61624	18.59424	19.01561
15	2021-09-08	18.88739	18.78662	18.77056	18.59424	19.00346
16	2021-09-09	18.86726	18.64593	18.63900	18.59424	18.99261
17	2021-09-10	18.84861	18.60678	18.60658	18.59424	18.98291
18	2021-09-11	18.83133	18.53706	18.54448	18.59424	18.97424
19	2021-09-12	18.81532	18.48088	18.49742	18.59424	18.96648
20	2021-09-13	18.80049	18.67483	18.62773	18.59424	18.95953
21	2021-09-14	18.78674	18.82779	18.78025	18.59424	18.95332
22	2021-09-15	18.77401	18.68453	18.64812	18.59424	18.94775
23	2021-09-16	18.76221	18.64416	18.61531	18.59424	18.94276
24	2021-09-17	18.75128	18.57284	18.55274	18.59424	18.93829
25	2021-09-18	18.74116	18.51527	18.50433	18.59424	18.93429
26	2021-09-19	18.73178	18.45634	18.59577	18.59424	18.93070
27	2021-09-20	18.72308	18.64998	18.75341	18.59424	18.92748
28	2021-09-21	18.71503	18.80214	18.62709	18.59424	18.92459
29	2021-09-22	18.70757	18.66050	18.59375	18.59424	18.92201
30	2021-09-23	18.70066	18.62088	18.53290	18.59424	18.91969

References

[1] Wikipedia contributors. "Kurtosis." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 25 Aug. 2021. Web. 26 Aug. 2021.

[2] Wikipedia contributors. "Cokurtosis." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 19 Jul. 2019. Web. 26 Aug. 2021.

[3] Stasinopoulos D. M. Rigby R.A. (2007) Generalized additive models for location scale and shape (GAMLSS) in R. Journal of Statistical Software, Vol. 23, Issue 7, Dec 2007, http://www.jstatsoft.org/v23/i07.

[4] Wikipedia contributors. "Hurst exponent." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 7 May. 2021. Web. 26 Aug. 2021.

[5] Wikipedia contributors. "Beta distribution." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 8 Aug. 2021. Web. 26 Aug. 2021.

[6] Wikipedia contributors. "Generalized beta distribution." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 14 Jan. 2021. Web. 8 Aug. 2021.

[900] Cromwell, JB (In Review) Financial Econometrics in the R Language for Pharmaceuticals: A Notebook of Articles for the Classroom Volume 1 Unpublished.

[1000] R Core Team (2015). R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria. URL https://www.R-project.org/.

[1001] Brian G. Peterson and Peter Carl (2018). PerformanceAnalytics: Econometric Tools for Performance and Risk Analysis. R package version 1.5.2. https://CRAN.R-project.org/package=PerformanceAnalytics

[1002] Martin Becker and Stefan Klößner (2017). PearsonDS: Pearson Distribution System. R package version 1.1. https://CRAN.R-project.org/package=PearsonDS

[1003] Antonio, Fabio Di Narzo, Jose Luis Aznarte and Matthieu Stigler (2009). tsDyn: Time series analysis based on dynamical systems theory

[1004] Matthieu Stigler (2010). tsDyn: Threshold cointegration: overview and implementation in R