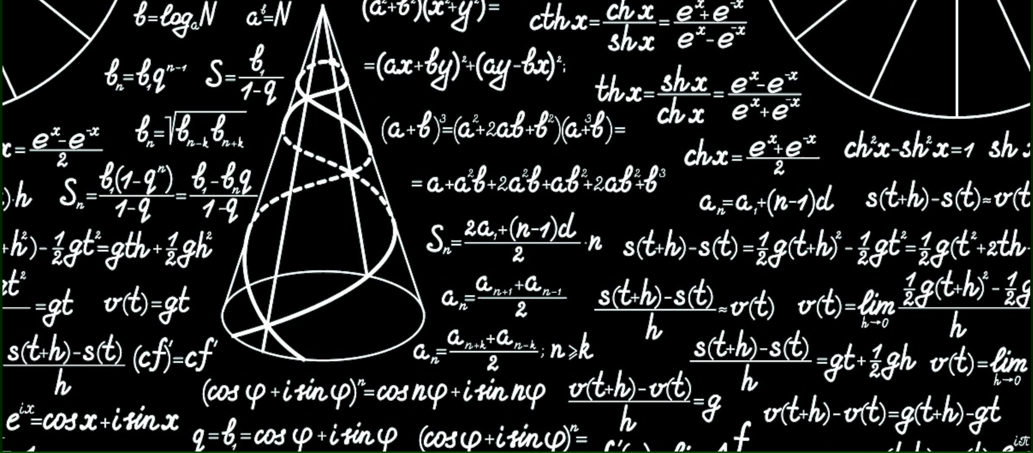


255RC4

Chengyang Shi, Ruizhi Deng, Haoran Shen, Jingfan Tang

Contents

1. Lagrange Multiplier

Problem

Let $f(x, y, z)$ be a differentiable function. Suppose that we wish to find the maximum or minimum value of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ where $k = \text{const} \in \mathbb{R}$. In other words, we wish to find the **maximum** or **minimum value** of $f(x, y, z)$ that lies on the curve C described by $g(x, y, z) = k$.

Solution

1. Find all values of x, y, z and λ such that

$$\boxed{\nabla f(x, y, z) = \lambda \nabla g(x, y, z), \quad g(x, y, z) = k.}$$

2. Evaluate f at all the points (x, y, z) that result from step (1). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Exercise

If $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, calculate the maximum of $f(x, y, z) = 8xyz$

Two Constraints

Problem

The goal is to find the **maximum and minimum values** of a three-variable function, $f(x, y, z)$, when the variables are subject to **two separate constraints**:

1. $g(x, y, z) = k$
2. $h(x, y, z) = c$

Geometrically, this means you're looking for the extreme values of f not just on a surface, but on the **curve C** where the two constraint surfaces intersect.

Solution

1. $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$
2. $g(x, y, z) = k$
3. $h(x, y, z) = c$

Exercises

Find the maximum and minimum values of f subject to the given constraints

$$f(x, y, z) = x^2 + y^2 + z^2; \quad x - y = 1, \quad y^2 - z^2 = 1$$

Solution

$f(x, y, z) = x^2 + y^2 + z^2$, $g(x, y, z) = x - y = 1$, $h(x, y, z) = y^2 - z^2 = 1$

$\Rightarrow \nabla f = \langle 2x, 2y, 2z \rangle$, $\lambda \nabla g = \langle \lambda, -\lambda, 0 \rangle$, and $\mu \nabla h = \langle 0, 2\mu y, -2\mu z \rangle$.

Then $2x = \lambda$, $2y = -\lambda + 2\mu y$, and $2z = -2\mu z \Rightarrow z = 0$ or $\mu = -1$.

If $z = 0$, then $y^2 - z^2 = 1$ implies $y^2 = 1 \Rightarrow y = \pm 1$. If $y = 1$, $x - y = 1$ implies $x = 2$, and if $y = -1$ we have $x = 0$, so possible points are $(2, 1, 0)$ and $(0, -1, 0)$.

If $\mu = -1$ then $2y = -\lambda + 2\mu y$ implies $4y = -\lambda$, but $\lambda = 2x$ so $4y = -2x \Rightarrow x = -2y$ and $x - y = 1$ implies $-3y = 1 \Rightarrow y = -\frac{1}{3}$. But then $y^2 - z^2 = 1$ implies $z^2 = -\frac{8}{9}$, an impossibility.

Thus the maximum value of f subject to the constraints is $f(2, 1, 0) = 5$ and the minimum is $f(0, -1, 0) = 1$.

Solution

Note: Since $x - y = 1 \Rightarrow x = y + 1$ is one of the constraints we could have solved the problem by solving $f(y, z) = (y + 1)^2 + y^2 + z^2$ subject to $y^2 - z^2 = 1$.

Bounded Region

Problem

Find the extreme values of f on the region described by the inequality.

Solution

1. Solve the Lagrange Multiplier Equation for the **Boundaries**
2. Find the critical point of f inside the boundary. i.e. $\nabla f(x_0, y_0, z_0) = 0$, (x_0, y_0, z_0) inside the bounded region.
3. Evaluate all the candidate points.

Exercises

Find the extreme values of f on the region described by the inequality.

1. $f(x, y) = 2x^2 + 3y^2 - 4x - 5, \quad x^2 + y^2 \leq 16$
2. $f(x, y) = e^{-xy}, \quad x^2 + 4y^2 \leq 1$

1. $f(x,y) = 2x^2 + 3y^2 - 4x - 5 \Rightarrow \nabla f = \langle 4x-4, 6y\rangle = \langle 0,0\rangle \Rightarrow x = 1, y = 0$. Thus $(1,0)$ is the only critical point of f, and it lies in the region $x^2 + y^2 < 16$. On the boundary, $g(x,y) = x^2 + y^2 = 16 \Rightarrow \lambda\nabla g = \langle 2\lambda x, 2\lambda y\rangle$, so $6y = 2\lambda y \Rightarrow$ either $y = 0$ or $\lambda = 3$. If $y = 0$, then $x = \pm 4$; if $\lambda = 3$, then $4x - 4 = 2\lambda x \Rightarrow x = -2$ and $y = \pm 2\sqrt{3}$. Now $f(1,0) = -7$, $f(4,0) = 11$, $f(-4,0) = 43$, and $f(-2, \pm 2\sqrt{3}) = 47$. Thus the maximum value of $f(x,y)$ on the disk $x^2 + y^2 \leq 16$ is $f(-2, \pm 2\sqrt{3}) = 47$, and the minimum value is $f(1,0) = -7$.

2. $f(x,y) = e^{-xy}$. For the interior of the region, we find the critical points: $f_x = -ye^{-xy}$, $f_y = -xe^{-xy}$, so the only critical point is $(0,0)$, and $f(0,0) = 1$. For the boundary, we use Lagrange multipliers. $g(x,y) = x^2 + 4y^2 = 1 \Rightarrow \lambda\nabla g = \langle 2\lambda x, 8\lambda y\rangle$, so setting $\nabla f = \lambda\nabla g$ we get $-ye^{-xy} = 2\lambda x$ and $-xe^{-xy} = 8\lambda y$. The first of these gives $e^{-xy} = -\frac{2\lambda x}{y}$, and then the second gives $-x(-\frac{2\lambda x}{y}) = 8\lambda y \Rightarrow x^2 = 4y^2$. Solving this last equation with the constraint $x^2 + 4y^2 = 1$ gives $x = \pm\frac{1}{\sqrt{2}}$ and $y = \pm\frac{1}{2\sqrt{2}}$. Now $f(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{2\sqrt{2}}) = e^{1/4} \approx 1.284$ and $f(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2\sqrt{2}}) = e^{-1/4} \approx 0.779$. The former are the maxima on the region and the latter are the minima.

2. Double Integrals

Definition

The **double integral** of f over the rectangle R is

$$\iint_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\Delta A$$

Remark

A function f is called **integrable** if the limit in the above Definition exists, which means $\forall \varepsilon > 0$ there is an integer N such that

$$\left| \iint_R f(x,y)dA - \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\Delta A \right| < \varepsilon, \quad \forall m > N, \forall n > N$$

Properties

1. $\iint_R [f(x,y) + g(x,y)]dA = \iint_R f(x,y)dA + \iint_R g(x,y)dA$
2. $\iint_R cf(x,y)dA = c\iint_R f(x,y)dA$, c is a constant
3. $f(x,y) \geq g(x,y)$ for all (x,y) in R, then $\iint_R f(x,y)dA \geq \iint_R g(x,y)dA$
4. $\left|\iint_R f(x,y)dA\right| \leq \iint_R |f(x,y)|dA$

Definition

The **iterated integral**

$$\int_a^b \int_c^d f(x,y)dydx = \int_a^b \left[\int_c^d f(x,y)dy\right]dx$$

is the integral of $\left[\int_c^d f(x,y)dy\right]$ between $x = a$ and $x = b$.

Fubini's Theorem

If f is continuous on the rectangle $R = \{(x,y)|a \leq x \leq b, c \leq y \leq d\}$, then

$$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

Definition

A plane region is said to be of **type I** if it lies between the graphs of two continuous functions of x , that is

$$D = \{(x, y) | a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

If f is continuous on a type I region D, then

$$\iint_D f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

A plane region is said to be of **type II** if

$$D = \{(x, y) | c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

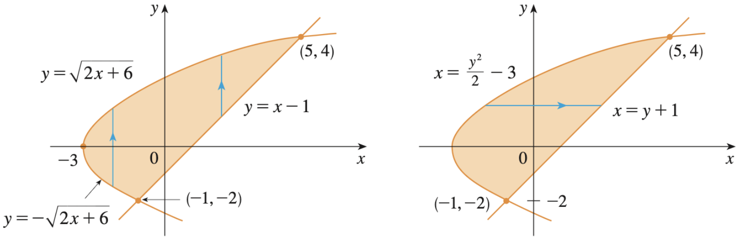
$$\iint_D f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

Example

V EXAMPLE 3 Evaluate $\iint_D xy\, dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

SOLUTION The region D is shown in Figure 12. Again D is both type I and type II, but the description of D as a type I region is more complicated because the lower boundary consists of two parts. Therefore we prefer to express D as a type II region:

$$D = \left\{(x, y) \mid -2 \leqslant y \leqslant 4,\ \tfrac{1}{2}y^2 - 3 \leqslant x \leqslant y + 1\right\}$$



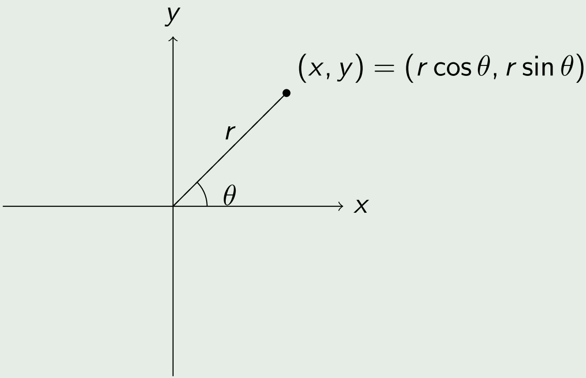
2 (a) D as a type I region (b) D as a type II region

The polar coordinate system is a two-dimensional coordinate system in which each point on a plane is determined by a distance from a reference point and an angle from a reference direction.

Conversion Formulas

The relationships between Cartesian coordinates (x, y) and polar coordinates (r, θ) are given by:

$$r^2 = x^2 + y^2$$
$$x = r\cos\theta$$
$$y = r\sin\theta$$



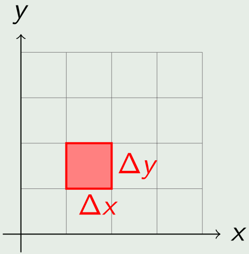
Typically, we consider $r \geq 0$ and $0 \leq \theta < 2\pi$.

Area Element: Cartesian vs. Polar

The differential element of area dA changes when we switch coordinate systems.

Cartesian Coordinates The area element is

a rectangle.

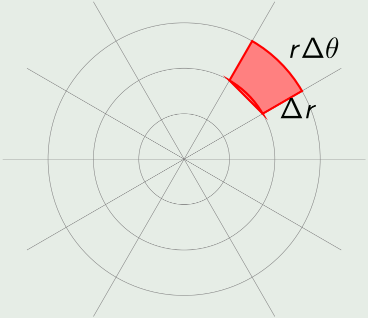


As $\Delta x \to 0$ and $\Delta y \to 0$:

$$dA = dx\,dy$$

Polar Coordinates The area element is a

"polar rectangle".

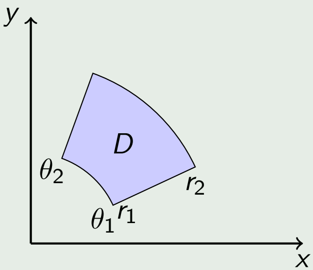


The area is approximately $A \approx (r\,\Delta\theta)\Delta r$. As differentials: $dA = r\,dr\,d\theta$

A region D defined in polar coordinates as:

$$D = \{(r, \theta) \mid r_1 \leq r \leq r_2,\ \theta_1 \leq \theta \leq \theta_2\}$$

is called a **polar rectangle**.



The integral over D becomes:

$$\iint_D f(x, y)\, dA = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Ex 6.1 Evaluate $\iint_R (5-x)dA$, $R=\{(x,y)|0\leq x\leq 8, 0\leq y\leq 2\}$
Ex 6.2 Evaluate $\iint_D xydA$, where D is the region bounded by the line $y=x-1$ and the parabola $y^2=2x+6$.
Ex 6.3 Express D as a union of regions of type I or type II and evaluate the integral $\iint_D ydA$., where D is the region bounded by the line $x=-1, y=-1, y=(x+1)^2, x=y-y^3$.
Ex 6.4 Use polar coordinates to combine the sum:
$\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} xydydx+\int_{1}^{\sqrt{2}}\int_{0}^{x} xydydx+\int_{\sqrt{2}}^{2}\int_{0}^{\sqrt{4-x^2}} xydydx$ into one double integral. Then evaluate the double integral.
Ex 6.5 Evaluate $\iint_R (x+y)e^{x^2-y^2}dA$ by making an appropriate change of variables. R is enclosed by the lines $x-y=0, x-y=2, x+y=0, x+y=3$.

Answers

6.1 16

6.2 36

6.3 $-\frac{2}{15}$

6.4 $\frac{15}{16}$

6.5 $\frac{1}{4}(e^6 - 7)$

3. Triple Integrals

1. **Domain:**

$$\mathcal{R} = [a, b] \times [c, d] \times [r, s].$$

2. **Partial Integral:**

$$\int_a^b f(x, y, z)\, dx$$

3. **Iterated Integral:**

$$\int_a^b \int_c^d \int_r^s f(x, y, z)\, dz\, dy\, dx = \int_a^b \left(\int_c^d \left(\int_r^s f(x, y, z)\, dz \right) dy \right) dx.$$

4. **Theorem (Fubini's Theorem):** The order of integration can be rearranged if f is integrable over the region $\mathcal{R}$:

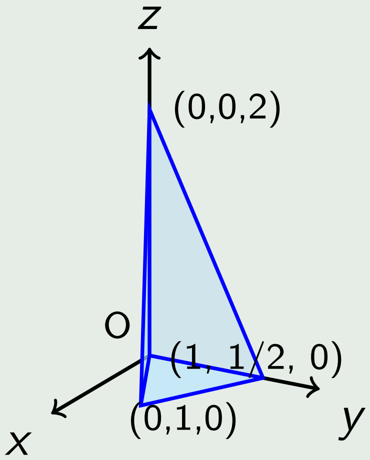
$$\int\int\int_{\mathcal{R}} f\, dV = \int_a^b \int_c^d \int_r^s f(x, y, z)\, dz\, dy\, dx.$$

Problem: Find the volume of the tetrahedron bounded by the planes:

- $x + 2y + z = 2$
- $x = 2y$
- $x = 0$
- $z = 0$

The volume V is given by the integral:

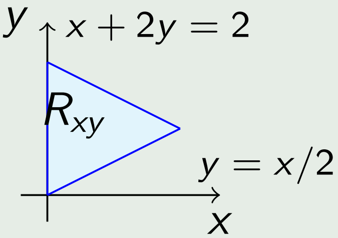
$$V = \iiint_T dV$$



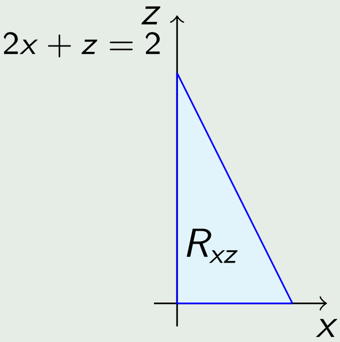
Strategy: Project the Solid onto 2D Planes

The key to setting up the integral is to understand the solid's "shadow" on the three coordinate planes. The complexity of these 2D regions often determines the difficulty of the integral.

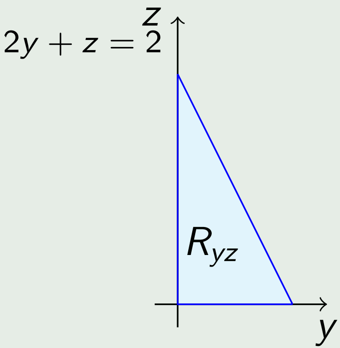
On xy-plane (R_{xy})



On xz-plane (R_{xz})



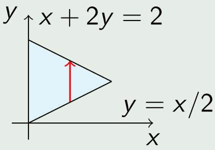
On yz-plane (R_{yz})



Orders based on dz (Projection on R_{xy})

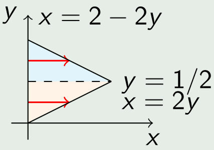
The inner integral is $\int_0^{2-x-2y} dz$. The outer double integral is over R_{xy}.

Order $dz\,dy\,dx$ (No Split)



$$\int_0^1 \int_{x/2}^{(2-x)/2} \int_0^{2-x-2y} dz\,dy\,dx$$

Order $dz\,dx\,dy$ (Requires Split)



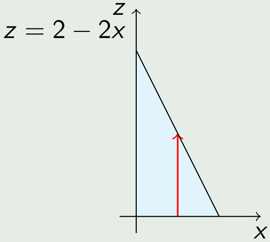
$$\int_0^{1/2} \int_0^{2y} \int_0^{2-x-2y} dz\,dx\,dy$$

$$+\int_{1/2}^1 \int_0^{2-2y} \int_0^{2-x-2y} dz\,dx\,dy$$

Orders based on dy (Projection on R_{xz})

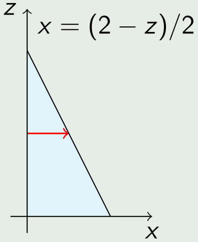
The inner integral is $\int_{x/2}^{(2-x-z)/2} dy$. The outer double integral is over R_{xz}.

Order $dy\,dz\,dx$ (No Split, Easiest)



$$\int_0^1 \int_0^{2-2x} \int_{x/2}^{(2-x-z)/2} dy\,dz\,dx$$

Order $dy\,dx\,dz$ (No Split)

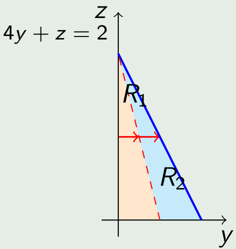


$$\int_0^2 \int_0^{(2-z)/2} \int_{x/2}^{(2-x-z)/2} dy\,dx\,dz$$

Complex Setup 1: Order $dx\,dy\,dz$

Integrating with respect to x first is complicated. Its upper bound is $\min(2y, 2-2y-z)$, forcing a split.

The projection on the yz-plane, R_{yz}, is split by the line $\mathbf{4y+z=2}$.



Setup for $dx\,dy\,dz$

This order splits the integral into two parts based on the horizontal integration strips.

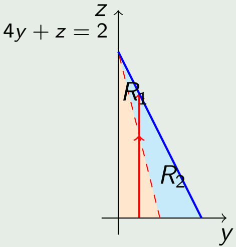
$$V = \underbrace{\int_0^2 \int_0^{(2-z)/4} \int_0^{2y} dx\,dy\,dz}_{\text{Volume over } R_1}$$

$$+ \underbrace{\int_0^2 \int_{(2-z)/4}^{(2-z)/2} \int_0^{2-2y-z} dx\,dy\,dz}_{\text{Volume over } R_2}$$

Complex Setup 2: Order $dx\,dz\,dy$

This order is even more complex, requiring the region to be broken into three separate triple integrals.

The vertical integration strips for dz cross over the splitting line $\mathbf{4y + z = 2}$.



Setup for $dx\,dz\,dy$

$$V = \underbrace{\int_0^{1/2}\int_0^{2-4y}\int_0^{2y} dx\,dz\,dy}_{\text{Bottom part of region}}$$

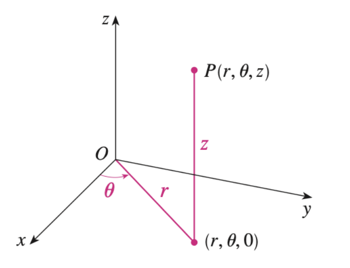
$$+ \underbrace{\int_0^{1/2}\int_{2-4y}^{2-2y}\int_0^{2-2y-z} dx\,dz\,dy}_{\text{Middle part of region}}$$

$$+ \underbrace{\int_{1/2}^{1}\int_0^{2-2y}\int_0^{2-2y-z} dx\,dz\,dy}_{\text{Top part of region}}$$

Cylindrical Coordinates

$$\begin{cases} r &= \sqrt{x^2+y^2} \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right) \\ z &= z \end{cases}$$

$$\begin{cases} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{cases}$$



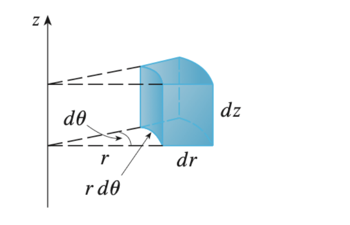


FIGURE 7

Example: Volume Using Cylindrical Coordinates

Problem

Calculate the volume of the solid enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

1. Coordinate Transformation

This solid is rotationally symmetric, making it ideal for cylindrical coordinates.

$$x^2 + y^2 = r^2$$
$$z = z$$
$$dV = r\,dz\,dr\,d\theta$$

2. Determine the Limits of Integration

- **Range of** z: $r^2 \leq z \leq 4$
- **Range of** r: $0 \leq r \leq 2$
- **Range of** θ: $0 \leq \theta \leq 2\pi$

Next Step

With the coordinate system and limits defined, we can now set up the integral.

Example: Volume Using Cylindrical Coordinates (Cont.)

3. Set Up and Evaluate the Integral

The volume V is given by the triple integral:

$$
\begin{aligned}
V &= \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r\, dz\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^2 r[z]_{r^2}^4\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^2 r(4 - r^2)\, dr\, d\theta \\
&= \int_0^{2\pi} \left[2r^2 - \frac{1}{4}r^4\right]_0^2 d\theta \\
&= \int_0^{2\pi} 4\, d\theta = 4[\theta]_0^{2\pi} = 8\pi
\end{aligned}
$$

Spherical Coordinates

$$\begin{cases} \rho &= \sqrt{x^2+y^2+z^2} \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right) \\ \phi &= \cos^{-1}\left(\frac{z}{\rho}\right) \end{cases}$$

$$\begin{cases} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \\ z &= \rho \cos\phi \end{cases}$$

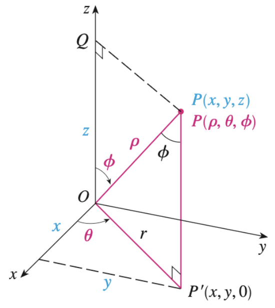


FIGURE 5

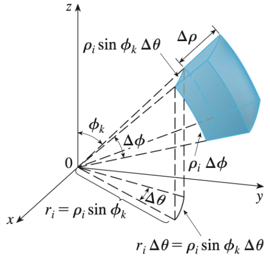


FIGURE 7

Example: Volume Using Spherical Coordinates

Problem

Calculate the volume of the part of the unit sphere $x^2 + y^2 + z^2 \leq 1$ that lies in the first octant $(x \geq 0, y \geq 0, z \geq 0)$.

1. Coordinate Transformation

This problem involves a sphere, a perfect scenario for spherical coordinates.

$$
\begin{aligned}
x^2 + y^2 + z^2 &= \rho^2 \\
dV &= \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta
\end{aligned}
$$

2. Determine the Limits of Integration

- **Range of** ρ:
  $0 \leq \rho \leq 1$
- **Range of** ϕ:
  $0 \leq \phi \leq \pi/2$
- **Range of** θ:
  $0 \leq \theta \leq \pi/2$

Next Step

With the spherical limits defined, we can proceed to the integration.

Example: Volume Using Spherical Coordinates (Cont.)

3. Set Up and Evaluate the Integral

The volume V is given by the triple integral:

$$
\begin{aligned}
V &= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{1} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
&= \int_0^{\pi/2} \int_0^{\pi/2} \sin\phi \left[\frac{1}{3}\rho^3\right]_0^1 \, d\phi \, d\theta \\
&= \frac{1}{3} \int_0^{\pi/2} \int_0^{\pi/2} \sin\phi \, d\phi \, d\theta \\
&= \frac{1}{3} \int_0^{\pi/2} [-\cos\phi]_0^{\pi/2} \, d\theta \\
&= \frac{1}{3} [\theta]_0^{\pi/2} = \frac{1}{3} \cdot \frac{\pi}{2} = \frac{\pi}{6}
\end{aligned}
$$

Exercise 6.6 Find the mass of the pyramid with base in the plane $z = -6$ and sides formed by the three planes $y = 0$, $y - x = 4$ and $2x + y + z = 4$ if the density of the solid is given by $\delta(x, y, z) = y$.

Exercise 6.7 Find the volume of the region bounded by $z = x + y$, $x + y = 5$, where $(x, y) \in [0, 5] \times [0, 5]$, and the planes $x = 0$, $y = 0$, and $z = 0$.

Exercise 6.8 Evaluate the integral

$$\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_{1}^{4-x^2-y^2} \frac{1}{z^2}\, dz\, dy\, dx.$$

We calculate the triple integral by

$$\int_0^6 \int_{y-4}^{5-\frac{y}{2}} \int_{-6}^{4-2x-y} y \, dz \, dx \, dy = 243.$$

Here we calculate the volume V of the region E using a triple integral:

$$V = \iiint_E dV = \int_0^5 \int_0^{5-x} \int_0^{x+y} dz\, dy\, dx = \frac{125}{3}.$$

Use cylindrical coordinates. The transformation is given by:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \Rightarrow \begin{cases} 0 \le r \le \sqrt{3} \\ 0 \le \theta \le 2\pi \\ 1 \le z \le 4 - r^2 \end{cases}$$

and $|J(r,\theta,z)| = r$.
Therefore, the integral

$$\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_{1}^{4-x^2-y^2} \frac{1}{z^2}\, dz\, dy\, dx$$

becomes

$$\int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{4-r^2} \frac{1}{z^2} r\, dz\, dr\, d\theta$$

$$= 2\pi \int_0^{\sqrt{3}} \left(r - \frac{r}{4-r^2} \right) dr = (3 - ln4)\pi.$$

Thank you!