255RC3

MATH2550J – Summer 2024

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1. Linear Independence

Linear Independence

Definition

Given a vector space V, a finite set $\{ v_1, v_2, \dots, v_k\} \subset V$ is linear independent if

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = \mathbf{0} \Leftrightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0.$$

A set of vectors is called **dependent** if it is not independent.

Remark

If $\{ v_1, v_2, \dots, v_k\} \in V$ is linear independent, then

$$\begin{aligned} \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k &= \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_k v_k \\ &\Leftrightarrow \forall i = 1, \dots, k, \alpha_i = \beta_i. \end{aligned}$$

Exercise 3.1 Prove that the vector set $\{v_1, v_2, \ldots v_n\}$ is linearly independent, where v_is are n-dimensional unit vectors.

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Solution 3.1 Suppose there exist $\alpha_1, \alpha_2, \dots \alpha_n$ such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \mathbf{0}$$

$$\alpha_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \alpha_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

We get

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

Therefore, vector set $\{v_1, v_2, \dots v_n\}$ is linearly independent.

2. Basis

Definition

Given a vector space V, a vector set $B = \{v_1, v_2, \ldots, v_n\}$ is a **basis** if it is linearly independent and spanning, that is, every $v \in V$ can be uniquely expressed as

$$v = \sum_{i=1}^{n} \alpha_i v_i,$$

where α_is are the *coordinates* of v with respect to the basis $\{v_1, v_2, \ldots, v_n\}$

Remark

- The dimension of V is denoted $dimV$.
- $dimV$ equals to the length of the basis of V.

Exercise 3.2 In the space $\mathbb{R}^{2\times 2}$, consisting of all second-order real square matrices, suppose

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Prove that $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ is a basis of space $\mathbb{R}^{2\times 2}$.

3. QR Decomposition

QR Decomposition

Definition

QR decomposition, also known as **QR factorization** or **QU factorization**, is a decomposition of a matrix A into a product

$$A = QR$$

of an **orthogonal matrix Q** and an **upper triangular matrix R**[1].

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How to find the QR decomposition of a matrix?

Definition

Consider the Gram-Schmidt procedure, with the vectors to be considered in the process as columns of the matrix A. That is,

$$A = \begin{bmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \\ | & | & & | \end{bmatrix}.$$

Then, the orthonormal basis can be found as follows:

$$\begin{aligned} \mathbf{u}_1 &= \mathbf{a}_1, & \mathbf{e}_1 &= \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|}, \\ \mathbf{u}_2 &= \mathbf{a}_2 - (\mathbf{a}_2 \cdot \mathbf{e}_1)\mathbf{e}_1, & \mathbf{e}_2 &= \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|}. \end{aligned}$$

Gram-Schmidt procedure

Definition

In general, for $k \geq 1$,

$$\mathbf{u}_{k+1} = \mathbf{a}_{k+1} - \sum_{i=1}^{k} (\mathbf{a}_{k+1} \cdot \mathbf{e}_i)\mathbf{e}_i, \quad \mathbf{e}_{k+1} = \frac{\mathbf{u}_{k+1}}{\|\mathbf{u}_{k+1}\|}.$$

Note that $\| \cdot \|$ is the L_2 norm.

Definition

$$
\begin{aligned}
A &= \begin{bmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \\ | & | & & | \end{bmatrix} \\
&= \begin{bmatrix} | & | & & | \\ \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \langle \mathbf{a}_1, \mathbf{e}_1 \rangle & \langle \mathbf{a}_2, \mathbf{e}_1 \rangle & \cdots & \langle \mathbf{a}_n, \mathbf{e}_1 \rangle \\ 0 & \langle \mathbf{a}_2, \mathbf{e}_2 \rangle & \cdots & \langle \mathbf{a}_n, \mathbf{e}_2 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \langle \mathbf{a}_n, \mathbf{e}_n \rangle \end{bmatrix} \\
&= QR.
\end{aligned} \tag{1}
$$

Note that once we find $\mathbf{e}_1, \ldots, \mathbf{e}_n$, it is not hard to write the QR factorization.

Exercise 3.3

Exercises

Consider the QR decomposition of matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Answer

Let's consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Using the Gram-Schmidt process, we obtain the orthonormal vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ as follows:

$$\mathbf{u}_1 = \mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix},$$

$$\mathbf{u}_2 = \mathbf{a}_2 - (\mathbf{a}_2 \cdot \mathbf{e}_1)\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix},$$

Answer

$$\mathbf{e}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \frac{1}{\sqrt{\frac{3}{2}}} \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix},$$

$$\mathbf{u}_3 = \mathbf{a}_3 - (\mathbf{a}_3 \cdot \mathbf{e}_1)\mathbf{e}_1 - (\mathbf{a}_3 \cdot \mathbf{e}_2)\mathbf{e}_2 =$$

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} - \frac{1}{\sqrt{6}} \begin{pmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix},$$

$$\mathbf{e}_3 = \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} = \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}.$$

Answer

Thus, we obtain the orthogonal matrix Q as:

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix},$$

and the upper triangular matrix R as:

$$R = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{\frac{3}{2}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \sqrt{\frac{2}{3}} \end{bmatrix}.$$

Exercise

Consider the QR decomposition of matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Exercise 3.4

Answer

$$A = QR = \begin{bmatrix} 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{42}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{5}{\sqrt{42}} \\ -1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{3}} & -\frac{4}{\sqrt{42}} \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & \sqrt{3} & \frac{1}{\sqrt{3}} \\ 0 & 0 & \sqrt{\frac{14}{3}} \end{bmatrix}.$$

4. Column Space and Null Space

Definition

Given a matrix $A_{m\times n}$, the set of all linear combinations of columns of A

$$\{y \in \mathbb{R}^m | y = Ax, \text{for some } x \in \mathbb{R}^n\}$$

is called **image** or **column space** of A, denoted by $im(A)$ or $C(A)$.

How to find column space

- Find the rref of A.
- For each non-zero row, find the column in which the first non-zero (pivot) number in the row resides.
- The columns found in the previous step correspond to the columns in the original matrix, which are the bases of the column space.

Definition

Set of solutions to the linear equations $Ax = 0$,

$$\{x \in \mathbb{R}^n | Ax = 0\}$$

is called the **kernel** or **null space** of matrix A, denoted by $ker(A)$ or $N(A)$.

How to find null space

- Use Gauss-Jordan Elimination to solve the equation $Av = 0$.

Exercise 4.1 Find C(A) and N(A) of A

$$A = \begin{pmatrix} 1 & 4 & 8 & -3 & -7 \\ -1 & 2 & 7 & 3 & 4 \\ -2 & 2 & 9 & 5 & 5 \\ 3 & 6 & 9 & -5 & 2 \end{pmatrix}$$

Solution

$$rref(A) = \begin{pmatrix} 1 & 0 & -2 & 0 & -3 \\ 0 & 1 & 5/2 & 0 & -1/2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C(A) = span(\begin{pmatrix} 1 \\ -1 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} -3 \\ 3 \\ 5 \\ -5 \end{pmatrix}), N(A) = span(\begin{pmatrix} 2 \\ -5/2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1/2 \\ 0 \\ -4 \\ 1 \end{pmatrix})$$

5. Orthogonal Projection Matrix

Orthogonal Projection Matrix

Definition

Project a vector v from space V to its subspace. The subspace has a group of basis

$$A = (\alpha_1, \alpha_2, \dots \alpha_n)$$

The orthogonal projection matrix is:

$$P = A(A^T A)^{-1} A^T$$

Remark

- v's projection is Pv
- properties: $P^2 = P, P^T = P$

Thank you!