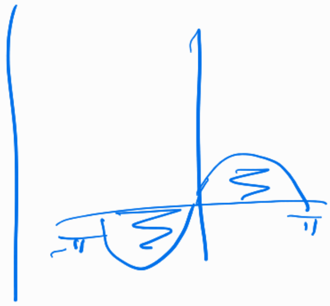
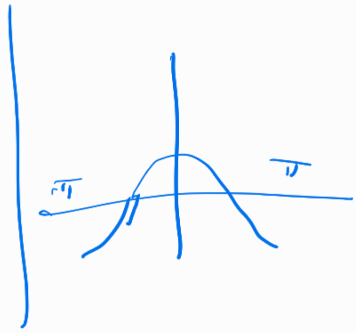
Begin:

1. $$\int_{-\pi}^{\pi} \sin mx \, dx = 0 \quad \forall m$$



2. $$\int_{-\pi}^{\pi} \cos mx \, dx = 0 \qquad m \neq 0$$
$$= 2\pi \qquad m = 0$$



3. $$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx$$
$$= \int_{-\pi}^{\pi} \left(\frac{\sin(m+n)x + \sin(m-n)x}{2} \right) dx$$
$$= \frac{1}{2} \left(\underbrace{\int_{-\pi}^{\pi} \sin(m+n)x \, dx}_{\to 0} + \underbrace{\int_{-\pi}^{\pi} \sin(m-n)x \, dx}_{\to 0} \right)$$
$$= 0$$

$$\boxed{\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0 \quad \forall m, n}$$

4. $$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \int_{-\pi}^{\pi} \left(\frac{\cos(m-n)x - \cos(m+n)x}{2} \right) dx$$

$m, n \geq 1$

$$= \frac{1}{2}\left[\int_{-\pi}^{\pi} \cos(m-n)x\,dx + \int_{-\pi}^{\pi} \cancel{\cos(m+n)x}\,dx\right]$$

$m = n \Rightarrow 2\pi$

$m \neq n \Rightarrow 0$

$$m = n \Rightarrow \quad = \frac{1}{2}\left[2\pi + 0\right] = \pi$$

$$m \neq n \Rightarrow \quad = \frac{1}{2}\left[0 + 0\right] = 0$$

5. $$\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = \int_{-\pi}^{\pi}\left(\frac{\cos(m+n)x + \cos(m-n)x}{2}\right)dx$$

$$= \frac{1}{2}\left[\int_{-\pi}^{\pi} \cancel{\cos(m+n)x}\,dx + \int_{-\pi}^{\pi} \cos(m-n)x\,dx\right]$$

$m = n \Rightarrow \pi$

$m \neq n \Rightarrow 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\int_{-\pi}^{\pi} f(x)\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi} 1\,dx + \sum_{n=1}^{\infty}\left[a_n \int_{-\pi}^{\pi} \cancel{\cos nx}\,dx + b_n \int_{-\pi}^{\pi} \cancel{\sin nx}\,dx\right]$$

$$\int_{-\pi}^{\pi} f(x)\,dx = \frac{a_0}{\cancel{2}}(\cancel{2}\pi) + 0$$

$$\boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx}$$

$$f(x)\cos mx = \frac{a_0}{2}\cos mx + \sum a_n \cos nx \cos mx$$

$$+ \sum b_n \sin nx \cos mx$$

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \frac{a_0}{2}\cancel{\int_{-\pi}^{\pi}\cos mx\,dx}^{\,0} +$$

$m \geq 1$

$$+ \sum a_n \int_{-\pi}^{\pi} \cos nx \cos mx\,dx$$

$$+ \sum b_n \cancel{\int_{-\pi}^{\pi} \sin nx \cos mx\,dx}^{\,0}$$

$$\int_{-\pi}^{\pi} \cos nx \cos mx\,dx$$

$n = m$

$= \pi$

$$= 0 + a_m(\pi) + 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$\int_{-\pi}^{\pi} f(x) \sin mx \, dx = \frac{a_0}{2} \cancelto{0}{\int_{-\pi}^{\pi} \sin mx \, dx}$$

$$+ \sum a_n \cancelto{0}{\int_{-\pi}^{\pi} \cos nx \sin mx \, dx}$$

$$+ \sum b_n \int_{-\pi}^{\pi} \sin nx \sin mx \, dx$$

$$= b_m (\pi)$$

$$\boxed{b_n = \frac{1}{\pi} \int f(x) \sin nx \, dx}$$

$$f(x) = \frac{a_0}{2} + \sum a_n \cos nx + \sum b_n \sin nx$$

$$[-\pi, \pi] \qquad [-L, L]$$

$$\downarrow \qquad\qquad \downarrow$$

$$x \qquad\qquad y$$

$$\frac{x}{\pi} \in [-1,1] \qquad \frac{y}{L} \in [-1,1]$$

$$\frac{x}{\pi} = \frac{y}{L} \Rightarrow y = \frac{Lx}{\pi}$$

$$\underline{\underline{x}} = \frac{y\pi}{L} \qquad dx = \underline{\underline{\frac{\pi}{L}(dy)}}$$

$$f(y) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi y}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi y}{L}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx \qquad y = \frac{Lx}{\pi}$$

$$= \frac{1}{\cancel{\pi}}\int_{-L}^{L} f(y) \frac{\cancel{\pi}}{L}\, dy \qquad y \in [-L, L]$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(y)\, dy$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos n\underline{x}\, dx$$

$$= \frac{1}{L}\int_{-L}^{L} f(y) \cos\left(\frac{n\pi y}{L}\right) dy$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(y)\sin\left(\frac{n\pi y}{L}\right)dy$$

$$\int_{-a}^{a} \text{odd fn } dx = \text{even}\Big|_{-a}^{a}$$

$$= \cancel{\text{even}(a)} - \cancel{\text{even}(a)}$$

$$= 0$$

$$\int_{-a}^{a} \text{even } dx = 2\int_{0}^{a} \text{even } dx$$

$f(x)$: even fn

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx$$

$f(x)\sin nx \rightarrow$ odd

even $\times$ odd $=$ odd

$b_n = 0 \quad \forall n$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \, dx$$

$f(x)$: odd fn

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \overbrace{f(x)\cos nx}^{\to \text{odd}}\, dx = 0$$

$$\underset{\text{odd}}{f(x)} \times \underset{\text{even}}{\cos nx} = \text{odd}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = 0$$

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\, dx$$

$$f(x) = \begin{cases} -1 & -\pi \le x < 0 \\ 1 & 0 \le x < \pi \end{cases}$$

↳ odd function

$x > 0$

$$f(x) = -f(-x)$$

$$1 \qquad -(-1)$$

$$\boxed{a_0, a_n = 0}$$

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\, dx$$

$$f(x) = 1$$

$$= \frac{2}{\pi} \int_0^{\pi} \sin nx \, dx$$

$$= \frac{2}{\pi} \left[-\frac{\cos nx}{n} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{-\cos n\pi + \cos 0}{n} \right]$$

$$b_n = \frac{2}{n\pi} \left[1 - \cos n\pi \right]$$

$\cos \pi = -1$

$\cos 2\pi = 1$

$\cos 3\pi = -1$

$\cos 4\pi = 1$

$n = \text{odd} \Rightarrow \cos n\pi = -1$

$n = \text{even} \Rightarrow \cos n\pi = 1$

$$n = \text{odd} \Rightarrow b_n = \frac{2}{n\pi} \left[\underbrace{1 - (-1)}_{2} \right] = \frac{4}{n\pi}$$

$$n = \text{even} \Rightarrow b_n = \frac{2}{n\pi} \left[\underbrace{1 - 1}_{0} \right] = 0$$

$a_0 = 0$

$a_n = 0$

$$b_n = \begin{cases} \frac{4}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

$$f(x) = \cancel{\frac{a_0}{2}}^{\,0} + \cancel{\sum a_n \cos nx}^{\,0} + \sum b_n \sin nx$$

$$= \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \sum_{\substack{n=1,3,\dots \\ \text{odd}}}^{\infty} \underbrace{b_n}_{\frac{4}{n\pi}} \sin nx + \cancel{\sum_{\substack{n=2,4,\dots \\ \text{even}}}^{\infty} b_n \sin nx}^{\,0}$$

$$f(x) = \frac{4}{\pi} \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{\sin nx}{n}$$

$n = 1$ to ∞ odd

$n = 2k+1$ $\quad$ $k = 0$ to ∞

$$\boxed{f(x) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1}}$$

Required: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$

$$x = \frac{\pi}{2} \Rightarrow \underbrace{f\left(\frac{\pi}{2}\right)}_{1} = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\frac{\pi}{2}}{(2n+1)}$$

$$1 = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\frac{\pi}{2}}{(2n+1)}$$

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{\sin(2n+1)\frac{\pi}{2}}{(2n+1)} \rightarrow (1)$$

$n=0$

$\sin\frac{\pi}{2} = 1$

$n=1$

$\sin 3\frac{\pi}{2} = -1$

$n=2$

$\sin 5\frac{\pi}{2} = 1$

$n=3$

$\sin 7\frac{\pi}{2} = -1$

$n = \text{even} \Rightarrow \sin = 1$

$n = \text{odd} \Rightarrow \sin = -1$

$$\Rightarrow \sin(2n+1)\frac{\pi}{2} = (-1)^n$$

$$(1) \Rightarrow \boxed{\boxed{\frac{\pi}{4}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}}$$