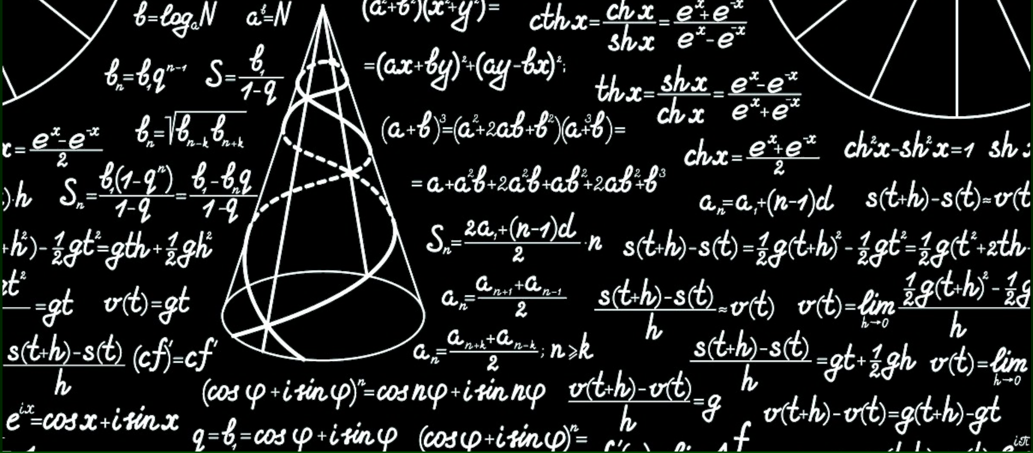

255RC2

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Contents

1. Transpose

Transpose

Definition

Given $A = (a_{ij})$, $A^T := (a_{ji})$

Proposition

Let A, B be matrices, $\lambda \in \mathbb{R}$.

1. $(A+B)^T = A^T + B^T$, $(\lambda A)^T = \lambda A^T$
2. $(AB)^T = B^T A^T$
3. $(A^T)^T = A$
4. If A is invertible, so is A^T, and $(A^T)^{-1} = (A^{-1})^T$

Definition

A matrix A is called **symmetric** if $A = A^T$
A matrix A is called **skew-symmetric** if $A^T = -A \iff A + A^T = 0$

Remark

1. A symmetric or skew-symmetric matrix is necessarily square.
2. For a skew-symmetric matrix, if $i = j$, then $a_{ij} = 0$

Example

$A = \begin{pmatrix} 1 & -4 & -2 \\ 4 & 1 & 3 \\ 2 & -3 & 2 \end{pmatrix}$ is not skew-symmetric.

2. Matrix as a Function

Matrix as a Function

Definition

An $n \times m$ Matrix can be interpreted as a function $F : M_{m\times p} \to M_{n\times p}$

Property

For $M \in M_{n\times m}, A, B \in M_{m\times p}, \alpha, \beta \in \mathbb{R}$
We have $M(\alpha A + \beta B) = \alpha MA + \beta MB$

Question

1. Do you know other function with such property?
2. Why we want such property?

3. Orthogonal Matrices and Orthonormal Vectors

Orthogonal Matrix

Definition

A matrix $A \in M_n(\mathbb{R})$ is called an orthogonal matrix iff $A^T = A^{-1}$

Properties.

- I_n is an orthogonal matrix.
- If A and B are orthogonal matrices, then AB is also orthogonal.
- If A is an orthogonal matrix, then A^{-1} is also orthogonal matrix.
- If A is an orthogonal matrix, then $det(A) = \pm 1$.

Examples.

- Rotation Matrix.

Rotation by θ in $\mathbb{R}^2$ is given by

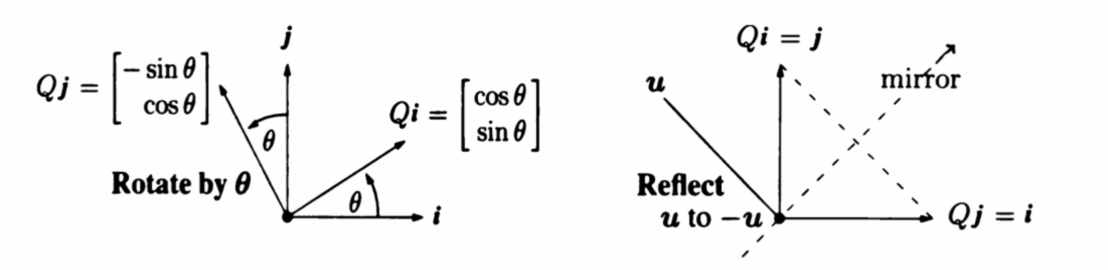
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

- Reflection Matrix.

Reflect (x_1, x_2) across $\theta/2$ in $\mathbb{R}^2$ is given by

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

You can treat such matrix as a function $Q : \mathbb{R}^2 \to \mathbb{R}^2$



Orthogonal Vectors and Orthonormal Vectors

Definition

Let $\{v_1, \cdots, v_k\} \in \mathbb{R}$ be a subset of k distinct vectors, then $\{v_1, \cdots, v_k\}$ is an orthogonal set of vectors if $\langle v_i, v_j \rangle = 0$ for all $1 \leq i, j \leq k$, $i \neq j$.
Also, $\{q_1, \cdots, q_k\}$ is an orthonormal set of vectors if it is an orthogonal set and all of its vectors are unit vectors (*i.e.*, $\|q_i\| = 1$ for $i \leq i \leq k$).

Remark

Any set containing a single vector is orthogonal; any set containing a single unit vector is orthonormal.

Example.

$\bar{i}, \bar{j}, \bar{k} \in \mathbb{R}^3$.

Proposition

Given vectors $q_i \in \mathbb{R}^n$, $i = 1, \cdots, m$ such that

$$\langle q_i, q_j \rangle = q_i^T q_j = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \quad (\delta_{ij} : \mathit{Kronecker})$$

then we call the set of vectors $\{q_i\}$ orthonormal, and

$$Q = \begin{bmatrix} | & & | \\ q_1 & \dots & q_m \\ | & & | \end{bmatrix} \Rightarrow Q^T Q = I_m.$$

Orthogonal Vectors and Orthonormal Vectors

Proof

$$Q^TQ = \begin{bmatrix} - & q_1^T & - \\ & \vdots & \\ - & q_m^T & - \end{bmatrix} \begin{bmatrix} | & & | \\ q_1 & \dots & q_m \\ | & & | \end{bmatrix} = \begin{bmatrix} q_1^Tq_1 & \dots & q_1^Tq_m \\ \vdots & \ddots & \vdots \\ q_m^Tq_1 & \dots & q_m^Tq_m \end{bmatrix} = I_m$$

Proposition

Let $Q = \begin{bmatrix} | & | & & | \\ \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \\ | & | & & | \end{bmatrix} \in M_n(\mathbb{R})$,

where $\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n\}$ is a set of orthonormal vectors in $\mathbb{R}^n$.

Then Q is an orthogonal matrix, $Q^T = Q^{-1}$.

We verify that $QQ^T = I_n$ and $Q^TQ = I_n$.

Exercise

Determine whether the following matrices are orthogonal matrices

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Proposition

The orthogonal matrices are precisely the matrices that preserve the inner product in $\mathbb{R}^n$ i.e., $\forall x, y \in \mathbb{R}^n$, $\langle x, y \rangle = \langle Ax, Ay \rangle$ or $x^T y = (Ax)^T Ay$.

Proof.

If A is orthogonal, i.e., $A^{-1} = A^T$, then $A^T A = I_n$, so
$(Ax)^T (Ay) = x^T A^T Ay = x^T I_n y = x^T y$. □

Why we need Orthonormal Matrix?

1. **Magnitude of Determinant and Column Norms:**
   - $|\det(A)| = 1$.
   - Column vectors $\mathbf{q}_i$ have unit norm: $||\mathbf{q}_i|| = 1$.
   - *Implication:* The transformation represented by A is a rotation or a reflection.
2. **Defining Property and Its Consequences:**
   - $A^T A = I_n$ (which means $A^{-1} = A^T$).
   - The original note mentions this "has very good symmetric properties."
   - *Implications include:*
     - Preservation of inner products (hence, lengths and angles are preserved: $\langle Ax, Ay \rangle = \langle x, y \rangle$).
     - The inverse A^{-1} is easy to compute (it's just the transpose A^T).
3. **Applications in Data Processing:**
   - Effective in removing redundant information from data

4. Kernel and Image

Definition

- Given matrix $A \in M_{m\times n}(\mathbb{R})$. The kernel, or nullspace of A, is defined as

$$\ker A = N(A) = \{x \in \mathbb{R}^n \mid Ax = \mathbf{0}\} \subseteq \mathbb{R}^n$$

  where $A : \mathbb{R}^n \to \mathbb{R}^n$.

- The image of A, or column space, is defined as

$$\begin{aligned} \operatorname{im} A = C(A) &= \{y \in \mathbb{R}^m \mid \exists x \in \mathbb{R}^n, y = Ax\} \\ &= \{Ax \mid x \in \mathbb{R}^n\} \end{aligned}$$

  (The image is a subspace of $\mathbb{R}^m$).

Definition

A function $f : A \to B$ is called injective, if
$\forall x_1, x_2 \in A, f(x_1) = f(x_2) \implies x_1 = x_2$
i.e., $\forall x_1, x_2 \in A, x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$

Proposition

Given $A \in M_{m \times n}(\mathbb{R})$, i.e., $A : \mathbb{R}^n \to \mathbb{R}^m$
then, A is injective $\iff \ker A = N(A) = \{0\}$

Injectivity

Proof

($\Rightarrow$) Assume A is injective, that is,
$\forall x_1, x_2 \in \mathbb{R}^n, Ax_1 = Ax_2 \implies x_1 = x_2$.
Take $v \in \ker A$, i.e., $Av = 0$. We also know that $A0 = 0$, hence $Av = A0$. By injectivity of A, $v = 0$. Therefore $\ker A = \{0\}$.
($\Leftarrow$) Conversely, suppose $\ker A = \{0\}$.
Take $v_1, v_2 \in \mathbb{R}^n$, s.t. $Av_1 = Av_2$. We want to show that $v_1 = v_2$. Indeed, we have $A(v_1 - v_2) = 0$, but $\ker A = \{0\}$, hence $v_1 - v_2 = 0$. So $v_1 = v_2$.

Proposition

Given $A \in M_{m\times n}(\mathbb{R})$, $\ker(A) = \ker(A^T A)$

Proof

$(\subseteq)$ Take $x \in \ker(A)$, i.e., $Ax = 0$. So $(A^T A)x = A^T(Ax) = A^T 0 = 0$. So $x \in \ker(A^T A)$.
$(\supseteq)$ Take $x \in \ker(A^T A)$, i.e., $A^T Ax = 0$. So $x^T A^T Ax = x^T 0 = 0$.
$(Ax)^T Ax = \|Ax\|^2 = 0. \implies Ax = 0 \implies x \in \ker A$.
$(v_1^2 + v_2^2 + \cdots + v_n^2 = 0 \implies v_i = 0)$

5. Projection Matrix

Projection Matrix

Definition

A matrix $P \in M_n(\mathbb{R})$ is a projection matrix, if $P^2 = P$.
It is called an orthogonal projection, if $P = P^T$.

Example

For an orthonormal matrix Q

- $(QQ^T)^T = (Q^T)^T Q^T = QQ^T$
- $(QQ^T)^2 = (QQ^T)(QQ^T) = Q(Q^T Q)Q^T = QI_k Q^T = QQ^T$

$QQ^T : \mathbb{R}^n \to \mathbb{R}^n$ is a projection

Orthogonal Projection Matrix

Proposition

A projection matrix P is an orthogonal projection if

$$\ker P \perp \operatorname{im} P$$

i.e., $\forall x \in \ker P$, $\forall y \in \operatorname{im} P$,

$$\langle x, y \rangle = x^T y = 0 \in \mathbb{R}$$

Exercise

Show that $\ker P \perp \operatorname{im} P$ if $P = P^T$

Example

$$\{q_1, q_2\} = \left\{ \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} \right\}$$

$$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 \end{bmatrix}, \quad Q^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$$

$$QQ^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

Reference

1. VV255, Lecture Notes 24SU, Runze Cai
2. VV255 RC2, Jiayue Huang, et al
3. Introduction to Linear Algebra, Sixth Edition, Gilbert Strang

Thank you and Enjoy your Summer Semester!