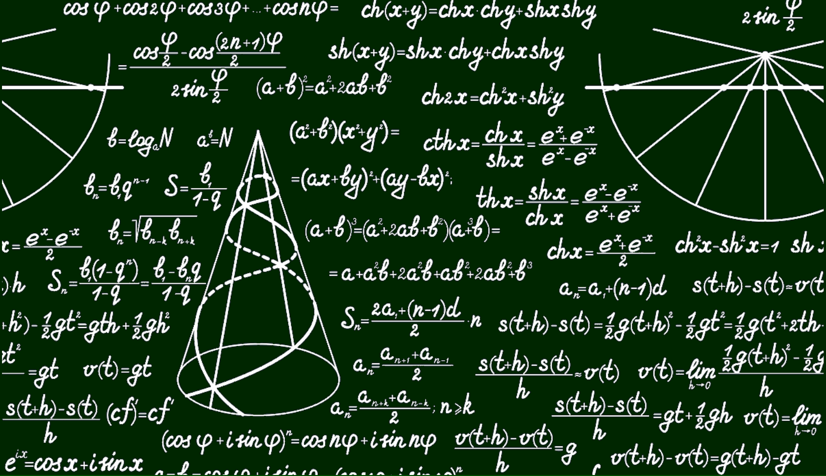


156RCs

MATH1560J – Fall 2023

Li Mingrui, Xia Yiwei, Zhang Haoran

Sincerest appreciation dedicated to
2024 VV156 TAs
Li Mingrui, Xia Yiwei, Zhang Haoran,
as well as all previous VV156 TAs.

1. About Honors Calculus

Contents

2024FA-VV156

1. Limits
2. Derivatives and Integrals
3. Series
4. Polar Coordinates
5. Basic Differential Equations

2024SU-VV255

1. Simple Linear Algebra
2. Partial Derivatives
3. Multiple Integrals

2024FA-VV256

1. Differential Equations
2. Deeper Linear Algebra
3. Fourier Transform and Laplace Transform

Other courses might contribute to Honors Calculus: VV214, VE203, etc.

Useful Software and Websites

MATLAB:

Mathematica:

Integration with Steps:

https://www.integral-calculator.com/

About This Slide

You had better understand the contents on green slides. They are relatively fundamental.

About This Slide

The content shown on blue slides may be relatively hard, but may contribute to getting high marks in the exams more or less. Do not worry when you cannot handle it.

About This Slide

Those on pink slides are just some interesting problems and crazy thoughts. Maybe they are of little application in exams, but they are quite interesting.

2. Functions

Three essential factors of a function:

- Domain: D
  A set containing the action objects of correspondence rule.
- Correspondence Rule
- Range: E
  A set of all images corresponding to all elements in the domain under a certain correspondence rule.

Definition

A **function** f is a rule that assigns to **each element** x in a set D exactly one element, called $f(x)$, in a set E.

The Vertical Line Test

A curve in the xy-plane is the graph of a function of x **iff** (if and only if) no vertical line intersects the curve more than once.

There are four ways to represent a function:

1. Verbally (by a description in words)
2. Numerically (by a table of values)
3. Visually (by a graph)
4. Algebraically (by an explicit formula)

Symmetry

- Even function: $f(-x) = f(x)$
- Odd function: $f(-x) = -f(x)$

Tip: Give priority to whether the domain D of the function is symmetrical.

Increasing & Decreasing property

A function f is called (strictly) **increasing** on an interval I if

$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

A function f is called (strictly) **decreasing** on an interval I if

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

Basic function types

Linear function

The graph of the function is a line:

$$y = f(x) = mx + b$$

m is the slope of the line and b is the y-intercept.

Polynomials

A function P is called a **polynomial** if

$$P(x) = \Sigma_{i=0}^{n} a_i x^i$$

a_i are **coefficients** and n is the **degree** of the polynomial if $a_n \neq 0$.

Quadratic function

A polynomial of degree 2 is of the form $P(x) = ax^2 + bx + c$ and is called a **quadratic function**.

Basic function types

Power function

A function of the form $f(x) = x^a$ is called a **power function**, where a is a constant. Consider an arbitary positive integer n:

- $a = n$:

$$y = x\text{: line} \qquad y = x^2\text{: parabola}$$

- $a = \frac{1}{n}$: **root function**
- $a = -1$: **reciprocal function**

Ratio function

A **Ratio function** f is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

The domain consists of all values of x such that $Q(x) \neq 0$.

Basic function types

Trigonometric function

$$\sin(x + 2\pi) = \sin x \qquad \cos(x + 2\pi) = \cos x \qquad \tan(x + \pi) = \tan x$$

Exponential function

The **exponential functions** are the functions of the form $f(x) = a^x$, where the base a is a positive constant.

Logarithmic function

The **logarithmic functions** $f(x) = \log_a x$, where the base a is a positive constant, are the inverse functions of the exponential functions.

Think about the question:
What condition does a meet when the exponential function increases in its domain? What about the logarithmic function?

How to draw a function

- You can simply trace the dots for simple functions.
- You may also try to use software:
  https://www.geogebra.org
  https://www.desmos.com

Dirichlet Function

$$1_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Special Functions

Impulse Function

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

Step Function

$$H[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

Ramp Function

$$R(x) := \begin{cases} x, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Special Functions

Hyperbolic Function

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \cosh(x) = \frac{e^x + e^{-x}}{2}, \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

Inverse trigonometric function

$$\arcsin(x), \arccos(x), \arctan(x)$$

$$\operatorname{arsinh}(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$$

$$\operatorname{arcosh}(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$$

$$\operatorname{artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

Function Transformation

Vertical and Horizontal Shifts, suppose $c > 0$

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward
$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward
$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right
$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

Vertical and Horizontal Stretching and Reflecting, suppose $c > 1$

$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c
$y = (1/c)f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c
$y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c
$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c
$y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis
$y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis

Definition

Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h, then g, and then f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

Exercise 1

Skip it if you find the exercise quite simple.

Find the domain of these functions:

1. $h(x) = \frac{1}{\sqrt[4]{x^2-5x}}$
2. $f(u) = \frac{u+1}{1+\frac{1}{u+1}}$
3. $F(p) = \sqrt{2-\sqrt{p}}$

Exercise 1

How to solve this kind of questions

- The denominator in the fractional function cannot be zero
- The quantity in the even root formula cannot take a negative value, that is, it should be greater than or equal to zero
- The antilogarithm of the logarithm cannot be negative and zero, that is, it must take a positive value
- The domain of the function $y = \arcsin x$, $y = \arccos x$ is $-1 \leqslant x \leqslant 1$
- $y = \tan x$, $x \neq k\pi + \pi/2$, $y = \cot x$, $x \neq k\pi$, k is integer

Exercise 2

Prove or Disprove

- If f and g are both even functions, is $f + g$ even? If f and g are both odd functions, is $f + g$ odd? What if f is even and g is odd? Justify your answers.
- If f and g are both even functions, is the product fg even? If f and g are both odd functions, is fg odd? What if f is even and g is odd? Justify your answers.

Exercise 3

Graph the functions step by step

$(1) y = 1 - 2\sqrt{x+3}$

$(2) y = |\cos \pi x|$

Exercise 4

Find the function (a) $f \circ g$, (b) $g \circ f$, (c) $f \circ f$, and (d) $g \circ g$ and their domains.

$f(x) = \frac{x}{1+x}, \quad g(x) = \sin 2x$

Exercise 5

Find $f \circ g \circ h$:

- $f(x) = \tan x$
- $g(x) = \frac{x}{x-1}$
- $h(x) = \sqrt[3]{x}$

(It is unnecessary to find the domain of this composite function in this exercise!)

Exercise 6

Composite Function

1. If $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 7$, find a function f such that $f \circ g = h$.
   (Think about what operations you would have to perform on the formula for g to end up with the formula for h.)
2. If $f(x) = 3x + 5$ and $h(x) = 3x^2 + 3x + 2$, find a function g such that $f \circ g = h$.

References

[1] Huang, Yucheng. VV156_RC1_updated.pdf. 2021.

[2] Cai, Runze. Chapter01.pdf. 2021.

[3] Zhou,Yishen.RC1. 2022.

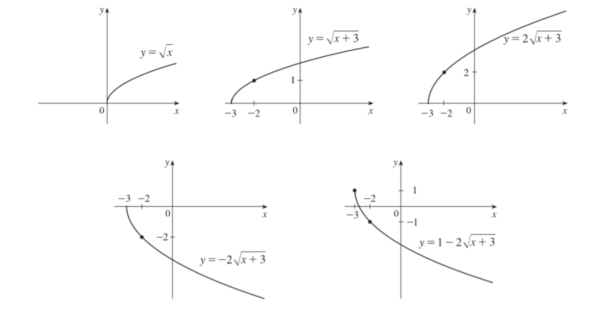
Exercise Answer 1-2

1. $x^2 - 5x > 0 \Rightarrow x \in (-\infty, 0) \cup (5, \infty)$
2. $1 + \frac{1}{u+1} \neq 0, u + 1 \neq 0 \Rightarrow u \in (\infty, -2) \cup (-2, -1) \cup (-1, \infty)$
3. $p \geq 0, 2 - \sqrt{q} \geq 0 \Rightarrow p \in [0, 4]$

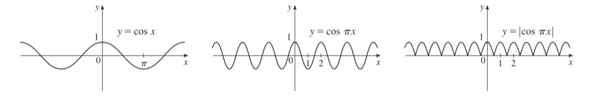
- Yes. Yes. Not necessarily.
- Yes. No.(fg is Even) fg is Odd.

Exercise Answer 3

(1)



(2)



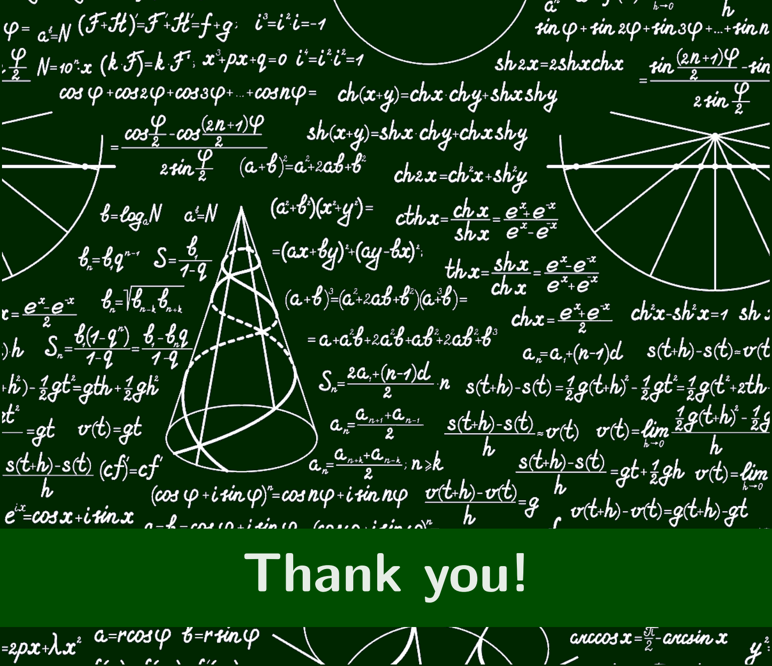
Exercise Answer 4-5

(a) $f \circ g(x) = \frac{sin2x}{1+sin2x}$. Domain: $x \in \{x | x \neq k\pi - \frac{\pi}{4}, k \in \mathbb{B}\}$
(b) $g \circ f(x) = sin(\frac{2x}{1+x})$. Domain: $x \in (\infty, -1) \cup (-1, \infty)$
(c) $f \circ f(x) = \frac{x}{1+2x}(x \neq -1)$.
Domain:$x \in (\infty, -1) \cup (-1, -\frac{1}{2}) \cup (-\frac{1}{2}, \infty)$
(d) $g \circ g(x) = sin(2sin(2x))$. Domain:$x \in \mathbb{R}$

$tan(\frac{\sqrt[3]{x}}{\sqrt[3]{x}-1})$

Exercise Answer 6

1. $x = \frac{g(x)-1}{2}$.
   Plug into h(x), $h(x) = 4(\frac{g(x)-1}{2})^2 + 4(\frac{g(x)-1}{2}) + 7 = g^2(x) + 6$.
   Also, $h(x) = f(g(x))$.
   Therefore, $f(x) = x^2 - 6$
2. $3g(x) + 5 = h(x)$, $g(x) = x^2 + x - 1$



Thank you!