255RC5

MATH2550J – Summer 2024

Li Mingrui, Xia Yiwei, Zhang Haoran, Huang Jiayue

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Notation

Exercises

- 1.1 Vector $v = (v_1, v_2, v_3, ...v_n)^T$ satisfies $vTv = 1$. Matrix H satisfies $H = I_n - 2XX^T$. Prove that H is a symmetric matrix and $HH^T = I_n$.
- 1.2 Square matrix $A_(n \times n)$ satisfies $A^2 + A = 4I_n$. Prove that matrix $A - I_n$ is invertible and find its inverse matrix.
- 1.3 We have matrices $A, PandB$.

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

, $P_{3\times 3}$ is invertible, $B = P^{-1}AP$. Find $B^{2016} - 2016A^2$.

Solutions

2. Linear System and Vector Space

3. Determinant

Definition of Determinant

Definition. The determinant $\det : M_{n\times n}(\mathbb{C}) \cong \underbrace{\mathbb{C}^n \times \cdots \times \mathbb{C}^n}_{n \text{ times}} \to \mathbb{C}$ is the unique function satisfying

- Alternating, i.e., for $v \in \mathbb{C}^n$, $\det(v_1, \cdots, v, \cdots, v, \cdots, v_n) = 0$, or equivalently skew-symmetric, i.e.,

$$\begin{aligned}&\det\left(v_1, \dots, v_{i-1}, v_i, v_{i+1}, \dots, v_{j-1}, v_j, v_{j+1}, \dots, v_n\right)\\ &= -\det\left(v_1, \dots, v_{i-1}, v_j, v_{i+1}, \dots, v_{j-1}, v_i, v_{j+1}, \dots, v_n\right)\end{aligned}$$

- Multilinear, i.e., for $\lambda, \mu \in \mathbb{C}$, $v_i, u \in \mathbb{C}^n$, $i = 1, \cdots, n$,

$$\begin{aligned}&\det\left(v_1, \dots, v_{i-1}, \lambda v_i + \mu u, v_{i+1}, \dots, v_n\right)\\ &= \lambda \det\left(v_1, \dots, v_{i-1}, v_i, v_{i+1}, \dots, v_n\right)\\ &+ \mu \det\left(v_1, \dots, v_{i-1}, u, v_{i+1}, \dots, v_n\right).\end{aligned}$$

- Unitary, i.e., $det I_n = 1$.

Determinant in $\mathbb{R}^2$.

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Determinant in $\mathbb{R}^3$.

$$A = \begin{bmatrix} u & v & w \end{bmatrix} \in M_{3\times 3}(\mathbb{R}) \Rightarrow \det A = u \cdot (v \times w)$$

Determinant Calculation

Suppose there is $n \times n$ matrix $A = (a_{ij})$

- $\det A = \sum_{(m_1,\ldots,m_n)\in perm(n)} (sign(m_1, \ldots, m_n)) a_{m_1 1} \ldots a_{m_n n}$

  where perm(n) the set off all permutations of (1,..., n). The sign of a permutation equals 1 if the natural order has been changed an even number of times and equals -1 if the natural order has been changed and odd number of times.

- cofactor $A_{ij} = (-1)^{i+j} \det(M_{ij})$

  where M_{ij} is the matrix generated by deleting A's ith colume and jth row, which is called the minor of a_{ij}

  $\det A = \sum_{i=1}^{n} a_{ij} A_{ij} = \sum_{j=1}^{n} a_{ij} A_{ij}$

Properties

- $\det(cA) = c^n \det(A)$ for $A \in M_{n\times n}(\mathbb{C})$ and $\lambda \in \mathbb{C}$.
- $\det(A) = \det\left(A^T\right)$.
- $\det(AB) = \det(A)\det(B)$ for $A, B \in M_{n\times n}(\mathbb{C})$.
- If $A = \left(a_{ij}\right)$ is a triangular matrix, i.e., $a_{ij} = 0$ for $i > j$ (or $i < j$), then

$$\det(A) = a_{11}a_{22} \cdot a_{nn} = \prod_{i=1}^{n} a_{ii}$$

4. Vector and Vector Functions

Vector

Definition. A **vector** is an object that captures a direction and a magnitude (length) in 2D/3D spaces. Geometrically, vectors are arrows in an arbitrary position in 2D/3D spaces.
Definition. The **tip** of the vector is the end with the arrow, while the **tail** is the end without it.
Definition. A vector drawn with its tail at the origin is called a **position vector**.

- Basis of $\mathbb{R}^3$: $\bar{e}_1, \bar{e}_2, \bar{e}_3$.
- Resolving a vector into components: $\bar{v} = v_1\bar{i} + v_2\bar{j} + v_3\bar{k}$.
- Magnitude: $|\bar{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$.
- n-dimensional vector $\bar{v} = (v_1, v_2, \cdots, v_n)$. $|\bar{v}| = \sqrt{\sum_{k=1}^{n} v_k^2}$.

Properties of Vectors

Let $\bar{a}$, $\bar{b}$ and $\bar{c}$ be n-dimensional vectors and α, β be real numbers (scalars). Then

1. $\bar{a} + \bar{b} = \bar{b} + \bar{a}$.
2. $\bar{a} + (\bar{b} + \bar{c}) = (\bar{a} + \bar{b}) + \bar{c}$.
3. $\bar{a} + \bar{0} = \bar{a}$.
4. $\bar{a} + (-\bar{a}) = \bar{0}$.
5. $\alpha(\bar{a} + \bar{b}) = \alpha\bar{a} + \alpha\bar{b}$.
6. $(\alpha\beta)\bar{a} = \alpha(\beta\bar{a})$.
7. $(\alpha + \beta)\bar{a} = \alpha\bar{a} + \beta\bar{a}$.
8. $1 \cdot \bar{a} = \bar{a}$.

Definition. A **vector-valued function** pr **vector function** is a function whose domain is a subset of the reals and range is a set of vectors, *i.e.*, we say that $\bar{r}$ is a **vector function** if $\bar{r} : A \to \mathbb{R}^3$ where $A \subseteq \mathbb{R}$.

For $\bar{r}(t) = f(t)\bar{i} + g(t)\bar{j} + h(t)\bar{k}$, we introduce the parametric equations

$$x = f(t), \;\; y = g(t), \;\; z = h(t)$$

Limit of a Vector Function

Definition. Let $\bar{r}(t) = f(t)\bar{i} + g(t)\bar{j} + h(t)\bar{k}$ be a vector function and $a \in \mathbb{R}$. If the limits $\lim\limits_{t \to a} f(t)$, $\lim\limits_{t \to a} g(t)$ and $\lim\limits_{t \to a} h(t)$ exist, then $\lim\limits_{t \to a} \bar{r}(t)$ exists and

$$\lim_{t \to a} \bar{r}(t) = \left(\lim_{t \to a} f(t)\right)\bar{i} + \left(\lim_{t \to a} g(t)\right)\bar{j} + \left(\lim_{t \to a} h(t)\right)\bar{k}.$$

Continuity of a Vector Function

Definition. Let $A \subseteq \mathbb{R}$. A vector function $\bar{r} : A \to \mathbb{R}^3$ is **continuous** at a point $a \in \mathbb{R}$ if $a \in A$ and

$$\lim_{t \to a} \bar{r}(t) = \bar{r}(a).$$

We say that $\bar{r} : A \to \mathbb{R}^3$ is **continuous on an interval** I if $\bar{r}$ is continuous at all points $a \in I$.
The continuity of $\bar{r}$ is equivalent to the continuity of $f(t)$, $g(t)$ and $h(t)$.

Differentiability

Definition. Let $A \subseteq \mathbb{R}$ and $\bar{r} : A \to \mathbb{R}^3$. Let $t \in A$. If the limit

$$\frac{d\bar{r}}{dt} = \bar{r}'(t) = \lim_{h \to 0} \frac{\bar{r}(t+h) - \bar{r}(t)}{h}$$

exists, then we say that $\bar{r}$ is **differentiable** at t.

Criteria for Differentiability

Theorem. If $\bar{r} = f(t)\bar{i} + g(t)\bar{j} + h(t)\bar{k}$ where f, g and h are functions differentiable on an interval I, then $\bar{r}$ is differentiable at every point in I and

$$\bar{r}'(t) = f'(t)\bar{i} + g'(t)\bar{j} + h'(t)\bar{k}.$$

Integration

Definition. Let $\bar{r}(t) = f(t)\bar{i} + g(t)\bar{j} + h(t)\bar{k}$ where f, g and h are functions that are integrable on $[a, b]$. Then

$$\int_a^b \bar{r}(t)dt = \left(\int_a^b f(t)dt\right)\bar{i} + \left(\int_a^b g(t)dt\right)\bar{j} + \left(\int_a^b h(t)dt\right)\bar{k}$$

$$\int \bar{r}(t)dt = \left(\int f(t)dt\right)\bar{i} + \left(\int g(t)dt\right)\bar{j} + \left(\int h(t)dt\right)\bar{k}$$

Let $\bar{r} : A \to \mathbb{R}^3$ be a vector function and $t \in A$. Let P be the point described by the vector $\bar{r}(t)$.

Tangent Vector and Tangent Line

Definition. If $\bar{r}'(t)$ exists and $\bar{r}'(t) \neq 0$, then $\bar{r}'(t)$ is called the **tangent vector** to the curve defined by $\bar{r}$ at the point P.

Definition. The **tangent line** to the curve described by $\bar{r}$ at the point P is the line that is parallel to the vector $\bar{r}'(t)$.

The **unit tangent vector**, sometimes denoted $\bar{T}(t)$, is the unit vector of $\bar{r}'(t)$

$$\bar{T}(t) = \frac{\bar{r}'(t)}{|\bar{r}'(t)|}.$$

Let $\bar{u}$ and $\bar{v}$ be differentiable vector functions. Let $c \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function.

Properties of Differentiability

1. $\dfrac{d}{dt}[\bar{u}(t) \pm \bar{v}(t)] = \bar{u}'(t) + \bar{v}'(t)$.
2. $\dfrac{d}{dt}[c\bar{u}(t)] = c\bar{u}'(t)$.
3. $\dfrac{d}{dt}[f(t)\bar{u}(t)] = f'(t)\bar{u}(t) + f(t)\bar{u}'(t)$.
4. $\dfrac{d}{dt}[\bar{u}(t) \cdot \bar{v}(t)] = \bar{u}'(t) \cdot \bar{v}(t) + \bar{u}(t) \cdot \bar{v}'(t)$.
5. $\dfrac{d}{dt}[\bar{u}(t) \times \bar{v}(t)] = \bar{u}'(t) \times \bar{v}(t) + \bar{u}(t) \times \bar{v}'(t)$.
6. $\dfrac{d}{dt}[\bar{u}(f(t))] = \bar{u}'(f(t))f'(t)$ (Chain Rule).

Property of the Tangent Vector

Theorem. Let $\bar{r}(t)$ be a vector function that is differentiable on an interval I. If for all $t \in I$, $|\bar{r}(t)|$ is constant, then for all $t \in I$, $\bar{r}(t)$ and $\bar{r}'(t)$ are perpendicular.

Proof.
Suppose that for all $t \in I$, $|\bar{r}(t)| = c$. Therefore

$$2\left(\bar{r}'(t) \cdot \bar{r}(t)\right) = \frac{d}{dt}[\bar{r}(t) \cdot \bar{r}(t)] = \frac{d}{dt}\left[|\bar{r}(t)|^2\right] = \frac{d}{dt}\left[c^2\right] = 0.$$

5. Functions of Several Variables

Definition. Let $n > 1$ be a natural number. A **real-valued function of n independent variables** or just a **function of n variables** is a function $f : D \to \mathbb{R}$ such that $D \subseteq \mathbb{R}^n$. We denote the function f as

$$f(x_1, \cdots, x_n)$$

- A real-valued function with $n > 1$ independent variables is a function that maps points in n-dimensional space to real numbers.
- A function of two variables $f : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}^2$ can be visualized in 3D space by $z = f(x, y)$. Functions of two variables often describe surfaces in $\mathbb{R}^3$.

Definition. Let $f : D \to \mathbb{R}$ be a function of n variables where $n \geq 1$. The **graph** of f is the collection of points in $\mathbb{R}^{n+1}$ defined by

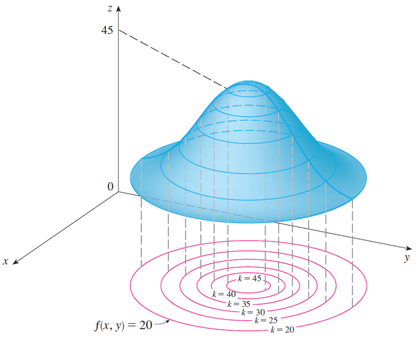
$$\{(x_1, \cdots, x_n, y) | y = f(x_1, \cdots, x_n)\}.$$

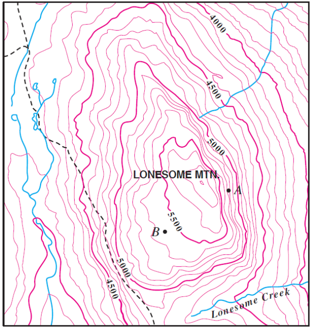
Definition. Let $f : D \to \mathbb{R}$ be a function of n variables where $n \geq 1$ with independent variables $x_1, \cdots, x_n$. The function f is **linear** if there exists $a_0, \cdots, a_n \in \mathbb{R}$ such that

$$f(x_1, \cdots, x_n) = a_0 + a_1 x_1 + \cdots + a_n x_n.$$

Linear functions of two variables specify planes in 3D space.

Definition. The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).Note that sometimes the equation will be in the form $f(x, y, z) = 0$ and in these cases the equations of the level curves are $f(x, y, k) = 0$.





Definition. Let $f : D \to \mathbb{R}$ be a function with $D \subseteq \mathbb{R}^n$. Let $\bar{a} \in \mathbb{R}^n$ and let $L \in \mathbb{R}$. We say that the **limit** of f at $\bar{x}$ approaches $\bar{a}$ is L and write

$$\lim_{\bar{x} \to \bar{a}} f(\bar{x}) = L$$

if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $\bar{x} \in \mathbb{R}^n$:

$$||\bar{x} - \bar{a}|| < \delta \Rightarrow |f(\bar{x}) - L| < \varepsilon.$$

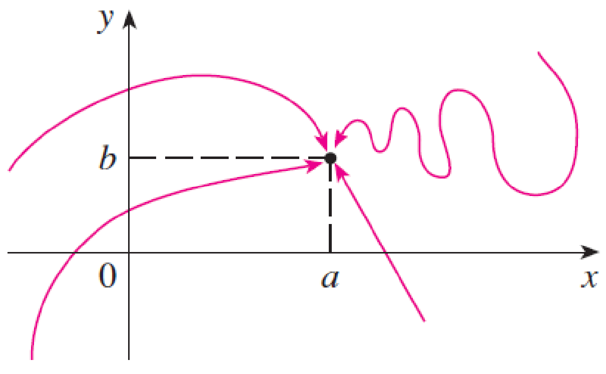


图: Different Directions to Approach (a, b).

Consider functions $f(x,y) = \dfrac{\sin(x^2+y^2)}{x^2+y^2}$ and $g(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$.

x \ y	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455
−0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
−0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0	0.841	0.990	1.000		1.000	0.990	0.841
0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455

图: Values of $f(x,y)$.

Consider functions $f(x, y) = \dfrac{\sin(x^2 + y^2)}{x^2 + y^2}$ and $g(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$.

x \ y	-1.0	-0.5	-0.2	0	0.2	0.5	1.0
-1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000
-0.5	-0.600	0.000	0.724	1.000	0.724	0.000	-0.600
-0.2	-0.923	-0.724	0.000	1.000	0.000	-0.724	-0.923
0	-1.000	-1.000	-1.000		-1.000	-1.000	-1.000
0.2	-0.923	-0.724	0.000	1.000	0.000	-0.724	-0.923
0.5	-0.600	0.000	0.724	1.000	0.724	0.000	-0.600
1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000

图: Values of $g(x, y)$.

Here we say that

$$\lim_{(x,y)\to(0,0)} \frac{\sin\left(x^2+y^2\right)}{x^2+y^2} = 1 \quad \text{and} \quad \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} \text{ does not exist.}$$

Criteria for Limit Not Existing

Formally, if $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a path $\mathcal{C}_1$ and $f(x,y) \to L_2$ as $f(x,y) \to (a,b)$ along a path $\mathcal{C}_2$, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

How to determine the existence of limit at the first look?

Three keys: by eye, by experience, by practice.

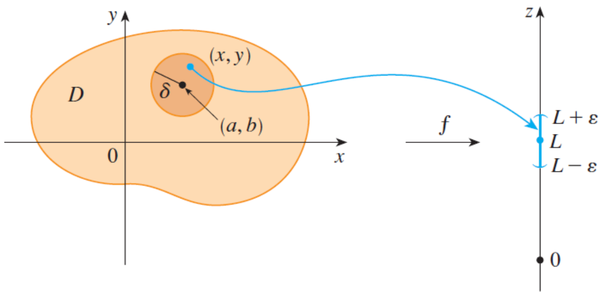


图: Illustration of the Definition of Limit.

Definition. **($\varepsilon\delta$)** Let $f : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}^n$ and let $\bar{a} \in D$. We say that f is **continuous** at $\bar{a}$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $\bar{x} \in D$:

$$||\bar{x} - \bar{a}|| < \delta \Rightarrow |f(\bar{x}) - f(\bar{a})| < \varepsilon.$$

Definition. **(Limit)** Let $f : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}^n$ and let $\bar{a} \in D$. We say that f is **continuous** at $\bar{a}$ if

$$\lim_{\bar{x} \to \bar{a}} f(\bar{x}) = f(\bar{a}).$$

Properties of Continuity. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}^n$ be functions that are continuous at $\bar{a} \in D$. Let $\alpha \in \mathbb{R}$. Then

1. $f + g$ is continuous at $\bar{a}$.
2. αf is continuous at $\bar{a}$.
3. fg is continuous at $\bar{a}$.
4. f/g is continuous at $\bar{a}$ if $g(\bar{a}) \neq 0$.

Theorem. Let $f : D \to \mathbb{R}$ and $g : E \to \mathbb{R}$ where $D \subseteq \mathbb{R}^n$ and $E \subseteq \mathbb{R}$ be functions. Let $\bar{a} \in D$ be such that $f(\bar{a}) \in E$. If f is continuous at $\bar{a}$ and g is continuous at $f(\bar{a})$, then $g \circ f : D \to \mathbb{R}$ is continuous at $\bar{a}$.

Definition. A function $f : \mathbb{R}^n \to \mathbb{R}$ that is the sum of the terms in the form $\alpha \prod_{1 \leq i \leq n} x_i^{k_i}$ where the x_i's are the independent variables, $k_i \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, is called a **polynomial function of n variables**.

Definition. A function that is the quotient of polynomial functions is called a **rational function**.

Theorem. A polynomial function of n variables is continuous at every point in $\mathbb{R}^n$. A rational function is continuous at every point in its domain.

Some tips

1.All the standard functions that we know to be continuous are still continuous even if we are plugging in more than one variable now. We just need to watch out for division by **zero, square roots of negative numbers, logarithms of zero or negative numbers,** etc.

2.Note that the idea about paths is one that we should not forget since it is a nice way to determine if a limit does not exist. If we can find two paths upon which the function approaches different values as we get near the point then we will know that the limit does not exist.

Exercise1.1 Determine if the following limit exist or not. If they do exist give the value of the limit.

$$\lim_{(x,y)\to(1,1)} \frac{2x^2 - xy - y^2}{x^2 - y^2}$$

Exercise1.2 Determine if the following limit exist or not. If they do exist give the value of the limit.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4 + 3y^4}$$

Definition. Let $f : D \to \mathbb{R}$ be a function where D is an open ball of $\mathbb{R}^n$. Let $\bar{a} \in D$. The function f is **differentiable** at $\bar{a}$ with **derivative** $Df(\bar{a}) \in \mathbb{R}^n$ if

$$\frac{||f(\bar{a}+\bar{h}) - f(\bar{a}) - Df(\bar{a}) \cdot \bar{h}||}{||\bar{h}||} \to 0 \text{ as } \bar{h} \to \bar{0}.$$

The derivative of the function f gives the best linear approximation of f at the point $\bar{a}$. The derivative of f of n variables is an n-dimensional **vector**!

Properties of Differentiation. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ where D is an open ball of $\mathbb{R}^n$ be functions that are differentiable at $\bar{a} \in D$. Let $\alpha \in \mathbb{R}$. Then

1. $f + g$ is differentiable at $\bar{a}$ with $D(f+g)(\bar{a}) = Df(\bar{a}) + Dg(\bar{a})$.
2. αf is differentiable at $\bar{a}$ with $D(\alpha f)(\bar{a}) = \alpha Df(\bar{a})$.

Definition. Let $f : D \to \mathbb{R}$ where D is an open ball of $\mathbb{R}^n$ be a function $f(x_1, \cdots, x_n)$ that is differentiable at $\bar{a} = (a_1, \cdots, a_n) \in D$. If

$$Df(\bar{a}) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

then we call α_i the **partial derivative of f with respect to x_i at** $\bar{a}$ and we write

$$\left.\frac{\partial f}{\partial x_i}\right|_{\bar{a}} \quad \text{or} \quad f_{x_i}(\bar{a}).$$

Geometric Interpretation. For functions of two variables $f(x, y)$ we can interpret the partial derivatives geometrically. Let (a, b, c) be a point such that $c = f(a, b)$ and $\mathcal{C}_1$ be the curve that is obtained by intersecting the graph $z = f(x, y)$ with the plane $y = b$ and $\mathcal{C}_2$ be the curve that is obtained by intersecting the graph $z = f(x, y)$ with the plane $x = a$. Then

1. The partial derivative $f_x(a, b) = \lim_{h \to 0} \dfrac{f(a + h, b) - f(a, b)}{h}$ is the slope of the tangent line of $\mathcal{C}_1$ in the plane $y = b$.
2. The partial derivative $f_y(a, b) = \lim_{h \to 0} \dfrac{f(a, b + h) - f(a, b)}{h}$ is the slope of the tangent line of $\mathcal{C}_2$ in the plane $x = a$.

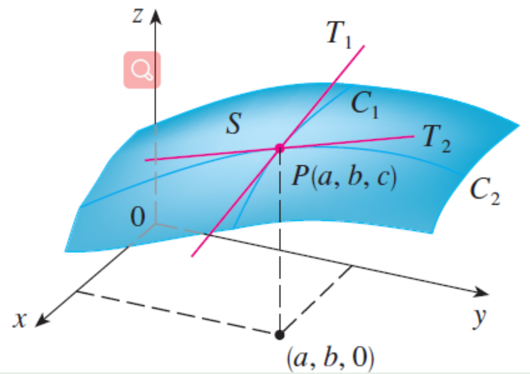


图: Geometric Interpretation of Partial Derivative.

Exercise 2.1 Find all of the first order partial derivatives for the following functions.

$$w = x^2y - 10y^2z^3 + 43x - 7tan(4y)$$

$$f(x, y) = cos(\frac{4}{x})e^{x^2y - 5y^3}$$

Exercise2.2 Find $\frac{\partial z}{\partial x}$and $\frac{\partial z}{\partial y}$for each of the following function.

$$x^3z^2 - 5xy^5z = x^2 + y^3$$

Definition. Let $f : D \to \mathbb{R}$ where D is an open ball of $\mathbb{R}^n$ be a function $f(x_1, \cdots, x_n)$ that is differentiable. If the partial derivative $f_{x_i}(x_1, \cdots, x_n)$ is differentiable, then we can find the partial derivative of it w.r.t one of the independent variables x_j. A partial derivative of a partial derivative is called a **second order partial derivative**. We write

$$\frac{\partial^2 f}{\partial x_i \partial x_j} \text{ or } f_{x_1 x_j}(x_1, \cdots, x_n)$$

for the partial derivative of $f_{x_i}(x_1, \cdots, x_n)$ w.r.t the variable x_j. It becomes $\frac{\partial^2 f}{\partial x_i^2}$ when we take the partial derivative twice with respect to the same variable.

Theorem. (Clairaut Theorem) Let $f : D \to \mathbb{R}$ where D is an open ball of $\mathbb{R}^2$ be a function $f(x_1, x_2)$ and let $(a, b) \in D$. If $f_{x_1x_2}$ and $f_{x_2x_1}$ are both continuous on D, then

$$f_{x_1x_2}(a, b) = f_{x_2x_1}(a, b).$$

Differentiable: Another Perspective

Definition. If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + +\varepsilon_1\Delta x + \varepsilon_2\Delta y,$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Theorem. If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Theorem. Let $f : D \to \mathbb{R}$ where D is an open ball of $\mathbb{R}^n$ be a function $f(x_1, \cdots, x_n)$. If for all $1 \leq i \leq n$, $\frac{\partial f}{\partial x_i}$ exists and is continuous on D, then

$$\Delta f = f_{x_1}(\bar{x})\Delta x_1 + \cdots + f_{x_n}(\bar{x})\Delta x_n + \varepsilon_1 \Delta x_1 + \cdots + \varepsilon_n \Delta x_n.$$

where $\varepsilon_i \to 0$, $i = 1, \cdots, n$ as $(\Delta x_1, \cdots, \Delta x_n) \to \bar{0}$.

Definition. The **total differential** of a function $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is

$$df = f_{x_1}(\bar{x})dx_1 + \cdots + f_{x_n}(\bar{x})dx_n$$

where $dx_1 = \Delta x_1, \cdots dx_n = \Delta x_n$.

Exercise 2.3 Find all the second order derivatives for

$$f(x, y) = cos(2x) - x^2 e^{5y} + 3y^2$$

Exercise 2.4 Find f_{xxyzz} for f(x,y,z) $= z^3 y^2 ln(x)$

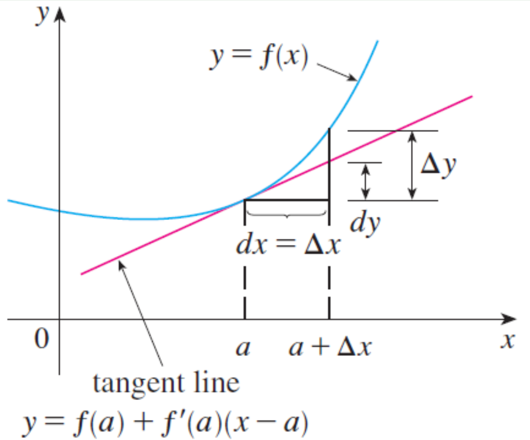


图: Differential dy of a Single Variate Function $y = f(x)$.

$x \backslash y$	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000
−0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
−0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0	−1.000	−1.000	−1.000		−1.000	−1.000	−1.000
0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000

图: Total Differential dz of Multivariate Function $z = f(x, y)$.

6. Tangent Planes and Linear Approximation

The equation of the tangent plane of $f(x, y)$ at (a, b) is

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b) \leftarrow \text{linear approximation!}$$

(Recall how we obtain the equation of the tangent line of a function $f(x)$ in VV156.

Linear Approximation: Important in Engineering.
Example Find the equation of the tangent plane to z = ln(2x+y) at (-1,3)

Tangent Planes

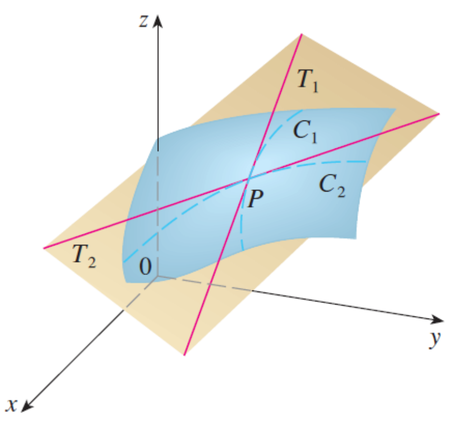


图: Example of Tangent Plane.

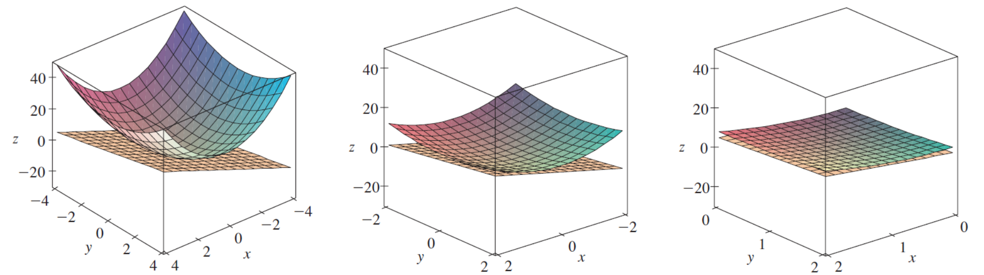


图: The Surface $z = 2x^2 + y^2$ Appears to Coincide with its Tangent Plane.

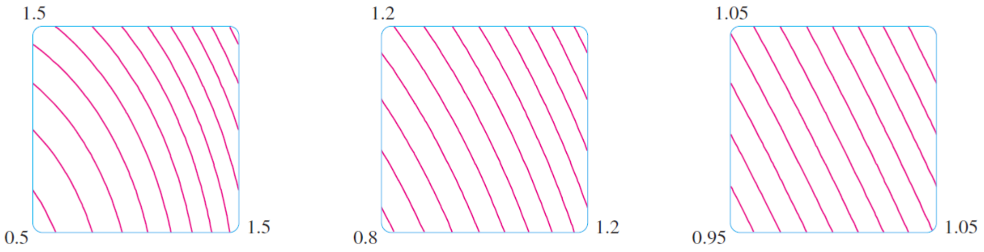


图: The Contour Map of $f(x, y) = 2x^2 + y^2$.

7. Chain Rule

Chain Rule: General Version

Theorem. Let u be a differentiable function of n variables $x_1, \cdots, x_n$ such that for all $1 \leq i \leq n$, x_i is a differentiable function of m variables $t_1, \cdots, t_m$. Then u is a function of $t_1, \cdots, t_m$ and for all $1 \leq j \leq m$,

$$\frac{\partial u}{\partial t_j} = \sum_{1 \leq i \leq n} \frac{\partial u}{\partial x_i} \frac{\partial x_i}{\partial t_j}.$$

Chain Rule I

Theorem. Let $x = g(t)$ and $y = h(t)$ be functions that are differentiable on an interval I. Let $z = f(x, y)$ be a function that is differentiable at the points with x-coordinate in the range of g restricted to I, and y-coordinates in the range of h restricted to I. Then $f(x, y)$ is differentiable with respect to t on the interval I and

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Exercise2.5 Find $\frac{dz}{dt}$ for $z = xe^{xy}, x = t^2, y = t^{-1}$

Chain Rule II

Theorem. Suppose that $z = f(x, y)$ is a differentiable function of x and y where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s},$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}.$$

Exercise 2.6 Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ for $z = e^{2r} sin(3\theta), r = st - t^2, \theta = \sqrt{s^2 + t^2}$

Assume that an equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x. If $F(x, y)$ is differentiable, then the chain rule tells us that

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0.$$

Therefore, if $\frac{\partial F}{\partial y} \neq 0$, then

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Similarly, if $F(x, y, z) = 0$, we have

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0.$$

Jacobian Matrix

Definition. Suppose $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a function that maps from an n-dimensional space to an m-dimensional space. This function consists of m functions: $f_1(x_1, x_2, \cdots, x_n), \cdots, f_m(x_1, x_2, \cdots, x_n)$. Then the Jacobian matrix is:

$$DF(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \dots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Properties of Differentiation. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ where D is an open ball of $\mathbb{R}^n$ be functions that are differentiable at $\bar{a} \in D$. Let $\alpha \in \mathbb{R}$. Then

1. $f + g$ is differentiable at $\bar{a}$ with $D(f+g)(\bar{a}) = Df(\bar{a}) + Dg(\bar{a})$.
2. αf is differentiable at $\bar{a}$ with $D(\alpha f)(\bar{a}) = \alpha Df(\bar{a})$.
3. $D(fg)(\bar{a}) = g(\bar{a})Df(\bar{a}) + f(\bar{a})Dg(\bar{a})$.
4. $D(f/g)(\bar{a}) = \frac{g(\bar{a})Df(\bar{a}) - f(\bar{a})Dg(\bar{a})}{g(\bar{a})^2}$

The Jacobian matrix gives the best linear approximation of F at the point x, when x is close to p.

$$F(x) \approx F(p) + DF(p) \cdot (x - p)$$