A New Mathematical Constant and Cubic Splines for Nonlinear Difference Polynomials

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Work in Progress Expect Updates **Introduction:** Nonlinear Difference functionals can have cubic second order recurrences from rational polynomials relationships. Here in the specification of the convergence of the J_1 constant, a rational polynomial is specified from convergent ratios and Mclaurin series.

1. **Objective 1:** Specification of a the convergents of the continued fraction for the J constant.
2. **Objective 2:** Estimation of the functiional specification based on a cubic spline representation of the convergents.
3. **Objective 3:** Design a smooth functional over th interval a,b and b, to infinity for convergence comparision within some epsilon.

Conclusion: In this mathematical note, a rational function of cubic polynomial was developed for the comparision test for convergence.

Keywords: Continued fractions, cubic irrational numbers, stability theory, cubic polynomials, Fibonacci numbers, rate of convergence

1 Introduction

Consider the $\varphi : \mathbb{N} \times X \to X$ defined as a recurrence relation for almost random sequences with [1] [2] [3] [4] [5]

$$u_n = \varphi(n, u_{n-1}) \quad \text{for} \quad n > 0 \tag{1}$$

Let I and I^2be a transformation to polar coordinates makes evident that θ is uniformly distributed (constant density) from 0 to 2 π

$$I = \int_{-\infty}^{\infty} e^{-x^2/2}\, dx \tag{2}$$

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\, dx\, dy \tag{3}$$

$$= \int_0^{2\pi}\int_0^{\infty} re^{-r^2/2}\, dr\, d\theta. \tag{4}$$

If u is uniformly distributed in the interval$0 \le u < 1$, then the point (cos(2nu),sin(2nu)) is uniformly distributed on the unit circumference $x^2 + y^2 = 1$, and multiplying that point by an independent random variable ρ whose distribution is

$$\Pr(\rho < a) = \int_0^a re^{-r^2/2}\, dr \tag{5}$$

$$(\rho\cos(2\pi u), \rho\sin(2\pi u)) \tag{6}$$

whose coordinates are jointly distributed as two independent standard normal random variables. This is Marsaglia polar method for random number sampling method for generating a pair of independent standard normal random variables. [6]

The curvature of a curve $y = f(x)$ is given by:

$$\kappa = \frac{y''}{(1 + y'^2)^{3/2}} \tag{7}$$

$$f'(a) = \lim_{h\to 0}\frac{f(a+h) - f(a)}{h} = \lim_{h\to 0}\frac{(a+h)^2 - a^2}{h} \tag{8}$$

$$= \lim_{h\to 0}\frac{a^2 + 2ah + h^2 - a^2}{h} = \lim_{h\to 0}\frac{2ah + h^2}{h} \tag{9}$$

$$= \lim_{h\to 0}(2a + h) = 2a \tag{10}$$

$$f''(x) = \lim_{h\to 0}\frac{f(x+h) - 2f(x) + f(x-h)}{h^2}. \tag{11}$$

Fits a piecewise interpolating or smoothing cubic polynomial to the x and y values. To make the solution well posed the the second and third derivatives are set to zero at the limits of the x values. Extrapolation outside the range of the x values will be a linear function. [7] [8]

Spline interpolation is often preferred over polynomial interpolation because the interpolation error can be made small even when using low degree polynomials for the spline. Spline interpolation avoids the problem of Runge's phenomenon, in which oscillation can occur between points when interpolating using high degree polynomials. [7] [8]

It is assumed that there are no repeated x values; use sreg followed by predict if you do have replicated data. Partial convergents of continued fractions or generalized continued fractions [7] [8]

2 Cubic Polynomials

A third-order polynomial $q(x)$ for which

$$q(x_1) = y_1 \tag{12}$$

$$q(x_2) = y_2 \tag{13}$$

$$q'(x_1) = k_1 \tag{14}$$

$$q'(x_2) = k_2 \tag{15}$$

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can be written in the symmetrical form

$$q(x) = \big(1 - t(x)\big)\, y_1 + t(x)\, y_2 + \tag{16}$$

$$t(x)\big(1 - t(x)\big)\Big(\big(1 - t(x)\big)\, a + t(x)\, b\Big) \tag{17}$$

where

$$t(x) = \frac{x - x_1}{x_2 - x_1} \tag{18}$$

$$a = k_1(x_2 - x_1) - (y_2 - y_1) \tag{19}$$

$$b = -k_2(x_2 - x_1) + (y_2 - y_1) \tag{20}$$

As

$$q' = \frac{dq}{dx} = \frac{dq}{dt}\frac{dt}{dx} = \frac{dq}{dt}\frac{1}{x_2 - x_1} \tag{21}$$

$$q = \frac{y_2 - y_1}{x_2 - x_1} + (1 - 2t)\frac{a(1 - t) + bt}{x_2 - x_1} + t(1 - t)\frac{b - a}{x_2 - x_1} \tag{22}$$

$$q' = 2\frac{b - 2a + (a - b)3t}{(x_2 - x_1)^2} \tag{23}$$

Setting $'t = 0$ and$t = 1$ respectively in equations the first derivatives $q(x1) = k1 and q(x2) = k2$ and also second derivatives

$$q''(x_1) = 2\frac{b - 2a}{(x_2 - x_1)^2} \tag{24}$$

$$q''(x_2) = 2\frac{a - 2b}{(x_2 - x_1)^2} \tag{25}$$

If now $''xi,'' yi''$ "i" $= 0$, 1, ..., "n" are $n + 1$ points and

$$q_i = (1 - t)\, y_{i-1} + t\, y_i + t(1 - t)\big((1 - t)\, a_i + t\, b_i\big) \tag{26}$$

where i = 1, 2, ..., "n" and $t = \frac{x - x_{i-1}}{x_i - x_{i-1}}$ are n third degree polynomials interpolating y in the interval $xi - 1 \le x \le xi$ for i = 1, ..., n such that $qi(xi) = qi + 1(xi)$ for i = 1, ..., n- 1 then the n polynomials together define a differentiable function in the interval $'x0 \le x \le'' xn$ and

$$a_i = k_{i-1}(x_i - x_{i-1}) - (y_i - y_{i-1}) \tag{27}$$

$$b_i = -k_i(x_i - x_{i-1}) + (y_i - y_{i-1}) \tag{28}$$

for "i" = 1, ..., "n" where

$$k_0 = q_1'(x_0) \tag{29}$$

$$k_i = q_i'(x_i) = q_{i+1}'(x_i) \qquad i = 1, \ldots, n - 1 \tag{30}$$

$$k_n = q_n'(x_n) \tag{31}$$

If the sequence $k0, k1, ..., kn$ is such that, in addition, $qi(xi) = qi + 1(xi)$ holds for i = 1, ..., n-1, then the resulting function will even have a continuous second derivative.

From follows that this is the case if and only if

$$\frac{k_{i-1}}{x_i - x_{i-1}} + \left(\frac{1}{x_i - x_{i-1}} + \frac{1}{x_{i+1} - x_i}\right) 2k_i + \tag{32}$$

$$\frac{k_{i+1}}{x_{i+1} - x_i} = \tag{33}$$

$$3\left(\frac{y_i - y_{i-1}}{(x_i - x_{i-1})^2} + \frac{y_{i+1} - y_i}{(x_{i+1} - x_i)^2}\right) \tag{34}$$

for i = 1, ..., n-1. The relations are n- 1 linear equations for the n + 1 values $k0, k1, ..., kn$. For the elastic rulers being the model for the spline interpolation one has that to the left of the left-most "knot" and to the right of the right-most "knot" the ruler can move freely and will therefore take the form of a straight line with $q = 0$. As q should be a continuous function of x one gets that for "Natural Splines" one in addition to the $n - 1$ linear equations should have that [7] [8]

$$q_1''(x_0) = 2\frac{3(y_1 - y_0) - (k_1 + 2k_0)(x_1 - x_0)}{(x_1 - x_0)^2} = 0 \tag{35}$$

$$q_n''(x_n) = -2\frac{3(y_n - y_{n-1}) - (2k_n + k_{n-1})(x_n - x_{n-1})}{(x_n - x_{n-1})^2} = 0 \tag{36}$$

$$\frac{2}{x_1 - x_0}k_0 + \frac{1}{x_1 - x_0}k_1 = 3\frac{y_1 - y_0}{(x_1 - x_0)^2} \tag{37}$$

$$\frac{1}{x_n - x_{n-1}}k_{n-1} + \frac{2}{x_n - x_{n-1}}k_n = 3\frac{y_n - y_{n-1}}{(x_n - x_{n-1})^2}. \tag{38}$$

Eventually, together constitute n + 1 linear equations that uniquely define the n + 1 parameters $k_0, k_1, ..., k_n$. [7] [8]

There exist other end conditions: "Clamped spline", that specifies the slope at the ends of the spline, and the popular "not-a-knot spline", that requires that the third derivative is also continuous at the $x1$ and $xN - 1$ points. For the "not-a-knot" spline, the additional equations [7] [8]

$$q_1'''(x_1) = q_2'''(x_1) \Rightarrow \frac{1}{\Delta x_1^2}k_0 + \tag{39}$$

$$\left(\frac{1}{\Delta x_1^2} - \frac{1}{\Delta x_2^2}\right)k_1 - \frac{1}{\Delta x_2^2}k_2 \tag{40}$$

$$= 2\left(\frac{\Delta y_1}{\Delta x_1^3} - \frac{\Delta y_2}{\Delta x_2^3}\right) \tag{41}$$

$$q_{n-1}'''(x_{n-1}) = q_n'''(x_{n-1}) \Rightarrow \frac{1}{\Delta x_{n-1}^2}k_{n-2} + \tag{42}$$

$$\left(\frac{1}{\Delta x_{n-1}^2} - \frac{1}{\Delta x_n^2}\right)k_{n-1} - \frac{1}{\Delta x_n^2}k_n \tag{43}$$

$$= 2\left(\frac{\Delta y_{n-1}}{\Delta x_{n-1}^3} - \frac{\Delta y_n}{\Delta x_n^3}\right) \tag{44}$$

where $\Delta x_i = x_i - x_{i-1}, \Delta y_i = y_i - y_{i-1}$.

3 Results

Consider three points the values for k_0, k_1, k_2 are found by solving the Tridiagonal matrix given by

$$\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} k_0 \\ k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{45}$$

with

$$a_{11} = \frac{2}{x_1 - x_0} \tag{46}$$

$$a_{12} = \frac{1}{x_1 - x_0} \tag{47}$$

$$a_{21} = \frac{1}{x_1 - x_0} \tag{48}$$

$$a_{22} = 2\left(\frac{1}{x_1 - x_0} + \frac{1}{x_2 - x_1}\right) \tag{49}$$

$$a_{23} = \frac{1}{x_2 - x_1} \tag{50}$$

$$a_{32} = \frac{1}{x_2 - x_1} \tag{51}$$

$$a_{33} = \frac{2}{x_2 - x_1} \tag{52}$$

$$b_1 = 3\,\frac{y_1 - y_0}{(x_1 - x_0)^2} \tag{53}$$

$$b_2 = 3\left(\frac{y_1 - y_0}{(x_1 - x_0)^2} + \frac{y_2 - y_1}{(x_2 - x_1)^2}\right) \tag{54}$$

$$b_3 = 3\,\frac{y_2 - y_1}{(x_2 - x_1)^2} \tag{55}$$

for the convergents in Table One

n	A	B
1	2.00	1.00
2	2.69	0.85
3	2.94	1.29
4	3.73	1.31
5	4.67	1.90
6	6.43	2.40
7	10.52	4.09
8	14.71	5.62
9	24.30	9.35
10	18.70	7.15
11	30.16	11.59
12	27.74	10.63
13	45.90	17.62
14	29.21	11.19
15	56.13	21.54
16	65.73	25.20
17	106.93	41.02
18	75.08	28.79
19	120.30	46.14
20	151.85	58.24
21	247.85	95.06

Figure One has the result of a cubic spline prediction based on the first twenty Convergents for the Continued Fraction with the Uniform Distribution samples. Here x has the x values that define the curve or a two column matrix of x and y values. y values are paired with the x's. xgrid is the grid to evaluate the fitted cubic interpolating curve and the derivative indicates whether the function or a a first or second derivative should be evaluated. The wt has Weights for different obsrevations in the scale of reciprocal variance with lam the value for smoothing parameter. Default value is zero giving interpolation. Lambda and df are the effective degrees of freedom with lambda =0 or a df equal to the number of observations. the spline function consisting of the two cubic polynomials $q_1(x)$ and $q_2(x)$

4 Conclusions

In this mathematical note, a cubic spline was fitted to the first N convergents of a continued fraction for prediction of the cyclical behavior of the mathematical constant. The convergence with P/Q as an approximation has an integer representation with both floor and ceiling transformations.

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