UM-SJTU Joint Institute

VV156 Worksheet 2

Basic Knowledge for Integral in Runze's Class

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0.Introduction

In this worksheet, I will try to help you find some exercises which are more related to this course according to the exams in 23FA and 21FA to reduce your workload on finding exercises to practice.

1 The Fundamental Theorem of Calculus

Although it hasn't been tested directly in VV156 in previous time, it's still of great importance as crz likes to mention it in the lecture, and it has been tested in the midterm of VV256 this semester. These two easy exercises are just to help you test whether yourself understand this thm.

Definition

1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.
2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

1.1

$$\text{If } f(x) = \int_0^{\sin x} \sqrt{1+t^2}\,dt \text{ and } g(y) = \int_3^y f(x)\,dx,\ \text{find } g''\left(\frac{\pi}{6}\right).$$

1.2

Find the derivative of

$$F(x) = \int_{\sqrt{x}}^{2x} \arctan t\,dt$$

2 Basic Integral

Although they aren't so difficult, but you have to master them because it's a foundation for all the other problems.

1. $\int_0^{2\pi} (\sin x + \cos x)^{11}\, dx$

2. $\int \csc^2(x)\, \tan^{2024}(x)\, \mathrm{d}x$

3. Known that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$,find $\int\limits_{-\infty}^{\infty} e^{-\frac{(x-2024)^2}{4}}\, \mathrm{d}x$

4. $\int \frac{1}{x\, \log(x) + 2\, x}\, \mathrm{d}x$

3 Trigonometric Integral

1. $\int \frac{x}{\sqrt{3-2x-x^2}} \mathbf{d}x$

2. $\int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^5\sqrt{9x^2-1}}$

3. $\int \frac{x^2+1}{(x^2-2x+2)^2}\, dx$

4. $\int \frac{x^2}{(3+4x-4x^2)^{3/2}}\, dx$

4 Integral by Parts

It's very important, you have to be a master of this method!

1.$\int_0^{\pi} e^{\cos t} \sin 2t\, dt$

2.$\int x^3\sqrt{1+x^2}\,dx$

3.$\int \ln(x^2+2x+2)\,dx$

4.Known that$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$, find $\int_{-\infty}^{\infty} \frac{e^{-x^2}}{(x^2+\frac{1}{2})^2}dx$ (Final 23FA)

5 Rational Function

It's also very important, you have to be a master of this method!

1.$\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy$

2.$\int \frac{x^2-3x+7}{(x^2-4x+6)^2}\,dx$

3.$\int \frac{x^3+2x^2+3x-2}{(x^2+2x+2)^2}\,dx$

4.$\int \frac{13+41x+23x^2+19x^3+4x^4}{(-1+x)^3(2+2x+x^2)^2}dx$ (Final 23FA)

6 Solution

6.1 The Fundamental Theorem of Calculus

6.1.1

$$g(y) = \int_3^y f(x)\,dx \implies g'(y) = f(y).$$

Since

$$f(x) = \int_0^{\sin x} \sqrt{1+t^2}\,dt,$$

$$g''(y) = f'(y) = \sqrt{1+\sin^2 y} \cdot \cos y,$$

so

$$g''\left(\frac{\pi}{6}\right) = \sqrt{1+\sin^2\left(\frac{\pi}{6}\right)} \cdot \cos\left(\frac{\pi}{6}\right) = \sqrt{1+\left(\frac{1}{2}\right)^2} \cdot \frac{\sqrt{3}}{2} = \sqrt{\frac{5}{2}} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{15}}{4}.$$

6.1.2

$$F(x) = \int_{\sqrt{x}}^{2x} \arctan t\,dt = \int_0^{2x} \arctan t\,dt + \int_{\sqrt{x}}^{0} \arctan t\,dt = -\int_0^{\sqrt{x}} \arctan t\,dt + \int_0^{2x} \arctan t\,dt \implies$$

$$F'(x) = -\arctan\sqrt{x} \cdot \frac{d}{dx}(\sqrt{x}) + \arctan 2x \cdot \frac{d}{dx}(2x) = -\frac{1}{2\sqrt{x}} \arctan\sqrt{x} + 2\arctan 2x.$$

6.2

6.2.1

$\int_0^{2\pi} (\sin x + \cos x)^{11}\,dx = \int_0^{2\pi} (\frac{\sqrt{2}}{2}\sin(x+\frac{\pi}{4}))^{11}\,dx = 0$

6.2.2

Integral Calculator is all you need

6.2.3

Integral Calculator is all you need

6.2.4

Integral Calculator is all you need

6.3

SOLUTION We can transform the integrand into a function for which trigonometric substitution is appropriate by first completing the square under the root sign:

$$3-2x-x^2 = 3-(x^2+2x) = 3+1-(x^2+2x+1)$$
$$= 4-(x+1)^2$$

This suggests that we make the substitution $u = x+1$. Then $du = dx$ and $x = u-1$, so

$$\int \frac{x}{\sqrt{3-2x-x^2}}\,dx = \int \frac{u-1}{\sqrt{4-u^2}}\,du$$

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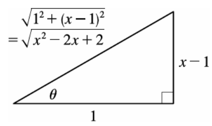
We now substitute $u = 2\sin\theta$, giving $du = 2\cos\theta\,d\theta$ and $\sqrt{4-u^2} = 2\cos\theta$, so

$$\begin{aligned}
\int \frac{x}{\sqrt{3-2x-x^2}}\,dx &= \int \frac{2\sin\theta-1}{2\cos\theta}\,2\cos\theta\,d\theta \\
&= \int (2\sin\theta-1)\,d\theta \\
&= -2\cos\theta-\theta+C \\
&= -\sqrt{4-u^2}-\sin^{-1}\left(\frac{u}{2}\right)+C \\
&= -\sqrt{3-2x-x^2}-\sin^{-1}\left(\frac{x+1}{2}\right)+C
\end{aligned}$$

16. Let $x = \frac{1}{3}\sec\theta$, so $dx = \frac{1}{3}\sec\theta\,\tan\theta\,d\theta$, $x = \sqrt{2}/3 \;\Rightarrow\; \theta = \frac{\pi}{4}$, $x = \frac{2}{3} \;\Rightarrow\; \theta = \frac{\pi}{3}$. Then

$$\begin{aligned}
\int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^5\sqrt{9x^2-1}} &= \int_{\pi/4}^{\pi/3} \frac{\frac{1}{3}\sec\theta\,\tan\theta\,d\theta}{\left(\frac{1}{3}\right)^5\sec^5\theta\,\tan\theta} = 3^4\int_{\pi/4}^{\pi/3}\cos^4\theta\,d\theta = 81\int_{\pi/4}^{\pi/3}\left[\tfrac{1}{2}(1+\cos 2\theta)\right]^2 d\theta \\
&= \tfrac{81}{4}\textstyle\int_{\pi/4}^{\pi/3}(1+2\cos 2\theta+\cos^2 2\theta)\,d\theta = \tfrac{81}{4}\int_{\pi/4}^{\pi/3}\left[1+2\cos 2\theta+\tfrac{1}{2}(1+\cos 4\theta)\right]d\theta \\
&= \tfrac{81}{4}\textstyle\int_{\pi/4}^{\pi/3}\left(\tfrac{3}{2}+2\cos 2\theta+\tfrac{1}{2}\cos 4\theta\right)d\theta = \tfrac{81}{4}\left[\tfrac{3}{2}\theta+\sin 2\theta+\tfrac{1}{8}\sin 4\theta\right]_{\pi/4}^{\pi/3} \\
&= \tfrac{81}{4}\left[\left(\tfrac{\pi}{2}+\tfrac{\sqrt{3}}{2}-\tfrac{\sqrt{3}}{16}\right)-\left(\tfrac{3\pi}{8}+1+0\right)\right] = \tfrac{81}{4}\left(\tfrac{\pi}{8}+\tfrac{7}{16}\sqrt{3}-1\right)
\end{aligned}$$

28. $x^2-2x+2 = (x^2-2x+1)+1 = (x-1)^2+1$. Let $x-1 = 1\tan\theta$, so $dx = \sec^2\theta\,d\theta$ and $\sqrt{x^2-2x+2} = \sec\theta$. Then

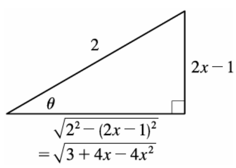


$$\begin{aligned}
\int \frac{x^2+1}{(x^2-2x+2)^2}\,dx &= \int \frac{(\tan\theta+1)^2+1}{\sec^4\theta}\sec^2\theta\,d\theta \\
&= \int \frac{\tan^2\theta+2\tan\theta+2}{\sec^2\theta}\,d\theta \\
&= \textstyle\int(\sin^2\theta+2\sin\theta\,\cos\theta+2\cos^2\theta)\,d\theta = \int(1+2\sin\theta\,\cos\theta+\cos^2\theta)\,d\theta \\
&= \textstyle\int\left[1+2\sin\theta\,\cos\theta+\tfrac{1}{2}(1+\cos 2\theta)\right]d\theta = \int\left(\tfrac{3}{2}+2\sin\theta\,\cos\theta+\tfrac{1}{2}\cos 2\theta\right)d\theta \\
&= \tfrac{3}{2}\theta+\sin^2\theta+\tfrac{1}{4}\sin 2\theta+C = \tfrac{3}{2}\theta+\sin^2\theta+\tfrac{1}{2}\sin\theta\,\cos\theta+C \\
&= \frac{3}{2}\tan^{-1}\left(\frac{x-1}{1}\right)+\frac{(x-1)^2}{x^2-2x+2}+\frac{1}{2}\,\frac{x-1}{\sqrt{x^2-2x+2}}\,\frac{1}{\sqrt{x^2-2x+2}}+C \\
&= \frac{3}{2}\tan^{-1}(x-1)+\frac{2(x^2-2x+1)+x-1}{2(x^2-2x+2)}+C = \frac{3}{2}\tan^{-1}(x-1)+\frac{2x^2-3x+1}{2(x^2-2x+2)}+C
\end{aligned}$$

$3+4x-4x^2 = -(4x^2-4x+1)+4 = 2^2-(2x-1)^2$.

Let $2x-1 = 2\sin\theta$, so $2\,dx = 2\cos\theta\,d\theta$ and $\sqrt{3+4x-4x^2} = 2\cos\theta$.

Then



$$\int \frac{x^2}{(3+4x-4x^2)^{3/2}}\,dx = \int \frac{\left[\frac{1}{2}(1+2\sin\theta)\right]^2}{(2\cos\theta)^3}\cos\theta\,d\theta$$

$$= \frac{1}{32}\int \frac{1+4\sin\theta+4\sin^2\theta}{\cos^2\theta}\,d\theta = \frac{1}{32}\int(\sec^2\theta+4\tan\theta\,\sec\theta+4\tan^2\theta)\,d\theta$$

$$= \tfrac{1}{32}\int[\sec^2\theta+4\tan\theta\,\sec\theta+4(\sec^2\theta-1)]\,d\theta$$

$$= \tfrac{1}{32}\int(5\sec^2\theta+4\tan\theta\,\sec\theta-4)\,d\theta = \tfrac{1}{32}(5\tan\theta+4\sec\theta-4\theta)+C$$

$$= \frac{1}{32}\left[5\cdot\frac{2x-1}{\sqrt{3+4x-4x^2}}+4\cdot\frac{2}{\sqrt{3+4x-4x^2}}-4\cdot\sin^{-1}\left(\frac{2x-1}{2}\right)\right]+C$$

$$= \frac{10x+3}{32\sqrt{3+4x-4x^2}}-\frac{1}{8}\sin^{-1}\left(\frac{2x-1}{2}\right)+C$$

. Let $x=\cos t$, so that $dx=-\sin t\,dt$. Thus,

$\int_0^\pi e^{\cos t}\sin 2t\,dt = \int_0^\pi e^{\cos t}(2\sin t\,\cos t)\,dt = \int_1^{-1} e^x\cdot 2x\,(-dx) = 2\int_{-1}^1 xe^x\,dx$. Now use parts with $u=x$, $dv=e^x\,dx$, $du=dx$, $v=e^x$ to get

$2\int_{-1}^1 xe^x\,dx = 2\left(\left[xe^x\right]_{-1}^1-\int_{-1}^1 e^x\,dx\right) = 2\left(e^1+e^{-1}-\left[e^x\right]_{-1}^1\right) = 2(e+e^{-1}-[e^1-e^{-1}]) = 2(2e^{-1}) = 4/e.$

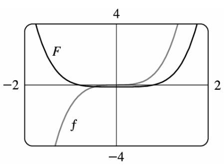
45. Let $u=\frac{1}{2}x^2$, $dv=2x\sqrt{1+x^2}\,dx \quad\Rightarrow\quad du=x\,dx$, $v=\frac{2}{3}(1+x^2)^{3/2}$.

Then

$$\int x^3\sqrt{1+x^2}\,dx = \tfrac{1}{2}x^2\left[\tfrac{2}{3}(1+x^2)^{3/2}\right]-\tfrac{2}{3}\int x(1+x^2)^{3/2}dx$$

$$= \tfrac{1}{3}x^2(1+x^2)^{3/2}-\tfrac{2}{3}\cdot\tfrac{2}{5}\cdot\tfrac{1}{2}(1+x^2)^{5/2}+C$$

$$= \tfrac{1}{3}x^2(1+x^2)^{3/2}-\tfrac{2}{15}(1+x^2)^{5/2}+C$$



We see from the graph that this is reasonable, since F increases where f is positive and F decreases where f is negative. Note also that f is an odd function and F is an even function.

Another method: Use substitution with $u=1+x^2$ to get $\frac{1}{5}(1+x^2)^{5/2}-\frac{1}{3}(1+x^2)^{3/2}+C$.

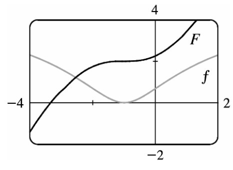
We first make the substitution $t=x+1$, so $\ln(x^2+2x+2)=\ln\left[(x+1)^2+1\right]=\ln(t^2+1)$. Then we use parts with $u=\ln(t^2+1)$, $dv=dt$:

$$\int\ln(t^2+1)\,dt = t\,\ln(t^2+1)-\int\frac{t(2t)\,dt}{t^2+1} = t\,\ln(t^2+1)-2\int\frac{t^2\,dt}{t^2+1} = t\,\ln(t^2+1)-2\int\left(1-\frac{1}{t^2+1}\right)dt$$

$$= t\,\ln(t^2+1)-2t+2\arctan t+C$$

$$= (x+1)\,\ln(x^2+2x+2)-2x+2\arctan(x+1)+K,\ \text{where } K=C-2$$

[Alternatively, we could have integrated by parts immediately with $u=\ln(x^2+2x+2)$.] Notice from the graph that $f=0$ where F has a horizontal tangent. Also, F is always increasing, and $f\geq 0$.



ex 5 Use integration by parts,

$$\int \frac{e^{-x^2}}{(x^2+\frac{1}{2})^2}\,dx = \frac{e^{-x^2}}{2x}\cdot\frac{-1}{x^2+\frac{1}{2}} - \int \frac{e^{-x^2}\,dx}{x^2}$$

$$= \frac{-e^{-x^2}}{2x(x^2+\frac{1}{2})} + \frac{e^{-x^2}}{x} + 2\int e^{-x^2}\,dx$$

$$= \frac{x e^{-x^2}}{x^2+\frac{1}{2}} + 2\int e^{-x^2}\,dx$$

Hence

$$\int_0^\infty \frac{e^{-x^2}\,dx}{(x^2+\frac{1}{2})^2} = \left.\frac{x e^{-x^2}}{x^2+\frac{1}{2}}\right|_0^\infty + 2\int_0^\infty e^{-x^2}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

17. $\dfrac{4y^2-7y-12}{y(y+2)(y-3)} = \dfrac{A}{y} + \dfrac{B}{y+2} + \dfrac{C}{y-3} \Rightarrow 4y^2-7y-12 = A(y+2)(y-3) + By(y-3) + Cy(y+2)$. Setting $y=0$ gives $-12=-6A$, so $A=2$. Setting $y=-2$ gives $18=10B$, so $B=\frac{9}{5}$. Setting $y=3$ gives $3=15C$, so $C=\frac{1}{5}$.

Now

$$\begin{aligned}\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy &= \int_1^2 \left(\frac{2}{y} + \frac{9/5}{y+2} + \frac{1/5}{y-3}\right) dy = \left[2\ln|y| + \tfrac{9}{5}\ln|y+2| + \tfrac{1}{5}\ln|y-3|\right]_1^2 \\ &= 2\ln 2 + \tfrac{9}{5}\ln 4 + \tfrac{1}{5}\ln 1 - 2\ln 1 - \tfrac{9}{5}\ln 3 - \tfrac{1}{5}\ln 2 \\ &= 2\ln 2 + \tfrac{18}{5}\ln 2 - \tfrac{1}{5}\ln 2 - \tfrac{9}{5}\ln 3 = \tfrac{27}{5}\ln 2 - \tfrac{9}{5}\ln 3 = \tfrac{9}{5}(3\ln 2 - \ln 3) = \tfrac{9}{5}\ln\tfrac{8}{3}\end{aligned}$$

37. $\dfrac{x^2-3x+7}{(x^2-4x+6)^2} = \dfrac{Ax+B}{x^2-4x+6} + \dfrac{Cx+D}{(x^2-4x+6)^2} \Rightarrow x^2-3x+7 = (Ax+B)(x^2-4x+6) + Cx + D \Rightarrow$ $x^2-3x+7 = Ax^3 + (-4A+B)x^2 + (6A-4B+C)x + (6B+D)$. So $A=0$, $-4A+B=1 \Rightarrow B=1$, $6A-4B+C=-3 \Rightarrow C=1$, $6B+D=7 \Rightarrow D=1$. Thus,

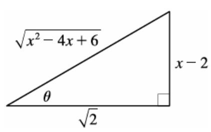
$$\begin{aligned}I &= \int \frac{x^2-3x+7}{(x^2-4x+6)^2}\,dx = \int\left(\frac{1}{x^2-4x+6} + \frac{x+1}{(x^2-4x+6)^2}\right) dx \\ &= \int \frac{1}{(x-2)^2+2}\,dx + \int \frac{x-2}{(x^2-4x+6)^2}\,dx + \int \frac{3}{(x^2-4x+6)^2}\,dx \\ &= I_1 + I_2 + I_3.\end{aligned}$$

$$I_1 = \int \frac{1}{(x-2)^2 + \left(\sqrt{2}\right)^2}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + C_1$$

$$I_2 = \frac{1}{2}\int \frac{2x-4}{(x^2-4x+6)^2}\,dx = \tfrac{1}{2}\int \frac{1}{u^2}\,du = \tfrac{1}{2}\left(-\frac{1}{u}\right) + C_2 = -\frac{1}{2(x^2-4x+6)} + C_2$$

$$I_3 = 3\int \frac{1}{\left[(x-2)^2 + \left(\sqrt{2}\right)^2\right]^2}\,dx = 3\int \frac{1}{[2(\tan^2\theta+1)]^2}\sqrt{2}\sec^2\theta\,d\theta \quad \begin{bmatrix} x-2=\sqrt{2}\tan\theta, \\ dx = \sqrt{2}\sec^2\theta\,d\theta \end{bmatrix}$$

$$\begin{aligned}&= \frac{3\sqrt{2}}{4}\int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta = \frac{3\sqrt{2}}{4}\int \cos^2\theta\,d\theta = \frac{3\sqrt{2}}{4}\int \tfrac{1}{2}(1+\cos 2\theta)\,d\theta \\ &= \frac{3\sqrt{2}}{8}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C_3 = \frac{3\sqrt{2}}{8}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{3\sqrt{2}}{8}\left(\tfrac{1}{2}\cdot 2\sin\theta\cos\theta\right) + C_3 \\ &= \frac{3\sqrt{2}}{8}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{3\sqrt{2}}{8}\cdot\frac{x-2}{\sqrt{x^2-4x+6}}\cdot\frac{\sqrt{2}}{\sqrt{x^2-4x+6}} + C_3 \\ &= \frac{3\sqrt{2}}{8}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{3(x-2)}{4(x^2-4x+6)} + C_3\end{aligned}$$



So $\quad I = I_1 + I_2 + I_3 \qquad [C = C_1 + C_2 + C_3]$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{-1}{2(x^2-4x+6)} + \frac{3\sqrt{2}}{8} \tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{3(x-2)}{4(x^2-4x+6)} + C$$

$$= \left(\frac{4\sqrt{2}}{8} + \frac{3\sqrt{2}}{8}\right) \tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{3(x-2)-2}{4(x^2-4x+6)} + C = \frac{7\sqrt{2}}{8} \tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{3x-8}{4(x^2-4x+6)} + C$$

38. $\dfrac{x^3+2x^2+3x-2}{(x^2+2x+2)^2} = \dfrac{Ax+B}{x^2+2x+2} + \dfrac{Cx+D}{(x^2+2x+2)^2} \quad \Rightarrow$

$x^3+2x^2+3x-2 = (Ax+B)(x^2+2x+2) + Cx + D \quad \Rightarrow$

$x^3+2x^2+3x-2 = Ax^3 + (2A+B)x^2 + (2A+2B+C)x + 2B + D.$

So $A = 1, 2A + B = 2 \quad \Rightarrow \quad B = 0, 2A + 2B + C = 3 \quad \Rightarrow \quad C = 1$, and $2B + D = -2 \quad \Rightarrow \quad D = -2$. Thus,

$$I = \int \frac{x^3+2x^2+3x-2}{(x^2+2x+2)^2}\,dx = \int \left(\frac{x}{x^2+2x+2} + \frac{x-2}{(x^2+2x+2)^2}\right) dx$$

$$= \int \frac{x+1}{x^2+2x+2}\,dx + \int \frac{-1}{x^2+2x+2}\,dx + \int \frac{x+1}{(x^2+2x+2)^2}\,dx + \int \frac{-3}{(x^2+2x+2)^2}dx$$

$$= I_1 + I_2 + I_3 + I_4.$$

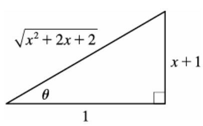
$$I_1 = \int \frac{x+1}{x^2+2x+2}\,dx = \int \frac{1}{u}\left(\frac{1}{2}\,du\right) \quad \left[\begin{matrix} u = x^2+2x+2, \\ du = 2(x+1)\,dx \end{matrix}\right] \quad = \frac{1}{2}\ln\left|x^2+2x+2\right| + C_1$$

$$I_2 = -\int \frac{1}{(x+1)^2+1}\,dx = -\frac{1}{1}\tan^{-1}\left(\frac{x+1}{1}\right) + C_2 = -\tan^{-1}(x+1) + C_2$$

$$I_3 = \int \frac{x+1}{(x^2+2x+2)^2}\,dx = \int \frac{1}{u^2}\left(\frac{1}{2}\,du\right) = -\frac{1}{2u} + C_3 = -\frac{1}{2(x^2+2x+2)} + C_3$$

$$I_4 = -3\int \frac{1}{[(x+1)^2+1]^2}\,dx = -3\int \frac{1}{(\tan^2\theta+1)^2}\sec^2\theta\,d\theta \quad \left[\begin{matrix} x+1 = 1\tan\theta, \\ dx = \sec^2\theta\,d\theta \end{matrix}\right]$$

$$= -3\int \frac{1}{\sec^2\theta}\,d\theta = -3\int \cos^2\theta\,d\theta = -\frac{3}{2}\int (1+\cos 2\theta)\,d\theta$$



$$= -\tfrac{3}{2}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C_4 = -\tfrac{3}{2}\theta - \tfrac{3}{2}\left(\tfrac{1}{2}\cdot 2\sin\theta\,\cos\theta\right) + C_4$$

$$= -\frac{3}{2}\tan^{-1}\left(\frac{x+1}{1}\right) - \frac{3}{2}\cdot\frac{x+1}{\sqrt{x^2+2x+2}}\cdot\frac{1}{\sqrt{x^2+2x+2}} + C_4$$

$$= -\frac{3}{2}\tan^{-1}(x+1) - \frac{3(x+1)}{2(x^2+2x+2)} + C_4$$

So $\quad I = I_1 + I_2 + I_3 + I_4 \qquad [C = C_1 + C_2 + C_3 + C_4]$

$$= \frac{1}{2}\ln(x^2+2x+2) - \tan^{-1}(x+1) - \frac{1}{2(x^2+2x+2)} - \frac{3}{2}\tan^{-1}(x+1) - \frac{3(x+1)}{2(x^2+2x+2)} + C$$

$$= \frac{1}{2}\ln(x^2+2x+2) - \frac{5}{2}\tan^{-1}(x+1) - \frac{3x+4}{2(x^2+2x+2)} + C$$

ex4 By partial fraction

$$f(x) = \frac{4}{(x-1)^3} + \frac{3}{(x^2+2x+2)^2}$$

$$\int \frac{4\,dx}{(x-1)^3} = \frac{4(x-1)^{-2}}{-2} + C$$

$$\int \frac{3\,dx}{(x^2+2x+2)^2} = 3\int \frac{d(x+1)}{[(x+1)^2+1]^2}$$

$$= \frac{3x}{2[(x^2+1)^2+1]} + \frac{3}{2}\int \frac{d(x+1)}{(x+1)^2+1}$$

$$= \frac{3}{2}\arctan(x+1) + \frac{3(x+1)}{2[(x+1)^2+1]} + C$$

So
$$\int f(x)\,dx = \frac{-2}{(x-1)^2} + \frac{3}{2}\arctan(x+1) + \frac{3(x+1)}{2[(x+1)^2+1]^2} + C$$