Math Notes:
Generalized Beta Type 2 Distribution for Lissajous Curve Heat Maps

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The Mathematical Learning Space Research Portfolio: Math Note 1

The generalized Beta probability density function has 30 distributions as limiting or special cases.[6]

$$GB(y;\mu,\sigma,c,\nu,\tau) = \frac{|\mu| y^{\mu\nu-1}(1-(1-c)(y/\sigma)^{\mu})^{\tau-1}}{\sigma^{\mu\nu} Beta(\nu,\tau)(1+c(y/\sigma)^{\mu})^{\nu+\tau}} \quad (1)$$

for $0 < y^{\mu} < \dfrac{\sigma^{\mu}}{1-c}$

where $0 \leq c \leq 1$. The probability density function of the Generalized Beta type 2, (GB2), is [1][2] [3] [4]

$$GB2(y;\mu,\sigma,\nu,\tau) = \frac{|\mu| y^{\mu\nu-1}}{\sigma^{\mu\nu} B(\nu,\tau)(1+(y/\sigma)^{\mu})^{\nu+\tau}} \quad (2)$$

where $y > 0, \mu > 0, -\infty < \sigma < \infty, \nu > 0, \tau > 0$. $GB2(y;\mu,\sigma,\nu,\tau) = GB(y;\mu,\sigma,c=1,\nu,\tau)$. The hth moment can be expressed as follows:[6]

$$\mathrm{E}_{GB}(Y^h) = \frac{\sigma^h B(\nu+h/\mu,\tau)}{B(\nu,\tau)} {}_2F_1\left[\begin{matrix}\nu+h/\mu, h/\mu; c\\ \nu+\tau+h/\mu;\end{matrix}\right], \quad (3)$$

where ${}_2F_1$ denotes the hypergeometric series. [5] Figure 1 provides the visual description of the pdf. Let $z = (x/\mu)^{\sigma}$ and $LL = \nu * log(z) + log(abs(\sigma)) - log(x) - |ln(\Gamma(\nu)| - |ln(\Gamma(\tau)| + |ln(\Gamma(\nu+\tau)| - (\nu+\tau) * log(1+z)$ where $|ln(\Gamma(x))| = \int_0^I nft^(x-1)exp(-t)dt$. [1] Consider a Lissajous curve [5] given by

$$x = a\cos(k_x t) \quad (4)$$
$$z = b\sin(k_z t) \quad (5)$$

where k_x and k_z are constants for the number of lobes in the function. Figure 2,3, and 4 have the heatmaps for the Lissajous curve for different parameter values from the Generalized Beta type 2 (GB2) where $A = GB2_1(y;\mu,\sigma,\nu,\tau) = 5,2,1,1$; $B = GB2_2(y;\mu,\sigma,\nu,\tau) = 5,2,1,2$ and $C = GB2_3(y;\mu,\sigma,\nu,\tau) = 5,2,2,1$.

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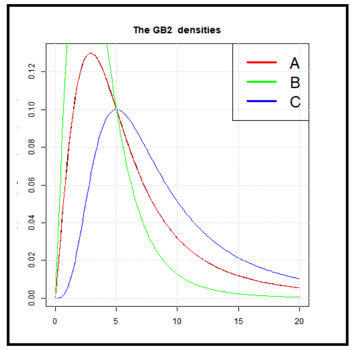

Figure 1: probability density function of the Generalized Beta type 2 (GB2) $A = GB2_1(y;\mu,\sigma,\nu,\tau) = 5,2,1,1$; $B = GB2_2(y;\mu,\sigma,\nu,\tau) = 5,2,1,2$ and $C = GB2_3(y;\mu,\sigma,\nu,\tau) = 5,2,2,1$

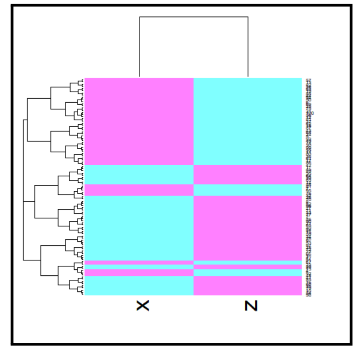

Figure 2: Lissajous Curve HeatMap a=1,b=1, $k_x = 1$, $k_z = 1$ and t=1:10

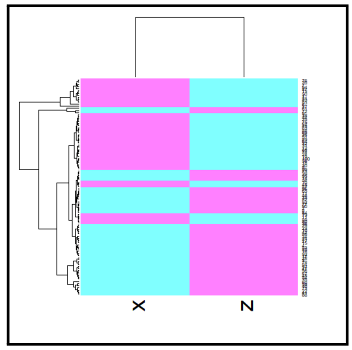

Figure 3: Lissajous Curve HeatMap a=$GB2_1(y;\mu,\sigma,\nu,\tau)$,b=1, $k_x = 1$, $k_z = 1$ and t=1:100 with $\mu = 5, \sigma = 2, \nu = 1, \tau = 1$

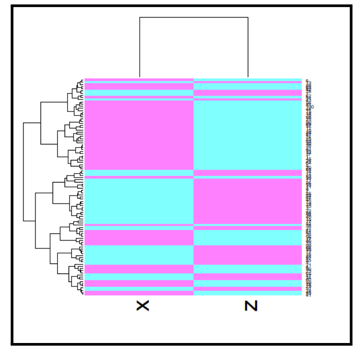

Figure 4: Lissajous Curve HeatMap a=1,b=1, $k_x = 1$, $k_z = GB2_2(y;\mu,\sigma,\nu,\tau)$ and t=1:100 $\mu = 5, \sigma = 2, \nu = 1, \tau = 1$

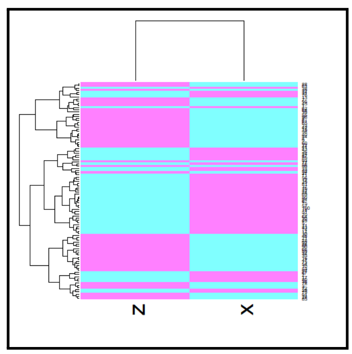

Figure 5: Lissajous Curve HeatMap a=1,b=1, $k_x = 1$, $k_z = GB2_3(y;\mu,\sigma,\nu,\tau)^3$ and t=1:100 $\mu = 5, \sigma = 2, \nu = 1, \tau = 2$

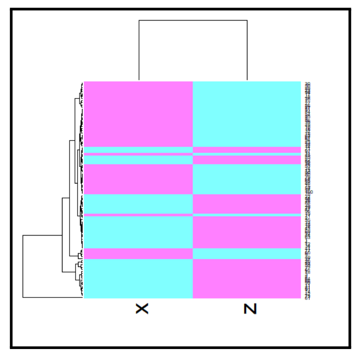

Figure 6: Lissajous Curve HeatMap a=$GB2_1(y;\mu,\sigma,\nu,\tau)$,b=1, $k_x = GB2_2(y;\mu,\sigma,\nu,\tau)$, $k_z = GB2_3(y;\mu,\sigma,\nu,\tau)$ and t=1:100 $\mu = 5, \sigma = 2, \nu = 1, \tau = 1$

References

[1] Rigby, R. A. and Stasinopoulos D. M. (2005). Generalized additive models for location, scale and shape,(with discussion), Appl. Statist., 54, part 3, pp 507-554.

[2] Stasinopoulos D. M. Rigby R. A. and Akantziliotou C. (2006) Instructions on how to use the GAMLSS package in R. Accompanying documentation in the current GAMLSS help files, (see also http://www.gamlss.org/).

[3] Stasinopoulos D. M. Rigby R.A. (2007) Generalized additive models for location scale and shape (GAMLSS) in R. Journal of Statistical Software, Vol. 23, Issue 7, Dec 2007, http://www.jstatsoft.org/v23/i07.

[4] Stasinopoulos D. M., Rigby R.A., Heller G., Voudouris V., and De Bastiani F., (2017) Flexible Regression and Smoothing: Using GAMLSS in R, Chapman and Hall/CRC.

[5] Wikipedia contributors. "Lissajous curve." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 10 Jun. 2021. Web. 7 Aug. 2021.

[6] Wikipedia contributors. "Generalized beta distribution." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 14 Jan. 2021. Web. 8 Aug. 2021.