MathLab: Problem Set 3

Solutions

Mathematics Club IITM

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- The solutions to the problems in Problem Set 3 can be found in this solution sheet.
- Feel free to reach out to us for doubts! Contact information of the problem-set creators:

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§1 The Basics of Manipulation

This section contains simple conceptual-level questions which test your understanding of the topic and prepare you for the upcoming sections. **Show your working in each question.**

1. *e^x and a constant were walking down the street one day. While e^x continued walking, oh dear had the constant run away. What operator in the path would cause the constant much dismay?*

   Find the solutions $y(x)$ to the following differential equations given the boundary conditions:

   a) $2\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} - 3y = 0$
   [$y(2) = e^{-2} + e^{12}$ and $y(16) = e^{-30} + e^5$]

   Final Expected Answer: y(6)

   Solution:

   Using the learnt method, we substitute e^{sx}. This gives us:

   $$2\frac{d^2e^{sx}}{dx^2} + 5\frac{de^{sx}}{dx} - 3e^{sx} = 0$$

   $$\implies (2s^2 + 5s - 3)e^{sx} = 0$$

   $$\implies (2s - 1)(s + 3) = 0$$

   $$\implies s = \frac{1}{2}, \ -3$$

$$\implies y = Ae^{\frac{1}{2}x} + Be^{-3x}$$

Substituting initial conditions and solving the system of 2 equations, we get:

$A = e^{-3},\ B = e^{18}$

$$\implies y = e^{\frac{1}{2}x-3} + e^{-3x+18}$$

$\therefore\ \ y(6) = 2$

b) $2\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} - 3y = 15\sqrt{2}e^{ix}$

[Assume only the forced part of the solution as discussed in the presentation i.e. system is at rest in the beginning]

Final Expected Answer: The magnitude of the solution (Since the solution will be complex)

Solution:

Since we already have an exponential driving force $f(x) = 15\sqrt{2}e^{ix}$, we assume the solution is of the form Ae^{ix}.

$$\implies 2\frac{d^2Ae^{ix}}{dx^2} + 5\frac{dAe^{ix}}{dx} - 3Ae^{ix} = 15\sqrt{2}e^{ix}$$

$$\implies Ae^{ix}(2(i)^2 + 5i - 3) = 15\sqrt{2}e^{ix}$$

$$\implies A(-5+5i) = 15\sqrt{2}$$

$$\implies A = \frac{15\sqrt{2}}{-5+5i}$$

$$\implies A = \frac{3\sqrt{2}}{i-1}$$

$$\therefore\ \ |A| = \frac{3\sqrt{2}}{\sqrt{2}} = 3$$

c) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 5y = \displaystyle\sum_{k=-\infty}^{\infty} 25\sqrt{13}k^2 e^{\frac{kix}{2}}$

[Assume only the forced part of the solution as discussed in the presentation i.e. system is at rest in the beginning]

Final Expected Answer: The magnitude of the coefficient of the $k = 2$ term in the solution series.

Solution:

Applying the usual method of solving a LCCDE with the driving force $f(x)$ being a

Fourier series, we assume the solution is of the form $\sum_{k=-\infty}^{\infty} A_k e^{\frac{kix}{2}}$.

$$\implies \frac{d^2}{dx^2}\sum_{k=-\infty}^{\infty} A_k e^{\frac{kix}{2}} - 2\frac{d}{dx}\sum_{k=-\infty}^{\infty} A_k e^{\frac{kix}{2}} - 5\sum_{k=-\infty}^{\infty} A_k e^{\frac{kix}{2}} = \sum_{k=-\infty}^{\infty} 25\sqrt{13}k^2 e^{\frac{kix}{2}}$$

$$\implies \sum_{k=-\infty}^{\infty} A_k \left(\frac{d^2 e^{\frac{kix}{2}}}{dx^2} - 2\frac{de^{\frac{kix}{2}}}{dx} - 5e^{\frac{kix}{2}}\right) = \sum_{k=-\infty}^{\infty} 25\sqrt{13}k^2 e^{\frac{kix}{2}}$$

$$\implies \sum_{k=-\infty}^{\infty} A_k \left(\left(\frac{ki}{2}\right)^2 + 2\frac{ki}{2} + 5\right) e^{\frac{kix}{2}} = \sum_{k=-\infty}^{\infty} 25\sqrt{13}k^2 e^{\frac{kix}{2}}$$

$$\implies \sum_{k=-\infty}^{\infty} -A_k \left(\frac{k^2}{4} + 5 + 2\frac{ki}{2}\right) e^{\frac{kix}{2}} = \sum_{k=-\infty}^{\infty} 25\sqrt{13}k^2 e^{\frac{kix}{2}}$$

Using the fact that exponentials of different frequencies are linearly independent, we equate coefficients to get:

$$-A_k\left(\frac{k^2}{4} + 5 + 2\frac{ki}{2}\right) = 25\sqrt{13}k^2$$

$$\implies A_k = \frac{-25\sqrt{13}\ k^2}{\frac{k^2}{4} + 5 + 2\frac{ki}{2}}$$

$$\therefore \quad A_2 = \frac{-100\sqrt{13}}{6+2i}$$

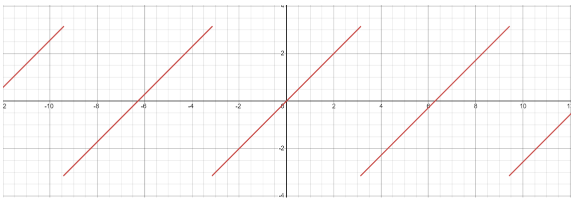
$$\implies |A_2| = \frac{50\sqrt{13}}{\sqrt{10}}$$

2. *Why did the function f go see a Fourier therapist? Because it wanted to break down its complex personality into a series of sines and cosines!*

Consider the function:

$$f(x) = x, -\pi \leq x < \pi$$

Here f has a period of 2π, that is $f(x+2\pi) = f(x)$. Find the Fourier series corresponding to $f(x)$.



A tiny plot of the periodic function $f(x)$

Final Expected Answer: If the coefficients of the series are a_n, b_n, then find the value of $a_2 + b_2$.

Solution:

We make an observation that the function $f(x) = x$ is an odd function. Thus, by applying the properties of Fourier series, we find that the Fourier Series corresponding to this function is a *sine series* and conclude that:

$$a_n = 0 \ \forall \ \ n$$

To find b_n, we can use: $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$

$\implies b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx$

$\implies b_n = \frac{1}{\pi}\left(\left[-\frac{x\cos(nx)}{n}\right]_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}\cos(nx)\,dx\right)$

$\implies b_n = \frac{1}{\pi}\left(\left[\frac{-\pi\cos(n\pi) + (-\pi\cos(-n\pi))}{n}\right] + \frac{1}{n}\left[\frac{\sin(nx)}{n}\right]_{-\pi}^{\pi}\right)$

Since $\sin(n\pi)$ is always 0 we get

$$b_n = -\frac{2}{n}\cos(n\pi)$$

$\therefore \quad a_2 = 0 \ \ \& \ \ b_2 = -1$

Hence, we get the final answer $a_2 + b_2$ as -1.

§2 The Hunt for Harmonics

§2.1 A janky spring

You are expected to use the Fourier series to solve this problem. Doing so requires that you have knowledge of both the exponential and trigonometric series. However, in order to help you out, we have provided the relevant conversion formulae to convert a given Fourier trigonometric series to an exponential series and vice versa in the appendix. We highly encourage you work them out for yourselves.

Achintya remembers learning about an interesting function called the ramp function $r(t)$ in his signal processing class. He remembers his teacher defining it as:

$$r(t) = \begin{cases} t & t \geq 0 \\ 0 & otherwise \end{cases}$$

On his class quiz, he comes across this differential equation when he attempts to model an old, rusty and stiff spring that is being slowly but steadily pulled by an insanely strong ant. However since he missed half of his math classes, he has no idea how to solve it. Can you help him?

$$5\frac{d^3x}{dt^3} + 3\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 2x = r_p(t)$$

Assume $r_p(t) = r(t)$ when $-\pi < t \leq \pi$ and is periodic with time period 2π. Also solve for only the particular solution (this is what we discussed in the session).

The insanely strong ant

Final Expected Answer: The solution to the differential equation given above using Fourier series methods.

Solution:

We begin by decomposing the given periodic ramp function into a trigonometric Fourier series.

$$\omega = \frac{2\pi}{t} = \frac{2\pi}{2\pi} = 1$$

Thus,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} r_p(t)dt = \frac{1}{\pi}\int_{0}^{\pi} tdt = \frac{\pi}{2}$$

$$a_k = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} r_p(t)\cos(kt)dt = \frac{1}{\pi}\int\limits_{0}^{\pi} t\cos(kt)dt = \frac{1}{\pi}\left(\frac{(-1)^k - 1}{k^2}\right)$$

$$b_k = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} r_p(t)\sin(kt)dt = \frac{1}{\pi}\int\limits_{0}^{\pi} t\sin(kt)dt = \frac{(-1)^{k+1}}{k}$$

$$\therefore \quad r_p(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos(kt) + \sum_{k=1}^{\infty} b_k\sin(kt)$$

$$\implies r_p(t) = \frac{\pi}{4} \ + \sum_{k=1}^{\infty}\frac{1}{\pi}\left(\frac{(-1)^k - 1}{k^2}\right)\cos(kt) \ + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin(kt)$$

Rearranging the formulae given in the Appendix, we get:
$A_k = a_k - ib_k \quad \& \quad A_{-k} = a_k + ib_k \quad \forall\ k > 0$

Substituting our values:

$$A_k = \frac{1}{2\pi}\left(\frac{i\pi\,(-1)^k}{k} + \frac{(-1)^k - 1}{k^2}\right) \qquad \forall\ k \neq 0$$

$$\therefore \quad r_p(t) = A_0 + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} A_k e^{ikt}$$

$$\implies r_p(t) = \frac{\pi}{4} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\frac{1}{2\pi}\left(\frac{i\pi\,(-1)^k}{k} + \frac{(-1)^k - 1}{k^2}\right)e^{ikt}$$

The differential equation now reduces to the standard LCCDE with an driving force in the form of a Fourier Series.

$$\frac{d^3x}{dt^3} + 3\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 2x = \sum_{k=-\infty}^{\infty} A_k e^{ikt}$$

As usual, we assume the solution is some series $\sum\limits_{k=-\infty}^{\infty} B_k e^{ikt}$. Substituting in the above differential equation and solving, we get:

$$\sum_{k=-\infty}^{\infty}\left(5\,(ik)^3 + 3\,(ik)^2 + 6ik + 2\right)B_k e^{ikt} = \sum_{k=-\infty}^{\infty} A_k e^{ikt}$$

Using our knowledge of Linear Independence, we get:

$$\left(5\,(ik)^3 + 3\,(ik)^2 + 6ik + 2\right)B_k = A_k$$

$$\implies B_k = \frac{A_k}{-5ik^3 - 3k^2 + 6ik + 2}$$

$$\therefore \quad x(t) = \sum_{k=-\infty}^{\infty} \frac{A_k}{-5ik^3 - 3k^2 + 6ik + 2}\, e^{ikt} \qquad A_k = \begin{cases} \dfrac{1}{2\pi}\left(\dfrac{j\pi\,(-1)^k}{k} + \dfrac{(-1)^k - 1}{k^2}\right) & k \neq 0 \\ \dfrac{\pi}{4} & k = 0 \end{cases}$$

Bonus Question 1: A recovering Ant [For those with 7/10 attendance]:

Consider the function $f(t) = |t|$ when $-\pi < t \leq \pi$ with the same period as the above question ($T = 2\pi$). Can you relate it somehow to the ramp function? Now use that knowledge to solve the above question but with the driving force replaced by $f(t)$ (with the same period) instead of the ramp function.

NOTE: *For those who have already met the attendance criteria, there is absolutely* ***NO*** *need to submit this bonus question. But I still ask you to try this out. Including it in your solution will make my day.*

Solution:

The function $f(t)$ can be represented as $r_p(t) + r_p(-t)$.

Thus,
$f(t) = r_p(t) + r_p(-t)$

$$= \frac{\pi}{4} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{2\pi}\left(\frac{i\pi\,(-1)^k}{k} + \frac{(-1)^k - 1}{k^2}\right) e^{ikt} + \frac{\pi}{4} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{2\pi}\left(\frac{i\pi\,(-1)^k}{k} + \frac{(-1)^k - 1}{k^2}\right) e^{-ikt}$$

Replacing k by $-k$ for the second term, we get:

$$= \frac{\pi}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left(\frac{1}{2\pi}\left(\frac{i\pi\,(-1)^k}{k} + \frac{(-1)^k - 1}{k^2}\right) e^{ikt} + \frac{1}{2\pi}\left(\frac{i\pi\,(-1)^{(-k)}}{-k} + \frac{(-1)^{(-k)} - 1}{(-k)^2}\right) e^{ikt}\right)$$

$$= \frac{\pi}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left(\frac{1}{2\pi}\left(\frac{i\pi\,(-1)^k}{k} + \frac{(-1)^k - 1}{k^2}\right) e^{ikt} + \frac{1}{2\pi}\left(\frac{i\pi\,(-1)^k}{-k} + \frac{(-1)^k - 1}{k^2}\right) e^{ikt}\right)$$

$$= \frac{\pi}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left(\frac{1}{2\pi}\left(\frac{i\pi\,(-1)^k}{k} + \frac{(-1)^k - 1}{k^2} + \frac{i\pi\,(-1)^k}{-k} + \frac{(-1)^k - 1}{k^2}\right) e^{ikt}\right)$$

$$= \frac{\pi}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{2\pi}\left(\frac{(-1)^k - 1}{k^2} + \frac{(-1)^k - 1}{k^2}\right) e^{ikt}$$

$$= \frac{\pi}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{\pi}\left(\frac{(-1)^k - 1}{k^2}\right) e^{ikt}$$

$$\therefore \quad A_k = \begin{cases} \dfrac{1}{\pi}\left(\dfrac{(-1)^k - 1}{k^2}\right) & k \neq 0 \\ \dfrac{\pi}{2} & k = 0 \end{cases}$$

The differential equation now reduces to the standard LCCDE with an driving force in the form of a Fourier Series.

$$\frac{d^3x}{dt^3} + 3\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 2x = \sum_{k=-\infty}^{\infty} A_k e^{ikt}$$

As usual, we assume the solution is some series $\sum_{k=-\infty}^{\infty} B_k e^{ikt}$. Substituting in the above differential equation and solving, we get:

$$\sum_{k=-\infty}^{\infty} \left(5\,(ik)^3 + 3\,(ik)^2 + 6ik + 2\right) B_k e^{ikt} = \sum_{k=-\infty}^{\infty} A_k e^{ikt}$$

Using our knowledge of Linear Independence, we get:

$$\left(5\,(ik)^3 + 3\,(ik)^2 + 6ik + 2\right) B_k = A_k$$

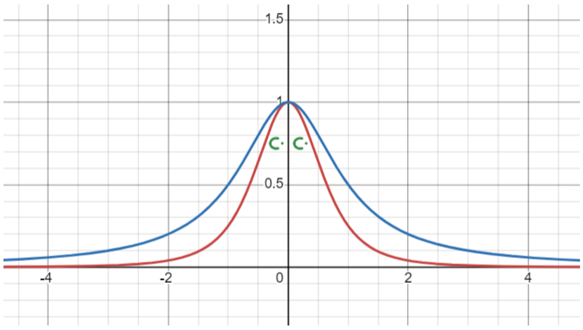
$$\implies B_k = \frac{A_k}{-5ik^3 - 3k^2 + 6ik + 2}$$

$$\therefore \quad x(t) = \sum_{k=-\infty}^{\infty} \frac{A_k}{-5ik^3 - 3k^2 + 6ik + 2}\, e^{ikt} \qquad A_k = \begin{cases} \dfrac{1}{\pi}\left(\dfrac{(-1)^k - 1}{k^2}\right) & k \neq 0 \\ \dfrac{\pi}{2} & k = 0 \end{cases}$$

§2.2 Hmm, integrals

The integral $\int \frac{1}{1+x^2}dx$ is something that you would have encountered a lot in your high school calculus and some of you might even scream tan in verse ex plus sea (yes, live with it) the moment that you see this fairly innocent integral.

But if I were to task you to find $\int \frac{1}{(1+x^2)^2}dx$ you would be walking into integration-by-parts and partial-fractions war-zone territory. Let us not ponder over that mess now!



A spooky figure made using $\frac{1}{1+x^2}$ *and* $\frac{1}{(1+x^2)^2}$

If I were to simplify things a bit, and ask you to find the value of the definite integral

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}dx$$

what would your approach be? There is a really elegant way to compute this integral using the concepts that were taught in the third session. Try to recall them and compute the value of this definite integral.

Hint: Think of a function whose Fourier transform is $\frac{1}{1+x^2}$ *or in other terms finding the inverse Fourier transform of* $\frac{1}{1+x^2}$ *might be helpful to you*

Final Expected Answer: The value of the definite integral $\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}dx$ computed using the concepts taught in the third session.

Solution:

The Fourier Transform of the function $e^{-|t|}$ is $\frac{2}{1+\omega^2}$. Using Parseval's Theorem along with this observation gives us the equation:

$$\int_{-\infty}^{\infty} \left(e^{-|t|}\right)^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{2}{1+\omega^2}\right)^2 d\omega$$

$$\implies \int_{-\infty}^{\infty} \frac{4}{(1+\omega^2)^2}d\omega = 2\pi \times \int_{-\infty}^{\infty} e^{-2|t|}dt = 4\pi \times \int_{0}^{\infty} e^{-2t}dt$$

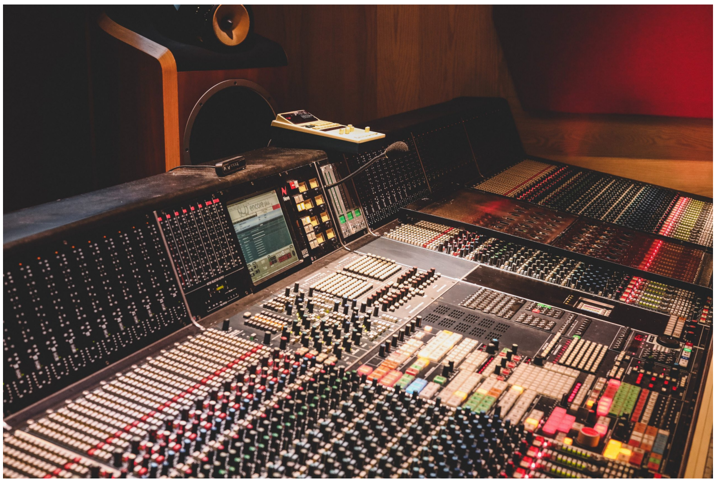
$$\implies \int_{-\infty}^{\infty} \frac{1}{(1+\omega^2)^2}d\omega = \pi \times \int_{0}^{\infty} e^{-2t}dt = \pi \times \frac{1}{2} = \frac{\pi}{2}$$

Thus, we have computed the value of the required integral; $\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}dx = \frac{\pi}{2}$

§3 A Tale of Musical Mismanagement

Welcome to the clumsy life of Navin!

This morning Navin was assigned the work of recording an orchestral performance that was being performed by the coordinators of the Mathematics Club. He was asked to record each instrument in a different channel, so that the audio can be equalized easily later on by our Music Directors - KK and Achintya. Unfortunately, Navin being the clumsy person that he is accidentally ended up recording all of the instruments *as a single audio file*.



The Music directors want to **tone down** (decrease the volume of) **the snare drums by 25%**. The snare drums used by the Mathematics Club produce sounds in the **frequency range of 150Hz to 250Hz**. Assume that no other musical instruments produce sounds in this audio range.
Luckily, he has *you* to help him! Navin hoping that you listened to his presentation attentively is asking you to write a program in `MATLAB` or `GNU Octave` which fulfills the directors' request and saves his job!

Final Expected Submission: A script written in `MATLAB` or `GNU Octave` that performs the following:

- Reads an audio file named *"i_hope_i_don't_lose_my_job.wav"* using the function `audioread` as an array along with the sample-rate of the audio
- Uses Fourier Transform methods to perform the request of reducing the volume of the drums by 25% in the audio data
- Writes the newly generated audio data using the function `audiowrite` to an audio file named *"thankyou_for_saving_my_job.wav"*

Note:

1. The functions `audioread`, `audiowrite` have been documented here: 1, 2, 3.
   Feel free to reach out to us in case you don't understand how these functions work!
2. If you are submitting your code via a Google Drive / GitHub link make sure to enable access for all, so that we will be able to access your submission.

Solution:

The GNU Octave / MATLAB code is given below. This code also accounts for multiple channels being present in the audio file.

```
clc;
clear;

% Parameters for our filter, you can play around with them
f1 = 150;
f2 = 250;
atten = 0.75;

% Read the audio file
[inputAudio, fs] = audioread('i_hope_i_dont_lose_my_job.wav');

% Get the number of channels in it
[numSamples, numChannels] = size(inputAudio);

% Run through each channel in the audio file and perform Fourier
↪  Transform
for i = 1:numChannels
    audioFT(:, i) = fft(inputAudio(:, i));
end

% Calculate the list of frequencies generated by the Fourier Transform
frequencies = (0:numSamples-1)*(fs/numSamples);

% Find indices corresponding to the 150Hz to 250Hz range
idx = (frequencies >= f1 & frequencies <= f2) | (frequencies >= (fs - f2)
↪  & frequencies <= (fs - f1));

% Reduce the corresponding frequencies by 25%
audioFT(idx, :, :) = audioFT(idx, :, :) * atten;

% Perform inverse Fourier Transform
for i = 1:numChannels
    modifiedAudio(:, i) = ifft(audioFT(:, i));
end

% Ensure the audio is real to account for computational errors
modifiedAudio = real(modifiedAudio);

% Write the modified audio to a new file
audiowrite('thankyou_for_saving_my_job.wav', modifiedAudio, fs);
```

If you have not accounted for multiple channels, your code might look like this:

```
clc;
clear;

% Parameters for our filter, you can play around with them
f1 = 150;
f2 = 250;
atten = 0.75;

% Read the audio file
[inputAudio, fs] = audioread('i_hope_i_dont_lose_my_job.wav');
numSamples = length(inputAudio);

% Perform Fourier Transform on the read audio file
audioFT = fft(inputAudio);

% Calculate the list of frequencies generated by the Fourier Transform
frequencies = (0:numSamples-1)*(fs/numSamples);

% Find indices corresponding to the 150Hz to 250Hz range
idx = (frequencies >= f1 & frequencies <= f2) | (frequencies >= (fs - f2)
↪  & frequencies <= (fs - f1));

% Reduce the corresponding frequencies by 25%
audioFT(idx, :) = audioFT(idx, :) * atten;

% Perform inverse Fourier Transform
modifiedAudio = ifft(audioFT);

% Ensure the audio is real to account for computational errors
modifiedAudio = real(modifiedAudio);

% Write the modified audio to a new file
audiowrite('thankyou_for_saving_my_job.wav', modifiedAudio, fs);
```

An alternative approach would be to use `for` loops instead of using the smart indexing features provided in `GNU Octave` / `MATLAB` to filter the frequencies from 150Hz to 250Hz. You can also use `fftshift` to reorder the frequencies if you wish to.

Your code doesn't need to be exactly the same as the solution given here. As long as your code has the same functionality as what is demanded by this problem you will get a good grade.

Bonus Question 2: Thinking in the Time Domain [For those with 7/10 attendance]:

NOTE: *For those who have already met the attendance criteria, you DO NOT need to submit this! You will be eligible for the certificate even if you don't attempt this question. This is just a bonus question. I will be very happy if you attempt this question, and that is a good thing!*

Can you think of a way to perform the same task of changing the contribution of some frequencies in a given function of time $f(t)$ **without** analysing the function in the frequency-domain using a Fourier Transform, instead working entirely in the time-domain. You can use the same example as the above question or use any example that you like.

Explain your algorithm.

Solution:

Let $X(\omega)$ be the Fourier Transform of your time-domain signal $x(t)$ and the filter $\mathcal{H}$ (attenuation of certain frequencies) that you wish apply be represented in the frequency-domain by $H(\omega)$. Let $h(t)$ be the inverse Fourier Transform of $H(\omega)$, that is the $h(t)$ is the time-domain representation of the filter.

You might have noticed that when you are applying the filter $\mathcal{H}$ to $x(t)$, what you are doing is: computing $X(\omega)$, multiplying it with $H(\omega)$ and then computing the inverse Fourier Transform of $X(\omega)H(\omega)$ to obtain $x_{new}(t)$. Let's try to perform this for a general x(t).

$$x_{new}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega t}X(\omega)H(\omega)d\omega$$

Writing $X(\omega)$ in terms of $x(t)$ using the expression for Fourier Transform, we get

$$2\pi x_{new}(t) = \int_{-\infty}^{+\infty} e^{i\omega t}\left(\int_{-\infty}^{+\infty} e^{-i\omega\tau}x(\tau)d\tau\right)H(\omega)d\omega = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i\omega(t-\tau)}x(\tau)H(\omega)d\tau d\omega$$

Now we change the order of the integration using Fubini's theorem from $d\tau d\omega$ to $d\omega d\tau$. Note that we are assuming here that the order of the integration can be reversed. This can be done as long as the inner integral exists and the functions inside are integrable, which is indeed true since we have assumed beforehand that $X(\omega)$ and every other quantity involved exist.

$$2\pi x_{new}(t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i\omega(t-\tau)}x(\tau)H(\omega)d\omega d\tau = \int_{-\infty}^{+\infty} x(\tau)\left(\int_{-\infty}^{+\infty} e^{i\omega t}H(\omega)e^{-i\omega\tau}d\omega\right)d\tau$$

$$\implies x_{new}(t) = \int_{-\infty}^{+\infty} x(\tau)\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega t}H(\omega)e^{-i\omega\tau}d\omega\right)d\tau$$

The expression $H(\omega)e^{-i\omega\tau}$ is simply the Fourier Transform of $h(t-\tau)$ and the expression $\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega t}(H(\omega)e^{-i\omega\tau})d\omega$ is nothing but the inverse Fourier Transform of $H(\omega)e^{-i\omega\tau}$. Using these observations,

$$x_{new}(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = (x * h)(t)$$

In the expression obtained above, we have shown the can result of the filtering process be obtained directly by convolving the original signal $x(t)$ with the time-domain representation of the required filter $h(t)$. The process of working in the frequency domain has been completely eliminated.

Bonus Question 3: A more sophisticated Filter [For those with 7/10 attendance]:

NOTE: *For those who have already met the attendance criteria, you DO NOT need to submit this! You will be eligible for the certificate even if you don't attempt this question. This is just a bonus question. I will be very happy if you attempt this question, and that is a good thing!*

Repeat Problem 3 (A Tale of Musical Mismanagement) but instead of toning down the amplitude of all frequencies in 150Hz to 250Hz by 25%, you have to tone down the frequencies:

- 150Hz to 175Hz by 20%,
- frequencies 175Hz to 225Hz by 25%,
- and frequencies 225Hz to 250Hz by 15%.

Follow the same instructions as given in Problem 3.

Solution:

The modifications to `GNU Octave` / `MATLAB` code in Problem 3 in order to make the filtering more sophisticated is given here. The variations in the code mentioned in the solution to Problem 3 apply here as well. You can work with mono or stereo audio files, use `fftshift` or not, and so on. The code snippet here shows only the modifications to the filtering process.

```
...

% Perform Fourier Transform on the read audio file
audioFT = fft(inputAudio);

% Parameters for our filter, you can play around with them
filterFreqs = [150, 175, 225, 250]
filterAttens = [0.80, 0.75, 0.85]

% Apply the filter to the given frequency ranges
for i = 1:length(filterAttens)
    f1 = filterFreqs(i);
    f2 = filterFreqs(i+1);

    % Find indices corresponding to each frequency range
    idx = (frequencies >= f1 & frequencies <= f2) | (frequencies >= (fs -
    ↪  f2) & frequencies <= (fs - f1));

    % Multiply the corresponding frequencies by their attenuation factor
    audioFT(idx, :) = audioFT(idx, :) * filterAttens(i);
end

...
```

Appendix

Let the exponential series be of the form $\sum_{k=-\infty}^{\infty} A_k e^{ik\omega_0 t}$ and the trigonometric series be of the form $\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k cos(\omega_0 t) + \sum_{k=1}^{\infty} b_k sin(\omega_0 t)$. Then, it can be shown that:

- $a_k = A_k + A_{-k}$
- $b_k = i(A_k - A_{-k})$

Note that $k \in \mathbb{Z}, k \geq 0$ or put simply k ranges from 0 (inclusive) to ∞ in integers.