A New Mathematical Constant with Continued Fraction Representation and Theorems

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Work in Progress Expect Updates The continued fraction representation of an irrational number is unique are present as cubic and quadratic irrational numbers in the roots of characteristic polynomials for the stability classification. The J_1 mathematical constant is presented as cubic irrational with periodic behavior with a pattern of 3n in the 2 place of the cycle. Here $J^a = 2.780776$ with n =100 and a=1. Decimal representations of quadratic irrationals are apparently random. The square roots of all (positive) integers, that are not perfect squares, are quadratic irrationals, hence are unique periodic continued fractions. Several theorems are presented with respect to the continued fraction representation.

Keywords: Continued fractions, cubic irrational numbers, stability theory, cubic polynomials, Fibonacci numbers, rate of convergence

1 Introduction

Every infinite continued fraction is irrational, and every irrational number has an infinite continued fraction where its initial segments provide rational approximations, i.e. convergents of the continued fraction. The larger a term is in the continued fraction, the closer the corresponding convergent is to the irrational number being approximated. Numbers like n have occasional large terms in their continued fraction with an easy approximate with rational numbers. Other numbers like e have only small terms early in their continued fraction and makes it more difficult to approximate rationally. [1]

The plastic number can be written using the hyperbolic cosine and its inverse from the roots of $x^3 - x - 1$. Cubic and quadratic irrational numbers occur frequenty in the roots of characteristic polynomials for the stability classification. Here the J_1 constant is presented as cubic irrational with periodic behavior with a pattern of 3n in the 2 place of the cycle. [1]

Convergent continued fraction maps the entire extended complex plane into a single point and when an infinite continued fraction converges, the corresponding sequence T_n of LFTs focuses the plane in the direction of x, the value of the continued fraction. At each stage of the process an increasingly larger region of the plane is mapped into a neighborhood of x, and the smaller and smaller region of the plane that's left over is stretched out ever more thinly to cover everything outside that neighborhood. [1]

Consider that $J^a = 2.780776$ with n =100 and a=1. Decimal representations of quadratic irrationals are apparently random and the square roots of all positiveintegers, that are not perfect squares, are quadratic irrationals, hence are unique periodic continued fractions. [1]

The continued fraction representation of an irrational number is unique are present as cubic and quadratic irrational numbers in the roots of characteristic polynomials for the stability classification. For example, a quadratic irrational number is an irrational real root of the quadratic equation [1]

$$ax^2 + bx + c = 0 \tag{1}$$

where the coefficients a,b, and c are integers, and the discriminant b2 minus 4ac, is greater than zero. By the quadratic formula every quadratic irrational can be written in the form [1]

$$\zeta = \frac{P + \sqrt{D}}{Q} \tag{2}$$

where P, D, and Q are integers, $D > 0$ is not a perfect square and Q divides the quantity P2 minus D. if x is a regular periodic continued fraction, then x is a quadratic irrational number. The continued fraction expansion in canonical form (S is any natural number that is not a perfect square): [1]

$$m_0 = 0 \tag{3}$$
$$d_0 = 1 \tag{4}$$
$$a_0 = \left\lfloor \sqrt{S} \right\rfloor \tag{5}$$
$$m_{n+1} = d_n a_n - m_n \tag{6}$$
$$d_{n+1} = \frac{S - m_{n+1}^2}{d_n} \tag{7}$$
$$a_{n+1} = \left\lfloor \frac{\sqrt{S} + m_{n+1}}{d_{n+1}} \right\rfloor = \left\lfloor \frac{a_0 + m_{n+1}}{d_{n+1}} \right\rfloor. \tag{8}$$

Notice that mn, dn, and an are always integers. The supergolden ratio roots of f $x^3 - x^2 - 1$ is also the fourth smallest Pisot number and the plastic number as the hyperbolic cosine and its inverse from the roots of $x^3 - x - 1$ [1]

$$\rho = \frac{1}{c}\cosh\left(\tfrac{1}{3}\cosh^{-1}(3c)\right), \qquad c = \cos\left(\tfrac{2\pi}{12}\right) = \sin\left(\tfrac{2\pi}{6}\right) = \tfrac{\sqrt{3}}{2}. \tag{9}$$

The integers a_0, a_1 etc., are called the "coefficients" or "terms" of the continued fraction [1]

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3}}} \tag{10}$$

in the notation of Gauss

$$x = a_0 + \underset{i=1}{\overset{n}{\mathrm{K}}} \frac{1}{a_i} \tag{11}$$

or as $x = [a_0; a_1, a_2, a_3]$ with the Euler identity

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$$a_0 + a_0a_1 + a_0a_1a_2 + \cdots + a_0a_1a_2\cdots a_n = \tag{12}$$
$$\frac{a_0}{1-}\frac{a_1}{1+a_1-}\frac{a_2}{1+a_2-}\cdots \tag{13}$$
$$\frac{a_n}{1+a_n}. \tag{14}$$

Examples of continued fraction representations of irrational numbers are: [1]

1. e = [2;1,2,1,1,4,1,1,6,1,1,8,...]The pattern repeats indefinitely with a period of 3 except that 2 is added to one of the terms in each cycle.
2. π = [3;7,15,1,292,1,1,1,2,1,3,1,...]. No pattern has ever been found in this representation.
3. golden ratio ϕ = [1;1,1,1,1,1,1,1,1,1,1,1,...].
4. $2^{1/2}$= [1;2,2,2,2,2,2,2,2,2,2,2,...].
5. $3^{1/2}$= [1;1,2,1,2,1,2,1,2,1,2,1,...].

and the constant e representations are [1]

$$\mathrm{EG}(a_n;x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}. \tag{15}$$

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \tag{16}$$

$$= 1 + \cfrac{x}{1 - \cfrac{1x}{2 + x - \cfrac{2x}{3 + x - \cfrac{3x}{4 + x - \ddots}}}} \tag{17}$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots \tag{18}$$

for x in this disk, f is given by a convergent power series [1]

$$f(x) = \sum_{n=0}^{\infty} a_n(x-b)^n. \tag{19}$$

Differentiating by x the above formula n times, then setting $x = b$ gives:

$$\frac{f^{(n)}(b)}{n!} = a_n \tag{20}$$

and so the power series expansion agrees with the Taylor series. [1]

Then the partial fraction decomposition of f(x) is the following:

$$f(x) = \frac{p(x)}{q(x)} = P(x) + \sum_{i=1}^{m}\sum_{r=1}^{j_i} \frac{A_{ir}}{(x-a_i)^r} + \sum_{i=1}^{n}\sum_{r=1}^{k_i} \frac{B_{ir}x + C_{ir}}{(x^2 + b_ix + c_i)^r} \tag{21}$$

Here, $P(x)$ is a (possibly zero) polynomial, and the A_{ir}, B_{ir}, and C_{ir} are real constants. [1]

2 The J_1 Mathematical Constant

Based on a pattern repeats indefinitely with a period of k except that I is added to one of the terms in each cycle. [1]

1. $2^{1/2}$= [1;2,2,2,2,2,2,2,2,2,2,2,...].
2. 2
3. $3^{1/2}$= [1;1,2,1,2,1,2,1,2,1,2,1,...].
4. e = [2;1,2,1,1,4,1,1,6,1,1,8,...]
5. j= [2;1,1,k,1,1,k,1,1,k,1,1,k,,...]
6. 3

The Thue-Siegel-Roth theorem states the irrationality measure of any algebraic number is exactly 2 with examples of $2^{1/2}$ and $3^{1/2}$.Given an algebraic number, there is a unique monic polynomial, i.e. rational coefficients of least degree that has the number as a root. This polynomial is called its minimal polynomial. If its minimal polynomial has degree n, then the algebraic number is said to be of degree n. For example, all rational numbers have degree 1, and an algebraic number of degree 2 is a quadratic irrational. If the elements $a_n$of the continued fraction expansion of an irrational number x satisfy $a_n < cn + d$ for positive c and d, the irrationality measure $\mu(x) = 2$. [1]

Numbers like e have only small terms early in their continued fraction, which makes them more difficult to approximate rationally. The golden ratio ϕ has terms equal to 1 everywhere, the smallest values possibl, which makes ϕ the most difficult number to approximate rationally. In this sense, therefore, it is the "most irrational" of all irrational numbers. Even-numbered convergents are smaller than the original number, while odd-numbered ones are larger. [1]

Consider the following constant

$$J^a = [2;1,1,3,1,1,3,1,1,3,1,1,3,,...] \tag{22}$$

where $a = 1$ and $J^a = 2.780776$ [1000] with n =100 for rep(c(1,3,1),100). The geometric mean of a data set $\{a_1, a_2, \ldots, a_n\}$ is given by:

$$\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} = \sqrt[n]{a_1a_2\cdots a_n}. \tag{23}$$

is the Khinchin's constant of 2.68. Here the constant is 1.442407 for n =1000. For any value μ less than this upper bound, the infinite set of all rationals p/q satisfying the above inequality yield an approximation of x. Conversely, if$\mu'$is greater than the upper bound, then there are at most finitely many (p, q) with $q > 0$ that satisfy the inequality; thus, the opposite inequality holds for all larger values of "q". In other words, given the irrationality measure$\mu$of a real number x, whenever a rational approximation xp/q,$p, q \in N'$ yields n+1 exact decimal digits, then [1]

$$\frac{1}{10^n} \geq \left|x - \frac{p}{q}\right| \geq \frac{1}{q^{\mu+\varepsilon}} \tag{24}$$

for any$\epsilon > 0$, except for at most a finite number of pairs (p, q). Almost all numbers have an irrationality measure equal to 2. [1]

$$J = a_0 + \mathop{\mathrm{K}}_{i=1}^{n} \frac{1}{a_i} \tag{25}$$

Let the Taylor series expansion of

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \tag{26}$$

with J^x defined $f'(a)$

$$\frac{d}{dx}J^x = \lim_{h\to 0}\frac{J^{x+h} - J^x}{h} = \lim_{h\to 0}\frac{J^xJ^h - J^x}{h} \tag{27}$$
$$= J^x \cdot \left(\lim_{h\to 0}\frac{J^h - 1}{h}\right). \tag{28}$$

Since e is

$$\left(1+\frac{1}{x}\right)^{x} < e < J < \left(1+\frac{1}{x}\right)^{x+a} \tag{29}$$

where$n! = \Gamma(n+1)$ and$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ and $e = \sum_{n=0}^{\infty}\frac{1}{n!}$

Proof:

Because $\lim_{n\to\infty}\frac{a_n}{b_n} = c$ we know that for all $\varepsilon > 0$ there is a positive integer $n_0$such that for all$n \geq n_0$ then $\left|\frac{a_n}{b_n} - c\right| < \varepsilon$, or equivalently

$$-\varepsilon < \frac{a_n}{b_n} - c < \varepsilon \tag{30}$$

$$c - \varepsilon < \frac{a_n}{b_n} < c + \varepsilon \tag{31}$$

$$(c-\varepsilon)b_n < a_n < (c+\varepsilon)b_n \tag{32}$$

As $c > 0$ we can chooseε to be sufficiently small such that $c - \varepsilon$ is positive. So $b_n < \frac{1}{c-\varepsilon}a_n$ and by the direct comparison test, if$\sum_n a_n$ converges then so does$\sum_n b_n$. Similarly $a_n < (c+\varepsilon)b_n$, so if $\sum_n a_n$diverges, again by the direct comparison test, so does $\sum_n b_n$.

Let x be

$$(1+x)e^x = e^x + xe^x \tag{33}$$

$$= \sum_{n=0}^{\infty}\frac{x^n}{n!} + \sum_{n=0}^{\infty}\frac{x^{n+1}}{n!} \tag{34}$$

$$= 1 + \sum_{n=1}^{\infty}\frac{x^n}{n!} + \sum_{n=0}^{\infty}\frac{x^{n+1}}{n!} \tag{35}$$

$$= 1 + \sum_{n=1}^{\infty}\frac{x^n}{n!} + \sum_{n=1}^{\infty}\frac{x^n}{(n-1)!} \tag{36}$$

$$= 1 + \sum_{n=1}^{\infty}\left(\frac{1}{n!} + \frac{1}{(n-1)!}\right)x^n \tag{37}$$

$$= 1 + \sum_{n=1}^{\infty}\frac{n+1}{n!}x^n \tag{38}$$

$$= \sum_{n=0}^{\infty}\frac{n+1}{n!}x^n. \tag{39}$$

Given an ordered sequence$\{a_n\}_{n=1}^{\infty}$of real numbers: the first difference $\Delta(a_n)$ is defined as

$$\Delta(a_n) = a_{n+1} - a_n. \tag{40}$$

The "'second difference"' $\Delta^2(a_n)$ is defined as

$$\Delta^2(a_n) = \Delta(a_{n+1}) - \Delta(a_n) \tag{41}$$

which can be simplified to

$$\Delta^2(a_n) = a_{n+2} - 2a_{n+1} + a_n. \tag{42}$$

$$a_{n+k} = \binom{k}{0}a_n + \binom{k}{1}\Delta(a_n) + \cdots + \binom{k}{k}\Delta^k(a_n) \tag{43}$$

More generally: the k-th difference of the sequence 'an written as $\Delta^k(a_n)$ is defined recursively as

$$\Delta^k(a_n) = \Delta^{k-1}(a_{n+1}) - \Delta^{k-1}(a_n) = \sum_{t=0}^{k}\binom{k}{t}(-1)^t a_{n+k-t}. \tag{44}$$

The sequence and its differences are related by a binomial transform. If $m > n$ then this remainder decreases and finally is less than any given quantity

$$\frac{k}{m+n} = \frac{k}{m} - \frac{kn}{m^2} + \frac{kn^2}{m^3} - \frac{kn^3}{m^4} + \cdots. \tag{45}$$

If m = n, then this equation becomes

$$\frac{k}{2m} = \frac{k}{m} - \frac{k}{m} + \frac{k}{m} - \frac{k}{m} + \cdots. \tag{46}$$

Bernoulli called this equation a "not inelegant paradox". The Euler transform with ordinary generating function and/or probability generating function can be generalized.

$$G(a_n; x) = \sum_{n=0}^{\infty} a_n x^n. \tag{47}$$

$$G(z) = \mathrm{E}(z^X) = \sum_{x=0}^{\infty} p(x) z^x \tag{48}$$

$$\sum_{n=0}^{\infty}(-1)^n a_n = \sum_{n=0}^{\infty}(-1)^n\frac{\Delta^n a_0}{2^{n+1}} \tag{49}$$

$$\sum_{n=0}^{\infty}(-1)^n\binom{n+p}{n}a_n = \sum_{n=0}^{\infty}(-1)^n\binom{n+p}{n}\frac{\Delta^n a_0}{2^{n+p+1}} \tag{50}$$

where "p" = 0, 1, 2,...

Here mathematical induction in the following form: $r \in \{0,1\}$ and the second step r we deduce validity for $r+2$. For $r=0$, $(1+x)^0 \geq 1+0x$ is equivalent to $1 \geq 1$ which is true. Similarly, for $r=1$, $(1+x)^r = 1+x \geq 1+x = 1+rx$. Now suppose the statement is true for $r=k$: then $(1+x)^k \geq 1+kx$. Then

$$(1+x)^{k+2} = (1+x)^k(1+x)^2 \tag{51}$$

$$\geq (1+kx)\left(1+2x+x^2\right) \qquad \text{by hypothesis}(1+x)^2 \geq 0 \tag{52}$$

$$= 1+2x+x^2+kx+2kx^2+kx^3 \tag{53}$$

$$= 1+(k+2)x+kx^2(x+2)+x^2 \tag{54}$$

$$\geq 1+(k+2)x \tag{55}$$

Consider$(1+x)^r = 1+rx+\binom{r}{2}x^2+...+\binom{r}{r}x^r$. Clearly $\binom{r}{2}x^2+...+\binom{r}{r}x^r \geq 0$, and hence $(1+x)^r \geq 1+rx$ and $e^x \geq x+a$ and $J^x \geq x+a$. The exponent r can be generalized to an arbitrary real number as follows: if $x > -1$, then

$$(1+x)^r \geq 1+rx \tag{56}$$

for$''r'' \leq 0$ or $''r' \geq 1$, and

$$(1+x)^r \leq 1+rx \tag{57}$$

for $0 \leq r \leq 1$. and

$$(1+x)^r = 1+rx+\binom{r}{2}x^2+...+\binom{r}{r}x^r \tag{58}$$

For example,

$$(1+x)^3 = 1+3x+\binom{3}{2}x^2+...+\binom{3}{3}x^3 \tag{59}$$

$$\geq 1+3x \tag{60}$$

$$J^x = \frac{x^0}{0!} + \cdots \tag{61}$$

$$= 1 + \ddots \tag{62}$$

The Glaisher–Gould sequence is 2-regular. The nth value in the sequence starting from 1="n" = 0 gives the highest power of 2 that divides the central binomial coefficient$\binom{2n}{n}$, and it gives the numerator of$2^n/n!$ (expressed as a fraction in lowest terms). The sequence, start with 1 and use the rule: If $k >= 0$ and $a(0), a(1), ..., a(2^k-1)$ are the first2^k terms, then the next2^k terms are $2*a(0), 2*a(1), ..., 2*$

$a(2^k-1)$. The Fibonacci numbers occur in the sums of "shallow" diagonals in Pascal's triangle with binomial coefficients

$$F_n = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-k-1}{k} \tag{63}$$

with

$$F_n = \frac{\varphi^n - \psi^n}{\varphi - \psi} = \frac{\varphi^n - \psi^n}{\sqrt{5}} < /math > \tag{64}$$

where $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.61803$ and the nth position in the Fibonacci sequence is given by

$$n = \log_\varphi \left(\frac{F_n\sqrt{5} + \sqrt{5F_n^2 \pm 4}}{2} \right) \tag{65}$$

The partial numerators and denominators of the fraction's successive convergents are related by the "fundamental recurrence formulas":

$$A_{-1} = 1 \qquad B_{-1} = 0 \tag{66}$$

$$A_0 = b_0 \qquad B_0 = 1 \tag{67}$$

$$A_{n+1} = b_{n+1}A_n + a_{n+1}A_{n-1} \qquad B_{n+1} = b_{n+1}B_n + a_{n+1}B_{n-1} \tag{68}$$

The continued fraction's successive convergents are then given by

$$x_n = \frac{A_n}{B_n}. \tag{69}$$

where continued fraction convergents can be taken to be Möbius transformations acting on the hyperbolic upper half-plane and leads to the fractal self-symmetry. Figure One has the example of the ratio of convergents with 11/4 14/5 53/19 and 67/24.

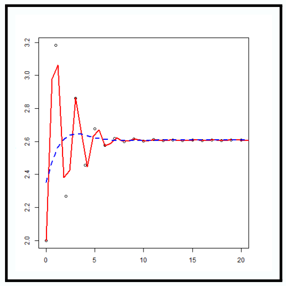

Figure 1: Convergents for J

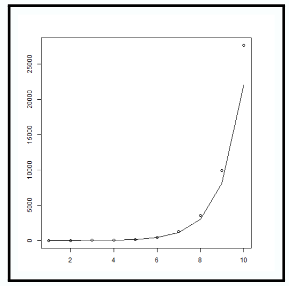

Figure 2: Plot of J^x and e^x

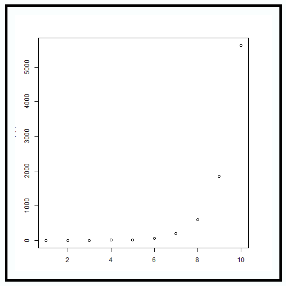

Figure 3: Plot of J^x - e^x

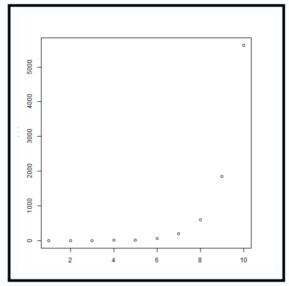

Figure 4: Plot of J^x - e^x

$$\ln 2 = [0; 1, 2, 3, 1, 6, 3, 1, 1, 2, 1, 1, 1, 1, 3, 10, 1, 1, 1, 2, 1, 1, 1, 1, 3, 2, 3, 1, ...] \tag{70}$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)} = \ln 2. \tag{71}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(4n^2-1)} = 2\ln 2 - 1. \tag{72}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(4n^2-1)} = \ln 2 - 1. \tag{73}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(9n^2-1)} = 2\ln 2 - \frac{3}{2}. \tag{74}$$

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-2n} = \ln 2. \tag{75}$$

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}(2n-1)+1}{8n^2-4n} = \ln 2. \tag{76}$$

if successive convergents are found, with numerators h1, h2, ... and denominators k1, k2, ... then the relevant recursive relation is:

$$h_n = a_n h_{n-1} + h_{n-2} \tag{77}$$

$$k_n = a_n k_{n-1} + k_{n-2} \tag{78}$$

The successive convergents are given by the formula

$$h_n/k_n = a_n h_{n-1} + h_{n-2}/a_n k_{n-1} + k_{n-2} \tag{79}$$

Table 1 has the coefficient values of a_n,h_n and k_n

Table 1: Values of a_n, h_n, k_n

-2	- 1	0	1	2
a_n	0	1	1	1
h_n	0	1	1	1
k_n	0	1	1	1

Let x be a real number to be approximated in continued fraction form and n be the number of partial denominators to evaluate then

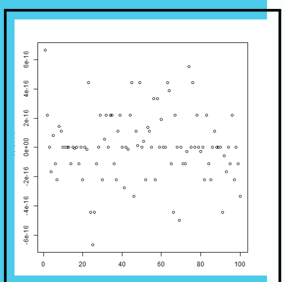

Figure 5: Standard Normal Variable

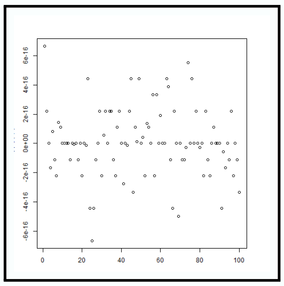

Figure 6: Standard Normal Variable

Theorem 2.1. The continued fraction is equivalent to a series of alternating terms:

$$a_0 + \sum_{n=0}^{\infty} \frac{(-1)^n}{k_n k_{n+1}} \tag{80}$$

Theorem 2.2. The matrix

$$\begin{bmatrix} h_n & h_{n-1} \\ k_n & k_{n-1} \end{bmatrix} \tag{81}$$

has determinant plus or minus one,
Theorem 2.3. Theorem One The J_1 mathematical constant is irrational.

Proof: Since every rational number can be represented as such c/d, for any positive integer n large enough that $2^{n-1} > d > 0$ no pair of integers (p, q) exists that simultaneously satisfies the two inequalities

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^n} \tag{82}$$

$$\left| x - \frac{p}{q} \right| = \left| \frac{c}{d} - \frac{p}{q} \right| = \frac{|cq - dp|}{dq} \tag{83}$$

$$\left| x - \frac{p}{q} \right| = \frac{|cq - dp|}{dq} = 0 \tag{84}$$

$$\left| x - \frac{p}{q} \right| = \frac{|cq - dp|}{dq} \geq \frac{1}{dq} \tag{85}$$

$$\left| x - \frac{p}{q} \right| \geq \frac{1}{dq} > \frac{1}{2^{n-1}q} \geq \frac{1}{q^n} \tag{86}$$

Hence the J_1 mathematical constant, if it exists, cannot be rational.

Theorem 1

$$\frac{1}{k_n(k_{n+1} + k_n)} < \left| x - \frac{h_n}{k_n} \right| < \frac{1}{k_n k_{n+1}}. \tag{87}$$

Corollary 1: A convergent is nearer to the limit of the continued fraction than any fraction whose denominator is less than that of the convergent.

Corollary 2: A convergent obtained by terminating the continued fraction just before a large term is a close approximation to the limit of the continued fraction.

3 Conclusion

In this short mathematical note, the J_1 mathematical constant was presented and examined with several properties and equation forms.

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