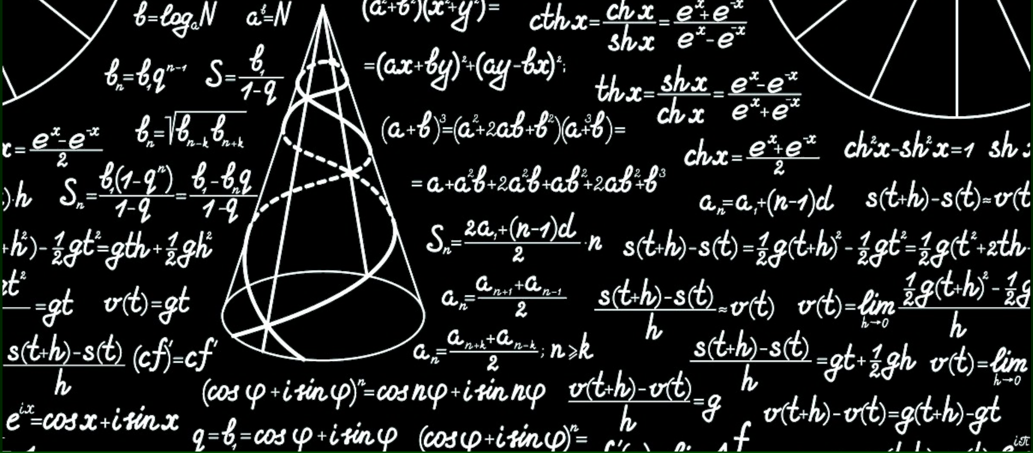

255RC4

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1. Arc Length

Arc Length

Thinking. Let's recast an old formula into terms of vector functions. We want to determine the length of a vector function,

$$\bar{r}(t) =< f(t), g(t), h(t) >$$

So, the length of the curve $\bar{r}(t)$ on the interval $a \leq t \leq b$ is

$$L = \int_a^b \sqrt{(\dot{f}(t))^2 + (\dot{g}(t))^2 + (\dot{h}(t))^2} dt$$

Think: what if the curve is 2D?

Notice that the integrand (the function we' re integrating) is nothing more than the magnitude of the tangent vector,

$$||\dot{\bar{r}}(t)|| = \sqrt{(\dot{f}(t))^2 + (\dot{g}(t))^2 + (\dot{h}(t))^2}$$

Therefore, the arc length can be written as,

$$L = \int_a^b ||\dot{\bar{r}}(t)|| dt$$

EX1.1. Determine the length of the curve $\bar{r}(t) = <2t, 3sin(2t), 3cos(2t)>$ on the interval $0 \leq t \leq 2\pi$

Arc length function and reparametrize

We need to take a quick look at another concept here. We define the arc length function as,

$$s(t) = \int_0^t ||\dot{\bar{r}}(u)||du$$

Okay, just why would we want to do this? Well let' s take the result of the arc length function above and solve it for t (t(s)).
Now, taking expression for t with respect to s (t(s)) and plugging it into the original vector function and we can reparametrize the function into the form, $\bar{r}(t(s))$

Example.Determine the arc length function for $\bar{r}(t) = <2t, 3sin(2t), 3cos(2t)>$and Reparametrize it.
Thinking Where on the curve $\bar{r}(t) = <2t, 3sin(2t), 3cos(2t)>$are we after traveling for a distance of $\frac{\pi\sqrt{10}}{3}$?

Exercise

Ex1.1 Find the arc length function for $\bar{r}(t) =< 4t, -2t, \sqrt{5}t^2 >$

Ex1.2 Determine where on the curve given by $\bar{r}(t) =< t^2, 2t^3, 1 - t^3 >$ we are after traveling a distance of 20.

2. Curvature

Tangent vector

Given the vector function $\bar{r}(t)$,we call $\bar{r}'(t)$ the tangent vector provided it exists and provided $\bar{r}'(t) \neq \bar{0}$ Also,theunit tangent vector to the curve is given by,

$$\bar{T}(t) = \frac{\bar{r}'(t)}{||\bar{r}'(t)||}$$

The tangent line to $\bar{r}(t)$ at P is then the line that passes through the point P and is parallel to the tangent vector.

Definition. The plane orthogonal to the (unit) tangent vector $\bar{T}$ of the curve at the point P is called the normal plane of the curve at the point P.

Definition. The plane that comes closest to containing the part of the curve near P is called the osculating plane of the curve at the point P.

The osculating plane contains the tangent vector $\bar{T}$ at the point P and the unit vector $\bar{N}(t) = \bar{T}'(t)/|\bar{T}'(t)|$ which indicates the direction in which the curve is turning at the point P.

Definition. The unit vector

$$\bar{N}(t) = \bar{T}'(t)/|\bar{T}'(t)|$$

which indicates the direction in which the curve is turning at the point P is called the principal unit normal vector. The binormal vector is defined to be

$$\bar{B}(t) = \bar{T}(t) \times \bar{N}(t)$$

Definition. The set of vectors $\bar{T}$, $\bar{B}$ and $\bar{N}$ which start at various points of the curve is called the TBN Frame.

Exercise

Ex2.1Find the vector equation of the tangent line to the curve given by $\bar{r}(t) = t^2\bar{i} + 2sin(t)\bar{j} + 2cos(t)\bar{k}$ at $t = \frac{\pi}{3}$

Ex2.2 Find the normal and binormal vectors for $\bar{r}(t) =< t, 3sint, 3cost >$

Definition. Let $A \subseteq \mathbb{R}$. We say that a vector function $\bar{r} : A \to \mathbb{R}^3$ is smooth on an interval $I \subseteq A$ if $\bar{r}'$ is continuous on I and for all $t \in I$, $\bar{r}'(t) \neq 0$. We say that a curve $\mathcal{C}$ is smooth if $\mathcal{C}$ can be described by a smooth vector function.

Definition. Let $\bar{R} : A \to \mathbb{R}$ be a vector function that is smooth on the interval I. The curvature of the curve $\mathcal{C}$ described by $\bar{r}$ is the function defined by

$$\kappa(t) = \left| \frac{d}{ds} \left[\widehat{\bar{r}'(t)} \right] \right| = || \frac{d\bar{T}}{ds} ||.$$

It measures the rate at which the direction of the vector function $\bar{r}$ is changing.

Theorem. Let $\bar{r} : A \to \mathbb{R}^3$ be a vector function that is smooth on the interval I and such that $\bar{r}'$ is differentiable on I. Then for all $t \in I$,

$$\kappa(t) = \frac{\left|\bar{r}'(t) \times \bar{r}''(t)\right|}{\left|\bar{r}'(t)\right|^3} = ||\frac{\bar{T}'(t)}{\bar{r}'(t)}||.$$

For a plane curve $\mathcal{C} : y = f(x)$, we can choose x as the parameter and write $\bar{r}(x) = x\bar{i} + f(x)\bar{j}$. Then $\bar{r}'(x) = \bar{i} + f'(x)\bar{j}$, $\bar{r}''(x) = f''(x)\bar{j} \Rightarrow \bar{r}'(x) \times \bar{r}''(x) = f''(x)\bar{k}$.

Theorem. Let $\mathcal{C} : y = f(x)$ where $f : D \to \mathbb{R}$. be a plane curve. Then for all $x \in D$

$$\kappa(x) = \frac{|f''(x)|}{\left[1 + \left(f'(x)\right)^2\right]^{\frac{3}{2}}}$$

Exercise

Ex2.3 Find the curvature of $\bar{r}(t) =< cos(2t), -sin(2t), 4t >$

Ex2.4 Find the curvature of $\bar{r}(t) =< 4t, -t^2, 2t^3 >$

3. Surfaces

Quadratic Surfaces

$$\Big\{(x, y, z)|Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Jz + K = 0\Big\}$$

where A to K are all constants and x, y, z being variables.
Please see the file "Surface.pdf".

4. Indefinite Integrals

Definition. The set of all antiderivatives of the function $f(x)$ is called the indefinite integral of $f(x)$ and is denoted by

$$\int f(x)\mathrm{d}x = F(x) \Leftrightarrow F'(x) = f(x)$$

Common Antiderivatives

- $\int x^\mu \mathrm{d}x = \frac{x^{\mu+1}}{\mu+1} + C, \mu \neq -1$
- $\int \frac{\mathrm{d}x}{x} = \ln|x| + C$
- $\int \sin x \ \mathrm{d}x = -\cos x + C$
- $\int \cos x \ \mathrm{d}x = \sin x + C$
- $\int e^x \ \mathrm{d}x = e^x + C$
- $\int a^x \ \mathrm{d}x = \frac{a^x}{\ln a} + C$
- $\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + C$
- $\int \frac{\mathrm{d}x}{1+x^2} = \arctan x + C$
- $\int \tan x \ \mathrm{d}x = -\ln|\cos x| + C$
- $\int \tan^2 x \ \mathrm{d}x = \tan x \ - x + C$
- $\int \sec x \ \mathrm{d}x = \ln|\sec x + \tan x| + C$
- $\int \sec^2 x \ \mathrm{d}x = \tan x + C$
- $\int \cot x \ \mathrm{d}x = \ln|\sin x| + C$
- $\int \cot^2 x \ \mathrm{d}x = -x - \cot x + C$
- $\int \csc x \ \mathrm{d}x = -\ln|\cot x + \csc x| + C$
- $\int \csc^2 x \ \mathrm{d}x = -\cot x + C$

Addition

$$\int \left(f_1(x) + f_2(x)\right) \mathrm{d}x = \int f_1(x)\mathrm{d}x + \int f_2(x)\mathrm{d}x$$

Example.

$$\int \tan^2 x \mathrm{d}x = \int \left(\frac{1}{\cos^2 x} - 1\right) \mathrm{d}x = \int \frac{1}{\cos^2 x}\mathrm{d}x - \int \mathrm{d}x = \tan x - x + C.$$

Multiplied by a Constant

$$\int (C \cdot f(x))\mathrm{d}x = C \int f(x)\mathrm{d}x, \;\; C = \text{const}$$

Example.

$$\int \frac{2\mathrm{d}x}{\sqrt{1-x^2}} = 2\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = 2\arcsin x + C.$$

Integration by Parts

Let $u(x), v(x)$ be differentiable on $\langle a, b\rangle$ and $H(x)$ be the antiderivative of $u'(x) \cdot v(x)$ on $\langle a, b\rangle$. Then $u(x)v(x) - H(x)$ is the antiderivative of $u'(x) \cdot v(x)$ on $\langle a, b\rangle$:

$$\int u'(x)v(x)\mathrm{d}x = u(x)v(x) - \int u(x)v'(x)\mathrm{d}x$$

Remark. Set the part with "friendilier derivative" to be $f(x)$.

Example.

$$\int xe^x\mathrm{d}x = \begin{bmatrix} u(x) = e^x \\ v(x) = x \end{bmatrix} = \int u'(x)v(x)\mathrm{d}x = xe^x - \int e^x\mathrm{d}x = xe^x - e^x + C.$$

Composition

Let $g(t) : \langle\alpha, \beta\rangle \mapsto \langle a, b\rangle$ be differentiable on $\langle\alpha, \beta\rangle$ and $F(x)$ be the antiderivative of $f(x)$ on $\langle a, b\rangle$. Then

$$\int f(g(t)) \cdot g'(t)\mathrm{d}t = \int f(g(t))\mathrm{d}(g(t)) = F(g(t)) + C.$$

Example.

$$\int xe^{x^2}\mathrm{d}x = \frac{1}{2}\int e^{x^2}(2x\mathrm{d}x) = \frac{1}{2}\int e^{x^2}\mathrm{d}(x^2) = \frac{1}{2}e^{x^2} + C,$$

$$b\int \frac{1}{x^2}\sin\frac{1}{x}\mathrm{d}x = -\int \sin\frac{1}{x}\mathrm{d}\left(\frac{1}{x}\right) = \cos\frac{1}{x} + C.$$

Substitution

Let $g(t) : \langle \alpha, \beta \rangle \mapsto \langle a, b \rangle$ be differentiable on $\langle \alpha, \beta \rangle$ and invertible with the inverse $t = t(x)$. If $H(x)$ if the antiderivative of $f(g(t)) \cdot g'(t)$ on $\langle \alpha, \beta \rangle$, then $H(t(x))$ is the antiderivative of $f(x)$ on $\langle a, b \rangle$:

$$\int f(x)\mathrm{d}x = \left[\begin{array}{c} x = g(t) \\ \mathrm{d}x = g'(t)\mathrm{d}t \end{array} \right] = \int f(g(t)) \cdot g'(t)\mathrm{d}t = H(t(x)) + C$$

Example.

$$\int \sqrt{1 - x^2}\mathrm{d}x = \left[\begin{array}{c} x = \sin t \\ \mathrm{d}x = \cos t\mathrm{d}t \\ -\frac{\pi}{2} \leqslant t \leqslant \frac{\pi}{2} \end{array} \right] = \int \cos t \cos t\mathrm{d}t = \int \frac{1 + \cos 2t}{2}\mathrm{d}t$$

(Continued.)

Example.

$$\int \sqrt{1-x^2}\mathrm{d}x = \begin{bmatrix} x = \sin t \\ \mathrm{d}x = \cos t\,\mathrm{d}t \\ -\frac{\pi}{2} \leqslant t \leqslant \frac{\pi}{2} \end{bmatrix} = \int \cos t \cos t\,\mathrm{d}t = \int \frac{1+\cos 2t}{2}\mathrm{d}t$$

$$= \int \frac{1}{2}\mathrm{d}t + \frac{1}{2}\int \cos 2t\,\mathrm{d}t = \left(\frac{t}{2} + \frac{1}{2}\cdot\frac{1}{2}\sin 2t + C\right)\bigg|_{t=\arcsin x}$$

$$= \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2} + C$$

Method 1: Trigonometric Substitution

Keys.

$$\sin^2\alpha + \cos^2\alpha = 1 \text{ and } \sec^2\alpha - \tan^2\alpha = 1$$

Detailed Substitution.

- $\sqrt{a^2 - x^2} \Rightarrow x = a\sin\alpha \Rightarrow \sqrt{a^2 - x^2} = a\cos\alpha;\ \mathrm{d}x = a\cos\alpha\ \mathrm{d}\alpha$
- $\sqrt{a^2 + x^2} \Rightarrow x = a\tan\alpha \Rightarrow \sqrt{a^2 + x^2} = a\sec\alpha;\ \mathrm{d}x = a\sec^2\alpha\ \mathrm{d}\alpha$
- $\sqrt{x^2 - a^2} \Rightarrow x = a\sec\alpha \Rightarrow \sqrt{x^2 - a^2} = a\tan\alpha;\ \mathrm{d}x = a\sec\alpha\tan\alpha\ \mathrm{d}\alpha$

Method 2: Rational Functions: $\frac{p}{q}$

Important Integrals.

- $\displaystyle\int \frac{\mathrm{d}x}{1+x^2} = \arctan x + C$
- $\displaystyle\int \frac{\mathrm{d}x}{a+bx} = \frac{1}{b}\ln|a+bx| + C$
- $\displaystyle\int \frac{2ax+b}{ax^2+bx+c}\mathrm{d}x = \ln\left|ax^2+bx+c\right| + C$

Remark. Actually, using these three formulas, you can solve all the questions that you are required to solve.

Substitution: Usual Steps to Calculate Integrals of $\frac{p}{q}$

Step 1: Change f into proper form ($\deg(p) < \deg(q)$)

Example.

$$f(x) = \frac{x^2}{(x-1)^2} = 1 + \frac{2x-1}{(x-1)^2}$$

Step 2: Partial fraction decomposition

Transfer $f(x)$ into the sum of following four classes of partial fractions

- $\dfrac{1}{x-a}$, $\dfrac{1}{(x-a)^n}$, $\dfrac{px+q}{ax^2+bx+c}$ or $\dfrac{px+q}{(ax^2+bx+c)^n}$.

Remark. For the first two fractions, their integral is easy to obtain. Here we mainly focus on the integral of the third fraction.

Substitution: Usual Steps to Calculate Integrals of $\frac{p}{q}$

Calculate the Integral of $\dfrac{px+q}{ax^2+bx+c}$

- Step 1: Take out the part which can be expressed as the derivative of denominator.

$$\frac{px+q}{ax^2+bx+c} = \frac{p}{2a} \cdot \frac{2ax+b}{ax^2+bx+c} + \frac{q-\dfrac{pb}{2a}}{ax^2+bx+c}$$

Calculate the Integral of $\dfrac{px+q}{ax^2+bx+c}$

- Step 1: Take out the part which can be expressed as the derivative of denominator.

$$\frac{px+q}{ax^2+bx+c} = \frac{p}{2a}\cdot\frac{2ax+b}{ax^2+bx+c} + \frac{q-\dfrac{pb}{2a}}{ax^2+bx+c}$$

  Then the integral of the first part can be calculated using the formula at the beginning of this section.

Calculate the Integral of $\dfrac{px+q}{ax^2+bx+c}$

- Step 1: Take out the part which can be expressed as the derivative of denominator.

$$\frac{px+q}{ax^2+bx+c} = \frac{p}{2a} \cdot \frac{2ax+b}{ax^2+bx+c} + \frac{q-\dfrac{pb}{2a}}{ax^2+bx+c}$$

  Then the integral of the first part can be calculated using the formula at the beginning of this section. Then we are going to explore the method to calculate the integral of

$$\frac{1}{ax^2+bx+c}$$

Calculate the Integral of $\dfrac{px+q}{ax^2+bx+c}$ (Continued.)

- Step 2: If $\Delta \geq 0$, then

$$\frac{1}{ax^2+bx+c}$$

can be expressed into the form

$$\frac{1}{a}\left(\frac{c_1}{x-x_1}+\frac{c_2}{x-x_2}\right)$$

whose integral is easy to compute.

Calculate the Integral of $\dfrac{px+q}{ax^2+bx+c}$ (Continued.)

If $\Delta < 0$, then the integral of $\dfrac{1}{ax^2+bx+c}$ can be expressed in the form of related to $f(x) = \arctan x$.

Example.

$$\int \frac{\mathrm{d}x}{x^2+x+1} = \int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\mathrm{d}x = \left(\frac{2}{\sqrt{3}}\right)^2 \int \frac{1}{\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]^2+1}\mathrm{d}x$$

$$= \left(\frac{2}{\sqrt{3}}\right)^2 \cdot \frac{\sqrt{3}}{2} \arctan\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right] + C.$$

Method 3: Substitute the Root

Generally, the steps of this method are

1. Rationalize the numerator or denominator.
2. Let $t = \sqrt[n]{ax+b}$ or $t = \sqrt[n]{\dfrac{ax+b}{cx+d}}$ and see what happens.

Example.

$$\int \frac{1}{\sqrt{1+x}+\sqrt[3]{1+x}}\mathrm{d}x = \left[\begin{array}{c} 1+x=t^6 \\ \mathrm{d}x = 6t^5\mathrm{d}t \end{array}\right] = \int \frac{6t^5}{t^3+t^2}\mathrm{d}t = \int \frac{6t^3}{t+1}\mathrm{d}t$$

$$= \int 6t^2 - 6t + 6 - \frac{6}{t+1}\mathrm{d}t$$

$$= 2t^3 - 3t^2 + 6t - 6\ln|t+1| + C$$

$$= 2\sqrt{1+x} - 3\sqrt[3]{1+x} + 6\sqrt[6]{1+x} - 6\ln\left|\sqrt[6]{1+x}+1\right| + C$$

Method 4: Substitute $t = \frac{1}{x}$

When the order of the denominator is much more greater than that of the numerator, we can try to apply the substitution $t = \dfrac{1}{x} \Rightarrow \mathrm{d}x = -\dfrac{\mathrm{d}t}{t^2}$.

Example.

$$\int \frac{\mathrm{d}x}{x^4\left(1+x^2\right)} = \frac{3x^2-1}{3x^3} - \arctan\frac{1}{x} + C$$

Case 1: The Integration of $\sin^n x$ or $\cos^n x$

$$\int \sin^n x \mathrm{d}x = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x \mathrm{d}x$$
$$\int \cos^n x \mathrm{d}x = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x \mathrm{d}x$$

Case 2: The Integration of $\sin^m x \cdot \cos^n x$

- At least one of m, n is odd.
  If both of m and n are odd, then substitute either $\sin x$ or $\cos x$ with t. Otherwise, substitute the one of even power with t.
- Both of m, n are even.
  Use $\sin^2 x + \cos^2 x = 1$ to transfer the integration into the form of $\sin^n x$ or $\cos^n x$.

Examples.

$$\int \sin^3 x \cos^2 x \, \mathrm{d}x = \left[\begin{array}{c} t = \cos x \\ \mathrm{d}t = -\sin x \mathrm{d}x \end{array} \right] = \int -(1-t^2)t^2 \, \mathrm{d}t,$$

$$\int \sin^2 x \cos^4 x \mathrm{d}x = \int (1-\cos^2 x)\cos^4 x \mathrm{d}x = \int \cos^4 x \mathrm{d}x - \int \cos^6 x \mathrm{d}x$$

$$= \frac{x}{16} + \frac{1}{32}\sin 2x - \frac{1}{6}\sin x \cos^5 x + \frac{1}{24}\sin x \cos^3 x + C.$$

Case 3: The Integration of $\sin nx \cdot \cos mx$

Use Sum-to-Product identities:

- $\sin nx \cos mx = \frac{1}{2}[\sin(n+m)x + \sin(n-m)x]$
- $\sin nx \sin mx = -\frac{1}{2}[\cos(n+m)x - \cos(n-m)x]$
- $\cos nx \cos mx = \frac{1}{2}[\cos(n+m)x + \cos(n-m)x]$

Example.

$$\int \sin 4x \cos 2x \cos 3x \mathrm{d}x = -\frac{1}{36}\cos 9x - \frac{1}{20}\cos 5x - \frac{1}{12}\cos 3x + \frac{1}{4}\cos x + C$$

Case 4: The Integration of $\tan^n x$ or $\sec^n x$

$$\int \tan^n x \mathrm{d}x = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \mathrm{d}x$$

$$\int \sec^n x \mathrm{d}x = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \mathrm{d}x$$

Example.

$$\int \sec^3 x \mathrm{d}x = \frac{1}{2} \left(\sec x \tan x + \ln \left| \sec x + \tan x \right| \right) + C.$$

Case 5: Substitute $t = \tan\dfrac{\pi}{2}$

Since

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}},\ \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}},\ \tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}$$

Substitute $t = \tan\dfrac{x}{2}$, we have

$$\sin x = \frac{2t}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2},\ \tan x = \frac{2t}{1-t^2},\ \mathrm{d}x = \frac{2}{t^2+1}\mathrm{d}t$$

Remark. This method works nearly for all rational functions containing trigonometric functions. However, sometimes, this will make the question become super complicated.

Example.

$$\int \frac{1}{4+\sin x}\mathrm{d}x = \frac{2}{\sqrt{15}} \arctan \frac{4\tan\dfrac{x}{2}+1}{\sqrt{15}} + C$$

Case 5: The Integration of $\dfrac{A_1 \cos\theta + A_2 \sin\theta}{B_1 \cos\theta + B_2 \sin\theta}$

Convert into the form of

$$\frac{C\left(B_1 \cos\theta + B_2 \sin\theta\right) + D\left[B_1(\cos\theta)' + B_2(\sin\theta)'\right]}{B_1 \cos\theta + B_2 \sin\theta}$$

Example.

$$\int \frac{\cos x}{2\sin x + \cos x}\mathrm{d}x = \frac{2}{5}\ln|2\tan x + 1| - \frac{1}{5}\ln\left|1 + \tan^2 x\right| + \frac{1}{5}x + C$$

$$\int \frac{\sin x}{\sin x + \cos x}\mathrm{d}x = \frac{x}{2} - \frac{1}{2}\ln|\sin x + \cos x| + C$$

Exercise

Evaluate the following integrals:

- $\int xe^{2x}\mathrm{d}x$
- $\int x^2 e^{ax}\mathrm{d}x$, a is a constant.
- $\int x \arctan x \mathrm{d}x$
- $\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}$
- $\int \frac{\mathrm{d}x}{x^2\sqrt{x^2+1}}$
- $\int \frac{1}{\sqrt{1+e^{2x}}}\mathrm{d}x$
- $\int \frac{\mathrm{d}x}{(1+\sqrt[3]{x})\sqrt{x}}$
- $\int \frac{\mathrm{d}x}{\sqrt[3]{(x+1)^2(x-1)^4}}$
- $\int \sin^2 x \cos^5 x \mathrm{d}x$

5. Vector and Vector Functions

Vector

Definition. A vector is an object that captures a direction and a magnitude (length) in 2D/3D spaces. Geometrically, vectors are arrows in an arbitrary position in 2D/3D spaces.
Definition. The tip of the vector is the end with the arrow, while the tail is the end without it.
Definition. A vector drawn with its tail at the origin is called a position vector.

- Basis of $\mathbb{R}^3$: $\bar{e}_1, \bar{e}_2, \bar{e}_3$.
- Resolving a vector into components: $\bar{v} = v_1\bar{i} + v_2\bar{j} + v_3\bar{k}$.
- Magnitude: $|\bar{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$.
- n-dimensional vector $\bar{v} = (v_1, v_2, \cdots, v_n)$. $|\bar{v}| = \sqrt{\sum_{k=1}^{n} v_k^2}$.

Properties of Vectors

Let $\bar{a}$, $\bar{b}$ and $\bar{c}$ be n-dimensional vectors and α, β be real numbers (scalars). Then

1. $\bar{a} + \bar{b} = \bar{b} + \bar{a}$.
2. $\bar{a} + (\bar{b} + \bar{c}) = (\bar{a} + \bar{b}) + \bar{c}$.
3. $\bar{a} + \bar{0} = \bar{a}$.
4. $\bar{a} + (-\bar{a}) = \bar{0}$.
5. $\alpha(\bar{a} + \bar{b}) = \alpha\bar{a} + \alpha\bar{b}$.
6. $(\alpha\beta)\bar{a} = \alpha(\beta\bar{a})$.
7. $(\alpha + \beta)\bar{a} = \alpha\bar{a} + \beta\bar{a}$.
8. $1 \cdot \bar{a} = \bar{a}$.

Definition. A vector-valued function pr vector function is a function whose domain is a subset of the reals and range is a set of vectors, *i.e.*, we say that $\bar{r}$ is a vector function if $\bar{r}: A \to \mathbb{R}^3$ where $A \subseteq \mathbb{R}$.

For $\bar{r}(t) = f(t)\bar{i} + g(t)\bar{j} + h(t)\bar{k}$, we introduce the parametric equations

$$x = f(t),\ y = g(t),\ z = h(t)$$

Limit of a Vector Function

Definition. Let $\bar{r}(t) = f(t)\bar{i} + g(t)\bar{j} + h(t)\bar{k}$ be a vector function and $a \in \mathbb{R}$. If the limits $\lim_{t\to a} f(t)$, $\lim_{t\to a} g(t)$ and $\lim_{t\to a} h(t)$ exist, then $\lim_{t\to a} \bar{r}(t)$ exists and

$$\lim_{t\to a} \bar{r}(t) = \left(\lim_{t\to a} f(t)\right)\bar{i} + \left(\lim_{t\to a} g(t)\right)\bar{j} + \left(\lim_{t\to a} h(t)\right)\bar{k}.$$

Continuity of a Vector Function

Definition. Let $A \subseteq \mathbb{R}$. A vector function $\bar{r} : A \to \mathbb{R}^3$ is continuous at a point $a \in \mathbb{R}$ if $a \in A$ and

$$\lim_{t \to a} \bar{r}(t) = \bar{r}(a).$$

We say that $\bar{r} : A \to \mathbb{R}^3$ is continuous on an interval I if $\bar{r}$ is continuous at all points $a \in I$. The continuity of $\bar{r}$ is equivalent to the continuity of $f(t)$, $g(t)$ and $h(t)$.

Differentiability

Definition. Let $A \subseteq \mathbb{R}$ and $\bar{r} : A \to \mathbb{R}^3$. Let $t \in A$. If the limit

$$\frac{d\bar{r}}{dt} = \bar{r}'(t) = \lim_{h \to 0} \frac{\bar{r}(t+h) - \bar{r}(t)}{h}$$

exists, then we say that $\bar{r}$ is differentiable at t.

Criteria for Differentiability

Theorem. If $\bar{r} = f(t)\bar{i} + g(t)\bar{j} + h(t)\bar{k}$ where f, g and h are functions differentiable on an interval I, then $\bar{r}$ is differentiable at every point in I and

$$\bar{r}'(t) = f'(t)\bar{i} + g'(t)\bar{j} + h'(t)\bar{k}.$$

Integration

Definition. Let $\bar{r}(t) = f(t)\bar{i} + g(t)\bar{j} + h(t)\bar{k}$ where f, g and h are functions that are integrable on $[a, b]$. Then

$$\int_a^b \bar{r}(t)dt = \left(\int_a^b f(t)dt\right)\bar{i} + \left(\int_a^b g(t)dt\right)\bar{j} + \left(\int_a^b h(t)dt\right)\bar{k}$$

$$\int \bar{r}(t)dt = \left(\int f(t)dt\right)\bar{i} + \left(\int g(t)dt\right)\bar{j} + \left(\int h(t)dt\right)\bar{k}$$

Let $\bar{r} : A \to \mathbb{R}^3$ be a vector function and $t \in A$. Let P be the point described by the vector $\bar{r}(t)$.

Tangent Vector and Tangent Line

Definition. If $\bar{r}'(t)$ exists and $\bar{r}'(t) \neq 0$, then $\bar{r}'(t)$ is called the tangent vector to the curve defined by $\bar{r}$ at the point P.

Definition. The tangent line to the curve described by $\bar{r}$ at the point P is the line that is parallel to the vector $\bar{r}'(t)$.

The unit tangent vector, sometimes denoted $\bar{T}(t)$, is the unit vector of $\bar{r}'(t)$

$$\bar{T}(t) = \frac{\bar{r}'(t)}{|\bar{r}'(t)|}.$$

Let $\bar{u}$ and $\bar{v}$ be differentiable vector functions. Let $c \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function.

Properties of Differentiability

1. $\dfrac{d}{dt}[\bar{u}(t) \pm \bar{v}(t)] = \bar{u}'(t) + \bar{v}'(t).$
2. $\dfrac{d}{dt}[c\bar{u}(t)] = c\bar{u}'(t).$
3. $\dfrac{d}{dt}[f(t)\bar{u}(t)] = f'(t)\bar{u}(t) + f(t)\bar{u}'(t).$
4. $\dfrac{d}{dt}[\bar{u}(t) \cdot \bar{v}(t)] = \bar{u}'(t) \cdot \bar{v}(t) + \bar{u}(t) \cdot \bar{v}'(t).$
5. $\dfrac{d}{dt}[\bar{u}(t) \times \bar{v}(t)] = \bar{u}'(t) \times \bar{v}(t) + \bar{u}(t) \times \bar{v}'(t).$
6. $\dfrac{d}{dt}[\bar{u}(f(t))] = \bar{u}'(f(t))f'(t)$ (Chain Rule).

Property of the Tangent Vector

Theorem. Let $\bar{r}(t)$ be a vector function that is differentiable on an interval I. If for all $t \in I$, $|\bar{r}(t)|$ is constant, then for all $t \in I$, $\bar{r}(t)$ and $\bar{r}'(t)$ are perpendicular.

Proof.
Suppose that for all $t \in I$, $|\bar{r}(t)| = c$. Therefore

$$2\left(\bar{r}'(t) \cdot \bar{r}(t)\right) = \frac{d}{dt}[\bar{r}(t) \cdot \bar{r}(t)] = \frac{d}{dt}\left[|\bar{r}(t)|^2\right] = \frac{d}{dt}\left[c^2\right] = 0.$$