The J_1 Mathematical Constant is a Transcendental Number

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1 Abstract

Transcendental number as mathematical constants include both Euler e and π used in mathematical biological modeling. Here a new mathematical constant is presented that is transcendental presented in three theorems. For $n = 2$ and $s = \frac{1}{6}$ then J_1 is transcendental.

2 Introduction

Mathematical constants play an important role in mathematical description and computation. [1][2][3][4][5][6][7][8]. Transcendental number as mathematical constants include both Euler e and π used in mathematical biological modeling. Here a new mathematical constant J_1 is presented that is transcendental.[1] [2] [3]

Let $s \in (\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6})$ and

$$\Gamma(n + 6/6) = n! \tag{1}$$

$$\Gamma(n + s_i) = k_i \tag{2}$$

where k_i is a transcendental number and algebraically dependent of π for any integer $n > 0$. [3] Consider a geometric series defined by [8]

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \tag{3}$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} \tag{4}$$

$$\frac{1}{(1-x)^3} = \sum_{n=2}^{\infty} \frac{(n-1)n}{2} x^{n-2}. \tag{5}$$

$$(1+x)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \tag{6}$$

Consider the arc length of a curve defined by

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{7}$$

$$L(f) = \lim_{N\to\infty} \sum_{i=1}^{N} \left| f(t_i) - f(t_{i-1}) \right| \tag{8}$$

$$\lim_{N\to\infty} \sum_{i=1}^{N} \left| f(t_i) - f(t_{i-1}) \right| \tag{9}$$

$$= \lim_{N\to\infty} \sum_{i=1}^{N} \left| \frac{f(t_i) - f(t_{i-1})}{\Delta t} \right| \Delta t \tag{10}$$

$$= \int_a^b \left| f'(t) \right| dt. \tag{11}$$

$$\arctan(x) = \int_0^x \frac{1}{z^2+1}\, dz\,, \tag{12}$$

$$\arctan(z) = i \sum_{n=1}^{\infty} \frac{1}{2n-1} \tag{13}$$

$$\left(\frac{1}{(1+2i/z)^{2n-1}} - \frac{1}{(1-2i/z)^{2n-1}} \right) \tag{14}$$

$$\pi = 20 \arctan \frac{1}{7} + 8 \arctan \frac{3}{79} \tag{15}$$

$$\pi = \sum_{n=1}^{\infty} \frac{n 2^n n!^2}{(2n)!} - 3 \tag{16}$$

$$\tag{17}$$

where $t_i = a + i(b-a)/N = a + i\Delta t$ for $i = 0, 1, \ldots, N$. Consider $\arctan \frac{1}{x} = \arctan \frac{1}{x+y} + \arctan \frac{y}{x^2+xy+1}$ with

$$\arctan x + \arctan \frac{1}{x} = \begin{cases} \frac{\pi}{2}, & \text{if } x > 0 \\ -\frac{\pi}{2}, & \text{if } x < 0 \end{cases} \tag{18}$$

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$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \tag{19}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad \text{for all } x \tag{20}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \tag{21}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \quad \text{for all } x \tag{22}$$

$$\tan x = \sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n (1-4^n)}{(2n)!} x^{2n-1} \tag{23}$$

$$= x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \quad \text{for } |x| < \frac{\pi}{2} \tag{24}$$

$$\sec x = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n} \tag{25}$$

$$= 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \cdots \quad \text{for } |x| < \frac{\pi}{2} \tag{26}$$

$$\arcsin x = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} \tag{27}$$

$$= x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots \quad \text{for } |x| \le 1 \tag{28}$$

$$\arccos x = \frac{\pi}{2} - \arcsin x \tag{29}$$

$$= \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} \tag{30}$$

$$= \frac{\pi}{2} - x - \frac{x^3}{6} - \frac{3x^5}{40} - \cdots \quad \text{for } |x| \le 1 \tag{31}$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \tag{32}$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \quad \text{for } |x| \le 1,\ x \ne \pm i \tag{33}$$

This definition is equivalent to the standard definition of arc length as an integral:

Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be the equation of an ellipse where the bounds on the circumference is given by

$$2\pi b \le C \le 2\pi a, \tag{34}$$

$$\pi(a+b) \le C \le 4(a+b), \tag{35}$$

$$4\sqrt{a^2+b^2} \le C \le \pi\sqrt{2(a^2+b^2)} \tag{36}$$

$$\pi = \frac{C}{d} \tag{37}$$

$$\int_{-\sqrt{2}/2}^{\sqrt{2}/2} \frac{1}{\sqrt{1-x^2}}\, dx \tag{38}$$

$$\pi = 2\Big[\arcsin x\Big]_{-\sqrt{2}/2}^{\sqrt{2}/2} \tag{39}$$

$$\tag{40}$$

where d is the diameter of the circle.

For number distribution of labels then the multinomial coefficients are from the binomial coefficients. Consider a number distribution n_i on a set of N total items, n_i is the number of items to be given the label i, then the number of choices at each step is given by [10]

$$\binom{N}{n_1}\binom{N-n_1}{n_2}\binom{N-n_1-n_2}{n_3}\cdots \tag{41}$$

$$= \frac{N!}{(N-n_1)!n_1!}\cdot \tag{42}$$

$$\frac{(N-n_1)!}{(N-n_1-n_2)!n_2!}\cdot \tag{43}$$

$$\frac{(N-n_1-n_2)!}{(N-n_1-n_2-n_3)!n_3!}\cdots. \tag{44}$$

The derivative of the polynomial $P = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0 = \sum_{i=0}^{n} a_i x^i$ with respect to x is the polynomial [11] $n a_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + 2a_2x + a_1 = \sum_{i=1}^{n} a_i \cdot i \cdot x^{i-1}$.

Similarly, the general indefinite integral of P is

$$\frac{a_n x^{n+1}}{n+1} + \frac{a_{n-1}x^n}{n} + \ldots + \frac{a_2x^3}{3} + \tag{45}$$

$$\frac{a_1x^2}{2} + a_0x + c \tag{46}$$

$$= c + \sum_{i=0}^{n} \frac{a_i x^{i+1}}{i+1} \tag{47}$$

where c is an arbitrary constant. Figure One has the R code listing for the power rule with the Caputo derivative. [9]

```
FD<-function(a,p){
  #a<-0.25
  #p = c(3, 2, 1);
  n=length(p)
  s=0
  s1<-NULL;j=1
  for(k in seq(1,2,0.01))
  {
    for(i in n:1){s=s+(gamma(k+1)/gamma(k-a+1))*p[i]*(k^(i-a))}
    s1[j]=s
    s=0
    j=j+1
  }
  return(s1)
}
```

3 J_1 constant and algebraic dependence

A subset S of a field L is algebraically independent over a subfield K if the elements of S do not satisfy any non-trivial polynomial equation with coefficients in K. A one element set α is algebraically independent over K if and only if α is transcendental over K. In general, all the elements of an algebraically independent set S over K are by necessity transcendental over K, and over all of the field extensions over K generated by the remaining elements of S. The numbers π and e^{π}, and $\Gamma(1/4)$, $e^{\pi\sqrt{3}}$, and $\Gamma(1/3)$ are algebraically independent over Q. For all positive integers n, π and $e^{\pi\sqrt{n}}$ are algebraically independent over Q. [5]

Consider the real value map defined on the interval [a,b] such that interval is tagged partitioned with $t_i \in [x_i, x_{i+1}]$ for $a = x_0 < x_1 < x_2 < ... < x_n = b$.

Theorem 3.1. Let $\pi = 2\int_{-1}^{1} \sqrt{1-x^2}\, dx$
where $\sum_{i=0}^{n-1} f(t_i)(x_{i+1} - x_i)$

Proof: Consider

$$f(x) = x^r \tag{48}$$

$$\int f(x)\, dx = \frac{x^{r+1}}{r+1} + C \tag{49}$$

$$I = \int \ln(x) \cdot 1\ dx \tag{50}$$

$$u = \ln(x) \ \Rightarrow\ du = \frac{dx}{x} \tag{51}$$

$$dv = dx \ \Rightarrow\ v = x \tag{52}$$

$$\int \ln(x)\ dx = x\ln(x) - \int \frac{x}{x}\ dx \tag{53}$$

$$= x\ln(x) - \int 1\ dx \tag{54}$$

$$= x\ln(x) - x + C \tag{55}$$

$$\tag{56}$$

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)dt\right) = \tag{57}$$

$$= f\big(x, b(x)\big) \cdot \frac{d}{dx} b(x) - f\big(x, a(x)\big) \cdot \tag{58}$$

$$\frac{d}{dx} a(x) + \tag{59}$$

$$\int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt \tag{60}$$

$$\varphi(\alpha) = \frac{x^2 - \alpha^2}{(x^2 + \alpha^2)^2} \tag{61}$$

$$\frac{d}{d\alpha}\varphi(\alpha) = \int_0^1 \frac{\partial}{\partial\alpha}\left(\frac{\alpha}{x^2+\alpha^2}\right)\ dx \tag{62}$$

$$= \int_0^1 \frac{x^2 - \alpha^2}{(x^2+\alpha^2)^2} dx \tag{63}$$

$$= -\frac{x}{x^2+\alpha^2}\bigg|_0^1 = -\frac{1}{1+\alpha^2} \tag{64}$$

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 4\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots\right) \tag{65}$$

$$= \cfrac{4}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \ddots}}}} \tag{66}$$

$$= \frac{2\sqrt{3}}{\sum_{n=0}^{\infty} \frac{(8n+1)\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3 9^n}} \tag{67}$$

$$= \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\ dx + \frac{22}{7} \tag{68}$$

where $S = \pi$ and $S^{\frac{1}{2}}$ is given by

$$a_n = \frac{S - x_n^2}{2x_n}, \tag{69}$$

$$b_n = x_n + a_n, \tag{70}$$

$$x_{n+1} = b_n - \frac{a_n^2}{2b_n} = (x_n + a_n) - \frac{a_n^2}{2(x_n + a_n)}. \tag{71}$$

and $\sqrt{S} \approx N + \frac{d}{2N} - \frac{d^2}{8N^3+4Nd} = \frac{8N^4+8N^2d+d^2}{8N^3+4Nd} = \frac{N^4+6N^2S+S^2}{4N^3+4NS} = \frac{N^2(N^2+6S)+S^2}{4N(N^2+S)}$ where N^2 is close to S and $d = S - N^2$.

Theorem 3.2. Let $n = 2$ and $s = \frac{1}{6}$ then J_1 is transcendental. [3]

Proof:

$$\Gamma(\frac{1}{2} + 0) = \pi^{\frac{1}{2}} \tag{72}$$

$$\Gamma(\frac{1}{4} + \frac{1}{4}) = \pi^{\frac{1}{2}} \tag{73}$$

$$\Gamma((\frac{1}{4} + \frac{1}{4}) + n) = \frac{(2n)!}{((2^2)^n)n!}\pi^{\frac{1}{2}} \tag{74}$$

$$\Gamma((\frac{1}{4} + \frac{1}{4}) - n) = \frac{(-2^2)^n n!}{(2^n)!}\pi^{\frac{1}{2}} \tag{75}$$

$$\Gamma((\frac{1}{4} + \frac{1}{4} + 0) + n) = \frac{(2n)!}{((2^2)^n)n!}\pi^{\frac{1}{2}} \tag{76}$$

$$\tag{77}$$

Let $k = (\frac{1}{4} + \frac{1}{4} + 0)$ Since $k \neq s \in (\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6})$. Let $n = 2$, then

$$\Gamma(\frac{1}{4} + \frac{1}{4} + \frac{1}{6} + \frac{11}{6}) = \frac{(2*2)!}{((2^2)^2)2!}\pi^{\frac{1}{2}} \tag{78}$$

$$\Gamma((\frac{1}{4} + \frac{1}{4} + \frac{11}{6}) + \frac{1}{6}) = \frac{(2*2)!}{((2^2)^2)2!}\pi^{\frac{1}{2}} \tag{79}$$

$$\Gamma((\frac{3}{12} + \frac{3}{12} + \frac{22}{12}) + \frac{1}{6}) = \frac{(2*2)!}{((2^2)^2)2!}\pi^{\frac{1}{2}} \tag{80}$$

$$\Gamma(\frac{28}{12} + \frac{1}{6}) = \frac{(2*2)!}{((2^2)^2)2!}\pi^{\frac{1}{2}} \tag{81}$$

$$\tag{82}$$

Since $a = \frac{24}{12} < \frac{28}{12} < b = \frac{36}{12}$ Since a and b both generate transcendental numbers, let $0 \leq \left\{\frac{28}{12}\right\} \leq 1 - \frac{1}{|12|}$ as $lim_{n->\infty} = 0$, then

$$\Gamma((1-\frac{1}{|n|})+\frac{1}{6})=\frac{(2*2)!}{((2^2)^2)2!}\pi^{\frac{1}{2}} \tag{83}$$

$$\tag{84}$$

then $1-\frac{1}{|n|}=1$ and $\frac{(2*2)!}{((2^2)^2)2!}\pi^{\frac{1}{2}}$ is transcendental.

Theorem 3.3. Consider

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(n+1)}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots. \tag{85}$$

$$\Gamma(n+\frac{1}{6})^x\approx\sum_{n=0}^{\infty}(1)^n\frac{1}{(n+2)\Gamma(n+1)}\frac{x^n}{g(n)} \tag{86}$$

$$\leq\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(n+1)}\qquad n<3 \text{ and } x=1 \tag{87}$$

$$\arctan(x)=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots \tag{88}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1};\qquad |x|\leq 1\qquad x\neq i,-i \tag{89}$$

$$\tag{90}$$

then $(1+0+x)e^x=\sum_{n=0}^{\infty}\frac{n+1+0}{n!}x^n$. where $n!=\Gamma(n+1)=D^n x^n=\frac{d^n}{dx^n}x^n$

Proof: Consider

$$\sum_{n=0}^{\infty}\frac{1}{\Gamma(n+1)}=\frac{1}{1}+\frac{1}{1}+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+\cdots=e \tag{91}$$

$$\sum_{n=0}^{\infty}\frac{1}{(n+2)\Gamma(n+1)}=\frac{1}{2}+\frac{1}{3}+\frac{1}{8}+\frac{1}{30}+\frac{1}{144}+\cdots=1 \tag{92}$$

$$\tag{93}$$

for $\Gamma(n+1)=n\Gamma(n)$ and $n!=\Gamma(n+1)$,

Let $\frac{a}{1-ax}=\sum_{n=0}^{\infty}(\frac{n!}{n!})^n x^n$ and $\frac{a}{(1-ax)^3}=\sum_{n=2}^{\infty}\frac{(n-1)n}{2}x^{n-2}$ where $a=1$ then

$$(1+0+x)e^x=1e^x+0e^x+xe^x \tag{94}$$

$$=\sum_{n=0}^{\infty}\frac{x^n}{n!}+0\sum_{n=0}^{\infty}\frac{x^n}{n!}+\sum_{n=0}^{\infty}\frac{x^{n+1}}{n!} \tag{95}$$

$$=1+\sum_{n=1}^{\infty}\frac{x^n}{n!}+0*(1+\sum_{n=1}^{\infty}\frac{x^n}{n!})+\sum_{n=0}^{\infty}\frac{x^{n+1}}{n!} \tag{96}$$

$$=1+ \tag{97}$$

$$\sum_{n=1}^{\infty}\frac{x^n}{n!} \tag{98}$$

$$+0*(1+\sum_{n=1}^{\infty}\frac{x^n}{(n-1)!}) \tag{99}$$

$$+\sum_{n=1}^{\infty}\frac{x^n}{(n-1)!} \tag{100}$$

$$=1+0*1+\sum_{n=1}^{\infty}\left(\frac{1}{n!}+0*\frac{1}{(n-1)!}+\frac{1}{(n-1)!}\right)x^n \tag{101}$$

$$=1+0*1+\sum_{n=1}^{\infty}\frac{n+1+0}{n!}x^n \tag{102}$$

$$=0*1+\sum_{n=0}^{\infty}\frac{n+1+0}{n!}x^n. \tag{103}$$

4 Conclusion

In this mathematical note, three theorems are presented with an examination of properties of a new mathematical constant for biological analysis. For $n=2$ and $s=\frac{1}{6}$ then J_1 is transcendental. [3]

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