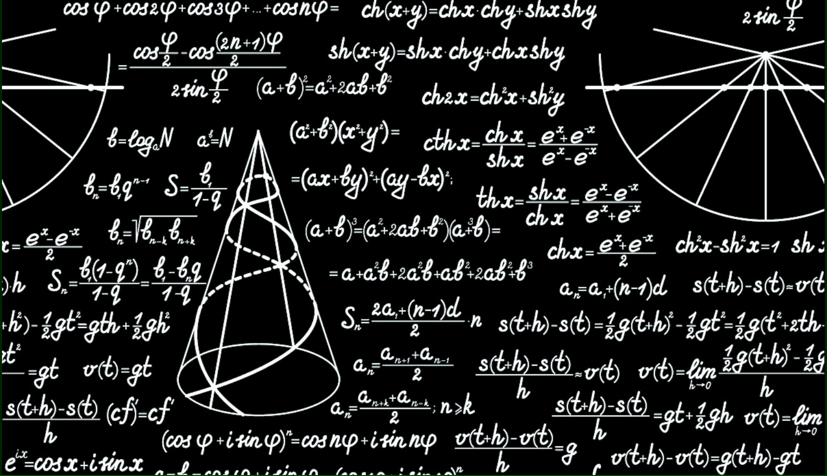

255RCs

Li Mingrui, Xia Yiwei, Zhang Haoran, Huang Jiayue

Sincerest appreciation dedicated to
2023 VV255 TAs
Jiani Jin, Dayong Wang and Yishen Zhou,
as well as all previous VV255 TAs.

1. Matrix Basics

Contents

Linear System of Equations

Definition

A linear system of m equations in n unknowns $x_1, x_2, \cdots, x_n \in V$ is a set of equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{1n}x_n = b_2, \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m. \end{cases}$$

It can also be represented in matrix form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{12} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Definition

If $b_1 = b_2 = \cdots = b_m = 0$, then it is called a **homogeneous system**. Otherwise, it is called an **inhomogeneous system**.

Norm of a Vector

Definition

Let $\bar{v} = (v_1, \cdots, v_n) \in \mathbb{R}^n$. The **norm** of $\bar{v}$ is given by

$$||\bar{v}|| = \sqrt{v \cdot v} = \sqrt{\sum_{i=1}^{n} v_i^2}$$

1. The **sum** of two matrices $A_{n\times m}$ and $B_{n\times m}$ is the matrix $C_{n\times m}$ such that

$$c_{ij} = a_{ij} + b_{ij},\ \ i = \overline{1,n},\ \ j = \overline{1,m}.$$

2. The **scalar product** $\alpha A_{n\times m} = (\alpha a_{ij}),\ \ i = \overline{1,n},\ \ j = \overline{1,m}.$

3. The **product** of a row-matrix $[a_1\ \ \cdots\ \ a_n]$ and a column matrix $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ is

$$[a_1\ \ \cdots\ \ a_n]\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + \cdots + a_n b_n.$$

It is the **inner product** of vectors
$\overline{a} = (a_1, \cdots, a_n), \overline{b} = (b_1, \cdots, b_n) \in \mathbb{R}^n$.

4. The product of a matrix $A_{n\times m}$ and a vector $\bar{x} \in \mathbb{R}^m$ is

$$A_{n\times m}\bar{x} = \begin{bmatrix} \bar{w}_1 \\ \bar{w}_2 \\ \vdots \\ \bar{w}_n \end{bmatrix} \bar{x} = (\text{def}) \begin{bmatrix} (\bar{w}_1, \bar{x}) \\ (\bar{w}_2, \bar{x}) \\ \vdots \\ (\bar{w}_n, \bar{x}) \end{bmatrix}$$

$$= (\text{prop}) \left(\bar{a}_1 \quad \bar{a}_2 \ldots \bar{a}_m\right) \bar{x} = x_1\bar{a}_1 + x_2\bar{a}_2 + \ldots + x_m\bar{a}_m$$

5. The product of a matrix $A_{n\times k}$ and a matrix $B_{k\times m}$ is

$$AB = \left[A\bar{b}_1 \; A\bar{b}_2 \; \cdots A\bar{b}_m\right]_{n\times m}.$$

6. Properties.
   - $A + B = B + A$. $AB \neq BA$!
   - $A(B + C) = AB + AC$, $(A + B)C = AC + BC$.

Identity Matrix and Zero Matrix

Definition

Identity matrix and **zero matrix**

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \quad O_n = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Properties

$A + O = O + A = A$, $AI = IA = A$.

Definition

Let $A_{n\times n} = (a_{ij})$ be a square matrix. The **trace** of A is defined as

$$\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

Theorem

Let $A_{m\times n} = (a_{ij})$ and $B_{n\times m} = (b_{ij})$ be two matrices.
Then $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

Definition

Let $A_{n\times n} = (a_{ij})$ be a square matrix. The **trace** of A is defined as

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

Theorem

Let $A_{m\times n} = (a_{ij})$ and $B_{n\times m} = (b_{ij})$ be two matrices.
Then $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Proof.

$$\begin{aligned}\operatorname{tr}(AB) &= \sum_{k=1}^{m} (AB)_{kk} = \sum_{k=1}^{m}\sum_{l=1}^{n} a_{kl} b_{lk} = \sum_{k=1}^{m}\sum_{l=1}^{n} b_{lk} a_{kl} \\ &= \sum_{l=1}^{n} (BA)_{ll} = \operatorname{tr}(BA).\end{aligned}$$

Trace

Properties

- $tr(A) = tr(A^T), \quad A_{n\times n} = (a_{ij})$
- $tr(kA) = k \cdot tr(A), \quad A_{n\times n} = (a_{ij})$, k is a scalar
- $tr(A+B) = tr(A) + tr(B), \quad A_{n\times n} = (a_{ij}), B_{n\times n} = (b_{ij})$
- $tr(ABC) = tr(BCA) = tr(CAB)$

Definition.

The **inverse** of A is A^{-1} only when

$$AA^{-1} = A^{-1}A = I.$$

In this case, we call matrix A **invertible**.

Theorems

- If matrix A is invertible, then $|A| \neq 0$.
- If $|A| \neq 0$, then A is invertible and $A^{-1} = \dfrac{1}{|A|}A^*$.

For now, you only need to know

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Definition

A matrix is in **reduced row-echelom form** (rref) if it satisfies all of the following conditions:

- If a row has nonzero entries, then the first nonzero entry is a 1, called the **leading 1** in this row.
- If a column contains a leading 1, then all the other entries in that column are 0.
- If a row contains a leading 1, then each row above it contains a leading 1 further to the left. That means, rows of 0's, if any, appear at the bottom of the matrix.

Definition

The number of leading 1's in the rref of a matrix A is called the **rank** of A.

How to find rref?

How to find rref?

Elementary Row Operations

1. $r_1 \leftrightarrow r_2$, swap two rows
2. $r_1 \to kr_1,\ k \in \mathbb{R}, k \neq 0$, multiplying a row by a non-zero scalar.
3. $r_1 \to r_1 + kr_2,\ k \in \mathbb{R}, k \neq 0$, add a multiple of a row to another row.

Exercise 1.1 Find rref of matrix

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 \\ -1 & -2 & -1 & 3 \\ -2 & -3 & 0 & 3 \\ 1 & 4 & 5 & -9 \end{bmatrix}.$$

Exercise 1.1 Find rref of matrix

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 \\ -1 & -2 & -1 & 3 \\ -2 & -3 & 0 & 3 \\ 1 & 4 & 5 & -9 \end{bmatrix}.$$

Solution 1.1

$$A \sim \begin{bmatrix} 1 & 4 & 5 & -9 \\ 0 & 2 & 4 & -6 \\ 0 & 5 & 10 & -15 \\ 0 & -3 & -6 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & -9 \\ 0 & 2 & 4 & -6 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & -9 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \operatorname{rank} A = 3.$

Augmented Matrix

Definition

An augmented matrix is a matrix obtained by adjoining a row or column vector, or sometimes another matrix with the same vertical dimension.

Application

- Solve a system of linear equations.
- Find the inverse of a matrix.

Exercise 1.2 Find Inverse of matrix

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 4 & -1 \\ 2 & 1 & 3 \end{bmatrix}.$$

Exercise 1.2 Find Inverse of matrix

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 4 & -1 \\ 2 & 1 & 3 \end{bmatrix}.$$

Solution 1.2

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 4 & -1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{13}{35} & -\frac{2}{35} & \frac{8}{35} \\ 0 & 1 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 1 & -\frac{11}{35} & -\frac{1}{35} & \frac{4}{35} \end{array}\right]$$

Gauss(-Jordan) Elimination

Definition

The process for finding the solution of $Ax = b$.

Gauss(-Jordan) Elimination

Definition

The process for finding the solution of $Ax = b$.

Steps

1. Find the augmented matrix $[A|b]$.
2. Find the rref of the augmented matrix.
3. Get the solution(s).

Theorem

- Appearing as $0 = d$ (d is not zero) and the original system of equations has no solution,
- If the number of non-zero rows = the number of unknowns, the original equation has a unique solution.
- If the number of non-zero rows < the number of unknowns, the original equation can be written with a general solution.

Exercise 1.3.1 Solving linear system of equations,

$$\begin{cases} x_1 - x_2 + x_3 = 1 \\ x_1 - x_2 - x_3 = 3 \\ 2x_1 - 2x_2 - x_3 = 3 \end{cases}$$

Exercise 1.3.2 Solving linear system of equations,

$$\begin{cases} x_1 - 3x_2 - 2x_3 - x_4 = 6 \\ 3x_1 - 8x_2 + x_3 + 5x_4 = 0 \\ -2x_1 + x_2 - 4x_3 + x_4 = -12 \\ -x_1 + 4x_2 - x_3 - 3x_4 = 2 \end{cases}$$

Solution 1.3.1 No solution.
Solution 1.3.2

$$x = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -3 \end{pmatrix}$$

Exercise 1.4 Solve matrix equations AX=B where,

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 4 & -1 \\ 2 & 1 & 3 \end{bmatrix}.$$

$$B = \begin{bmatrix} 5 & -1 \\ -2 & 3 \\ 1 & 4 \end{bmatrix}.$$

Solution 1.4

$$x = \begin{pmatrix} \frac{11}{5} & \frac{13}{35} \\ \frac{4}{5} & \frac{6}{5} \\ -\frac{7}{5} & \frac{24}{35} \end{pmatrix}$$

Thank you!