255RC6

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Contents

1. Curl and Divergence

Definition

$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the **curl** of $\mathbf{F}$ is the vector field on defined by

$$\begin{aligned}\nabla \times \mathbf{F} = &\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ =&(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})\mathbf{i} + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\mathbf{j} + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\mathbf{k} \\ =&\text{curl } \mathbf{F}\end{aligned}$$

Theorem

If f is a function of three variables that has continuous second-order partial derivatives, then

$$curl(\nabla f) = \mathbf{0}$$

Definition

$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and $\partial P/\partial x$,$\partial Q/\partial y$, and $\partial R/\partial z$ exist, then the **divergence** of $\mathbf{F}$ is the function of three variables defined by

$$\nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Theorem

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q , and R have continuous second-order partial derivatives, then

$$\text{div curl } \mathbf{F} = 0$$

Ex 8.1 Vector field $\mathbf{F} = (x^2y, y^2z, z^2x)$, evaluate the divergence of the vector field at point (2,1,-2).
Ex 8.2 Given a rigid body rotating about the z-axis with angular velocity $\boldsymbol{\omega} = (0, 0, \omega)$, find the curl of the linear velocity $\boldsymbol{\nu}$ at any point M on the rigid body.

Solution 8.1 -8

Solution 8.2 $\nabla \times \boldsymbol{\nu} = 2\mathbf{w}$

Ex 8.3 Prove the following identities. Assuming that the appropriate partial derivatives exist and are continuous. f is a scalar field and $\mathbf{F}$, $\mathbf{G}$ are vector fields.

1. div$(f\mathbf{F})$=f div $\mathbf{F}+\mathbf{F}\cdot\nabla f$
2. curl$(f\mathbf{F})$=f curl $\mathbf{F}+(\nabla f)\times\mathbf{F}$
3. div$(\mathbf{F}\times\mathbf{G})$=$\mathbf{G}\cdot$ curl $\mathbf{F}$-$\mathbf{F}\cdot$ curl $\mathbf{G}$

If f is a scalar field and $\mathbf{F}$, $\mathbf{G}$ are vector fields, then $f\mathbf{F}$,$\mathbf{F} \cdot \mathbf{G}$, and $\mathbf{F} \times \mathbf{G}$ are defined by

$$(f\mathbf{F})(x, y, z) = f(x, y, z)\mathbf{F}(x, y, z)$$
$$(\mathbf{F} \cdot \mathbf{G})(x, y, z) = \mathbf{F}(x, y, z) \cdot \mathbf{G}(x, y, z)$$
$$(\mathbf{F} \times \mathbf{G})(x, y, z) = \mathbf{F}(x, y, z) \times \mathbf{G}(x, y, z)$$

2. Parametric Surfaces and Areas

Definition

We suppose that

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

is a vector-valued function defined on a region D in the uv-plane. The set of all points (x,y,z) in $\mathbb{R}^3$ is called a **parametric surface** S. Equations

$$x = x(u, v) \qquad y = y(u, v) \qquad z = z(u, v)$$

.are called **parametric equations** of S.

Definition

If a smooth parametric surface S is given by the equation

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \qquad (u, v) \in D$$

and S is covered just once as (u, v) ranges throughout the parameter domain D, then the **surface area** of S is

$$A = \iint_D \left|\mathbf{r}_u \times \mathbf{r}_v\right| dA$$

where

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \qquad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

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f the surface can be expressed with equation $z = f(x, y)$, let $u = x, v = y$, we have

$$x = x \qquad y = y \qquad z = f(x, y)$$

then

$$\mathbf{r}_x = \mathbf{i} + (\frac{\partial f}{\partial x})\mathbf{k} \qquad \mathbf{r}_y = \mathbf{j} + (\frac{\partial f}{\partial y})\mathbf{k}$$

We have

$$\left|\mathbf{r}_x \times \mathbf{r}_y\right| = \left| -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}\right| = \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}$$

Ex 8.4 Find the area of the following surfaces

- The part of the plane $3x + 2y + z = 6$ that lies in the first octant.
- The helicoid (or spiral ramp) with vector equation
  $\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + v\mathbf{k}, 0 \le u \le 1, 0 \le v \le \pi$

Solution 8.4

- $3\sqrt{14}$
- $\frac{\pi}{2}[\sqrt{2} + \mathit{ln}(1 + \sqrt{2})]$

3. Surface Integrals

Types

1. surface integral of f over the surface S

$$\iint_S f(x,y,z)dS = \iint_D f(\mathbf{r}(u,v))\left|\mathbf{r}_U \times \mathbf{r}_v\right| dA$$

2. surface integral of $\mathbf{F}$ over an oriented surface S is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot \mathbf{n}dS$$

This integral is also called the **flux** of $\mathbf{F}$ across S.

Oriented Surface Integral

Suppose the surface is parametrized by $r(u, v)$ and the unit normal is given by $\hat{n} = \frac{r_u \times r_v}{||r_u \times r_v||}$.
Then $\iint_S F \cdot dS = \iint_S (F \cdot \hat{n})\, d\sigma = \iint_D F \cdot \frac{r_u \times r_v}{||r_u \times r_v||} ||r_u \times r_v||\, dA = \iint_D F \cdot (r_u \times r_v)\, dA$.

Ex 8.5 Evaluate the following surface integrals.

1. Calculate the surface integral
$$\iint_S (x + y^2)\, dS,$$
where S is the surface of the cylinder given by $x^2 + y^2 = 4$ and $0 \le z \le 3$.

2. Calculate the surface integral:
$$\iint_S (x^2 - z)\, dS,$$
where S is the surface with parameterization
$$\mathbf{r}(u, v) = \langle v, u^2 + v^2, 1\rangle, \quad 0 \le u \le 2, \quad 0 \le v \le 3.$$

Exercises

1. $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + 4x^2\,\mathbf{j} + yz\,\mathbf{k}$, $\quad S$ is the surface $z = xe^y$, $\quad 0 \leq x \leq 1, 0 \leq y \leq 1$, with upward orientation
2. $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$, $\quad S$ is the hemisphere $x^2 + y^2 + z^2 = 25, y \geq 0$, oriented in the direction of the positive y-axis

Answers

1. 24π
2. 24

Answers

$\mathbf{F}(x,y,z) = xy\,\mathbf{i} + 4x^2\,\mathbf{j} + yz\,\mathbf{k}$, $\quad z = g(x,y) = xe^y$, and D is the square $[0,1]\times[0,1]$, so by Equation 10

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D [-xy(e^y) - 4x^2(xe^y) + yz]\,dA \\ &= \int_0^1\int_0^1 (-xye^y - 4x^3e^y + xye^y)\,dy\,dx \\ &= \int_0^1 \left[-4x^3e^y\right]_{y=0}^{y=1} dx = (e-1)\int_0^1 (-4x^3)\,dx = 1-e\end{aligned}$$

Answers

$\mathbf{F}(x,y,z) = x\,\mathbf{i} - z\,\mathbf{j} + y\,\mathbf{k}$, $z = g(x,y) = \sqrt{4 - x^2 - y^2}$ and D is the quarter disk $\left\{(x,y)\,\middle|\,0 \le x \le 2, 0 \le y \le \sqrt{4-x^2}\right\}$. S has downward orientation, so by Formula 10,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -\iint_D \left[-x \cdot \tfrac{1}{2}(4-x^2-y^2)^{-1/2}(-2x) - (-z)\cdot \tfrac{1}{2}(4-x^2-y^2)^{-1/2}(-2y) + y\right] dA$$

$$= -\iint_D \left(\frac{x^2}{\sqrt{4-x^2-y^2}} - \sqrt{4-x^2-y^2}\cdot\frac{y}{\sqrt{4-x^2-y^2}} + y\right) dA$$

$$= -\iint_D x^2(4-(x^2+y^2))^{-1/2}\,dA = -\int_0^{\pi/2}\int_0^2 (r\cos\theta)^2(4-r^2)^{-1/2}\,r\,dr\,d\theta$$

$$= -\int_0^{\pi/2}\cos^2\theta\,d\theta \int_0^2 r^3(4-r^2)^{-1/2}\,dr \qquad \left[\text{let } u = 4-r^2 \;\Rightarrow\; r^2 = 4-u \text{ and } -\tfrac{1}{2}\,du = r\,dr\right]$$

$$= -\int_0^{\pi/2}\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta\right)d\theta \int_4^0 -\tfrac{1}{2}(4-u)(u)^{-1/2}\,du$$

$$= -\left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/2}\left(-\tfrac{1}{2}\right)\left[8\sqrt{u} - \tfrac{2}{3}u^{3/2}\right]_4^0 = -\tfrac{\pi}{4}\left(-\tfrac{1}{2}\right)\left(-16+\tfrac{16}{3}\right) = -\tfrac{4}{3}\pi$$

Thank you!