A New Mathematical Constant and the Fibonacci Sequence: Properties and Directions

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Work in Progress Expect Updates **Introduction:** Fibonacci sequences appear in biological settings (a) tree branching, (b) leaf arrangement, (c) pineapple fruitlets , (d) artichoke flowering, (e) fern uncurling and (f) pine cone arrangement.

1. **Objective 1:**The plastic number can be written using the hyperbolic cosine and its inverse from the roots of $x^3 - x - 1$
2. **Objective 2:** Cubic and quadratic irrational numbers occur frequenty in the roots of characteristic polynomials for the stability classification. Here the J_1 constant is presented as cubic irrational with periodic behavior with a pattern of 3n in the 2 place of the cycle.
3. **Objective 3:**

Conclusion: $J^a = 2.780776$ with n =100. Decimal representations of quadratic irrationals are apparently random. The square roots of all (positive) integers, that are not perfect squares, are quadratic irrationals, hence are unique periodic continued fractions.

Keywords: Continued fractions, cubic irrational numbers, stability theory, cubic polynomials, Fibonacci numbers, rate of convergence

1 Introduction

Fibonacci sequences appear in biological settings (a) tree branching, (b) leaf arrangement, (c) pineapple fruitlets , (d) artichoke flowering, (e) fern uncurling and (f) pine cone arrangement. [1]

An order-d homogeneous linear recurrence with constant coefficients is an equation of the form

$$s(n) = c_1 s(n-1) + c_2 s(n-2) + \cdots + c_d s(n-d), \tag{1}$$

where the "d" coefficients $c_1, c_2, \dots, c_d$ are constants. A sequence $s(0), s(1), s(2), \dots$ is a "'constant-recursive sequence"' if there is an order-"d" homogeneous linear recurrence with constant coefficients that it satisfies for all $n \geq d$. The geometric sequence $a, ar, ar^2, \dots$ is constant-recursive, since it satisfies the recurrence $s(n) = rs(n-1)$ for all $n \geq 1$. A sequence that is eventually periodic with period lengthℓ is constant-recursive, since it satisfies $s(n) = s(n-\ell)$for all $n \geq d$ for some d. [1]

The characteristic polynomial, i.e. auxiliary polynomial of the recurrence is the polynomial

$$x^d - c_1 x^{d-1} - \cdots - c_{d-1} x - c_d \tag{2}$$

whose coefficients are the same as those of the recurrence. The nth term s(n) of a constant-recursive sequence can be written in terms of the roots of its characteristic polynomial. If the d roots $r_1, r_2, \dots, r_d$ are all distinct, then the "n"th term of the sequence is [1]

$$s(n) = k_1 r_1^n + k_2 r_2^n + \cdots + k_d r_d^n \tag{3}$$

where the coefficients "ki' are constants that can be determined from the initial conditions. [1] The Fibonacci numbers occur in the sums of "shallow" diagonals in Pascal's triangle with binomial coefficients [1]

$$F_n = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-k-1}{k} \tag{4}$$

with

$$F_n = \frac{\varphi^n - \psi^n}{\varphi - \psi} = \frac{\varphi^n - \psi^n}{\sqrt{5}} \tag{5}$$

where $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.61803$ and the nth position in the Fibonacci sequence is given by

$$n = \log_\varphi \left(\frac{F_n\sqrt{5} + \sqrt{5F_n^2 \pm 4}}{2} \right) \tag{6}$$

The differential equation

$$ay'' + by' + cy = 0 \tag{7}$$

has solution

$$y = e^{ax}. \tag{8}$$

The conversion of the differential equation to a difference equation of the Taylor coefficients is

$$af[n+2] + bf[n+1] + cf[n] = 0. \tag{9}$$

It is easy to see that the nth derivative of e^{ax} evaluated at 0 is an. Consider

$$w_{t+1} = \frac{aw_t + b}{cw_t + d}. \tag{10}$$

with solution for$ad - bc \neq 0$ is

$$y_{t+1} = \alpha - \frac{\beta}{y_t} \tag{11}$$

where $\alpha = (a+d)/c$ and $\beta = (ad-bc)/c^2$and where $w_t = y_t - d/c$. Furthermore $y_t = x_{t+1}/x_t$ then

$$x_{t+2} - \alpha x_{t+1} + \beta x_t = 0. \tag{12}$$

In the solution equation

$$x_t = c_1 \lambda_1^t + \cdots + c_n \lambda_n^t \tag{13}$$

If the absolute value of the characteristic root is less than 1, a term with real characteristic roots converges to 0 as t grows indefinitely large . If the absolute value equals 1, the term remains constant as t grows if the root is +1 yet if the root is minus 1 fluctuation between two values. If the absolute value of the root is greater than 1 the term will increase over time. If the absolute value of the modulus M of the roots is less than 1,a pair of terms with complex conjugate characteristic roots will

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converge to 0 with dampening fluctuations. If the modulus equals 1 then constant amplitude fluctuations in the combined terms will persist and if the modulus is greater than 1, the combined terms have fluctuations of ever-increasing magnitude. Therefore, the evolving variable x will converge to 0 if all of the characteristic roots have magnitude less than 1. [1]

If the largest root has absolute value 1, neither convergence to 0 nor divergence to infinity will occur. If all roots with magnitude 1 are real and positive, x will converge to the sum of their constant terms $c - i$, different from the stable case, this converged value has initial condition dependence and different starting points that lead to different points in the long run. If any root is minus 1, term contributes permanent fluctuations between two values. If any of the unit-magnitude roots are complex then constant-amplitude fluctuations of x will persist. [1] If any characteristic root has magnitude greater than 1, then x will diverge to infinity as time goes to infinity, or have fluctuations between increasingly large positive and negative values. [1]

Table One has the order n + 1 square Fibonacci matrix which is derived from a Fibonacci sequence.

	1	2	3	4	5	6	7	8	9	10	11
1	1.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2	2.00	1.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
3	3.00	2.00	1.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
4	5.00	3.00	2.00	1.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00
5	8.00	5.00	3.00	2.00	1.00	1.00	0.00	0.00	0.00	0.00	0.00
6	13.00	8.00	5.00	3.00	2.00	1.00	1.00	0.00	0.00	0.00	0.00
7	21.00	13.00	8.00	5.00	3.00	2.00	1.00	1.00	0.00	0.00	0.00
8	34.00	21.00	13.00	8.00	5.00	3.00	2.00	1.00	1.00	0.00	0.00
9	55.00	34.00	21.00	13.00	8.00	5.00	3.00	2.00	1.00	1.00	0.00
10	89.00	55.00	34.00	21.00	13.00	8.00	5.00	3.00	2.00	1.00	1.00
11	144.00	89.00	55.00	34.00	21.00	13.00	8.00	5.00	3.00	2.00	1.00

For the Fibonacci sequence, the characteristic polynomial is $x^2 - x - 1$, whose roots $\phi = \frac{1+\sqrt{5}}{2}$ and $\bar{\phi} = \frac{1-\sqrt{5}}{2}$ appear in Binet's formula

$$F_n = \frac{\phi^n - \bar{\phi}^n}{\sqrt{5}}. \tag{14}$$

$$F_{kn+c} = \sum_{i=0}^{k} \binom{k}{i} F_{c-i} F_n^i F_{n+1}^{k-i} \tag{15}$$

or alternatively

$$F_{kn+c} = \sum_{i=0}^{k} \binom{k}{i} F_{c+i} F_n^i F_{n-1}^{k-i} \tag{16}$$

A sequence is constant-recursive precisely when its generating function

$$\sum_{n\geq 0} s(n)x^n = s(0) + s(1)x^1 + s(2)x^2 + s(3)x^3 + \cdots \tag{17}$$

is a rational function. The denominator is the polynomial obtained from the auxiliary polynomial by reversing the order of the coefficients, and the numerator is determined by the initial values of the sequence. The generating function of the Fibonacci sequence is

$$\frac{x}{1 - x - x^2}. \tag{18}$$

In general, multiplying a generating function by the polynomial

$$1 - c_1x^1 - c_2x^2 - \cdots - c_dx^d \tag{19}$$

yields a series

$$\left(s(0) + s(1)x^1 + s(2)x^2 + \cdots\right)\left(1 - c_1x^1 - c_2x^2 - \cdots - c_dx^d\right) \tag{20}$$

$$= \left(b_0 + b_1x^1 + b_2x^2 + \cdots\right), \tag{21}$$

where

$$b_n = s(n) - c_1s(n-1) - c_2s(n-2) - \cdots - c_ds(n-d). \tag{22}$$

If , $s(n)$, satisfies the recurrence relation

$$s(n) = c_1s(n-1) + c_2s(n-2) + \cdots + c_ds(n-d) \tag{23}$$

then , $b_n = 0$ for all , $n \geq d$,.

In other words,

$$\left(s(0) + s(1)x^1 + s(2)x^2 + \cdots\right)\left(1 - c_1x^1 - c_2x^2 - \cdots - c_dx^d\right) = \tag{24}$$

$$\left(b_0 + b_1x^1 + b_2x^2 + \cdots + b_{d-1}x^{d-1}\right) \tag{25}$$

therefore the rational function is

$$\sum_{n\geq 0} s(n)x^n = \frac{b_0 + b_1x^1 + b_2x^2 + \cdots + b_{d-1}x^{d-1}}{1 - c_1x^1 - c_2x^2 - \cdots - c_dx^d}. \tag{26}$$

2 J_1 Mathematical Constant

Consider the number J_1 such that [1]

$$(x+y)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^{r-k} y^k \tag{27}$$

$$= x^r + rx^{r-1}y + \frac{r(r-1)}{2!}x^{r-2}y^2 \tag{28}$$

$$+ \frac{r(r-1)(r-2)}{3!}x^{r-3}y^3 + \cdots. \tag{29}$$

and in more dimensions, it is often useful to deal with products of binomial expressions. By the binomial theorem this is equal to

$$(x_1+y_1)^{n_1} \cdots (x_d+y_d)^{n_d} = \tag{30}$$

$$\sum_{k_1=0}^{n_1} \cdots \sum_{k_d=0}^{n_d} \binom{n_1}{k_1} x_1^{k_1} y_1^{n_1-k_1} \ldots \tag{31}$$

$$\binom{n_d}{k_d} x_d^{k_d} y_d^{n_d-k_d}. \tag{32}$$

Let e be a number such that

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \tag{33}$$

Applying the binomial theorem to this expression yields the usual infinite series for e. In particular:

$$\left(1+\frac{1}{n}\right)^n = 1+\binom{n}{1}\frac{1}{n}+\binom{n}{2}\frac{1}{n^2} \tag{34}$$

$$+\binom{n}{3}\frac{1}{n^3}+\cdots+\binom{n}{n}\frac{1}{n^n}. \tag{35}$$

The kth term of this sum is

$$\binom{n}{k}\frac{1}{n^k}=\frac{1}{k!}\cdot\frac{n(n-1)(n-2)\cdots(n-k+1)}{n^k} \tag{36}$$

As n goes to ∞ the rational expression on the right approaches 1, and therefore

$$\lim_{n\to\infty}\binom{n}{k}\frac{1}{n^k}=\frac{1}{k!}. \tag{37}$$

This indicates that e can be written as a series:

$$e=\sum_{k=0}^{\infty}\frac{1}{k!}=\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots \tag{38}$$

Indeed, since each term of the binomial expansion is an increasing function of n, it follows from the monotone convergence theorem for series that the sum of this infinite series is equal to e. [1] This may be written more concisely, by multi-index notation as

$$(x+y)^\alpha=\sum_{\nu\le\alpha}\binom{\alpha}{\nu}x^\nu y^{\alpha-\nu}. \tag{39}$$

with

$$(1+x)^{-1}=\frac{1}{1+x}=1-x+x^2-x^3+x^4-x^5+\cdots \tag{40}$$

3 Algorithm

Consider Fibonacci Search for a univariate function minimum in a bounded interval presented in Algorithm 1.

Algorithm 1 Fibonacci Search for a univariate function minimum in a bounded interval.

F = c(1, 2); n = 2

while (F[n] $\le$ 2 * (b - a)/tol) ;

$$F[n+1]=F[n]+F[n-1]; n=n+1 \tag{41}$$
$$x1=a \tag{42}$$
$$x2=b \tag{43}$$
$$xa=a+(b-a)*F[n-2]/F[n] \tag{44}$$
$$fxa=f(xa,\ldots) \tag{45}$$
$$xb=a+(b-a)*F[n-1]/F[n] \tag{46}$$
$$fxb=f(xb,\ldots) \tag{47}$$
$$k=1 \tag{48}$$

while (k $\le$n - 3 and xa ¡ xb and (x2 - x1) $\ge$ tol)

$$if(fxa>fxb) \tag{49}$$
$$x1=xa \tag{50}$$
$$xa=xb \tag{51}$$
$$xb=x1+(x2-x1)*F[n-k-1]/F[n-k] \tag{52}$$
$$fxa=fxb \tag{53}$$
$$fxb=f(xb,\ldots) \tag{54}$$

else

$$x2=xb \tag{55}$$
$$xb=xa \tag{56}$$
$$xa=x1+(x2-x1)*F[n-k-2]/F[n-k] \tag{57}$$
$$fxb=fxa \tag{58}$$
$$fxa=f(xa,\ldots) \tag{59}$$

k = k + 1

xmin = (xa + xb)/2

fmin = f(xmin, ...)

if (endp) fa = f(a, ...);fb = f(b, ...)

if (xmin - a ¡ tol and fa ¡ fmin) xmin = a;fmin = fa

else if (b - xmin ¡ tol and fb ¡ fmin) xmin = b;fmin = fb

estim.prec = max(xmin - x1, x2 - xmin

return(list(xmin = xmin, fmin = fmin, niter = k, estim.prec = estim.prec))

4 Conclusion

In this brief mathematical note the J_1 mathematical constant is defined as an alternative to oth constants for classication of properties for cubic polynomials. Here the continued fraction is specified with convergents and its relationship to (a) super golden ratio, (b) plastic number and the (c) transcendental e to the Fibonacci sequence.

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