VV156 Final RC Part III

Sequence and Series

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VV156 RC
FINAL

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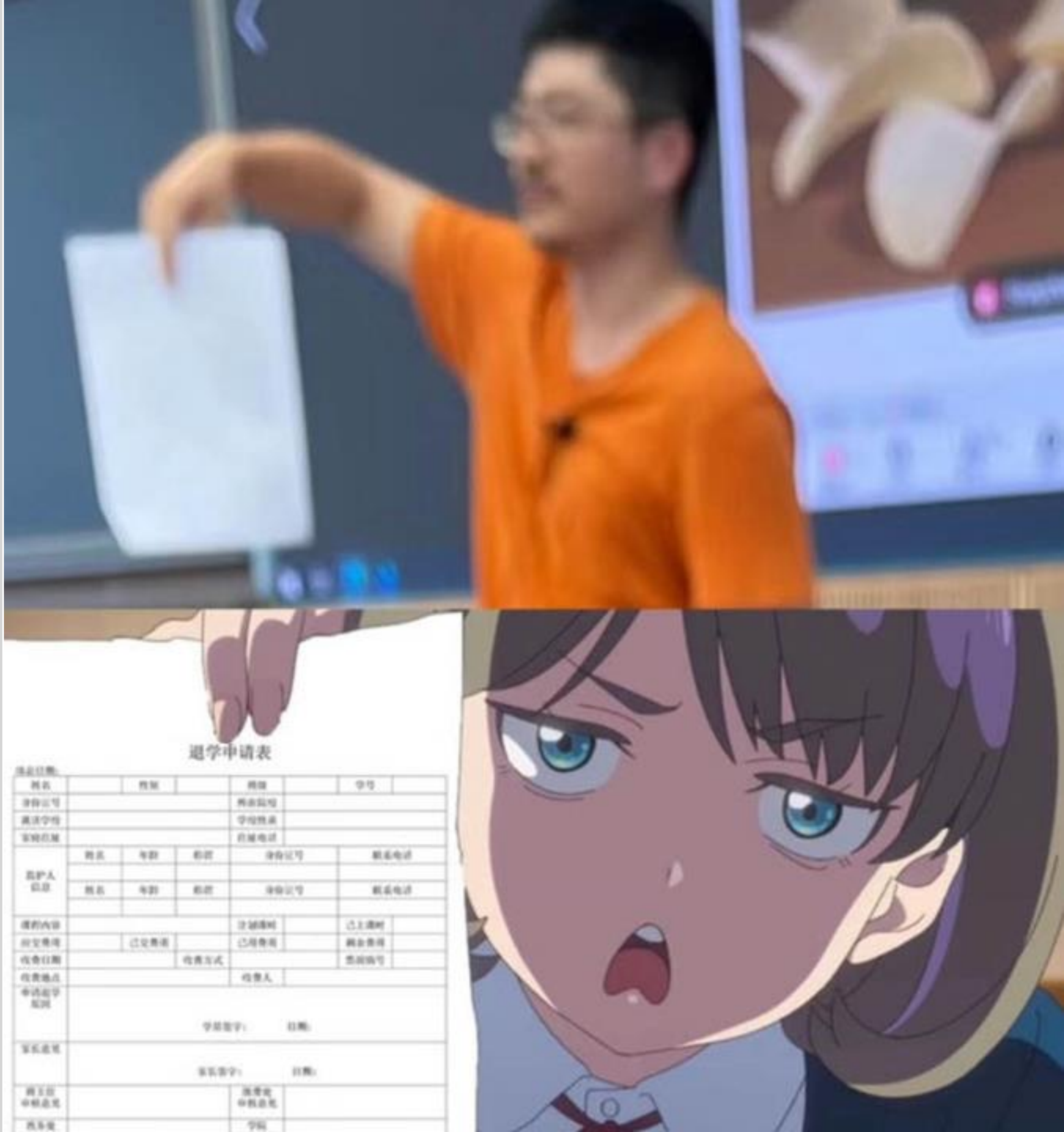

Sequences

Overview

A ***sequence*** is an infinite list of numbers listed in a definite order, e.g.,

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

Definition

A ***sequence*** is a **function** whose domain is the set of positive integers $\mathbb{N} \setminus \{0\}$, or natural numbers $\mathbb{N}$, e.g.,

$$\begin{aligned} a : \mathbb{N} &\to \mathbb{R} \\ n &\mapsto a_n \end{aligned}$$

Often denoted by $\{a_n\}$, $\{a_n\}_{n=1}^{\infty}$, (a_n), or $(a_n)_{n=1}^{\infty}$. Note that $a = (a_n) \in \mathbb{R}^{\mathbb{N}}$.

Convergence and Divergence

Definition

A sequence (a_n) has the ***limit*** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.
If $\lim_{n\to\infty} a_n$ exists, we say that the sequence ***converges*** (or is ***convergent***).
Otherwise, we say the sequence ***diverges*** (or is ***divergent***).

Sequences

Convergence and Divergence

Theorem

Let (a_n), (b_n), and (c_n) be real sequences such that $a_n \leq b_n \leq c_n$ for all $n > N$, where N is some positive integer. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L \Rightarrow \lim_{n\to\infty} b_n = L.$$

Theorem

Let f be a ***continuous*** *function defined for all $x \geq N$ for some positive integer N. Suppose that (a_n) is a real sequence such that $a_n = f(n)$ for $n \geq N$. then*

$$\lim_{x\to\infty} f(x) = L \Rightarrow \lim_{n\to\infty} a_n = L$$

Sequences

Basic Properties

Theorem

Let $\lim_{n\to\infty} a_n = A$ *and* $\lim_{n\to\infty} b_n = B$*, where* $A, B \in \mathbb{R}$*. Then*

1. $\lim_{n\to\infty} (a_n \pm b_n) = A \pm B$
2. $\lim_{n\to\infty} (a_n b_n) = AB$
3. $\lim_{n\to\infty} (ka_n) = kA$ *for any constant* k*.*
4. $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{A}{B}$ *if* $B \neq 0$*.*
5. $\lim_{n\to\infty} (a_n^p) = A^p$ *if* $p > 0$ *and* $a_n > 0$*.*

Series

Overview

A ***series*** is a sum of infinitely many numbers.

Definition

A ***series*** is an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots ,$$

where $a_1, a_2, \ldots$ are the terms. We say that a_n is the n-th or general term of the series.

Partial Sums

Definition

A ***partial sum*** of a series is the sum of all terms up to a_k,

$$s_k = a_0 + a_1 + \cdots + a_{k-1} + a_k = \sum_{n=0}^{k} a_n$$

Theorem/Definition

The series $\sum_{n=0}^{\infty} a_n = s$ iff the sequence of partial sums (s_k) converges to s.

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Series

Geometric Series

Definition

A series is ***geometric*** if each term (except for the first) can be obtained from the previous by multiplying it by a constant r, called the ***ratio*** of the series.

Theorem

A geometric series

$$\sum_{n=0}^{\infty} ar^n = ar^0 + ar^1 + ar^2 + ar^3 + \cdots$$

where $a \neq 0$ is a fixed constant, is

1. *convergent if $|r| < 1$, in which case $\displaystyle\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$,*
2. *divergent if $|r| \geq 1$.*

Series

Theorem

If $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

Corollary ("Divergence Test")

If $\lim_{n\to\infty} a_n \neq 0$ *or does not exist* (*i.e.*, $a_n \not\to 0$ *as* $n \to \infty$), *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Series

Basic Properties

Thoerem

If $\sum_{n\geq 0} a_n < \infty$ and $\sum_{n\geq 0} b_n < \infty$, then for any $c \in \mathbb{R}$

- $\sum_{n\geq 0} ca_n = c\sum_{n\geq 0} a_n < \infty$
- $\sum_{n\geq 0}(a_n + b_n) = \sum_{n\geq 0} a_n + \sum_{n\geq 0} b_n$

Theorem (Baby Rudin, Thoerem 3.51)

If the series $\sum a_n$, $\sum b_n$, $\sum c_n$ converge to A, B, C, and $c_n = a_0 b_n + \cdots + a_n b_0$, then $C = AB$. That is,

$$\left(\sum_{n\geq 0} a_n\right)\left(\sum_{n\geq 0} b_n\right) = \sum_{n\geq 0}\sum_{k=0}^{n} a_k b_{n-k}$$

provided that all series converge.

Test of Convergence (Very Important!!!)

Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

The converse of this statement fails: a series may diverge even though $a_n \to 0$ as $n \to \infty$.

Example

Harmonic series: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges while $\lim_{n\to\infty} \frac{1}{n} = 0$.

EXAMPLE 9 Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2 + 4}$ diverges.

SOLUTION

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{5n^2 + 4} = \lim_{n\to\infty} \frac{1}{5 + 4/n^2} = \frac{1}{5} \neq 0$$

So the series diverges by the Test for Divergence.

Test of Convergence (Very Important!!!)

Integral Test

Suppose f is a continuous, positive, decreasing function for all $x \geq N$, where N is a positive integer. If $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n < \infty \Leftrightarrow \int_N^{\infty} f(x)\, dx < \infty$$

Example Time

EXAMPLE 1 Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence.

SOLUTION The function $f(x) = 1/(x^2 + 1)$ is continuous, positive, and decreasing on $[1, \infty)$ so we use the Integral Test:

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \tan^{-1}x\Big]_1^t$$

$$= \lim_{t\to\infty}\left(\tan^{-1}t - \frac{\pi}{4}\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Thus $\int_1^{\infty} 1/(x^2+1)\,dx$ is a convergent integral and so, by the Integral Test, the series $\Sigma\, 1/(n^2+1)$ is convergent.

Example Time

V EXAMPLE 2 For what values of p is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

SOLUTION If $p < 0$, then $\lim_{n\to\infty} (1/n^p) = \infty$. If $p = 0$, then $\lim_{n\to\infty} (1/n^p) = 1$. In either case $\lim_{n\to\infty} (1/n^p) \neq 0$, so the given series diverges by the Test for Divergence (11.2.7).

If $p > 0$, then the function $f(x) = 1/x^p$ is clearly continuous, positive, and decreasing on $[1, \infty)$. We found in Chapter 7 [see (7.8.2)] that

$$\int_1^{\infty} \frac{1}{x^p}\, dx \quad \text{converges if } p > 1 \text{ and diverges if } p \leqslant 1$$

It follows from the Integral Test that the series $\Sigma\, 1/n^p$ converges if $p > 1$ and diverges if $0 < p \leqslant 1$. (For $p = 1$, this series is the harmonic series discussed in Example 8 in Section 11.2.)

The series in Example 2 is called the ***p*-series**. It is important in the rest of this chapter, so we summarize the results of Example 2 for future reference as follows.

1 The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

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Test of Convergence (Very Important!!!)

Limit Comparison Test

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. Let $\lim_{n\to\infty} \frac{a_n}{b_n} = L$. Then

- If $0 < L < \infty$, then both series converge or both diverge.
- If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
- If $L = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Otherwise, it's inconclusive, try other tests!

How to choose $\sum b_n$

To find b_n, simplify the ratio of the largest power of n in the numerator and the largest power of n in the denominator.

At most time, such test is used to test rational functions!

Example Time

EXAMPLE 4 Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ converges or diverges.

SOLUTION The dominant part of the numerator is $2n^2$ and the dominant part of the denominator is $\sqrt{n^5} = n^{5/2}$. This suggests taking

$$a_n = \frac{2n^2 + 3n}{\sqrt{5 + n^5}} \qquad b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}} \cdot \frac{n^{1/2}}{2} = \lim_{n\to\infty} \frac{2n^{5/2} + 3n^{3/2}}{2\sqrt{5 + n^5}}$$

$$= \lim_{n\to\infty} \frac{2 + \dfrac{3}{n}}{2\sqrt{\dfrac{5}{n^5} + 1}} = \frac{2 + 0}{2\sqrt{0 + 1}} = 1$$

Since $\Sigma\, b_n = 2\, \Sigma\, 1/n^{1/2}$ is divergent $\big(p$-series with $p = \frac{1}{2} < 1\big)$, the given series diverges by the Limit Comparison Test.

Notice that in testing many series we find a suitable comparison series $\Sigma\, b_n$ by keeping only the highest powers in the numerator and denominator.

Test of Convergence (Very Important!!!)

Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series and let $r = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$. Then

- $\sum a_n$ converges if $r < 1$.
- $\sum a_n$ diverges if $r > 1$ or $r = \infty$.
- inconclusive if $r = 1$.

Example Time

EXAMPLE 4 Test the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ for absolute convergence.

SOLUTION We use the Ratio Test with $a_n = (-1)^n n^3/3^n$:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\dfrac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\dfrac{(-1)^n n^3}{3^n}}\right| = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$$

$$= \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3}\left(1 + \frac{1}{n}\right)^3 \to \frac{1}{3} < 1$$

Thus, by the Ratio Test, the given series is absolutely convergent and therefore convergent.

Test of Convergence (Very Important!!!)

Root Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with nonnegative terms and let $r = \lim_{n\to\infty} \sqrt[n]{a_n}$. then

- $\sum a_n$ converges if $0 \leq r < 1$.
- $\sum a_n$ diverges if $r > 1$ or $r = \infty$.
- inconclusive if $r = 1$.

Example Time

V EXAMPLE 6 Test the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2} \right)^n$.

SOLUTION

$$a_n = \left(\frac{2n+3}{3n+2} \right)^n$$

$$\sqrt[n]{|a_n|} = \frac{2n+3}{3n+2} = \frac{2 + \dfrac{3}{n}}{3 + \dfrac{2}{n}} \to \frac{2}{3} < 1$$

Thus the given series converges by the Root Test.

Test of Convergence (Very Important!!!)

Direct Comparison Test

Suppose $\sum a_n$ and $\sum b_n$ are series such that $0 \le a_n \le b_n$ for all $n > N$, where $N \in \mathbb{N}$ is fixed, then

- $\sum b_n$ converges $\Rightarrow \sum a_n$ converges;
- $\sum a_n$ diverges $\Rightarrow \sum b_n$ diverges.

Example

- $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2}$. Note that $0 < \dfrac{\cos^2 n}{n^2} < \dfrac{1}{n^2}$ for all n, and $\sum \dfrac{1}{n^2} < \infty$. Hence the series is convergent.

Test of Convergence (Very Important!!!)

Consider the ***alternating harmonic series***

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \pm \cdots$$

In general, a series of the form $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ with $b_n > 0$ is called an ***alternating series***.

Alternating Series/Leibniz Test

Given $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ with $b_n > 0$. Then the series is convergent if

(1) $\lim_{n\to\infty} b_n = 0$, and

(2) $b_{n+1} \le b_n$ for all $n \ge N$ for some positive integer N.

Remark

Note that (1) and (2) are often summarized as $b_n \downarrow 0$ (or $b_n \searrow 0$) as $n \to \infty$.

Example Time

EXAMPLE 3 Test the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ for convergence or divergence.

SOLUTION The given series is alternating so we try to verify conditions (i) and (ii) of the Alternating Series Test.

Unlike the situation in Example 1, it is not obvious that the sequence given by $b_n = n^2/(n^3+1)$ is decreasing. However, if we consider the related function $f(x) = x^2/(x^3+1)$, we find that

$$f'(x) = \frac{x(2 - x^3)}{(x^3+1)^2}$$

Since we are considering only positive x, we see that $f'(x) < 0$ if $2 - x^3 < 0$, that is, $x > \sqrt[3]{2}$. Thus f is decreasing on the interval $(\sqrt[3]{2}, \infty)$. This means that $f(n+1) < f(n)$ and therefore $b_{n+1} < b_n$ when $n \geqslant 2$. (The inequality $b_2 < b_1$ can be verified directly but all that really matters is that the sequence $\{b_n\}$ is eventually decreasing.)

Condition (ii) is readily verified:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^3+1} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1 + \frac{1}{n^3}} = 0$$

Thus the given series is convergent by the Alternating Series Test.

Test of Convergence (Very Important!!!)

Definition

A series $\sum a_n$ is called

- absolutely convergent if $\sum |a_n|$ is convergent.
- conditionally convergent if $\sum a_n$ is convergent by not absolutely convergent.

Absolute Convergence Test

Theorem

If $\sum a_n$ is absolutely convergent, then it is convergent. That is,

$$\sum |a_n| < \infty \Rightarrow \sum a_n < \infty$$

(*Equivalently, if $\sum a_n$ diverges, then $\sum |a_n|$ diverges.*)

Strategy for testing of convergence

1. If the series is of the form $\sum 1/n^p$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leqslant 1$.

2. If the series has the form $\sum ar^{n-1}$ or $\sum ar^n$, it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geqslant 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.

3. If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. In particular, if a_n is a rational function or an algebraic function of n (involving roots of polynomials), then the series should be compared with a p-series. Notice that most of the series in Exercises 11.4 have this form. (The value of p should be chosen as in Section 11.4 by keeping only the highest powers of n in the numerator and denominator.) The comparison tests apply only to series with positive terms, but if $\sum a_n$ has some negative terms, then we can apply the Comparison Test to $\sum |a_n|$ and test for absolute convergence.

4. If you can see at a glance that $\lim_{n\to\infty} a_n \neq 0$, then the Test for Divergence should be used.

5. If the series is of the form $\sum (-1)^{n-1}b_n$ or $\sum (-1)^n b_n$, then the Alternating Series Test is an obvious possibility.

6. Series that involve factorials or other products (including a constant raised to the nth power) are often conveniently tested using the Ratio Test. Bear in mind that $|a_{n+1}/a_n| \to 1$ as $n \to \infty$ for all p-series and therefore all rational or algebraic functions of n. Thus the Ratio Test should not be used for such series.

7. If a_n is of the form $(b_n)^n$, then the Root Test may be useful.

8. If $a_n = f(n)$, where $\int_1^\infty f(x)\,dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

Strategy for testing of convergence

If you find the strategy listed in calculus is too long and you don't want to read it, you can refer to this

PROBLEM-SOLVING STRATEGY: CHOOSING A CONVERGENCE TEST FOR A SERIES

Consider a series $\sum_{n=1}^{\infty} a_n$. In the steps below, we outline a strategy for determining whether the series converges.

1. Is $\sum_{n=1}^{\infty} a_n$ a familiar series? For example, is it the harmonic series (which diverges) or the alternating harmonic series (which converges)? Is it a $p-\text{series}$ or geometric series? If so, check the power p or the ratio r to determine if the series converges.
2. Is it an alternating series? Are we interested in absolute convergence or just convergence? If we are just interested in whether the series converges, apply the alternating series test. If we are interested in absolute convergence, proceed to step 3, considering the series of absolute values $\sum_{n=1}^{\infty} |a_n|$.
3. Is the series similar to a $p-\text{series}$ or geometric series? If so, try the comparison test or limit comparison test.
4. Do the terms in the series contain a factorial or power? If the terms are powers such that $a_n = b_n^n$, try the root test first. Otherwise, try the ratio test first.
5. Use the divergence test. If this test does not provide any information, try the integral test.

Power Series

Definition

Let x be a variable and $a \in \mathbb{R}$ some constant. Then the series

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

is called a ***power series centered at*** a.

In particular, $\sum_{n=0}^{\infty} c_n x^n$ is a ***power series centered at*** 0.

Power Series, IOC & ROC (Very Important!!!)

Convergence of Power Series

Theorem

For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, *there are only three possibilites:*

(i) *The series **converges** only when* $x = a$;

(ii) *The series **converges** for all* x;

(iii) *There is a positive number* R *such that the series **converges** if* $|x-a| < R$, *and **diverges** if* $|x-a| > R$.

The number R in (iii) is called the ***radius of convergence (ROC)*** of the power series. By convention, we have $R = 0$ in (i) and $R = \infty$ in (ii).

Definition

The ***interval of convergence (IOC)*** of a power series is the interval that consists of all values of x for which the series converges. We have in

(i) IOC $= [a, a]$ or $\{a\}$;

(ii) IOC $= (-\infty, \infty)$;

(iii) Four possibilities, IOC $=$

- $(a-R, a+R)$
- $(a-R, a+R]$
- $[a-R, a+R)$
- $[a-R, a+R]$

Check the boundary points separately!

Example Time

V EXAMPLE 5 Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

SOLUTION If $a_n = n(x+2)^n/3^{n+1}$, then

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+2)^n} \right|$$

$$= \left(1 + \frac{1}{n}\right) \frac{|x+2|}{3} \to \frac{|x+2|}{3} \qquad \text{as } n \to \infty$$

Using the Ratio Test, we see that the series converges if $|x+2|/3 < 1$ and it diverges if $|x+2|/3 > 1$. So it converges if $|x+2| < 3$ and diverges if $|x+2| > 3$. Thus the radius of convergence is $R = 3$.

The inequality $|x+2| < 3$ can be written as $-5 < x < 1$, so we test the series at the endpoints -5 and 1. When $x = -5$, the series is

$$\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n n$$

which diverges by the Test for Divergence [$(-1)^n n$ doesn't converge to 0]. When $x = 1$, the series is

$$\sum_{n=0}^{\infty} \frac{n(3)^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} n$$

which also diverges by the Test for Divergence. Thus the series converges only when $-5 < x < 1$, so the interval of convergence is $(-5, 1)$.

Power Series

Theorem

A real power series may be integrated or differentiated any number of times within the interval of convergence. In particular, a function represented by a power series has derivatives of all orders. Specifically, If the power series $\sum c_n(x-a)^n$ *has radius of convergence* $R > 0$,

$$\text{(i)}\quad \frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}[c_n(x-a)^n] = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

$$\text{(ii)}\quad \int \left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \int c_n(x-a)^n = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The ***radii of convergence*** *of the power series in Equations* (i) *and* (ii) *are both* R.

Remark

The **radius of convergence** remains the same when a power series is differentiated or integrated, this does not mean that the **interval of convergence**

Taylor Series

Letting $n \to \infty$ yields a power series called a ***Taylor series***:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

We say that this is the ***series of*** $f(x)$ if it converges to $f(x)$.
If $a = 0$, then a Taylor series is called a ***Maclaurin series***.

If you find it's hard for you to understand it, refer to
https://www.bilibili.com/video/BV1qW411N7FU/?p=11&share_source=copy_web&vd_source=3ec21da555ff4c3e17bc07dc4fde8604

Taylor Series

Important Maclaurin Series and Their Radii of Convergence (p.762)

$$\frac{1}{1-x} = \sum_{n\geq 0} x^n \qquad R = 1$$

$$e^x = \sum_{n\geq 0} \frac{x^n}{n!} \qquad R = \infty$$

$$\sin x = \sum_{n\geq 0} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad R = \infty$$

$$\cos x = \sum_{n\geq 0} (-1)^n \frac{x^{2n}}{(2n)!} \qquad R = \infty$$

$$\arctan x = \sum_{n\geq 0} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad R = 1$$

$$\log(1+x) = \sum_{n\geq 1} (-1)^{n-1} \frac{x^n}{n} \qquad R = 1$$

$$(1+x)^\alpha = \sum_{n\geq 0} \binom{\alpha}{n} x^n \qquad R = 1$$

Example Time

EXAMPLE 6 Find a power series representation for $\ln(1 + x)$ and its radius of convergence.

SOLUTION We notice that the derivative of this function is $1/(1 + x)$. From Equation 1 we have

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \cdots \qquad |x| < 1$$

Integrating both sides of this equation, we get

$$\begin{aligned}
\ln(1+x) &= \int \frac{1}{1+x}\,dx = \int (1 - x + x^2 - x^3 + \cdots)\,dx \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + C \\
&= \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n} + C \qquad |x| < 1
\end{aligned}$$

To determine the value of C we put $x = 0$ in this equation and obtain $\ln(1 + 0) = C$. Thus $C = 0$ and

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n} \qquad |x| < 1$$

The radius of convergence is the same as for the original series: $R = 1$.

If you find it hard to do such problems at first glance, you can try to find the derivate or integral of it.

Then you may find that you can use Taylor Series to expand the new function in series.

Finally, you can use the series you have got to get the series expression of the original function. (refer to slide30 and the example in the left)

Exercises!

Determine whether the series is convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$$

$$\sum_{k=1}^{\infty} \frac{\sqrt[3]{k} - 1}{k\left(\sqrt{k} + 1\right)}$$

$$\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$$

$$\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)$$

Solution

Let $f(x) = \dfrac{1}{x\sqrt{\ln x}}$. Then f is positive, continuous, and decreasing on $[2, \infty)$, so we can apply the Integral Test.

Since $\displaystyle\int \frac{1}{x\sqrt{\ln x}}\,dx \begin{bmatrix} u = \ln x, \\ du = dx/x \end{bmatrix} = \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{\ln x} + C$, we find

$\displaystyle\int_2^\infty \frac{dx}{x\sqrt{\ln x}} = \lim_{t\to\infty} \int_2^t \frac{dx}{x\sqrt{\ln x}} = \lim_{t\to\infty} \left[2\sqrt{\ln x}\right]_2^t = \lim_{t\to\infty} \left(2\sqrt{\ln t} - 2\sqrt{\ln 2}\right) = \infty$. Since the integral diverges, the given series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ diverges.

$\displaystyle a_k = \frac{\sqrt[3]{k} - 1}{k(\sqrt{k} + 1)} < \frac{\sqrt[3]{k}}{k(\sqrt{k} + 1)} < \frac{\sqrt[3]{k}}{k\sqrt{k}} = \frac{k^{1/3}}{k^{3/2}} = \frac{1}{k^{7/6}}$, so the series $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt[3]{k} - 1}{k(\sqrt{k} + 1)}$ converges by comparison with the convergent p-series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{7/6}}$ $\left(p = \frac{7}{6} > 1\right)$.

Solution

. Use the Ratio Test. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{e^{(n+1)^2}}\cdot\frac{e^{n^2}}{n!}\right| = \lim_{n\to\infty}\frac{(n+1)n!\cdot e^{n^2}}{e^{n^2+2n+1}n!} = \lim_{n\to\infty}\frac{n+1}{e^{2n+1}} = 0 < 1$, so $\sum_{n=1}^{\infty}\frac{n!}{e^{n^2}}$ converges.

Use the Limit Comparison Test with $a_n = \sqrt[n]{2} - 1$ and $b_n = 1/n$. Then

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{2^{1/n}-1}{1/n} = \lim_{x\to\infty}\frac{2^{1/x}-1}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{2^{1/x}\cdot\ln 2\cdot(-1/x^2)}{-1/x^2} = \lim_{x\to\infty}(2^{1/x}\cdot\ln 2) = 1\cdot\ln 2 = \ln 2 > 0.$$

So since $\sum_{n=1}^{\infty} b_n$ diverges (harmonic series), so does $\sum_{n=1}^{\infty}\left(\sqrt[n]{2}-1\right)$.

Alternate solution: $\sqrt[n]{2} - 1 = \dfrac{1}{2^{(n-1)/n} + 2^{(n-2)/n} + 2^{(n-3)/n} + \cdots + 2^{1/n} + 1}$ [rationalize the numerator] $\geq \dfrac{1}{2n}$,

and since $\sum_{n=1}^{\infty}\frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (harmonic series), so does $\sum_{n=1}^{\infty}\left(\sqrt[n]{2}-1\right)$ by the Comparison Test.

Exercises!

3–28 Find the radius of convergence and interval of convergence of the series.

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$$

$$\sum_{n=2}^{\infty} \frac{(x+2)^n}{2^n \ln n}$$

$$\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n \sqrt{n}}$$

$$\sum_{n=2}^{\infty} \frac{b^n}{\ln n}(x-a)^n, \quad b>0$$

Solution

If $a_n = \dfrac{(x-2)^n}{n^2+1}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(x-2)^{n+1}}{(n+1)^2+1}\cdot\dfrac{n^2+1}{(x-2)^n}\right| = |x-2|\lim\limits_{n\to\infty}\dfrac{n^2+1}{(n+1)^2+1} = |x-2|$. By the Ratio Test, the series $\sum\limits_{n=0}^{\infty}\dfrac{(x-2)^n}{n^2+1}$ converges when $|x-2|<1 \quad [R=1] \quad \Leftrightarrow \quad -1<x-2<1 \quad \Leftrightarrow \quad 1<x<3$. When $x=1$, the series $\sum\limits_{n=0}^{\infty}(-1)^n\dfrac{1}{n^2+1}$ converges by the Alternating Series Test; when $x=3$, the series $\sum\limits_{n=0}^{\infty}\dfrac{1}{n^2+1}$ converges by comparison with the p-series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ $[p=2>1]$. Thus, the interval of convergence is $I=[1,3]$.

If $a_n = \dfrac{(x+2)^n}{2^n\ln n}$, then $\lim\limits_{n\to\infty}\left|\dfrac{(x+2)^{n+1}}{2^{n+1}\ln(n+1)}\cdot\dfrac{2^n\ln n}{(x+2)^n}\right| = \lim\limits_{n\to\infty}\dfrac{\ln n}{\ln(n+1)}\cdot\dfrac{|x+2|}{2} = \dfrac{|x+2|}{2}$ since

$$\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)} = \lim_{x\to\infty}\frac{\ln x}{\ln(x+1)} \overset{\mathrm{H}}{=} \lim_{x\to\infty}\frac{1/x}{1/(x+1)} = \lim_{x\to\infty}\frac{x+1}{x} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right) = 1.$$

By the Ratio Test, the series $\sum\limits_{n=2}^{\infty}\dfrac{(x+2)^n}{2^n\ln n}$ converges when $\dfrac{|x+2|}{2}<1 \quad \Leftrightarrow \quad |x+2|<2 \quad [R=2] \quad \Leftrightarrow \quad -2<x+2<2 \quad \Leftrightarrow \quad -4<x<0$.

When $x=-4$, the series $\sum\limits_{n=2}^{\infty}\dfrac{(-1)^n}{\ln n}$ converges by the Alternating Series Test. When $x=0$, the series $\sum\limits_{n=2}^{\infty}\dfrac{1}{\ln n}$ diverges by the Limit Comparison Test with $b_n=\dfrac{1}{n}$ (or by comparison with the harmonic series). Thus, the interval of convergence is $[-4,0)$.

Solution

If $a_n = \dfrac{(2x-1)^n}{5^n\sqrt{n}}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2x-1)^{n+1}}{5^{n+1}\sqrt{n+1}}\cdot\frac{5^n\sqrt{n}}{(2x-1)^n}\right| = \lim_{n\to\infty}\frac{|2x-1|}{5}\sqrt{\frac{n}{n+1}} = \lim_{n\to\infty}\frac{|2x-1|}{5}\sqrt{\frac{1}{1+1/n}} = \frac{|2x-1|}{5}.$$

By the Ratio Test, the series $\sum_{n=1}^{\infty}\dfrac{(2x-1)^n}{5^n\sqrt{n}}$ converges when $\dfrac{|2x-1|}{5} < 1 \iff |2x-1| < 5 \iff \left|x-\frac{1}{2}\right| < \frac{5}{2} \iff$ $-\frac{5}{2} < x - \frac{1}{2} < \frac{5}{2} \iff -2 < x < 3$, so $R = \frac{5}{2}$. When $x = 3$, the series $\sum_{n=1}^{\infty}\dfrac{1}{\sqrt{n}}$ is a divergent p-series $\left(p = \frac{1}{2} \le 1\right)$.

When $x = -2$, the series $\sum_{n=1}^{\infty}\dfrac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test. Thus, the interval of convergence is $I = [-2, 3)$.

$a_n = \dfrac{b^n}{\ln n}(x-a)^n$, where $b > 0$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{b^{n+1}(x-a)^{n+1}}{\ln(n+1)}\cdot\frac{\ln n}{b^n(x-a)^n}\right| = \lim_{n\to\infty}\frac{\ln n}{\ln(n+1)}\cdot b\,|x-a| = b\,|x-a| \text{ since}$$

$$\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)} = \lim_{x\to\infty}\frac{\ln x}{\ln(x+1)} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{1/x}{1/(x+1)} = \lim_{x\to\infty}\frac{x+1}{x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{1}{1} = 1.$$ By the Ratio Test, the series $\sum_{n=2}^{\infty}\dfrac{b^n}{\ln n}(x-a)^n$ converges when $b\,|x-a| < 1 \iff |x-a| < \dfrac{1}{b} \iff -\dfrac{1}{b} < x-a < \dfrac{1}{b} \iff a-\dfrac{1}{b} < x < a+\dfrac{1}{b}$, so $R = \dfrac{1}{b}$. When $x = a + \dfrac{1}{b}$, the series $\sum_{n=2}^{\infty}\dfrac{1}{\ln n}$ diverges by comparison with the divergent p-series $\sum_{n=2}^{\infty}\dfrac{1}{n}$ since $\dfrac{1}{\ln n} > \dfrac{1}{n}$ for $n \ge 2$. When $x = a - \dfrac{1}{b}$, the series $\sum_{n=2}^{\infty}\dfrac{(-1)^n}{\ln n}$ converges by the Alternating Series Test. Thus, the interval of convergence is $I = \left[a-\dfrac{1}{b}, a+\dfrac{1}{b}\right)$.

Exercises!

Find the IOC, ROC of the series $\sum \frac{2(1+x)^k}{5^k log(2k+3)}$ (Final 23FA)

Solution

ex2 (i) the general term is $c_k = \dfrac{2}{5^k \log(2k+3)}$

$$R = \lim_{k\to\infty} \frac{|c_k|}{|c_{k+1}|} = \lim_{k\to\infty} \left| \frac{2}{5^k \log(2k+3)} \cdot \frac{5^{k+1}\log(2k+5)}{2} \right|$$

$$= 5 \lim_{k\to\infty} \frac{\log(2k+5)}{\log(2k+3)}$$

$$= 5 \lim_{k\to\infty} \frac{\frac{2}{2k+5}}{\frac{2}{2k+3}}$$

$$= 5$$

So the ROC is $R = 5$

Solution

(ii) Let $|x+1|<5$, so $-6<x<4$

when $x=-6$

$$\sum_{k\geq 0} \frac{2\,(-5)^k}{5^k \log(2k+3)} = \sum_{k\geq 0} \frac{2}{\log(2k+3)}(-1)^k$$

we see that $\frac{2}{\log(2k+3)} \downarrow 0$ as $k\to\infty$,

So the series converges by alternating series test

when $x=4$

$$\sum_{k\geq 0} \frac{2\,(+5)^k}{5^k \log(2k+3)} = \sum_{k\geq 0} \frac{2}{\log(2k+3)}$$

by limit comparison test w/ $\sum_{k\geq 1} \frac{1}{k}$

$$\lim_{k\to\infty} \frac{\frac{2}{\log(2k+3)}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{2k}{\log(2k+3)} = \infty$$

so the series diverges since $\sum_{k\geq 1} \frac{1}{k}$ diverges.

Hence the IOC is $[-6,4)$.

Exercises!

In last year, such problems haven't appeared in the exams, but this year Prof.Cai said that it might appear in the exam ,I'm not quite sure how it will be like in the exam.
These exercises are what I have found from calculus, you can refer to if you want to do some exercises.

21–24 Find a power series representation for f, and graph f and several partial sums $s_n(x)$ on the same screen. What happens as n increases?

21. $f(x) = \dfrac{x^2}{x^2 + 1}$

22. $f(x) = \ln(1 + x^4)$

23. $f(x) = \ln\left(\dfrac{1 + x}{1 - x}\right)$

24. $f(x) = \tan^{-1}(2x)$

25–28 Evaluate the indefinite integral as a power series. What is the radius of convergence?

25. $\displaystyle\int \frac{t}{1 - t^8}\,dt$

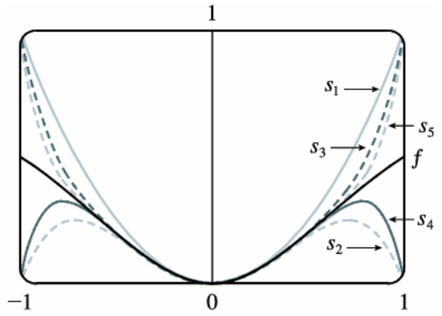
26. $\displaystyle\int \frac{t}{1 + t^3}\,dt$

27. $\displaystyle\int x^2 \ln(1 + x)\,dx$

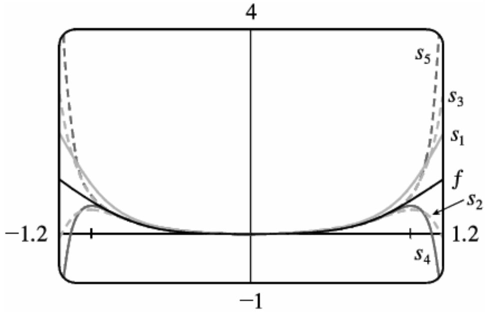
28. $\displaystyle\int \frac{\tan^{-1}x}{x}\,dx$

Solution

21. $f(x) = \dfrac{x^2}{x^2+1} = x^2\left(\dfrac{1}{1-(-x^2)}\right) = x^2 \sum\limits_{n=0}^{\infty} (-x^2)^n = \sum\limits_{n=0}^{\infty} (-1)^n\, x^{2n+2}$. This series converges when $\left|-x^2\right| < 1 \;\Leftrightarrow\; x^2 < 1 \;\Leftrightarrow\; |x| < 1$, so $R = 1$. The partial sums are $s_1 = x^2$, $s_2 = s_1 - x^4$, $s_3 = s_2 + x^6$, $s_4 = s_3 - x^8$, $s_5 = s_4 + x^{10}$, $\ldots$. Note that s_1 corresponds to the first term of the infinite sum, regardless of the value of the summation variable and the value of the exponent. As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-1, 1)$.



22. From Example 6, we have $\ln(1+x) = \sum\limits_{n=1}^{\infty} (-1)^{n-1}\dfrac{x^n}{n}$ with $|x| < 1$, so $f(x) = \ln(1+x^4) = \sum\limits_{n=1}^{\infty} (-1)^{n-1}\dfrac{x^{4n}}{n}$ with $\left|x^4\right| < 1 \;\Leftrightarrow\; |x| < 1$ $\;[R = 1]$. The partial sums are $s_1 = x^4$, $s_2 = s_1 - \frac{1}{2}x^8$, $s_3 = s_2 + \frac{1}{3}x^{12}$, $s_4 = s_3 - \frac{1}{4}x^{16}$, $s_5 = s_4 + \frac{1}{5}x^{20}$, $\ldots$. Note that s_1 corresponds to the first term of the infinite sum, regardless of the value of the summation variable and the value of the exponent. As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $[-1, 1]$. (When $x = \pm 1$, the series is the convergent alternating harmonic series.)



Solution

23. $f(x) = \ln\left(\dfrac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = \displaystyle\int \frac{dx}{1+x} + \int \frac{dx}{1-x} = \int \frac{dx}{1-(-x)} + \int \frac{dx}{1-x}$

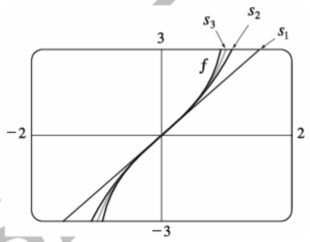
$$= \int \left[\sum_{n=0}^{\infty} (-1)^n x^n + \sum_{n=0}^{\infty} x^n\right] dx = \int \left[(1 - x + x^2 - x^3 + x^4 - \cdots) + (1 + x + x^2 + x^3 + x^4 + \cdots)\right] dx$$

$$= \int (2 + 2x^2 + 2x^4 + \cdots)\, dx = \int \sum_{n=0}^{\infty} 2x^{2n}\, dx = C + \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}$$

But $f(0) = \ln \frac{1}{1} = 0$, so $C = 0$ and we have $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}$ with $R = 1$. If $x = \pm 1$, then $f(x) = \pm 2 \displaystyle\sum_{n=0}^{\infty} \frac{1}{2n+1}$,

which both diverge by the Limit Comparison Test with $b_n = \dfrac{1}{n}$.

The partial sums are $s_1 = \dfrac{2x}{1}$, $s_2 = s_1 + \dfrac{2x^3}{3}$, $s_3 = s_2 + \dfrac{2x^5}{5}$, ...



As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-1, 1)$.

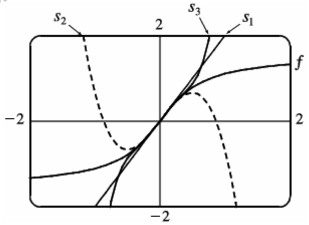
24. $f(x) = \tan^{-1}(2x) = 2\displaystyle\int \frac{dx}{1+4x^2} = 2\int \sum_{n=0}^{\infty} (-1)^n \left(4x^2\right)^n dx = 2\int \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n}\, dx$

$$= C + 2\sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1} \qquad [f(0) = \tan^{-1} 0 = 0, \text{ so } C = 0]$$

The series converges when $|4x^2| < 1 \;\Leftrightarrow\; |x| < \frac{1}{2}$, so $R = \frac{1}{2}$. If $x = \pm\frac{1}{2}$, then $f(x) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$ and

$f(x) = \displaystyle\sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{2n+1}$, respectively. Both series converge by the Alternating Series Test. The partial sums are

$s_1 = \dfrac{2x}{1}$, $s_2 = s_1 - \dfrac{2^3 x^3}{3}$, $s_3 = s_2 + \dfrac{2^5 x^5}{5}$, ...



As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

Solution

25. $\dfrac{t}{1-t^8} = t \cdot \dfrac{1}{1-t^8} = t\sum\limits_{n=0}^{\infty}(t^8)^n = \sum\limits_{n=0}^{\infty} t^{8n+1} \quad\Rightarrow\quad \displaystyle\int \frac{t}{1-t^8}\,dt = C + \sum_{n=0}^{\infty}\frac{t^{8n+2}}{8n+2}$. The series for $\dfrac{1}{1-t^8}$ converges when $|t^8| < 1 \quad\Leftrightarrow\quad |t| < 1$, so $R = 1$ for that series and also the series for $t/(1-t^8)$. By Theorem 2, the series for $\displaystyle\int \frac{t}{1-t^8}\,dt$ also has $R = 1$.

26. $\dfrac{t}{1+t^3} = t \cdot \dfrac{1}{1-(-t^3)} = t\sum\limits_{n=0}^{\infty}(-t^3)^n = \sum\limits_{n=0}^{\infty}(-1)^n t^{3n+1} \quad\Rightarrow\quad \displaystyle\int \frac{t}{1+t^3}\,dt = C + \sum_{n=0}^{\infty}(-1)^n\frac{t^{3n+2}}{3n+2}$. The series for $\dfrac{1}{1+t^3}$ converges when $|-t^3| < 1 \quad\Leftrightarrow\quad |t| < 1$, so $R = 1$ for that series and also for the series $\dfrac{t}{1+t^3}$. By Theorem 2, the series for $\displaystyle\int \frac{t}{1+t^3}\,dt$ also has $R = 1$.

27. From Example 6, $\ln(1+x) = \sum\limits_{n=1}^{\infty}(-1)^{n-1}\dfrac{x^n}{n}$ for $|x| < 1$, so $x^2\ln(1+x) = \sum\limits_{n=1}^{\infty}(-1)^{n-1}\dfrac{x^{n+2}}{n}$ and $\displaystyle\int x^2\ln(1+x)\,dx = C + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{n+3}}{n(n+3)}$. $R = 1$ for the series for $\ln(1+x)$, so $R = 1$ for the series representing $x^2\ln(1+x)$ as well. By Theorem 2, the series for $\displaystyle\int x^2\ln(1+x)\,dx$ also has $R = 1$.

28. From Example 7, $\tan^{-1}x = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{2n+1}$ for $|x| < 1$, so $\dfrac{\tan^{-1}x}{x} = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{2n}}{2n+1}$ and $\displaystyle\int \frac{\tan^{-1}x}{x}\,dx = C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)^2}$. $R = 1$ for the series for $\tan^{-1}x$, so $R = 1$ for the series representing $\dfrac{\tan^{-1}x}{x}$ as well. By Theorem 2, the series for $\displaystyle\int \frac{\tan^{-1}x}{x}\,dx$ also has $R = 1$.

Reference

- James Stewart Calculus Early Transcendentals 7th Edition
- VV156 24FA Lecture Slides, Runze Cai

THANK YOU AND GOOD LUCK！