Part III Surface Integral, Stokes' Theorem, Divergence Theorem

I. Parametric Surface and Their Area

Parametric surface

In much the same way that we describe a space curve by a vector function of a single parameter $\mathbf{r}(t)=x(t)\hat{\mathbf{i}}+y(t)\hat{\mathbf{j}}$, we can describe a surface by a vector function $\mathbf{r}(u,v)$ of two parameters u and v:

$$\mathbf{r}(u,v)=x(u,v)\hat{\mathbf{i}}+y(u,v)\hat{\mathbf{j}}+z(u,v)\hat{\mathbf{k}}$$

$\mathbf{r}(u,v)$ is a vector-valued function defined on a region D in the uv -plane. So x, y, and z, the component functions of $\mathbf{r}$, are functions of the two variables u and v with domain D. The set of all points (x,y,z) in $\mathcal{R}^3$ such that

$$\begin{aligned} x&=x(u,v)\\ y&=y(u,v)\\ z&=z(u,v) \end{aligned} \tag{1}$$

when (u,v) varies throughout D, is called ***a parametric surface*** S, and Equations (1) are called ***parametric equations of*** S. Each choice of u and v gives a point on S; by making all choices, we get all of S. In other words, the surface S is traced out by the tip/end of the position vector $\mathbf{r}(u,v)$ as (u,v) moves throughout the region D.

Problem 1

Identify and sketch the surface with vector equation

$$\mathbf{r}(u,v)=2\cos u\,\hat{\mathbf{i}}+v\,\hat{\mathbf{j}}+2\sin u\,\hat{\mathbf{k}}$$

Solution

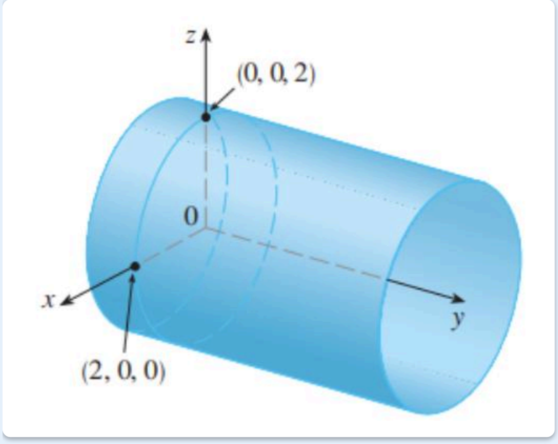
So for any point on the surface, we have

$$x+z=2\cos u+2\sin u=2(\cos u+\sin u).$$

Given the above expression, if we square both sides and apply the Pythagorean identity for trigonometric functions over the interval (u), we get

$$x^2+z^2=(2\cos u)^2+(2\sin u)^2=4(\cos^2 u+\sin^2 u)=4.$$

This means that vertical cross-sections parallel to the -plane (that is, with constant) are all circles with radius 2. Since and no restriction is placed on , the surface is a circular cylinder with radius 2 whose axis is the -axis (see Figure).



Surfaces of Revolution

Surfaces of revolution can be represented **parametrically** and thus graphed using a computer. For instance, let's consider the surface obtained by rotating the curve $y = f(x)$ about the x -axis, where $a \leq x \leq b$. Let θ be the angle of rotation. If (x, y_0) is a point on $y = f(x)$, then the curve traced out by the point (x, y_0) as θ varies from 0 to 2π is given by the parametric equations:

$$\begin{aligned} x &= x \\ y &= y_0 \cos\theta = f(x)\cos\theta \\ z &= y_0 \sin\theta = f(x)\sin\theta \end{aligned} \tag{2}$$

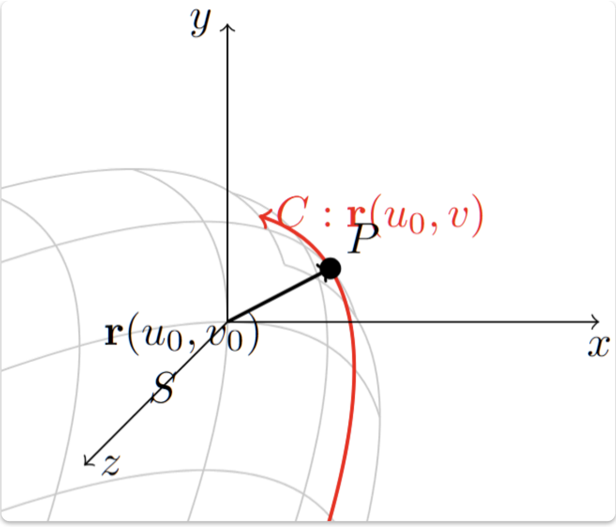
Therefore, we take x and θ as parameters and regard Equations (2) as parametric equations of the surface of revolution. The parameter domain is given by $D = \{(x, \theta) \,|\, a \leq x \leq b,\ 0 \leq \theta \leq 2\pi\}$.

Tangent Planes

We now need to find **the tangent plane to a parametric surface** S traced out by a vector function

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

at a point P with position vector $\mathbf{r}(u_0, v_0)$. If we keep u constant by putting $u = u_0$, then $\mathbf{r}(u_0, v)$ becomes **a vector function of the single parameter** v and defines a grid curve C lying on S.



The tangent vector to C at P is obtained by taking the partial derivative of $\mathbf{r}$ with respect to v :

$$\mathbf{T}_v = \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial x(u_0, v)}{\partial v}\mathbf{i} + \frac{\partial y(u_0, v)}{\partial v}\mathbf{j} + \frac{\partial z(u_0, v)}{\partial v}\mathbf{k}.$$

Surface Area

Definition

If a smooth parametric surface S is given by the equation

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}, \quad (u, v) \in D$$

and S is covered just once as (u, v) ranges throughout the parameter domain D, then the surface area S of is

$$A(\mathbf{S}) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| dA$$

Where

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}, \quad \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}.$$

Problem 2

Find the surface area of a sphere of radius a.

Solution

In the parametric representation of a sphere of radius a is:

$$\begin{aligned} x &= a \sin\phi \cos\theta \\ y &= a \sin\phi \sin\theta \\ z &= a \cos\phi \end{aligned}$$

With the parameter domain being

$$D = \{(\phi, \theta) \mid 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi\}.$$

We first compute the cross product of the tangent vectors:

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

Expanding the determinant, we get

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \big(a^2\sin^2\phi\cos\theta\big)\mathbf{i} + \big(a^2\sin^2\phi\sin\theta\big)\mathbf{j} + (a^2\sin\phi\cos\phi)\mathbf{k}$$

Thus

$$\begin{aligned} |\mathbf{r}_\phi \times \mathbf{r}_\theta| &= \sqrt{(a^2\sin^2\phi\cos\theta)^2 + (a^2\sin^2\phi\sin\theta)^2 + (a^2\sin\phi\cos\phi)^2} \\ &= \sqrt{a^4\sin^4\phi\cos^2\theta + a^4\sin^4\phi\sin^2\theta + a^4\sin^2\phi\cos^2\phi} \\ &= \sqrt{a^4\sin^4\phi(\cos^2\theta + \sin^2\theta) + a^4\sin^2\phi\cos^2\phi} \\ &= \sqrt{a^4\sin^4\phi + a^4\sin^2\phi\cos^2\phi} \\ &= a^2\sin\phi\sqrt{\sin^2\phi + \cos^2\phi} \\ &= a^2\sin\phi \end{aligned}$$

for $0 \le \phi \le \pi$
Thus, the area of the sphere is given by

$$A = \iint_D |\mathbf{r}_\phi \times \mathbf{r}_\theta| dA = \iint_D a^2\sin\phi\, d\theta\, d\phi = a^2\int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi\, d\phi = 4\pi a^2$$

Theorem

For the special case of a surface S with the equation $\mathbf{z = f(x, y)}$, where (x, y) lies in D and f has continuous partial derivatives, we take x and y as parameters. The parametric equations are

$$x = x, \quad y = y, \quad z = f(x, y)$$

Therefore, the parametric representation can be written as

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}.$$

To find the surface area of S , we need to compute the cross product and then find its magnitude:

$$\mathbf{r}_x = \frac{\partial \mathbf{r}}{\partial x} = \mathbf{i} + \frac{\partial f}{\partial x}\mathbf{k}, \quad \mathbf{r}_y = \frac{\partial \mathbf{r}}{\partial y} = \mathbf{j} + \frac{\partial f}{\partial y}\mathbf{k}.$$

Computing the cross product, we have

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial y}\mathbf{i} - \frac{\partial f}{\partial x}\mathbf{j} + \mathbf{k}.$$

Thus,

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(-\frac{\partial f}{\partial y}\right)^2 + \left(-\frac{\partial f}{\partial x}\right)^2 + 1^2} = \sqrt{\left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial x}\right)^2 + 1}$$

Therefore, the surface area of is given by

$$A(\mathbf{S}) = \iint_D \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1 dA}.$$

Problem 3

Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

Solution

The plane intersects the paraboloid in the circle $x^2 + y^2 = 9$, $z = 9$. Using Formula, we have

$$A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \iint_D \sqrt{1 + (2x)^2 + (2y)^2}\, dA = \iint_D \sqrt{1 + 4(x^2 + y$$

Converting to polar coordinates, we obtain

$$A = \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^3 r\sqrt{1 + 4r^2}\, dr = \frac{\pi}{6}\left(37\sqrt{37} - 1\right).$$

Use substitution when calculatin the integral.

II. Surface Integral

Parametric Surface

To compute the surface integral of $f(x, y, z)$ over a parametric surface S, we have

$$\iint_{\mathbf{S}} f(x, y, z)\, \mathrm{d}S$$

In most cases we use the parameterization expression of that surface, then the integral of a function over is given by:

$$\iint_{\mathbf{S}} f(x, y, z) dS = \iint_D f(\mathbf{r}(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| dA$$

where $\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}$ and $\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}$.

Special Case

When the surface is given by a graph $z = f(x, y)$, we take x and y as parameters. The parametric equations are

$$\iint_S f(x, y, z)\, \mathrm{d}S = \iint_D f(x, y, f(x, y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, \mathrm{d}A,$$

Oriented Surfaces

Definition

If $\mathbf{F}$ is a continuous vector field defined on an oriented surface $\mathbf{S}$ with unit normal vector n, then the surface integral of $\mathbf{F}$ over $\mathbf{S}$ is

$$\iint_{\mathbf{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_{\mathbf{S}} \mathbf{F} \cdot \mathbf{n}\, \mathrm{d}S$$

This integral is also called the ***flux*** $\mathbf{F}$ of across S.

The ***surface integral of a vector field*** can be expressed over a parametric surface as follows:

$$\iint_{\mathbf{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_{\mathbf{S}} \mathbf{F} \cdot \mathbf{n}\, \mathrm{d}S = \iint_{\mathbf{D}} \mathbf{F}(\mathbf{r}(u, v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} \|\mathbf{r}_u \times \mathbf{r}_v\|\, \mathrm{d}A = \iint_{\mathbf{D}} \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, \mathrm{d}$$

where D is the parameter domain of the surface $\mathbf{S}$ and $\mathbf{r}(u, v)$ is the parameterization of the surface.

Difference with Common Surface Integral

1. 什么是“有向面”？

- **普通的面**（surface）：只关心形状和位置，不关心“哪一面朝外”。
- **有向面**（oriented surface）：不仅有形状和位置，还规定了“正方向”——即每一点都指定了一个**单位法向量**（normal vector），通常用 $\mathbf{n}$ 表示

2. 为什么要“有向”？

- 在很多物理和数学问题中（如流量、磁通量、斯托克斯定理、高斯定理等），**方向**很重要
- 只有规定了方向，才能明确“流进”还是“流出”，“正通量”还是“负通量”

3. 在积分中的区别

- **有向面积分**：$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, \mathrm{d}S$
- **普通面积分**：$\iint_S f(x, y, z)\, \mathrm{d}S$

Special Case: $z = f(x, y)$

In this special case where $\mathbf{F} = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ and $\mathbf{S}$ is given by $z = f(x,y)$, we have:

$$\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot \left(-\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}\right).$$

Thus, the dot product simplifies to

$$\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = P\left(-\frac{\partial z}{\partial x}\right) + Q\left(-\frac{\partial z}{\partial y}\right) + R.$$

So the formula becomes

$$\iint_{\mathbf{S}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathbf{S}} \left(P\left(-\frac{\partial z}{\partial x}\right) + Q\left(-\frac{\partial z}{\partial y}\right) + R\right) dS.$$

or in terms of the parameter domain D,

$$\iint_{\mathbf{S}} \mathbf{F} \cdot d\mathbf{S} = \iint_{D} \left(-P\frac{\partial z}{\partial x} - Q\frac{\partial z}{\partial y} + R\right) dA.$$

Problem 4

Compute the surface integral $\iint_{\mathbf{S}} x^2 \cdot \mathrm{d}S$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution

we use the parametric representation

$$0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi,$$

given by

$$x = \sin\phi\cos\theta, \quad y = \sin\phi\sin\theta, \quad z = \cos\phi,$$

that is,

$$\mathbf{r}(\phi,\theta) = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}.$$

To compute the surface integral

$$\iint_{\mathbf{S}} x^2\,dS = \iint_D (\sin\phi\cos\theta)^2\,|\mathbf{r}_\phi \times \mathbf{r}_\theta|\,dA,$$

we use the parametric representation

$$\mathbf{r}(\phi,\theta) = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k},$$

with ϕ and θ as parameters, and

$$0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi.$$

The magnitude of the cross product of the tangent vectors is

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sin\phi.$$

Thus, the integral becomes

$$\iint_{\mathbf{S}} x^2\,dS = \int_0^{2\pi}\int_0^{\pi} (\sin\phi\cos\theta)^2 \sin\phi\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi} \sin^3\phi\cos^2\theta\,d\phi\,d\theta.$$

Integrating with respect to ϕ and θ, we get

$$\int_0^{2\pi} \cos^2\theta\,d\theta \int_0^{\pi} \sin^3\phi\,d\phi = \frac{4\pi}{3}$$

Problem 5

Evaluate $\iint_S y \cdot dS$, where S is the surface $z = x + y^2$, $0 \le x \le 1$, $0 \le y \le 2$.

Solution

Since $\frac{\partial z}{\partial x} = 1$, $\frac{\partial z}{\partial y} = 2y$

$$\iint_{\mathbf{S}} y \cdot d\mathbf{S} = \iint_{\mathbf{D}} y\sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2} \cdot d\mathbf{A} = \int_0^1 \int_0^2 y\sqrt{1+1+4y^2}dydx = \frac{13\sqrt{2}}{3}$$

Problem 6

Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = -yj + xj + zk$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

Solution

We have

$$\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

$$= \iint_D (-[y(-2x) - x(-2y)] + 1 - x^2 - y^2) dA$$

$$= \iint_D (1 + 4xy - x^2 - y^2) dA$$

$$= \iint_D (1 + 4r^2\cos\theta\sin\theta - r^2) r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 (r - r^3 + 4r^2\cos\theta\sin\theta) dr\, d\theta$$

$$= \int_0^{2\pi}\left(\frac{1}{4} + \cos\theta\sin\theta\right) d\theta = \frac{1}{4}(2\pi) + 0 = \frac{\pi}{2}$$

The disk S_2 is oriented downward, so its unit normal vector is $\mathbf{n} = -\mathbf{k}$ and we have

$$\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot(-\mathbf{k})\, dS = \iint_D (-z)\, dA = 0$$

since $z = 0$ on S_2. Finally, we compute, by definition, $\iint_S \mathbf{F}\cdot d\mathbf{S}$ as the sum of the surface integrals of $\mathbf{F}$ over the pieces S_1 and S_2:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} \mathbf{F}\cdot d\mathbf{S} + \iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \frac{\pi}{2} + 0 = \frac{\pi}{2}$$

Extra Exercises

1. Find the area of surface:

- The surface $z = \frac{2}{3}(x^{\frac{3}{2}} + y^{\frac{3}{2}}), 0 \le x \le 1, 0 \le y \le 1$.
- The part of the plane with vector equation $r(u,v) = (u+v, 2-3u, 1+u-v)$ that is given by $0 \le u \le 2, -1 \le v \le 1$.
- The part of the plane $x + 2y + 3z = 1$ that lies inside the cylinder $x^2 + y^2 = 3$.
- The part of surface $z = cos(x^2 + y^2)$ that lies inside the cylinder $x^2 + y^2 = 1$.(Integral form can be remain

2.

- $\iint_{\mathbf{S}} xzdS$, S is the boundary of the region enclosed by the cylinder, $y^2 + z^2 = 9$ and the planes $x = 0$ and $x + y = 5$
- Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2 + y^2}$, $1 \le z \le 4$, if its density function is $\rho(x,y,z) = 10 - z$.
- $\iint_{\mathbf{S}} x^2 + y^2 + z^2 dS$, S is the part of the cyclinder $x^2 + y^2 = 9$ between the planes $z = 0$ and $z = 2$, togetl the top and bottom disks.
- $F(x,y,z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$, S is the part of the para boloid $z = 4 - x^2 - y^2$ that lies above the square $0 \le x \le 1, 0 \le y \le 1$, and has upward orientation.

Answers

1.

- $\frac{4}{15}\left(3^{5/2} - 2^{7/2} + 1\right)$
- $4\sqrt{22}$
- $\sqrt{14}\pi$
- $2\pi \int_0^1 r\sqrt{1 + 4r^2 \sin^2(r)}\, dr$

2.

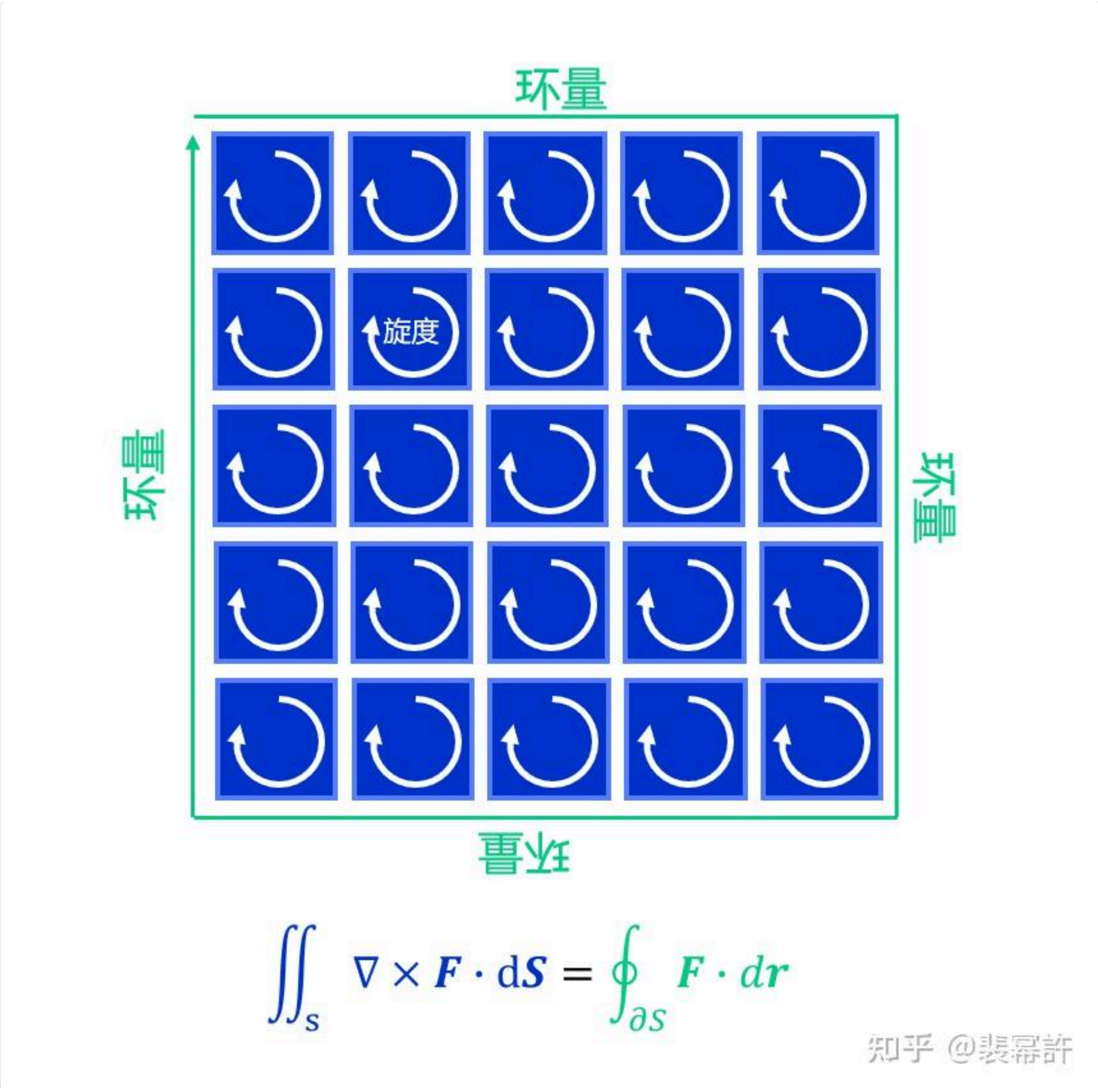
- 0
- $108\sqrt{2}\pi$
- 241π
- $\frac{1}{2}\pi^2$

III. Stokes' Theorem 化旋为环

Theorem

- Let C be a positively oriented, piecewise-smooth, simple, closed curve in $\mathbb{R}^3$ and let S be a surface whose boundary is C oriented with respect to the orientation of C according to the right-hand rule. Let $\mathbf{F}$ be a vector field on $\mathbb{R}^3$ whose component have continuous partial derivatives on a domain that contains S, then

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S \mathrm{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \iint_S \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

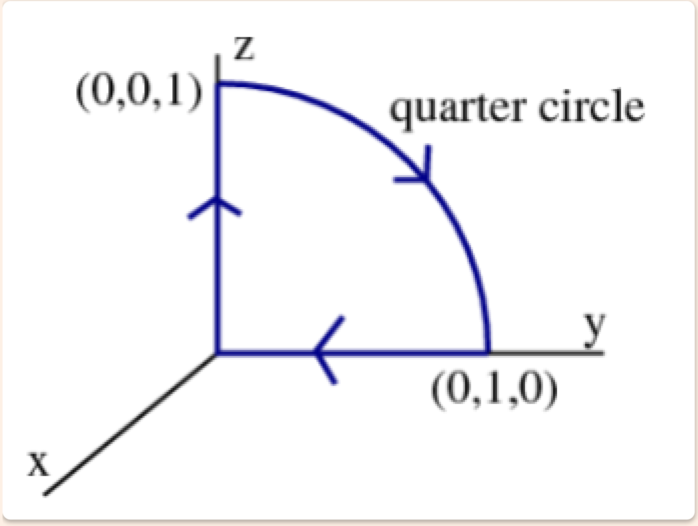


- In general, if $\mathbf{S}_1$ and $\mathbf{S}_2$ are oriented surfaces with the same oriented boundary curve C and both satisfy the hypotheses of Stokes' Theorem, then

$$\iint_{\mathcal{S}_1} \operatorname{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \int_{C} \mathbf{F} \cdot \mathrm{dr} = \iint_{\mathcal{S}_2} \operatorname{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S}$$

ⓘ Problem 7

Let C be the closed curve illustrated below



For $F(x,y,z) = (y,z,x)$, compute $\int_C F \cdot ds$

Solution

Stokes' Theorem $\int_C F\,ds = \iint_S \text{curl}\, F \cdot dS$

$$\text{curl}(F) = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = (-1,-1,-1)$$

Choose S the quarter disk of yz-plane

Parameterize the surface: $\Phi(r,\theta) = (0, r\cos\theta, r\sin\theta)$

$0 \le r \le 1 \quad 0 \le \theta \le \frac{\pi}{2}$

$\boxed{\text{Normal vector}}$ $(-1, 0, 0)$ 乘surface 的方向向量

$$\iint_S \text{curl}(F) \cdot dS = \iint_S (-1,-1,-1)\boxed{(-1,0,0)}\,dS = \iint_S dS$$

求的是通过面的flux

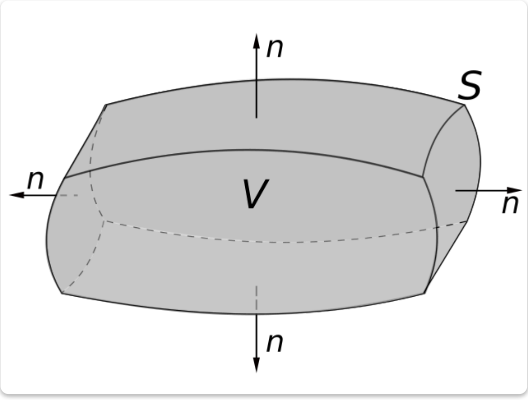
$$= \int_0^1 \int_0^{\frac{\pi}{2}} r\,d\theta\,dr = \frac{\pi}{4}$$

IV. Divergence Theorem 化散为通

Theorem

- Let $\mathcal{S}$ be a piecewise-smooth surface that encloses a solid $\mathcal{V}$ that is oriented so that the normal vectors point away from the interior of $\mathcal{S}$. Let F be a vector field on $\mathbb{R}^3$ whose components have continuous partial derivatives on an open region that contains $\mathcal{V}$. Then

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iiint_{\mathcal{V}} \mathrm{div}(\mathbf{F})\, \mathrm{d}V = \iiint_{\mathcal{V}} \nabla \cdot \mathbf{F}\, \mathrm{d}V$$



Problem 8

Calculate $\iint_S = \frac{x}{r^3} dydz + \frac{y}{r^3} dxdz + \frac{z}{r^3} dxdy$, where $r = \sqrt{x^2 + y^2 + z^2}$, and S is:
(1) the outside of the ball $x^2 + y^2 + z^2 = a^2$
(2) the outside of the ball $(x - 114)^2 + (y - 514)^2 + (z - 1919)^2 = 810^2$
(3) the upper side of the surface $1 - \frac{z}{7} = \frac{(x-2)^2}{25} + \frac{(y-1)^2}{16}$, where $z \geq 0$

Solution

(1) $r=\sqrt{x^2+y^2+z^2}=a$

$\frac{1}{r^3}$ 可提出来

$I=\frac{1}{a^3}\iint x\,dy\,dz+y\,dz\,dx+z\,dx\,dy$

$=\frac{1}{a^3}\iiint_V \operatorname{div}(x,y,z)\,dV$

$=\frac{1}{a^3}\iiint_V 3\,dV$

$=\frac{3}{a^3}\cdot\frac{4\pi a^3}{3}=4\pi$

(2) $R=\sqrt{(114^2+514^2+1919)^2}>810$

The pole (0, 0, 0) is not inside the volumn　　r 不可以提出来

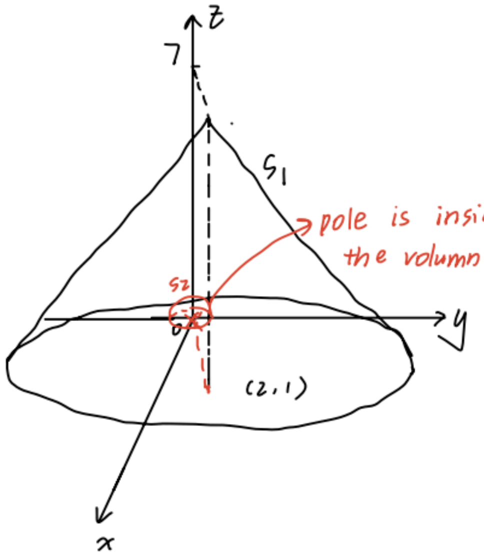
$\Rightarrow$ We can use divergence theorem on the whole volumn

$\frac{\partial \frac{x}{r^3}}{\partial x}=\frac{r^2-3x^2}{r^5}$　　$\frac{\partial \frac{y}{r^3}}{\partial y}=\frac{r^2-3y^2}{r^5}$　　$\frac{\partial \frac{z}{r^3}}{\partial z}=\frac{r^2-3z^2}{r^5}$

$\operatorname{div}V=\frac{3r^2-3(x^2+y^2+z^2)}{r^5}=0$

$I=\iiint_V 0\,dV=0$

(3)



pole is inside the volumn $\Rightarrow$ Dig it out

$\iint_S F\,ds=\iint_{S_1}F\,ds-\iint_{S_2}F\,ds=\iiint_V \operatorname{div}F\,dV=$

$\iint_{S_1}F\,ds=\iint_{S_2}F\,ds$

let S_2 be upper hemisph

with $R=\varepsilon\to 0$　　$\frac{1}{r^3}$

$\iint_{S_2}F\,ds=\boxed{\frac{1}{\varepsilon^3}\iiint_{V_2}3\,dV_2}=\frac{1}{\varepsilon^3}\cdot 3\cdot\frac{2\pi\varepsilon^3}{3}=2\pi$

$\Rightarrow \iint_{S_1}F\,ds=2\pi$