MathLab: Problem Set 1

Solutions

Mathematics Club IITM

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- The solutions to the 4 problems in Problem Set 1 can be found in this solution sheet.
- Feel free to reach out to us for doubts! Contact information of the solution-set creators:

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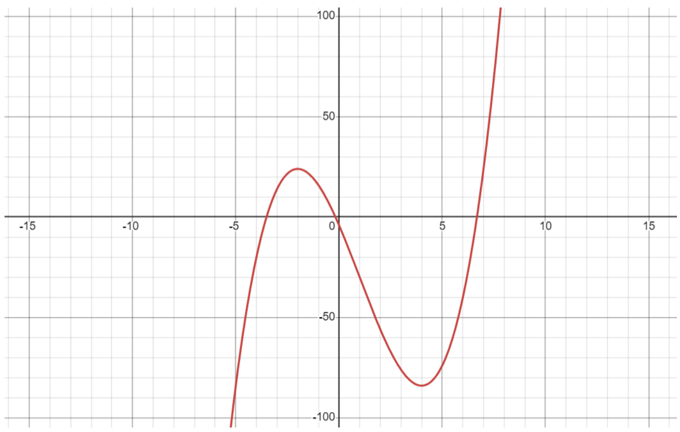
Problem 1: The Newton-Raphson Method



A plot of the function $f(x)$

- In the first session, we have covered the Newton-Raphson method which can be used to find roots of a function. Keeping that in mind answer the following questions.
- Our beloved coordinator Deena tries to solve the polynomial

$$f(x) = x^3 - 3x^2 - 24x - 4$$

  with the initial guess $x_1 = 4$. Can he find the root of the polynomial?
  - If yes, give the first two iterations i.e., x_2 and x_3
  - If no, mention the reason

A plot of $f(x)$ has been given in the first page for your reference.

- Deena gives up on the initial value of $x_1 = 4$ and **chooses another** x_1. He finds out that he cannot get a solution through Newton-Raphson method for the newly chosen value of x_1. What is the value of the **newly chosen** x_1 ?

Solution:

The iterative statement for Newton Raphson's method is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The function given to us is $f(x) = x^3 - 3x^2 - 24x - 4 \implies f'(x) = 3x^2 - 6x - 24$

For $x_1 = 4$,

$$f(4) = 4^3 - 3 \times 4^2 - 24 \times 4 - 4 = 64 - 48 - 96 - 4 = 0$$

and

$$f'(4) = 3(4^2 - 2 \times 4 - 8) = 0$$

Since $f'(4) = 0$, the tangent will never cut the x-axis and plugging the values obtained above in the iterative scheme will involve division by 0 which is not allowed. This implies that the Newton-Raphson method can't be used to find a solution.

This is because the initial point is an extrema/inflection point i.e $f'(x_1) = 0$

As Deena is unable to find a solution even with the new value of x_1, the new x_1 must also be an extrema or an inflection point. On factorising $f'(x)$, we get $f'(x) = 3(x-4)(x+2)$.On equating $f'(x_1)$ to 0, we get $3(x_1 - 4)(x_1 + 2) = 0 \implies x_1 = 4$ or $x_1 = -2$

Therefore the newly chosen $x_1 = -2$

Problem 2: On Orders and Degrees

Find the Order and Degree of the following Differential Equations:

1. $\left(\frac{d^3y}{dx^3}\right)^2 + 2\left(\frac{d^2y}{dx^2}\right)^3 + 3\left(\frac{dy}{dx}\right)^4 = 0$
2. $\left(\frac{d^4y}{dx^4}\right)^{\frac{5}{7}} = 4\left(\frac{d^3y}{dx^3}\right)^2 - \left(\frac{dy}{dx}\right)$
3. $\sin\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^2 + y = 0$
4. $\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^5 - e^{\frac{d^3y}{dx^3}} = 0$

Final Expected Answer: State the Order and Degree of the given Differential Equations and mention "Not defined" with an explanation whenever the Order or Degree is not defined.

Solution:

1. $$\left(\frac{d^3y}{dx^3}\right)^2 + 2\left(\frac{d^2y}{dx^2}\right)^3 + 3\left(\frac{dy}{dx}\right)^4 = 0$$
   - The order is 3 because the highest derivative involved is the third derivative.
   - The degree is 2 because the highest exponent of the third derivative is 2.
2. $$\left(\frac{d^4y}{dx^4}\right)^{\frac{5}{7}} = 4\left(\frac{d^3y}{dx^3}\right)^2 - \left(\frac{dy}{dx}\right)$$
   - The order is 4 because the highest derivative involved is the fourth derivative.
   - The differential equation is not in polynomial form. It can be converted to polynomial form by taking power 7 on both sides. Now the exponent of fourth derivative is 5 hence, the degree is 5
3. $$\sin\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^2 + y = 0$$
   - The order is 2 as highest derivative involved is 2.
   - The degree is not defined because it cannot be written in polynomial form because a transcendental function ($\sin(\cdot)$) is involved.
4. $$\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^5 - e^{\frac{d^3y}{dx^3}} = 0$$
   - The order is 3 because the highest derivative involved is the third derivative.
   - The degree is not defined as there is a transcendental function ($e^{(\cdot)}$) involved

Problem 3: Classification Conundrums

Classify the following Differential Equations based on Linearity and Homogeneity.

1. $\frac{d^2u}{dx^2} + \frac{u}{1+x^2} = 0$

2. $y'' + 3y' - 4y = \sin(x)$

3. $y'' + e^y y' + \cos(y) = \cos(x)$

4. $y'' + \sin(y)y' + y^3 = 0$

Final Expected Answer: State if the given Differential Equations are i) Linear or Non-linear and ii) Homogenous or Non-homogenous.

Solution:

1. $$\frac{d^2u}{dx^2} + \frac{u}{1+x^2} = 0$$

   • This is a linear differential equation because it can be written in the form of $a_2(x)u'' + a_1(x)u' + a_0(x)u + b(x) = 0$
   • This is also a homogeneous differential equation because we can apply the transformation $\overline{y} = ay$ and $\overline{x} = x$ to get the same differential equation again.

2. $y'' + 3y' - 4y = \sin(x)$

   • This is a linear differential equation because it is in the form $a_2(x)y'' + a_1(x)y' + a_0(x)y + b(x) = 0$.
   • This is not a homogeneous differential equation because of the terms $sin(x)$ due to which there is no invariance in the differential equation no matter which allowed transformation (scaling x, y and derivatives of y) we apply.

3. $y'' + e^y y' + \cos(y) = \cos(x)$

   • This is not a linear differential equation due to the terms e^y and $\cos(y)$. It cannot be written in the form $a_2(x)y'' + a_1(x)y' + a_0(x)y + b(x) = 0$.
   • This is not a homogeneous differential equation because of the cosine and exponential terms due to which there is no invariance in the differential equation no matter which allowed transformation we apply.

4. $y'' + \sin(y)y' + y^3 = 0$

   • This is not a Linear differential equation due to the terms $\sin(y)$ and y^3. It cannot be written in the form $a_2(x)y'' + a_1(x)y' + a_0(x)y + b(x) = 0$.
   • But this is a homogeneous differential equation because we can apply the transformation $\overline{y} = ay$ and $\overline{x} = x$ to obtain the same differential equation again.

Problem 4: Pradyumnan's Interests

Pradyumnan borrowed Rs. 6,40,000 to buy a car. The lender charges interest at an annual rate of 10%. Assuming that interest is compounded continuously and that the borrower makes payments continuously at a constant annual rate k, determine the payment rate k that is required to pay off the loan in 3 years. Also determine how much interest is paid during the 3-year period.

Note that by "continuously" we mean every passing instant of time. Every nanosecond counts!

Set up equations and show your working clearly while solving this question.

Solution:

Let the amount to be given by Pradyumnan as the dependent variable as A. We know that the rate of change of A is a function of compound interest rate and annual return rate.
A increases on account of accumulating interest and decreases with the annual return rate.
Now writing this as an equation gives us:

$$\frac{dA}{dt} = 0.1 \times A - k$$

$$\frac{dA}{dt} - 0.1A = k$$

We have a first order linear differential equation with us. Multiplying with the integrating factor, we get:

$$e^{-0.1t}\frac{dA}{dt} - 0.1e^{-0.1t}A = -ke^{-0.1t} \implies \frac{d(Ae^{-0.1t})}{dt} = -ke^{-0.1t}$$

$$\implies Ae^{-0.1t} = \int -ke^{-0.1t}\,dt = 10ke^{-0.1t} + C$$

Thus, we obtain $A(t) = 10k + Ce^{0.1t}$ as the solution to the differential equation.

On applying the boundary conditions, $A(0) = 6,40,000$ and $A(3) = 0$ we get

$$10k + C = 640000 \text{ (From } t = 0) \tag{1}$$

$$10k + Ce^{0.3} = 0 \implies C = -10ke^{-0.3} \text{ (From } t = 3) \tag{2}$$

On substituting (2) in (1)

$$10k(1 - e^{-0.3}) = 640000 \implies k = \frac{64000}{1 - e^{-0.3}} \implies k \approx \text{Rs. } 2{,}46{,}930$$

The interest paid in 3 years is nothing but **Amount returned - Amount borrowed**

$$\textbf{Interest} = 3k - A(0)$$

$$\textbf{Interest} \approx 3 \times 246930 - 640000$$

$$\textbf{Interest} \approx 740790 - 640000$$

Therefore the interest paid by Pradyumnan is $\approx$ Rs. 1,00,790