

$\frac{2 \sin \frac{\varphi}{2}}{(a+b)^2 = a^2 + 2ab + b^2}$ $\operatorname{ch}^2 x = \operatorname{ch}^2 x + \operatorname{sh}^2 x$
 $b = \log_a N \quad a^t = N$ $(a^2 + b^2)(x^2 + y^2) = \operatorname{cth} x = \frac{\operatorname{ch} x}{\operatorname{sh} x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$
 $b_n = b, q^{n-1} \quad S_n = \frac{b}{1-q}$ $= (ax + by)^2 + (ay - bx)^2; \quad \operatorname{th} x = \frac{\operatorname{sh} x}{\operatorname{ch} x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
 $t = \frac{e^x - e^{-x}}{2} \quad b_n = \sqrt{b_{n-k} b_{n+k}}$ $(a+b)^3 = (a^2 + 2ab + b^2)(a^3 + b^3) = \operatorname{ch} x = \frac{e^x + e^{-x}}{2} \quad \operatorname{ch}^2 x - \operatorname{sh}^2 x = 1 \quad \operatorname{sh} x =$
 $) h \quad S_n = \frac{b(1-q^n)}{1-q} = \frac{b - b_n q}{1-q}$ $= a + ab + 2a^2b + ab^2 + 2ab^2 + b^3 \quad a_n = a_1 + (n-1)d \quad s(t+h) - s(t) = v(t)$
 $\frac{1}{2}gt^2 - \frac{1}{2}gt^2 = gth + \frac{1}{2}gh^2$ $S_n = \frac{2a_1 + (n-1)d}{2} \cdot n \quad s(t+h) - s(t) = \frac{1}{2}g(t+h)^2 - \frac{1}{2}gt^2 = \frac{1}{2}g(t^2 + 2th +$
 $\frac{it^2}{2} = gt \quad v(t) = gt$ $a_n = \frac{a_{n+1} + a_{n-1}}{2} \quad \frac{s(t+h) - s(t)}{h} = v(t) \quad v(t) = \lim_{h \rightarrow 0} \frac{\frac{1}{2}g(t+h)^2 - \frac{1}{2}gt^2}{h}$
 $s(t+h) - s(t) \quad (cf)' = cf'$ $a_n = \frac{a_{n+k} + a_{n-k}}{2}; n \geq k \quad \frac{s(t+h) - s(t)}{h} = gt + \frac{1}{2}gh \quad v(t) = \lim$

156 RCs

$z = r(\cos \varphi + i \sin \varphi) = \cos n\varphi + i \sin n\varphi$
 $a+bi \quad r e^{i\varphi}, r e^{i\varphi_2} = r_1 r_2 e^{i(\varphi_1 + \varphi_2)}$ $M(a,b)$

MATH1560J Fall 2025 $a = r \cos \varphi \quad b = r \sin \varphi$

$\Delta x \rightarrow 0 \quad \Delta x \quad \omega(v) \quad h \rightarrow 0 \quad h$
 $s(t) = \frac{1}{2}gt^2 / h \quad y = f(x_0) + f'(x_0)(x - x_0) \quad z = re$
 $\arccos x = \frac{\pi}{2} - \arcsin x \quad y^2$

Sincerest appreciation dedicated to
2024 VV156 TAs
Li Mingrui, Xia Yiwei, Zhang Haoran,
as well as all previous VV156 TAs.

1. About Honors Calculus

2024FA-VV156

- ① Limits
- ② Derivatives and Integrals
- ③ Series
- ④ Polar Coordinates
- ⑤ Basic Differential Equations

2024SU-VV255

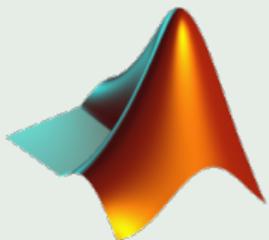
- ① Simple Linear Algebra
- ② Partial Derivatives
- ③ Multiple Integrals

2024FA-VV256

- ① Differential Equations
- ② Deeper Linear Algebra
- ③ Fourier Transform and Laplace Transform

Other courses might contribute to Honors Calculus: VV214, VE203, etc.

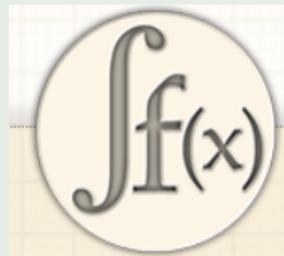
MATLAB:



Mathematica:



Integration with Steps:



<https://www.integral-calculator.com/>

You had better understand the contents on green slides. They are relatively fundamental.

The content shown on blue slides may be relatively hard, but may contribute to getting high marks in the exams more or less. Do not worry when you cannot handle it.

Those on pink slides are just some interesting problems and crazy thoughts. Maybe they are of little application in exams, but they are quite interesting.

2. Functions

Three essential factors of a function:

- Domain: D
- Correspondence Rule
- Range: E

A set containing the action objects of correspondence rule.
A set of all images corresponding to all elements in the domain under a certain correspondence rule.

Definition

A **function** f is a rule that assigns to **each element** x in a set D exactly one element, called $f(x)$, in a set E .

The Vertical Line Test

A curve in the xy -plane is the graph of a function of x **iff** (if and only if) no vertical line intersects the curve more than once.

There are four ways to represent a function:

- ① Verbally (by a description in words)
- ② Numerically (by a table of values)
- ③ Visually (by a graph)
- ④ Algebraically (by an explicit formula)

Symmetry

- Even function: $f(-x) = f(x)$
- Odd function: $f(-x) = -f(x)$

Tip: Give priority to whether the domain D of the function is symmetrical.

Increasing & Decreasing property

A function f is called (strictly) **increasing** on an interval I if

$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

A function f is called (strictly) **decreasing** on an interval I if

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

Linear function

The graph of the function is a line:

$$y = f(x) = mx + b$$

m is the slope of the line and b is the y-intercept.

Polynomials

A function P is called a **polynomial** if

$$P(x) = \sum_{i=0}^n a_i x^i$$

a_i are **coefficients** and n is the **degree** of the polynomial if $a_n \neq 0$.

Quadratic function

A polynomial of degree 2 is of the form $P(x) = ax^2 + bx + c$ and is called a **quadratic function**.

Power function

A function of the form $f(x) = x^a$ is called a **power function**, where a is a constant.

Consider an arbitrary positive integer n :

- $a = n$:

$$y = x: \text{ line} \quad y = x^2: \text{ parabola}$$

- $a = \frac{1}{n}$: **root function**
- $a = -1$: **reciprocal function**

Ratio function

A **Ratio function** f is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

The domain consists of all values of x such that $Q(x) \neq 0$.

Trigonometric function

$$\sin(x + 2\pi) = \sin x \quad \cos(x + 2\pi) = \cos x \quad \tan(x + \pi) = \tan x$$

Exponential function

The **exponential functions** are the functions of the form $f(x) = a^x$, where the base a is a positive constant.

Logarithmic function

The **logarithmic functions** $f(x) = \log_a x$, where the base a is a positive constant and $a \neq 1$, are the inverse functions of the exponential functions.

Think about the question:

What condition does a meet when the exponential function increases in its domain? What about the logarithmic function?

- You can simply trace the dots for simple functions.
- You may also try to use software:
<https://www.geogebra.org>
<https://www.desmos.com>

Dirichlet Function

$$1_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Impulse Function

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

Step Function

$$H[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

Ramp Function

$$\begin{cases} x, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Hyperbolic Function

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \cosh(x) = \frac{e^x + e^{-x}}{2}, \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

Inverse trigonometric function

$$\arcsin(x), \arccos(x), \arctan(x)$$

$$\text{arsinh}(x) = \ln \left(x + \sqrt{x^2 + 1} \right)$$

$$\text{arcosh}(x) = \ln \left(x + \sqrt{x^2 - 1} \right)$$

$$\text{artanh}(x) = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$$

Vertical and Horizontal Shifts, suppose $c > 0$

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward

$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward

$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right

$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

Vertical and Horizontal Stretching and Reflecting, suppose $c > 1$

$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c

$y = (1/c)f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c

$y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c

$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c

$y = -f(x)$, reflect the graph of $y = f(x)$ about the x -axis

$y = f(-x)$, reflect the graph of $y = f(x)$ about the y -axis

Definition

Given two functions f and g , the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h , then g , and then f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

Skip it if you find the exercise quite simple.

Find the domain of these functions:

$$\textcircled{1} \quad h(x) = \frac{1}{\sqrt[4]{x^2 - 5x}}$$

$$\textcircled{2} \quad f(u) = \frac{u+1}{1 + \frac{1}{u+1}}$$

$$\textcircled{3} \quad F(p) = \sqrt{2 - \sqrt{p}}$$

How to solve this kind of questions

- The denominator in the fractional function cannot be zero
- The quantity in the even root formula cannot take a negative value, that is, it should be greater than or equal to zero
- The antilogarithm of the logarithm cannot be negative and zero, that is, it must take a positive value
- The domain of the function $y = \arcsin x, y = \arccos x$ is $-1 \leq x \leq 1$
- $y = \tan x, x \neq k\pi + \pi/2, y = \cot x, x \neq k\pi, k$ is integer

Prove or Disprove

- If f and g are both even functions, is $f + g$ even? If f and g are both odd functions, is $f + g$ odd? What if f is even and g is odd? Justify your answers.
- If f and g are both even functions, is the product fg even? If f and g are both odd functions, is fg odd? What if f is even and g is odd? Justify your answers.

Graph the functions step by step

$$(1) y = 1 - 2\sqrt{x + 3}$$

$$(2) y = |\cos \pi x|$$

Find the function (a) $f \circ g$, (b) $g \circ f$, (c) $f \circ f$, and (d) $g \circ g$ and their domains.

$$f(x) = \frac{x}{1+x}, \quad g(x) = \sin 2x$$

Find $f \circ g \circ h$:

- $f(x) = \tan x$
- $g(x) = \frac{x}{x-1}$
- $h(x) = \sqrt[3]{x}$

(It is unnecessary to find the domain of this composite function in this exercise!)

Composite Function

- ① If $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 7$, find a function f such that $f \circ g = h$.
(Think about what operations you would have to perform on the formula for g to end up with the formula for h .)
- ② If $f(x) = 3x + 5$ and $h(x) = 3x^2 + 3x + 2$, find a function g such that $f \circ g = h$.

- [1] Huang, Yucheng. VV156_RC1_updated.pdf. 2021.
- [2] Cai, Runze. Chapter01.pdf. 2021.
- [3] Zhou, Yishen.RC1. 2022.

① $x^2 - 5x > 0 \Rightarrow x \in (-\infty, 0) \cup (5, \infty)$

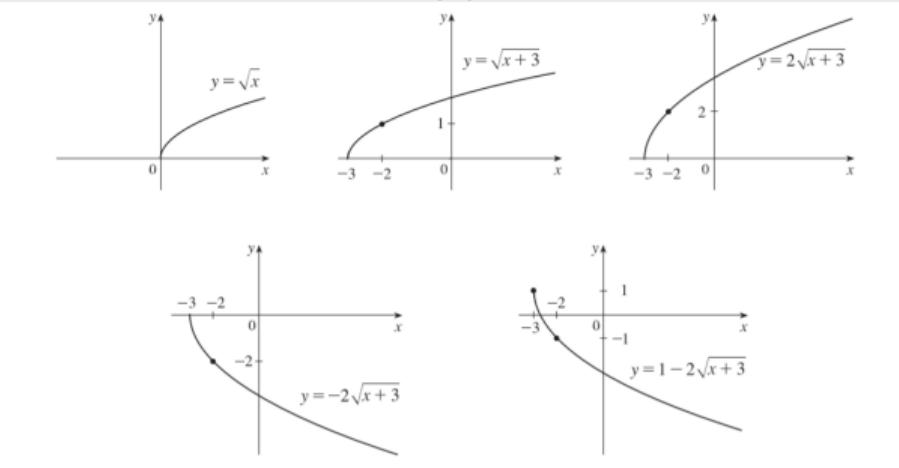
② $1 + \frac{1}{u+1} \neq 0, u + 1 \neq 0 \Rightarrow u \in (\infty, -2) \cup (-2, -1) \cup (-1, \infty)$

③ $p \geq 0, 2 - \sqrt{q} \geq 0 \Rightarrow p \in [0, 4]$

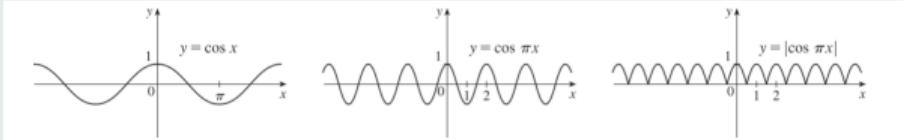
- Yes. Yes. Not necessarily.
- Yes. No.(fg is Even) fg is Odd.

Exercise Answer 3

(1)



(2)



(a) $f \circ g(x) = \frac{\sin 2x}{1 + \sin 2x}$. Domain: $x \in \{x | x \neq k\pi - \frac{\pi}{4}, k \in \mathbb{B}\}$

(b) $g \circ f(x) = \sin(\frac{2x}{1+x})$. Domain: $x \in (\infty, -1) \cup (-1, \infty)$

(c) $f \circ f(x) = \frac{x}{1+2x} (x \neq -1)$. Domain: $x \in (\infty, -1) \cup (-1, -\frac{1}{2}) \cup (-\frac{1}{2}, \infty)$

(d) $g \circ g(x) = \sin(2\sin(2x))$. Domain: $x \in \mathbb{R}$

$$\tan\left(\frac{\sqrt[3]{x}}{\sqrt[3]{x}-1}\right)$$

$$\textcircled{1} \quad x = \frac{g(x)-1}{2}.$$

Plug into $h(x)$, $h(x) = 4\left(\frac{g(x)-1}{2}\right)^2 + 4\left(\frac{g(x)-1}{2}\right) + 7 = g^2(x) + 6$.

Also, $h(x) = f(g(x))$.

Therefore, $f(x) = x^2 - 6$

$$\textcircled{2} \quad 3g(x) + 5 = h(x), \quad g(x) = x^2 + x - 1$$

3. Derivatives

Tangent Line

The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

Instantaneous rate of change

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

$$\text{instantaneous rate of change} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \rightarrow x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Try to define velocity in this way.

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{h}$$

$$v(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

Derivative

The **derivative of a function f at a number a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

if this limit exists.

The slope of the tangent line of a function is the corresponding derivative.

Notations

Newton:

$$\dot{y}$$

Leibniz:

$$\frac{dy}{dx}$$

Lagrange:

$$f'(x)$$

Jacobi: (Partial Derivatives)

$$\frac{\partial f}{\partial x}$$

Differentiable

A function f is differentiable at a if $f'(a)$ exists. It is differentiable on an open interval (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

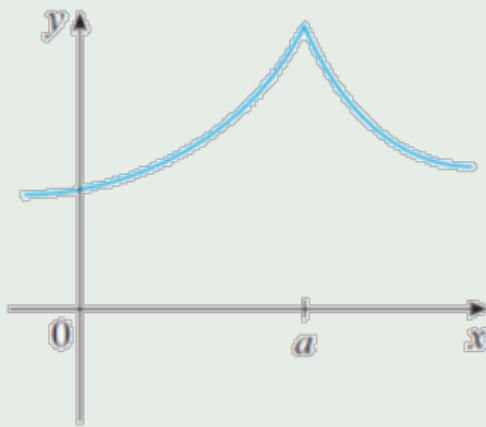
Differentiable and Continuity

If f is differentiable at a , then f is continuous at a .

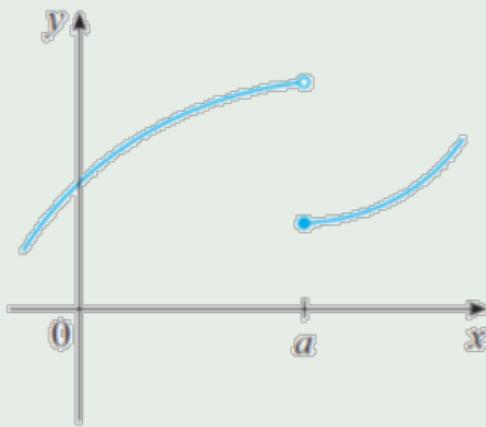
NOTE The converse of Theorem is false; that is, there are functions that are continuous but not differentiable. For instance, the function $f(x) = |x|$ is continuous at 0 because

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} |x| = 0 = f(0)$$

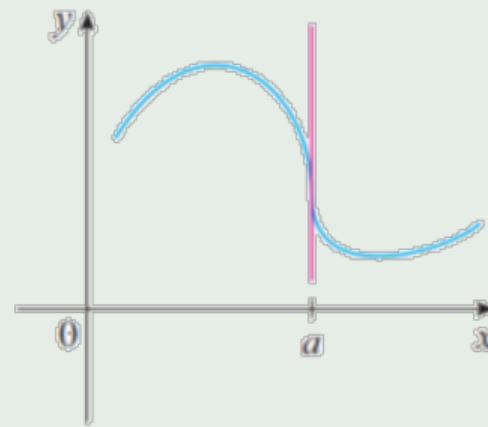
The function is not differentiable at these points:



(a) A corner



(b) A discontinuity



(c) A vertical tangent

$$(f')' = f'' \quad \text{second derivative}$$

$$(f'')' = f''' \quad \text{third derivative}$$

Example: still consider the position function

position-velocity-acceleration-jerk-snap...

$$x - \frac{dy}{dt} - \frac{d^2y}{dt^2} - \frac{d^3y}{dt^3} - \frac{d^4y}{dt^4} \dots$$

微积分初步

Saturday, June 27, 2020 8:04 PM

微分

初等函数微分表

1.	$y = c$	$dy = 0$	
2.	$y = x^\mu$	$dy = \mu x^{\mu-1} \cdot dx$	
	$y = \frac{1}{x}$	$dy = -\frac{dx}{x^2}$	
	$y = \sqrt{x}$	$dy = \frac{dx}{2\sqrt{x}}$	
3.	<u>$y = a^x$</u>	<u>$dy = a^x \cdot \ln a \cdot dx$</u>	
4.	$y = e^x$	$dy = e^x \cdot dx$	
	$y = \log_a x$	$dy = \frac{\log_e e \cdot dx}{x} = \frac{dx}{x \ln a}$	
	$y = \ln x$	$dy = \frac{dx}{x}$	
5.	$y = \sin x$	$dy = \cos x \cdot dx$	$y = \sin x + \frac{1}{\cos x} \quad dy = -\cos x \csc x dx - \sin x \sec x dx$
6.	$y = \cos x$	$dy = -\sin x \cdot dx$	$y = \cos x + \frac{1}{\sin x} \quad dy = \sin x \sec^2 x dx + \cos x \tan x dx$
7.	$y = \operatorname{tg} x$	$dy = \sec^2 x \cdot dx = \frac{dx}{\cos^2 x}$	
8.	$y = \operatorname{ctg} x$	$dy = -\csc^2 x dx = -\frac{dx}{\sin^2 x}$	$[\sinh(x)]' = \cosh(x)$ $[\cosh(x)]' = \sinh(x)$ $[\tanh(x)]' = \operatorname{sech}^2(x)$ $[\coth(x)]' = -\operatorname{csch}^2(x)$ $[\operatorname{sech}(x)]' = -\operatorname{sech}(x) \tanh(x)$ $[\operatorname{csch}(x)]' = -\operatorname{csch}(x) \coth(x)$ $[\operatorname{ar sinh}(x)]' = \frac{1}{\sqrt{x^2+1}}$ $[\operatorname{ar cosh}(x)]' = \frac{1}{\sqrt{x^2+1}} \quad (x > 1)$ $[\operatorname{ar tanh}(x)]' = \frac{1}{1-x^2} \quad (x < 1)$ $[\operatorname{ar coth}(x)]' = \frac{1}{1-x^2} \quad (x > 1)$
9.	$y = \arcsin x$	$dy = \frac{dx}{\sqrt{1-x^2}}$	
10.	$y = \arccos x$	$dy = -\frac{dx}{\sqrt{1-x^2}}$	
11.	$y = \arctg x$	$dy = \frac{dx}{1+x^2}$	
12.	$y = \operatorname{arcctg} x$	$dy = -\frac{dx}{1+x^2}$	

微分法则

I. $d(cu) = c \cdot du,$

II. $d(u \pm v) = du \pm dv,$

III. $d(uv) = u \cdot dv + v \cdot du,$

IV. $d\left(\frac{u}{v}\right) = \frac{v \cdot du - u \cdot dv}{v^2}.$

$$\frac{d}{dx}(C) = 0$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$(cf)' = cf'$$

$$(f + g)' = f' + g'$$

$$(f - g)' = f' - g'$$

$$(fg)' = fg' + gf'$$

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} (\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx} (\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} (\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx} (\cot^{-1} x) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx} (\log_a x) = \frac{1}{x \ln a}$$

$$\frac{d}{dx} (\ln x) = \frac{1}{x}$$

$$\frac{d}{dx} \ln |x| = \frac{1}{x}$$

Find an equation of the tangent line to the curve $y = 2x \sin x$ at the point $(\frac{\pi}{2}, \pi)$.

Find an equation of the tangent line to the curve $y = 2x \sin x$ at the point $(\frac{\pi}{2}, \pi)$.

Solution: $y' = 2(\sin x + x \cos x)$

$$k = y'\left(\frac{\pi}{2}\right) = 2$$

$$y - \pi = 2\left(x - \frac{\pi}{2}\right)$$

$$y - 2x = 0$$

Composite Function

- ① If $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 7$, find a function f such that $f \circ g = h$.
(Think about what operations you would have to perform on the formula for g to end up with the formula for h .)
- ② If $f(x) = 3x + 5$ and $h(x) = 3x^2 + 3x + 2$, find a function g such that $f \circ g = h$.

Differentiate:

$$\textcircled{1} \quad y = x^3 + \frac{7}{x^4} - \frac{2}{x} + 12$$

$$\textcircled{2} \quad y = \sin x \cos x$$

$$\textcircled{3} \quad y = \sqrt{x} \sin x$$

$$\textcircled{4} \quad y = 3e^x \cos x$$

$$\textcircled{5} \quad y = \frac{x \sin x}{1+x}$$

$$\textcircled{6} \quad y = \frac{1 - \sec x}{\tan x}$$

$$\textcircled{7} \quad y = x^2 \ln x \cos x$$

$$\textcircled{8} \quad y = \ln 3 + \frac{e^x}{x^2}$$

Solution:

① $3x^2 - \frac{28}{x^5} + \frac{2}{x^2}$

② $\cos 2x$

③ $\frac{1}{2}x^{-\frac{1}{2}}\sin x + x^{\frac{1}{2}}\cos x$

④ $3e^x(-\sin x + \cos x)$

⑤ $\frac{dy}{dx} = \frac{(x\sin x)'(1+x) - x\sin x}{(1+x)^2} = \frac{\sin x + (1+x)x\cos x}{(1+x)^2}$

⑥ $y = \frac{\cos x - 1}{\sin x} \rightarrow \frac{dy}{dx} = \frac{1}{1 + \cos x}$

⑦ $\frac{dy}{dx} = 2x\ln x \cos x + x \cos x - x^2 \ln x \sin x$

⑧ $(-2x^{-3} + x^{-2})e^x$

Let $y = \log_{\varphi(x)} f(x)$ ($\varphi(x) > 0$, $\varphi(x) \neq 1$, $f(x) > 0$). Suppose that both $\varphi(x)$ and $f(x)$ are differentiable. Calculate $\frac{dy}{dx}$.

Solution:

$$y = \frac{\ln f(x)}{\ln \varphi(x)}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{1}{f(x)} f'(x) \ln \varphi(x) - \frac{1}{\varphi(x)} \varphi'(x) \ln f(x)}{[\ln \varphi(x)]^2} \\ &= \frac{f'(x)}{f(x) \ln \varphi(x)} - \frac{\varphi'(x) \ln f(x)}{\varphi(x) [\ln \varphi(x)]^2}\end{aligned}$$

Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

The Power Rule Combined with the Chain Rule

If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$$

Alternatively,

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

Find the derivative of the function:

$$\textcircled{1} \quad y = (4x - x^2)^{100}$$

$$\textcircled{2} \quad y = 5^{-\frac{1}{x}}$$

$$\textcircled{3} \quad y = e^{-2x} \cos 4x$$

$$\textcircled{4} \quad y = \left(\frac{x^2 + 1}{x^2 - 1} \right)^3$$

$$\textcircled{5} \quad y = \frac{\arcsin x}{\arccos x}$$

$$\textcircled{6} \quad y = [\sin(e^{(\sin x)^2})]^2$$

$$\textcircled{7} \quad y = \arcsin \sqrt{\frac{1-x}{1+x}}$$

$$\textcircled{8} \quad y = n^{n^x} + x^{n^n} + n^{x^n} \quad (n > 0, n \neq 1)$$

$$\textcircled{1} \quad 200(2-x)(4x-x^2)^{99}$$

$$\textcircled{2} \quad \ln 5 \cdot 5^{-\frac{1}{x}} \cdot x^{-2}$$

$$\textcircled{3} \quad -2e^{-2x}(\cos 4x + 2\sin 4x)$$

$$\textcircled{4} \quad y' = 3\left(\frac{x^2+1}{x^2-1}\right)^2\left(-\frac{2}{(x^2-1)^2}\right) \cdot 2x = -\frac{12x(x^2+1)^2}{(x^2-1)^3}$$

$$\textcircled{5} \quad y' = \frac{(\arcsinx)' \arccos x - (\arccos x)' \arcsinx}{(\arccos x)^2} = \frac{\arccos x + \arcsinx}{\sqrt{1-x^2}(\arccos x)^2}$$

$$\textcircled{6} \quad y' = 2[\sin(e^{(\sin x)^2})] \cdot \cos(e^{(\sin x)^2}) \cdot e^{(\sin x)^2} \cdot 2\sin x \cos x = \sin 2x \cdot \sin(2e^{(\sin x)^2}) \cdot e^{(\sin x)^2}$$

$$\textcircled{7} \quad y' = \left(1 - \frac{1-x}{1+x}\right)^{-\frac{1}{2}} \cdot \frac{1}{2} \left(\frac{1-x}{1+x}\right)^{-\frac{1}{2}} \left[-\frac{2}{(x+1)^2}\right] = -\frac{1}{\sqrt{2x(1-x)} \cdot (x+1)}$$

$$\textcircled{8} \quad y' = n^{n^x+x}(\ln n)^2 + n^n x^{n^n-1} + n^{x^n+1} x^{(n-1)} \ln n$$

Find the second derivative of the following function:

(**Warning:** Don't forget to double check your first derivative!)

① $y = \tan x$

② $y = \frac{1}{x^3 + 1}$

③ $y = x \cos x$

$$\textcircled{1} \quad y' = \sec^2 x$$

$$y'' = 2\sec x \cdot (\tan x \cdot \sec x) = 2\tan x \cdot \sec^2 x$$

$$\textcircled{2} \quad y' = -\frac{1}{(x^3 + 1)^2} \cdot 3x^2$$

$$y'' = -3 \cdot \frac{2x(x^3 + 1)^2 - 2(x^3 + 1)3x^4}{(x^3 + 1)^4} = \frac{6x(2x^3 - 1)}{(x^3 + 1)^3}$$

$$\textcircled{3} \quad y' = \cos x - x \sin x$$

$$y'' = -2\sin x - x\cos x$$

Answer the following three questions based on $\frac{dy}{dx} = y'$:

- ① Express $\frac{dx}{dy}$ with y'
- ② Express $\frac{d^2x}{dy^2}$ with y' and y''
- ③ Express $\frac{d^3x}{dy^3}$ with y' , y'' and y'''

Solution:

$$\textcircled{1} \quad \frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{y'}$$

$$\textcircled{2} \quad \frac{d^2x}{dy^2} = \frac{d}{dy} \cdot \frac{dx}{dy} = \frac{d}{dx} \frac{1}{y'} \frac{dx}{dy} = -\frac{y''}{(y')^3}$$

$$\textcircled{3} \quad \frac{d^3x}{dy^3} = \frac{d}{dy} \frac{d^2x}{dy^2} = \frac{d}{dx} \frac{d^2x}{dy^2} \frac{dx}{dy} = -\frac{y'''(y')^3 - 3(y')^2(y'')^2}{(y')^6} \cdot \frac{1}{y'} = \frac{3(y'')^2 - y'(y''')^2}{(y')^5}$$

Find y' if $\sin(x + y) = y^2 \cos x$

Differentiating implicitly with respect to x and remembering that y is a function of x , we get

$$\cos(x + y) \cdot (1 + y') = y^2(-\sin x) + (\cos x)(2yy')$$

(Note that we have used the Chain Rule on the left side and the Product Rule and Chain Rule on the right side.) If we collect the terms that involve y' , we get

$$\cos(x + y) + y^2 \sin x = (2y \cos x)y' - \cos(x + y) \cdot y'$$

So

$$y' = \frac{y^2 \sin x + \cos(x + y)}{2y \cos x - \cos(x + y)}$$

Calculate the derivative of the following implicit function:

① $y^2 - 2xy + 9 = 0$

② $xy = e^{xy}$

$$\textcircled{1} \quad 2yy' - 2y - 2xy' = 0, y' = \frac{y}{y-x}$$

\textcircled{2} The derivative does not exist!

Use **logarithmic differentiation** to calculate the derivative of the following implicit function:

$$\textcircled{1} \quad y = \left(\frac{x}{x+1}\right)^x$$

$$\textcircled{2} \quad y = \sqrt[5]{\frac{x-5}{\sqrt[5]{x^2 + 2}}}$$

Solution:

$$\textcircled{1} \quad \ln y = x[\ln x - \ln(x+1)]$$

$$\frac{1}{y}y' = [\ln x - \ln(x+1)] + x\left[\frac{1}{x} - \frac{1}{x+1}\right]$$

$$y' = \left(\frac{x}{x+1}\right)^x \left[\ln\left|\frac{x}{x+1}\right| + \frac{1}{x+1}\right]$$

$$\textcircled{2} \quad \ln y = \frac{1}{5}\ln|x-5| - \frac{1}{25}\ln(x^2+2)$$

$$y' \frac{1}{y} = \frac{1}{5(x-5)} - \frac{2x}{25(x^2+2)}$$

$$y' = y\left[\frac{1}{5(x-5)} - \frac{2x}{25(x^2+2)}\right]$$

The *Bessel function* of order 0, $y = J(x)$, satisfies the differential equation $xy'' + y' + xy = 0$ for all values of x and its value at 0 is $J(0) = 1$.

- (a) Find $J'(0)$.
- (b) Use implicit differentiation to find $J''(0)$.

Solution:

① Take $x=0$, $0 + J'(0) + 0 = 0 \rightarrow J'(0) = 0$

② $xy''' + 2y'' + y + xy' = 0$

At $x = 0$, $2y'' + 1 = 0$, $y'' = -\frac{1}{2}$

10.

Find the derivative of the function $f(x) = (x \sin^2(x))^{\frac{1}{3}}$.

11.

The equation of the tangent to $f(x)$ at $x = 1$ is $y = -2(x - 3)$, and knowing that $g(x) = f^{-1} + 3$, find the equation of the tangent to $g(x)$ at $x = 3$.

12.

Find the slope of the tangent to the curve $xy = \arctan(3y)$ when $y = \frac{1}{3}$.

Definition

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the **linear approximation** or **tangent line approximation** of f at a . The linear function whose graph is this tangent line, that is,

$$L(x) = f(a) + f'(a)(x - a)$$

is called the linearization of f at a .

Use linear approximation to estimate $2.0006^{1.9998}$

Solution:

$$f(x) = 2^{x+2}$$

$$f'(x) = 2^{x+2} \ln 2$$

$$2^{1.9998} = f(0) + f'(0) \times (-0.0002) = 2^2 + 4\ln 2 \times (-0.0002) = 3.9994$$

$$g(x) = (2+x)^{1.9998}$$

$$g'(x) = 1.9998(2+x)^{0.9998}$$

$$2.0006^{1.9998} = g(0) + g'(0) \times 0.0006 = 2^{1.9998} + 1.9998 \times 2^{0.9998} \times 0.0006 = 4.0018$$

Absolute Maximum and Absolute Minimum

Let c be a number in the domain D of a function f . Then $f(c)$ is the

- **absolute maximum** value of f on D if $f(c) \geq f(x)$ for all x in D .
- **absolute minimum** value of f on D if $f(c) \leq f(x)$ for all x in D .

Local Maximum and Local Minimum

The number $f(c)$ is a

- **local maximum** value of f if $f(c) \geq f(x)$ when x is near c .
- **local minimum** value of f if $f(c) \leq f(x)$ when x is near c .

Critical number

A **critical number** of a function f is a number c in the domain such that either $f'(c) = 0$ or $f'(c)$ does not exist, or the end points of a region if it is bounded.

local maximum/minimum → critical number

The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Fermat's Theorem

If f has a local maximum or minimum at c , and if $f'(c)$ exists, then $f'(c) = 0$.

Rolle's Theorem

Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b) .
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

Lagrange Mean Value Theorem

Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b) .

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently, $f(b) - f(a) = f'(c)(b - a)$

Cauchy Mean Value Theorem (Extended Mean Value Theorem)

Let f, g be two functions that satisfy the following hypotheses:

1. f, g is continuous on the closed interval $[a, b]$.
2. f, g is differentiable on the open interval (a, b) .
3. $x \in (a, b), g'(x) \neq 0$

Then there is a number c in (a, b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

or, equivalently, $(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$

Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

① $f(x) = x^3 - 3x + 2, \quad [-2, 2]$

② $f(x) = \ln x, \quad [1, 4]$

Solutions:

① $f(-2) = 0, f(2) = 4$

$$k = \frac{f(2) - f(-2)}{4} = 1$$

$$f'(x) = 3x^2 - 3, 3c^2 - 3 = 1$$

$$c = \frac{2\sqrt{3}}{3}$$

② $k = \frac{f(4) - f(1)}{3} = \frac{2\ln 2}{3}$

$$f'(x) = \frac{1}{x}$$

$$d = \frac{3}{2\ln 2}$$

Increasing/Decreasing Test

If $f'(x) > 0$ on an interval, then f is increasing on that interval.

If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

The First Derivative Test

Suppose that c is a critical number of a continuous function f .

If f' changes from positive to negative at c , f has a local maximum at c .

If f' changes from negative to positive at c , f has a local minimum at c .

If f' does not change sign at c , f has no local maximum or minimum at c .

Concave upward/downward

If the graph of f lies above all of its tangents on an interval I , then it is called **concave upward** on I . If the graph of f lies below all of its tangents on an interval I , then it is called **concave downward** on I .

Concavity Test

If $f''(x) > 0$ for all x in I , then the graph of f is concave upward on I .

If $f''(x) < 0$ for all x in I , then the graph of f is concave downward on I .

Inflection point

A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P .

The Second Derivative Test

Suppose f'' is continuous near c .

If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c .

If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c .

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4. Limits

"Rough" Definition of a Limit

Suppose $f(x)$ is defined when x is near the number a . (This means that f is defined on some open interval that contains a , **except possibly** at a itself.)

Then we write

$$\lim_{x \rightarrow a} f(x) = L$$

and say

"the limit if $f(x)$, as x approaches a , equals L "

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be **sufficiently close to** a (on either side of a) but **not equal to** a .

Considering a function called *Heaviside Function*

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \quad (1)$$

Does $\lim_{t \rightarrow 0} H(t)$ exists?

We write

$$\lim_{x \rightarrow a^-} f(x) = L$$

and say the left-hand limit of $f(x)$ as x approaches a is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be **sufficiently close to a** and x less than a .

When calculating $\lim_{x \rightarrow a^-} f(x)$, we consider only $x < a$.

Similarly, we can get the right-hand limit of $f(x)$ as x approaches a .

When does $\lim_{x \rightarrow a} f(x)$ exists?

$$\lim_{x \rightarrow a} f(x) = L$$

if and only if

$$\lim_{x \rightarrow a^-} f(x) = L \text{ and } \lim_{x \rightarrow a^+} f(x) = L$$

Can we directly regard L as $f(a)$?

Let f be a function defined on both sides of a , **except possibly** at a itself. Then

$$\lim_{x \rightarrow a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x **sufficiently close to** a , but **not equal to** a .

Let f be a function defined on both sides of a , **except possibly** at a itself. Then

$$\lim_{x \rightarrow a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x **sufficiently close to** a , but **not equal to** a .

Warning:

$\lim_{x \rightarrow a} f(x) = (-)\infty$ does not mean that we are regarding $(-)\infty$ as a number. Nor does it mean that the limit exists!

- ① Let f be a function defined on some interval (a, ∞) . Then

$$\lim_{x \rightarrow \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

- ② Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \rightarrow -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.

$$\textcircled{1} \quad \lim_{x \rightarrow \infty} f(x) = \infty$$

$$\textcircled{2} \quad \lim_{x \rightarrow \infty} f(x) = -\infty$$

$$\textcircled{3} \quad \lim_{x \rightarrow -\infty} f(x) = \infty$$

$$\textcircled{4} \quad \lim_{x \rightarrow -\infty} f(x) = -\infty$$

Five basic laws:

Suppose that c is a constant and the limits

$$\lim_{x \rightarrow a} f(x) \text{ and } \lim_{x \rightarrow a} g(x)$$

exists. Then

$$\textcircled{1} \quad \lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

$$\textcircled{2} \quad \lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$$

$$\textcircled{3} \quad \lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x)$$

$$\textcircled{4} \quad \lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$$

$$\textcircled{5} \quad \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \quad (\text{if } \lim_{x \rightarrow a} g(x) \neq 0)$$

Another six laws:

$$\text{① } \lim_{x \rightarrow a} [f(x)]^n = [\lim_{x \rightarrow a} f(x)]^n \text{ (where } n \text{ is a positive integer)}$$

$$\text{② } \lim_{x \rightarrow a} c = c$$

$$\text{③ } \lim_{x \rightarrow a} x = a$$

$$\text{④ } \lim_{x \rightarrow a} x^n = a^n \text{ (where } n \text{ is a positive integer)}$$

$$\text{⑤ } \lim_{x \rightarrow a} \sqrt[n]{x} = \sqrt[n]{a} \text{ (where } n \text{ is a positive integer)}$$

if n is even, we assume that $a > 0$

$$\text{⑥ } \lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)} \text{ (where } n \text{ is a positive integer)}$$

if n is even, we assume that $\lim_{x \rightarrow a} f(x) > 0$

For composite functions, we have

Law for Composite Functions

$$\lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x))$$

for all the limits and functions exist.

E.g. Solve $\lim_{x \rightarrow \infty} \cos \frac{1}{x}$.

Warning:

The Limit Laws can't be applied to infinite limits because $(-) \infty$ is not a number!

即极限进行运算时，必须保证运算前的极限存在（不是未定式/无穷），而且极限为有限个

Two additional properties of limits:

- ① if $f(x) \leq g(x)$ when x is near a (**except possibly** at a) and the limits of f and g both exists as x approaches a , then

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x)$$

- ② (**The Squeeze Theorem**) if $f(x) \leq g(x) \leq h(x)$ when x is near a (**except possibly** at a) and

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

then

$$\lim_{x \rightarrow a} g(x) = L$$

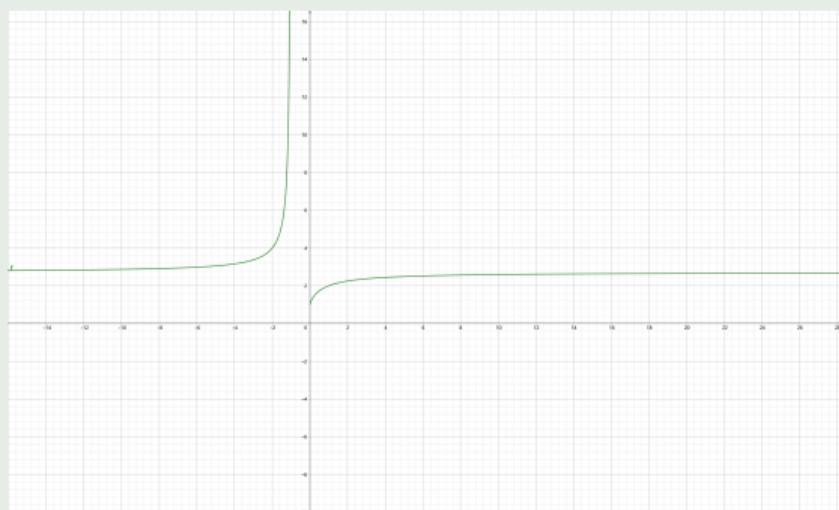
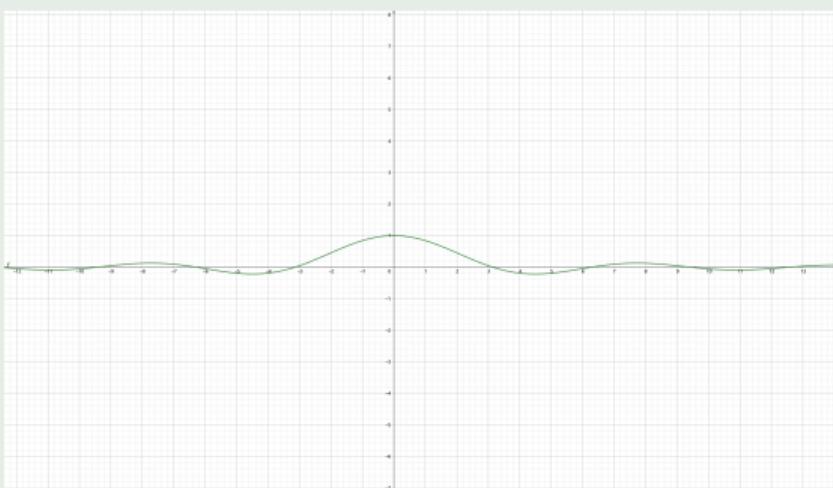
Be sure to keep these two limits in mind!

① $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

(How to prove it? Considering $\sin x$, x and $\tan x$. Then, use the squeeze theorem.)

② $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e = 2.718281828459045 \dots$

(Proof: <https://www.zhihu.com/question/277272238>)



$$\textcircled{1} \quad \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

相关变形:

- $\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0$

- $\lim_{f(x) \rightarrow 0} \frac{\sin f(x)}{f(x)} = 1$

$$\textcircled{1} \quad \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$$

相关变形：

- $\lim_{x \rightarrow 0^+} \left(1 + \frac{1}{x}\right)^x = 1$

- $\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e$

If $u \rightarrow 1, v \rightarrow \infty$, then we have

- $\lim_{x \rightarrow +\infty} (1+x)^{\frac{1}{x}} = 1$

$$\lim u^v = "1^\infty" = e^{\lim v(u-1)}$$

- $\lim_{x \rightarrow 0^-} \left(1 - \frac{1}{x}\right)^x = 1$

也就是幂指函数（后面会讲）

- $\lim_{x \rightarrow \infty} \left(1 - \frac{1}{x}\right)^x = \frac{1}{e}$

常见极限计算方法:

- ① 四则运算与函数性质
- ② 导数定义
- ③ 两个重要极限
- ④ 洛必达
- ⑤ 等价无穷小
- ⑥ 泰勒展开 (了解即可)

四则运算与函数性质

有理式函数极限：

对有理式 $F(x) = \frac{p(x)}{q(x)} = \frac{a_0 + a_1x_1 + \cdots + a_mx^m}{b_0 + b_1x_1 + \cdots + b_nx^n}$, 有

$$\lim_{x \rightarrow \infty} F(x) = \begin{cases} \frac{a_m}{b_n} & m = n \\ 0 & m < n \\ \infty & m > n \end{cases}$$

1. 计算下列极限：

$$① \lim_{x \rightarrow 0} \frac{4x^3 - 2x^2 + x}{2x + 3x^2}$$

$$② \lim_{x \rightarrow \infty} \frac{3x^3 + 4x^2 + 2}{7x^3 + 5x^2 - 3}$$

$$③ \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right) \left(2 - \frac{1}{x^2}\right)$$

四则运算与函数性质

2. 设 $\lim_{x \rightarrow -1} \frac{x^3 - ax^2 - x + 4}{x + 1}$ 存在极限值且为 m , 试求 a 和 m 的值。

有根号：分子有理化或根式整体换元

3. 计算下列极限：

$$\textcircled{1} \quad \lim_{x \rightarrow 0} \frac{-1 + \sqrt[n]{x+1}}{x}$$

$$\textcircled{2} \quad \lim_{x \rightarrow -4} \frac{\sqrt{9+x^2} - 5}{x+4}$$

$$\textcircled{3} \quad \lim_{x \rightarrow -\infty} (\sqrt{x^2 + 2x} + x)$$

$$\textcircled{4} \quad \lim_{x \rightarrow 0} \frac{\sqrt{1 + \tan^2 x} - \sqrt{1 + \sin^2 x}}{(5^x - 1) \arctan^3 x}$$

导数定义

计算下列极限：

$$\textcircled{1} \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h}$$

$$(\text{What if } \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h}?)$$

两个重要极限

计算下列极限：

① $\lim_{x \rightarrow 0} \frac{\sin \omega x}{x}$

② $\lim_{x \rightarrow \infty} \left(1 - \frac{1}{x}\right)^{kx} \quad (k \text{ is a positive integer})$

③ $\lim_{x \rightarrow \frac{\pi}{2}} (\sin x)^{\tan x}$

④ $\lim_{x \rightarrow 0^+} (\cos \sqrt{x})^{\frac{\pi}{x}}$

⑤ $\lim_{x \rightarrow \infty} \left(\frac{x+3}{x+6}\right)^{\frac{x-1}{2}}$

L'Hospital's Rule

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \rightarrow a} f(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = 0$$

or that

$$\lim_{x \rightarrow a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞ .) Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Warning: Always judge whether l'Hôpital's rule can be applied before you use it, and don't neglect those basic methods of finding the limit.

Evaluate the following limits:

$$\textcircled{1} \quad \lim_{x \rightarrow 0^+} x^{\sin x}$$

$$\textcircled{2} \quad \lim_{x \rightarrow 1} \frac{\ln x}{x - 1}$$

$$\textcircled{3} \quad \lim_{x \rightarrow \infty} x^3 e^{-x^2}$$

$$\textcircled{4} \quad \lim_{x \rightarrow \infty} [x - x^2 \ln \frac{x+1}{x}]$$



$$\textcircled{5} \quad \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}}$$

Tip: You're highly recommended to remember this part!

When $x \rightarrow 0$

$$a^x - 1 \sim x \ln a$$

$$\arcsin(a)x \sim \sin(a)x \sim (a)x$$

$$\arctan(a)x \sim \tan(a)x \sim (a)x$$

$$\ln(1 + x) \sim x$$

$$\sqrt{1 + x} - \sqrt{1 - x} \sim x$$

$$(1 + ax)^b - 1 \sim abx$$

$$\sqrt[b]{1 + ax} - 1 \sim \frac{a}{b}x$$

$$1 - \cos x \sim \frac{x^2}{2}$$

$$x - \ln(1 + x) \sim \frac{x^2}{2}$$

When $x \rightarrow 0$

$$\tan x - \sin x \sim \frac{x^3}{2}$$

$$\tan x - x \sim \frac{x^3}{3}$$

$$x - \arctan x \sim \frac{x^3}{3}$$

$$x - \sin x \sim \frac{x^3}{6}$$

$$\arcsin x - x \sim \frac{x^3}{6}$$

Example: Solve the limit

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\ln(1 + 4x)}{\sin(3x)} &= \lim_{x \rightarrow 0} \frac{\ln(1 + 4x)}{\sin(3x)} \lim_{x \rightarrow 0} \frac{4x}{\ln(1 + 4x)} \lim_{x \rightarrow 0} \frac{\sin(3x)}{3x} \\ &= \lim_{x \rightarrow 0} \frac{4x}{3x} = 4/3\end{aligned}$$

For more exercise regarding to equivalent infinitesimal, please refer to Worksheet 1.

注意：加减法中不可使用部分的等价无穷小代换！只有乘除形式的才可以代换

注意：加减法中不可使用部分的等价无穷小代换！

Example: Solve the limit

- $\lim_{x \rightarrow 0} \frac{x - \sin x \cos x}{x^3}$

- $\lim_{x \rightarrow \infty} \frac{(1 + \frac{1}{x})^{x^2}}{e^x}$

Definition

Taylor expansion around $x = x_0$:

$$f(x) = f(x_0) + \sum_{i=1}^n \frac{f^{(i)}(x_0)}{i!} (x - x_0)^i + R_n, \text{ where } R_n = o[(x - x_0)^n]$$

It simulates a function around a point with a polynomial function.

Taylor expansion of some polynomials when x is around 0:

$$\textcircled{1} \quad e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$$

$$\textcircled{2} \quad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$$

$$\textcircled{3} \quad \sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)$$

$$\textcircled{4} \quad \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)$$

$$\textcircled{5} \quad \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + o(x^5)$$

Tip

$o(x^n)$ means the order of the polynomial is larger than n ;

$O(x^n)$ means the order of the polynomial is larger than or equal to n .

The transformation of Taylor Expansion:

Example

The Taylor expansion of e^{x^2} around $x=0$:

$$e^{x^2} = 1 + x^2 + \frac{(x^2)^2}{2} + \frac{(x^2)^3}{6} + o((x^2)^3) = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + o(x^6)$$

Some methods to calculate the limits:

- ① Use those limit laws directly
- ② Exchange the order of functions and limit symbols based on the continuity of composite function. (Will be mentioned later)
- ③ Do factorization, denominator rationalization or numerator rationalization.
- ④ If a factor approaching zero is find in the denominator, try to eliminate it.
- ⑤ Translate the formula into the form of "two important limits"
- ⑥ **The method to solve those formulas having the form of $u(x)^{v(x)}$ will be discussed at a deeper level after the differentiation and l'Hôpital's rule are taught.**

$$\textcircled{1} \quad \lim_{x \rightarrow 0} \frac{4x^3 - 2x^2 + x}{2x + 3x^2} = \lim_{x \rightarrow 0} \frac{4x^2 - 2x + 1}{2 + 3x} = 0.5$$

$$\textcircled{2} \quad \lim_{x \rightarrow 0} \frac{\frac{1}{3+x} - \frac{1}{3}}{x} = \lim_{x \rightarrow 0} \frac{3 - 3 - x}{3x(3+x)} = \lim_{x \rightarrow 0} \frac{-1}{3(3+x)} = -\frac{1}{9}$$

$$\textcircled{3} \quad \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right) \left(2 - \frac{1}{x^2}\right) = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right) \cdot \lim_{x \rightarrow \infty} \left(2 - \frac{1}{x^2}\right) = 1 \cdot 2 = 2$$

$$\textcircled{4} \quad \text{Let } t^n - 1 := x, \lim_{x \rightarrow 0} \frac{-1 + \sqrt[n]{x+1}}{x} = \lim_{t \rightarrow 1} \frac{t-1}{t^n - 1} = \lim_{t \rightarrow 1} \frac{1}{1+t+t^2+\cdots+t^{n-1}} = \frac{1}{n}$$

$$\textcircled{1} \quad \lim_{x \rightarrow -4} \frac{\sqrt{9+x^2} - 5}{x+4} = \lim_{x \rightarrow -4} \frac{x-4}{\sqrt{9+x^2} + 5} = -\frac{4}{5}$$

$$\textcircled{2} \quad \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} = \lim_{h \rightarrow 0} nx^{n-1} + h \cdot (\dots) = nx^{n-1}$$

$$\textcircled{3} \quad \lim_{x \rightarrow 0} \frac{\sin \omega x}{x} = \omega \cdot \lim_{x \rightarrow 0} \frac{\sin \omega x}{\omega x} = \omega$$

$$\textcircled{4} \quad \lim_{x \rightarrow \infty} \left(1 - \frac{1}{x}\right)^{kx} = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{-x}\right)^{(-x)(-k)} = e^{-k}$$

$$\textcircled{5} \quad \lim_{\substack{x \rightarrow \frac{\pi}{2}}} (\sin x)^{\tan x} = \lim_{\substack{x \rightarrow \frac{\pi}{2}}} (1 + (\sin x - 1))^{\frac{1}{\sin x - 1} \cdot (\sin x - 1) \tan x} = e^{\lim_{x \rightarrow \frac{\pi}{2}} \frac{(\sin x - 1) \tan x}{2}} =$$

$$e^{\lim_{x \rightarrow \frac{\pi}{2}} \frac{(\sin x - 1) \sin x}{\sqrt{1 - \sin^2 x}}} = e^{-\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sqrt{1 - \sin x} \sin x}{\sqrt{1 + \sin x}}} = e^0 = 1$$

Solution:

① 1

② 1

③ $\lim_{x \rightarrow \infty} \frac{x^3}{e^{x^2}} = \lim_{x \rightarrow \infty} \frac{3x^2}{2xe^{x^2}} = \lim_{x \rightarrow \infty} \frac{6x}{(2 + 4x^2)e^{x^2}} = \lim_{x \rightarrow \infty} \frac{6}{12x + 8x^2} = 0$

④ $u = \frac{1}{x}$

$$\lim_{u \rightarrow 0} \frac{u - \ln(u+1)}{u^2} = \lim_{u \rightarrow 0} \frac{1 - \frac{1}{u+1}}{2u} = \lim_{u \rightarrow 0} \frac{\frac{1}{(u+1)^2}}{2} = \frac{1}{2}$$

⑤ $\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 1}}{x} = 1$

- [1] Huang, Yucheng. VV156_RC2.pdf. 2021.
- [2] Cai, Runze. Chapter01.pdf. 2021.
- [3] Department of mathematics, Tongji University. Advanced Mathematics (7th Edition). 2014.
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- [6] Zhou, Yishen. RC2. 2022.

5. Continuity

A function f is continuous at a number a if

$$\lim_{x \rightarrow a} = f(a)$$

This definition actually implicitly requires three things:

① $f(a)$ is defined

② $\lim_{x \rightarrow a} f(x)$ exists

③ $\lim_{x \rightarrow a} f(x) = f(a)$

A function f is continuous from the right at a number a if

$$\lim_{x \rightarrow a^+} = f(a)$$

A function f is continuous from the left at a number a if

$$\lim_{x \rightarrow a^-} = f(a)$$

A function is continuous on an interval if it is continuous at **every number in the interval**. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left*.)

If f and g are continuous at a and c is a constant, then the following functions are also continuous at a :

- ① $f + g$
- ② $f - g$
- ③ cf
- ④ fg
- ⑤ $\frac{f}{g}$ (if $g(a) \neq 0$)

The following types of functions are continuous at every number in their domain(s):

- ① polynomials
- ② rational functions
- ③ root functions
- ④ (inverse) trigonometric functions
- ⑤ exponential functions
- ⑥ logarithmic functions

If f is continuous at b and $\lim_{x \rightarrow a} g(x) = b$, then $\lim_{x \rightarrow a} f(g(x)) = f(b)$.

In other words,

$$\lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x))$$

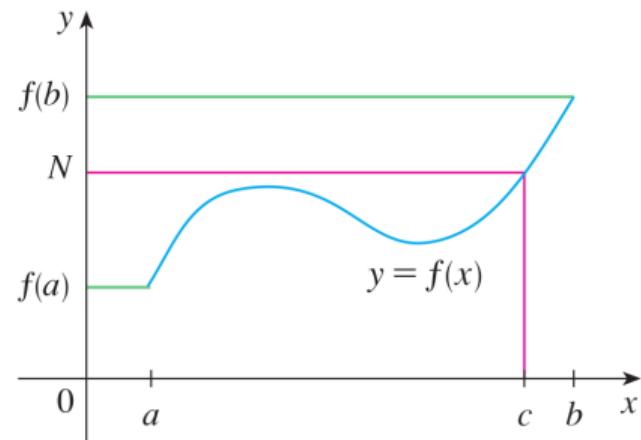
If g is continuous at a and f is continuous at $g(a)$, then $f(g(x))$ is continuous at a .

"A continuous function of a continuous function is a continuous function."

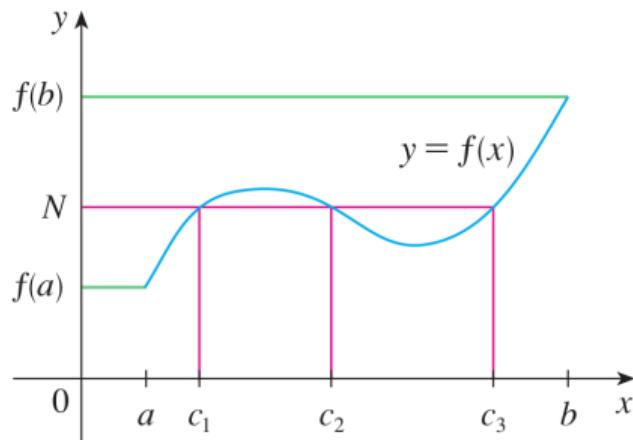
Theorem 5-The Intermediate Value Theorem

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

Note that the value N can be taken on once or more than once.



(a)



(b)

- ① removable discontinuity
- ② infinite discontinuity
- ③ jump discontinuity

Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

$$\sqrt[3]{x} = 1 - x, \quad (0, 1)$$

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$$\sqrt[3]{x} = 1 - x, \quad (0, 1)$$

Solution:

$\sqrt[3]{x} = 1 - x$ has a root on $(0, 1)$ equals to $f(x) = \sqrt[3]{x} + x - 1 = 0$ has a solution on $(0, 1)$. Since $f(0) = -1$, $f(1) = 1$, and 0 is between -1 and 1, there must be a point c such that $f(c) = 0$.

Let $f(x) = \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$. What kind of discontinuity is $x = 0$?

Let $f(x) = \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$. What kind of discontinuity is $x = 0$?

Solution:

$$\lim_{x \rightarrow 0^-} f(x) = \frac{0 - 1}{0 + 1} = -1$$

$$\lim_{x \rightarrow 0^+} f(x) = \frac{\infty - 1}{\infty + 1} = 1$$

Jump discontinuity.

Find the values of a and b that make f continuous everywhere.

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2} & x < 2 \\ ax^2 - bx + 3 & 2 \leq x < 3 \\ 2x - a - b & x \leq 3 \end{cases} \quad (2)$$

Find the values of a and b that make f continuous everywhere.

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2} & x < 2 \\ ax^2 - bx + 3 & 2 \leq x < 3 \\ 2x - a - b & x \leq 3 \end{cases} \quad (3)$$

Solution:

$$\lim_{x \rightarrow 2^-} f(x) = \frac{(x - 2)(x + 2)}{(x - 2)} = 4$$

$$f(2) = 4a - 2b + 3$$

$$\lim_{x \rightarrow 3^-} f(x) = 9a - 3b + 3$$

$$f(3) = 6 - a - b$$

Find a and b that make $f(x) = \lim_{n \rightarrow +\infty} \frac{x^{2n-1} + ax^2 + bx}{x^{2n} + 1}$ continuous on $(-\infty, \infty)$.

Find a and b that make $f(x) = \lim_{n \rightarrow +\infty} \frac{x^{2n-1} + ax^2 + bx}{x^{2n} + 1}$ continuous on $(-\infty, \infty)$.

Solution:

$$f(x) = \begin{cases} ax^2 + bx & : -1 < x < 1 \\ \frac{a - b - 1}{2} & : x = -1 \\ \frac{a + b + 1}{2} & : x = 1 \\ \frac{1}{x} & : x > 1 \text{ or } x < -1 \end{cases}$$

$$\text{At } x = 1, \begin{cases} \frac{a + b + 1}{2} = a + b \\ a + b = 1 \end{cases}$$

$$\text{At } x = -1, a - b = -1$$

Thus, we have $a = 0, b = 1$

There are two functions:

$f(x)$ is continuous on $(-\infty, \infty)$, and $f(x) \neq 0$.

$\varphi(x)$ is defined on $(-\infty, \infty)$, but $\varphi(x)$ has discontinuity.

Judge whether the following four statements are correct:

- ① $\varphi[f(x)]$ must have discontinuity.
- ② $[\varphi(x)]^2$ must have discontinuity.
- ③ Whether $f[\varphi(x)]$ has discontinuity is uncertain.
- ④ $\frac{\varphi(x)}{f(x)}$ must have discontinuity.

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- ④ $\frac{\varphi(x)}{f(x)}$ must have discontinuity.

Solution:

- ① No
- ② No
- ③ Yes
- ④ Yes

$$\textcircled{1} \quad \lim_{x \rightarrow a} c = c$$

$$\textcircled{2} \quad \lim_{x \rightarrow a} x^n = a^n$$

$$\textcircled{3} \quad \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\textcircled{4} \quad \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$$

$$\textcircled{5} \quad \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

$$\textcircled{6} \quad \lim_{x \rightarrow 0} \frac{\arcsin x}{x} = 1$$

$$\textcircled{7} \quad \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$$

$$\textcircled{8} \quad \lim_{x \rightarrow 0^+} x^x = 1$$

$$\textcircled{1} \quad \lim_{x \rightarrow 0^+} x \ln x = 0$$

$$\textcircled{2} \quad \lim_{x \rightarrow 0} \frac{\sqrt[n]{x+1} - 1}{x} = \frac{1}{n}$$

$$\textcircled{3} \quad \lim_{x \rightarrow \infty} x^{\frac{1}{x}} = 1$$

$$\textcircled{4} \quad \lim_{x \rightarrow \pm\infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$\textcircled{5} \quad \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e$$

$$\textcircled{6} \quad \lim_{x \rightarrow 0} (1 + \sin x)^{\frac{1}{x}} = e$$

$$\textcircled{7} \quad \lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 - a^2}\right) = 0$$

$$\textcircled{8} \quad \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$$

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6. Integral

Definition

A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I .

Theorem

If F is an antiderivative of f on an interval I , then the most general antiderivative of f on I is

$$F(x) + C$$

, where C is an arbitrary constant.

Function	Antiderivative	Function	Antiderivative
$cf(x)$	$cF(x)$	$\sec^2 x$	$\tan x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sec x \tan x$	$\sec x$
$x^n (n \neq -1)$	$\frac{x^{n+1}}{n+1}$	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
$\frac{1}{x}$	$\ln x $	$\frac{1}{1+x^2}$	$\tan^{-1} x$
e^x	e^x	$\cosh x$	$\sinh x$
$\cos x$	$\sin x$	$\sinh x$	$\cosh x$
$\sin x$	$-\cos x$		

notation

$$\int f(x)dx = F(x) \quad \text{means} \quad F'(x) = f(x)$$

How to understand this notation?

Theorem

If the antiderivatives of functions $f(x)$ and $g(x)$ exist, then for any constant p and q , the antiderivative of $pf(x)+qg(x)$ also exists, and we have

$$\int [pf(x) + qg(x)]dx = p \int f(x)dx + q \int g(x)dx$$

Keep the following indefinite integrals firmly in mind!

$$\int cf(x)dx = c \int f(x)dx$$

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

$$\int kdx = kx + C$$

$$\int \frac{1}{x}dx = \ln|x| + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int e^x dx = e^x + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \csc^2 x dx = -\cot x + C$$

$$\int \sec^2 x dx = \tan x + C$$

$$\int \csc x \cot x dx = -\csc x + C$$

$$\int \sec x \tan x dx = \sec x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$$

$$\int \frac{1}{x^2+1} dx = \arctan x + C$$

$$\int \cosh x dx = \sinh x + C$$

$$\int \sinh x dx = \cosh x + C$$

Definition

$$\int f(x)g'(x)dx = \int f(x)dg(x)$$

How to memorize?

$$\frac{dg(x)}{dx} = g'(x)$$

$$\Rightarrow dx = \frac{dg(x)}{g'(x)}$$

$$\Rightarrow \int f(x)g'(x)dx = \int f(x)g'(x) \cdot \frac{dg(x)}{g'(x)} = \int f(x)dg(x)$$

Type 1: Direct Substitution ($u = g(x)$)

Substitution Rule for Direct Substitution

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I , then

$$\int f(g(x))g'(x)dx = \int f(g(x))dg(x) = \int f(u)du = F(u) + C$$

Type 2: Inverse Substitution ($x = \varphi(t)$)

Substitution Rule for Inverse Substitution

If $x = \varphi(t)$ is an invertible function, then

$$\int f(x)dx = \int f(\varphi(t))d\varphi(t) = \int f(\varphi(t))\varphi'(t)dt = \tilde{F}(t) = \tilde{F}(\varphi^{-1}(x)) + C$$

$$\textcircled{1} \int \frac{3x^3}{1-x^4} dx$$

$$\textcircled{2} \int \frac{\ln x}{x} dx$$

$$\textcircled{3} \int \frac{\sec^2 x}{\tan^{10} x} dx$$

An Important Application: Trigonometric Substitutions:

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2 \theta = \cos^2 \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2 \theta = \sec^2 \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta, \quad 0 \leq \theta < \frac{\pi}{2}$	$\sec^2 \theta - 1 = \tan^2 \theta$

$$\int \frac{1}{x^2\sqrt{x^2 + 4}}$$

$$\int u dv = uv - \int v du$$

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

How to choose $f(x)$ and $g'(x)$?

Among the following function types:

the more to the left, the more suitable to be $f(x)$

the more to the right, the more suitable to be $g'(x)$

(left) inverse trigonometric function, logarithm function, power function, trigonometric function, exponential function **(right)**

When should we consider applying Integration by Parts?

- ① If the integrand is the product of **inverse trigonometric function, logarithm function or power function** and another function **whose antiderivative is easy to calculate**.
- ② If the integrand is the product of **trigonometric function and exponential function**, apply Integration by Part twice in order to find an identical equation about the original integral, and then solve that equation.
- ③ If the integrand contains n (or a relatively large number), apply Integration by Part to **find the recursion formula**.

$$\textcircled{1} \int x \sin x \, dx$$

$$\textcircled{2} \int \ln^2 x \, dx$$

$$\textcircled{3} \int x^2 e^x \, dx$$

$$\int \sin^m x \cos^n x \, dx$$

- ① If at least m and n is odd, adopt substitution rule. $\int \sin^m x \cos^{2n+1} x \, dx =$

$$\int \sin^m x \cos^{2n} x \, d(\sin x) = \int \sin^m x \cdot (1 - \sin^2 x)^n \, d(\sin x) \stackrel{u=\sin x}{=} \int u^m (1 - u^2)^n \, du$$

$$\begin{aligned} \int \cos^m x \sin^{2n+1} x \, dx &= - \int \cos^m x \sin^{2n} x \, d(\cos x) = \\ &- \int \cos^m x \cdot (1 - \cos^2 x)^n \, d(\cos x) \stackrel{u=\cos x}{=} - \int u^m (1 - u^2)^n \, du \end{aligned}$$

- ② Else, use half-angle identities that $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$.

$$\int \sec^m x \tan^n x dx$$

- ① If m is even (not 0), use $d(\tan x) = \sec^2 x dx$ to transform it into the integration of $\tan x$.

$$\int \sec^{2k} x \tan^n x dx = \int (\tan^2 x + 1)^k \tan^n x d(\tan x) \stackrel{u=\tan x}{=} \int (u^2 + 1)^k u^n du.$$

- ② If n is odd, used $d(\sec x) = \sec x \tan x dx$ to transform it into the integration of $\sec x$.

$$\int \tan^{2k+1} x \sec^n dx = \int (\sec^2 x - 1)^k \sec^{n-1} d(\sec x) = \dots$$

Though not as useful, you can also explore $\int \csc^m x \cot^n x dx$.

$$\int \sin mx \cos nx \, dx$$

Use Product-to-Sum formula:

$$① \sin \alpha \cos \beta = \frac{1}{2}(\sin(\alpha + \beta) + \sin(\alpha - \beta))$$

$$② \cos \alpha \sin \beta = \frac{1}{2}(\sin(\alpha + \beta) - \sin(\alpha - \beta))$$

$$③ \sin \alpha \sin \beta = \frac{1}{2}(\cos(\alpha + \beta) + \cos(\alpha - \beta))$$

$$④ \cos \alpha \cos \beta = -\frac{1}{2}(\cos(\alpha + \beta) - \cos(\alpha - \beta))$$

$$\textcircled{1} \quad \int_0^{\pi/2} \sin^7 x \cos^5 x \, dx$$

Exercise 1:

$$\textcircled{1} \quad \text{ans} = 0.75 \int \frac{1}{1-x^4} dx^4 = -\frac{3}{3} \int \frac{1}{1-x^4} d(1-x^4) = -\frac{3}{4} \ln|1-x^4| + C$$

$$\textcircled{2} \quad d(\ln x) = dx \cdot \frac{1}{x} \Rightarrow \text{ans} = \int \ln x d(\ln x) = \frac{1}{2} \ln^2 x + C$$

$$\textcircled{3} \quad d(\tan x) = \sec^2 x \cdot dx \Rightarrow \text{ans} = \int \frac{d(\tan x)}{\tan^{10} x} = -\frac{1}{9} \tan^{-9} x + C$$

Exercise 2:

$$\text{Let } x = 2\tan\theta, \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}), \frac{dx}{d\theta} = 2\sec^2\theta, \sqrt{x^2+4} = 2\sec\theta, \text{ans} = \int \frac{\cos\theta}{4\sin^2\theta} d\theta.$$

$$\text{Then } u := \sin\theta, \text{ans} = \int \frac{du}{4u^2} = -\frac{1}{4u} + C = -\frac{1}{4\sin\theta} + C = \frac{-\sqrt{x^2+4}}{4x} + C$$

Exercise 3:

$$\textcircled{1} \quad \text{ans} = - \int x d(\cos x) = -x \cos x + \int \cos x dx = -x \cos x + \sin x + C$$

$$\textcircled{2} \quad \text{ans} = x \ln^2 x - \int 2 \ln x dx = x \ln^2 x - 2x \ln x + \int 2 dx = x \ln^2 x - 2x \ln x + 2x + C$$

$$\textcircled{3} \quad \text{ans} = x^2 e^x - 2 \int x e^x dx = x^2 e^x - 2x e^x + 2 \int e^x dx = (x^2 - 2x + 2)e^x + C$$

Exercise 4:

$$\textcircled{1} \quad \int_0^{\pi/2} \sin^7 x \cos^5 x dx = \int_0^1 \sin^7 x (1 - \sin^2 x)^2 d(\sin x) = [\frac{1}{8} u^8 - \frac{1}{5} u^{10} + \frac{1}{12} u^{12}]_0^1 = \frac{1}{120}$$

$$\int e^{ax} \sin^n x \ dx$$

Use induction.

$$\begin{aligned}
 & \int e^{ax} \sin^n x \ dx \\
 &= \frac{1}{a} \int \sin^n x \ d(e^{ax}) \\
 &= \frac{1}{a} e^{ax} \sin^n x - \frac{n}{a} \int e^{ax} \sin^{n-1} x \cos x \ dx \\
 &= \frac{1}{a} e^{ax} \sin^n x - \frac{n}{a^2} e^{ax} \sin^{n-1} x \cos x + \frac{n}{a^2} \int e^{ax} ((n-1) \sin^{n-2} \cos^2 x - \sin^n x) \ dx \\
 &= \frac{1}{a} e^{ax} \sin^n x - \frac{n}{a^2} e^{ax} \sin^{n-1} x \cos x + \frac{n}{a^2} \int e^{ax} ((n-1) \sin^{n-2} - n \sin^n x) \ dx
 \end{aligned}$$

\Rightarrow

$$\frac{a^2 + n^2}{a^2} \int e^{ax} \sin^n x \ dx = \frac{e^{ax} \sin^{n-1} x}{a} (\sin x - \frac{n}{a} \cos x) + \frac{n(n-1)}{a^2} \int e^{ax} \sin^{n-2} x \ dx$$

\Rightarrow

$$\int e^{ax} \sin^n x \ dx = \frac{e^{ax} \sin^{n-1} x}{a^2 + n^2} (a \sin x - n \cos x) + \frac{n(n-1)}{a^2 + n^2} \int e^{ax} \sin^{n-2} x \ dx$$

Use similar method,

$$\begin{aligned} ① \int e^{ax} \sin^n x \, dx \\ = \frac{e^{ax} \cos^{n-1} x}{a^2 + n^2} (a \cos x + n \sin x) + \frac{n(n-1)}{a^2 + n^2} \int e^{ax} \cos^{n-2} x \, dx \end{aligned}$$

$$② \int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

$$③ \int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

$$④ \int \sin^{-n} x \, dx = -\frac{1}{n-1} \cdot \frac{\cos x}{\sin^{n-1} x} + \frac{n-2}{n-1} \cdot \int \sin^{-(n-2)} x \, dx$$

$$⑤ \int \cos^{-n} x \, dx = \frac{1}{n-1} \cdot \frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1} \cdot \int \cos^{-(n-2)} x \, dx$$

- ① Try to prove that $\int (\sec x)^{2n+1} dx = \frac{\tan x \cdot (\sec x)^{2n-1}}{2n} + \frac{2n-1}{2n} \int (\sec x)^{2n-1} dx$ if $n \geq 1$
- ② Calculate $\int \sec^3 x dx$

Solution:

$$\begin{aligned}(1) \quad & \int (\sec x)^{2n+1} dx \\&= \int (\sec x)^{2n-1} d(\tan x) \\&= \tan x \cdot (\sec x)^{2n-1} - (2n-1) \int \tan^2 x \sec^{2n-1} x dx \\&= \tan x \cdot (\sec x)^{2n-1} - (2n-1) \int \sec^{2n-1} x dx + (2n-1) \int \sec^{2n+1} x dx\end{aligned}$$

$$(2) \quad \frac{1}{2} (\ln |\tan x + \sec x| + \sec x \tan x) + C$$

Theorem

The antiderivatives of rational function are always elementary functions.

Here, rational functions are in the form that $\frac{p_m(x)}{q_n(x)}$, where $p_m(x)$ and $q_n(x)$ are polynomials in the degree of m and n.

Basic rational functions

$$\textcircled{1} \quad \int \frac{dx}{ax + b} = \frac{1}{a} \ln|ax + b| + C$$

$$\textcircled{2} \quad \int \frac{dx}{(ax + b)^n} = \int \frac{d(ax + b)}{a(ax + b)^n} = \frac{1}{a(n - 1)(ax + b)^{n-1}} + C$$

$$\textcircled{3} \quad \int \frac{xdx}{ax^2 + bx + c} = \frac{1}{2a} \ln|ax^2 + bx + c| - \frac{b}{2a} \int \frac{dx}{ax^2 + bx + c}$$

$$\textcircled{4} \quad \int \frac{dx}{ax^2 + bx + c} \stackrel{\Delta=b^2-4ac<0}{=} \frac{2}{\sqrt{-\Delta}} \arctan \frac{2ax + b}{\sqrt{4ac - b^2}} + C$$

$$\textcircled{5} \quad \int \frac{dx}{ax^2 + bx + c} \stackrel{\Delta=b^2-4ac>0}{=} \frac{1}{\sqrt{\Delta}} \ln \left| \frac{2ax + b - \sqrt{\Delta}}{2ax + b + \sqrt{\Delta}} \right| + C$$

Remark. What if $\Delta = 0$?

Then, what about $\int \frac{x+d}{(ax^2+bx+c)^n} dx$?

$$\textcircled{1} \quad \int \frac{x+d}{(ax^2+bx+c)^n} dx = \frac{1}{2a} \int \frac{2ax+b}{(ax^2+bx+c)^n} dx + \left(d - \frac{b}{2a}\right) \int \frac{1}{(ax^2+bx+c)^n} dx$$

$$\textcircled{2} \quad \frac{1}{2a} \int \frac{2ax+b}{(ax^2+bx+c)^n} dx = \frac{1}{2a} \int \frac{d(ax^2+bx+c)}{(ax^2+bx+c)^n} = \dots$$

$$\textcircled{3} \quad \int \frac{1}{(ax^2+bx+c)^n} dx = 16a^2 \int \frac{dx}{((2ax+b)^2 - b^2 + 4ac)^n} = 8a \int \frac{du}{(u^2 - b^2 + 4ac)^n}$$

$$\textcircled{4} \quad \int \frac{du}{(au^2+b)^n} = \frac{2n-3}{2b(n-1)} \int \frac{du}{(au^2+b)^{n-1}} + \frac{u}{2b(n-1)(au^2+b)^{n-1}}$$

5 Combine all tools above up!

$$\text{Form: } \frac{f(x)}{[A(x)]^a[B(x)]^b\dots} = \frac{f_a(x)}{[A(x)]^a} + \frac{f_b(x)}{[B(x)]^b} + \dots$$

Purpose: Split a complex rational function into several functions whose forms are familiar and easy to integrate, or change the integration into familiar forms with trigonometric substitution

Critical skill: **Undetermined coefficient method, Trigonometrical Substitution, Reciprocal Substitution.**

Also, as rational functions are easier to deal with, we can carry out **Rationalization**.

Key character: Whether the denominator of the integrand can be factorized or not

If the degree of the polynomial (or other type of functions) in the numerator is greater than or equal to that in the denominator, **extract a polynomial (or other type of functions)**.

If the degree of the polynomial in the denominator is too high, in order to reduce the workload, it's recommended to **apply substitution rule before attempting to split the function.**

Exercise 6-Undetermined coefficient method

$$\textcircled{1} \int \frac{4y^2 - 7y - 12}{y(y+2)(y+3)} dy$$

$$\textcircled{2} \int \frac{x^4 + x^3 + 3x^2 - 1}{(x^2 + 1)^2(x - 1)} dx$$

Exercise 7-Reciprocal Substitution

$$\textcircled{1} \int \frac{dx}{x^8(1+x^2)}$$

Exercise 8-Rationalization

$$\textcircled{1} \int \frac{dx}{\sqrt{1-x} + \sqrt[3]{1-x}}$$

When encountering complex combination of trigonometric functions, we can substitute $t = \tan\frac{x}{2}$.

Then there is $\sin x = \frac{2t}{t^2 + 1}$, $\cos x = \frac{1 - t^2}{t^2 + 1}$, $\tan x = \frac{2t}{1 - t^2}$, $\frac{dx}{dt} = \frac{2}{t^2 + 1}$.
It is thus converted into rational functions.

Exercise 9—"Universal Substitution"

$$\textcircled{1} \int \frac{dx}{1 + \cos x + \sin^2(\frac{x}{2})}$$

Definition

If a function $f(x)$ is defined on the interval $[a, b]$, insert several points randomly, and we will get:

$$a = x_0 < x_1 < \dots < x_n = b$$

which defines n intervals whose corresponding lengths are

$$\Delta x_k = x_k - x_{k-1}$$

The for every interval select $\xi_k \in [x_{k-1}, x_k]$, the Riemann Sum of $f(x)$ on the interval $[a, b]$ is:

$$S_n = \sum_{k=1}^n f(\xi_k) \Delta_k$$

Riemann Sum is the sum of areas of rectangles.

In order to calculate the area of a figure:

- ① Cut that irregular figure into small strips, and then regard the strip as a rectangle approximately.
- ② Measure the lengths of these small rectangles respectively and calculate each of their area.
- ③ Add up all the rectangular areas to get the total area.

Only applicable to one situation: $\int e^{x^2} dx$.

For simplicity, let us suggest $x \geq 0$.

$$\int e^{x^2} dx = \sqrt{\int e^{x^2} dx \cdot \int e^{y^2} dy} = \sqrt{\iint e^{x^2+y^2} dxdy} = \sqrt{\iint r e^{r^2} drd\theta} = \sqrt{\pi e^{x^2}}.$$

Example 1

$$\begin{aligned} \int \sin(\ln x) dx &= x \sin(\ln x) - \int x \cos(\ln x) \cdot \frac{1}{x} dx = x \sin(\ln x) - x \cos(\ln x) - \int \sin(\ln x) dx \\ \Rightarrow \int \sin(\ln x) dx &= \frac{x \sin(\ln x) - x \cos(\ln x)}{2} \end{aligned}$$

Example 2

$$I = \int \frac{\sin^2 x}{\sin x + \sqrt{3} \cos x} dx$$

$$\text{define } J := \int \frac{\cos^2 x}{\sin x + \sqrt{3} \cos x} dx, I + J = \int \frac{1}{\sin x + \sqrt{3} \cos x} dx = \frac{1}{2} \int \frac{1}{\sin(x + \frac{\pi}{3})} dx =$$

$$\frac{1}{2} \ln |\csc(x + \frac{\pi}{3}) - \cot(x + \frac{\pi}{3})| + C$$

$$I - 3J = \int \sin x - \sqrt{3} \cos x dx = -\cos x - \sqrt{3} \sin x + C$$

$$\text{Thus } I = \frac{3}{8} \ln |\csc(x + \frac{\pi}{3}) - \cot(x + \frac{\pi}{3})| - \frac{1}{4} \cos x - \frac{\sqrt{3}}{4} \sin x + C$$

Definition

If f is a function defined for $a \leq x \leq b$, we divide the integral $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0 (= a), x_1, x_2, \dots, x_n (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \dots, x_n^*$ be any sample points in these subintervals, so x_i^* lies in the i th subinterval. Then the definite integral of f from a to b is

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*)\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is integrable on $[a,b]$.

Definite integral can also be intuitively understood as the net area between the curve $(f(x), x)$, line $x = a$, $x = b$, and x axis.

Properties

① Linearity

$$\int_a^b [k_1 f(x) + k_2 g(x)] dx = k_1 \int_a^b f(x) dx + k_2 \int_a^b g(x) dx$$

② Internal Additivity

$$\int_a^c f(x) dx + \int_c^b f(x) dx = \int_a^b f(x) dx$$

③ Comparison Properties

If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x) dx \geq \int_a^b g(x) dx$

If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$

Properties

④ Absolute Integrability

If $f(x)$ is integrable on $[a,b]$, then $|f(x)|$ is also integrable on $[a,b]$, and we have

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx$$

Note

If $f(x)$ and $g(x)$ are integrable on $[a,b]$, then $f(x) \cdot g(x)$ is also integrable on $[a,b]$, but generally

$$\int_a^b f(x)g(x)dx \neq \left(\int_a^b f(x)dx \right) \cdot \left(\int_a^b g(x)dx \right)$$

Calculate the definite integral by definition

$$\textcircled{1} \quad \int_{-3}^0 (1 + \sqrt{9 - x^2}) dx$$

$$\textcircled{2} \quad \int_{\pi}^{\pi} \sin^2 x \cos^4 x dx$$

Solutions:

- ① $\int_{-3}^0 (1 + \sqrt{9 - x^2}) dx$ can be interpreted as the area under the graph of $f(x) = 1 + \sqrt{9 - x^2}$ between $x = -3$ and $x = 0$. This is equal to one-quarter the area of the circle with radius 3, plus the area of the rectangle, so

$$\int_{-3}^0 (1 + \sqrt{9 - x^2}) dx = \frac{1}{4}\pi \cdot 3^2 + 1 \cdot 3 = 3 + \frac{9}{4}\pi$$

- ② $\int_{\pi}^{\pi} \sin^2 x \cos^4 x dx = 0$
since the limits of integral are equal

Newton-Leibniz Formula

Suppose f is continuous on $[a,b]$.

- If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$
- If F is any antiderivative of f (i.e. $F' = f$), then
$$\int_a^b f(x)dx = F(b) - F(a)$$

$$\textcircled{1} \int_0^{\pi} \sqrt{\sin^3 x - \sin^5 x} dx$$

$$\textcircled{2} \int_0^3 \frac{x^2}{(x^2 - 3x + 3)^2} dx$$

Solutions: 1. $\int_0^\pi \sqrt{\sin^3 x - \sin^5 x} dx$

Since $\sqrt{\sin^3 x - \sin^5 x} = \sqrt{\sin^3 x(1 - \sin^2 x)} = \sin^{\frac{3}{2}} \cdot |\cos x|$, where $|\cos x| = \cos x$ on $[0, \frac{\pi}{2}]$;

$|\cos x| = -\cos x$ on $[\frac{\pi}{2}, \pi]$,

$$\begin{aligned} & \int_0^\pi \sqrt{\sin^3 x - \sin^5 x} dx \\ &= \int_0^{\frac{\pi}{2}} \sin^{\frac{3}{2}} x \cos x dx + \int_{\frac{\pi}{2}}^\pi \sin^{\frac{3}{2}} x (-\cos x) dx \\ &= \int_0^{\frac{\pi}{2}} \sin^{\frac{3}{2}} x d(\sin x) - \int_{\frac{\pi}{2}}^\pi \sin^{\frac{3}{2}} x d(\sin x) \\ &= \left[\frac{2}{5} \sin^{\frac{5}{2}} x \right]_0^{\frac{\pi}{2}} - \left[\frac{2}{5} \sin^{\frac{5}{2}} x \right]_{\frac{\pi}{2}}^\pi \\ &= \frac{2}{5} - \left(-\frac{2}{5} \right) = \frac{4}{5} \end{aligned}$$

Solutions: 2. $\int_0^3 \frac{x^2}{(x^2 - 3x + 3)^2} dx$

$x^2 - 3x + 3 = (x - \frac{3}{2})^2 + \frac{3}{4}$, let $x - \frac{3}{2} = \frac{\sqrt{3}}{2} \tan u$ ($|u| < \frac{\pi}{2}$), such that

$$(x^2 - 3x + 3)^2 = (\frac{3}{4} \sec^2 u)^2, dx = \frac{\sqrt{3}}{2} \sec^2 u du.$$

$$x = 0 \rightarrow u = -\frac{\pi}{3}; x = 3 \rightarrow u = \frac{\pi}{3}.$$

$$\int_0^3 \frac{x^2}{(x^2 - 3x + 3)^2} dx = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (\frac{3}{4} \tan^2 u + \frac{3\sqrt{3}}{2} \tan u + \frac{9}{4}) \cdot \frac{16}{9} \cdot \frac{\sqrt{3}}{2} \cos^2 u du$$

$$= \frac{8}{3\sqrt{3}} \cdot 2 \int_0^{\frac{\pi}{3}} (\frac{3}{4} \tan^2 u + \frac{9}{4}) \cos^u du$$

$$= \frac{4}{\sqrt{3}} \int_0^{\frac{\pi}{3}} (\sin^2 u + 3\cos^2 u) du = \frac{4}{\sqrt{3}} \int_0^{\frac{\pi}{3}} (2 + \cos 2u) du$$

$$= \frac{4}{\sqrt{3}} [2u + \frac{1}{2} \sin 2u]_0^{\frac{\pi}{3}}$$

$$= \frac{8\pi}{3\sqrt{3}} + 1$$

Definition

Type1: Unboundedness

- ① If $f(x)$ is continuous on $[a, +\infty]$, for any $t > a$, the improper integral of $f(x)$ on $[a, +\infty]$ is

$$\int_a^{+\infty} f(x)dx = \lim_{t \rightarrow +\infty} \int_a^t f(x)dx$$

- ② If $f(x)$ is continuous on $[-\infty, a]$, for any $t < a$, the improper integral of $f(x)$ on $[-\infty, a]$ is

$$\int_a^{+\infty} f(x)dx = \lim_{t \rightarrow -\infty} \int_t^a f(x)dx$$

Definition

Type1: Unboundedness

- ③ If both $\int_a^{\infty} f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent

$$\int_{-\infty}^{+\infty} f(x)dx = \int_a^{+\infty} f(x)dx + \int -\infty^a f(x)dx$$

Definition

Type2: Discontinuous

- ① If f is continuous on $[a,b)$ and is discontinuous at b , then the improper integral on is

$$\int_a^b f(x)dx = \lim_{t \rightarrow b^-} \int_a^t f(x)dx$$

- ② If f is continuous on $(a,b]$ and is discontinuous at a , then the improper integral on is

$$\int_a^b f(x)dx = \lim_{t \rightarrow a^+} \int_t^b f(x)dx$$

Definition

Type2: Discontinuous

- ③ If f has a discontinuity at c ($a < c < b$), and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Convergence and divergence of improper integrals

If the limits mentioned above are exists (as a finite number), then the improper integral is called convergent, otherwise it's divergent.

$$\textcircled{1} \int_e^\infty \frac{1}{x(\ln x)^3} dx$$

$$\textcircled{2} \int_0^3 \frac{1}{x^2 - 6x + 5} dx$$

Solutions: 1.

$$\int_e^\infty \frac{1}{x(\ln x)^3} dx = \lim_{t \rightarrow \infty} \int_e^t \frac{1}{x(\ln x)^3} dx$$

$$= \lim_{t \rightarrow \infty} \int_1^{Int} u^{-3} du$$

$$(u = \ln x, du = dx/x)$$

$$= \lim_{t \rightarrow \infty} \left[-\frac{1}{2u^2} \right]_1^{Int}$$

$$= \lim_{t \rightarrow \infty} \left[-\frac{1}{2(Int)^2} + \frac{1}{2} \right]$$

$$= 0 + \frac{1}{2} = \frac{1}{2}$$

Convergent

Solutions:

$$\begin{aligned}
 2. \int_0^3 \frac{dx}{x^2 - 6x + 5} &= \int_0^3 \frac{dx}{(x-1)(x-5)} \\
 &= \int_0^1 \frac{dx}{(x-1)(x-5)} + \int_1^3 \frac{dx}{(x-1)(x-5)} \\
 &= I_1 + I_2 \\
 I_1 &= \frac{1}{4} \lim_{t \rightarrow 1^-} \int_0^t \left(\frac{1}{x-5} - \frac{1}{x-1} \right) dx \\
 &= \lim_{t \rightarrow 1^-} \left[-\frac{1}{4} \ln|t-1| + \frac{1}{4} \ln|t-5| \right]_0^t \\
 &= \lim_{t \rightarrow 1^-} \left[\left(-\frac{1}{4} \ln|t-1| + \frac{1}{4} \ln|t-5| \right) - \left(-\frac{1}{4} \ln|-1| + \frac{1}{4} \ln|-5| \right) \right]_0^t
 \end{aligned}$$

$$= \infty$$

Since I_1 is divergent, I is divergent.

Theorem 1

Suppose $f(x)$ is continuous on $[a, \infty)$, and $f(x) \geq 0$. If

$$F(x) = \int_a^x f(t)dt$$

has an upper bound on $[a, \infty)$, then the improper integral $\int_a^\infty f(x)dx$ converges.

Theorem 2: Comparison Theorem

Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

- (a) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.
- (b) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

Judge whether the following improper integrals converge or not. If some of them converge, calculate their value.

$$\textcircled{1} \quad \int_1^{\infty} \frac{1}{x^4} dx$$

$$\textcircled{2} \quad \int_0^2 \frac{dx}{(1-x)^2}$$

$$\textcircled{3} \quad \int_1^{\infty} \frac{1}{\sqrt{x}} dx$$

$$\textcircled{4} \quad \int_0^1 \frac{x}{\sqrt{1-x^2}} dx$$

Solutions:

$$\textcircled{1} \quad \int_0^\infty \frac{dx}{x^4} = - \left[\frac{1}{3x^3} \right] = \frac{1}{3}$$

$$\textcircled{2} \quad \int_0^t \frac{dx}{(1-x)^2} = \left[\frac{1}{1-x} \right]_0^t = \frac{1}{1-t} - 1 \text{ Diverge}$$

$$\textcircled{3} \quad \int_1^t \frac{1}{\sqrt{x}} dx = [2\sqrt{x}]_1^t = 2\sqrt{t} - 2 \text{ Diverge}$$

$$\textcircled{4} \quad \int_0^1 \frac{x dx}{1-x^2} = -[\sqrt{1-x^2}]_0^1 = 1$$

Calculate the following improper integral:

$$\int_0^\infty \frac{1 - e^{-x^2}}{x^2} dx$$

You can directly use the conclusion that $\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$.
(How to calculate this integral will be taught in VV255.)

Solutions:

$$\int_0^\infty \frac{1 - 3^{-x^2}}{x^2} dx = \int_0^\infty (e^{-x^2} - 1) d\left(\frac{1}{x}\right)$$

$$= \left[\frac{e^{-x^2} - 1}{x} \right]_0^\infty - \int_0^\infty \frac{1}{x} d(e^{-x^2} - 1)$$

$$= \left[\frac{e^{-x^2} - 1}{x} \right]_0^\infty + 2 \int_0^\infty e^{-x^2} dx$$

$$\lim_{x \rightarrow \infty} \frac{e^{-x^2} - 1}{x} = 0, \lim_{x \rightarrow 0} \frac{e^{-x^2} - 1}{x} = \lim_{x \rightarrow 0} \frac{-2xe^{-x^2}}{1} = 0$$

$$2 \int_0^\infty e^{-x^2} dx = \int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$$

$$\text{Therefore, } \int_0^\infty \frac{1 - 3^{-x^2}}{x^2} dx = \sqrt{\pi}$$

Theorem

$$\frac{d}{dt} \int_{\alpha(t)}^{\beta(t)} f(x)dx = f(\beta(t))\beta'(t) - f(\alpha(t))\alpha'(t)$$

How to prove it?

Use the integration to deal with the combination of limits and series.

Example

$$\textcircled{1} \quad \lim_{n \rightarrow \infty} \sum_{i=0}^n \frac{i}{n^2}$$

$$\textcircled{2} \quad \lim_{n \rightarrow \infty} \sqrt{n} \left(\frac{1}{n} - \sum_{i=0}^n \frac{1}{n + \sqrt{i}} \right)$$

Method

① Simply Transform into Integration. $\lim_{n \rightarrow \infty} \sum_{i=0}^n \frac{i}{n^2} = \int_0^n \frac{i}{n^2} di = \frac{1}{2}$

② Use Squeeze Theorem. $ans = \lim_{n \rightarrow \infty} \sqrt{n} \sum_{i=0}^n \frac{\sqrt{i}}{n(n + \sqrt{i})}$

$$ans \leq \lim_{n \rightarrow \infty} \sqrt{n} \sum_{i=0}^n \frac{\sqrt{i}}{n^2} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^n \sqrt{\frac{k}{n}} = \int_0^1 \sqrt{x} dx = \frac{2}{3}.$$

$$ans \geq \lim_{n \rightarrow \infty} \sqrt{n} \sum_{i=0}^n \frac{\sqrt{i}}{n(n + \sqrt{n})} = \lim_{n \rightarrow \infty} \frac{n}{n + \sqrt{n}} \frac{1}{n} \sum_{i=0}^n \sqrt{\frac{k}{n}} = \frac{2}{3}.$$

Exercise: Refer to Worksheet 1 1.3,1.4

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7. Parametric Equations & Polar Coordinates

Definition

Suppose that x and y are both given as functions of a third variable t (called a parameter) by the equations

$$x = f(t) \quad y = g(t)$$

$$x = \cos t \quad y = \sin t \quad (0 \leq t \leq 2\pi)$$

If we plot points, it appears that the curve is a circle. We can confirm this impression by eliminating t . Observe that

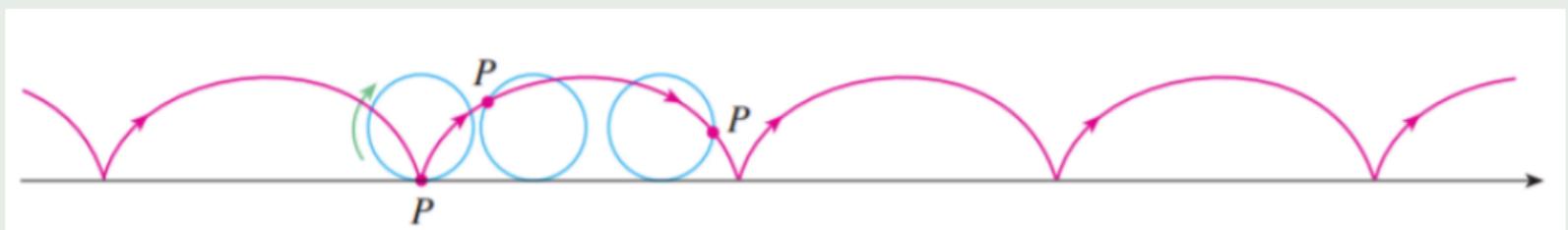
$$x^2 + y^2 = \sin^2 t + \cos^2 t = 1$$

Thus the point (x, y) moves on the unit circle $x^2 + y^2 = 1$. Notice that in this example the parameter t can be interpreted as the angle (in radians). As t increases from 0 to 2π , the point $(x, y) = (\cos t, \sin t)$ moves once around the circle in the counterclockwise direction starting from the point $(1, 0)$.

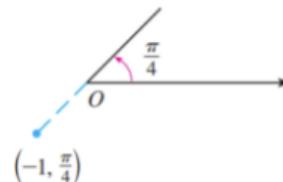
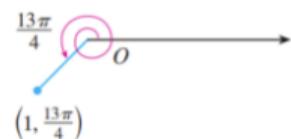
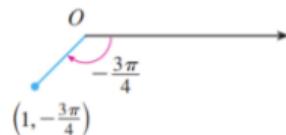
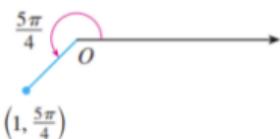
Definition

The curve traced out by a point on the circumference of a circle as the circle rolls along a straight line is called a cycloid . Therefore parametric equations of the cycloid are

$$x = r(\theta - \sin\theta) \quad y = r(1 - \cos\theta) \quad \theta \in R$$



We choose a point in the plane that is called the pole (or origin) and is labeled O. Then we draw a ray (half-line) starting at O called the polar axis. This axis is usually drawn horizontally to the right and corresponds to the positive x-axis in Cartesian coordinates. If P is any other point in the plane, let r be the distance from O to P and let θ be the angle (usually measured in radians) between the polar axis and the line OP as in Figure 1 . Then the point P is represented by the ordered pair (r, θ) and r, θ are called polar coordinates of P. We use the convention that an angle is positive if measured in the counterclockwise direction from the polar axis and negative in the clockwise direction. If $P = O$, then $r = 0$ and we agree that $(0, \theta)$ represents the pole for any value of θ .



In fact, since a complete counterclockwise rotation is given by an angle 2π , the point represented by polar coordinates (r, θ) is also represented by

$$(r, \theta + 2n\pi) \text{ and } (-r, \theta + (2n + 1)\pi)$$

Relationship between Polar Coordinates and Cartesian Coordinates

$$\begin{aligned}x &= r\cos\theta \quad y = r\sin\theta \\r^2 &= x^2 + y^2 \quad \tan\theta = \frac{y}{x}\end{aligned}$$

cylindrical to Cartesian coordinates

$$x = r\cos\phi$$

$$y = r\sin\phi$$

$$z = z$$

inverse relations (from Cartesian to cylindrical coordinates)

$$r = \sqrt{x^2 + y^2}$$

$$\tan\phi = \frac{y}{x}$$

$$z = z$$

spherical to Cartesian coordinates

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

inverse relations (from Cartesian to spherical coordinates)

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\tan\phi = \frac{y}{x}$$

$$\tan\theta = \frac{\sqrt{x^2+y^2}}{z}$$

Definition

- parametric representation:

$$\begin{aligned}x(\theta) &= 2a(1 - \cos\theta)\cos\theta \\y(\theta) &= 2a(1 - \cos\theta)\sin\theta \\&\quad (0 \leq \theta \leq 2\pi)\end{aligned}$$

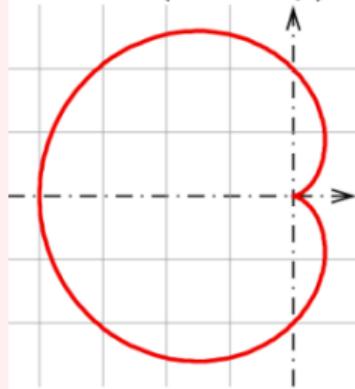
- polar coordinates:

$$\begin{aligned}r &= 2a(1 - \cos\theta) \\&\quad (0 \leq \theta \leq 2\pi)\end{aligned}$$

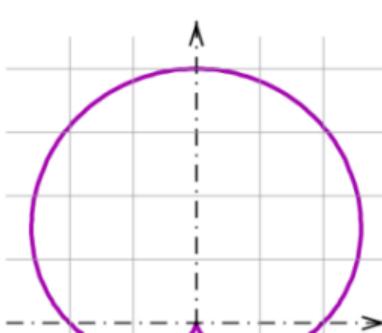
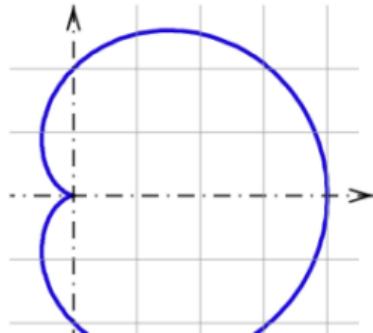
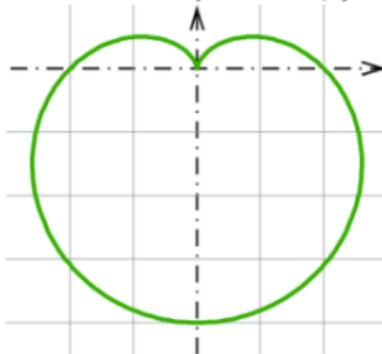
Cardioid is special Cycloid and special Limaçon

Cardioid

$$r = 2a(1 - \cos \varphi)$$

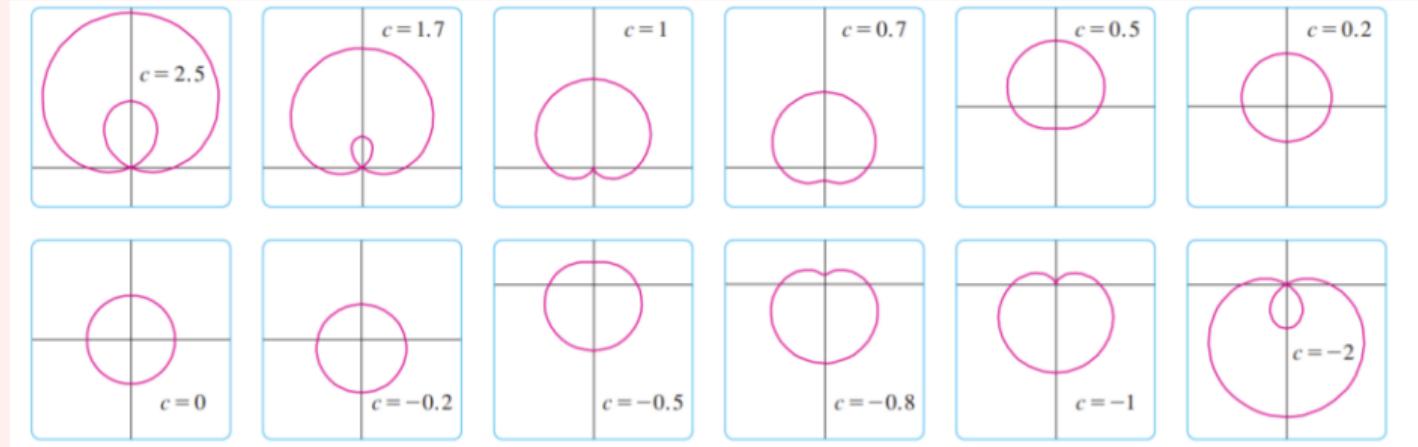


$$r = 2a(1 - \sin \varphi)$$



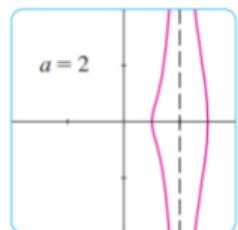
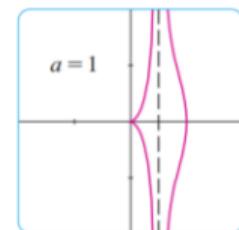
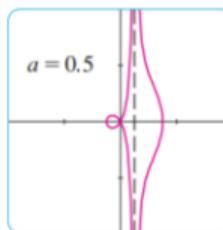
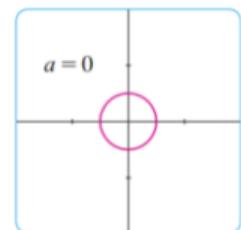
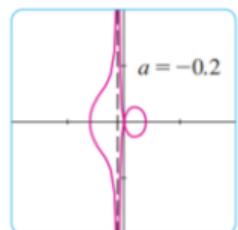
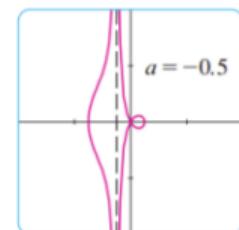
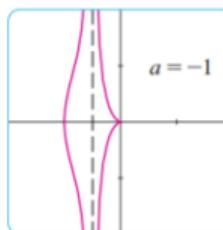
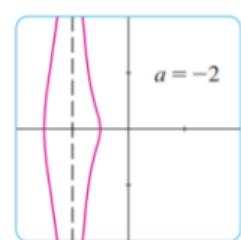
Definition

$$r = 1 + c \sin \theta$$



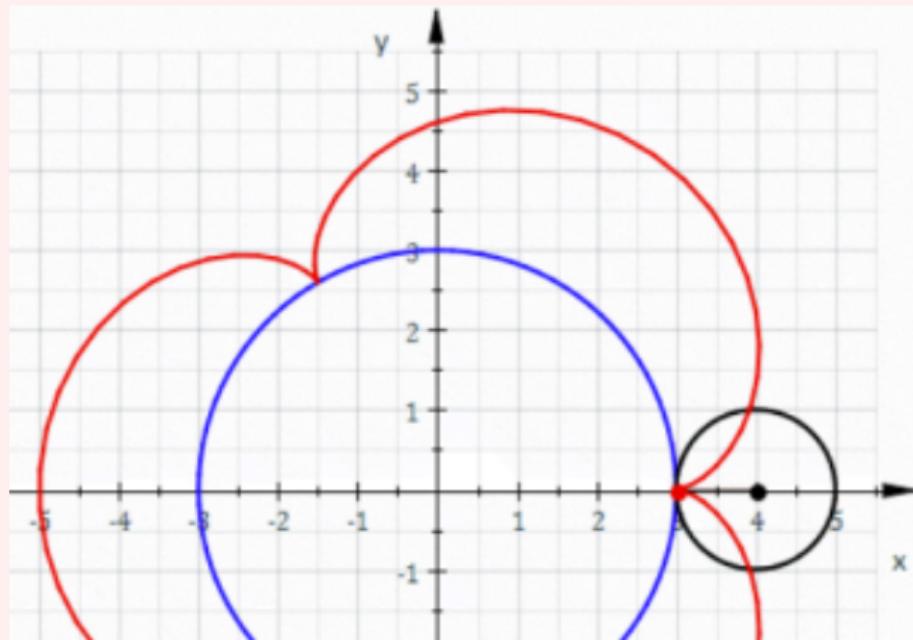
Definition

$$r = 1 + c \sec \theta$$



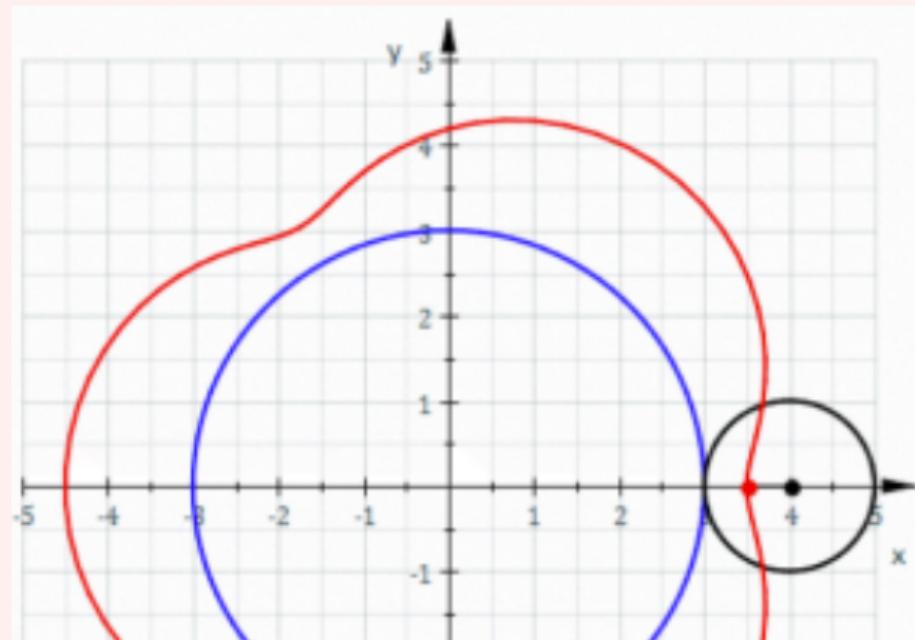
Definition

$$x(\theta) = (R + r)\cos\theta - r\cos\left(\frac{R+r}{r}\theta\right)$$
$$y(\theta) = (R + r)\sin\theta - r\sin\left(\frac{R+r}{r}\theta\right)$$



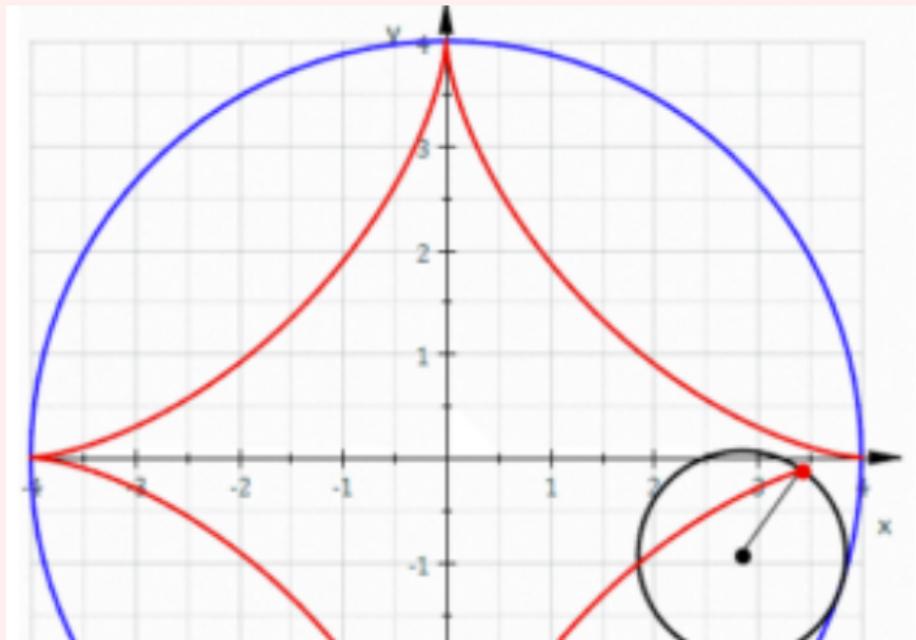
Definition

$$x(\theta) = (R + r)\cos\theta - d\cos\left(\frac{R+r}{r}\theta\right)$$
$$y(\theta) = (R + r)\sin\theta - d\sin\left(\frac{R+r}{r}\theta\right)$$



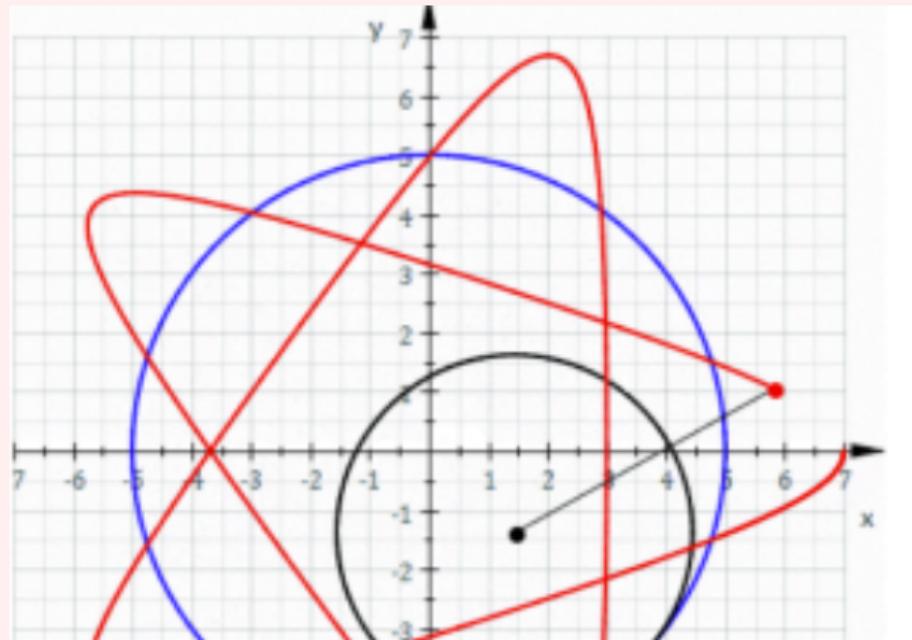
Definition

$$x(\theta) = (R - r)\cos\theta - r\cos\left(\frac{R-r}{r}\theta\right)$$
$$y(\theta) = (R - r)\sin\theta - r\sin\left(\frac{R-r}{r}\theta\right)$$



Definition

$$x(\theta) = (R - r)\cos\theta - d\cos\left(\frac{R-r}{r}\theta\right)$$
$$y(\theta) = (R - r)\sin\theta - d\sin\left(\frac{R-r}{r}\theta\right)$$



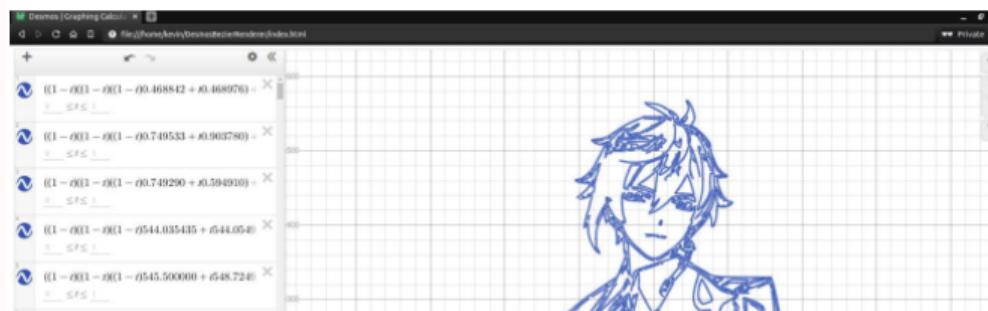
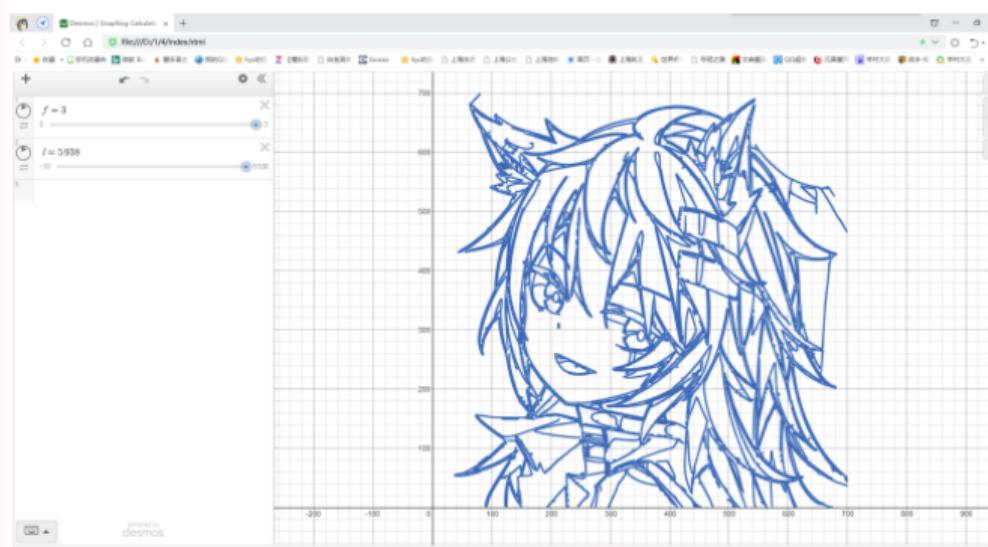
Definition

Bézier curves are used in computer-aided design and are named after the French mathematician Pierre Bézier (1910-1999), who worked in the automotive industry. A cubic Bézier curve is determined by four control points, $P_0(x_0, y_0), P_1(x_1, y_1), P_2(x_2, y_2)$, and $P_3(x_3, y_3)$, and is defined by the parametric equations:

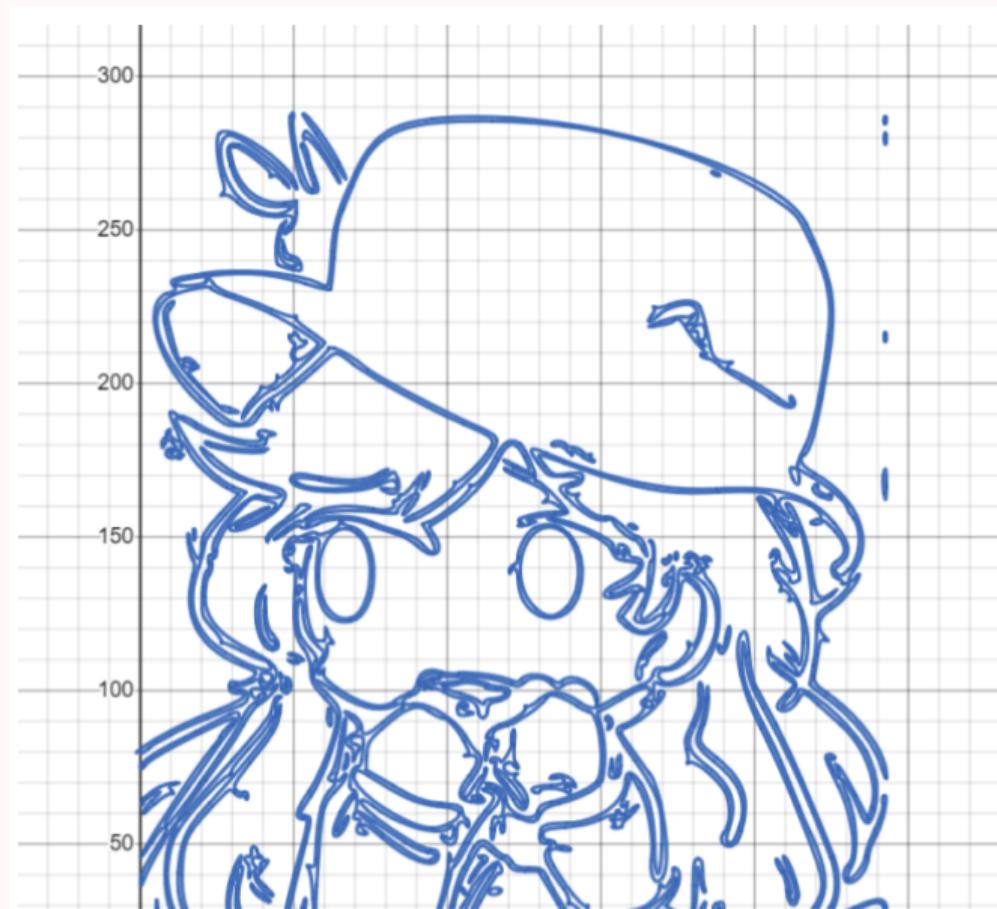
$$\begin{aligned}x &= x_0(1-t)^3 + 3x_1t(1-t)^2 + 3x_2t^2(1-t) + x_3t^3 \\y &= y_0(1-t)^3 + 3y_1t(1-t)^2 + 3y_2t^2(1-t) + y_3t^3\end{aligned}$$

More Details: <https://zhuanlan.zhihu.com/p/471457420>

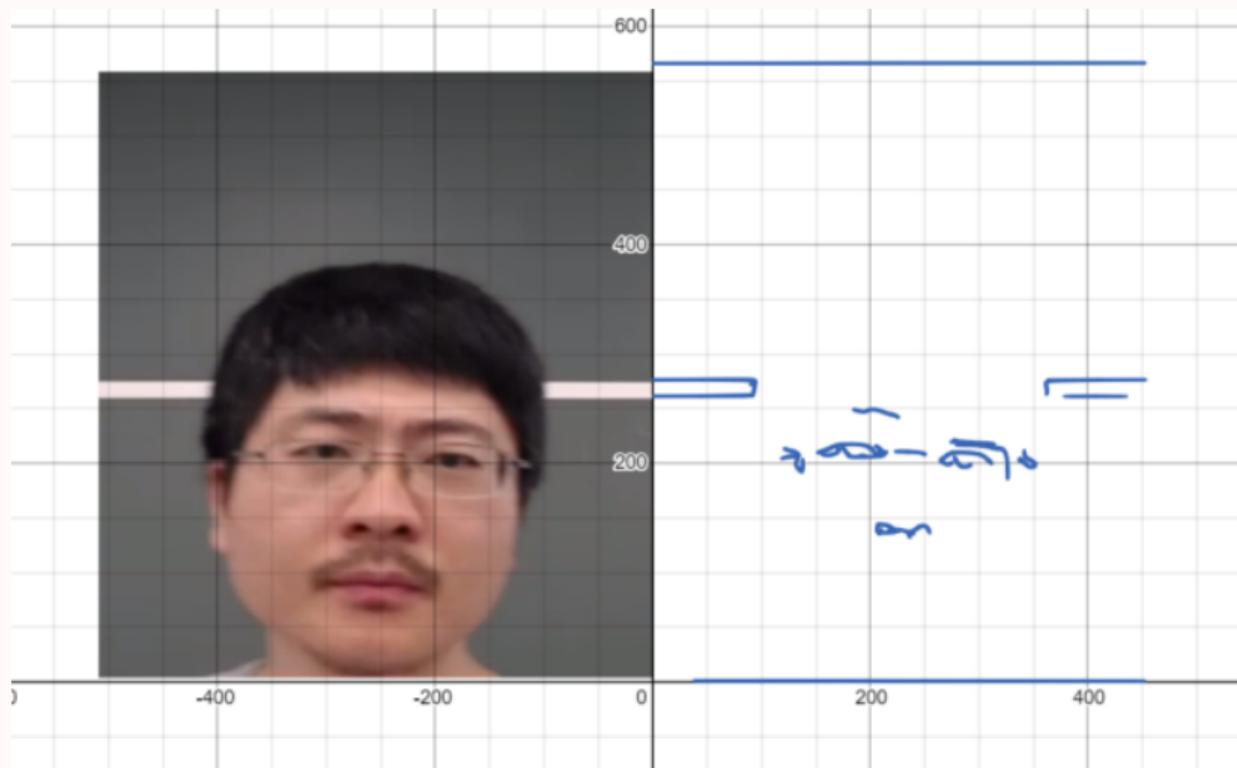
Application of Bézier Curves in desmos



Application of Bézier Curves in desmos



Application of Bézier Curves in desmos



Slope of the Tangent Line with Parametric Curves

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

For the second order derivative, we have:

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \frac{dy}{dx} = \frac{\frac{d}{dt} \frac{dy}{dx}}{\frac{dx}{dt}}$$

Area

We know that the area under a curve $y = F(x)$ from a to b is $A = \int_a^b F(x)dx$, where $F(x) \geq 0$. If the curve is traced out once by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, then we can calculate an area formula by using the Substitution Rule for Definite Integrals as follows:

$$A = \int_a^b ydx = \int_{\alpha}^{\beta} g(t)f'(t)dt$$

or

$$= \int_{\beta}^{\alpha} g(t)f'(t)dt$$

Arc Length

If a curve C is described by the parametric equations $x = f(t), y = g(t), \alpha \leq t \leq \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β , then the length of C is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Surface Area

In the same way as for arc length, we can adapt to obtain a formula for surface area. If the curve given by the parametric equations $x = f(t), y = g(t), \alpha \leq t \leq \beta$ is rotated about the x-axis, where f' and g' are continuous and $g(t) \geq 0$, then the area of the resulting surface is given by

$$S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Slope of the Tangent Line with Polar Coordinates

$$x = r\cos\theta, y = r\sin\theta$$

where r can be regarded as a function of θ . Hence we have

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dy}{d\theta}\sin\theta + r\cos\theta}{\frac{dy}{d\theta}\cos\theta - r\sin\theta}$$

Suppose we have the function in polar coordinates:

$$r = f(\theta), a \leq \theta \leq b$$

For the area enclosed by this function, we have:

$$A = \int_a^b \frac{1}{2} f^2(\theta) d\theta$$

For the area enclosed by this function, we have:

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

tangents

Find equations of the tangents to the curve $x = 3t^2 + 1, y = 2t^3 + 1$ that pass through the point $(4, 3)$.

- ① Find the exact length of the curve

$$x = 1 + 3t^2, y = 4 + 2t^3, 0 \leq t \leq 1$$

- ② Find the area enclosed by the x-axis and the curve

$$x = 1 + e^t, y = t - t^2$$

- ③ Find the exact area of the surface obtained by rotating the given curve about the x-axis.

$$x = 3t - t^3, y = 3t^2, 0 \leq t \leq 1$$

- ① Find the slope of the tangent line to the given polar curve at the point specified by the value of θ

$$r = 2 - \sin\theta, \theta = \frac{\pi}{3}$$

- ② Find the area of the region enclosed by one loop of the curve:

$$r = 2\sin 5\theta$$

exercises 1:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t^2}{6t} = t$$

when pass through $(4, 3)$, we have $3 - (2t^3 + 1) = t(4 - (3t^2 + 1)) \Rightarrow t = 1$ or $t = -2$

$$\Rightarrow y = x - 1 \text{ or } y = -2x + 11$$

exercises 2:

① $L = \int_0^1 \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} dt = \int_0^1 6t\sqrt{1+t^2} dt = \int_1^2 \sqrt{u} \left(\frac{1}{2} du\right) = 3 \cdot \frac{3}{2} [u^{\frac{3}{2}}] \Big|_1^2 = 2(2\sqrt{2} - 1)$

② The curve $x = 1 + e^t, y = t - t^2$ intersects the x-axis when $y = 0$, that is, when $t = 0$ and $t = 1$. The corresponding values of x are 2 and $1 + e$. The shaded area is given by $\int_2^{1+e} (y_T - y_B) dx = \int_0^1 (y(t) - 0) x'(t) dt = \int_0^1 (t - t^2) e^t dt = \int_0^1 te^t dt - \int_0^1 t^2 \cdot e^t dt = 3 \int_0^1 te^t dt - t^2 e^t \Big|_0^1 = 3((t-1)e^t) \Big|_0^1 - e = 3 - e$

③ $x = 3t - t^3, y = 3t^2, 0 \leq t \leq 1$.

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (3 - 3t^2)^2 + (6t)^2 = (3(1+t^2))^2$$

$$S = \int_0^1 2\pi \cdot 3t^2 \cdot (1+t^2) dt$$

$$= 18\pi \left(\frac{1}{3}t^3 + \frac{1}{5}t^5\right) \Big|_0^1$$

$$= \frac{48}{5}\pi$$

① $x = r\cos\theta = (2 - \sin\theta)\cos\theta, y = r\sin\theta = (2 - \sin\theta)\sin\theta$

$$\frac{dy}{dx} = \frac{2\cos\theta - \sin 2\theta}{-2\sin\theta - \cos 2\theta}$$

when $\theta = \frac{\pi}{3}$, $\frac{dy}{dx} = \frac{2 - \sqrt{3}}{1 - 2\sqrt{3}}$

② $\sin 5\theta = 0 \Rightarrow 5\theta = n\pi \Rightarrow \theta = \frac{\pi}{5}n$

$$\begin{aligned} A &= \int_0^{\frac{\pi}{5}} \frac{1}{2}(2\sin 5\theta)^2 d\theta \\ &= 2 \int_0^{\frac{\pi}{5}} \frac{1}{2}(1 - \cos 10\theta) \Big|_0^{\frac{\pi}{5}} = \frac{\pi}{5} \end{aligned}$$

1. Huang, Jiahe. VV156 RC4.pdf.2022.
2. Huang, Yucheng. VV156 RC6.pdf. 2021.
3. Chen, Jixiu et al. Mathematical Analysis (3rd Version). 2019
4. Li, Junhao. VV156 RC5.pdf.2022.

8. Series

A sequence a_n has the limit L and we write:

$$\lim_{n \rightarrow \infty} a_n = L \text{ or } a_n \rightarrow L \text{ as } n \rightarrow \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n \rightarrow \infty} a_n$ exists, we say the sequence converges (or is convergent). Otherwise, we say the sequence diverges (or is divergent).

Definition

Suppose that there is a sequence a_n . If for any fixed positive number ε , there exists a positive integer N such that for any $n > N$, we have

$$|a_n - L| < \varepsilon$$

Then we say that the sequence a_n has the limit L .

Theorems

- ① If $\lim_{n \rightarrow \infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n \rightarrow \infty} a_n = L$.
- ② $\lim_{n \rightarrow \infty} a_n = \infty$ means that for every positive number M there is an integer N such that if $n > N$ then $a_n > M$.
- ③ (Squeeze theorem) If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$, then $\lim_{n \rightarrow \infty} b_n = L$.
- ④ If $\lim_{n \rightarrow \infty} |a_n| = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.
- ⑤ Every bounded, monotonic sequence is convergent.
- ⑥ *(Bolzano Weierstrass Theorem) A Bounded sequence must have a convergent subsequence.

Properties

If a_n and b_n are convergent sequences and c is a constant, then:

$$\text{① } \lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$$

$$\text{② } \lim_{n \rightarrow \infty} (a_n - b_n) = \lim_{n \rightarrow \infty} a_n - \lim_{n \rightarrow \infty} b_n$$

$$\text{③ } \lim_{n \rightarrow \infty} ca_n = c \lim_{n \rightarrow \infty} a_n$$

$$\text{④ } \lim_{n \rightarrow \infty} (a_n b_n) = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n$$

$$\text{⑤ } \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n}, (\lim_{n \rightarrow \infty} b_n \neq 0)$$

$$\text{⑥ } \lim_{n \rightarrow \infty} a_n^p = (\lim_{n \rightarrow \infty} a_n)^p, (p > 0, a_n > 0)$$

Definition

Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{\infty} a_i = a_1 + a_2 + \dots + a_n$$

If the sequence s_n is convergent and $\lim_{n \rightarrow \infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called convergent and we write:

$$a_1 + a_2 + \dots + a_n = s \text{ or } \sum_{i=1}^{\infty} a_i = s$$

The number s is called the sum of the series. If the sequence s_n is divergent, then the series is called divergent.

The geometric series:

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, |r| < 1$$

If $|r| \geq 1$, the geometric series is divergent.

Definition

P-series is defined as

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

If $p = 1$, then we also call it harmonic series:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

It is divergent.

Definition

$$s_N(x) = \frac{a_0}{2} + \sum_{n=1}^N (a_n \cos(\frac{2\pi n x}{P}) + b_n \sin(\frac{2\pi n x}{P}))$$

Requirement for Convergent Series

Suppose the series $\sum_{n=1}^{\infty} x_n$ is convergent, then the sequence x_n is an infinitesimal, which means

$$\lim_{n \rightarrow \infty} x_n = 0$$

This can be used to test if a series is divergent.

Linearity for Convergent Series

Suppose $\sum_{n=1}^{\infty} a_n = A$, $\sum_{n=1}^{\infty} b_n = B$, and α, β are two constants, then

$$\sum_{n=1}^{\infty} (\alpha a_n + \beta b_n) = \alpha A + \beta B$$

- ① Divergence Test Theorem (Requirement for Convergent Series)
- ② Integral Test
- ③ Comparison Test
- ④ Cauchy Test (Root Test)
- ⑤ d' Alembert Test (Ratio Test)
- ⑥ Leibniz Test
- ⑦ Absolute Convergence Test

Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent **if and only if** the improper integral is convergent. In other words:

- (i) If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
- (ii) If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Integral Test: Important Conclusions for p-series

For what values of p is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

SOLUTION:

(i) If $p < 0$, then $\lim_{n \rightarrow \infty} \left(\frac{1}{n^p} \right) = \infty$

(ii) If $p = 0$ $\lim_{n \rightarrow \infty} \left(\frac{1}{n^p} \right) = 1$

In either case $\lim_{n \rightarrow \infty} \left(\frac{1}{n^p} \right) \neq 0$

(iii) $p > 0$, $f(x) = \frac{1}{x^p}$ is clearly continuous, positive, and decreasing on $[1, \infty]$. And we have: $\int_1^{\infty} \frac{1}{x^p} dx$ converges if $p > 1$ and diverges if $p \leq 1$

Conclusion

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n , then $\sum a_n$ is also convergent. (ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n , then $\sum a_n$ is also divergent. In using the Comparison Test we must, of course, have some known series $\sum b_n$ for the purpose of comparison. Most of the time we use one of these series:

- A p-series $[\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1]$ - A geometric series $\sum ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1]$

Theorem

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Remark: Usually, this is more convenient to use.

- (i) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$ then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
- (ii) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- (iii) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

- (i) If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
- (ii) If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- (iii) If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L = 1$, the Root Test is inconclusive.

Definition

If the series satisfies

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} (-1)^{n+1} u_n$$

Then we call it an alternating series.

Further, if the series

$$\sum_{n=1}^{\infty} (-1)^{u+1} u_n$$

satisfies: (i) $u_{n+1} \leq u_n$ for all n (ii) $\lim_{n \rightarrow \infty} u_n = 0$ Then the series is convergent.
We call it Leibniz series.

Conclusion

The convergence and divergence property of a series has nothing to do with the first N terms, where N is a finite number.

So, we can write:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^N a_n + \sum_{n=N+1}^{\infty}$$

Then, if the series

$$\sum_{n=N+1}^{\infty} a_n$$

satisfies the conditions of Leibniz Series, we can still conclude that the series is convergent.

Definition

Suppose that $\sum_{n=1}^{\infty} x_n$ is a convergent series. Then if $\sum_{n=1}^{\infty} |x_n|$ is convergent, $\sum_{n=1}^{\infty} x_n$ is absolutely convergent. Else $\sum_{n=1}^{\infty} x_n$ is a conditionally convergent.

Theorem

If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Methods

The convergence and divergence property of $\sum_{n=1}^{\infty} |x_n|$ can be determined by the criterion mentioned before.

Typically, if $\sum_{n=1}^{\infty} |x_n|$ diverges, $\sum_{n=1}^{\infty} x_n$ does not necessarily diverges.

However, if the divergence property is determined by Ratio Test or Root Test, then the series $\sum_{n=1}^{\infty} x_n$ also diverges. That's because these two criterion are based on the fact that the sequence is not approaches 0 ($x \rightarrow \infty$).

For each series $\sum_{n=0}^{\infty} a_n$, we can form the sequence of partial sums

$$A_n = \sum_{k=0}^n a_k$$

and

$$S_n = \frac{A_{n+1}A_{(n-1)} - A_n^2}{A_{n+1} + A_{n-1} - 2A_n}$$

This new sequence, called the Shanks transformation of the series, will usually converge faster than the original series. It is denoted by $S(A_n)$, and works particular well on alternating series.

Now let's expand the concept of series to functions.

Series with Function Terms

Suppose $u_n(x)$ is a function sequence with common domain E , then the sum of these infinite numbers of function terms

$$u_1(x) + u_2(x) + \cdot + \cdot + \cdot + u_n(x) + \cdot + \cdot + \cdot$$

is called function series, denoted as

$$\sum_{n=1}^{\infty} u_n(x)$$

Different from series of number terms, function series has the concept of convergence point and convergence domain.

Convergence Point

For a fixed $x_0 \in E$, if the series

$$\sum_{n=1}^{\infty} u_n(x_0)$$

is convergent, then we say that the function series

$$\sum_{n=1}^{\infty} u_n(x)$$

is convergent at x_0 .

Convergence Domain

The set that includes all the convergence point of the given function series is called the convergence domain.

Power Series is a special kind of function series.

Definition

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + \cdots$$

This kind of function series is called power series.

Cauchy-Hadamard Theorem

The power series

$$\sum_{n=0}^{\infty} a_n x^n$$

is absolutely convergent when $|x| < R$, and it is divergent when $|x| > R (R > 0)$. R is called the radius of convergence.

Note: at the endpoints $x = \pm R$, the convergence and divergence property of the function series should be judged by other methods.

Cauchy-Hadamard Theorem: for General Cases

The power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is absolutely convergent when $|x - x_0| < R$, and it is divergent when $|x - x_0| > R (R > 0)$.

Cauchy Test

For the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

If

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = A$$

Then the radius of convergence of this power series is $\frac{1}{A}$. Specially, If $A = 0$, then $R = +\infty$; if $A = +\infty$, then $R = 0$.

d' Alembert Test

For the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

If

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = A$$

Then the radius of convergence of this power series is $\frac{1}{A}$. Specially, If $A = 0$, then $R = +\infty$; if $A = +\infty$, then $R = 0$.

Integrals Term by Term

We can take the integrals of a power series term by term, if the interval lies in its domain of convergence.

That means, if $a, b \in D$ (D is the domain of convergence), then

$$\int_a^b \sum_{n=0}^{\infty} a_n x^n dx = \sum_{n=0}^{\infty} \int_a^b a_n x^n dx$$

If we take $a = 0$ and $b = x$, then

$$\int_0^x \sum_{n=0}^{\infty} a_n x^n dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

Derivatives Term by Term

Suppose the power series $\sum_{n=0}^{\infty} a_n x^n$ has the radius of convergence R. Then we can take the derivatives term by term on $(-R, R)$.

$$\frac{d}{dx} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{d}{dx} a_n x^n = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$\frac{d}{dx} \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} \frac{d}{dx} a_n (x - x_0)^n = \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1}$$

Shift the Index of Summation

We can shift the " starting point" of summation. General Case:

$$\sum_{n=m}^{\infty} a_n(x - x_0)^n = \sum_{n=m+k}^{\infty} a_{n-k}(x - x_0)^{n-k}$$

$$\sum_{n=m}^{\infty} a_n(x - x_0)^n = \sum_{n=m-k}^{\infty} a_{n+k}(x - x_0)^{n+k}$$

Taylor Expansion of Elementary Functions

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots, x \in R$$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots, x \in (-1, 1]$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots, x \in R$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^{2n}}{(2n)!} x^{2n} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots, x \in R$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{2n-1} x^{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots, x \in [-1, 1]$$

Taylor Expansion of Elementary Functions

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \frac{(\alpha(\alpha-1)\dots(\alpha-n+1))}{n!} x^n$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots, x \in (-1, 1)$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - \dots, x \in (-1, 1)$$

Sequences and Series

Let $a_n = \frac{2n}{3n+1}$

1. Determine whether a_n is convergent.
2. Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent.

Convergence and Divergence

Find the values of x for which the series converges. Find the sum of the series for those values of x .

1.

$$\sum_{n=0}^{\infty} (-4)^n (x - 5)^n$$

2.

$$\sum_{n=0}^{\infty} \frac{\sin^n x}{3^n}$$

Determine whether the series is convergent or divergent

1.

$$\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}$$

2.

$$\sum_{n=1}^{\infty} \frac{n}{n^4 + 1}$$

Determine whether the series is convergent or divergent

1.

$$\sum_{n=1}^{\infty} \frac{1}{(n^2 + 2n + 2)^2}$$

2.

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

Determine whether the series is convergent or divergent

1.

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

2.

$$\sum_{n=1}^{\infty} \left(\frac{-2n}{n+1} \right)^{5n}$$

1. Determine the Power Series Expansion of $f(x) = \frac{1}{3+5x-2x^2}$ at $x = 0$.
2. Determine the Power Series Expansion of $f(x) = \ln\left(\frac{\sin x}{x}\right)$ at $x = 0$.

9. Differential Equation

Definition

Equations representing the relationship between some unknown functions, the derivative of those functions and the independent variable.

Order: The order of the highest derivative of the unknown function is called the order of the differential equation.

Linear: The highest degree of the unknown function and its derivatives of any order is 1. Generally, n^{th} order differential equations have the form of $F(x, y, y', \dots, y^{(n)}) = 0$

In Vv156, we only need to solve some special types of ODEs.

Type 1: $y' + p(x)y = q(x)$

Step1: Find the general solution to the corresponding homogeneous ode $y' + p(x)y = 0$

$$\frac{dy}{dx} = -p(x)y$$

$$\frac{dy}{y} = -p(x)dx$$

$$\ln|y| = - \int p(x)dx + c_1$$

$$y_g = Ce^{-\int p(x)dx} (C = e^{\pm c_1})$$

Step2: Find one special solution to $y' + p(x)y = q(x)$

Variation of constants:

$$Ce^{-\int p(x)dx} \rightarrow C(x)e^{-\int p(x)dx}$$

$$y' = C'(x)e^{-\int p(x)dx} - C(x)p(x)e^{-\int p(x)dx}$$

$$C'(x)e^{-\int p(x)dx} - C(x)p(x)e^{-\int p(x)dx} + C(x)p(x)e^{\int p(x)dx} = q(x)$$

$$C'(x)e^{-\int p(x)dx} = q(x)$$

$$C'(x) = q(x)e^{\int p(x)dx}$$

$$C(x) = \int q(x)e^{\int p(x)dx} dx + C_2$$

Since we only need to find one special solution, we assume $C_2 = 0$

$$y_s = e^{-\int p(x)dx} \cdot \int q(x)e^{\int p(x)dx} dx$$

Step 3: Combine y_g and y_s together

$$y = Ce^{-\int p(x)dx} + e^{-\int p(x)dx} \cdot \int q(x)e^{p(x)dx} dx$$

$$y' + p(x)y = q(x)y^n$$

When $n \neq 0, 1$ the equation is not linear.

However, we can do some transformation:

① $y^{-n} \cdot y' + p(x) \cdot y^{1-n} = q(x)$

② $z = y^{1-n}$

Then, $\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$

Thus, $z' + (1-n)p(x)z = (1-n)q(x)$

which is in the form of $y' + p(x)y = q(x)$

Type 2: $y'' + py' + qy = f(x)$

Step 1: Find the general solution to the corresponding homogeneous ode
 $y'' + py' + qy = 0$ Solve the characteristic equation: $\lambda^2 + p\lambda + q = 0$

$$\begin{cases} \text{Two different real roots } \lambda_1, \lambda_2 \quad y_g = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \\ \text{Two equal real roots } \lambda \quad y_g = C_1 e^{\lambda x} + C_2 x e^{\lambda x} \\ \text{Two different complex roots } \alpha \pm \beta i \quad y_g = C_1 e^{\alpha x} \sin \beta x + C_2 e^{\alpha x} \cos \beta x \end{cases}$$

Step 2: Find one special solution to $y'' + py' + qy = f(x)$

In Vv156, we only have two types of $f(x)$:

① $f(x) = e^{\lambda x} P_m(x)$

② $f(x) = e^{\lambda x} [P_n(x)\cos(\omega x) + P_l(x)\sin(\omega x)]$

For both 1, 2, we first need to check whether $\lambda(\pm\omega i)$ is the root of the characteristic equation $\lambda^2 + p\lambda + q = 0$ or not

Type 2: $y'' + py' + qy = f(x)$

$f(x) = e^{\lambda x} P_m(x)$:

1	λ is one of the different real roots of $\lambda^2 + p\lambda + q = 0$	k
		1
2	λ is the identical real roots of $\lambda^2 + p\lambda + q = 0$	2
0	λ is not the real root of $\lambda^2 + p\lambda + q = 0$	0

$f(x) = e^{\lambda x}[P_n(x)\cos(\omega x) + P_I(x)\sin(\omega x)]$:

2	$\lambda \pm \omega i$ are the complex roots of $\lambda^2 + p\lambda + q = 0$	k
		1
0	$\lambda \pm \omega i$ are not the complex roots of $\lambda^2 + p\lambda + q = 0$	0

Apply undetermined coefficient method:

- For 1: $y_s = x^k e^{\lambda x} \cdot Q_m(x)$
- For 2: $y_s = x^k e^{\lambda x} [Q_m(x) \cos(\omega x) + R_m(x) \sin(\omega x)]$ ($m = \text{MAX}\{n, l\}$)

Q_m and R_m are another polynomial of degree m

→ Calculate y'_s and y''_s to solve Q_m and R_m

Type 2: $y'' + py' + qy = f(x)$

Step 3: Combine y_g and y_s together

If $f(x) = f_1(x) + \dots + f_n(x)$, and $f_i(x)$ is the form of 1, 2, we can calculate y_{si} separately.

$$\textcircled{1} \quad y' = \frac{y}{2x} + \frac{x^2}{2y}$$

$$\textcircled{2} \quad y'' + 6y' + 10y = e^{-3x} \sin x$$

Solution 1:

$$\textcircled{1} \quad y' + \left(-\frac{1}{2x}\right)y = 0.5x^2y^{-1}$$

$$yy' + \left(-\frac{1}{2x}\right)y^2 = 0.5x^2$$

$$\textcircled{2} \quad \text{Let } z = y^2 \rightarrow \frac{dz}{dx} = 2y \frac{dy}{dx}$$

$$z' - \frac{z}{x} = x^2$$

$$\text{Now } p(x) = -\frac{1}{x}, q(x) = x^2$$

$$\textcircled{3} \quad \text{Apply } y = Ce^{-\int p(x)dx} + e^{-\int p(x)dx} \cdot \int q(x)e^{p(x)dx} dx$$

$$\text{We can get } z = Cx + \frac{1}{2}x^3$$

$$\text{Thus, } y = \pm \sqrt{Cx + \frac{1}{2}x^3}$$

Solution 2:

① $\lambda^2 + 6\lambda + 10 = 0 \quad \lambda = -3 \pm i$

Thus, the general solution is $y_g = e^{-3x}(C_1 \cos x + C_2 \sin x)$

② Observe $e^{-3x} \sin x \rightarrow \lambda = -3, \omega = 1$

Since $-3 \pm i$ are complex roots of $\lambda^2 + 6\lambda + 10 = 0$, there is $k = 1$

So $y_s = e^{-3x}x(a \cos x + b \sin x)$

$$y'_s = e^{-3x}x[(b - 3a)\cos x + (-3b - a)\sin x] + e^{-3x}(a \cos x + b \sin x)$$

$$y''_s = e^{-3x}x[(8a - 6b)\cos x + (6a + 8b)\sin x] + e^{-3x}[(2b - 6a)\cos x - (2a + 6b)\sin x]$$

Since $y'' + 6y' + 10y = e^{-3x} \sin x$

Check the coefficient, $a = -0.5, b = 0$

$$y_s = e^{-3x}x(-0.5 \cos x)$$

③ Combine y_g and y_s

$$y = y_g + y_s = C_1 e^{-3x} \cos x + C_2 e^{-3x} \sin x - 0.5 x e^{-3x} \cos x$$

$$\log N - \log_{10} N = f(kx+b), \quad a+b, b=-1$$

$$\varphi = a^i N (F+H) = F+H = f+g, \quad i^2 = i^2 \cdot i^2 = 1$$

$$\frac{\varphi}{2} N = 10^i \cdot x \quad (kF) = kF, \quad x^3 + px + q = 0 \quad i^2 = i^2 \cdot i^2 = 1$$

$$\cos \varphi + \cos 2\varphi + \cos 3\varphi + \dots + \cos n\varphi = \operatorname{ch}(x+y) = \operatorname{ch}x \operatorname{ch}y + \operatorname{sh}x \operatorname{sh}y$$

$$= \frac{\cos \frac{\varphi}{2} - \cos \frac{(2n+1)\varphi}{2}}{2 \sin \frac{\varphi}{2}} \quad \operatorname{sh}(x+y) = \operatorname{sh}x \operatorname{ch}y + \operatorname{ch}x \operatorname{sh}y \\ (a+b)^2 = a^2 + 2ab + b^2 \quad \operatorname{ch}2x = \operatorname{ch}^2 x + \operatorname{sh}^2 y$$

$$b = \log_a N \quad a^i N$$

$$b_n = b \cdot q^{n-1} \quad S_n = \frac{b}{1-q}$$

$$r = \frac{e^x - e^{-x}}{2} \quad b_n = \sqrt{b_{n-k} b_{n+k}}$$

$$h \quad S_n = \frac{b(1-q^n)}{1-q} = \frac{b - b_n q}{1-q}$$

$$+ h^2 - \frac{1}{2}gt^2 = gth + \frac{1}{2}gh^2$$

$$\frac{dt^2}{t^2} = gt \quad v(t) = gt$$

$$\frac{s(t+h) - s(t)}{h} (cf) = cf$$

$$e^{ix} = \cos x + i \sin x$$

$$(\cos \varphi + i \sin \varphi)^n = \cos n\varphi + i \sin n\varphi$$

$$a = r \cos \varphi \quad b = r \sin \varphi$$

$$a_n = \frac{a_{n-1} + a_{n-2}}{2}, \quad n > k$$

$$s(t+h) - s(t) = \frac{v(t+h) - v(t)}{h} h$$

$$v(t) = \lim_{h \rightarrow 0} \frac{v(t+h) - v(t)}{h}$$

$$v(t+h) - v(t) = g(t+h) - gt$$

$$\arccos x = \frac{\pi}{2} - \arcsin x \quad y^2$$

$$\frac{1}{a^n} = a^{-n} \quad f(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\sin \varphi + \sin 2\varphi + \sin 3\varphi + \dots + \sin n\varphi$$

$$\operatorname{sh}2x = 2\operatorname{sh}x \operatorname{ch}x = \frac{\sin \frac{(2n+1)\varphi}{2}}{2 \sin \frac{\varphi}{2}}$$

$$\operatorname{ch}2x = \operatorname{ch}^2 x + \operatorname{sh}^2 y$$

$$\operatorname{ch}x = \frac{e^x + e^{-x}}{2}$$

$$\operatorname{sh}x = \frac{e^x - e^{-x}}{2}$$

$$\operatorname{th}x = \frac{\operatorname{sh}x}{\operatorname{ch}x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\operatorname{ch}x = \frac{e^x + e^{-x}}{2} \quad \operatorname{ch}^2 x - \operatorname{sh}^2 x = 1 \quad \operatorname{sh}:$$

$$= a + ab + 2ab + ab^2 + 2ab^2 + b^3 \quad a_n = a + (n-1)d \quad s(t+h) - s(t) = v(t)$$

$$S_n = \frac{2a + (n-1)d}{2} n \quad s(t+h) - s(t) = \frac{1}{2}g(t+h)^2 - \frac{1}{2}gt^2 = \frac{1}{2}g(t^2 + 2th)$$

$$a_n = \frac{a_{n-1} + a_{n-2}}{2} \quad \frac{s(t+h) - s(t)}{h} = v(t) \quad v(t) = \lim_{h \rightarrow 0} \frac{1}{2}g(t+h)^2 - \frac{1}{2}gt^2$$

$$s(t+h) - s(t) = \frac{v(t+h) - v(t)}{h} h$$

$$v(t) = \lim_{h \rightarrow 0} v(t+h) - v(t)$$

$$v(t+h) - v(t) = g(t+h) - gt$$

$$v(t) = \lim_{h \rightarrow 0} v(t+h) - v(t)$$

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$$v(t) = \lim_{h \rightarrow 0} v(t+h) - v(t)$$

Thank you!