VV156 Mid RC Part II

Derivative and Optimization

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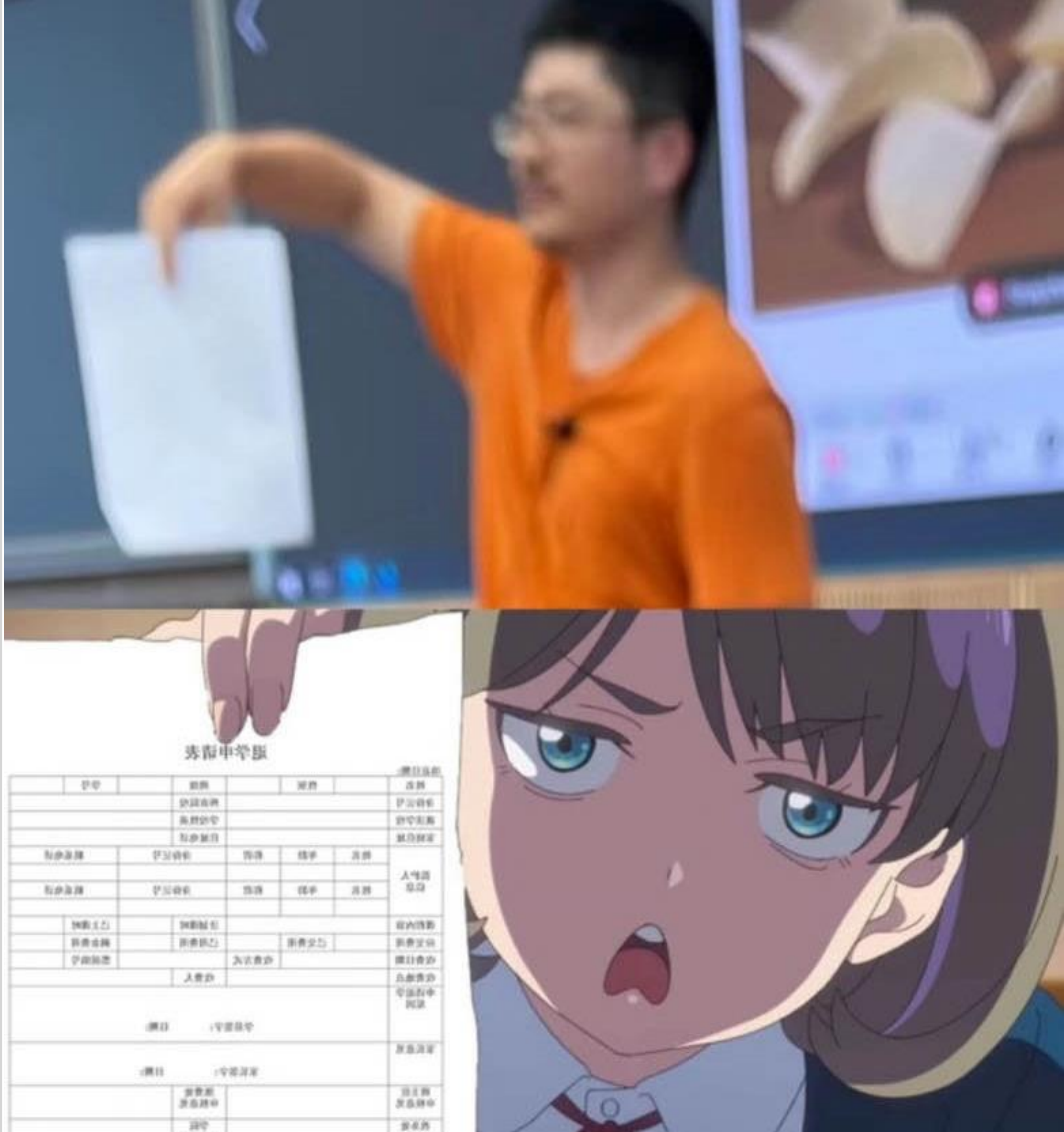
Contents

- Definition of Derivatives
- Differential Formulas
- Chain Rule
- Implicit Differentiation
- Logarithmatic Differentiation
- Inverse Function Differentiation
- Optimization Problems
- Mean Value Theorem
- L’Hospital’s Rule



Definition of Derivatives

- Geometry Interpretation: Tangent Line

  The slope of the tangent line of a function is the corresponding derivative.

- Physical Interpretation: Instantaneous rate of change
- Definition:

Derivative

The **derivative of a function *f* at a number *a***, denoted by $f'(a)$, is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

if this limit exists.

- Notations:

Notations

$$\dot{y}, \frac{dy}{dx}, f'(x), \frac{\partial f}{\partial x}$$

Differentiable

Differentiable

A function f is differentiable at $\boldsymbol{a}$ if $f'(a)$ exists. It is differentiable on an open interval (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

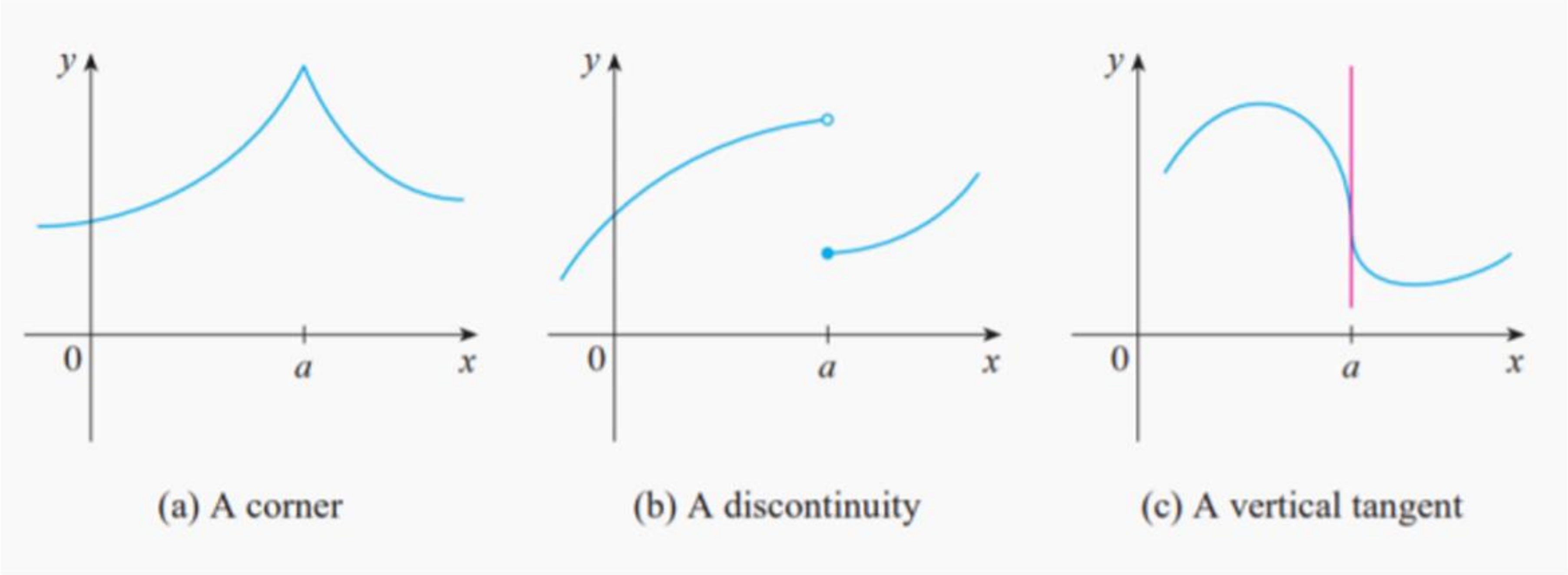
Differentiable and Continuity

If f is differentiable at a, then f is continuous at a.
NOTE The converse of Theorem is false; that is, there are functions that are continuous but not differentiable. For instance, the function $f(x) = |x|$ is continuous at 0 because

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} |x| = 0 = f(0)$$

Differentiable

The function is not differentiable at these points:



Differential Formulas

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$(cf)' = cf'$$

$$(f+g)' = f' + g'$$

$$(f-g)' = f' - g'$$

$$(fg)' = fg' + gf'$$

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$$

Chain Rule

Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Implicit Differentiation

Find y' if $\sin(x + y) = y^2 \cos x$

Differentiating implicitly with respect to x and remembering that y is a function of x, we get

$$\cos(x + y) \cdot (1 + y') = y^2(-\sin x) + (\cos x)\,(2yy')$$

(Note that we have used the Chain Rule on the left side and the Product Rule and Chain Rule on the right side.) If we collect the terms that involve y', we get

$$\cos(x + y) + y^2 \sin x = (2y \cos x)y' - \cos(x + y) \cdot y'$$

So

$$y' = \frac{y^2 \sin x + \cos(x + y)}{2y \cos x - \cos(x + y)}$$

25–32 Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

$$\sin(x + y) = 2x - 2y, \quad (\pi, \pi)$$

$$x^2 + y^2 = (2x^2 + 2y^2 - x)^2$$
$$\left(0, \tfrac{1}{2}\right)$$

Solutions

$\sin(x+y) = 2x - 2y \quad \Rightarrow \quad \cos(x+y)\cdot(1+y') = 2 - 2y' \quad \Rightarrow \quad \cos(x+y)\cdot y' + 2y' = 2 - \cos(x+y) \quad \Rightarrow$

$y'[\cos(x+y)+2] = 2-\cos(x+y) \quad \Rightarrow \quad y' = \dfrac{2-\cos(x+y)}{\cos(x+y)+2}$. When $x=\pi$ and $y=\pi$, we have $y' = \dfrac{2-1}{1+2} = \dfrac{1}{3}$, so

an equation of the tangent line is $y-\pi = \frac{1}{3}(x-\pi)$, or $y = \frac{1}{3}x + \frac{2\pi}{3}$.

$x^2+y^2 = (2x^2+2y^2-x)^2 \quad \Rightarrow \quad 2x+2y\,y' = 2(2x^2+2y^2-x)(4x+4y\,y'-1)$. When $x=0$ and $y=\frac{1}{2}$, we have

$0+y' = 2(\frac{1}{2})(2y'-1) \quad \Rightarrow \quad y' = 2y'-1 \quad \Rightarrow \quad y'=1$, so an equation of the tangent line is $y-\frac{1}{2} = 1(x-0)$

or $y = x+\frac{1}{2}$.

Logarithmatic Differentiation

Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

Prof.Cai has mentioned the example in the lecture. If you have difficulty with this problem, do check your notes before the exam.

$$y = \sqrt{x}\, e^{x^2-x}(x+1)^{2/3}$$

$$y = (\sin x)^{\ln x}$$

Solutions

SOLUTION We take logarithms of both sides of the equation and use the Laws of Logarithms to simplify:

$$\ln y = \tfrac{3}{4}\ln x + \tfrac{1}{2}\ln(x^2 + 1) - 5\ln(3x + 2)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2 + 1} - 5\cdot\frac{3}{3x + 2}$$

Solving for dy/dx, we get

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2 + 1} - \frac{15}{3x + 2}\right)$$

Because we have an explicit expression for y, we can substitute and write

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2 + 1}}{(3x + 2)^5}\left(\frac{3}{4x} + \frac{x}{x^2 + 1} - \frac{15}{3x + 2}\right)$$

Solutions

42. $y = \sqrt{x}\, e^{x^2-x}(x+1)^{2/3} \quad \Rightarrow \quad \ln y = \ln\left[x^{1/2}e^{x^2-x}(x+1)^{2/3}\right] \quad \Rightarrow$

$$\ln y = \tfrac{1}{2}\ln x + (x^2 - x) + \tfrac{2}{3}\ln(x+1) \quad \Rightarrow \quad \frac{1}{y}y' = \frac{1}{2}\cdot\frac{1}{x} + 2x - 1 + \frac{2}{3}\cdot\frac{1}{x+1} \quad \Rightarrow$$

$$y' = y\left(\frac{1}{2x} + 2x - 1 + \frac{2}{3x+3}\right) \quad \Rightarrow \quad y' = \sqrt{x}\, e^{x^2-x}(x+1)^{2/3}\left(\frac{1}{2x} + 2x - 1 + \frac{2}{3x+3}\right)$$

48. $y = (\sin x)^{\ln x} \quad \Rightarrow \quad \ln y = \ln(\sin x)^{\ln x} \quad \Rightarrow \quad \ln y = \ln x \cdot \ln\sin x \quad \Rightarrow \quad \frac{1}{y}y' = \ln x \cdot \frac{1}{\sin x}\cdot\cos x + \ln\sin x \cdot \frac{1}{x} \quad \Rightarrow$

$$y' = y\left(\ln x \cdot \frac{\cos x}{\sin x} + \frac{\ln\sin x}{x}\right) \quad \Rightarrow \quad y' = (\sin x)^{\ln x}\left(\ln x \cot x + \frac{\ln\sin x}{x}\right)$$

Implicit Differentiation

Inverse function theorem

- $$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

- $$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Implicit Differentiation

Find out the expression of $\frac{d^3y}{dx^3}$ with $\frac{dy}{dx}$.

One way of defining $\sec^{-1}x$ is to say that $y = \sec^{-1}x \iff \sec y = x$ and $0 \leqslant y < \pi/2$ or $\pi \leqslant y < 3\pi/2$. Show that, with this definition,

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2 - 1}}$$

Solution

$$\frac{\mathrm{dx}}{\mathrm{dy}}=\frac{1}{\frac{\mathrm{dy}}{\mathrm{dx}}}=\frac{1}{y'}$$

$$\frac{d^2x}{\mathrm{dy}^2}=\frac{d}{\mathrm{dy}}\left(\frac{\mathrm{dx}}{\mathrm{dy}}\right)=\frac{d}{\mathrm{dy}}\left(\frac{1}{y'}\right)=\frac{d}{\mathrm{dx}}\left(\frac{1}{y'}\right)\frac{\mathrm{dx}}{\mathrm{dy}}=-\frac{y''}{y'^2}\left(\frac{1}{y'}\right)=-\frac{y''}{y'^3}$$

$$\frac{d^3x}{\mathrm{dy}^3}=\frac{d}{\mathrm{dy}}\left(\frac{d^2x}{\mathrm{dy}^2}\right)=\frac{d}{\mathrm{dy}}(-\frac{y''}{y'^3})=\frac{d}{\mathrm{dx}}(-\frac{y''}{y'^3})\frac{\mathrm{dx}}{\mathrm{dy}}=\frac{-y'''y'^3+y''\cdot 3y'^2}{(y'^3)^2}\left(\frac{1}{y'}\right)=\frac{3y''^2-y'''y'}{y'^5}$$

注意记号 $y'=\frac{d}{\mathrm{dx}}(y)=\frac{\mathrm{dy}}{\mathrm{dx}}$，$y''=\frac{d}{\mathrm{dx}}(y')=\frac{d^2y}{\mathrm{dx}^2}$，$y'''=\frac{d}{\mathrm{dx}}(y'')=\frac{d^3y}{\mathrm{dx}^3}$都是对x求导

y''·3y'^2处应为y''·3y'^2·y''=3(y''^2)(y'^2)【太不细心了】

(a) Let $y=\sec^{-1}x$. Then $\sec y=x$ and $y\in\left(0,\frac{\pi}{2}\right]\cup\left(\pi,\frac{3\pi}{2}\right]$. Differentiate with respect to x: $\sec y\ \tan y\left(\frac{dy}{dx}\right)=1\quad\Rightarrow$

$$\frac{dy}{dx}=\frac{1}{\sec y\ \tan y}=\frac{1}{\sec y\sqrt{\sec^2 y-1}}=\frac{1}{x\sqrt{x^2-1}}.$$ Note that $\tan^2 y=\sec^2 y-1\quad\Rightarrow\quad\tan y=\sqrt{\sec^2 y-1}$

since $\tan y>0$ when $0<y<\frac{\pi}{2}$ or $\pi<y<\frac{3\pi}{2}$.

Optimization Problems

1 Definition Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **absolute maximum** value of f on D if $f(c) \geqslant f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leqslant f(x)$ for all x in D.

2 Definition The number $f(c)$ is a

- **local maximum** value of f if $f(c) \geqslant f(x)$ when x is near c.
- **local minimum** value of f if $f(c) \leqslant f(x)$ when x is near c.

3 The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Optimization Problems

<u>Def</u> A <u>critical point</u> of a function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is a number $c \in D$ s.t. either $f'(c) = 0$ or $f'(c)$ does not exist.

Optimization Problems

Closed Interval Method p. 278

Let $f : [a,b] \to \mathbb{R}$ be a continuous function.

To find the global max/min of f over $[a,b]$, we do the following.

① We find the candidate points $x \in [a,b]$

(i) $f'(x) = 0$

(ii) $f'(x)$ does not exist

$\Big\}$ critical points

(iii) $x = a$, $x = b$.

② Evaluate f at all the candidate points and pick the global max/min.

Optimization Problems

47–62 Find the absolute maximum and absolute minimum values of f on the given interval.

$$f(x) = \frac{x}{x^2 - x + 1}, \quad [0, 3]$$

$$f(x) = \ln(x^2 + x + 1), \quad [-1, 1]$$

$$f(x) = x - 2\tan^{-1}x, \quad [0, 4]$$

Solution

54. $f(x) = \dfrac{x}{x^2 - x + 1}$, $[0, 3]$.

$$f'(x) = \frac{(x^2 - x + 1) - x(2x - 1)}{(x^2 - x + 1)^2} = \frac{x^2 - x + 1 - 2x^2 + x}{(x^2 - x + 1)^2} = \frac{1 - x^2}{(x^2 - x + 1)^2} = \frac{(1 + x)(1 - x)}{(x^2 - x + 1)^2} = 0 \quad \Leftrightarrow$$

$x = \pm 1$, but $x = -1$ is not in the given interval, $[0, 3]$. $f(0) = 0$, $f(1) = 1$, and $f(3) = \frac{3}{7}$. So $f(1) = 1$ is the absolute maximum value and $f(0) = 0$ is the absolute minimum value.

61. $f(x) = \ln(x^2 + x + 1)$, $[-1, 1]$. $f'(x) = \dfrac{1}{x^2 + x + 1} \cdot (2x + 1) = 0 \quad \Leftrightarrow \quad x = -\frac{1}{2}$. Since $x^2 + x + 1 > 0$ for all x, the domain of f and f' is $\mathbb{R}$. $f(-1) = \ln 1 = 0$, $f\left(-\frac{1}{2}\right) = \ln \frac{3}{4} \approx -0.29$, and $f(1) = \ln 3 \approx 1.10$. So $f(1) = \ln 3 \approx 1.10$ is the absolute maximum value and $f\left(-\frac{1}{2}\right) = \ln \frac{3}{4} \approx -0.29$ is the absolute minimum value.

Solution

62. $f(x) = x - 2\tan^{-1} x$, $[0, 4]$. $f'(x) = 1 - 2 \cdot \dfrac{1}{1+x^2} = 0 \;\Leftrightarrow\; 1 = \dfrac{2}{1+x^2} \;\Leftrightarrow\; 1 + x^2 = 2 \;\Leftrightarrow\; x^2 = 1 \;\Leftrightarrow\; x = \pm 1$. $f(0) = 0$, $f(1) = 1 - \frac{\pi}{2} \approx -0.57$, and $f(4) = 4 - 2\tan^{-1} 4 \approx 1.35$. So $f(4) = 4 - 2\tan^{-1} 4$ is the absolute maximum value and $f(1) = 1 - \frac{\pi}{2}$ is the absolute minimum value.

Optimization Problems

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

The First Derivative Test Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.

$\boxed{4}$ **Fermat's Theorem** If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Optimization Problems

a function $f: D \subseteq \mathbb{R} \to \mathbb{R}$ is convex if

$\forall x, y \in D$, $\lambda \in [0,1]$,

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$$

$$\Leftrightarrow \quad f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$$

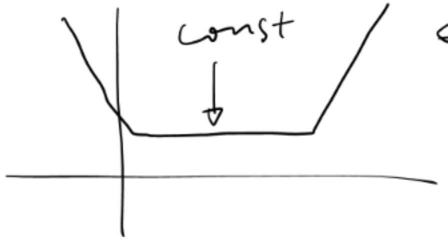
A funtion $f: D \subseteq \mathbb{R} \to \mathbb{R}$ is concave if

$\forall x, y \in D$, $\lambda \in [0,1]$,

$$f(\lambda x + (1-\lambda) y) \geq \lambda f(x) + (1-\lambda) f(y)$$

$$\Leftrightarrow \quad f\left(\frac{x+y}{2}\right) \geq \frac{f(x)+f(y)}{2}$$

f is call strictly convex/concave if we have $<$ or $>$ instead of $\leq$ or $\geq$ in the def'n



$\leftarrow$ convex but not strictly convex.

Optimization Problems

A function f is convex $\iff$ $-f$ is concave.

If a function $f: \mathbb{R} \to \mathbb{R}$

is both convex and concave, then

f is an affine function, i.e., $f(x) = ax+b$

for some const $a, b \in \mathbb{R}$. In engineering

context, it is usually called a "linear fcn"

In mathematics, a linear fcn usually has

zero offset, i.e., $b=0$.

Optimization Problems

What Does f'' Say About f?

Definition If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.

Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Definition A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

The Second Derivative Test Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

conVex

conCAVE:

Note that, “concave upward” is equal to “convex”, “concave downward” is equal to “concave”. And Prof.Cai will prefer the notation of “convex” and “concave”

Optimization Problems

33–44

(a) Find the intervals of increase or decrease.
(b) Find the local maximum and minimum values.
(c) Find the intervals of concavity and the inflection points.

$$C(x) = x^{1/3}(x + 4) \qquad\qquad f(x) = \ln(x^2 + 9)$$

45–52

(a) Find the vertical and horizontal asymptotes.
(b) Find the intervals of increase or decrease.
(c) Find the local maximum and minimum values.
(d) Find the intervals of concavity and the inflection points.

$$f(x) = \ln(1 - \ln x) \qquad\qquad f(x) = e^{\arctan x}$$

Solution

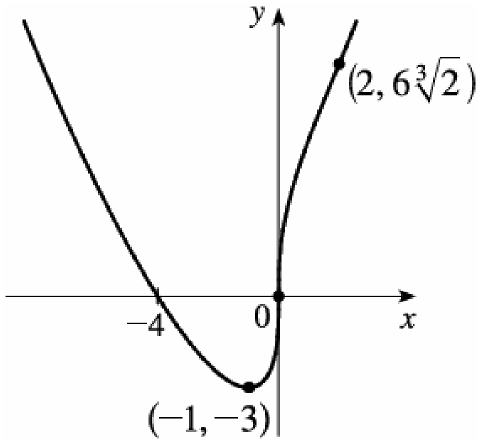
45. (a) $C(x) = x^{1/3}(x+4) = x^{4/3} + 4x^{1/3} \quad \Rightarrow \quad C'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3} = \frac{4}{3}x^{-2/3}(x+1) = \dfrac{4(x+1)}{3\sqrt[3]{x^2}}$. $C'(x) > 0$ if $-1 < x < 0$ or $x > 0$ and $C'(x) < 0$ for $x < -1$, so C is increasing on $(-1, \infty)$ and C is decreasing on $(-\infty, -1)$.

(b) $C(-1) = -3$ is a local minimum value.

(c) $C''(x) = \frac{4}{9}x^{-2/3} - \frac{8}{9}x^{-5/3} = \frac{4}{9}x^{-5/3}(x-2) = \dfrac{4(x-2)}{9\sqrt[3]{x^5}}$.

$C''(x) < 0$ for $0 < x < 2$ and $C''(x) > 0$ for $x < 0$ and $x > 2$, so C is concave downward on $(0, 2)$ and concave upward on $(-\infty, 0)$ and $(2, \infty)$. There are inflection points at $(0, 0)$ and $\left(2, 6\sqrt[3]{2}\right) \approx (2, 7.56)$.

(d)



Solution

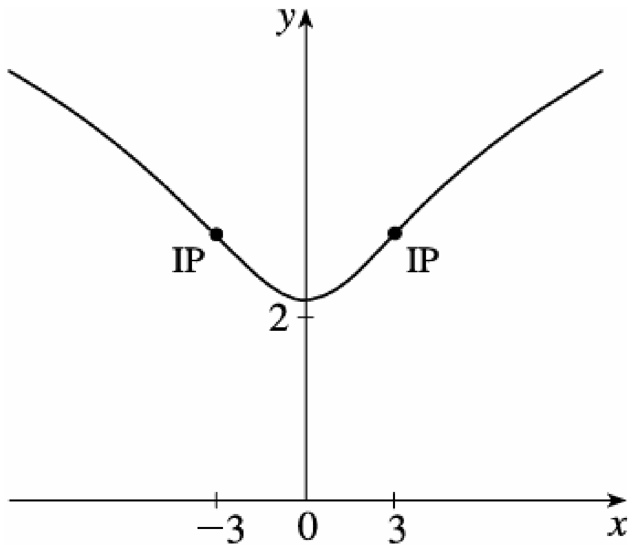
46. (a) $f(x) = \ln(x^2+9) \quad \Rightarrow \quad f'(x) = \dfrac{1}{x^2+9} \cdot 2x = \dfrac{2x}{x^2+9}$. $\quad f'(x) > 0 \quad \Leftrightarrow \quad 2x > 0 \quad \Leftrightarrow \quad x > 0$ and $f'(x) < 0 \quad \Leftrightarrow$ $x < 0$. So f is increasing on $(0, \infty)$ and f is decreasing on $(-\infty, 0)$.

(b) f changes from decreasing to increasing at $x = 0$, so $f(0) = \ln 9$ is a local minimum value. There is no local maximum value.

(c) $f''(x) = \dfrac{(x^2+9) \cdot 2 - 2x(2x)}{(x^2+9)^2} = \dfrac{18 - 2x^2}{(x^2+9)^2} = \dfrac{-2(x+3)(x-3)}{(x^2+9)^2}$.

$f''(x) = 0 \quad \Leftrightarrow \quad x = \pm 3$. $\; f''(x) > 0$ on $(-3, 3)$ and $f''(x) < 0$ on $(-\infty, -3)$ and $(3, \infty)$. So f is CU on $(-3, 3)$, and f is CD on $(-\infty, -3)$ and $(3, \infty)$. There are inflection points at $(\pm 3, \ln 18)$.

(d)



Solution

55. $f(x) = \ln(1 - \ln x)$ is defined when $x > 0$ (so that $\ln x$ is defined) and $1 - \ln x > 0$ [so that $\ln(1 - \ln x)$ is defined].

The second condition is equivalent to $1 > \ln x \quad \Leftrightarrow \quad x < e$, so f has domain $(0, e)$.

(a) As $x \to 0^+$, $\ln x \to -\infty$, so $1 - \ln x \to \infty$ and $f(x) \to \infty$. As $x \to e^-$, $\ln x \to 1^-$, so $1 - \ln x \to 0^+$ and $f(x) \to -\infty$. Thus, $x = 0$ and $x = e$ are vertical asymptotes. There is no horizontal asymptote.

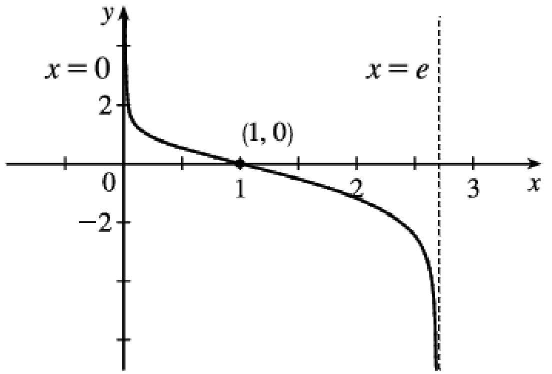
(b) $f'(x) = \dfrac{1}{1 - \ln x}\left(-\dfrac{1}{x}\right) = -\dfrac{1}{x(1 - \ln x)} < 0$ on $(0, e)$. Thus, f is decreasing on its domain, $(0, e)$.

(c) $f'(x) \neq 0$ on $(0, e)$, so f has no local maximum or minimum value.

(d) $$f''(x) = -\frac{-[x(1 - \ln x)]'}{[x(1 - \ln x)]^2} = \frac{x(-1/x) + (1 - \ln x)}{x^2(1 - \ln x)^2} = -\frac{\ln x}{x^2(1 - \ln x)^2}$$

so $f''(x) > 0 \quad \Leftrightarrow \quad \ln x < 0 \quad \Leftrightarrow \quad 0 < x < 1$. Thus, f is CU on $(0, 1)$ and CD on $(1, e)$. There is an inflection point at $(1, 0)$.

(e)



Solution

56. (a) $\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$, so $\lim_{x\to\infty} e^{\arctan x} = e^{\pi/2}$ [≈ 4.81], so $y = e^{\pi/2}$ is a HA.

$\lim_{x\to-\infty} e^{\arctan x} = e^{-\pi/2}$ [≈ 0.21], so $y = e^{-\pi/2}$ is a HA. No VA.

(b) $f(x) = e^{\arctan x} \quad \Rightarrow \quad f'(x) = e^{\arctan x} \cdot \dfrac{1}{1+x^2} > 0$ for all x. Thus, f is increasing on $\mathbb{R}$.

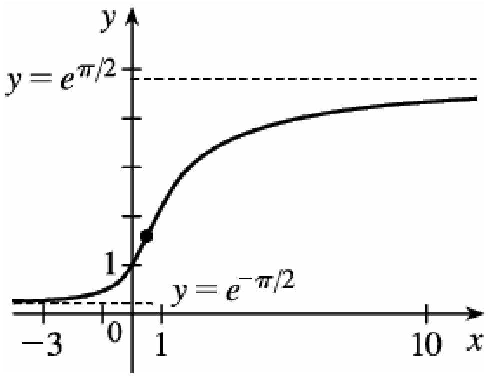
(c) There is no local maximum or minimum.

(d) $f''(x) = e^{\arctan x}\left[\dfrac{-2x}{(1+x^2)^2}\right] + \dfrac{1}{1+x^2} \cdot e^{\arctan x} \cdot \dfrac{1}{1+x^2}$

$= \dfrac{e^{\arctan x}}{(1+x^2)^2}(-2x+1)$

$f''(x) > 0 \quad \Leftrightarrow \quad -2x+1 > 0 \quad \Leftrightarrow \quad x < \frac{1}{2}$ and $f''(x) < 0 \quad \Leftrightarrow$ $x > \frac{1}{2}$, so f is CU on $\left(-\infty, \frac{1}{2}\right)$ and f is CD on $\left(\frac{1}{2}, \infty\right)$. There is an inflection point at $\left(\frac{1}{2}, f\left(\frac{1}{2}\right)\right) = \left(\frac{1}{2}, e^{\arctan(1/2)}\right) \approx \left(\frac{1}{2}, 1.59\right)$.

(e)



Mean Value Theorem

Rolle's Theorem Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$\boxed{1} \qquad f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$\boxed{2} \qquad f(b) - f(a) = f'(c)(b - a)$$

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L'Hospital's Rule

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

L'Hospital's Rule

NOTE 1 L'Hospital's Rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the given conditions are satisfied. It is especially important to verify the conditions regarding the limits of f and g before using l'Hospital's Rule.

NOTE 2 L'Hospital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity; that is, "$x \longrightarrow a$" can be replaced by any of the symbols $x \longrightarrow a^+$, $x \longrightarrow a^-$, $x \longrightarrow \infty$, or $x \longrightarrow -\infty$.

L'Hospital's Rule

Indeterminate Products

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ (or $-\infty$), then it isn't clear what the value of $\lim_{x\to a} [f(x)\,g(x)]$, if any, will be. There is a struggle between f and g. If f wins, the answer will be 0; if g wins, the answer will be ∞ (or $-\infty$). Or there may be a compromise where the answer is a finite nonzero number. This kind of limit is called an **indeterminate form of type $0 \cdot \infty$**. We can deal with it by writing the product fg as a quotient:

$$fg = \frac{f}{1/g} \qquad \text{or} \qquad fg = \frac{g}{1/f}$$

This converts the given limit into an indeterminate form of type $\frac{0}{0}$ or ∞/∞ so that we can use l'Hospital's Rule.

Example

V EXAMPLE 6 Evaluate $\lim_{x \to 0^+} x \ln x$.

SOLUTION The given limit is indeterminate because, as $x \to 0^+$, the first factor (x) approaches 0 while the second factor ($\ln x$) approaches $-\infty$. Writing $x = 1/(1/x)$, we have $1/x \to \infty$ as $x \to 0^+$, so l'Hospital's Rule gives

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0$$

NOTE In solving Example 6 another possible option would have been to write

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{x}{1/\ln x}$$

This gives an indeterminate form of the type $0/0$, but if we apply l'Hospital's Rule we get a more complicated expression than the one we started with. In general, when we rewrite an indeterminate product, we try to choose the option that leads to the simpler limit.

L'Hospital's Rule

Indeterminate Differences

If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, then the limit

$$\lim_{x \to a} [f(x) - g(x)]$$

is called an **indeterminate form of type $\infty - \infty$**. Again there is a contest between f and g. Will the answer be ∞ (f wins) or will it be $-\infty$ (g wins) or will they compromise on a finite

number? To find out, we try to convert the difference into a quotient (for instance, by using a common denominator, or rationalization, or factoring out a common factor) so that we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.

L'Hospital's Rule

EXAMPLE 7 Compute $\lim_{x \to (\pi/2)^-} (\sec x - \tan x)$.

SOLUTION First notice that $\sec x \to \infty$ and $\tan x \to \infty$ as $x \to (\pi/2)^-$, so the limit is indeterminate. Here we use a common denominator:

$$\lim_{x \to (\pi/2)^-} (\sec x - \tan x) = \lim_{x \to (\pi/2)^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right)$$

$$= \lim_{x \to (\pi/2)^-} \frac{1 - \sin x}{\cos x} = \lim_{x \to (\pi/2)^-} \frac{-\cos x}{-\sin x} = 0$$

Note that the use of l'Hospital's Rule is justified because $1 - \sin x \to 0$ and $\cos x \to 0$ as $x \to (\pi/2)^-$.

L'Hospital's Rule

Indeterminate Powers

Several indeterminate forms arise from the limit

$$\lim_{x \to a} [f(x)]^{g(x)}$$

1. $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ type 0^0
2. $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = 0$ type ∞^0
3. $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \pm\infty$ type 1^∞

Each of these three cases can be treated either by taking the natural logarithm:

$$\text{let} \quad y = [f(x)]^{g(x)}, \quad \text{then} \quad \ln y = g(x) \ln f(x)$$

or by writing the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x) \ln f(x)}$$

(Recall that both of these methods were used in differentiating such functions.) In either method we are led to the indeterminate product $g(x) \ln f(x)$, which is of type $0 \cdot \infty$.

L'Hospital's Rule

EXAMPLE 8 Calculate $\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x}$.

SOLUTION First notice that as $x \to 0^+$, we have $1 + \sin 4x \to 1$ and $\cot x \to \infty$, so the given limit is indeterminate. Let

$$y = (1 + \sin 4x)^{\cot x}$$

Then $$\ln y = \ln[(1 + \sin 4x)^{\cot x}] = \cot x \ln(1 + \sin 4x)$$

so l'Hospital's Rule gives

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \frac{\ln(1 + \sin 4x)}{\tan x} = \lim_{x \to 0^+} \frac{\dfrac{4 \cos 4x}{1 + \sin 4x}}{\sec^2 x} = 4$$

So far we have computed the limit of $\ln y$, but what we want is the limit of y. To find this we use the fact that $y = e^{\ln y}$:

$$\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x} = \lim_{x \to 0^+} y = \lim_{x \to 0^+} e^{\ln y} = e^4$$

L'Hospital's Rule

36. $\lim_{x \to 0^+} \dfrac{x^x - 1}{\ln x + x - 1}$

38. $\lim_{x \to 0} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$

40. $\lim_{x \to a^+} \dfrac{\cos x \ln(x - a)}{\ln(e^x - e^a)}$

49. $\lim_{x \to 1} \left(\dfrac{x}{x - 1} - \dfrac{1}{\ln x} \right)$

51. $\lim_{x \to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{e^x - 1} \right)$

L'Hospital's Rule

$$\lim_{x \to 0+} \left(\frac{1}{x} - \frac{1}{\tan^{-1} x} \right)$$

60. $\lim_{x \to \infty} x^{(\ln 2)/(1 + \ln x)}$

64. $\lim_{x \to 1} (2 - x)^{\tan(\pi x/2)}$

66. $\lim_{x \to \infty} \left(\frac{2x - 3}{2x + 5} \right)^{2x+1}$

68. $\lim_{x \to 0} \frac{5^x - 4^x}{3^x - 2^x}$

Solution

$\lim_{x \to 0^+} \dfrac{x^x - 1}{\ln x + x - 1}$. From Example 9, $\lim_{x \to 0^+} x^x = 1$, so $\lim_{x \to 0^+} (x^x - 1) = 0$. As $x \to 0^+$, $\ln x \to -\infty$, so

$\ln x + x - 1 \to -\infty$ as $x \to 0^+$. Thus, $\lim_{x \to 0^+} \dfrac{x^x - 1}{\ln x + x - 1} = 0$.

. This limit has the form $\frac{0}{0}$. $\lim_{x \to 0} \dfrac{e^x - e^{-x} - 2x}{x - \sin x} \stackrel{\text{H}}{=} \lim_{x \to 0} \dfrac{e^x + e^{-x} - 2}{1 - \cos x} \stackrel{\text{H}}{=} \lim_{x \to 0} \dfrac{e^x - e^{-x}}{\sin x} \stackrel{\text{H}}{=} \lim_{x \to 0} \dfrac{e^x + e^{-x}}{\cos x} = \dfrac{1+1}{1} = 2$

42. This limit has the form $\frac{\infty}{\infty}$.

$$\lim_{x \to a^+} \frac{\cos x \ln(x-a)}{\ln(e^x - e^a)} = \lim_{x \to a^+} \cos x \lim_{x \to a^+} \frac{\ln(x-a)}{\ln(e^x - e^a)} \stackrel{\text{H}}{=} \cos a \lim_{x \to a^+} \frac{\dfrac{1}{x-a}}{\dfrac{1}{e^x - e^a} \cdot e^x}$$

$$= \cos a \lim_{x \to a^+} \frac{1}{e^x} \cdot \lim_{x \to a^+} \frac{e^x - e^a}{x - a} \stackrel{\text{H}}{=} \cos a \cdot \frac{1}{e^a} \lim_{x \to a^+} \frac{e^x}{1} = \cos a \cdot \frac{1}{e^a} \cdot e^a = \cos a$$

Solution

This limit has the form $\infty - \infty$.

$$\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)=\lim_{x\to 1}\frac{x\ln x-(x-1)}{(x-1)\ln x}\overset{\text{H}}{=}\lim_{x\to 1}\frac{x(1/x)+\ln x-1}{(x-1)(1/x)+\ln x}=\lim_{x\to 1}\frac{\ln x}{1-(1/x)+\ln x}$$

$$\overset{\text{H}}{=}\lim_{x\to 1}\frac{1/x}{1/x^2+1/x}\cdot\frac{x^2}{x^2}=\lim_{x\to 1}\frac{x}{1+x}=\frac{1}{1+1}=\frac{1}{2}$$

53. This limit has the form $\infty - \infty$.

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)=\lim_{x\to 0^+}\frac{e^x-1-x}{x(e^x-1)}\overset{\text{H}}{=}\lim_{x\to 0^+}\frac{e^x-1}{xe^x+e^x-1}\overset{\text{H}}{=}\lim_{x\to 0^+}\frac{e^x}{xe^x+e^x+e^x}=\frac{1}{0+1+1}=\frac{1}{2}$$

Solution

. This limit has the form $\infty - \infty$.

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\tan^{-1}x}\right)=\lim_{x\to 0^+}\frac{\tan^{-1}x-x}{x\tan^{-1}x}\overset{\text{H}}{=}\lim_{x\to 0^+}\frac{1/(1+x^2)-1}{x/(1+x^2)+\tan^{-1}x}=\lim_{x\to 0^+}\frac{1-(1+x^2)}{x+(1+x^2)\tan^{-1}x}$$

$$=\lim_{x\to 0^+}\frac{-x^2}{x+(1+x^2)\tan^{-1}x}\overset{\text{H}}{=}\lim_{x\to 0^+}\frac{-2x}{1+(1+x^2)(1/(1+x^2))+(\tan^{-1}x)(2x)}$$

$$=\lim_{x\to 0^+}\frac{-2x}{2+2x\tan^{-1}x}=\frac{0}{2+0}=0$$

. $y=x^{(\ln 2)/(1+\ln x)} \quad\Rightarrow\quad \ln y=\dfrac{\ln 2}{1+\ln x}\ln x \quad\Rightarrow$

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{(\ln 2)(\ln x)}{1+\ln x}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{(\ln 2)(1/x)}{1/x}=\lim_{x\to\infty}\ln 2=\ln 2,\text{ so }\lim_{x\to\infty}x^{(\ln 2)/(1+\ln x)}=\lim_{x\to\infty}e^{\ln y}=e^{\ln 2}=2.$$

Solution

$$y = (2-x)^{\tan(\pi x/2)} \quad \Rightarrow \quad \ln y = \tan\left(\frac{\pi x}{2}\right) \ln(2-x) \quad \Rightarrow$$

$$\lim_{x\to 1} \ln y = \lim_{x\to 1} \left[\tan\left(\frac{\pi x}{2}\right) \ln(2-x)\right] = \lim_{x\to 1} \frac{\ln(2-x)}{\cot\left(\frac{\pi x}{2}\right)} \overset{\mathrm{H}}{=} \lim_{x\to 1} \frac{\frac{1}{2-x}(-1)}{-\csc^2\left(\frac{\pi x}{2}\right)\cdot\frac{\pi}{2}} = \frac{2}{\pi}\lim_{x\to 1} \frac{\sin^2\left(\frac{\pi x}{2}\right)}{2-x}$$

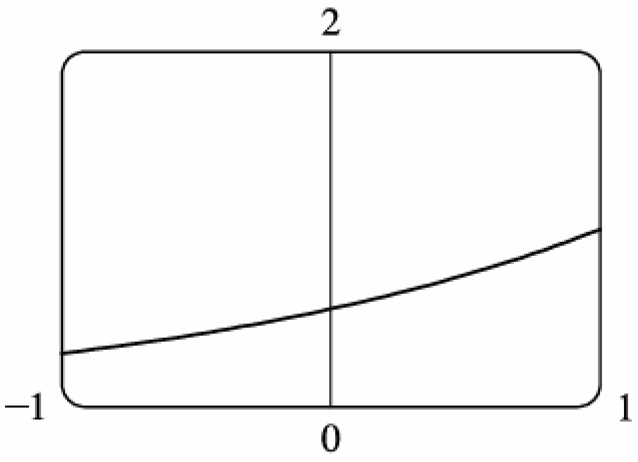
$$= \frac{2}{\pi}\cdot\frac{1^2}{1} = \frac{2}{\pi} \quad \Rightarrow \quad \lim_{x\to 1} (2-x)^{\tan(\pi x/2)} = \lim_{x\to 1} e^{\ln y} = e^{(2/\pi)}$$

$$y = \left(\frac{2x-3}{2x+5}\right)^{2x+1} \quad \Rightarrow \quad \ln y = (2x+1)\ \ln\left(\frac{2x-3}{2x+5}\right) \quad \Rightarrow$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(2x-3)-\ln(2x+5)}{1/(2x+1)} \overset{\mathrm{H}}{=} \lim_{x\to\infty} \frac{2/(2x-3)-2/(2x+5)}{-2/(2x+1)^2} = \lim_{x\to\infty} \frac{-8(2x+1)^2}{(2x-3)(2x+5)}$$

$$= \lim_{x\to\infty} \frac{-8(2+1/x)^2}{(2-3/x)(2+5/x)} = -8 \quad \Rightarrow \quad \lim_{x\to\infty} \left(\frac{2x-3}{2x+5}\right)^{2x+1} = e^{-8}$$

Solution



From the graph, as $x \to 0$, $y \approx 0.55$. The limit has the form $\frac{0}{0}$.

$$\lim_{x\to 0} \frac{5^x - 4^x}{3^x - 2^x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{5^x \ln 5 - 4^x \ln 4}{3^x \ln 3 - 2^x \ln 2} = \frac{\ln 5 - \ln 4}{\ln 3 - \ln 2} = \frac{\ln \frac{5}{4}}{\ln \frac{3}{2}} \quad [\approx 0.55]$$

Reference

- VV156 23FA Mid RC Part 3, Mingrui Li
- James Stewart Calculus Early Transcendentals 7th Edition
- VV156 24FA Lecture Notes, Runze Cai

THANK YOU AND GOOD LUCK！