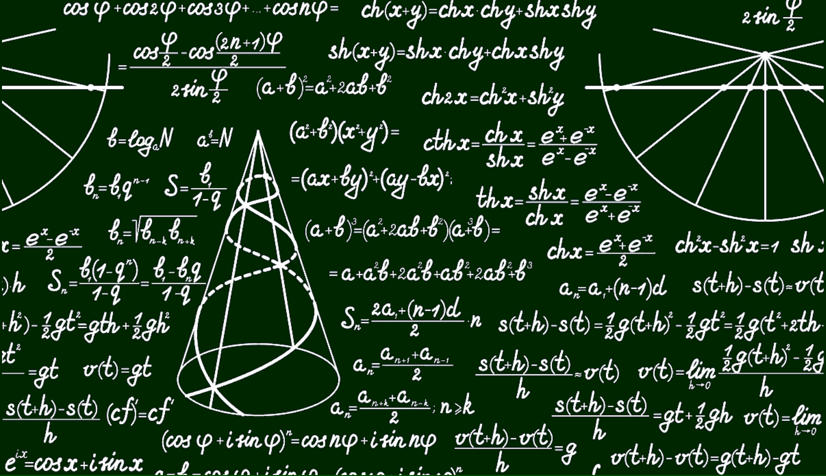


156 RCs

1. Functions

Three essential factors of a function:

- Domain: D
  A set containing the action objects of correspondence rule.
- Correspondence Rule
- Range: E
  A set of all images corresponding to all elements in the domain under a certain correspondence rule.

Definition

A **function** f is a rule that assigns to **each element** x in a set D exactly one element, called $f(x)$, in a set E.

The Vertical Line Test

A curve in the xy-plane is the graph of a function of x **iff** (if and only if) no vertical line intersects the curve more than once.

Definitions & Properties

There are four ways to represent a function:

1. Verbally (by a description in words)
2. Numerically (by a table of values)
3. Visually (by a graph)
4. Algebraically (by an explicit formula)

Symmetry

- Even function: $f(-x) = f(x)$
- Odd function: $f(-x) = -f(x)$

Tip: Give priority to whether the domain D of the function is symmetrical.

Increasing & Decreasing property

A function f is called (strictly) **increasing** on an interval I if

$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

A function f is called (strictly) **decreasing** on an interval I if

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

Linear function

The graph of the function is a line:

$$y = f(x) = mx + b$$

m is the slope of the line and b is the y-intercept.

Polynomials

A function P is called a **polynomial** if

$$P(x) = \Sigma_{i=0}^{n} a_i x^i$$

a_i are **coefficients** and n is the **degree** of the polynomial if $a_n \neq 0$.

Quadratic function

A polynomial of degree 2 is of the form $P(x) = ax^2 + bx + c$ and is called a **quadratic function**.

Basic function types

Power function

A function of the form $f(x) = x^a$ is called a **power function**, where a is a constant. Consider an arbitary positive integer n:

- $a = n$:

  $$y = x\text{: line} \qquad y = x^2\text{: parabola}$$

- $a = \frac{1}{n}$: **root function**
- $a = -1$: **reciprocal function**

Ratio function

A **Ratio function** f is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

The domain consists of all values of x such that $Q(x) \neq 0$.

Basic function types

Trigonometric function

$$\sin(x + 2\pi) = \sin x \qquad \cos(x + 2\pi) = \cos x \qquad \tan(x + \pi) = \tan x$$

Exponential function

The **exponential functions** are the functions of the form $f(x) = a^x$, where the base a is a positive constant.

Logarithmic function

The **logarithmic functions** $f(x) = \log_a x$, where the base a is a positive constant and $a \neq 1$, are the inverse functions of the exponential functions.

Think about the question:
What condition does a meet when the exponential function increases in its domain? What about the logarithmic function?

How to draw a function

- You can simply trace the dots for simple functions.
- You may also try to use software:
  https://www.geogebra.org
  https://www.desmos.com

Dirichlet Function

$$1_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Special Functions

Impulse Function

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

Step Function

$$H[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

Ramp Function

$$R(x) := \begin{cases} x, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Hyperbolic Function

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \cosh(x) = \frac{e^x + e^{-x}}{2}, \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

Inverse trigonometric function

$$\arcsin(x), \arccos(x), \arctan(x)$$

$$\operatorname{arsinh}(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$$

$$\operatorname{arcosh}(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$$

$$\operatorname{artanh}(x) = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right)$$

Function Transformation

Vertical and Horizontal Shifts, suppose $c > 0$

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward
$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward
$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right
$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

Vertical and Horizontal Stretching and Reflecting, suppose $c > 1$

$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c
$y = (1/c)f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c
$y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c
$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c
$y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis
$y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis

Definition

Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h, then g, and then f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

Exercise 1

Skip it if you find the exercise quite simple.

Find the domain of these functions:

1. $h(x) = \frac{1}{\sqrt[4]{x^2-5x}}$
2. $f(u) = \frac{u+1}{1+\frac{1}{u+1}}$
3. $F(p) = \sqrt{2 - \sqrt{p}}$

How to solve this kind of questions

- The denominator in the fractional function cannot be zero
- The quantity in the even root formula cannot take a negative value, that is, it should be greater than or equal to zero
- The antilogarithm of the logarithm cannot be negative and zero, that is, it must take a positive value
- The domain of the function $y = \arcsin x, y = \arccos x$ is $-1 \leqslant x \leqslant 1$
- $y = \tan x$, $x \neq k\pi + \pi/2, y = \cot x$, $x \neq k\pi$, k is integer

Exercise 2

Prove or Disprove

- If f and g are both even functions, is $f + g$ even? If f and g are both odd functions, is $f + g$ odd? What if f is even and g is odd? Justify your answers.
- If f and g are both even functions, is the product fg even? If f and g are both odd functions, is fg odd? What if f is even and g is odd? Justify your answers.

Exercise 3

Graph the functions step by step

(1)$y = 1 - 2\sqrt{x+3}$
(2)$y = |\cos \pi x|$

Exercise 4

Find the function (a) $f \circ g$, (b) $g \circ f$, (c) $f \circ f$, and (d) $g \circ g$ and their domains.

$f(x) = \frac{x}{1+x}, \quad g(x) = \sin 2x$

Exercise 5

Find $f \circ g \circ h$:

- $f(x) = \tan x$
- $g(x) = \frac{x}{x-1}$
- $h(x) = \sqrt[3]{x}$

(It is unnecessary to find the domain of this composite function in this exercise!)

Exercise 6

Composite Function

1. If $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 7$, find a function f such that $f \circ g = h$.
   (Think about what operations you would have to perform on the formula for g to end up with the formula for h.)
2. If $f(x) = 3x + 5$ and $h(x) = 3x^2 + 3x + 2$, find a function g such that $f \circ g = h$.

References

[1] Huang, Yucheng. VV156_RC1_updated.pdf. 2021.

[2] Cai, Runze. Chapter01.pdf. 2021.

[3] Zhou,Yishen.RC1. 2022.

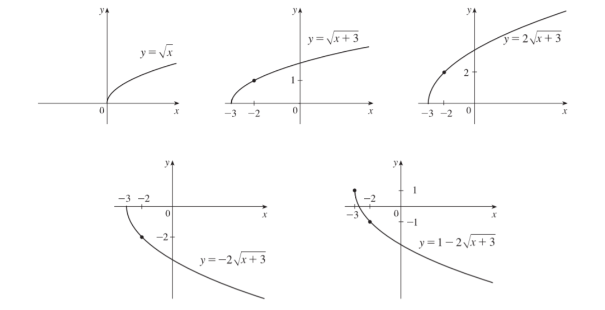
Exercise Answer 1-2

1. $x^2 - 5x > 0 \Rightarrow x \in (-\infty, 0) \cup (5, \infty)$
2. $1 + \frac{1}{u+1} \neq 0, u + 1 \neq 0 \Rightarrow u \in (\infty, -2) \cup (-2, -1) \cup (-1, \infty)$
3. $p \geq 0, 2 - \sqrt{q} \geq 0 \Rightarrow p \in [0, 4]$

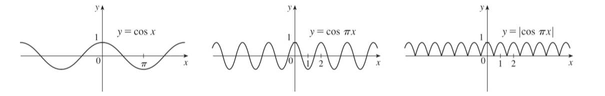
- Yes. Yes. Not necessarily.
- Yes. No.(fg is Even) fg is Odd.

Exercise Answer 3

(1)



(2)



Exercise Answer 4-5

(a) $f \circ g(x) = \frac{sin2x}{1+sin2x}$. Domain: $x \in \{x|x \neq k\pi - \frac{\pi}{4}, k \in \mathbb{B}\}$
(b) $g \circ f(x) = sin(\frac{2x}{1+x})$. Domain: $x \in (\infty, -1) \cup (-1, \infty)$
(c) $f \circ f(x) = \frac{x}{1+2x}(x \neq -1)$.
Domain:$x \in (\infty, -1) \cup (-1, -\frac{1}{2}) \cup (-\frac{1}{2}, \infty)$
(d) $g \circ g(x) = sin(2sin(2x))$. Domain:$x \in \mathbb{R}$

$tan(\frac{\sqrt[3]{x}}{\sqrt[3]{x}-1})$

Exercise Answer 6

1. $x = \frac{g(x)-1}{2}$.
   Plug into h(x), $h(x) = 4(\frac{g(x)-1}{2})^2 + 4(\frac{g(x)-1}{2}) + 7 = g^2(x) + 6$.
   Also, $h(x) = f(g(x))$.
   Therefore, $f(x) = x^2 - 6$
2. $3g(x) + 5 = h(x)$, $g(x) = x^2 + x - 1$

2. Derivatives

Tangent Line

The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

Instantaneous rate of change

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$\text{instantaneous rate of change } = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Try to define velocity in this way.

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{h}$$

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Derivative

The **derivative of a function f at a number a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

if this limit exists.

The slope of the tangent line of a function is the corresponding derivative.

Notations

Newton:

$$\dot{y}$$

Leibniz:

$$\frac{dy}{dx}$$

Lagrange:

$$f'(x)$$

Jacobi: (Partial Derivatives)

$$\frac{\partial f}{\partial x}$$

Differentiable

A function f is differentiable at **a** if $f'(a)$ exists. It is differentiable on an open interval (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

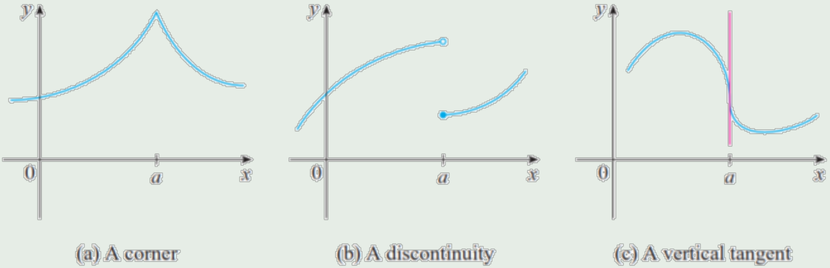
Differentiable and Continuity

If f is differentiable at a, then f is continuous at a.
NOTE The converse of Theorem is false; that is, there are functions that are continuous but not differentiable. For instance, the function $f(x) = |x|$ is continuous at 0 because

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} |x| = 0 = f(0)$$

Differentiable

The function is not differentiable at these points:



$$(f')' = f'' \qquad \text{second derivative}$$

$$(f'')' = f''' \qquad \text{third derivative}$$

Example: still consider the position function

position-velocity-acceleration-jerk-snap$\cdots$

$$x - \frac{dy}{dt} - \frac{d^2y}{dt^2} - \frac{d^3y}{dt^3} - \frac{d^4y}{dt^4} \cdots$$

微积分初步

Saturday, June 17, 2023 8:38 PM

微分

初等函数微分表

1.	$y = c$	$dy = 0$
2.	$y = x^{\mu}$	$dy = \mu x^{\mu-1} \cdot dx$
	$y = \dfrac{1}{x}$	$dy = -\dfrac{dx}{x^2}$
	$y = \sqrt{x}$	$dy = \dfrac{dx}{2\sqrt{x}}$
3.	$y = a^x$	$dy = a^x \cdot \ln a \cdot dx$
	$y = e^x$	$dy = e^x \cdot dx$
4.	$y = \log_a x$	$dy = \dfrac{\log_a e \cdot dx}{x} = \dfrac{dx}{x \ln a}$
	$y = \ln x$	$dy = \dfrac{dx}{x}$
5.	$y = \sin x$	$dy = \cos x \cdot dx$
6.	$y = \cos x$	$dy = -\sin x \cdot dx$
7.	$y = \mathrm{tg} x$	$dy = \sec^2 x \cdot dx = \dfrac{dx}{\cos^2 x}$
8.	$y = \mathrm{ctg} x$	$dy = -\csc^2 x dx = -\dfrac{dx}{\sin^2 x}$
9.	$y = \arcsin x$	$dy = \dfrac{dx}{\sqrt{1-x^2}}$
10.	$y = \arccos x$	$dy = -\dfrac{dx}{\sqrt{1-x^2}}$
11.	$y = \mathrm{arctg} x$	$dy = \dfrac{dx}{1+x^2}$
12.	$y = \mathrm{arcctg} x$	$dy = -\dfrac{dx}{1+x^2}$

$y = \csc x = \frac{1}{\sin x}$ $dy = -\cos x \csc^2 x \, dx = -\csc x \cot x \, dx$

$y = \sec x = \frac{1}{\cos x}$ $dy = \sin x \sec^2 x \, dx = \sec x \tan x \, dx$

$[\sinh(x)]' = \cosh(x)$

$[\cosh(x)]' = \sinh(x)$

$[\tanh(x)]' = \operatorname{sech}^2(x)$

$[\coth(x)]' = -\operatorname{csch}^2(x)$

$[\operatorname{sech}(x)]' = -\operatorname{sech}(x)\tanh(x)$

$[\operatorname{csch}(x)]' = -\operatorname{csch}(x)\coth(x)$

$[\operatorname{ar}\sinh(x)]' = \frac{1}{\sqrt{x^2+1}}$

$[\operatorname{ar}\cosh(x)]' = \frac{1}{\sqrt{x^2-1}} \quad (x > 1)$

$[\operatorname{ar}\tanh(x)]' = \frac{1}{1-x^2} \quad (|x| < 1)$

$[\operatorname{ar}\coth(x)]' = \frac{1}{1-x^2} \quad (|x| > 1)$

微分法则

I. $d(cu) = c \cdot du,$

II. $d(u \pm v) = du \pm dv,$

III. $d(uv) = u \cdot dv + v \cdot du,$

IV. $d\left(\dfrac{u}{v}\right) = \dfrac{v \cdot du - u \cdot dv}{v^2}.$

Differential formulas

$$\frac{d}{dx}(C) = 0$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$(cf)' = cf'$$

$$(f+g)' = f' + g'$$

$$(f-g)' = f' - g'$$

$$(fg)' = fg' + gf'$$

$$(\frac{f}{g})' = \frac{gf' - fg'}{g^2}$$

Differential formulas

$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

Differential formulas

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$$

Differential formulas

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$
$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$
$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$

Exercise 1

Find an equation of the tangent line to the curve $y = 2x \sin x$ at the point $(\frac{\pi}{2}, \pi)$.

Find an equation of the tangent line to the curve $y = 2x \sin x$ at the point $(\frac{\pi}{2}, \pi)$.

Solution: $y' = 2(sinx + xcosx)$
$k = y'(\frac{\pi}{2}) = 2$
$y - \pi = 2(x - \frac{\pi}{2})$
$y - 2x = 0$

Composite Function

1. If $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 7$, find a function f such that $f \circ g = h$.
   (Think about what operations you would have to perform on the formula for g to end up with the formula for h.)
2. If $f(x) = 3x + 5$ and $h(x) = 3x^2 + 3x + 2$, find a function g such that $f \circ g = h$.

Exercise 2

Differentiate:

1. $y = x^3 + \frac{7}{x^4} - \frac{2}{x} + 12$
2. $y = \sin x \cos x$
3. $y = \sqrt{x} \sin x$
4. $y = 3e^x \cos x$
5. $y = \frac{x \sin x}{1 + x}$
6. $y = \frac{1 - \sec x}{\tan x}$
7. $y = x^2 \ln x \cos x$
8. $y = \ln 3 + \frac{e^x}{x^2}$

Solution:

1. $3x^2 - \dfrac{28}{x^5} + \dfrac{2}{x^2}$
2. cos2x
3. $\dfrac{1}{2}x^{-\frac{1}{2}} sinx + x^{\frac{1}{2}} cosx$
4. $3e^x(-sinx + cosx)$
5. $\dfrac{dy}{dx} = \dfrac{(xsinx)'(1+x) - xsinx}{(1+x)^2} = \dfrac{sinx + (1+x)xcosx}{(1+x)^2}$
6. $y = \dfrac{cosx - 1}{sinx} \to \dfrac{dy}{dx} = \dfrac{1}{1 + cosx}$
7. $\dfrac{dy}{dx} = 2xlnxcosx + xcosx - x^2 lnxsinx$
8. $(-2x^{-3} + x^{-2})e^x$

Exercise 3

Let $y = \log_{\varphi(x)} f(x)$ $(\varphi(x) > 0,\ \varphi(x) \neq 1,\ f(x) > 0)$. Suppose that both $\varphi(x)$ and $f(x)$ are differentiable. Calculate $\dfrac{dy}{dx}$.

Exercise 3

Solution:

$$y = \frac{lnf(x)}{ln\varphi(x)}$$

$$\frac{dy}{dx} = \frac{\frac{1}{f(x)}f'(x)ln\varphi(x) - \frac{1}{\varphi(x)}\varphi'(x)lnf(x)}{[ln\varphi(x)]^2}$$

$$= \frac{f'(x)}{f(x)ln\varphi(x)} - \frac{\varphi'(x)lnf(x)}{\varphi(x)[ln\varphi(x)]^2}$$

Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

The Power Rule Combined with the Chain Rule

If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}\left(u^n\right) = nu^{n-1}\frac{du}{dx}$$

Alternatively,

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

Exercise 4

Find the derivative of the function:

1. $y = (4x - x^2)^{100}$
2. $y = 5^{-\frac{1}{x}}$
3. $y = e^{-2x} \cos 4x$
4. $y = (\frac{x^2+1}{x^2-1})^3$
5. $y = \frac{\arcsin x}{\arccos x}$
6. $y = [\sin(e^{(\sin x)^2})]^2$
7. $y = \arcsin \sqrt{\frac{1-x}{1+x}}$
8. $y = n^{n^x} + x^{n^n} + n^{x^n} \; (n > 0,\ n \neq 1)$

Exercise 4

1. $200(2-x)(4x-x^2)^{99}$
2. $ln5 \cdot 5^{-\frac{1}{x}} \cdot x^{-2}$
3. $-2e^{-2x}(cos4x + 2sin4x)$
4. $y' = 3(\frac{x^2+1}{x^2-1})^2(-\frac{2}{(x^2-1)^2}) \cdot 2x = -\frac{12x(x^2+1)^2}{(x^2-1)^3}$
5. $y' = \frac{(arcsinx)'arccosx - (arccosx)'arcsinx}{(arccosx)^2} = \frac{arccosx + arcsinx}{\sqrt{1-x^2}(arccosx)^2}$
6. $y' = 2[sin(e^{(sinx)^2})] \cdot cos(e^{(sinx)^2}) \cdot e^{(sinx)^2} \cdot 2sinxcosx =$ $sin2x \cdot sin(2e^{(sinx)^2}) \cdot e^{(sinx)^2}$
7. $y' = (1 - \frac{1-x}{1+x})^{-\frac{1}{2}} \cdot \frac{1}{2}(\frac{1-x}{1+x})^{-\dfrac{1}{2}}[-\frac{2}{(x+1)^2}] =$ $-\frac{1}{\sqrt{2x(1-x)} \cdot (x+1)}$
8. $y' = n^{n^x+x}(lnn)^2 + n^n x^{n^n-1} + n^{x^n+1}x^{(n-1)}lnn$

Exercise 5

Find the second derivative of the following function:
(**Warning**: Don't forget to double check your first derivative!)

1. $y = \tan x$
2. $y = \dfrac{1}{x^3 + 1}$
3. $y = x \cos x$

Exercise 5

1. $y' = sec^2x$
   $y'' = 2secx \cdot (tanx \cdot secx) = 2tanx \cdot sec^2x$

2. $y' = -\dfrac{1}{(x^3+1)^2} \cdot 3x^2$
   $$y'' = -3 \cdot \frac{2x(x^3+1)^2 - 2(x^3+1)3x^4}{(x^3+1)^4} = \frac{6x(2x^3-1)}{(x^3+1)^3}$$

3. $y' = cosx - xsinx$
   $y'' = -2sinx - xcosx$

Exercise 6

Answer the following three questions based on $\dfrac{dy}{dx} = y'$:

1. Express $\dfrac{dx}{dy}$ with y'
2. Express $\dfrac{d^2x}{dy^2}$ with y' and y''
3. Express $\dfrac{d^3x}{dy^3}$ with y' ,y'' and y'''

Exercise 6

Solution:

1. $$\frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{y'}$$

2. $$\frac{d^2x}{dy^2} = \frac{d}{dy} \cdot \frac{dx}{dy} = \frac{d}{dx}\frac{1}{y'}\frac{dx}{dy} = -\frac{y''}{(y')^3}$$

3. $$\frac{d^3x}{dy^3} = \frac{d}{dy}\frac{d^2x}{dy^2} = \frac{d}{dx}\frac{d^2x}{dy^2}\frac{dx}{dy} = -\frac{y'''(y')^3 - 3(y')^2(y'')^2}{(y')^6} \cdot \frac{1}{y'} = \frac{3(y'')^2 - y'(y''')^2}{(y')^5}$$

Find y' if $\sin(x+y) = y^2 \cos x$

Differentiating implicitly with respect to x and remembering that y is a function of x, we get

$$\cos(x+y) \cdot \left(1 + y'\right) = y^2(-\sin x) + (\cos x)\left(2yy'\right)$$

(Note that we have used the Chain Rule on the left side and the Product Rule and Chain Rule on the right side.) If we collect the terms that involve y', we get

$$\cos(x+y) + y^2 \sin x = (2y\cos x)y' - \cos(x+y) \cdot y'$$

So

$$y' = \frac{y^2 \sin x + \cos(x+y)}{2y\cos x - \cos(x+y)}$$

Exercise 7

Calculate the derivative of the following implicit function:

1. $y^2 - 2xy + 9 = 0$
2. $xy = e^{xy}$

Solution

1. $2yy' - 2y - 2xy' = 0, y' = \dfrac{y}{y-x}$
2. The derivative does not exist!

Exercise 8

Use **logarithmatic differentiation** to calculate the derivative of the following implicit function:

1. $y = (\frac{x}{x+1})^x$
2. $y = \sqrt[5]{\frac{x-5}{\sqrt[5]{x^2+2}}}$

Exercise 8

Solution:

1. $lny = x[lnx - ln(x+1)]$
   $\frac{1}{y}y' = [lnx - ln(x+1)] + x[\frac{1}{x} - \frac{1}{x+1}]$
   $y' = (\frac{x}{x+1})^x[ln|\frac{x}{x+1}| + \frac{1}{x+1}]$
2. $lny = \frac{1}{5}ln|x-5| - \frac{1}{25}ln(x^2+2)$
   $y'\frac{1}{y} = \frac{1}{5(x-5)} - \frac{2x}{25(x^2+2)}$
   $y' = y[\frac{1}{5(x-5)} - \frac{2x}{25(x^2+2)}]$

Exercise 9

The *Bessel function* of order 0, $y = J(x)$, satisfies the differential equation $xy'' + y' + xy = 0$ for all values of x and its value at 0 is $J(0) = 1$.

(a) Find $J'(0)$.

(b) Use implicit differentiation to find $J''(0)$.

Exercise 9

Solution:

1. Take x=0, $0 + J'(0) + 0 = 0 \to J'(0) = 0$
2. $xy''' + 2y'' + y + xy' = 0$
   At $x = 0$, $2y'' + 1 = 0$, $y'' = -\frac{1}{2}$

10.

Find the derivative of the function $f(x) = (x\sin^2(x))^{\frac{1}{3}}$.

11.

The equation of the tangent to $f(x)$ at $x = 1$ is $y = -2(x-3)$, and knowing that $g(x) = f^{-1} + 3$, find the equation of the tangent to $g(x)$ at $x = 3$.

12.

Find the slope of the tangent to the curve $xy = \arctan(3y)$ when $y = \dfrac{1}{3}$.

Definition

$$f(x) \approx f(a) + f'(a)(x-a)$$

is called the **linear approximation** or **tangent line approximation** of f at a. The linear function whose graph is this tangent line, that is,

$$L(x) = f(a) + f'(a)(x-a)$$

is called the linearization of f at a.

Exercise 10

Use linear approximation to estimate $2.0006^{1.9998}$

Exercise 10

Solution:
$f(x) = 2^{x+2}$
$f'(x) = 2^{x+2} ln2$
$2^{1.9998} = f(0) + f'(0) \times (-0.0002) = 2^2 + 4ln2 \times (-0.0002) = 3.9994$
$g(x) = (2+x)^{1.9998}$
$g'(x) = 1.9998(2+x)^{0.9998}$
$2.0006^{1.9998} = g(0) + g'(0) \times 0.0006 =$
$2^{1.9998} + 1.9998 \times 2^{0.9998} \times 0.0006 = 4.0018$

Absolute Maximum and Absolute Minimum

Let c be a number in the domain D of a function f. Then $f(c)$ is the
- **absolute maximum** value of f on D if $f(c) \geqslant f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leqslant f(x)$ for all x in D.

Local Maximum and Local Minimum

The number $f(c)$ is a
- **local maximum** value of f if $f(c) \geqslant f(x)$ when x is near c.
- **local minimum** value of f if $f(c) \leqslant f(x)$ when x is near c.

Critical number

A **critical number** of a function f is a number c in the domain such that either $f'(c) = 0$ or $f'(c)$ does not exist, or the end points of a region if it is bounded.

local maximum/minimum $\rightarrow$ critical number

Related Theorems

The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Fermat's Theorem

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Rolle's Theorem

Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

Lagrange Mean Value Theorem

Let f be a function that satisfies the following hypotheses:
1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently, $f(b) - f(a) = f'(c)(b - a)$

Cauchy Mean Value Theorem (Extended Mean Value Theorem)

Let f, g be two functions that satisfy the following hypotheses:
1. f, g is continuous on the closed interval $[a, b]$.
2. f, g is differentiable on the open interval (a, b).
3.$x \in (a, b), g'(x) \neq 0$
Then there is a number c in (a, b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

or, equivalently, $(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$

Exercise 12

Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

1. $f(x) = x^3 - 3x + 2, \quad [-2, 2]$
2. $f(x) = \ln x, \quad [1, 4]$

Exercise 12

Solutions:

1. f(-2) = 0, f(2) = 4

   $$k = \frac{f(2) - f(-2)}{4} = 1$$

   $$f'(x) = 3x^2 - 3, 3c^2 - 3 = 1$$

   $$c = \frac{2\sqrt{3}}{3}$$

2. $$k = \frac{f(4) - f(1)}{3} = \frac{2ln2}{3}$$

   $$f'(x) = \frac{1}{x}$$

   $$d = \frac{3}{2ln2}$$

Relationship between f' and f

Increasing/Decreasing Test

If $f'(x) > 0$ on an interval, then f is increasing on that interval.
If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

The First Derivative Test

Suppose that c is a critical number of a continuous function f.
If f' changes from positive to negative at c, f has a local maximum at c.
If f' changes from negative to positive at c, f has a local minimum at c.
If f' does not change sign at c, f has no local maximum or minimum at c.

Relationship between f'' and f

Concave upward/downward

If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on an interval I, then it is called **concave downward** on I.

Concavity Test

If $f''(x) > 0$ for all x in I, then the graph of I is concave upward on I. If $f''(x) < 0$ for all x in I, then the graph of I is concave downward on I.

Relationship between f'' and f

Inflection point

A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

The Second Derivative Test

Suppose f'' is continuous near c.
If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

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3. Limits

"Rough" Definition of a Limit

Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, **except possibly** at a itself.)
Then we write

$$\lim_{x \to a} f(x) = L$$

and say

"the limit if $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be **sufficiently close to** a (on either side of a) but **not equal to** a.

Considering a function called *Heaviside Function*

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \tag{1}$$

Does $\lim_{t \to 0} H(t)$ exists?

One-sided Limits

We write

$$\lim_{x \to a^-} f(x) = L$$

and say the left-hand limit of $f(x)$ as x approaches a is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be **sufficiently close to** a and x less than a.

When calculating $\lim\limits_{x \to a^-} f(x)$, we consider only $x < a$.

Similarly, we can get the right-hand limit of $f(x)$ as x approaches a.

When does $\lim_{x \to a} f(x)$ exists?

$$\lim_{x \to a} f(x) = L$$

if and only if

$$\lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L$$

Can we directly regard L as $f(a)$?

Infinite Limits

Let f be a function defined on both sides of a, **except possibly** at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x **sufficiently close to** a, but **not equal to** a.
Let f be a function defined on both sides of a, **except possibly** at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x **sufficiently close to** a, but **not equal to** a.

Warning:

$\lim_{x \to a} f(x) = (-)\infty$ does not mean that we are regarding $(-)\infty$ as a number. Nor does it mean that the limit exists!

Limits at Infinity

1. Let f be a function defined on some interval (a, ∞). Then
$$\lim_{x\to\infty} f(x) = L$$
means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

2. Let f be a function defined on some interval $(-\infty, a)$. Then
$$\lim_{x\to-\infty} f(x) = L$$
means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.

Infinite Limits at Infinity

1. $\lim_{x \to \infty} f(x) = \infty$
2. $\lim_{x \to \infty} f(x) = -\infty$
3. $\lim_{x \to -\infty} f(x) = \infty$
4. $\lim_{x \to -\infty} f(x) = -\infty$

Five basic laws:
Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \text{ and } \lim_{x\to a} g(x)$$

exists. Then

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$
4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ **(if $\lim_{x\to a} g(x) \neq 0$)**

Another six laws:

1. $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$ (where n is a positive integer)
2. $\lim_{x \to a} c = c$
3. $\lim_{x \to a} x = a$
4. $\lim_{x \to a} x^n = a^n$ (where n is a positive integer)
5. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ (where n is a positive integer)
   if n is even, we assume that $a > 0$
6. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ (where n is a positive integer)
   if n is even, we assume that $\lim_{\mathbf{x} \to a} f(x) > 0$

Warning:

The Limit Laws can't be applied to infinite limits because $(-)\infty$ is not a number!

即极限进行运算时，必须保证运算前后得到的极限存在（不是未定式/无穷）

Two additional properties of limits:

1. if $f(x) \leq g(x)$ when x is near a (**except possibly** at a) and the limits of f and g both exists as x approaches a, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$$

2. **(The Squeeze Theorem)** if $f(x) \leq g(x) \leq h(x)$ when x is near a (**except possibly** at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

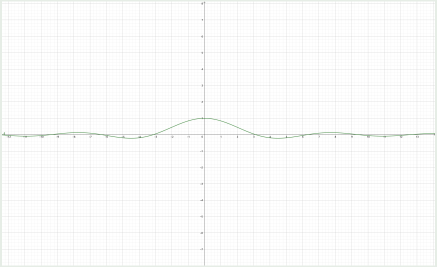
$$\lim_{x \to a} g(x) = L$$

Two Important Limits

Be sure to keep these two limits in mind!

1. $\lim_{x\to 0} \frac{\sin x}{x} = 1$
   (How to prove it? Considering $\sin x$, x and $\tan x$. Then, use the squeeze theorem.)

2. $\lim_{x\to\infty} (1 + \frac{1}{x})^x = e = 2.718281828459045\cdots$
   (Proof: https://www.zhihu.com/question/277272238)



The Precise Definition of a Limit

Let f be a function defined on some open interval that contains the number a, **except possibly** at a itself. Then we say that the limit if $f(x)$ as x approaches a is L, and we write

$$\lim_{x \to a} f(x) = L$$

if for **every** number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon$$

Left-hand limits:

$$\lim_{x \to a^-} f(x) = L$$

if for **every** number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$\text{if} \quad a - \delta < x < a \quad \text{then} \quad |f(x) - L| < \varepsilon$$

Right-hand limits:

$$\lim_{x \to a^+} f(x) = L$$

if for **every** number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$\text{if} \quad a < x < a + \delta \quad \text{then} \quad |f(x) - L| < \varepsilon$$

Infinite limits:

1. Let f be a function defined on some open interval that contains the number a, **except possibly** at a itself. Then
$$\lim_{x \to a} f(x) = \infty$$
means that for **every positive** number M, there is a number $\delta > 0$ such that
$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad f(x) > M$$

2. Let f be a function defined on some open interval that contains the number a, **except possibly** at a itself. Then
$$\lim_{x \to a} f(x) = -\infty$$
means that for **every negative** number N, there is a number $\delta > 0$ such that
$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad f(x) < N$$

Limits at Infinity:

1. Let f be a function defined on some interval (a, ∞). Then
$$\lim_{x\to\infty} f(x) = L$$
means that for every $\varepsilon > 0$, there is a corresponding number N such that
$$\text{if} \quad x > N \quad \text{then} \quad |f(x) - L| < \varepsilon$$

2. Let f be a function defined on some interval $(-\infty, a)$. Then
$$\lim_{x\to-\infty} f(x) = L$$
means that for every $\varepsilon > 0$, there is a corresponding number N such that
$$\text{if} \quad x < N \quad \text{then} \quad |f(x) - L| < \varepsilon$$

Exercise 1

Evaluate the following limits

1. $\lim\limits_{x\to 0} \dfrac{4x^3 - 2x^2 + x}{2x + 3x^2}$
2. $\lim\limits_{x\to 0} \dfrac{\dfrac{1}{3+x} - \dfrac{1}{3}}{x}$
3. $\lim\limits_{x\to \infty} (1 + \dfrac{1}{x})(2 - \dfrac{1}{x^2})$
4. $\lim\limits_{x\to 0} \dfrac{-1 + \sqrt[n]{x+1}}{x}$

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Exercise 2

Evaluate the following limits

1. $\lim_{x \to -4} \frac{\sqrt{9+x^2}-5}{x+4}$
2. $\lim_{\mathbf{h} \to 0} \frac{(x+h)^3-x^3}{h}$(What if $\lim_{\mathbf{h} \to 0} \frac{(x+h)^n-x^n}{h}$?)
3. $\lim_{x \to 0} \frac{\sin \omega x}{x}$
4. $\lim_{x \to \infty} (1-\frac{1}{x})^{kx}$ (k is a positive integer)
5. $\lim_{x \to \frac{\pi}{2}} (\sin x)^{\tan x}$

Tip: You're highly recommended to remember this part!

When $x \to 0$

$$a^x - 1 \sim x \ln a$$

$$\arcsin(a)x \sim \sin(a)x \sim (a)x$$

$$\arctan(a)x \sim \tan(a)x \sim (a)x$$

$$\ln(1+x) \sim x$$

$$\sqrt{1+x} - \sqrt{1-x} \sim x$$

$$(1+ax)^b - 1 \sim abx$$

$$\sqrt[b]{1+ax} - 1 \sim \frac{a}{b}x$$

$$1 - \cos x \sim \frac{x^2}{2}$$

$$x - \ln(1+x) \sim \frac{x^2}{2}$$

When $x \to 0$

$$\tan x - \sin x \sim \frac{x^3}{2}$$

$$\tan x - x \sim \frac{x^3}{3}$$

$$x - \arctan x \sim \frac{x^3}{3}$$

$$x - \sin x \sim \frac{x^3}{6}$$

$$\arcsin x - x \sim \frac{x^3}{6}$$

Equivalent Infinitesimal

Example: Solve the limit

$$\begin{aligned}\lim_{x\to 0}\frac{\ln(1+4x)}{\sin(3x)} &= \lim_{x\to 0}\frac{\ln(1+4x)}{\sin(3x)}\lim_{x\to 0}\frac{4x}{\ln(1+4x)}\lim_{x\to 0}\frac{\sin(3x)}{3x} \\ &= \lim_{x\to 0}\frac{4x}{3x} = 4/3\end{aligned}$$

For more exercise regarding to equivalent infinitesimal, please refer to Worksheet 1.

Taylor Expansion

Definition

Taylor expansion around $x = x_0$:

$f(x) = f(x_0) + \sum_{i=1}^{n} \frac{f^{(i)}(x)}{i!}(x - x_0)^i + R_n$, where $R_n = o[(x - x_0)^n]$

It simulates a function around a point with a polynomial function.

Taylor Expansion

Taylor expansion of some polynomials when x is around 0:

1. $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$
2. $ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$
3. $sinx = x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)$
4. $cosx = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)$
5. $tanx = x + \frac{x^3}{3} + \frac{2x^5}{15} + o(x^5)$

Tip

$o(x^n)$ *means the order of the polynomial is* **larger** *than* n;

$O(x^n)$ *means the order of the polynomial is* **larger than or equal to** n.

The transformation of Taylor Expansion:

Example

The Taylor expansion of e^{x^2} around x=0:

$e^{x^2} = 1 + x^2 + \frac{(x^2)^2}{2} + \frac{(x^2)^3}{6} + o((x^2)^3) = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + o(x^6)$

L'Hospital's Rule

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Exercise 13

Warning: Always judge whether l'Hôpital's rule can be applied before you use it, and **don't neglect those basic methods of finding the limit**.

Evaluate the following limits:

1. $\lim_{x \to 0^+} x^{\sin x}$
2. $\lim_{x \to 1} \frac{\ln x}{x - 1}$
3. $\lim_{x \to \infty} x^3 e^{-x^2}$
4. $\lim_{x \to \infty} [x - x^2 \ln \frac{x+1}{x}]$

5. $\lim_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}}$

Some methods to calculate the limits:

1. Use those limit laws directly
2. Exchange the order of functions and limit symbols based on the continuity of composite function. (Will be mentioned later)
3. Do factorization, denominator rationalization or numerator rationalization.
4. If a factor approaching zero is find in the denominator, try to eliminate it.
5. Translate the formula into the form of "two important limits"
6. **The method to solve those formulas having the form of $u(x)^{v(x)}$ will be discussed at a deeper level after the differentiation and l'Hôpital's rule are taught.**

Exercise Answer 1

1. $$\lim_{x\to 0}\frac{4x^3-2x^2+x}{2x+3x^2}=\lim_{x\to 0}\frac{4x^2-2x+1}{2+3x}=0.5$$
2. $$\lim_{x\to 0}\frac{\frac{1}{3+x}-\frac{1}{3}}{x}=\lim_{x\to 0}\frac{3-3-x}{3x(3+x)}=\lim_{x\to 0}\frac{-1}{3(3+x)}=-\frac{1}{9}$$
3. $$\lim_{x\to\infty}(1+\frac{1}{x})(2-\frac{1}{x^2})=\lim_{x\to\infty}(1+\frac{1}{x})\cdot\lim_{x\to\infty}(2-\frac{1}{x^2})=1\cdot 2=2$$
4. Let $t^n-1:=x$, $\lim_{x\to 0}\frac{-1+\sqrt[n]{x+1}}{x}=\lim_{t\to 1}\frac{t-1}{t^n-1}=$ $\lim_{t\to 1}\frac{1}{1+t+t^2+\cdots+t^{n-1}}=\frac{1}{n}$

Exercise Answer 2

1. $\lim\limits_{x \to -4} \dfrac{\sqrt{9+x^2}-5}{x+4} = \lim\limits_{x \to -4} \dfrac{x-4}{\sqrt{9+x^2}+5} = -\dfrac{4}{5}$

2. $\lim\limits_{\mathbf{h} \to 0} \dfrac{(x+h)^n - x^n}{h} = \lim\limits_{\mathbf{h} \to 0} nx^{n-1} + h \cdot (\cdots) = nx^{n-1}$

3. $\lim\limits_{x \to 0} \dfrac{\sin \omega x}{x} = \omega \cdot \lim\limits_{x \to 0} \dfrac{\sin \omega x}{\omega x} = \omega$

4. $\lim\limits_{x \to \infty} (1 - \dfrac{1}{x})^{kx} = \lim\limits_{x \to \infty} (1 + \dfrac{1}{-x})^{(-x)(-k)} = e^{-k}$

5. $\lim\limits_{x \to \frac{\pi}{2}} (\sin x)^{\tan x} = \lim\limits_{x \to \frac{\pi}{2}} (1 + (\sin x - 1))^{\dfrac{1}{\sin x - 1} \cdot (\sin x - 1)\tan x} =$
$e^{\lim\limits_{x \to \frac{\pi}{2}} (\sin x - 1)\tan x} = e^{\lim\limits_{x \to \frac{\pi}{2}} \dfrac{(\sin x - 1)\sin x}{\sqrt{1-\sin^2 x}}} = e^{-\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\sqrt{1-\sin x}\sin x}{\sqrt{1+\sin x}}} =$
$e^0 = 1$

Solutions to Exercise 13

Solution:

1. 1
2. 1
3. $\lim_{x\to\infty} \frac{x^3}{e^{x^2}} = \lim_{x\to\infty} \frac{3x^2}{2xe^{x^2}} = \lim_{x\to\infty} \frac{6x}{(2+4x^2)e^{x^2}} = \lim_{x\to\infty} \frac{6}{12x+8x^2} =$ 0
4. $u = \frac{1}{x}$

   $$\lim_{u\to 0} \frac{u - ln(u+1)}{u^2} = \lim_{u\to 0} \frac{1 - \frac{1}{u+1}}{2u} = \lim_{u\to 0} \frac{\frac{1}{(u+1)^2}}{2} = \frac{1}{2}$$
5. $\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x} = 1$

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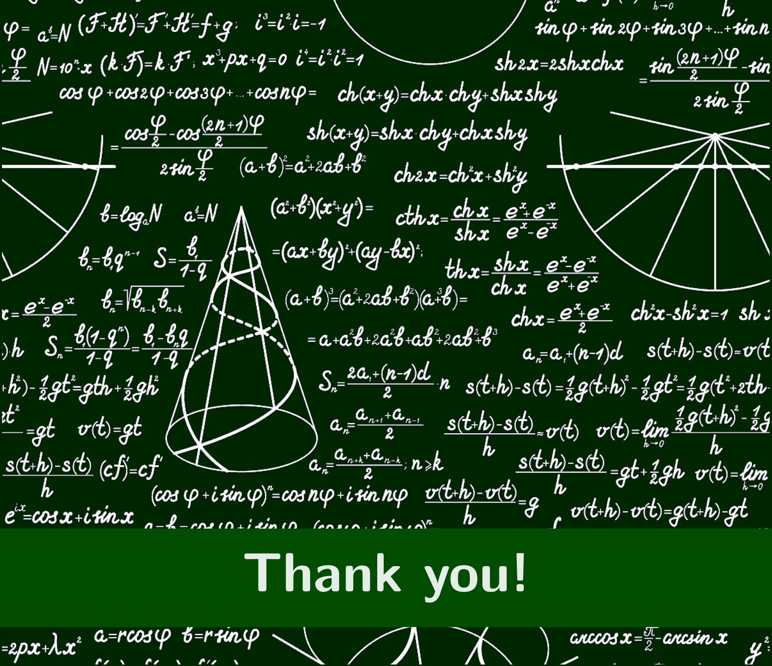

Thank you!