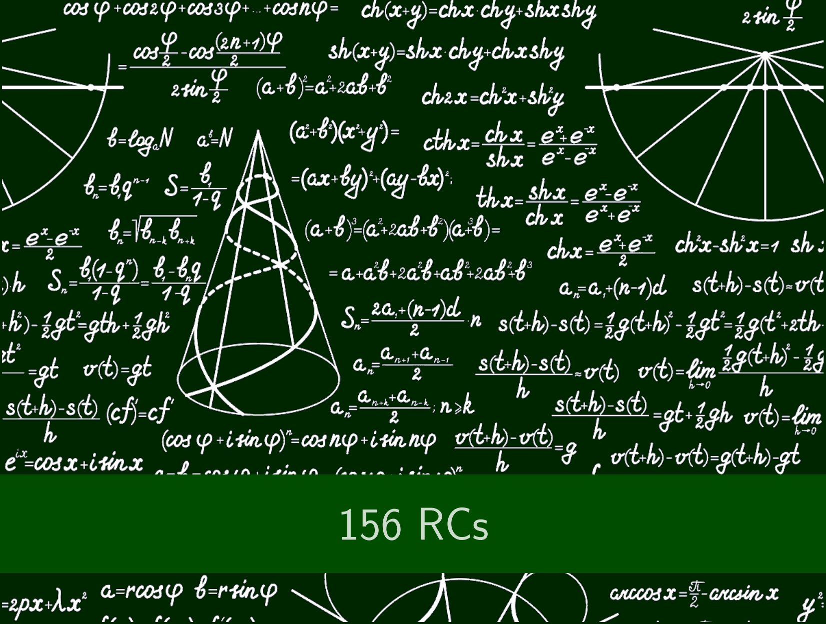


156 RCs

1. Limits

"Rough" Definition of a Limit

Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, **except possibly** at a itself.)
Then we write

$$\lim_{x \to a} f(x) = L$$

and say

"the limit if $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be **sufficiently close to** a (on either side of a) but **not equal to** a.

Considering a function called *Heaviside Function*

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \tag{1}$$

Does $\lim_{t \to 0} H(t)$ exists?

We write

$$\lim_{x \to a^-} f(x) = L$$

and say the left-hand limit of $f(x)$ as x approaches a is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be **sufficiently close to** a and x less than a.

When calculating $\lim_{x \to a^-} f(x)$, we consider only $x < a$.

Similarly, we can get the right-hand limit of $f(x)$ as x approaches a.

When does $\lim_{x \to a} f(x)$ exists?

$$\lim_{x \to a} f(x) = L$$

if and only if

$$\lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L$$

Can we directly regard L as $f(a)$?

Let f be a function defined on both sides of a, **except possibly** at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x **sufficiently close to** a, but **not equal to** a.

Let f be a function defined on both sides of a, **except possibly** at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x **sufficiently close to** a, but **not equal to** a.

Warning:

$\lim_{x \to a} f(x) = (-)\infty$ does not mean that we are regarding $(-)\infty$ as a number. Nor does it mean that the limit exists!

1. Let f be a function defined on some interval (a, ∞). Then
$$\lim_{x\to\infty} f(x) = L$$
means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

2. Let f be a function defined on some interval $(-\infty, a)$. Then
$$\lim_{x\to-\infty} f(x) = L$$
means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.

Infinite Limits at Infinity

1. $\lim_{x\to\infty} f(x) = \infty$
2. $\lim_{x\to\infty} f(x) = -\infty$
3. $\lim_{x\to-\infty} f(x) = \infty$
4. $\lim_{x\to-\infty} f(x) = -\infty$

Five basic laws:
Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \text{ and } \lim_{x \to a} g(x)$$

exists. Then

1. $\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. $\lim_{x \to a}[f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. $\lim_{x \to a}[cf(x)] = c \lim_{x \to a} f(x)$
4. $\lim_{x \to a}[f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
5. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ **(if $\lim_{x \to a} g(x) \neq 0$)**

Another six laws:

1. $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$ (where n is a positive integer)
2. $\lim_{x\to a} c = c$
3. $\lim_{x\to a} x = a$
4. $\lim_{x\to a} x^n = a^n$ (where n is a positive integer)
5. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ (where n is a positive integer)
   if n is even, we assume that $a > 0$
6. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ (where n is a positive integer)
   if n is even, we assume that $\lim_{\mathbf{x}\to a} f(x) > 0$

For composite functions, we have

Law for Composite Functions

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$$

for all the limits and functions exist.

E.g. Solve $\lim_{x \to \infty} \cos \dfrac{1}{x}$.

Warning:

The Limit Laws can't be applied to infinite limits because $(-)\infty$ is not a number!

即极限进行运算时，必须保证运算前的极限存在（不是未定式/无穷），而且极限为有限个

Two additional properties of limits:

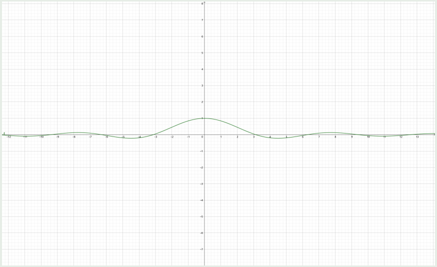
1. if $f(x) \leq g(x)$ when x is near a (**except possibly** at a) and the limits of f and g both exists as x approaches a, then
$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$$
2. **(The Squeeze Theorem)** if $f(x) \leq g(x) \leq h(x)$ when x is near a (**except possibly** at a) and
$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$
then
$$\lim_{x \to a} g(x) = L$$

Two Important Limits

Be sure to keep these two limits in mind!

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1$
   (How to prove it? Considering $\sin x$, x and $\tan x$. Then, use the squeeze theorem.)

2. $\lim_{x \to \infty} (1 + \frac{1}{x})^x = e = 2.718281828459045 \cdots$
   (Proof: https://www.zhihu.com/question/277272238)



1. $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$

相关变形：

- $\lim\limits_{x\to\infty}\dfrac{\sin x}{x}=0$
- $\lim\limits_{f(x)\to 0}\dfrac{\sin f(x)}{f(x)}=1$

1. $\lim\limits_{x\to\infty}(1+\frac{1}{x})^x = e$

相关变形:

- $\lim\limits_{x\to 0^+}(1+\frac{1}{x})^x = 1$
- $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e$
- $\lim\limits_{x\to +\infty}(1+x)^{\frac{1}{x}} = 1$
- $\lim\limits_{x\to 0^-}(1-\frac{1}{x})^x = 1$
- $\lim\limits_{x\to\infty}(1-\frac{1}{x})^x = \frac{1}{e}$

If $u \to 1$, $v \to \infty$, then we have

$$\lim \mathbf{u}^{\mathbf{v}} = "\mathbf{1}^{\infty}" = \mathbf{e}^{\lim \mathbf{v}(\mathbf{u}-\mathbf{1})}$$

也就是幂指函数（后面会讲）

常见极限计算方法:

1. 四则运算与函数性质
2. 导数定义
3. **两个重要极限**
4. **洛必达**
5. 等价无穷小
6. 泰勒展开（了解即可）

四则运算与函数性质

有理式函数极限：

对有理式$F(x)=\dfrac{p(x)}{q(x)}=\dfrac{a_0+a_1x_1+\cdots+a_mx^m}{b_0+b_1x_1+\cdots+b_nx^n}$，有

$$\lim_{x\to\infty}F(x)=\begin{cases}\dfrac{a_m}{b_n} & m=n\\ 0 & m<n\\ \infty & m>n\end{cases}$$

1. 计算下列极限：

1. $\lim\limits_{x\to 0}\dfrac{4x^3-2x^2+x}{2x+3x^2}$
2. $\lim\limits_{x\to\infty}\dfrac{3x^3+4x^2+2}{7x^3+5x^2-3}$
3. $\lim\limits_{x\to\infty}(1+\dfrac{1}{x})(2-\dfrac{1}{x^2})$

四则运算与函数性质

2. 设 $\lim_{x\to -1} \dfrac{x^3 - ax^2 - x + 4}{x+1}$存在极限值且为$m$, 试求$a$和$m$的值。

有根号：分子有理化或根式整体换元

3. 计算下列极限：

1. $\lim_{x\to 0} \dfrac{-1 + \sqrt[n]{x+1}}{x}$
2. $\lim_{x\to -4} \dfrac{\sqrt{9+x^2} - 5}{x+4}$
3. $\lim_{x\to -\infty} (\sqrt{x^2+2x} + x)$
4. $\lim_{x\to 0} \dfrac{\sqrt{1+\tan^2 x} - \sqrt{1+\sin^2 x}}{(5^x - 1)\arctan^3 x}$

Exercise 2

导数定义

计算下列极限:

1. $\lim\limits_{x\to 0}\dfrac{-1+\sqrt[n]{x+1}}{x}$
2. $\lim\limits_{\mathbf{h}\to 0}\dfrac{(x+h)^3-x^3}{h}$

   (What if $\lim\limits_{\mathbf{h}\to 0}\dfrac{(x+h)^n-x^n}{h}$?)

两个重要极限

计算下列极限：

1. $\lim\limits_{x \to 0} \frac{\sin \omega x}{x}$
2. $\lim\limits_{x \to \infty} (1 - \frac{1}{x})^{kx}$ (k is a positive integer)
3. $\lim\limits_{x \to \frac{\pi}{2}} (\sin x)^{\tan x}$
4. $\lim\limits_{x \to 0^+} (\cos \sqrt{x})^{\frac{\pi}{x}}$
5. $\lim\limits_{x \to \infty} \left(\frac{x+3}{x+6} \right)^{\frac{x-1}{2}}$

L'Hospital's Rule

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Exercise 13

Warning: Always judge whether l'Hôpital's rule can be applied before you use it, and **don't neglect those basic methods of finding the limit**.

Evaluate the following limits:

1. $\lim\limits_{x \to 0^+} x^{\sin x}$
2. $\lim\limits_{x \to 1} \dfrac{\ln x}{x - 1}$
3. $\lim\limits_{x \to \infty} x^3 e^{-x^2}$
4. $\lim\limits_{x \to \infty} [x - x^2 \ln \dfrac{x + 1}{x}]$

5. $\lim\limits_{x \to \infty} \dfrac{x}{\sqrt{x^2 + 1}}$

Tip: You're highly recommended to remember this part!

When $x \to 0$

$$a^x - 1 \sim x \ln a$$

$$\arcsin(a)x \sim \sin(a)x \sim (a)x$$

$$\arctan(a)x \sim \tan(a)x \sim (a)x$$

$$\ln(1+x) \sim x$$

$$\sqrt{1+x} - \sqrt{1-x} \sim x$$

$$(1+ax)^b - 1 \sim abx$$

$$\sqrt[b]{1+ax} - 1 \sim \frac{a}{b}x$$

$$1 - \cos x \sim \frac{x^2}{2}$$

$$x - \ln(1+x) \sim \frac{x^2}{2}$$

When $x \to 0$

$$\tan x - \sin x \sim \frac{x^3}{2}$$

$$\tan x - x \sim \frac{x^3}{3}$$

$$x - \arctan x \sim \frac{x^3}{3}$$

$$x - \sin x \sim \frac{x^3}{6}$$

$$\arcsin x - x \sim \frac{x^3}{6}$$

Example: Solve the limit

$$\lim_{x\to 0}\frac{\ln(1+4x)}{\sin(3x)} = \lim_{x\to 0}\frac{\ln(1+4x)}{\sin(3x)}\lim_{x\to 0}\frac{4x}{\ln(1+4x)}\lim_{x\to 0}\frac{\sin(3x)}{3x}$$
$$= \lim_{x\to 0}\frac{4x}{3x} = 4/3$$

For more exercise regarding to equivalent infinitesimal, please refer to Worksheet 1.

注意：加减法中不可使用部分的等价无穷小代换！只有乘除形式的才可以代换

注意：加减法中不可使用部分的等价无穷小代换!

Example: Solve the limit

- $\lim\limits_{x\to 0} \dfrac{x - \sin x \cos x}{x^3}$
- $\lim\limits_{x\to\infty} \dfrac{(1+\dfrac{1}{x})^{x^2}}{e^x}$

Taylor Expansion

Definition

Taylor expansion around $x = x_0$:

$$f(x) = f(x_0) + \sum_{i=1}^{n} \frac{f^{(i)}(x)}{i!}(x - x_0)^i + R_n, \text{ where } R_n = o[(x - x_0)^n]$$

It simulates a function around a point with a polynomial function.

Taylor Expansion

Taylor expansion of some polynomials when x is around 0:

1. $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$
2. $ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$
3. $sinx = x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)$
4. $cosx = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)$
5. $tanx = x + \frac{x^3}{3} + \frac{2x^5}{15} + o(x^5)$

Tip

$o(x^n)$ *means the order of the polynomial is* **larger** *than* n;

$O(x^n)$ *means the order of the polynomial is* **larger than or equal to** n.

The transformation of Taylor Expansion:

Example

The Taylor expansion of e^{x^2} around x=0:
$e^{x^2} = 1 + x^2 + \frac{(x^2)^2}{2} + \frac{(x^2)^3}{6} + o((x^2)^3) = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + o(x^6)$

Conclusions

Some methods to calculate the limits:

1. Use those limit laws directly
2. Exchange the order of functions and limit symbols based on the continuity of composite function. (Will be mentioned later)
3. Do factorization, denominator rationalization or numerator rationalization.
4. If a factor approaching zero is find in the denominator, try to eliminate it.
5. Translate the formula into the form of "two important limits"
6. **The method to solve those formulas having the form of $u(x)^{v(x)}$ will be discussed at a deeper level after the differentiation and l'Hôpital's rule are taught.**

Exercise Answer 1

1. $\lim\limits_{x\to 0} \dfrac{4x^3 - 2x^2 + x}{2x + 3x^2} = \lim\limits_{x\to 0} \dfrac{4x^2 - 2x + 1}{2 + 3x}$=0.5

2. $\lim\limits_{x\to 0} \dfrac{\dfrac{1}{3+x} - \dfrac{1}{3}}{x} = \lim\limits_{x\to 0} \dfrac{3 - 3 - x}{3x(3+x)} = \lim\limits_{x\to 0} \dfrac{-1}{3(3+x)} = -\dfrac{1}{9}$

3. $\lim\limits_{x\to\infty} (1 + \dfrac{1}{x})(2 - \dfrac{1}{x^2}) = \lim\limits_{x\to\infty} (1 + \dfrac{1}{x}) \cdot \lim\limits_{x\to\infty} (2 - \dfrac{1}{x^2}) = 1 \cdot 2 = 2$

4. Let $t^n - 1 := x$, $\lim\limits_{x\to 0} \dfrac{-1 + \sqrt[n]{x+1}}{x} = \lim\limits_{t\to 1} \dfrac{t-1}{t^n - 1} =$
   $\lim\limits_{t\to 1} \dfrac{1}{1 + t + t^2 + \cdots + t^{n-1}} = \dfrac{1}{n}$

Exercise Answer 2

1. $$\lim_{x\to -4}\frac{\sqrt{9+x^2}-5}{x+4}=\lim_{x\to -4}\frac{x-4}{\sqrt{9+x^2}+5}=-\frac{4}{5}$$

2. $$\lim_{\mathbf{h}\to 0}\frac{(x+h)^n-x^n}{h}=\lim_{\mathbf{h}\to 0} nx^{n-1}+h\cdot(\cdots)=nx^{n-1}$$

3. $$\lim_{x\to 0}\frac{\sin\omega x}{x}=\omega\cdot\lim_{x\to 0}\frac{\sin\omega x}{\omega x}=\omega$$

4. $$\lim_{x\to\infty}(1-\frac{1}{x})^{kx}=\lim_{x\to\infty}(1+\frac{1}{-x})^{(-x)(-k)}=e^{-k}$$

5. $$\lim_{x\to\frac{\pi}{2}}(\sin x)^{\tan x}=\lim_{x\to\frac{\pi}{2}}(1+(\sin x-1))^{\frac{1}{\sin x-1}\cdot(\sin x-1)\tan x}=$$
$$e^{\lim_{x\to\frac{\pi}{2}}(\sin x-1)\tan x}=e^{\lim_{x\to\frac{\pi}{2}}\frac{(\sin x-1)\sin x}{\sqrt{1-\sin^2 x}}}=e^{-\lim_{x\to\frac{\pi}{2}}\frac{\sqrt{1-\sin x}\sin x}{\sqrt{1+\sin x}}}=$$
$$e^0=1$$

Solutions to Exercise 13

Solution:

1. 1
2. 1
3. $\lim_{x\to\infty} \frac{x^3}{e^{x^2}} = \lim_{x\to\infty} \frac{3x^2}{2xe^{x^2}} = \lim_{x\to\infty} \frac{6x}{(2+4x^2)e^{x^2}} = \lim_{x\to\infty} \frac{6}{12x+8x^2} =$
   0
4. $u = \frac{1}{x}$

   $$\lim_{u\to 0} \frac{u - ln(u+1)}{u^2} = \lim_{u\to 0} \frac{1 - \frac{1}{u+1}}{2u} = \lim_{u\to 0} \frac{\frac{1}{(u+1)^2}}{2} = \frac{1}{2}$$
5. $\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x} = 1$

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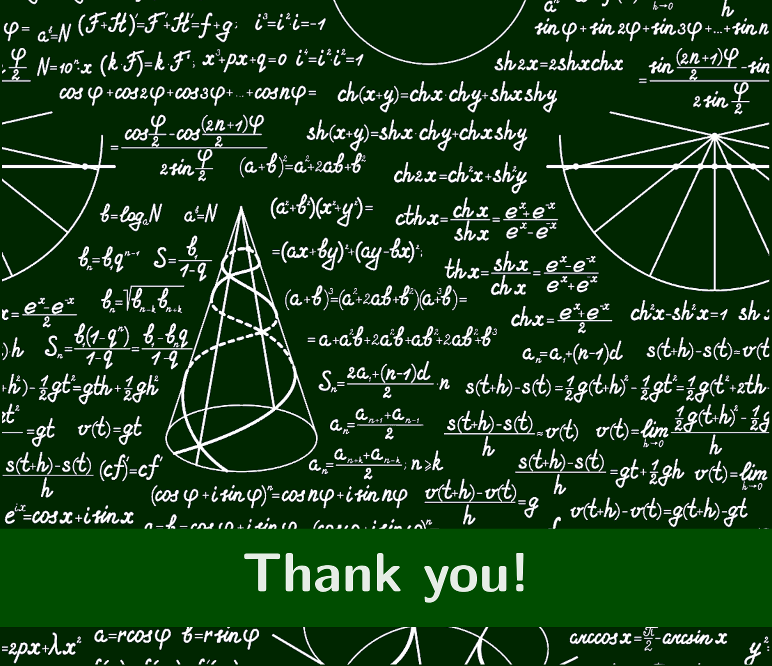
ch(x+y)=chx chy+shx shy
sh(x+y)=shx chy+chx shy
sh2x=2shx chx
ch2x=ch²x+sh²y
b=log_a N
a^b=N
(a+b)²=a²+2ab+b²
ch²x-sh²x=1
a_n=a_1+(n-1)d
e^{ix}=cos x+i sin x
a=r cos φ
b=r sin φ

Thank you!