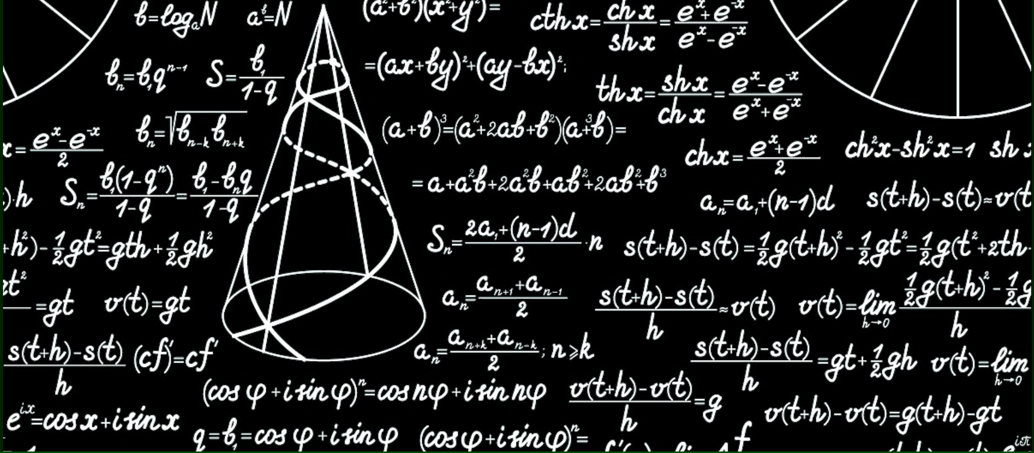

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1. Double Integrals

Definition

The **double integral** of f over the rectangle R is

$$\iint_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\Delta A$$

Remark

A function f is called **integrable** if the limit in the above Definition exists, which means $\forall \varepsilon > 0$ there is an integer N such that

$$\left| \iint_R f(x,y)dA - \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\Delta A \right| < \varepsilon, \quad \forall m > N, \forall n > N$$

Properties

1. $\iint_R [f(x,y) + g(x,y)]dA = \iint_R f(x,y)dA + \iint_R g(x,y)dA$
2. $\iint_R cf(x,y)dA = c \iint_R f(x,y)dA$, c is a constant
3. $f(x,y) \geq g(x,y)$ for all (x,y) in R, then $\iint_R f(x,y)dA \geq \iint_R g(x,y)dA$
4. $\left| \iint_R f(x,y)dA \right| \leq \iint_R |f(x,y)|dA$

Definition

The **iterated integral**

$$\int_a^b \int_c^d f(x,y)dydx = \int_a^b \left[\int_c^d f(x,y)dy \right] dx$$

is the integral of $\left[\int_c^d f(x,y)dy \right]$ between $x = a$ and $x = b$.

Fubini's Theorem

If f is continuous on the rectangle $R = \{(x,y) | a \leq x \leq b, c \leq y \leq d\}$, then

$$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

Definition

A plane region is said to be of **type I** if it lies between the graphs of two continuous functions of x , that is

$$D = \{(x, y)|a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

If f is continuous on a type I region D, then

$$\iint_D f(x, y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx$$

A plane region is said to be of **type II** if

$$D = \{(x, y)|c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

$$\iint_D f(x, y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)dxdy$$

Definition

The **Jacobian** of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Suppose that T is a transformation whose **Jacobian** is **nonzero** and that maps a region Sin the uv-plane onto a region R in the xy-plane. f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint_R f(x, y)dA = \iint_S f\left[x(u, v), y(u, v)\right]\left|\frac{\partial(x, y)}{\partial(u, v)}\right|dudv$$

Ex 6.1 Evaluate $\iint_R (5-x)dA$, $R = \{(x,y)|0 \leq x \leq 8, 0 \leq y \leq 2\}$
Ex 6.2 Evaluate $\iint_D xydA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.
Ex 6.3 Express D as a union of regions of type I or type II and evaluate the integral $\iint_D ydA.$, where D is the region bounded by the line $x = -1, y = -1, y = (x+1)^2, x = y - y^3$.
Ex 6.4 Use polar coordinates to combine the sum $\int_{1/\sqrt{2}}^{1} \int_{\sqrt{1-x^2}}^{x} xydydx + \int_{1}^{\sqrt{2}} \int_{0}^{x} xydydx + \int_{\sqrt{2}}^{2} \int_{0}^{\sqrt{4-x^2}} xydydx$ into one double integral. Then evaluate the double integral.
Ex 6.5 Evaluate $\iint_R (x+y)e^{x^2-y^2} dA$ by making an appropriate change of variables. R is enclosed by the lines $x - y = 0, x - y = 2, x + y = 0, x + y = 3$.

Answers

6.1 16

6.2 36

6.3 $-\frac{2}{15}$

6.4 $\frac{15}{16}$

6.5 $\frac{1}{4}(e^6 - 7)$

2. Triple Integrals

1. **Domain:**

$$\mathcal{R} = [a, b] \times [c, d] \times [r, s].$$

2. **Partial Integral:**

$$\int_a^b f(x, y, z)\, dx$$

3. **Iterated Integral:**

$$\int_a^b \int_c^d \int_r^s f(x, y, z)\, dz\, dy\, dx = \int_a^b \left(\int_c^d \left(\int_r^s f(x, y, z)\, dz \right) dy \right) dx.$$

4. **Theorem (Fubini' s Theorem):** The order of integration can be rearranged if f is integrable over the region $\mathcal{R}$:

$$\int\int\int_{\mathcal{R}} f\, dV = \int_a^b \int_c^d \int_r^s f(x, y, z)\, dz\, dy\, dx.$$

Theorem

Let $\mathcal{R}$ and $\mathcal{S}$ be bounded regions in $\mathbb{R}^3$ that contain all of their boundary points. Let $T : \mathcal{D} \to \mathbb{R}^3$ where $\mathcal{D} \subset \mathbb{R}^3$ be an injective map that is onto and maps $\mathcal{S}$ in the *uvw*-plane to $\mathcal{R}$ in the *xyz*-plane. If $f(x, y, z)$ is continuous on $\mathcal{R}$, all of the components of T have continuous partial derivatives and the Jacobian $JT(u, v, w)$ is never 0 on $\mathcal{S}$, then

$$\iiint_{\mathcal{R}} f(x, y, z)\, dx\, dy\, dz = \iiint_{\mathcal{S}} f(T(u, v, w))\, \left|JT(u, v, w)\right|\, du\, dv\, dw.$$

The transformation T from cylindrical coordinates to rectangular coordinates is given by:

$$T(r, \theta, s) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ s \end{bmatrix}^T.$$

The transformation T from spherical coordinates to rectangular coordinates is given by:

$$T(\rho, \theta, \phi) = \begin{bmatrix} \rho\sin\phi\cos\theta \\ \rho\sin\phi\sin\theta \\ \rho\cos\phi \end{bmatrix}^T.$$

Remark

The Jacobian J of the transformation from cylindrical to rectangular coordinates is:

$$J(r, \theta, s) = r.$$

The Jacobian J of the transformation from spherical to rectangular coordinates is:

$$J(\rho, \theta, \phi) = -\rho^2 \sin \phi.$$

Exercise 6.6 Find the mass of the pyramid with base in the plane $z = -6$ and sides formed by the three planes $y = 0$, $y - x = 4$ and $2x + y + z = 4$ if the density of the solid is given by $\delta(x, y, z) = y$. **Exercise 6.7** Find the volume of the region bounded by $z = x + y$, $x + y = 5$, where $(x, y) \in [0, 5] \times [0, 5]$, and the planes $x = 0$, $y = 0$, and $z = 0$.
Exercise 6.8 Evaluate the integral

$$\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_{1}^{4-x^2-y^2} \frac{1}{z^2}\, dz\, dy\, dx.$$

We calculate the triple integral by

$$\int_0^6 \int_{y-4}^{5-\frac{y}{2}} \int_{-6}^{4-2x-y} y \, dz \, dx \, dy = 243.$$

Here we calculate the volume V of the region E using a triple integral:

$$V = \iiint_E dV = \int_0^5 \int_0^{5-x} \int_0^{x+y} dz\, dy\, dx = \frac{125}{3}.$$

Use cylindrical coordinates. The transformation is given by:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \Rightarrow \begin{cases} 0 \leq r \leq \sqrt{3} \\ 0 \leq \theta \leq 2\pi \\ 1 \leq z \leq 4 - r^2 \end{cases}$$

and $|J(r,\theta,z)| = r$.
Therefore, the integral

$$\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_{1}^{4-x^2-y^2} \frac{1}{z^2}\, dz\, dy\, dx$$

becomes

$$\int_{0}^{2\pi} \int_{0}^{\sqrt{3}} \int_{1}^{4-r^2} \frac{1}{z^2} r\, dz\, dr\, d\theta$$

$$= 2\pi \int_{0}^{\sqrt{3}} \left(r - \frac{r}{4-r^2} \right) dr = (3 - ln4)\pi.$$

3. Vector Field

Definition

A vector field on two (or three) dimensional space is a function $\mathbf{F}$ that assigns to each point (x, y) (or (x, y, z)) a two (or three) dimensional vector given by $\mathbf{F}(x, y)$ (or $\mathbf{F}(x, y, z)$).

The standard notation for the function $\mathbf{F}$ is,

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$$

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

Definition

If f is a scalar function of two variables, then its gradient ∇f (or grad f) is defined by

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

Ex 6.9 Find the gradient vector field of following functions.

1. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$
2. $f(x, y) = ln(1 + x^2 + y^2)$

Thank you!