Lagrange Multiplier.

Method of Lagrange Multipliers To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:

(a) Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and $$g(x, y, z) = k$$

(b) Evaluate f at all the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

$$\boxed{\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)} \tag{16}$$

In this case Lagrange's method is to look for extreme values by solving five equations in the five unknowns x, y, z, λ, and μ. These equations are obtained by writing Equation 16 in terms of its components and using the constraint equations:

$$f_x = \lambda g_x + \mu h_x$$
$$f_y = \lambda g_y + \mu h_y$$
$$f_z = \lambda g_z + \mu h_z$$
$$g(x, y, z) = k$$
$$h(x, y, z) = c$$

Bounded Region

Problem

Find the extreme values of f on the region described by the inequality.

Solution

1. Solve the Lagrange Multiplier Equation for the **Boundaries**
2. Find the critical point of f inside the boundary. i.e. $\nabla f(x_0, y_0, z_0) = 0$, (x_0, y_0, z_0) inside the bounded region.
3. Evaluate all the candidate points.

例题见 RC4.

* we can't use such method when
  - ∇f or ∇g DNE
  - $\nabla g = 0$.

Polar Coordinates.

$dA = r\,dr\,d\theta$.

The surface area $z = f(x,y)$ over D (f_x, f_y, const) is given by.

$$A = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\,dx\,dy$$

Cylindrical Coordinate.

$dA = r\,dr\,d\theta\,dz$.

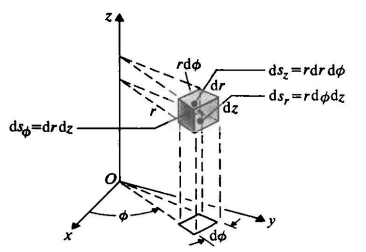


图 2-15 圆柱坐标系中的一个体积元

Spherical Coordinates.

$$dv = \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi.$$

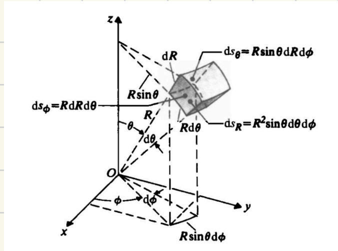


图 2-19 球坐标系中的一个体积元

坐标转换.

for $(u,v) \longmapsto (x,y) = (g(u,v), h(u,v))$.
$\quad (r(u,v))$

$$dx\,dy = \left|\det\begin{pmatrix} g_u & g_v \\ h_u & h_v \end{pmatrix}\right| du\,dv.$$

$$= \left|\det\left(\frac{\partial(x,y)}{\partial(u,v)}\right)\right| du\,dv.$$

(if $\det\left(\frac{\partial(x,y)}{\partial(u,v)}\right)$ is hard to calculate, try $\frac{\partial(u,v)}{\partial(x,y)}$).

Line Integral.

$$L = \int_a^b ds = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

Let F be a continuous vector field, defined on a piecewise smooth curve parametrised by $r(t)$, $a \le t \le b$

$$W = \int_C \langle F, dr \rangle = \int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t)\, dt.$$

$$= \int_C (F \cdot T)\, ds$$
T ← unit tangent vector.

⇐ 计算线积分可考虑参数化轨迹方程，个人认为可以简化运算(?).

$$\int_C \nabla f \cdot dr = f(r(b)) - f(r(a)).$$

$\int_C F\,dr$ is independent of path in some domain D $\Leftrightarrow \oint_C F\,dr = 0$ $\forall$ closed path C.

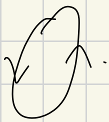
$\Rightarrow$ F is a conservative field on D. i.e. $\exists$ potential function f s.t. $F = \nabla f$.

if $F(x,y) = (P(x,y), Q(x,y))$. is conservative & P, Q. 可一阶导. then.

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

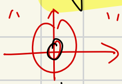
If D is simply connected, the converse also holds.

Green's theorem.



for closed C. ($C = \partial D$.) . 中间没有洞！

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA. \quad \text{(positively oriented)}$$

常用于构造 “  ” 类图形，通过计算黑色小圆求红色大圆.

$\text{curl}(\text{grad } f) = 0$ i.e. $\nabla \times (\nabla f) = 0$.

If $F = \nabla f$, i.e. F is a conservative vector field, then $\nabla \times F = 0$.
(vice versa).

$\text{div}(\text{curl } F) = 0$ i.e. $\nabla \cdot (\nabla \times F) = 0$.

$f: \mathbb{R}^n \to \mathbb{R}$ $\quad \nabla^2 f = \text{div}(\text{grad } f) = \nabla \cdot \nabla f = \Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$.

Surface Integral.

S: paramatrized by $r(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\cdot\vec{k}$

求表面积、...

$$S = \iint_D \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| dA. \qquad d\sigma = \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| dA.$$

$\Leftarrow$ $dA: du\,dv$.

$d\sigma: \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| du\,dv$.

for $r(x,y) = (x, y, f(x,y))$.

$$d\sigma = \sqrt{1 + f_x^2 + f_y^2}\, dA$$

$$\iint_S f(x,y,z)\, d\sigma = \iint_D f(r(u,v)) \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| du\, dv$$

Surface $z = f(x,y)$.

$$\hat{n} = \frac{-\frac{\partial g}{\partial x}\hat{i} - \frac{\partial g}{\partial y}\hat{j} + \hat{k}}{\sqrt{1 + (\frac{\partial g}{\partial x})^2 + (\frac{\partial g}{\partial y})^2}}.$$

Surface parametrized by $r(u,v)$.

$$\hat{n} = \frac{r_u \times r_v}{\|r_u \times r_v\|}.$$

Flux / Surface Integral of $F: \mathbb{R}^3 \to \mathbb{R}^3$ over S is given by.

求通量. $$\iint_S F \cdot dS = \iint_S \langle F, dS \rangle = \iint_S \langle F, \hat{n} \rangle\, d\sigma = \iint_S (F \cdot \hat{n})\, d\sigma$$

$$= \iint_D F \cdot (r_u \times r_v)\, dA$$

$$= \iint_D (-P g_x - Q g_y + R)\, dA.$$

(Suppose $S \subset \mathbb{R}^3$ is the graph of $g(x,y)$).

$$F = \begin{pmatrix} P \\ Q \\ R \end{pmatrix}. \quad r_u \times r_v = \begin{pmatrix} 1 \\ 0 \\ g_x \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ g_y \end{pmatrix} = \begin{pmatrix} -g_x \\ -g_y \\ 1 \end{pmatrix}.$$

Stokes' Theorem.

有方向的, i.e. 通量.

把$\hat{n}$拿出去, 让$(\nabla \times F)$点乘$\hat{n}$;

$$\int_{C = \partial S} F \cdot dr = \iint_S (\nabla \times F)\, dS$$

$$\int_{\partial S} \langle F, T \rangle\, ds \qquad \iint_S \langle \nabla \times F, \hat{n} \rangle\, d\sigma = \iint_S \langle \nabla \times F, r_x \times r_y \rangle\, dA.$$

设方向的面积元.

$\partial S = \partial\partial E$

If S is a closed surface in $\mathbb{R}^3$, (i.e. $S = \partial E$ for some solid region $E \subseteq \mathbb{R}^3$)

w/ $\hat{n}$ pointing outward, then, For smooth vector over $\mathbb{R}^3$.

$$\iint_S \nabla \times F \, ds \overset{\text{Stokes'}}{=\!=} \int_{\partial S = \emptyset} F \cdot dr = 0 .$$

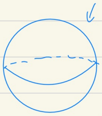
Gauß's theorem / divergence theorem.

$E \subseteq \mathbb{R}^3$ solid region $S = \partial E$, which is a closed surface w/ normal vector pointing outward, then the flux of a vector field is

!!! 注意算的时候不要搞错符号

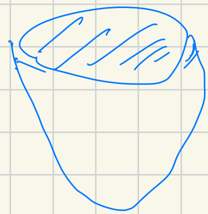
$$\iint_{S=\partial E} F \cdot ds = \iiint_E \nabla \cdot F \, dV .$$

closed surface S.



S is a closed surface in $\mathbb{R}^3$, satisfying the hypothesis of stokes' thm, F is smooth in $\mathbb{R}^3$, e.t.c. then

$$0 = \iiint_E \underbrace{(\nabla \cdot \nabla \times F)}_{0} dV \overset{\text{Gauß}}{=} \iint_S (\nabla \times F) \cdot ds \overset{\text{Stokes'}}{=\!=} \int_{\partial S = \emptyset} F \cdot dr = 0 .$$



声明：标黄的是我个人认为比较重要的内容。

此份资料可能有些混乱，仅做辅助使用。

（此份资料纯根据去年 lecture notes 整理，当时我也不知道考什么，红笔与荧光笔部分是我做了TA依据个人理解和大家的问题新加的批注）。