Session 3.1 - Appendix

Achintya Raghavan

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§1 Intro (Slides 1-3)

¶ What are linear constant coefficient differential equations?

Breaking it down:

1. **Linear DE:** DEs where the differential terms are not part of some other function. Like $sin(\frac{dy}{dx})$ or $\frac{dy}{dx2}^2$.
2. **Constant Coefficient DE:** DEs where the attached scaling for the derivative term is independent of x, i.e. it is a numerical constant. Eg: $5\frac{dy}{dx}$ is const. coeff. while $x\frac{dy}{dx}$ is not.

Problem 1.1. (Slide 3)
$\frac{dy}{dx} = ky$
$\Rightarrow \frac{dy}{ky} = dx$
$\Rightarrow \frac{1}{k}ln(y) = x + c$
$\Rightarrow y = Ae^{kx}$ where A is some constant.

¶ Another way to write a known solution to the spring (Slide 3)

$$A\ sin(\omega t) + B\ cos(\omega t)$$

$$\implies A\left(\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\right) + B\left(\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right)$$

$$\implies \left(\frac{A}{2i} + \frac{B}{2}\right)e^{i\omega t} + \left(\frac{A}{2i} - \frac{B}{2}\right)e^{-i\omega t}$$

$$\implies \alpha e^{i\omega t} + \beta e^{-i\omega t}$$

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§2 The Exponential Function (Slide 4-6)

¶ What is it? (Slide 4)

e^{sx} where:
$s = k + i\omega$ contains 2 parts: The imaginary part (which oscillates) and the real part (which decays or grows).

¶ Solving homogeneous DE using the exponential (Slide 5)

The derivation is in the slides so I won't rewrite it here:

- Note that the characteristic polynomial has n roots.
- It can be shown that linear combinations of these solutions are also solutions. Thus the general solution is:

$$\sum_{k=1}^{n} A_i e^{s_i x}$$

where s_i are the roots of the characteristic polynomial.

Problem 2.1. (Slide 6)

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega^2 x = 0$$

$$Substituting\ e^{st},\ we\ get:$$

$$\frac{d^2e^{st}}{dt^2} + 2\gamma\frac{de^{st}}{dt} + \omega^2 e^{st} = 0$$

$$\implies e^{st}\left(\ s^2 + 2\gamma s + \omega^2\right) = 0$$

$$\implies s = \ -\gamma \pm \sqrt{\gamma^2 - \omega^2}$$

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§3 Non-Homogeneous Equations (Slide 7-12)

§3.1 A special case (Slide 7-9)

¶ Solution when $f(x) = e^{sx}$ (Slide 8)

Solution to damped driven oscillator has been given in the slides.

For a **general DE**:

$$\sum_{k=1}^{n} a_k \frac{d^k y}{dx^k} = e^{s_0 x}$$

Assuming $Ae^{s_0 x}$ as a solution,

$$\sum_{k=1}^{n} a_k \frac{d^k Ae^{s_0 x}}{dx^k} = e^{s_0 x}$$

$$\implies \sum_{k=1}^{n} a_k A s_0^k e^{s_0 x} = e^{s_0 x}$$

$$\implies \sum_{k=1}^{n} a_k A s_0^k = 1$$

$$\implies A = \frac{1}{\sum_{k=1}^{n} a_k s_0^k}$$

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§3.2 A very special function (Slide 11-12)

¶ Intro (Slide 11)

- The previous sections/slides dealt with a special case of $f(x)$.
- We'd like to deal with the general case (or at least, the periodic case).
- We thus analyse the series $\sum_{k=-\infty}^{\infty} A_k e^{ik\omega_0 t}$
- Why this series? Because:

> **Theorem**
>
> Every periodic function can be represented in terms of its Fourier Series $\sum_{k=-\infty}^{\infty} A_k e^{ik\omega_0 t}$ (**The Fourier Theorem**)

- It is equivalent to a sine form which is given on Slide 14

Problem 3.1. (Slide 11)

We assume the solution to $\dfrac{d^2x}{dt^2} + 2\gamma\dfrac{dx}{dt} + \omega^2 x = \displaystyle\sum_{k=-\infty}^{\infty} a_k\, e^{ik\omega_0 t}$ is $\displaystyle\sum_{k=-\infty}^{\infty} b_k e^{ik\omega_0 t}$

$$\frac{d^2 \sum_{k=-\infty}^{\infty} b_k e^{ik\omega_0 t}}{dt^2} + 2\gamma \frac{d \sum_{k=-\infty}^{\infty} b_k e^{ik\omega_0 t}}{dt} + \omega^2 \sum_{k=-\infty}^{\infty} b_k e^{ik\omega_0 t} \; = \sum_{k=-\infty}^{\infty} \; a_k e^{ik\omega_0 t}$$

$$\implies \quad \sum_{k=-\infty}^{\infty} -k^2\omega_0^2 \; b_k e^{ik\omega_0 t} + \sum_{k=-\infty}^{\infty} 2i\gamma k\omega_0 \; b_k e^{ik\omega_0 t} + \sum_{k=-\infty}^{\infty} \omega^2 \; b_k e^{ik\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{ik\omega_0 t}$$

$$\implies \quad \sum_{k=-\infty}^{\infty} b_k \left(-k^2\omega_0^2 + 2i\gamma k\omega_0 + \omega^2\right) e^{ik\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{ik\omega_0 t}$$

Applying the Linear Independence Property (*Section* 4)

$$\implies \quad b_k = \frac{a_k}{2i\gamma k\omega_0 + \omega^2 - \omega_0^2}$$

§4 Linear Independence of Exponentials with different frequencies (Slide 10)

I felt this topic warranted its own Section.

To summerize:

- Exponentials with different discrete frequencies like $e^{ik_1\omega t}$ and $e^{ik_2\omega t}$ are not scaled versions of each other, i.e. $e^{ik_2\omega t} = Ae^{ik_1\omega t}$ has no solution for A
- This property is called linear independence.
- Further. linear independence forces $Ae^{ik_1\omega t} + Be^{ik_2\omega t} = Ce^{ik_1\omega t} + De^{ik_2\omega t}$ to imply $A = C \quad \& \quad B = D$

Lemma 4.1

$e^{ik_2\omega t} = Ae^{ik_1\omega t}$ has no solution for A

Proof.

We take magnitude on both sides. This gives:

$|e^{ik_2\omega t}| = |A|\ |e^{ik_1\omega t}|$

$\implies |A| = 1$

$\implies A = e^{i\theta}$

Now we substitute $A = e^{i\theta}$ in $e^{ik_2\omega t} = Ae^{ik_1\omega t}$ giving us:

$e^{ik_2\omega t} = e^{i(k_1\omega t+\theta)}$

$\implies k_2\omega t = k_1\omega t + \theta + 2n\pi$

$\implies (k_2 - k_1)\omega t - 2n\pi = \theta$

But we expect A to be independent of t. Therefore the only solution is $k_2 = k_1$ which contradicts the assumption that the exponents have 2 different frequencies. □

Lemma 4.2

$Ae^{ik_1\omega t} + Be^{ik_2\omega t} = Ce^{ik_1\omega t} + De^{ik_2\omega t} \implies A = C \ \& \ B = D$

Proof.

We know that $e^{ik_2\omega t} = \phi e^{ik_1\omega t}$ has no solution for ϕ.

Thus, the only solution to $\alpha e^{ik_2\omega t} = \beta e^{ik_1\omega t}$ is $\alpha = \beta = 0$.

Let:

$\alpha = A - C \quad \& \quad \beta = D - B$

Then the equation becomes:

$Ae^{ik_1\omega t} + Be^{ik_2\omega t} = Ce^{ik_1\omega t} + De^{ik_2\omega t}$

with the only solution being $A - C = 0 \ \ \& \ \ B - D = 0$ □

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