Techsoc Freshie Roadmaps

Mathematics Club x Programming Club

6 August 2024

Instructions

- Welcome to the first session of the Freshie Roadmaps! The goal is for you to have as much fun as possible.
- All 8 problems are worth 10 marks and the bonus problem is worth 5 marks.
- Try all questions even if you can't solve them completely. Write your answers clearly and concisely.
- Don't use unfair means like the internet because all of you are here to learn and enjoy.

Problem 1: An Army of Unfair Dice

a) A blindfolded Navin throws two identical six-sided dice which have been carefully constructed by Aditi in such a way that $P(X = 1) = P(X = 3) = P(X = 5) = 2/9$, and $P(X = 2) = P(X = 4) = P(X = 6) = 1/9$ (where X is the number that shows up on the top face). Aditi calculates the sum of the numbers that appear and asks Navin to guess the sum. What should Navin say to maximize his chances of being right?

b) Now, Navin throws 50 such dice together and painstakingly calculates the sum of the numbers and asks a blindfolded Aditi to find the remainder on dividing the sum by 4. What would be Aditi's best guess?

Problem 2: A Tale of Two Cities

In Mathland, a country of 101 cities live a few coordinators of the Mathematics Club. Their cities are connected by one-way roads.

a) There are 50 roads going into and coming out of each city. What's the minimum number of roads one of them would have to travel to reach a fellow coordinator's city?

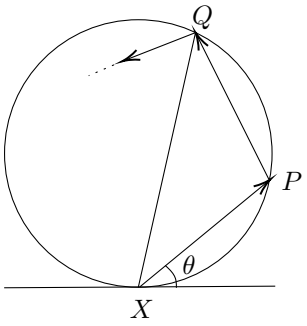
b) The coordinators are lazy. They don't like travelling more than 3 roads to reach each other. In order to optimize this, the heads decide that n roads in and out of each city are sufficient to make them meet and get them to do work. Help Atreya and Pradyumnan find n.

Problem 3: Stumbling Across a Grid

Pranjal starts at the point $(4, 4)$ on a grid of lattice points and moves until he reaches the coordinate axes for the 1st time. At any point Pranjal can move either left, down or left-down with equal chance, independent of his previous moves. What is the probability that he reaches the coordinate axes at $(0, 0)$? Let the probability be $\frac{a}{3^b}$, where a and b re positive integers, and 3 does not divide a. Find $a + b$.

Problem 4: The Circle of Lif- Polynomials

Standing at the edge of the circle of life, Prasanna and KK are pondering over the intricacies of reflection and polynomials. As such, they set up an experiment where they stand on the boundary of the circle of life and shoot out lasers.



While KK shoots out laser over $\theta \in [0, \pi]$, Prasanna measures the distances $p = |XP|$, $q = |XQ|$. Now they construct a polynomial

$$f(z) = pz^2 + qz + 1$$

What is the probability that this polynomial has real roots? Keep in mind that a wise man once said the circle of life has a radius of 1/2.

Problem 5: Skill Issues

For both parts of this question, you have to find the error(s) that the algorithm can run into, and also suggest a way to fix them.

a) Parijat gave Yukash a sequence of n positive integers $a_1, a_2, ..., a_n$. Since Yukash likes to think backwards, he would like to know if the sequence is a palindrome[†] or not. He wrote the following logic to solve this problem, but since he suffers from skill issues there is(are) some mistake(s). Help Yukash fix his problems.

[†]A sequence is called a palindrome if it reads the same forwards and backwards. For example, $(1, 2, 3, 2, 1)$ is a palindrome but $(1, 2, 3, 2)$ is not.

- **Step 1:** Start the algorithm.
- **Step 2:** Set x to n.
- **Step 3:** If $x > 0$, go to **Step 4**. Otherwise, go to **Step 7**.
- **Step 4:** If $a_x \neq a_{n-x}$, go to **Step 5**. Otherwise, go to **Step 6**.
- **Step 5:** Output that "The given sequence is not a palindrome". Go to **Step 8**.
- **Step 6:** Increase x by 1. Go to **Step 3**.
- **Step 7:** Output that "The given sequence is a palindrome".
- **Step 8:** End the algorithm.

b) To take revenge, Yukash gave Parijat a whole number n and asked him to check whether it's a perfect square or not. Parijat was very confident but he got infected with Yukash's skill issues and made some error(s) himself. Help him fix them.

- **Step 1:** Start the algorithm.
- **Step 2:** Set x to 0.
- **Step 3:** If $x^2 = n$, go to **Step 4**. Otherwise, go to **Step 5**.
- **Step 4:** Output that "The given number is a perfect square". Go to **Step 7**.
- **Step 5:** Increase x by 1. Go to **Step 3**.
- **Step 6:** Output that "The given number is not a perfect square".
- **Step 7:** End the algorithm.

Problem 6: Financial Foresight: Broke

Tushar and Parva are quite rich at the moment. As they have no care for money, they decide to play a game with their entire worth at stake. Initially, let's say Tushar has a coins and Parva has b coins in their wallets. Each person takes turns to play, with Tushar starting first.

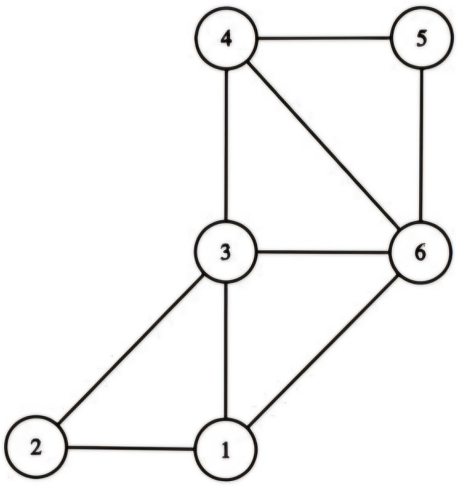
In each turn, a player performs both the following steps in order:

1. Exchange wallets with the other player, or decide to keep his current wallet.
2. Throw away 1 coin from his wallet. A player loses if he is unable to remove any coins.

If both players play optimally (which means that Tushar and Parva are both strategic geniuses), when would Tushar win and when would Parva win?

Problem 7: Euler Cameo

Arya wants to place his home on one of the vertices of the following graph so that he can roam around all day on different roads (edges) and still come back home. But right now he isn't able to do that. To help him, add exactly one edge to the following graph such that after adding it, you will be able to draw all the edges of the graph *without lifting up the pen from the paper or drawing any edge multiple times*, and *starting and ending at the same point on the paper*. To answer, write down the two distinct vertices joined by the new edge.



Problem 8: Conflict Management

Karthik has 10 intervals on the number line: $(6,9),(1,3),(3,7),(2,6),(7,9),(2,5),(5,7),(4,6),(6,8),(1,10)$. He wants to choose as many intervals as possible, such that no point on the number line belongs to more than 2 intervals. Let k be the maximum number of such intervals. Help Karthik by giving any k intervals satisfying the condition. For a few extra points, try to explain why you think your value of k is the highest!

Example: $(1,3),(2,4),(3,5)$ satisfies the condition as no point lies in more than 2 intervals (note that 3 lies in only $(2,4)$), but $(1,4),(2,5),(3,6)$ doesn't satisfy the condition as 3.5 lies in three intervals.

Bonus Problem! Thinkception

Answer any positive integer between 1 and 100. Let's call your answer x. We will calculate $\bar{x}$, the average of all x values answered by everyone giving this paper right now. If x is between $\frac{\bar{x}}{2} \pm 5\%$, you will get points for this question. Have fun :)

Answer Sheet

Team Name: MCxPC
Participant 1 Name: Mathematics Club
Participant 1 Roll Number: MC23B001
Participant 1 Contact Number: 1
Participant 2 Name: Programming Club
Participant 2 Roll Number: PC23B001
Participant 2 Contact Number: 0

1. a) 6.
   If the dice were fair, the answer would be 7. On an average we can expect the sum of our rolls to be around

$$E = 2\left(1 \times \frac{2}{9} + 2 \times \frac{1}{9} + 3 \times \frac{2}{9} + 4 \times \frac{1}{9} + 5 \times \frac{2}{9} + 6 \times \frac{1}{9}\right) \approx 6.67$$

   A good guess would be to check 7 and numbers around 7. Now intuitively getting a pair of odd numbers is more likely than a pair of even numbers or an even number and an odd number. Moreover, 6 has one extra (odd, odd) pair because it splits into (3, 3), while 8 splits into (4, 4), and 7 can never split into an (odd, odd) pair. It is most likely to obtain 6.

   b) 3.
   An incorrect approach to this would be to use the answer of the previous question and follow up with 8 mod 4 = 0 or 7 mod 4 = 3 or 6 mod 4 = 2. An obvious counter-argument to this approach can be constructed by throwing a die only once and figuring out the probabilities of the number of the top face modulo 4 (which has been done below for you).
   First, we construct the probability distributions of the random variable $X_k = (\#(\text{Die}_k) \bmod 4)$ where $k = 1, 2, ..., 50$ represent the 50 dice. It follows that $P(X_k = 0) = 1/9, P(X_k = 1) = 4/9, P(X_k = 2) = 2/9, P(X_k = 3) = 2/9$ and this distribution has a maximum at $X_k = 1$.
   When two dice are rolled, then the probability distribution of the resultant sum $Y = (X_1 + X_2) \bmod 4$ would be $P(Y = j) = \sum_{i=0}^{3} P(X_1 = i) \times P(X_2 = (j - i) \bmod 4)$ where $j = 0, 1, 2, 3$. The maximum boost to the probability occurs when the peaks of the distributions line up exactly (Try to plot the two distributions X_1 and X_2 - one as is and the other being flipped and shifted along and then being multiplied - to understand what "line up" means). The maximum probability would occur at $j = 3$ (in Y) when the identical X have a peak at 1. Let $Z = (Y_1 + Y_2) \bmod 4$ (think of this as 4 dice being thrown); here the identical Y have a peak at 3, Z will have a peak at 1.
   What we have done here is group the rolling of the 50 dice into sets of 2. Walking along the trail of the groups, the best guess for Aditi alternates between 1 and 3. At the 50^{th} roll, 3 will have the maximum chance of occurring. (Albeit when you throw so many dice together, the probability distribution becomes almost uniform)

2. a) 2, partial marks have been awarded for answering 1 as well.
   Each city is connected by 50 roads leading both into and out of it, which means you can reach 50 different cities from any given city. Similarly, 50 different cities can lead to your destination city. This suggests a total of 102 distinct cities. However, since there are only 101 cities in total, there must be at least one city that is common in the paths connecting the two required cities. Therefore, the minimum number of roads you would need to travel to get from one required city to (any of) the other using the shortest path is 2.

   b) 40.
   Let us assume that the required number of roads going in and out of each city is x. Traveling 3 roads means stopping at 2 intermediate cities.

   - The first intermediate city should be approachable from the roads going out of city 1

- The second intermediate city should be approachable from the roads leading into city 2
- The second intermediate city should be accessible from the first intermediate city.

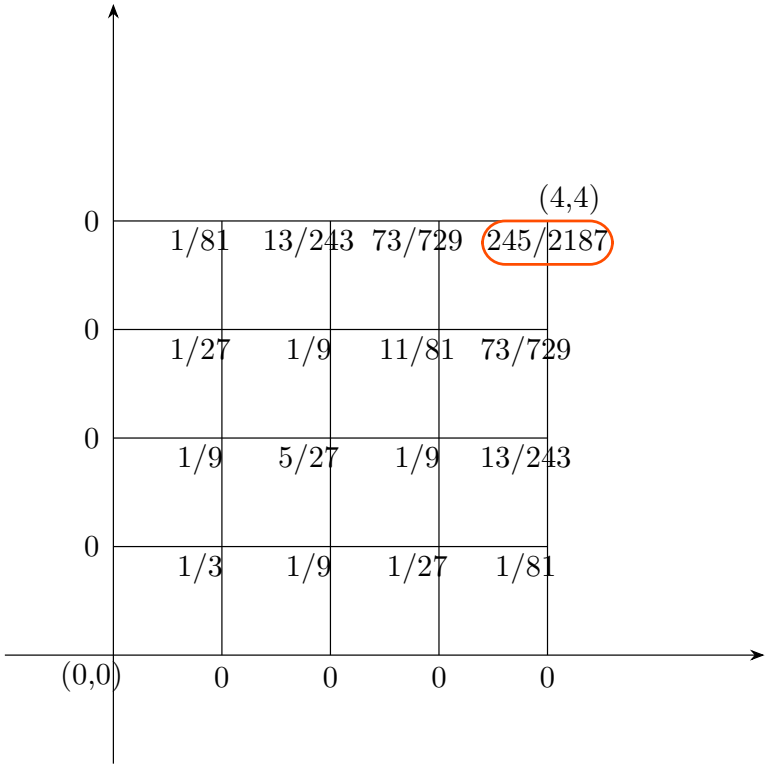
x^2 roads lead out of the first set of possible intermediate cities. Out of these x^2 roads, a maximum of $\frac{x(x-1)}{2}$ form connections within these x cities, whereas $x(101-(2x+2))$ connect this first intermediate city to cities that do not directly lead into city 2. The remaining roads adhere to the conditions listed above.
$x^2 > \frac{x(x-1)}{2} + x(101-(2x+2))$ is the equation that we are required to solve. Its smallest integral solution is 40.

3. 252.

We will recursively calculate the probability of reaching $(0,0)$ from (x,y).

$$P(x,y) = \frac{1}{3}P(x-1,y) + \frac{1}{3}P(x-1,y-1) + \frac{1}{3}P(x,y-1)$$

for $x, y \geq 1$. The base cases are $P(0,0) = 1, P(x,0) = P(0,y) = 0$ for x, y not equal to 0.
Hence we can recursively find that $P(4,4) = \dfrac{245}{2187}$



4. 0.

Using simple geometry it can be shown that $p = \sin(\theta), q = \sin(2\theta)$ and for the quadratic equation to have real roots we require $q^2 - 4p \geq 0$. Taking into account the symmetry of the problem, the required probability boils down to calculating $P(\sin^2(x)\cos^2(x) \geq \sin(x), x \in [0, \pi/2])$. The inequality is satisfied only at a point $x = 0$ over the entire interval, making the probability 0. (Is $\sin(x)\cos^2(x) > 1$ possible?)

5. a) Step 4: Should be "$a_x \neq a_{n+1-x}$". Otherwise, the wrong elements are compared (for example, a_1 vs $a_n - 1$ instead of a_1 vs a_n).
   Step 6: Should be "Decrease x by 1". Otherwise, x keeps increasing without limit, resulting in an infinite loop.

   b) Step 5: Should be "If $x^2 \leq n$ go to Step 3, otherwise go to Step 6". Otherwise x keeps increasing and gives an infinite loop. Alternate answers to achieve the same thing were also accepted.

6. Tushar wins if $a + b$ is odd. Parva wins if $a + b$ is even.

The problem is essentially the same as removing 1 coin each turn from a wallet with $a + b$ coins. Since each player is removing coins from the wallet of his choice, it doesn't matter how much the individual wallets have. The only way the game ends is if both wallets have 0 coins, i.e. the total amount is 0. Clearly, if $a + b$ is odd then Tushar will remove the last coin and if $a + b$ is even then Parva will.

7. The new edge joins vertices 1 and 4.
   The path described in the question is called a Eulerian Cycle (look it up!). To satisfy the given conditions, for every edge (road) used to arrive at a particular vertex (house), there will be a different edge used to leave the vertex. So each vertex must have an even number of edges (so that they can be divided into pairs like this). The only vertices with an odd number of edges are 1 and 4. Joining them results in a graph that has an Eulerian Cycle (one such cycle is $4-6-5-4-3-6-1-3-2-1-4$)

8. One set of k $(=6)$ intervals is $(1,3),(2,5),(4,6),(5,7),(6,8),(7,9)$. Other valid answers will also get points.
   One algorithm to get the answer works as follows: Arrange all the intervals in increasing order of their end point, as it is always advantageous to choose the interval with lower end point (because this leaves more space after this for further intervals). Now start including intervals one by one from left to right, and exclude an interval whenever it clashes with two or more previous intervals at the same point.
   Sorted order: $(1,3),(2,5),(2,6),(4,6),(3,7),(5,7),(6,8),(6,9),(7,9),(1,10)$
   - $(2,6)$ is removed as 2.5 would overlap with $(1,3),(2,5),(2,6)$
   - $(3,7)$ is removed as 4.5 would overlap with $(2,5),(4,6),(3,7)$
   - $(6,9)$ is removed as 5.5 would overlap with $(5,7),(6,8),(6,9)$
   - $(1,10)$ is removed as 2.5 would overlap with $(1,3),(2,5),(1,10)$

   All other intervals are kept, giving us $(1,3),(2,5),(4,6),(5,7),(6,8),(7,9)$.

Bonus:
The average was 30.68067227. Points for anyone who answered 15 or 16. If you're interested, here's the data!
In terms of strategy, there's a variation to this question where the allowed range is 0 to 100. Then, it is most advantageous for everyone to pick 0, so that $\bar{x}=0$ and everyone gets the points. However, what makes this question interesting is that everyone picking the same number always results in everyone losing. So one might expect people to choose a random number, giving an average of 50 (which is why 25 has a peak in the graph). However, if enough people think this way and choose $25=\frac{50}{2}$, the average shifts to 25 and the new correct answer is 12 or 13. This continues for as long as you can go, but it's likely people will stop at 25. Another factor is the presence of favourite numbers like 13, 37 (Veritasium) and 69, which skew the expected distribution. There's no accurate way (that we know of) to solve this, other than to put your best estimate from what others are answering. However, if you have a strategy, do share it in the group!