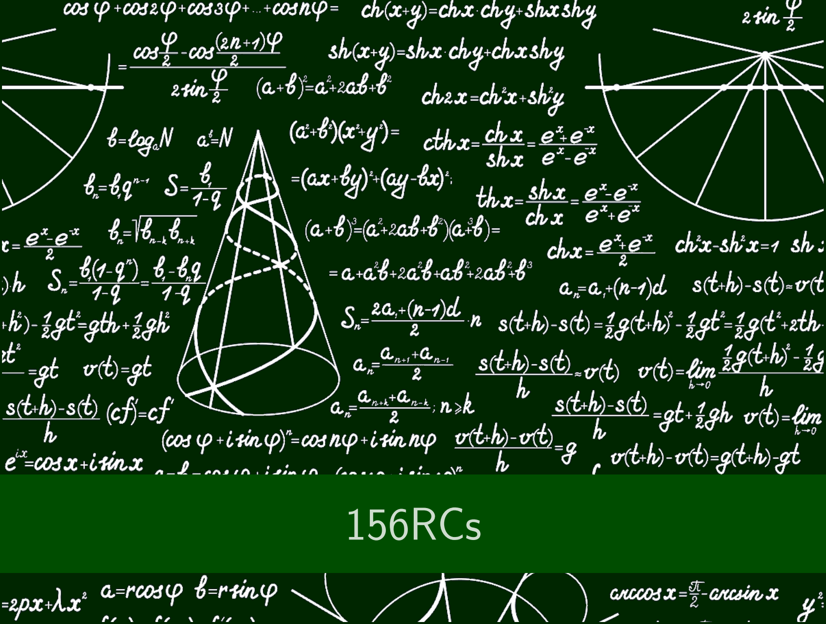


156RCs

Li Mingrui, Xia Yiwei, Zhang Haoran

Sincerest appreciation dedicated to
2024 VV156 TAs
Li Mingrui, Xia Yiwei, Zhang Haoran,
as well as all previous VV156 TAs.

1. About Honors Calculus

Contents

2024FA-VV156

1. Limits
2. Derivatives and Integrals
3. Series
4. Polar Coordinates
5. Basic Differential Equations

2024SU-VV255

1. Simple Linear Algebra
2. Partial Derivatives
3. Multiple Integrals

2024FA-VV256

1. Differential Equations
2. Deeper Linear Algebra
3. Fourier Transform and Laplace Transform

Other courses might contribute to Honors Calculus: VV214, VE203, etc.

Useful Software and Websites

MATLAB:

Mathematica:

Integration with Steps:

https://www.integral-calculator.com/

About This Slide

You had better understand the contents on green slides. They are relatively fundamental.

About This Slide

The content shown on blue slides may be relatively hard, but may contribute to getting high marks in the exams more or less. Do not worry when you cannot handle it.

About This Slide

Those on pink slides are just some interesting problems and crazy thoughts. Maybe they are of little application in exams, but they are quite interesting.

2. Functions

Three essential factors of a function:

- Domain: D
  A set containing the action objects of correspondence rule.
- Correspondence Rule
- Range: E
  A set of all images corresponding to all elements in the domain under a certain correspondence rule.

Definition

A **function** f is a rule that assigns to **each element** x in a set D exactly one element, called $f(x)$, in a set E.

The Vertical Line Test

A curve in the xy-plane is the graph of a function of x **iff** (if and only if) no vertical line intersects the curve more than once.

There are four ways to represent a function:

1. Verbally (by a description in words)
2. Numerically (by a table of values)
3. Visually (by a graph)
4. Algebraically (by an explicit formula)

Symmetry

- Even function: $f(-x) = f(x)$
- Odd function: $f(-x) = -f(x)$

Tip: Give priority to whether the domain D of the function is symmetrical.

Increasing & Decreasing property

A function f is called (strictly) **increasing** on an interval I if

$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

A function f is called (strictly) **decreasing** on an interval I if

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

Basic function types

Linear function

The graph of the function is a line:

$$y = f(x) = mx + b$$

m is the slope of the line and b is the y-intercept.

Polynomials

A function P is called a **polynomial** if

$$P(x) = \Sigma_{i=0}^{n} a_i x^i$$

a_i are **coefficients** and n is the **degree** of the polynomial if $a_n \neq 0$.

Quadratic function

A polynomial of degree 2 is of the form $P(x) = ax^2 + bx + c$ and is called a **quadratic function**.

Basic function types

Power function

A function of the form $f(x) = x^a$ is called a **power function**, where a is a constant. Consider an arbitary positive integer n:

- $a = n$:

$$y = x\text{: line} \qquad y = x^2\text{: parabola}$$

- $a = \frac{1}{n}$: **root function**
- $a = -1$: **reciprocal function**

Ratio function

A **Ratio function** f is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

The domain consists of all values of x such that $Q(x) \neq 0$.

Basic function types

Trigonometric function

$$\sin(x+2\pi)=\sin x \qquad \cos(x+2\pi)=\cos x \qquad \tan(x+\pi)=\tan x$$

Exponential function

The **exponential functions** are the functions of the form $f(x)=a^x$, where the base a is a positive constant.

Logarithmic function

The **logarithmic functions** $f(x)=\log_a x$, where the base a is a positive constant, are the inverse functions of the exponential functions.

Think about the question:
What condition does a meet when the exponential function increases in its domain? What about the logarithmic function?

How to draw a function

- You can simply trace the dots for simple functions.
- You may also try to use software:
  https://www.geogebra.org
  https://www.desmos.com

Dirichlet Function

$$1_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Impulse Function

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

Step Function

$$H[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

Ramp Function

$$R(x) := \begin{cases} x, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Hyperbolic Function

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \cosh(x) = \frac{e^x + e^{-x}}{2}, \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

Inverse trigonometric function

$$\arcsin(x), \arccos(x), \arctan(x)$$
$$\operatorname{arsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right)$$
$$\operatorname{arcosh}(x) = \ln\left(x + \sqrt{x^2-1}\right)$$
$$\operatorname{artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

Vertical and Horizontal Shifts, suppose $c > 0$

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward
$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward
$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right
$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

Vertical and Horizontal Stretching and Reflecting, suppose $c > 1$

$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c
$y = (1/c)f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c
$y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c
$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c
$y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis
$y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis

Definition

Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h, then g, and then f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

Exercise 1

Skip it if you find the exercise quite simple.

Find the domain of these functions:

1. $h(x) = \frac{1}{\sqrt[4]{x^2 - 5x}}$
2. $f(u) = \frac{u+1}{1+\frac{1}{u+1}}$
3. $F(p) = \sqrt{2 - \sqrt{p}}$

Exercise 1

How to solve this kind of questions

- The denominator in the fractional function cannot be zero
- The quantity in the even root formula cannot take a negative value, that is, it should be greater than or equal to zero
- The antilogarithm of the logarithm cannot be negative and zero, that is, it must take a positive value
- The domain of the function $y = \arcsin x$, $y = \arccos x$ is $-1 \leqslant x \leqslant 1$
- $y = \tan x$, $x \neq k\pi + \pi/2$, $y = \cot x$, $x \neq k\pi$, k is integer

Exercise 2

Prove or Disprove

- If f and g are both even functions, is $f + g$ even? If f and g are both odd functions, is $f + g$ odd? What if f is even and g is odd? Justify your answers.
- If f and g are both even functions, is the product fg even? If f and g are both odd functions, is fg odd? What if f is even and g is odd? Justify your answers.

Exercise 3

Graph the functions step by step

$(1) y = 1 - 2\sqrt{x+3}$
$(2) y = |\cos \pi x|$

Exercise 4

Find the function (a) $f \circ g$, (b) $g \circ f$, (c) $f \circ f$, and (d) $g \circ g$ and their domains.

$f(x) = \frac{x}{1+x}, \quad g(x) = \sin 2x$

Exercise 5

Find $f \circ g \circ h$:

- $f(x) = \tan x$
- $g(x) = \frac{x}{x-1}$
- $h(x) = \sqrt[3]{x}$

(It is unnecessary to find the domain of this composite function in this exercise!)

Exercise 6

Composite Function

1. If $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 7$, find a function f such that $f \circ g = h$.
   (Think about what operations you would have to perform on the formula for g to end up with the formula for h.)
2. If $f(x) = 3x + 5$ and $h(x) = 3x^2 + 3x + 2$, find a function g such that $f \circ g = h$.

References

[1] Huang, Yucheng. VV156_RC1_updated.pdf. 2021.

[2] Cai, Runze. Chapter01.pdf. 2021.

[3] Zhou,Yishen.RC1. 2022.

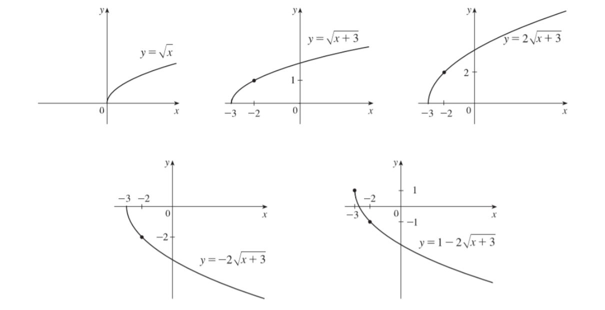
Exercise Answer 1-2

1. $x^2 - 5x > 0 \Rightarrow x \in (-\infty, 0) \cup (5, \infty)$
2. $1 + \frac{1}{u+1} \neq 0, u + 1 \neq 0 \Rightarrow u \in (\infty, -2) \cup (-2, -1) \cup (-1, \infty)$
3. $p \geq 0, 2 - \sqrt{q} \geq 0 \Rightarrow p \in [0, 4]$

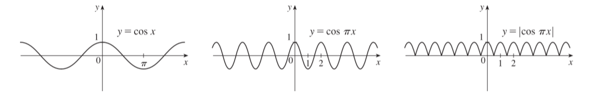
- Yes. Yes. Not necessarily.
- Yes. No.(fg is Even) fg is Odd.

Exercise Answer 3

(1)



(2)



Exercise Answer 4-5

(a) $f \circ g(x) = \frac{sin2x}{1+sin2x}$. Domain: $x \in \{x|x \neq k\pi - \frac{\pi}{4}, k \in \mathbb{B}\}$
(b) $g \circ f(x) = sin(\frac{2x}{1+x})$. Domain: $x \in (\infty, -1) \cup (-1, \infty)$
(c) $f \circ f(x) = \frac{x}{1+2x}(x \neq -1)$.
Domain:$x \in (\infty, -1) \cup (-1, -\frac{1}{2}) \cup (-\frac{1}{2}, \infty)$
(d) $g \circ g(x) = sin(2sin(2x))$. Domain:$x \in \mathbb{R}$

$tan(\frac{\sqrt[3]{x}}{\sqrt[3]{x}-1})$

Exercise Answer 6

1. $x = \frac{g(x)-1}{2}$.
   Plug into h(x), $h(x) = 4(\frac{g(x)-1}{2})^2 + 4(\frac{g(x)-1}{2}) + 7 = g^2(x) + 6$.
   Also, $h(x) = f(g(x))$.
   Therefore, $f(x) = x^2 - 6$
2. $3g(x) + 5 = h(x)$, $g(x) = x^2 + x - 1$

3. Limits

"Rough" Definition of a Limit

Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, **except possibly** at a itself.)
Then we write

$$\lim_{x \to a} f(x) = L$$

and say

"the limit if $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be **sufficiently close to** a (on either side of a) but **not equal to** a.

Considering a function called *Heaviside Function*

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \tag{1}$$

Does $\lim_{t \to 0} H(t)$ exists?

We write

$$\lim_{x \to a^-} f(x) = L$$

and say the left-hand limit of $f(x)$ as x approaches a is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be **sufficiently close to** a and x less than a.

When calculating $\lim\limits_{x \to a^-} f(x)$, we consider only $x < a$.

Similarly, we can get the right-hand limit of $f(x)$ as x approaches a.

When does $\lim_{x \to a} f(x)$ exists?

$$\lim_{x \to a} f(x) = L$$

if and only if

$$\lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L$$

Can we directly regard L as $f(a)$?

Let f be a function defined on both sides of a, **except possibly** at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x **sufficiently close to** a, but **not equal to** a.
Let f be a function defined on both sides of a, **except possibly** at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x **sufficiently close to** a, but **not equal to** a.

Warning:

$\lim_{x \to a} f(x) = (-)\infty$ does not mean that we are regarding $(-)\infty$ as a number. Nor does it mean that the limit exists!

Limits at Infinity

1. Let f be a function defined on some interval (a, ∞). Then
$$\lim_{x \to \infty} f(x) = L$$
means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

2. Let f be a function defined on some interval $(-\infty, a)$. Then
$$\lim_{x \to -\infty} f(x) = L$$
means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.

Infinite Limits at Infinity

1. $\lim\limits_{x \to \infty} f(x) = \infty$
2. $\lim\limits_{x \to \infty} f(x) = -\infty$
3. $\lim\limits_{x \to -\infty} f(x) = \infty$
4. $\lim\limits_{x \to -\infty} f(x) = -\infty$

The Limit Laws

Five basic laws:
Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \text{ and } \lim_{x\to a} g(x)$$

exists. Then

1. $\lim\limits_{x\to a}[f(x)+g(x)] = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$
2. $\lim\limits_{x\to a}[f(x)-g(x)] = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x)$
3. $\lim\limits_{x\to a}[cf(x)] = c\lim\limits_{x\to a} f(x)$
4. $\lim\limits_{x\to a}[f(x)g(x)] = \lim\limits_{x\to a} f(x) \cdot \lim\limits_{x\to a} g(x)$
5. $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ **(if $\lim\limits_{\mathbf{x}\to a} g(x) \neq 0$)**

Another six laws:

1. $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$ (where n is a positive integer)
2. $\lim_{x \to a} c = c$
3. $\lim_{x \to a} x = a$
4. $\lim_{x \to a} x^n = a^n$ (where n is a positive integer)
5. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ (where n is a positive integer)
   if n is even, we assume that $a > 0$
6. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ (where n is a positive integer)
   if n is even, we assume that $\lim_{\mathbf{x} \to a} f(x) > 0$

Warning:
The Limit Laws can't be applied to infinite limits because $(-)\infty$ is not a number!

Two additional properties of limits:

1. if $f(x) \leq g(x)$ when x is near a (**except possibly** at a) and the limits of f and g both exists as x approaches a, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$$

2. **(The Squeeze Theorem)** if $f(x) \leq g(x) \leq h(x)$ when x is near a (**except possibly** at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

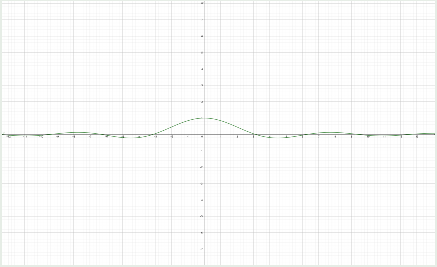
$$\lim_{x \to a} g(x) = L$$

Two Important Limits

Be sure to keep these two limits in mind!

1. $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$
   (How to prove it? Considering $\sin x$, x and $\tan x$. Then, use the squeeze theorem.)

2. $\lim_{x \to \infty} (1 + \dfrac{1}{x})^x = e = 2.718281828459045 \cdots$



Let f be a function defined on some open interval that contains the number a, **except possibly** at a itself. Then we say that the limit if $f(x)$ as x approaches a is L, and we write

$$\lim_{x \to a} f(x) = L$$

if for **every** number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon$$

The Precise Definition of a Limit

Left-hand limits:

$$\lim_{x \to a^-} f(x) = L$$

if for **every** number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$\text{if} \quad a - \delta < x < a \quad \text{then} \quad |f(x) - L| < \varepsilon$$

Right-hand limits:

$$\lim_{x \to a^+} f(x) = L$$

if for **every** number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$\text{if} \quad a < x < a + \delta \quad \text{then} \quad |f(x) - L| < \varepsilon$$

Infinite limits:

1. Let f be a function defined on some open interval that contains the number a, **except possibly** at a itself. Then
$$\lim_{x\to a} f(x) = \infty$$
means that for **every positive** number M, there is a number $\delta > 0$ such that
$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad f(x) > M$$

2. Let f be a function defined on some open interval that contains the number a, **except possibly** at a itself. Then
$$\lim_{x\to a} f(x) = -\infty$$
means that for **every negative** number N, there is a number $\delta > 0$ such that
$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad f(x) < N$$

The Precise Definition of a Limit

Limits at Infinity:

1. Let f be a function defined on some interval (a, ∞). Then
$$\lim_{x \to \infty} f(x) = L$$
means that for every $\varepsilon > 0$, there is a corresponding number N such that
$$\text{if} \quad x > N \quad \text{then} \quad |f(x) - L| < \varepsilon$$

2. Let f be a function defined on some interval $(-\infty, a)$. Then
$$\lim_{x \to -\infty} f(x) = L$$
means that for every $\varepsilon > 0$, there is a corresponding number N such that
$$\text{if} \quad x < N \quad \text{then} \quad |f(x) - L| < \varepsilon$$

Exercise 1

Evaluate the following limits

1. $\lim\limits_{x\to 0} \dfrac{4x^3 - 2x^2 + x}{2x + 3x^2}$
2. $\lim\limits_{x\to 0} \dfrac{\dfrac{1}{3+x} - \dfrac{1}{3}}{x}$
3. $\lim\limits_{x\to\infty} (1 + \dfrac{1}{x})(2 - \dfrac{1}{x^2})$
4. $\lim\limits_{x\to 0} \dfrac{-1 + \sqrt[n]{x+1}}{x}$

Exercise 2

Evaluate the following limits

1. $\lim\limits_{x \to -4} \dfrac{\sqrt{9+x^2}-5}{x+4}$
2. $\lim\limits_{\mathbf{h} \to 0} \dfrac{(x+h)^3 - x^3}{h}$(What if $\lim\limits_{\mathbf{h} \to 0} \dfrac{(x+h)^n - x^n}{h}$?)
3. $\lim\limits_{x \to 0} \dfrac{\sin \omega x}{x}$
4. $\lim\limits_{x \to \infty} (1 - \dfrac{1}{x})^{kx}$ (k is a positive integer)
5. $\lim\limits_{x \to \frac{\pi}{2}} (\sin x)^{\tan x}$

Some methods to calculate the limits:

1. Use those limit laws directly
2. Exchange the order of functions and limit symbols based on the continuity of composite function. (Will be mentioned later)
3. Do factorization, denominator rationalization or numerator rationalization.
4. If a factor approaching zero is find in the denominator, try to eliminate it.
5. Translate the formula into the form of "two important limits"
6. **The method to solve those formulas having the form of $u(x)^{v(x)}$ will be discussed at a deeper level after the differentiation and l'Hôpital's rule are taught.**

Exercise Answer 1

1. $\lim\limits_{x\to 0}\dfrac{4x^3-2x^2+x}{2x+3x^2}=\lim\limits_{x\to 0}\dfrac{4x^2-2x+1}{2+3x}$=0.5

2. $\lim\limits_{x\to 0}\dfrac{\dfrac{1}{3+x}-\dfrac{1}{3}}{x}=\lim\limits_{x\to 0}\dfrac{3-3-x}{3x(3+x)}=\lim\limits_{x\to 0}\dfrac{-1}{3(3+x)}=-\dfrac{1}{9}$

3. $\lim\limits_{x\to\infty}(1+\dfrac{1}{x})(2-\dfrac{1}{x^2})=\lim\limits_{x\to\infty}(1+\dfrac{1}{x})\cdot\lim\limits_{x\to\infty}(2-\dfrac{1}{x^2})=1\cdot 2=2$

4. Let $t^n-1:=x$, $\lim\limits_{x\to 0}\dfrac{-1+\sqrt[n]{x+1}}{x}=\lim\limits_{t\to 1}\dfrac{t-1}{t^n-1}=$
$\lim\limits_{t\to 1}\dfrac{1}{1+t+t^2+\cdots+t^{n-1}}=\dfrac{1}{n}$

Exercise Answer 2

1. $$\lim_{x\to -4} \frac{\sqrt{9+x^2}-5}{x+4} = \lim_{x\to -4} \frac{x-4}{\sqrt{9+x^2}+5} = -\frac{4}{5}$$

2. $$\lim_{\mathbf{h}\to 0} \frac{(x+h)^n - x^n}{h} = \lim_{\mathbf{h}\to 0} nx^{n-1} + h\cdot(\cdots) = nx^{n-1}$$

3. $$\lim_{x\to 0} \frac{\sin\omega x}{x} = \omega \cdot \lim_{x\to 0} \frac{\sin\omega x}{\omega x} = \omega$$

4. $$\lim_{x\to\infty} (1-\frac{1}{x})^{kx} = \lim_{x\to\infty} (1+\frac{1}{-x})^{(-x)(-k)} = e^{-k}$$

5. $$\lim_{x\to\frac{\pi}{2}} (\sin x)^{\tan x} = \lim_{x\to\frac{\pi}{2}} (1+(\sin x-1))^{\frac{1}{\sin x - 1}\cdot(\sin x-1)\tan x} =$$
$$e^{\lim\limits_{x\to\frac{\pi}{2}} (\sin x-1)\tan x} = e^{\lim\limits_{x\to\frac{\pi}{2}} \frac{(\sin x-1)\sin x}{\sqrt{1-\sin^2 x}}} = e^{-\lim\limits_{x\to\frac{\pi}{2}} \frac{\sqrt{1-\sin x}\sin x}{\sqrt{1+\sin x}}} =$$
$$e^0 = 1$$

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[1] Huang, Yucheng. VV156_RC2.pdf. 2021.

[2] Cai, Runze. Chapter01.pdf. 2021.

[3] Department of mathematics, Tongji University. Advanced Mathematics (7th Edition). 2014.

[4] James Stewart. Calculus (7th Edition). 2014.

[5] Department of mathematics, Tongji University. Learning Guidance of Advanced Mathematics (7th Edition). 2014.

[6]Zhou,Yishen.RC2. 2022.

4. Continuity

Definition of Continuity

A function f is continuous at a number a if

$$\lim_{x \to a} = f(a)$$

This definition actually implicitly requires three things:

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

A function f is continuous from the right at a number a if

$$\lim_{x \to a^+} = f(a)$$

A function f is continuous from the left at a number a if

$$\lim_{x \to a^-} = f(a)$$

Definition of Continuity

A function is continuous on an interval if it is continuous at **every number in the interval**. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left*.)

Theorem 1

If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\dfrac{f}{g}$ **(if $g(a) \neq 0$)**

Theorem 2

The following types of functions are continuous at every number in their domain(s):

1. polynomials
2. rational functions
3. root functions
4. (inverse) trigonometric functions
5. exponential functions
6. logarithmic functions

Theorem 3

If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$.

In other words,

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$$

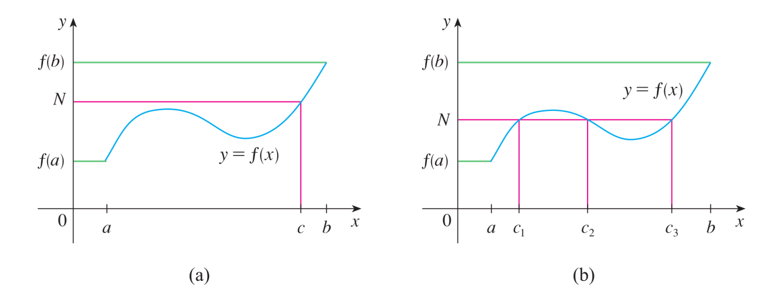
Theorem 4

If g is continuous at a and f is continuous at $g(a)$, then $f(g(x))$ is continuous at a.

"A continuous function of a continuous function is a continuous function."

Theorem 5-The Intermediate Value Theorem

Suppose that f **is continuous on the closed interval** $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.
Note that the value N can be taken on once or more than once.



Types of Discontinuities

1. removable discontinuity
2. infinite discontinuity
3. jump discontinuity

Exercise 1

Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

$$\sqrt[3]{x} = 1 - x, \quad (0,\ 1)$$

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$$\sqrt[3]{x} = 1 - x, \quad (0, 1)$$

Solution:
$\sqrt[3]{x} = 1 - x$ has a root on $(0, 1)$ equals to $f(x) = \sqrt[3]{x} + x - 1 = 0$ has a solution on $(0, 1)$.
Since $f(0) = -1$, $f(1) = 1$, and 0 is between -1 and 1, there must be a point c such that $f(c) = 0$.

Exercise 2

Let $f(x) = \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$. What kind of discontinuity is $x = 0$?

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Solution:

$$\lim_{x \to 0^-} f(x) = \frac{0 - 1}{0 + 1} = -1$$

$$\lim_{x \to 0^+} f(x) = \frac{\infty - 1}{\infty + 1} = 1$$

Jump discontinuity.

Exercise 3

Find the values of a and b that make f continuous everywhere.

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & x < 2 \\ ax^2 - bx + 3 & 2 \leq x < 3 \\ 2x - a - b & x \leq 3 \end{cases} \tag{2}$$

Exercise 3

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$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & x < 2 \\ ax^2 - bx + 3 & 2 \leq x < 3 \\ 2x - a - b & x \leq 3 \end{cases} \tag{3}$$

Solution:

$\lim_{x \to 2^-} f(x) = \dfrac{(x-2)(x+2)}{(x-2)} = 4$

$f(2) = 4a - 2b + 3$

$\lim_{x \to 3^-} f(x) = 9a - 3b + 3$

$f(3) = 6 - a - b$

Exercise 4

Find a and b that make $f(x) = \lim_{n \to +\infty} \dfrac{x^{2n-1} + ax^2 + bx}{x^{2n} + 1}$ continuous on $(-\infty, \infty)$.

Exercise 4

Find a and b that make $f(x) = \lim_{n\to+\infty} \frac{x^{2n-1} + ax^2 + bx}{x^{2n} + 1}$ continuous on $(-\infty, \infty)$.

Solution:

$$f(x) = \begin{cases} ax^2 + bx : -1 < x < 1 \\ \frac{a-b-1}{2} : x = -1 \\ \frac{a+b+1}{2} : x = 1 \\ \frac{1}{x} : x > 1 \ or \ x < -1 \end{cases}$$

At x = 1, $\begin{cases} \frac{a+b+1}{2} = a + b \\ a + b = 1 \end{cases}$

At x = -1, $a - b = -1$

Thus, we have a = 0, b = 1

Exercise 5

There are two functions:
$f(x)$ is continuous on $(-\infty, \infty)$, and $f(x) \neq 0$.
$\varphi(x)$ is defined on $(-\infty, \infty)$, but $\varphi(x)$ has discontinuity.

Judge whether the following four statements are correct:

1. $\varphi[f(x)]$ must have discountinuity.
2. $[\varphi(x)]^2$ must have discountinuity.
3. Whether $f[\varphi(x)]$ has discountinuity is uncertain.
4. $\dfrac{\varphi(x)}{f(x)}$ must have discountinuity.

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Solution:

1. No
2. No
3. Yes
4. Yes

List of Limits

1. $\lim_{x \to a} c = c$
2. $\lim_{x \to a} x^n = a^n$
3. $\lim_{x \to 0} \frac{\sin x}{x} = 1$
4. $\lim_{x \to 0} \frac{\tan x}{x} = 1$
5. $\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$
6. $\lim_{x \to 0} \frac{\arcsin x}{x} = 1$
7. $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$
8. $\lim_{x \to 0^+} x^x = 1$

List of Limits

1. $\lim\limits_{x\to 0^+} x\ln x = 0$

2. $\lim\limits_{x\to 0} \dfrac{\sqrt[n]{x+1}-1}{x} = \dfrac{1}{n}$

3. $\lim\limits_{x\to\infty} x^{\frac{1}{x}} = 1$

4. $\lim\limits_{x\to\pm\infty} \left(1+\dfrac{1}{x}\right)^x = e$

5. $\lim\limits_{x\to 0} (1+x)^{\frac{1}{x}} = e$

6. $\lim\limits_{x\to 0} (1+\sin x)^{\frac{1}{x}} = e$

7. $\lim\limits_{x\to\infty} \left(x-\sqrt{x^2-a^2}\right) = 0$

8. $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x} = 1$

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5. Derivatives

Tangent Line

The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

Instantaneous rate of change

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$\text{instantaneous rate of change } = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Try to define velocity in this way.

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{h}$$

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Derivative

The **derivative of a function f at a number a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

if this limit exists.

The slope of the tangent line of a function is the corresponding derivative.

Definition

Notations

Newton:

$$\dot{y}$$

Leibniz:

$$\frac{dy}{dx}$$

Lagrange:

$$f'(x)$$

Jacobi: (Partial Derivatives)

$$\frac{\partial f}{\partial x}$$

Differentiable

A function f is differentiable at **a** if $f'(a)$ exists. It is differentiable on an open interval (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

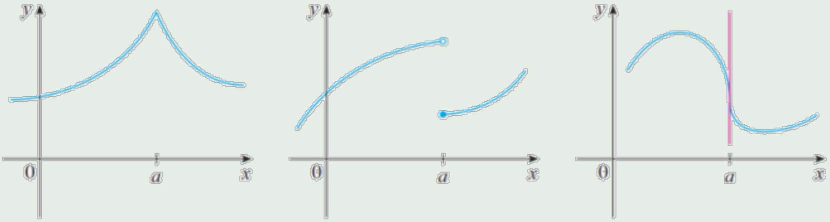
Differentiable and Continuity

If f is differentiable at a, then f is continuous at a.
NOTE The converse of Theorem is false; that is, there are functions that are continuous but not differentiable. For instance, the function $f(x) = |x|$ is continuous at 0 because

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} |x| = 0 = f(0)$$

Differentiable

The function is not differentiable at these points:



(a) A corner (b) A discontinuity (c) A vertical tangent

Higher derivatives

$$(f')' = f'' \qquad \text{second derivative}$$

$$(f'')' = f''' \qquad \text{third derivative}$$

Example: still consider the position function

position-velocity-acceleration-jerk-snap$\cdots$

$$x - \frac{dy}{dt} - \frac{d^2y}{dt^2} - \frac{d^3y}{dt^3} - \frac{d^4y}{dt^4}\cdots$$

Differential formulas

$$\frac{d}{dx}(C) = 0$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$(cf)' = cf'$$

$$(f+g)' = f' + g'$$

$$(f-g)' = f' - g'$$

$$(fg)' = fg' + gf'$$

$$(\frac{f}{g})' = \frac{gf' - fg'}{g^2}$$

$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{x\sqrt{x^2-1}}$$
$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$$
$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$
$$\frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$$

Differential formulas

$$\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x \ln a}$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx} \ln |x| = \frac{1}{x}$$

Exercise 1

Find an equation of the tangent line to the curve $y = 2x \sin x$ at the point $(\frac{\pi}{2}, \pi)$.

Exercise 1

Find an equation of the tangent line to the curve $y = 2x \sin x$ at the point $(\frac{\pi}{2}, \pi)$.

Solution:

$y' = 2(sinx + xcosx)$

$k = y'(\frac{\pi}{2}) = 2$

$y - \pi = 2(x - \frac{\pi}{2})$

$y - 2x = 0$

Exercise 2

Differentiate:

1. $y = x^3 + \dfrac{7}{x^4} - \dfrac{2}{x} + 12$
2. $y = \sin x \cos x$
3. $y = \sqrt{x} \sin x$
4. $y = 3e^x \cos x$
5. $y = \dfrac{x \sin x}{1 + x}$
6. $y = \dfrac{1 - \sec x}{\tan x}$
7. $y = x^2 \ln x \cos x$
8. $y = \ln 3 + \dfrac{e^x}{x^2}$

Exercise 2

Solution:

1. $3x^2 - \frac{28}{x^5} + \frac{2}{x^2}$
2. cos2x
3. $\frac{1}{2}x^{-\frac{1}{2}} sinx + x^{\frac{1}{2}} cosx$
4. $3e^x(-sinx + cosx)$
5. $\frac{dy}{dx} = \frac{(xsinx)'(1+x) - xsinx}{(1+x)^2} = \frac{sinx + (1+x)xcosx}{(1+x)^2}$
6. $y = \frac{cosx - 1}{sinx} \to \frac{dy}{dx} = \frac{1}{1 + cosx}$
7. $\frac{dy}{dx} = 2xlnxcosx + xcosx - x^2 lnxsinx$
8. $(-2x^{-3} + x^{-2})e^x$

Exercise 3

Let $y = \log_{\varphi(x)} f(x)$ $(\varphi(x) > 0,\ \varphi(x) \neq 1,\ f(x) > 0)$. Suppose that both $\varphi(x)$ and $f(x)$ are differentiable. Calculate $\dfrac{dy}{dx}$.

Exercise 3

Solution:

$$y = \frac{lnf(x)}{ln\varphi(x)}$$

$$\frac{dy}{dx} = \frac{\frac{1}{f(x)}f'(x)ln\varphi(x) - \frac{1}{\varphi(x)}\varphi'(x)lnf(x)}{[ln\varphi(x)]^2}$$

$$= \frac{f'(x)}{f(x)ln\varphi(x)} - \frac{\varphi'(x)lnf(x)}{\varphi(x)[ln\varphi(x)]^2}$$

Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

The Power Rule Combined with the Chain Rule

If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Alternatively,

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

Exercise 4

Find the derivative of the function:

1. $y = (4x - x^2)^{100}$
2. $y = 5^{-\frac{1}{x}}$
3. $y = e^{-2x} \cos 4x$
4. $y = (\frac{x^2+1}{x^2-1})^3$
5. $y = \frac{\arcsin x}{\arccos x}$
6. $y = [\sin(e^{(\sin x)^2})]^2$
7. $y = \arcsin \sqrt{\frac{1-x}{1+x}}$
8. $y = n^{n^x} + x^{n^n} + n^{x^n} \ (n > 0,\ n \neq 1)$

Exercise 4

1. $200(2-x)(4x-x^2)^{99}$
2. $ln5 \cdot 5^{-\frac{1}{x}} \cdot x^{-2}$
3. $-2e^{-2x}(cos4x + 2sin4x)$
4. $y' = 3(\frac{x^2+1}{x^2-1})^2(-\frac{2}{(x^2-1)^2}) \cdot 2x = -\frac{12x(x^2+1)^2}{(x^2-1)^3}$
5. $y' = \frac{(arcsinx)'arccosx - (arccosx)'arcsinx}{(arccosx)^2} = \frac{arccosx + arcsinx}{\sqrt{1-x^2}(arccosx)^2}$
6. $y' = 2[sin(e^{(sinx)^2})] \cdot cos(e^{(sinx)^2}) \cdot e^{(sinx)^2} \cdot 2sinxcosx =$ $sin2x \cdot sin(2e^{(sinx)^2}) \cdot e^{(sinx)^2}$
7. $y' = (1-\frac{1-x}{1+x})^{-\frac{1}{2}} \cdot \frac{1}{2}(\frac{1-x}{1+x})^{-\frac{1}{2}}[-\frac{2}{(x+1)^2}] =$ $-\frac{1}{\sqrt{2x(1-x)} \cdot (x+1)}$
8. $y' = n^{n^x+x}(lnn)^2 + n^n x^{n^n-1} + n^{x^n+1}x^{(n-1)}lnn$

Exercise 5

Find the second derivative of the following function:
(**Warning**: Don't forget to double check your first derivative!)

1. $y = \tan x$
2. $y = \dfrac{1}{x^3 + 1}$
3. $y = x \cos x$

Exercise 5

1. $y' = sec^2x$
   $y'' = 2secx \cdot (tanx \cdot secx) = 2tanx \cdot sec^2x$
2. $y' = -\dfrac{1}{(x^3+1)^2} \cdot 3x^2$
   $$y'' = -3 \cdot \frac{2x(x^3+1)^2 - 2(x^3+1)3x^4}{(x^3+1)^4} = \frac{6x(2x^3-1)}{(x^3+1)^3}$$
3. $y' = cosx - xsinx$
   $y'' = -2sinx - xcosx$

Exercise 6

Answer the following three questions based on $\dfrac{dy}{dx} = y'$:

1. Express $\dfrac{dx}{dy}$ with y'
2. Express $\dfrac{d^2x}{dy^2}$ with y' and y''
3. Express $\dfrac{d^3x}{dy^3}$ with y' ,y'' and y'''

Exercise 6

Solution:

1. $$\frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{y'}$$
2. $$\frac{d^2x}{dy^2} = \frac{d}{dy} \cdot \frac{dx}{dy} = \frac{d}{dx}\frac{1}{y'}\frac{dx}{dy} = -\frac{y''}{(y')^3}$$
3. $$\frac{d^3x}{dy^3} = \frac{d}{dy}\frac{d^2x}{dy^2} = \frac{d}{dx}\frac{d^2x}{dy^2}\frac{dx}{dy} = -\frac{y'''(y')^3 - 3(y')^2(y'')^2}{(y')^6} \cdot \frac{1}{y'} = \frac{3y''' - y'(y'')^2}{(y')^5}$$

Find y' if $\sin(x+y) = y^2 \cos x$

Differentiating implicitly with respect to x and remembering that y is a function of x, we get

$$\cos(x+y) \cdot \left(1+y'\right) = y^2(-\sin x) + (\cos x)\left(2yy'\right)$$

(Note that we have used the Chain Rule on the left side and the Product Rule and Chain Rule on the right side.) If we collect the terms that involve y', we get

$$\cos(x+y) + y^2 \sin x = (2y\cos x)y' - \cos(x+y) \cdot y'$$

So

$$y' = \frac{y^2 \sin x + \cos(x+y)}{2y\cos x - \cos(x+y)}$$

Exercise 7

Calculate the derivative of the following implicit function:

1. $y^2 - 2xy + 9 = 0$
2. $xy = e^{xy}$

Solution

1. $2yy' - 2y - 2xy' = 0, y' = \dfrac{y}{y - x}$
2. The derivative does not exist!

Exercise 8

Use **logarithmatic differentiation** to calculate the derivative of the following implicit function:

1. $y = (\frac{x}{x+1})^x$
2. $y = \sqrt[5]{\frac{x-5}{\sqrt[5]{x^2+2}}}$

Exercise 8

Solution:

1. $lny = x[lnx - ln(x+1)]$
   $\frac{1}{y}y' = [lnx - ln(x+1)] + x[\frac{1}{x} - \frac{1}{x+1}]$
   $y' = (\frac{x}{x+1})^x[ln|\frac{x}{x+1}| + \frac{1}{x+1}]$

2. $lny = \frac{1}{5}ln|x-5| - \frac{1}{25}ln(x^2+2)$
   $y'\frac{1}{y} = \frac{1}{5(x-5)} - \frac{2x}{25(x^2+2)}$
   $y' = y[\frac{1}{5(x-5)} - \frac{2x}{25(x^2+2)}]$

Exercise 9

The *Bessel function* of order 0, $y = J(x)$, satisfies the differential equation $xy'' + y' + xy = 0$ for all values of x and its value at 0 is $J(0) = 1$.

(a) Find $J'(0)$.

(b) Use implicit differentiation to find $J''(0)$.

Solution:

1. Take x=0, $0 + J'(0) + 0 = 0 \rightarrow J'(0) = 0$
2. $xy''' + 2y'' + y + xy' = 0$
   At $x = 0$, $2y'' + 1 = 0$, $y'' = -\frac{1}{2}$

Linear approximation

Definition

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the **linear approximation** or **tangent line approximation** of f at a. The linear function whose graph is this tangent line, that is,

$$L(x) = f(a) + f'(a)(x - a)$$

is called the linearization of f at a.

Exercise 10

Use linear approximation to estimate $2.0006^{1.9998}$

Solution: $f(x) = 2^{x+2}$
$f'(x) = 2^{x+2} ln2$
$2^{1.9998} = f(0) + f'(0) \times (-0.0002) = 2^2 + 4ln2 \times (-0.0002) = 3.9994$
$g(x) = (2 + x)^{1.9998}$
$g'(x) = 1.9998(2 + x)^{0.9998}$
$2.0006^{1.9998} = g(0) + g'(0) \times 0.0006 =$
$2^{1.9998} + 1.9998 \times 2^{0.9998} \times 0.0006 = 4.0018$

Tip: You're highly recommended to remember this part!

When $x \to 0$

$$a^x - 1 \sim x \ln a$$

$$\arcsin(a)x \sim \sin(a)x \sim (a)x$$

$$\arctan(a)x \sim \tan(a)x \sim (a)x$$

$$\ln(1+x) \sim x$$

$$\sqrt{1+x} - \sqrt{1-x} \sim x$$

$$(1+ax)^b - 1 \sim abx$$

$$\sqrt[b]{1+ax} - 1 \sim \frac{a}{b}x$$

$$1 - \cos x \sim \frac{x^2}{2}$$

$$x - \ln(1+x) \sim \frac{x^2}{2}$$

When $x \to 0$

$$\tan x - \sin x \sim \frac{x^3}{2}$$

$$\tan x - x \sim \frac{x^3}{3}$$

$$x - \arctan x \sim \frac{x^3}{3}$$

$$x - \sin x \sim \frac{x^3}{6}$$

$$\arcsin x - x \sim \frac{x^3}{6}$$

Example: Solve the limit

$$
\begin{aligned}
\lim_{x\to 0} \frac{\ln(1+4x)}{\sin(3x)} &= \lim_{x\to 0} \frac{\ln(1+4x)}{\sin(3x)} \lim_{x\to 0} \frac{4x}{\ln(1+4x)} \lim_{x\to 0} \frac{\sin(3x)}{3x} \\
&= \lim_{x\to 0} \frac{4x}{3x} = 4/3
\end{aligned}
$$

For more exercise regarding to equivalent infinitesimal, please refer to Worksheet 1.

Definition

Taylor expansion around $x = x_0$:

$f(x) = f(x_0) + \sum_{i=1}^{n} \frac{f^{(i)}(x)}{i!}(x - x_0)^i + R_n$, where $R_n = o[(x - x_0)^n]$

It simulates a function around a point with a polynomial function.

Taylor Expansion

Taylor expansion of some polynomials when x is around 0:

1. $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$
2. $ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$
3. $sinx = x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)$
4. $cosx = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)$
5. $tanx = x + \frac{x^3}{3} + \frac{2x^5}{15} + o(x^5)$

Tip

$o(x^n)$ *means the order of the polynomial is* **larger** *than* n;

$O(x^n)$ *means the order of the polynomial is* **larger than or equal to** n.

The transformation of Taylor Expansion:

Example

The Taylor expansion of e^{x^2} around x=0:

$e^{x^2} = 1 + x^2 + \frac{(x^2)^2}{2} + \frac{(x^2)^3}{6} + o((x^2)^3) = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + o(x^6)$

Exercise 11

1. Calculate The Taylor expansion of:
   1. e^{sinx} around x=0 (below degree 4)
   2. $ln(2+x)$ around x=-1 (below degree 4)
2. Calculate the limit

$$\lim_{x\to 0} \frac{ln(e^{sinx}+sinx)-ln(e^{tanx}+tanx)}{x^2\cdot tanx}$$

Exercise 11

Solution:

(1)

1. $e^{sinx} = 1 + (x - \frac{1}{6}x^3 + O(x^5)) + \frac{1}{2}(x - \frac{1}{6}x^3 + O(x^5))^2 + \frac{1}{6}(x + O(x^3))^3 + \frac{1}{24}(x + O(x^3))^4 = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + O(x^5)$
2. $ln(2+x) = ln(1+(1+x)) = ln(1+x) = (1+x) - \frac{(1+x)^2}{2} + \frac{(1+x)^3}{3} - \frac{(1+x)^4}{4} + O((1+x)^5)$

Solution:

(2)

The denominator's Taylor expansion is $x^2 tanx = x^3 + O(x^5)$, thus for the numerator we can ignore elements of degree 4 or higher.

1. $e^x + x = 1 + 2x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4)$
2. $ln(e^x + x) = 2x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{1}{2}(2x + \frac{x^2}{2} + \frac{x^3}{6})^2 + \frac{1}{3}(2x + \frac{x^2}{2} + \frac{x^3}{6})^3 + O(x^4) = 2x - \frac{3}{2}x^2 - \frac{1}{2}x^3 + O(x^4)$
3. $ln(e^{sinx} + sinx) = 2x - \frac{1}{3}x^3 - \frac{3}{2}x^2 - \frac{1}{2}x^3 + O(x^4)$
4. $ln(e^{tanx} + tanx) = 2x + \frac{2}{3}x^3 - \frac{3}{2}x^2 - \frac{1}{2}x^3 + O(x^4)$

$$ans = \frac{-x^3}{x^3} = -1$$

Absolute Maximum and Absolute Minimum

Let c be a number in the domain D of a function f. Then $f(c)$ is the
- **absolute maximum** value of f on D if $f(c) \geqslant f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leqslant f(x)$ for all x in D.

Local Maximum and Local Minimum

The number $f(c)$ is a
- **local maximum** value of f if $f(c) \geqslant f(x)$ when x is near c.
- **local minimum** value of f if $f(c) \leqslant f(x)$ when x is near c.

Critical number

A **critical number** of a function f is a number c in the domain such that either $f'(c) = 0$ or $f'(c)$ does not exist.

local maximum/minimum $\rightarrow$ critical number

Related Theorems

The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Fermat's Theorem

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Rolle's Theorem

Let f be a function that satisfies the following three hypotheses:
1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

Lagrange Mean Value Theorem

Let f be a function that satisfies the following hypotheses:
1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently, $f(b) - f(a) = f'(c)(b - a)$

Cauchy Mean Value Theorem (Extended Mean Value Theorem)

Let f, g be two functions that satisfy the following hypotheses:
1. f, g is continuous on the closed interval $[a, b]$.
2. f, g is differentiable on the open interval (a, b).
3.$x \in (a, b), g'(x) \neq 0$
Then there is a number c in (a, b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

or, equivalently, $(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$

Exercise 12

Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

1. $f(x) = x^3 - 3x + 2, \quad [-2, 2]$
2. $f(x) = \ln x, \quad [1, 4]$

Exercise 12

Solutions:

1. f(-2) = 0, f(2) = 4

   $k = \dfrac{f(2) - f(-2)}{4} = 1$

   $f'(x) = 3x^2 - 3, 3c^2 - 3 = 1$

   $c = \dfrac{2\sqrt{3}}{3}$

2. $k = \dfrac{f(4) - f(1)}{3} = \dfrac{2ln2}{3}$

   $f'(x) = \dfrac{1}{x}$

   $d = \dfrac{3}{2ln2}$

Relationship between f' and f

Increasing/Decreasing Test

If $f'(x) > 0$ on an interval, then f is increasing on that interval.
If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

The First Derivative Test

Suppose that c is a critical number of a continuous function f.
If f' changes from positive to negative at c, f has a local maximum at c.
If f' changes from negative to positive at c, f has a local minimum at c.
If f' does not change sign at c, f has no local maximum or minimum at c.

Relationship between f'' and f

Concave upward/downward

If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on an interval I, then it is called **concave downward** on I.

Concavity Test

If $f''(x) > 0$ for all x in I, then the graph of I is concave upward on I. If $f''(x) < 0$ for all x in I, then the graph of I is concave downward on I.

Relationship between f'' and f

Inflection point

A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

The Second Derivative Test

Suppose f'' is continuous near c.
If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

L'Hospital's Rule

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Exercise 13

Warning: Always judge whether l'Hôpital's rule can be applied before you use it, and **don't neglect those basic methods of finding the limit**.

Evaluate the following limits:

1. $\lim_{x \to 0^+} x^{\sin x}$
2. $\lim_{x \to 1} \frac{\ln x}{x - 1}$
3. $\lim_{x \to \infty} x^3 e^{-x^2}$
4. $\lim_{x \to \infty} [x - x^2 \ln \frac{x+1}{x}]$

5. $\lim_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}}$

Exercise 13

Solution:

1. 1
2. 1
3. $\lim_{x\to\infty}\frac{x^3}{e^{x^2}} = \lim_{x\to\infty}\frac{3x^2}{2xe^{x^2}} = \lim_{x\to\infty}\frac{6x}{(2+4x^2)e^{x^2}} = \lim_{x\to\infty}\frac{6}{12x+8x^2} = 0$
4. $u = \frac{1}{x}$

   $$\lim_{u\to 0}\frac{u - ln(u+1)}{u^2} = \lim_{u\to 0}\frac{1-\frac{1}{u+1}}{2u} = \lim_{u\to 0}\frac{\frac{1}{(u+1)^2}}{2} = \frac{1}{2}$$
5. $\lim_{x\to\infty}\frac{\sqrt{x^2+1}}{x} = 1$

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6. Integral

Antiderivatives

Definition

A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

Theorem

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

, where C is an arbitrary constant.

Function	Antiderivative	Function	Antiderivative
$cf(x)$	$cF(x)$	$\sec^2 x$	$\tan x$
$f(x)+g(x)$	$F(x)+G(x)$	$\sec x \tan x$	$\sec x$
$x^n (n \neq -1)$	$\frac{x^{n+1}}{n+1}$	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
$\frac{1}{x}$	$\ln \lvert x \rvert$	$\frac{1}{1+x^2}$	$\tan^{-1} x$
e^x	e^x	$\cosh x$	$\sinh x$
$\cos x$	$\sin x$	$\sinh x$	$\cosh x$
$\sin x$	$-\cos x$		

notation

$$\int f(x)dx = F(x) \quad \text{means} \quad F'(x) = f(x)$$

How to understand this notation?

Theorem

If the antiderivatives of functions *f(x)* and *g(x)* exist, then for any constant p and q, the antiderivate of *pf(x)+qg(x)* also exists, and we have

$$\int [pf(x) + qg(x)]dx = p\int f(x)dx + q\int g(x)dx$$

Indefinite Integrals

Keep the following indefinite integrals firmly in mind!

$$\int cf(x)dx = c\int f(x)dx \qquad \int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$$

$$\int kdx = kx + C \qquad \int \frac{1}{x}dx = \ln|x| + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad \int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int e^x dx = e^x + C \qquad \int \cos x dx = \sin x + C$$

$$\int \sin x dx = -\cos x + C \qquad \int \csc^2 x dx = -\cot x + C$$

$$\int \sec^2 x dx = \tan x + C \qquad \int \csc x \cot x dx = -\csc x + C$$

$$\int \sec x \tan x dx = \sec x + C \qquad \int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C$$

$$\int \frac{1}{x^2+1}dx = \arctan x + C \qquad \int \cosh x dx = \sinh x + C$$

$$\int \sinh x dx = \cosh x + C$$

Substitution Rule

Definition

$$\int f(x)g'(x)dx = \int f(x)dg(x)$$

How to memorize?

$\dfrac{dg(x)}{dx} = g'(x)$

$\Rightarrow dx = \dfrac{dg(x)}{g'(x)}$

$\Rightarrow \int f(x)g'(x)dx = \int f(x)g'(x) \cdot \dfrac{dg(x)}{g'(x)} = \int f(x)dg(x)$

Type 1: Direct Substitution ($u = g(x)$)

Substitution Rule for Direct Substitution

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(g(x))dg(x) = \int f(u)du = F(u) + C$$

Type 2: Inverse Substitution ($x = \varphi(t)$)

Substitution Rule for Inverse Substitution

If $x = \varphi(t)$ is an invertible function, then

$$\int f(x)dx = \int f(\varphi(t))d\varphi(t) = \int f(\varphi(t))\varphi'(t)dt = \widetilde{F}(t) = \widetilde{F}(\varphi^{-1}(x)) + C$$

Exercise 1

1. $\int \frac{3x^3}{1-x^4}dx$

2. $\int \frac{lnx}{x}dx$

3. $\int \frac{sec^2x}{tan^{10}x}dx$

An Important Application: Trigonometric Substitutions:

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Exercise 2

$$\int \frac{1}{x^2\sqrt{x^2+4}}$$

$$\int u dv = uv - \int v du$$

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

How to choose $f(x)$ and $g'(x)$?

Among the following function types:
the more to the left, the more suitable to be $f(x)$
the more to the right, the more suitable to be $g'(x)$

(left) inverse trigonometric function, logarithm function, power function, trigonometric function, exponential function **(right)**

Applicable Situations

When should we consider applying Integration by Parts?

1. If the integrand is the product of **inverse trigonometric function, logarithm function or power function** and another function **whose antiderivative is easy to calculate**.

2. If the integrand is the product of **trignometric function and exponential function**, apply Integration by Part twice in order to find an identical equation about the original integral, and then solve that equation.

3. If the integrand contains n (or a relatively large number), apply Integration by Part to **find the recursion formula**.

Exercise 3

1. $\int xsinx \ dx$
2. $\int ln^2x \ dx$
3. $\int x^2e^x \ dx$

1. If at least m and n is odd, adopt substitution rule.
$\int sin^m x cos^{2n+1} x dx = \int sin^m x cos^{2n} x d(sinx) =$

$\int sin^m x \cdot (1 - sin^2 x)^n d(sinx) \overset{\text{u=sinx}}{=} \int u^m (1 - u^2)^n du$

$\int cos^m x sin^{2n+1} x dx = -\int cos^m x sin^{2n} x d(cosx) =$
$-\int cos^m x \cdot (1 - cos^2 x)^n d(cosx) \overset{\text{u=cosx}}{=} -\int u^m (1 - u^2)^n du$

2. Else, use half-angle identities that $sin^2 x = \frac{1}{2}(1 - cos2x)$,
$cos^2 x = \frac{1}{2}(1 + cos2x)$.

1. If m is even (not 0), use $d(tanx) = sec^2 xdx$ to transform it into the integration of tanx.

   $\int sec^{2k} xtan^n xdx = \int (tan^2 x + 1)^k tan^n xd(tanx) \overset{\text{u=tanx}}{=}$
   $\int (u^2 + 1)^k u^n du.$

2. If n is odd, use$d(secx) = secxtanxdx$ to transform it into the integration of secx.

   $\int tan^{2k+1} xsec^n dx = \int (sec^2 x - 1)^k sec^{n-1} d(secx) = \cdots$

Though not as useful, you can also explore $\int csc^m xcot^n xdx$.

Use Product-to-Sum formula:

1. $sin\alpha cos\beta = \frac{1}{2}(sin(\alpha+\beta) + sin(\alpha-\beta))$
2. $cos\alpha sin\beta = \frac{1}{2}(sin(\alpha+\beta) - sin(\alpha-\beta))$
3. $sin\alpha sin\beta = \frac{1}{2}(cos(\alpha+\beta) + cos(\alpha-\beta))$
4. $cos\alpha cos\beta = -\frac{1}{2}(cos(\alpha+\beta) - cos(\alpha-\beta))$

Exercise 4

1. $\int_{0}^{\pi/2} sin^7 x cos^5 x \; dx$

Exercise 1-4 Answer

Exercise 1:

1. ans=$0.75\int \frac{1}{1-x^4}dx^4 = -\frac{3}{3}\int \frac{1}{1-x^4}d(1-x^4) = -\frac{3}{4}ln|1-x^4|+C$
2. $d(lnx) = dx \cdot \frac{1}{x} \Rightarrow ans = \int lnxd(lnx) = \frac{1}{2}ln^2x + C$
3. $d(tanx) = sec^2x \cdot dx \Rightarrow ans = \int \frac{d(tanx)}{tan^{10}x} = -\frac{1}{9}tan^{-9}x + C$

Exercise 2:

Let $x =: 2tan\theta$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\frac{dx}{d\theta} = 2sec^2\theta, \sqrt{x^2+4} = 2sec\theta$, ans=$\int \frac{cos\theta}{4sin^2\theta}d\theta$.

Then $u := sin\theta$, ans=$\int \frac{du}{4u^2} = -\frac{1}{4u} + C = -\frac{1}{4sin\theta} + \mathrm{C} = \frac{-\sqrt{x^2+4}}{4x} + C$

Exercise 3:

1. ans=$-\int xd(cosx) = -xcosx + \int cosxdx = -xcosx + sinx + C$
2. ans=$xln^2x - \int 2lnxdx = xln^2x - 2xlnx + \int 2dx = xln^2x - 2xlnx + 2x + C$
3. ans=$x^2e^x - 2\int xe^xdx = x^2e^x - 2xe^x + 2\int e^xdx = (x^2-2x+2)e^x + C$

Exercise 4:

1. $\int_0^{\pi/2} sin^7xcos^5x\ dx = \int_0^1 sin^7x(1-sin^2x)^2d(sinx) = [\frac{1}{8}u^8 - \frac{1}{5}u^{10} + \frac{1}{12}u^{12}]_0^1 = \frac{1}{120}$

Use **induction**.

$$\int e^{ax} sin^n x\ dx$$
$$= \frac{1}{a} \int sin^n x\ d(e^{ax})$$
$$= \frac{1}{a} e^{ax} sin^n x - \frac{n}{a} \int e^{ax} sin^{n-1} x cos x\ dx$$
$$= \frac{1}{a} e^{ax} sin^n x - \frac{n}{a^2} e^{ax} sin^{n-1} x cos x + \frac{n}{a^2} \int e^{ax} ((n-1) sin^{n-2} cos^2 x - sin^n x)\ dx$$
$$= \frac{1}{a} e^{ax} sin^n x - \frac{n}{a^2} e^{ax} sin^{n-1} x cos x + \frac{n}{a^2} \int e^{ax} ((n-1) sin^{n-2} - n sin^n x)\ dx$$

$\Rightarrow$

$$\frac{a^2+n^2}{a^2} \int e^{ax} sin^n x\ dx = \frac{e^{ax} sin^{n-1} x}{a} (sin x - \frac{n}{a} cos x) + \frac{n(n-1)}{a^2} \int e^{ax} sin^{n-2}\ dx$$

$\Rightarrow$

$$\int e^{ax} sin^n x\ dx = \frac{e^{ax} sin^{n-1} x}{a^2+n^2} (a sin x - n cos x) + \frac{n(n-1)}{a^2+n^2} \int e^{ax} sin^{n-2}\ dx$$

Use similar method,

1. $\int e^{ax} sin^n x \ dx$
$= \dfrac{e^{ax} cos^{n-1} x}{a^2 + n^2}(acosx + nsinx) + \dfrac{n(n-1)}{a^2+n^2} \int e^{ax} cos^{n-2} \ dx$

2. $\int sin^n x dx = -\dfrac{1}{n} sin^{n-1} x cosx + \dfrac{n-1}{n} \int sin^{n-2} x dx$

3. $\int cos^n x dx = \dfrac{1}{n} cos^{n-1} x sinx + \dfrac{n-1}{n} \int cos^{n-2} x dx$

4. $\int sin^{-n} x dx = -\dfrac{1}{n-1} \cdot \dfrac{cosx}{sin^{n-1} x} + \dfrac{n-2}{n-1} \cdot \int sin^{-(n-2)} x dx$

5. $\int cos^{-n} x dx = \dfrac{1}{n-1} \cdot \dfrac{sinx}{cos^{n-1} x} + \dfrac{n-2}{n-1} \cdot \int cos^{-(n-2)} x dx$

Exercise 5

1. Try to prove that $\int (secx)^{2n+1}dx = \frac{tanx \cdot (secx)^{2n-1}}{2n} + \frac{2n-1}{2n}\int (secx)^{2n-1}dx$ if $n \geq 1$
2. Calculate $\int sec^3 xdx$

Solution:

(1)

$$\int (secx)^{2n+1}dx$$
$$= \int (secx)^{2n-1}d(tanx)$$
$$= tanx \cdot (secx)^{2n-1} - (2n-1)\int tan^2 x sec^{2n-1} x dx$$
$$= tanx \cdot (secx)^{2n-1} - (2n-1)\int sec^{2n-1} x dx + (2n-1)\int sec^{2n+1} x dx$$

(2)

$$\frac{1}{2}(ln|tanx + secx| + secxtanx) + C$$

Theorem

The antiderivatives of rational function are always elementary functions.
Here, rational functions are in the form that $\frac{p_m(x)}{q_n(x)}$, where $p_m(x)$ and $q_n(x)$ are polynomials in the degree of m and n.

Basic rational functions

1. $$\int \frac{dx}{ax+b} = \frac{1}{a} ln|ax+b| + C$$
2. $$\int \frac{dx}{(ax+b)^n} = \int \frac{d(ax+b)}{a(ax+b)^n} = \frac{1}{a(n-1)(ax+b)^{n-1}} + C$$
3. $$\int \frac{xdx}{ax^2+bx+c} = \frac{1}{2a} ln|ax^2+bx+c| - \frac{b}{2a} \int \frac{dx}{ax^2+bx+c}$$
4. $$\int \frac{dx}{ax^2+bx+c} \overset{\Delta=b^2-4ac<0}{=} \frac{2}{\sqrt{-\Delta}} arctan \frac{2ax+b}{\sqrt{4ac-b^2}} + C$$
5. $$\int \frac{dx}{ax^2+bx+c} \overset{\Delta=b^2-4ac>0}{=} \frac{1}{\sqrt{\Delta}} ln\left|\frac{2ax+b-\sqrt{\Delta}}{2ax+b+\sqrt{\Delta}}\right| + C$$

Remark. What if $\Delta = 0$?

Then, what about $\int \frac{x+d}{(ax^2+bx+c)^n}dx$?

1. $\int \frac{x+d}{(ax^2+bx+c)^n}dx = \frac{1}{2a}\int \frac{2ax+b}{(ax^2+bx+c)^n}dx + (d - \frac{b}{2a})\int \frac{1}{(ax^2+bx+c)^n}dx$
2. $\frac{1}{2a}\int \frac{2ax+b}{(ax^2+bx+c)^n}dx = \frac{1}{2a}\int \frac{d(ax^2+bx+c)}{(ax^2+bx+c)^n} = \cdots$
3. $\int \frac{1}{(ax^2+bx+c)^n}dx = 16a^2\int \frac{dx}{((2ax+b)^2-b^2+4ac)^n} = 8a\int \frac{du}{(u^2-b^2+4ac)^n}$
4. $\int \frac{du}{(au^2+b)^n} = \frac{2n-3}{2b(n-1)}\int \frac{du}{(au^2+b)^{n-1}} + \frac{u}{2b(n-1)(au^2+b)^{n-1}}$
5. Combine all tools above up!

Form: $\dfrac{f(x)}{[A(x)]^a[B(x)]^b \ldots} = \dfrac{f_a(x)}{[A(x)]^a} + \dfrac{f_b(x)}{[B(x)]^b} + \ldots$

Purpose: Split a complex rational function into several functions whose forms are familiar and easy to integrate, or change the integration into familiar forms with trignometric substitution

Critical skill: **Undetermined coefficient method, Trigonometrical Substitution, Reciprocal Substitution.**

Also, as rational functions are easier to deal with, we can carry out **Rationalization.**

Key chracter: Whether the denominator of the integrand can be factorized or not

If the degree of the polynomial (or other type of functions) in the numerator is greater than or equal to that in the denominator, **extract a polynomial (or other type of functions)**.

If the degree of the polynomial in the denominator is too high, in order to reduce the workload, it's recommended to **apply substitution rule before attempting to split the function**.

Exercise 6-Undetermined coefficient method

1. $\int \frac{4y^2 - 7y - 12}{y(y+2)(y+3)} dy$
2. $\int \frac{x^4 + x^3 + 3x^2 - 1}{(x^2+1)^2(x-1)} dx$

Exercise 7-Reciprocal Substitution

1. $\int \frac{dx}{x^8(1+x^2)}$

Exercise 8-Rationalization

1. $\int \frac{dx}{\sqrt{1-x} + \sqrt[3]{1-x}}$

When encountering complex conbination of trigonometric functions, we can substitute $t = tan\frac{x}{2}$.
Then there is
$sinx = \frac{2t}{t^2+1}, cosx = \frac{1-t^2}{t^2+1}, tanx = \frac{2t}{1-t^2}, \frac{dx}{dt} = \frac{2}{t^2+1}$.
It is thus converted into rational functions.

Exercise 9-"Universal Substitution"

1. $\int \frac{dx}{1 + cosx + sin^2(\frac{x}{2})}$

Definition

If a function f(x) is defined on the interval [a, b], insert several points randomly, and we will get:

$$a = x_0 < x_1 < \cdots < x_n = b$$

which defines n intervals whose corresponding lengthes are

$$\Delta x_k = x_k - x_{k-1}$$

The for every interval select $\xi_k \in [x_{k-1}, x_k]$, the Riemann Sum of f(x) on the interval [a, b] is:

$$S_n = \sum_{k=1}^{n} f(\xi_k)\Delta_k$$

Riemann Sum is the sum of areas of rectangles.

In order to calculate the area of a figure:

1. Cut that irregular figure into small strips, and then regard the strip as a rectangle approximately.
2. Measure the lengths of these small rectangles respectively and calculate each of their area.
3. Add up all the rectangular areas to get the total area.

Only applicable to one situation: $\int e^{x^2} dx$.
For simplicity, let us suggest $x \geq 0$.
$\int e^{x^2} dx = \sqrt{\int e^{x^2} dx \cdot \int e^{y^2} dy} = \sqrt{\iint e^{x^2+y^2} dxdy} = \sqrt{\iint re^{r^2} drd\theta} = \sqrt{\pi e^{x^2}}$.

Example 1

$\int sin(lnx)dx =$

$xsin(lnx) - \int xcos(lnx) \cdot \frac{1}{x} dx = xsin(lnx) - xcos(lnx) - \int sin(lnx)dx$

$\Rightarrow \int sin(lnx)dx = \frac{xsin(lnx) - xcos(lnx)}{2}$

Example 2

$I = \int \frac{sin^2 x}{sinx + \sqrt{3}cosx} dx$

define $J := \frac{cos^2 x}{sinx + \sqrt{3}cosx} dx$, $I + J = \int \frac{1}{sinx + \sqrt{3}cosx} dx =$

$\frac{1}{2} \int \frac{1}{sin(x + \frac{\pi}{3})} dx = \frac{1}{2} ln|csc(x + \frac{\pi}{3}) - cot(x + \frac{\pi}{3})| + C$

$I - 3J = \int sinx - \sqrt{3}cosxdx = -cosx - \sqrt{3}sinx + C$

Thus I $= \frac{3}{8} ln|csc(x + \frac{\pi}{3}) - cot(x + \frac{\pi}{3})| - \frac{1}{4} cosx - \frac{\sqrt{3}}{4} sinx + C$

Definition

If f is a function defined for $a \leq x \leq b$, we divide the integral [a, b] into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0(= a), x_1, x_2, ..., x_n(= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, ..., x_n^*$ ne any sample points in these subintervals, so x_i^* lies in the ith subinterval. Then the definite integral of f from a to b is

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is integrable on [a,b].

Definite integral can also be intuitively understood as the net area between the curve (f(x), x), line x = a, x = b, and x axis.

Properties

1. **Linearity**

$$\int_a^b [k_1 f(x) + k_2 g(x)]dx = k_1 \int_a^b f(x)dx + k_2 \int_a^b g(x)dx$$

2. **Internal Additivity**

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$$

3. **Comparison Properties**
   If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
   If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

Properties

4. **Absolute Integrability**
   If f(x) is intergrable on [a,b], then $|f(x)|$ is also integrable on [a,b], and we have

$$|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$$

Note

If f(x) and g(x) are intergrable on [a,b], then f(x)·g(x) is also intergrable on [a,b], but generally

$$\int_a^b f(x)g(x)dx \neq (\int_a^b f(x)dx) \cdot (\int_a^b g(x)dx)$$

Exercise 10

Calculate the definite integral by definition

1. $\int_{-3}^{0}(1+\sqrt{9-x^2})dx$
2. $\int_{\pi}^{\pi} sin^2 x cos^4 x dxx$

Exercise 10

Solutions:

1. $\int_{-3}^{0}(1+\sqrt{9-x^2})dx$ can be interpreted as the area under the graph of $f(x) = 1+\sqrt{9-x^2}$ between x = -3 and x = 0. This is equal to one-quarter the area of the circle with radius 3, plus the area of the rectangle, so
   $\int_{-3}^{0}(1+\sqrt{9-x^2})dx = \frac{1}{4}\pi \cdot 3^2 + 1 \cdot 3 = 3 + \frac{9}{4}\pi$
2. $\int_{\pi}^{\pi} sin^2 x cos^4 x dx = 0$
   since the limits of intergral are equal

Newton-Leibniz Formula

Suppose f is continuous on [a,b].

- If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$
- If F is any antiderivative of f(i.e. $F' = f$), then $\int_a^b f(x)dx = F(b) - F(a)$

Exercise 11

1. $\int_0^{\pi} \sqrt{sin^3 x - sin^5 x} dx$
2. $\int_0^3 \frac{x^2}{(x^2 - 3x + 3)^2} dx$

Solutions: 1. $\int_0^{\pi} \sqrt{sin^3x - sin^5x}dx$
Since $\sqrt{sin^3x - sin^5x} = \sqrt{sin^3x(1 - sin^2x)} = sin^{\frac{3}{2}} \cdot |cosx|$, where
$|cosx| = cosx$ on $[0, \frac{\pi}{2}]$; $|cosx| = -cosx$ on $[\frac{\pi}{2}, \pi]$,
$\int_0^{\pi} \sqrt{sin^3x - sin^5x}dx$
$= \int_0^{\frac{\pi}{2}} sin^{\frac{3}{2}}xcosxdx + \int_{\frac{\pi}{2}}^{\pi} sin^{\frac{3}{2}}x(-cosx)dx$
$= \int_0^{\frac{\pi}{2}} sin^{\frac{3}{2}}xd(sinx) - \int_{\frac{\pi}{2}}^{\pi} sin^{\frac{3}{2}}xd(sinx)$
$= [\frac{2}{5}sin^{\frac{5}{2}}x]_0^{\frac{\pi}{2}} - [\frac{2}{5}sin^{\frac{5}{2}}x]_{\frac{\pi}{2}}^{\pi}$
$= \frac{2}{5} - (-\frac{2}{5}) = \frac{4}{5}$

Exercise 11

Solutions: 2. $\int_0^3 \frac{x^2}{(x^2-3x+3)^2}dx$

$x^2-3x+3=(x-\frac{3}{2})^2+\frac{3}{4}, letx-\frac{3}{2}=\frac{\sqrt{3}}{2}tanu(|u|<\frac{\pi}{2})$, such that

$(x^2-3x+3)^2=(\frac{3}{4}sec^2u)^2,\ dx=\frac{\sqrt{3}}{2}sec^2udu.$

$x=0\rightarrow u=-\frac{\pi}{3};\ x=3\rightarrow u=\frac{\pi}{3}.$

$$\int_0^3 \frac{x^2}{(x^2-3x+3)^2}dx=\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}(\frac{3}{4}tan^2u+\frac{3\sqrt{3}}{2}tanu+\frac{9}{4})\cdot\frac{16}{9}\cdot\frac{\sqrt{3}}{2}cos^2udu$$

$$=\frac{8}{3\sqrt{3}}\cdot 2\int_0^{\frac{\pi}{3}}(\frac{3}{4}tan^2u+\frac{9}{4})cos^udu$$

$$=\frac{4}{\sqrt{3}}\int_0^{\frac{\pi}{3}}(sin^2u+3cos^2u)du=\frac{4}{\sqrt{3}}\int_0^{\frac{\pi}{3}}(2+cos2u)du$$

$$=\frac{4}{\sqrt{3}}[2u+\frac{1}{2}sin2u]_0^{\frac{\pi}{3}}$$

$$=\frac{8\pi}{3\sqrt{3}}+1$$

Definition

Type1: Unboundedness

1. If $f(x)$ is continuous on $[a, +\infty]$, for any $t > a$, the improper integral of f(x) on $[a, +\infty]$ is

$$\int_a^{+\infty} f(x)dx = \lim_{t \to +\infty} \int_a^t f(x)dx$$

2. If $f(x)$ is continuous on $[-\infty, a]$, for any $t < a$, the improper integral of f(x) on $[-\infty, a]$ is

$$\int_a^{+\infty} f(x)dx = \lim_{t \to -\infty} \int_t^a f(x)dx$$

Definition

Type1: Unboundedness

3. If both $\int_a^{\infty} f(x)dx$ and $\int_{-\infty}^{a} f(x)dx$ are convergent

$$\int_{-\infty}^{+\infty} f(x)dx = \int_a^{+\infty} f(x)dx + \int -\infty^a f(x)dx$$

Definition

Type2: Discontinuous

1. If f is continuous on [a,b) and is discontinuous at b, then the improper integral on is

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

2. If f is continuous on (a,b] and is discontinuous at a, then the improper integral on is

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$$

Definition

Type2: Discontinuous

3. If f has a discontinuity at c (a¡c¡b), and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Convergence and divergence of improper integrals

If the limits mentioned above are exists (as a finite number), then the improper integral is called convergent, otherwise it's divergent.

Exercise 12

1. $\int_e^\infty \frac{1}{x(ln x)^3} dx$
2. $\int_0^3 \frac{1}{x^2 - 6x + 5} dx$

Exercise 12

Solutions: 1.

$$\int_e^{\infty} \frac{1}{x(lnx)^3}dx = \lim_{t\to\infty}\int_e^{t} \frac{1}{x(lnx)^3}dx$$

$$= \lim_{t\to\infty}\int_1^{lnt} u^{-3}du$$

$(u = lnx, du = dx/x)$

$$= \lim_{t\to\infty}\left[-\frac{1}{2u^2}\right]_1^{lnt}$$

$$= \lim_{t\to\infty}\left[-\frac{1}{2(lnt)^2}+\frac{1}{2}\right]$$

$$= 0 + \frac{1}{2} = \frac{1}{2}$$

Convergent

Solutions:

2. $\int_0^3 \frac{dx}{x^2-6x+5} = \int_0^3 \frac{dx}{(x-1)(x-5)}$

$= \int_0^1 \frac{dx}{(x-1)(x-5)} + \int_1^3 \frac{dx}{(x-1)(x-5)}$

$= I_1 + I_2$

$I_1 = \frac{1}{4} \lim_{t\to 1^-} \int_0^t \left(\frac{1}{x-5} - \frac{1}{x-1} \right) dx$

$= \lim_{t\to 1^-} \left[-\frac{1}{4} ln|t-1| + \frac{1}{4} ln|t-5| \right]_0^t$

$$= \lim_{t\to 1^-} \left[\left(-\frac{1}{4} ln|t-1| + \frac{1}{4} ln|t-5| \right) - \left(-\frac{1}{4} ln|-1| + \frac{1}{4} ln|-5| \right) \right]_0^t$$

$= \infty$

Since I_1 is divergent, I is divergent.

Theorem 1

Suppose $f(x)$ is continuous on $[a, \infty)$, and $f(x) \geq 0$. If

$$F(x) = \int_a^x f(t)dt$$

has an upper bound on $[a, \infty)$, then the improper integral $\int_a^\infty f(x)dx$ converges.

Theorem 2: Comparison Theorem

Suppose that f and g are continuous functions with $f(x) \geqslant g(x) \geqslant 0$ for $x \geqslant a$.
(a) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.
(b) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

Exercise 13

Judge whether the following improper integrals converge or not. If some of them converge, calculate their value.

1. $\int_1^{\infty} \frac{1}{x^4} dx$
2. $\int_0^{2} \frac{dx}{(1-x)^2}$
3. $\int_1^{\infty} \frac{1}{\sqrt{x}} dx$
4. $\int_0^{1} \frac{x}{\sqrt{1-x^2}} dx$

Exercise 13

Solutions:

1. $\int_0^\infty \frac{dx}{x^4} = -\left[\frac{1}{3x^3}\right] = \frac{1}{3}$
2. $\int_0^t \frac{dx}{(1-x)^2} = \left[\frac{1}{1-x}\right]_0^t = \frac{1}{1-t} - 1$ Diverge
3. $\int_1^t \frac{1}{\sqrt{x}} dx = [2\sqrt{x}]_1^t = 2\sqrt{t} - 2$ Diverge
4. $\int_0^1 \frac{xdx}{1-x^2} = -[\sqrt{1-x^2}]_0^1 = 1$

Exercise 14

Calculate the following improper integral:

$$\int_0^\infty \frac{1 - e^{-x^2}}{x^2} dx$$

You can directly use the conclusion that $\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$.
(How to calculate this integral will be taught in VV255.)

Solutions:

$$\int_0^\infty \frac{1-3^{-x^2}}{x^2}dx = \int_0^\infty (e^{-x^2}-1)d(\frac{1}{x})$$

$$= \left[\frac{e^{-x^2}-1}{x}\right]_0^\infty - \int_0^\infty \frac{1}{x}d(e^{-x^2}-1)$$

$$= \left[\frac{e^{-x^2}-1}{x}\right]_0^\infty + 2\int_0^\infty e^{-x^2}dx$$

$$\lim_{x\to\infty}\frac{e^{-x^2}-1}{x} = 0, \lim_{x\to 0}\frac{e^{-x^2}-1}{x} = \lim_{x\to 0}\frac{-2xe^{-x^2}}{1} = 0$$

$$2\int_0^\infty e^{-x^2}dx = \int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}$$

Therefore, $\int_0^\infty \frac{1-3^{-x^2}}{x^2}dx = \sqrt{\pi}$

Differenciation of definite integral

Theorem

$$\frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} f(x)dx = f(\beta(t))\beta'(t) - f(\alpha(t))\alpha'(t)$$

How to prove it?

Use the integration to deal with the combination of limits and series.

Example

1. $\lim_{n\to\infty} \sum_{i=0}^{n} \frac{i}{n^2}$
2. $\lim_{n\to\infty} \sqrt{n}(\frac{1}{n} - \sum_{i=0}^{n} \frac{1}{n+\sqrt{i}})$

Method

1. Simply Transform into Integration. $\lim\limits_{n\to\infty}\sum\limits_{i=0}^{n}\frac{i}{n^2}=\int\limits_0^n\frac{i}{n^2}di=\frac{1}{2}$

2. Use Squeeze Theorem. $ans=\lim\limits_{n\to\infty}\sqrt{n}\sum\limits_{i=0}^{n}\frac{\sqrt{i}}{n(n+\sqrt{i})}$

$$ans\le\lim_{n\to\infty}\sqrt{n}\sum_{i=0}^{n}\frac{\sqrt{i}}{n^2}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n}\sqrt{\frac{k}{n}}=\int\limits_0^1\sqrt{x}dx=\frac{2}{3}.$$

$$ans\ge\lim_{n\to\infty}\sqrt{n}\sum_{i=0}^{n}\frac{\sqrt{i}}{n(n+\sqrt{n})}=\lim_{n\to\infty}\frac{n}{n+\sqrt{n}}\frac{1}{n}\sum_{i=0}^{n}\sqrt{\frac{k}{n}}=\frac{2}{3}.$$

Exercise: Refer to Worksheet 1 1.3,1.4

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7. Parametric Equations & Polar Coordinates

Definition

Suppose that x and y are both given as functions of a third variable t (called a parameter) by the equations

$$x = f(t)\ y = g(t)$$

$$x = cost\ y = sint\ (0 \leqslant t \leqslant 2\pi)$$

If we plot points, it appears that the curve is a circle. We can confirm this impression by eliminating t. Observe that

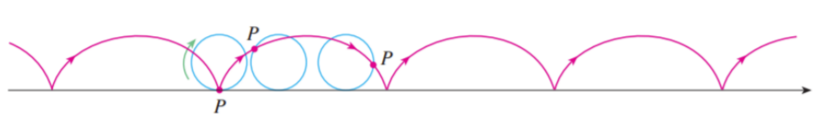
$$x^2 + y^2 = sin^2 t + cos^2 t = 1$$

Thus the point (x, y) moves on the unit circle $x^2 + y^2 = 1$. Notice that in this example the parameter t can be interpreted as the angle (in radians). As t increases from 0 to 2π, the point $(x, y) = (cost, sint)$ moves once around the circle in the counterclockwise direction starting from the point $(1, 0)$.

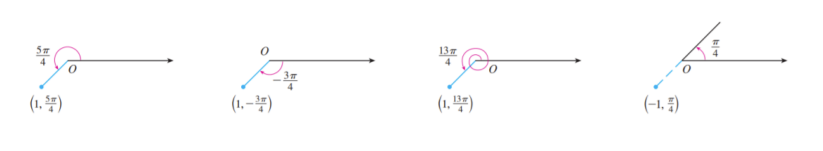
Definition

The curve traced out by a point on the circumference of a circle as the circle rolls along a straight line is called a cycloid . Therefore parametric equations of the cycloid are

$$x = r(\theta - sin\theta) \; y = r(1 - cos\theta) \; \theta \in R$$



We choose a point in the plane that is called the pole (or origin) and is labeled O. Then we draw a ray (half-line) starting at O called the polar axis. This axis is usually drawn horizontally to the right and corresponds to the positive x-axis in Cartesian coordinates.
If P is any other point in the plane, let r be the distance from O to P and let θ be the angle (usually measured in radians) between the polar axis and the line OP as in Figure 1 . Then the point P is represented by the ordered pair (r, θ) and r,θ are called polar coordinates of P. We use the convention that an angle is positive if measured in the counterclockwise direction from the polar axis and negative in the clockwise direction. If P = O, then $r = 0$ and we agree that $(0, \theta)$ represents the pole for any value of θ.



In fact, since a complete counterclockwise rotation is given by an angle 2π, the point represented by polar coordinates (r, θ) is also represented by

$$(r, \theta + 2n\pi) \text{ and } (-r, \theta + (2n+1)\pi)$$

Polar Coordinates and Cartesian Coordinates

Relationship between Polar Coordinates and Cartesian Coordinates

$$x = rcos\theta \ y = rsin\theta$$
$$r^2 = x^2 + y^2 \ tan\theta = \frac{y}{x}$$

Cartesian coordinates and Cylindrical coordinates

cylindrical to Cartesian coordinates

$$x = rcos\phi$$
$$y = rsin\phi$$
$$z = z$$

inverse relations (from Cartesian to cylindrical coordinates)

$$r = \sqrt{x^2 + y^2}$$
$$tan\phi = \frac{y}{x}$$
$$z = z$$

Cartesian coordinates and Spherical coordinates

spherical to Cartesian coordinates

$$x = rsin\theta cos\phi$$
$$y = rsin\theta sin\phi$$
$$z = rcos\theta$$

inverse relations (from Cartesian to spherical coordinates)

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$tan\phi = \frac{y}{x}$$
$$tan\theta = \frac{\sqrt{x^2+y^2}}{z}$$

Definition

- parametric representation:

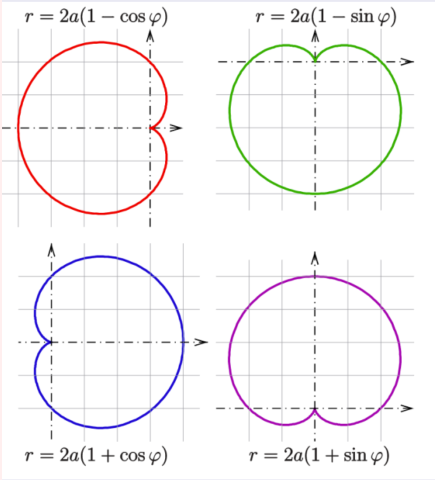
$$x(\theta) = 2a(1 - cos\theta)cos\theta$$
$$y(\theta) = 2a(1 - cos\theta)sin\theta$$
$$(0 \leqslant \theta \leqslant 2\pi)$$

- polar coordinates:

$$r = 2a(1 - cos\theta)$$
$$(0 \leqslant \theta \leqslant 2\pi)$$

Cardioid is special Cycloid and special Limaçon

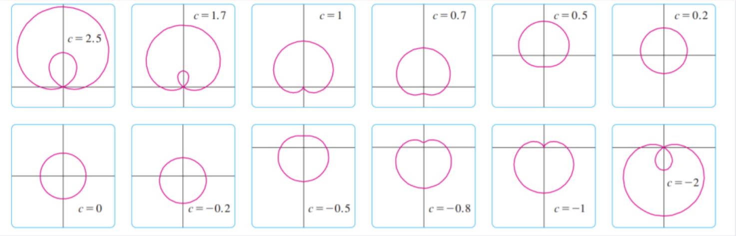
Cardioid



$r = 2a(1 - \cos\varphi)$
$r = 2a(1 - \sin\varphi)$
$r = 2a(1 + \cos\varphi)$
$r = 2a(1 + \sin\varphi)$

Definition

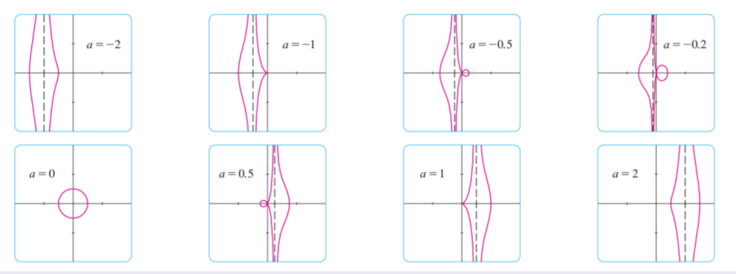
$$r = 1 + c\sin\theta$$



Conchoid

Definition

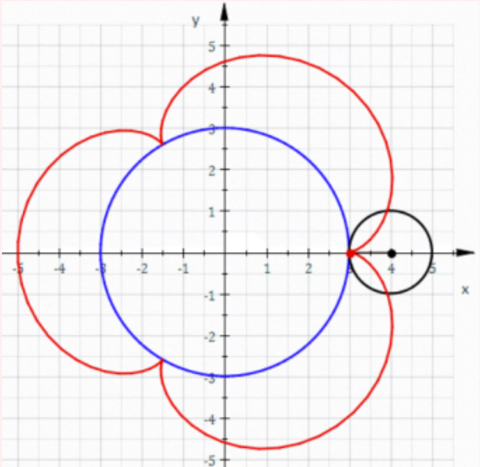
$$r = 1 + c\sec\theta$$



Epicycloid

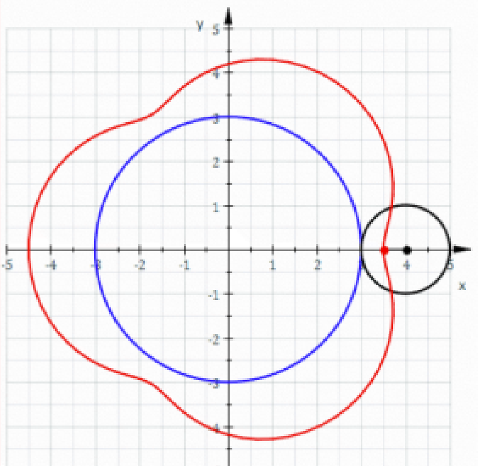
Definition

$$x(\theta) = (R + r)cos\theta - rcos(\frac{R+r}{r}\theta)$$
$$y(\theta) = (R + r)sin\theta - rsin(\frac{R+r}{r}\theta)$$



Definition

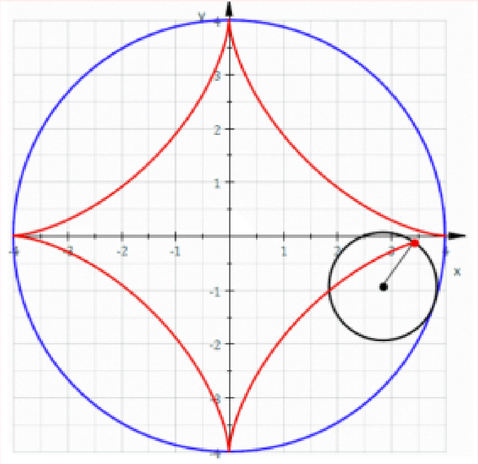
$$x(\theta) = (R + r)cos\theta - dcos(\frac{R+r}{r}\theta)$$
$$y(\theta) = (R + r)sin\theta - dsin(\frac{R+r}{r}\theta)$$



Hypocycloid

Definition

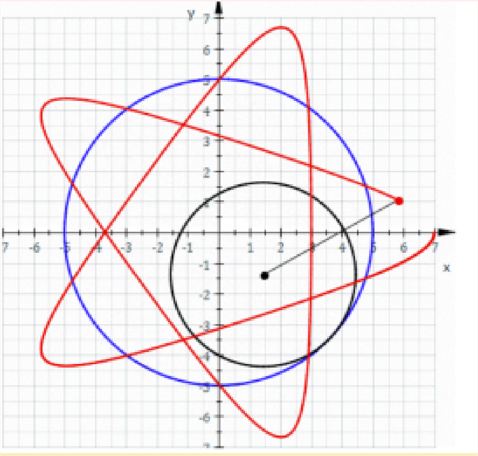
$$x(\theta) = (R - r)cos\theta - rcos(\frac{R-r}{r}\theta)$$
$$y(\theta) = (R - r)sin\theta - rsin(\frac{R-r}{r}\theta)$$



Hypotrochoid

Definition

$$x(\theta) = (R - r)cos\theta - dcos(\frac{R-r}{r}\theta)$$
$$y(\theta) = (R - r)sin\theta - dsin(\frac{R-r}{r}\theta)$$

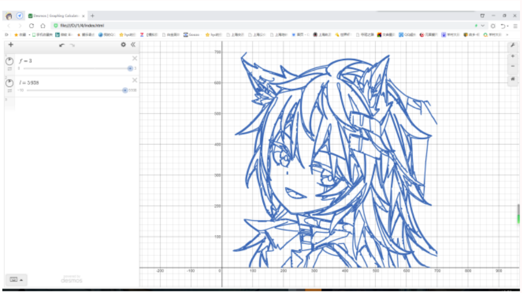


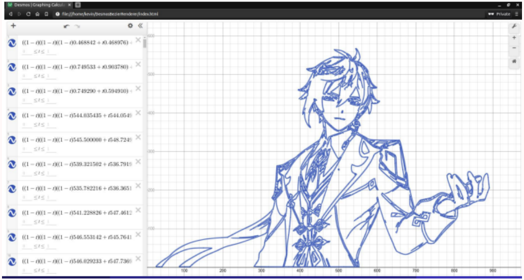
Definition

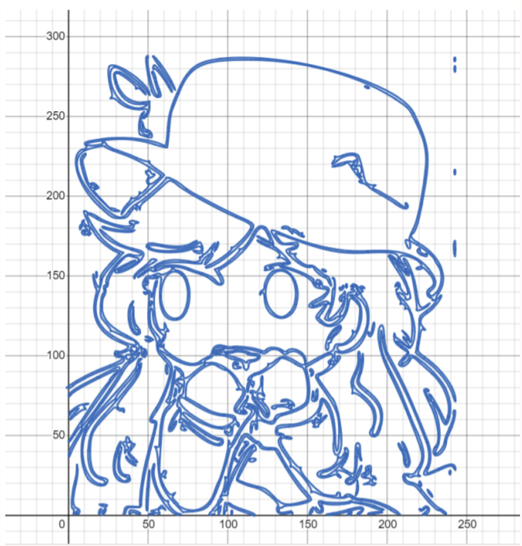
Bézier curves are used in computer-aided design and are named after the French mathematician Pierre Bézier (1910-1999), who worked in the automotive industry. A cubic Bézier curve is determined by four control points, $P_0(x_0, y_0)$,$P_1(x_1, y_1)$,$P_2(x_2, y_2)$, and $P_3(x_3, y_3)$, and is defined by the parametric equations:

$$
\begin{aligned}
x &= x_0(1-t)^3 + 3x_1 t(1-t)^2 + 3x_2 t^2(1-t) + x_3 t^3 \\
y &= y_0(1-t)^3 + 3y_1 t(1-t)^2 + 3x_2 t^2(1-t) + y_3 t^3
\end{aligned}
$$

More Details: https://zhuanlan.zhihu.com/p/471457420






300
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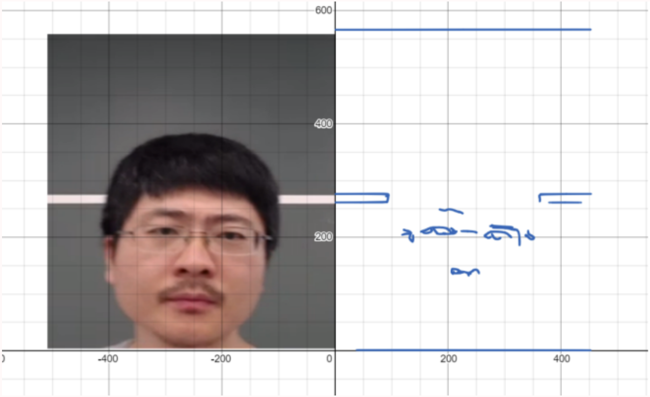
Application of Bézier Curves in desmos



Slope of the Tangent Line with Parametric Curves

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

For the second order derivative, we have:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{\frac{d}{dt}\frac{dy}{dx}}{\frac{dx}{dt}}$$

Area

We know that the area under a curve $y = F(x)$ from a to b is $A = \int_a^b F(x)dx$, where $F(x) \geqslant 0$. If the curve is traced out once by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, then we can calculate an area formula by using the Substitution Rule for Definite Integrals as follows:

$$A = \int_a^b y dx = \int_\alpha^\beta g(t)f'(t)dt$$

or

$$= \int_\beta^\alpha g(t)f'(t)dt$$

Arc Length

If a curve C is described by the parametric equations
$x = f(t)$,$y = g(t)$,$\alpha \leqslant t \leqslant \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases fromα to β, then the length of C is

$$L = \int_{\alpha}^{\beta} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$$

Surface Area

In the same way as for arc length, we can adapt to obtain a formula for surface area. If the curve given by the parametric equations $x = f(t)$,$y = g(t)$,$\alpha \leqslant t \leqslant \beta$ is rotated about the x-axis, where f' and g' are continuous and $g(t) \geqslant 0$, then the area of the resulting surface is given by

$$S = \int_{\alpha}^{\beta} 2\pi y \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$$

Slope of the Tangent Line with Polar Coordinates

$$x = rcos\theta, y = rsin\theta$$

where r can be regarded as a function of θ. Hence we have

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dy}{d\theta}sin\theta + rcos\theta}{\frac{dy}{d\theta}cos\theta - rsin\theta}$$

Suppose we have the function in polar coordinates:

$$r = f(\theta), a \leqslant \theta \leqslant b$$

For the area enclosed by this function, we have:

$$A = \int_a^b \frac{1}{2} f^2(\theta) d\theta$$

For the area enclosed by this function, we have:

$$L = \int_a^b \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta$$

Exercise 1

tangents

Find equations of the tangents to the curve $x = 3t^2 + 1, y = 2t^3 + 1$ that pass through the point $(4, 3)$.

Exercise 2

1. Find the exact length of the curve
$$x = 1 + 3t^2,\ y = 4 + 2t^3,\ 0 \leqslant t \leqslant 1$$
2. Find the area enclosed by the x-axis and the curve
$$x = 1 + e^t,\ y = t - t^2$$
3. Find the exact area of the surface obtained by rotating the given curve about the x-axis.
$$x = 3t - t^3,\ y = 3t^2,\ 0 \leqslant t \leqslant 1$$

Exercise 3

1. Find the slope of the tangent line to the given polar curve at the point specified by the value of θ
$$r = 2 - sin\theta,\ \theta = \frac{\pi}{3}$$
2. Find the area of the region enclosed by one loop of the curve:
$$r = 2sin5\theta$$

exercises 1:

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t^2}{6t} = t$

when pass through $(4, 3)$, we have

$3 - (2t^3 + 1) = t(4 - (3t^2 + 1)) \Rightarrow t = 1$ or $t = -2$

$\Rightarrow y = x - 1$ or $y = -2x + 11$

exercises 2:

1. $L = \int_0^1 \sqrt{(\frac{dy}{dt})^2 + (\frac{dx}{dt})^2} dt = \int_0^1 6t\sqrt{1+t^2} dt = \int_1^2 \sqrt{u}(\frac{1}{2}du) =$ $3 \cdot \frac{3}{2}[u^{\frac{3}{2}}]|_1^2 = 2(2\sqrt{2}-1)$
2. The curve $x = 1 + e^t$,$y = t - t^2 = t(1-t)$ intersects the x-axis when $y = 0$, that is, when $t = 0$ and $t = 1$. The corresponding values of x are 2 and $1 + e$. The shaded area is given by
$\int_2^{1+e}(y_T - y_B)dx = \int_0^1(y(t)-0)x'(t)dt = \int_0^1(t-t^2)e^t dt$
$= \int_0^1 te^t dt - \int_0^1 t^2 \cdot e^t dt = 3\int_0^1 te^t dt - t^2e^t|_0^1$
$= 3((t-1)e^t)|_0^1 - e = 3 - e$
3. $x = 3t - t^3$,$y = 3t^2$,$0 \leqslant t \leqslant 1$.
$(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = (3-3t^2)^2 + (6t)^2 = (3(1+t)^2)^2$
$S = \int_0^1 2\pi \cdot 3t^2 \cdot (1+t^2)dt$
$= 18\pi(\frac{1}{3}t^3 + \frac{1}{5}t^5)|_0^1$
$= \frac{48}{5}\pi$

Answers

1. $x = rcos\theta = (2 - sin\theta)cos\theta, y = rsin\theta = (2 - sin\theta)sin\theta$
   $\frac{dy}{dx} = \frac{2cos\theta - sin2\theta}{-2sin\theta - cos2\theta}$
   when $\theta = \frac{\pi}{3}$, $\frac{dy}{dx} = \frac{2-\sqrt{3}}{1-2\sqrt{3}}$
2. $sin5\theta = 0 \Rightarrow 5\theta = n\pi \Rightarrow \theta = \frac{\pi}{5}n$
   $A = \int_0^{\frac{\pi}{5}} \frac{1}{2}(2sin5\theta)^2 d\theta$
   $= 2\int_0^{\frac{\pi}{5}} \frac{1}{2}(1 - cos10\theta)|_0^{\frac{\pi}{5}} = \frac{\pi}{5}$

Reference

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2. Huang, Yucheng. VV156 RC6.pdf. 2021.
3. Chen, Jixiu et al. Mathematical Analysis (3rd Version). 2019
4. Li, Junhao. VV156 RC5.pdf.2022.

8. Series

A sequence a_n has the limit L and we write:

$$\lim_{n \to \infty} a_n = L \text{ or } a_n \to L \text{ as n} \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n \to \infty} a_n$ exists, we say the sequence converges (or is convergent). Otherwise, we say the sequence diverges (or is divergent).

Limits of Sequences: Precise Definition

Definition

Suppose that there is a sequence a_n. If for any fixed positive number ε, there exits a positive integer N such that for any $n > N$, we have

$$|a_n - L| < \varepsilon$$

Then we say that the sequence an has the limit L.

Theorems

1. If $lim_{n\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $lim_{n\to\infty} a_n = L$.
2. $lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that if $n > N$ then $a_n > M$.
3. (Squeeze theorem) If $a_n \leqslant b_n \leqslant c_n$ for $n \geqslant n_0$ and $lim_{n\to\infty} a_n = lim_{n\to\infty} c_n = L$, then $lim_{n\to\infty} b_n = L$.
4. If $lim_{n\to\infty} |a_n| = 0$, then $lim_{n\to\infty} a_n = 0$.
5. Every bounded, monotonic sequence is convergent.
6. *(Bolzano Weierstrass Theorem) A Bounded sequence must have a convergent subsequence.

Properties

If a_n and b_n are convergent sequences and c is a constant, then:

1. $\lim\limits_{n\to\infty}(a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$
2. $\lim\limits_{n\to\infty}(a_n - b_n) = \lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} b_n$
3. $\lim\limits_{n\to\infty} ca_n = c \lim\limits_{n\to\infty} a_n$
4. $\lim\limits_{n\to\infty}(a_n b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$
5. $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}, (\lim\limits_{n\to\infty} b_n \neq 0)$
6. $\lim\limits_{n\to\infty} a_n^p = (\lim\limits_{n\to\infty} a_n)^p, (p > 0, a_n > 0)$

Definition

Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ...$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{\infty} a_i = a_1 + a_2 + ... + a_n$$

If the sequence s_n is convergent and $\lim\limits_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called convergent and we write:

$$a_1 + a_2 + ... + a_n = s \text{ or } \sum_{i=1}^{\infty} a_i = s$$

The number s is called the sum of the series. If the sequence s_n is divergent, then the series is called divergent.

The geometric series:

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \ldots$$

s convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, |r| < 1$$

if If $|r| \geqslant 1$, the geometric series is divergent.

Definition

P-series is defied as

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

If $p = 1$, then we also call it harmonic series:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

It is divergent.

Definition

$$s_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} (a_n cos(\frac{2\pi nx}{P}) + b_n sin(\frac{2\pi nx}{P}))$$

Fundamental Properties of Series

Requirement for Convergent Series

Suppose the series $\sum_{n=1}^{\infty} x_n$ is convergent, then the sequence x_n is an infinitesimal, which means

$$\lim_{n\to\infty} x_n = 0$$

This can be used to test if a series is divergent.

Linearity for Convergent Series

Suppose $\sum_{n=1}^{\infty} a_n = A, \sum_{n=1}^{\infty} b_n = B$, and α, β are two constants, then

$$\sum_{n=1}^{\infty} (\alpha a_n + \beta b_n) = \alpha A + \beta B$$

Several Methods

1. Divergence Test Theorem (Requirement for Convergent Series)
2. Integral Test
3. Comparison Test
4. Cauchy Test (Root Test)
5. d'Alembert Test (Ratio Test)
6. Leibniz Test
7. Absolute Convergence Test

Integral Test

Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_1^\infty a_n$ an is convergent **if and only if** the improper integral is convergent. In other words:
(i) If $\int_1^\infty f(x)dx$ is convergent, then $\sum_{n=1}^\infty a_n$ is convergent. (ii) If $\int_1^\infty f(x)dx$ is divergent, then $\sum_{n=1}^\infty a_n$ is divergent.

Integral Test: Important Conclusions for p-series

For what values of p is the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

SOLUTION:
(i) If $p < 0$, then $lim_{n\to\infty}(\frac{1}{n^p}) = \infty$
(ii) If $p = 0$ $lim_{n\to\infty}(\frac{1}{n^p}) = 1$
In either case $lim_{n\to\infty}(\frac{1}{n^p}) \neq 0$
(iii)$p > 0, f(x) = \frac{1}{x^p}$ is clearly continuous, positive, and decreasing on $[1, \infty]$. And we have: $\int_1^{\infty} \frac{1}{x^p} dx$ converges if $p > 1$ and diverges if $p \leqslant 1$

Conclusion

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \geqslant 1$.

Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.
(i) If $\sum b_n$ is convergent and $a_n \leqslant b_n$ for all n, then $\sum a_n$ is also convergent. (ii) If $\sum b_n$ is divergent and $a_n \geqslant b_n$ for all n, then $\sum a_n$ is also divergent. In using the Comparison Test we must, of course, have some known series $\sum b_n$ for the purpose of comparison. Most of the time we use one of these series:
- A p-series $[\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leqslant 1]$ - A geometric series $\sum ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geqslant 1]$

Comparison Test: Expressed by Limits

Theorem

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\underset{n\to\infty}{limits} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Remark:Usually, this is more convenient to use.

d'Alembert Test (Ratio Test)

(i) If $lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| = L < 1$ then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| = L > 1$ or $lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| = \infty$, then the series $\sum_{n=1}^{\infty}$ is divergent.

(iii) If $lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| = L = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

Cauchy Test (Root Test)

(i) If $lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $lim_{n\to\infty} \sqrt[n]{|a_n|} = L = 1$, the Root Test is inconclusive.

Definition

If the series satisfies

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} (-1)^{u+1} u_n$$

Then we call it an alternating series.

Further, if the series

$$\sum_{n=1}^{\infty}(-1)^{u+1}u_n$$

satisfies: $(i)u_{n+1} \leqslant u_n$ for all n $(ii)lim_{n\to\infty}u_n = 0$ Then the series is convergent.
We call it Leibniz series.

Conclusion

The convergence and divergence property of a series has nothing to do with the first N terms, where N is a finite number.

So, we can write:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty}$$

Then, if the series

$$\sum_{n=N+1}^{\infty} a_n$$

satisfies the conditions of Leibniz Series, we can still conclude that the series is convergent.

Absolute Convergence and Conditional Convergence

Definition

Suppose that $\sum_{n=1}^{\infty} x_n$ is a convergent series. Then if $\sum_{n=1}^{\infty} |x_n|$ is convergent, $\sum_{n=1}^{\infty} x_n$ is absolutely convergent. Else $\sum_{n=1}^{\infty} x_n$ is a conditionally convergent.

Theorem

If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Absolute Convergence and Conditional Convergence

Methods

The convergence and divergence property of $\sum_{n=1}^{\infty} |x_n|$ can be determined by the criterion mentioned before.

Typically, if$\sum_{n=1}^{\infty} |x_n|$ diverges, $\sum_{n=1}^{\infty} x_n$ does not necessarily diverges. However, if the divergence property is determined by Ratio Test or Root Test, then the series$\sum_{n=1}^{\infty} x_n$ also diverges.That's because these two criterion are based on the fact that the sequence is not approaches 0 $(x \to \infty)$.

Shanks Transformation

For each series $\sum_{n=0}^{\infty} a_n$, we can form the sequence of partial sums

$$A_n = \sum_{k=0}^{n} a_n$$

and

$$S_n = \frac{A_{n+1}A_(n-1) - A_n^2}{A_{n+1} + A_{n-1} - 2A_n}$$

This new sequence, called the Shanks transformation of the series, will usually converge faster than the original series. It is denoted by S (A_n), and works particular well on alternating series.

Now let's expand the concept of series to functions.

Series with Function Terms

Suppose un(x) is a function sequence with common domain E, then the sum of these infinite numbers of function terms

$$u_1(x) + u_2(x) + \cdot + \cdot + \cdot + u_n(x) + \cdot + \cdot + \cdot$$

is called function series, denoted as

$$\sum_{n=1}^{\infty} u_n(x)$$

Convergence Point and Convergence Domain

Different from series of number terms, function series has the concept of convergence point and convergence domain.

Convergence Point

For a fixed $x_0 \in E$, if the series

$$\sum_{n=1}^{\infty} u_n(x_0)$$

is convergent, then we say that the function series

$$\sum_{n=1}^{\infty} u_n(x)$$

is convergent at x_0.

Convergence Point and Convergence Domain

Convergence Domain

The set that includes all the convergence point of the given function series is called the convergence domain.

Power Series is a special kind of function series.

Definition

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + \cdots$$

This kind of function series is called power series.

Radius of Convergence

Cauchy-Hadamard Theorem

The power series

$$\sum_{n=0}^{\infty} a_n x^n$$

is absolutely convergent when $|x| < R$, and it is divergent when $|x| > R (R > 0)$. R is called the radius of convergence.

Note: at the endpoints $x = \pm R$, the convergence and divergence property of the function series should be judged by other methods.

Radius of Convergence

Cauchy-Hadamard Theorem: for General Cases

The power series

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n$$

is absolutely convergent when $|x - x_0| < R$, and it is divergent when $|x - x_0| > R(R > 0)$.

Radius of Convergence: Cauchy Test

Cauchy Test

For the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

If

$$\lim_{n \to \infty} \sqrt[n]{|a_n|} = A$$

Then the radius of convergence of this power series is $\frac{1}{A}$ Specially, If $A = 0$, then $R = +\infty$; if $A = +\infty$, then $R = 0$.

Radius of Convergence: d'Alembert Test

d'Alembert Test

For the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

If

$$\underset{n\to\infty}{lim} \frac{a_{n+1}}{a_n} = A$$

Then the radius of convergence of this power series is $\frac{1}{A}$ Specially, If $A = 0$, then $R = +\infty$; if $A = +\infty$, then $R = 0$.

Integrals Term by Term

We can take the integrals of a power series term by term, if the interval lies in its domain of convergence.

That means, if $a, b \in D$(D is the domain of convergence), then

$$\int_a^b \sum_{n=0}^{\infty} a_n x^n dx = \sum_{n=0}^{\infty} \int_a^b a_n x^n dx$$

If we take $a = 0$ and $b = x$, then

$$\int_0^x \sum_{n=0}^{\infty} a_n x^n dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} dx$$

Derivatives Term by Term

Suppose the power series $\sum_{n=0}^{\infty} a_n x^n$ has the radius of convergence R. Then we can take the derivatives term by term on $(-R, R)$.

$$\frac{d}{dx}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{d}{dx} a_n x^n = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$\frac{d}{dx}\sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{n=0}^{\infty} \frac{d}{dx} a_n (x-x_0)^n = \sum_{n=0}^{\infty} n a_n (x-x_0)^{n-1}$$

Shift the Index of Summation

We can shift the "starting point" of summation. General Case:

$$\sum_{n=m}^{\infty} a_n (x - x_0)^n = \sum_{n=m+k}^{\infty} a_{n-k} (x - x_0)^{n-k}$$

$$\sum_{n=m}^{\infty} a_n (x - x_0)^n = \sum_{n=m-k}^{\infty} a_{n+k} (x - x_0)^{n+k}$$

Taylor Expansion of Elementary Functions

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^3 + \frac{1}{6}x^3 + ..., x \in R$$

$$ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - ..., x \in (-1, 1]$$

$$sinx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ..., x \in R$$

$$cosx = \sum_{n=0}^{\infty} \frac{(-1)^{2n}}{(2n)!} x^{2n} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - ..., x \in R$$

$$arctanx = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{2n-1} x^{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - ..., x \in [-1, 1]$$

Taylor Expansion of Elementary Functions

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \frac{(\alpha(\alpha-1)...(\alpha-n+1))}{n!} x^n$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + ..., x \in (-1, 1)$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - ..., x \in (-1, 1)$$

Sequences and Series

Let $a_n = \frac{2n}{3n+1}$

1. Determine whether a_n is convergent.
2. Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent.

Convergence and Divergence

Find the values of x for which the series converges. Find the sum of the series for those values of x.

1.

$$\sum_{n=0}^{\infty}(-4)^n(x-5)^n$$

2.

$$\sum_{n=0}^{\infty}\frac{sin^n x}{3^n}$$

Ex3: Integral Test

Determine whether the series is convergent or divergent

1. $$\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}$$

2. $$\sum_{n=1}^{\infty} \frac{n}{n^4 + 1}$$

Ex4: Comparison Test

Determine whether the series is convergent or divergent

1. $$\sum_{n=1}^{\infty} \frac{1}{(n^2+2n+2)^2}$$
2. $$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

Ex5: Ratio & Root

Determine whether the series is convergent or divergent

1. $$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

2. $$\sum_{n=1}^{\infty} (\frac{-2n}{n+1})^{5n}$$

Ex6

1. Determine the Power Series Expansion of $f(x) = \frac{1}{3+5x-2x^2}$ at $x = 0$.
2. Determine the Power Series Expansion of $f(x) = ln(\frac{sinx}{x})$ at $x = 0$.

9. Differential Equation

Definition

Equations representing the relationship between some unknown functions, the derivative of those functions and the independent variable.

Order: The order of the highest derivative of the unknown function is called the order of the differential equation.

Linear: The highest degree of the unknown function and its derivatives of any order is 1.

Generally, $n^t h$ order differential equations have the form of
$F(x, y, y', ..., y^{(n)}) = 0$

In Vv156, we only need to solve some special types of ODEs.

Type 1: y' + p(x) y = q(x)

Step1: Find the general solution to the corresponding homogeneous ode $y' + p(x)y = 0$

$$\frac{dy}{dx} = -p(x)y$$
$$\frac{dy}{y} = -p(x)dx$$
$$ln|y| = -\int p(x)dx + c_1$$
$$y_g = Ce^{-\int p(x)dx}(C = e^{\pm c_1})$$

Type 1: $y' + p(x)y = q(x)$

Step2: Find one special solution to $y' + p(x)y = q(x)$

Variation of constants:

$$Ce^{-\int p(x)dx} \to C(x)e^{-\int p(x)dx}$$

$$y' = C'(x)e^{-\int p(x)dx} - C(x)p(x)e^{-\int p(x)dx}$$

$$C'(x)e^{-\int p(x)dx} - C(x)p(x)e^{-\int p(x)dx} + C(x)p(x)e^{\int p(x)dx} = q(x)$$

$$C'(x)e^{-\int p(x)dx} = q(x)$$

$$C'(x) = q(x)e^{\int p(x)dx}$$

$$C(x) = \int q(x)e^{\int p(x)dx}dx + C_2$$

Since we only need to find one special solution, we assume $C_2 = 0$

$$y_s = e^{-\int p(x)dx} \cdot \int q(x)e^{p(x)dx}dx$$

Step 3: Combine y_g and y_s together

$$y = Ce^{-\int p(x)dx} + e^{-\int p(x)dx} \cdot \int q(x)e^{p(x)dx}dx$$

Bernoulli's equation

$y' + p(x)y = q(x)y^n$

When $n \pm 0, 1$ the equation is not linear.
However, we can do some transformation:

1. $y^{-n} \cdot y' + p(x) \cdot y^{1-n} = q(x)$
2. $z = y^{1-n}$

Then, $\dfrac{dz}{dx} = (1-n)y^{-n}\dfrac{dy}{dx}$
Thus, $z' + (1-n)p(x)z = (1-n)q(x)$
which is in the form of $y' + p(x)y = q(x)$

Type 2: $y'' + py' + qy = f(x)$

Step 1: Find the general solution to the corresponding homogeneous ode $y'' + py' + qy = 0$ Solve the characteristic equation: $\lambda^2 + p\lambda + q = 0$

$$\begin{cases} \text{Two different real roots } \lambda_1, \lambda_2 \quad y_g = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \\ \text{Two equal real roots } \lambda \quad y_g = C_1 e^{\lambda x} + C_2 x e^{\lambda x} \\ \text{Two different complex roots } \alpha \pm \beta i \quad y_g = C_1 e^{\alpha x} sin\beta x + C_2 e^{\alpha x} cos\beta x \end{cases}$$

Type 2: $y'' + py' + qy = f(x)$

Step 2: Find one special solution to $y'' + py' + qy = f(x)$
In Vv156, we only have two types of $f(x)$:

1. $f(x) = e^{\lambda x} P_m(x)$
2. $f(x) = e^{\lambda x}[P_n(x) cos(\omega x) + P_l(x) sin(\omega x)]$

For both 1, 2, we first need to check whether $\lambda(\pm\omega i)$ is the root of the characteristic equation $\lambda^2 + p\lambda + q = 0$ or not

Type 2: $y'' + py' + qy = f(x)$

$f(x) = e^{\lambda x} P_m(x)$**:**

1	k
λ is one of the different real roots of $\lambda^2 + p\lambda + q = 0$	1
λ is the identical real roots of $\lambda^2 + p\lambda + q = 0$	2
λ is not the real root of $\lambda^2 + p\lambda + q = 0$	0

$f(x) = e^{\lambda x}[P_n(x) cos(\omega x) + P_l(x) sin(\omega x)]$**:**

2	k
$\lambda \pm \omega i$ are the complex roots of $\lambda^2 + p\lambda + q = 0$	1
$\lambda \pm \omega i$ are not the complex roots of $\lambda^2 + p\lambda + q = 0$	0

Type 2: $y'' + py' + qy = f(x)$

Apply undetermined coefficient method:

- For 1: $y_s = x^k e^{\lambda x} \cdot Q_m(x)$
- For 2: $y_s = x^k e^{\lambda x}[Q_m(x)cos(\omega x) + R_m(x)sin(\omega x)]$
  $(m = MAX\{n, l\})$

Q_m and R_m are another polynomial of degree m
$\rightarrow$ Calculate y_s' and y_s'' to solve Q_m and R_m

Type 2: $y'' + py' + qy = f(x)$

Step 3: Combine y_g and y_s together
If $f(x) = f_1(x) + ... + f_n(x)$, and $f_i(x)$ is the form of 1, 2, we can calculate y_{si} separately.

1. $y' = \frac{y}{2x} + \frac{x^2}{2y}$
2. $y'' + 6y' + 10y = e^{-3x} sinx$

Solution 1:

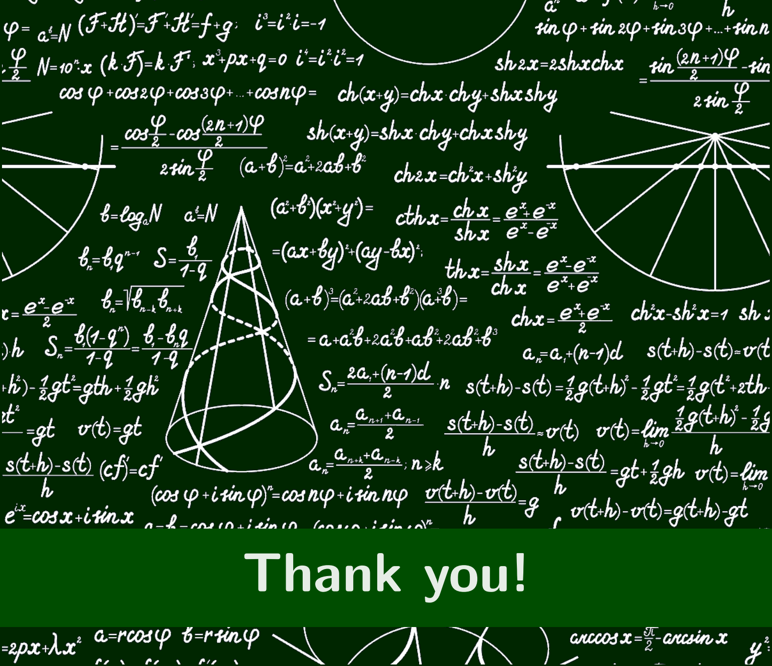
1. $y' + (-\frac{1}{2x})y = 0.5x^2y^{-1}$
   $yy' + (-\frac{1}{2x})y^2 = 0.5x^2$
2. Let $z = y^2 \rightarrow \frac{dz}{dx} = 2y\frac{dy}{dx}$
   $z' - \frac{z}{x} = x^2$
   Now $p(x) = -\frac{1}{x}\ q(x) = x^2$
3. Apply $y = Ce^{-\int p(x)dx} + e^{-\int p(x)dx} \cdot \int q(x)e^{p(x)dx}dx$
   We can get $z = Cx + \frac{1}{2}x^3$
   Thus, $y = \pm\sqrt{Cx + \frac{1}{2}x^3}$

Solution 2:

1. $\lambda^2 + 6\lambda + 10 = 0 \quad \lambda = -3 \pm i$
   Thus, the general solution is $y_g = e^{-3x}(C_1 cosx + C_2 sinx)$

2. Observe $e^{-3x}sinx \to \lambda = -3, \omega = 1$
   Since $-3 \pm i$ are complex roots of $\lambda^2 + 6\lambda + 10 = 0$, there is k $= 1$
   So $y_s = e^{-3x}x(acosx + bsinx)$
   $y_s' = e^{-3x}x[(b - 3a)cosx + (-3b - a)sinx] + e^{-3x}(acosx + bsinx)$
   $y_s'' = e^{-3x}x[(8a - 6b)cosx + (6a + 8b)sinx] + e^{-3x}[(2b - 6a)cosx - (2a + 6b)sinx]$
   Since $y'' + 6y' + 10y = e^{-3x}sinx$
   Check the coefficient, $a = -0.5$, $b = 0$
   $y_s = e^{-3x}x(-0.5cosx)$

3. Combine y_g and y_s
   $y = y_g + y_s = C_1 e^{-3x}cosx + C_2 e^{-3x}sinx - 0.5xe^{-3x}cosx$



Thank you!