VV255/MATH2550J Honors Calculus III
Recitation Class 1

Zhang Fan

University of Michigan - Shanghai Jiao Tong University Joint Institute

Spring Term 2022

Linear System of Equations

Definition. A linear system of m equations in n unknowns $x_1, x_2, \cdots, x_n \in V$ is a set of equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{1n}x_n = b_2, \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m. \end{cases}$$

It can also be represented in matrix form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{12} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Definition. If $b_1 = b_2 = \cdots = b_m = 0$, then it is called a homogeneous system. Otherwise, it is called an inhomogeneous system.

Matrix Algebra

1. The sum of two matrices $A_{n\times m}$ and $B_{n\times m}$ is the matrix $C_{n\times m}$ such that

$$c_{ij} = a_{ij} + b_{ij},\ \ i = \overline{1,n},\ \ j = \overline{1,m}.$$

2. The scalar product $\alpha A_{n\times m} = (\alpha a_{ij}),\ \ i = \overline{1,n},\ \ j = \overline{1,m}.$

3. The product of a row-matrix $[a_1 \ \cdots \ a_n]$ and a column matrix $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ is

$$[a_1 \ \cdots \ a_n] \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + \cdots + a_n b_n.$$

It is the inner product of vectors $\bar{a} = (a_1, \cdots, a_n), \bar{b} = (b_1, \cdots, b_n) \in \mathbb{R}^n$.

4. The product of a matrix $A_{n\times m}$ and a vector $\bar{x} \in \mathbb{R}^m$ is

$$A_{n\times m}\bar{x} = \begin{bmatrix} \bar{w}_1 \\ \bar{w}_2 \\ \vdots \\ \bar{w}_n \end{bmatrix} \bar{x} = (\text{def}) \begin{bmatrix} (\bar{w}_1, \bar{x}) \\ (\bar{w}_2, \bar{x}) \\ \vdots \\ (\bar{w}_n, \bar{x}) \end{bmatrix}$$

$$= (\text{prop}) \begin{pmatrix} \bar{a}_1 & \bar{a}_2 \ldots \bar{a}_m \end{pmatrix} \bar{x} = x_1\bar{a}_1 + x_2\bar{a}_2 + \ldots + x_m\bar{a}_m$$

5. The product of a matrix $A_{n\times k}$ and a matrix $B_{k\times m}$ is

$$AB = \begin{bmatrix} A\bar{b}_1 & A\bar{b}_2 & \cdots A\bar{b}_m \end{bmatrix}_{n\times m}.$$

6. Properties.
   - $A + B = B + A$. $AB \neq BA$!
   - $A(B + C) = AB + AC$, $(A + B)C = AC + BC$.

Norm, Identity Matrix and Zero Matrix

Definition. Let $\bar{v} = (v_1, \cdots, v_n) \in \mathbb{R}^n$. The norm of $\bar{v}$ is given by

$$||\bar{v}|| = \sqrt{v \cdot v} = \sqrt{\sum_{i=1}^{n} v_i^2}$$

Definition. Identity matrix and zero matrix

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \quad O_n = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Properties. $A + O = O + A = A$, $AI = IA = A$.

Trace

Definition. Let $A_{n\times n} = (a_{ij})$ be a square matrix. The trace of A is defined as

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

Theorem. Let $A_{m\times n} = (a_{ij})$ and $B_{n\times m} = (b_{ij})$ be two matrices. Then $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Trace

Definition. Let $A_{n\times n} = (a_{ij})$ be a square matrix. The trace of A is defined as

$$\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

Theorem. Let $A_{m\times n} = (a_{ij})$ and $B_{n\times m} = (b_{ij})$ be two matrices. Then $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

Proof.

$$\mathrm{tr}(AB) = \sum_{k=1}^{m} (AB)_{kk} = \sum_{k=1}^{m}\sum_{l=1}^{n} a_{kl}b_{lk} = \sum_{k=1}^{m}\sum_{l=1}^{n} b_{lk}a_{kl} = \sum_{l=1}^{n} (BA)_{ll} = \mathrm{tr}(BA).$$

Reduced Row-Echelom Form

Definition. A matrix is in reduced row-echelom form (rref) if it satisfies all of the following conditions:

- If a row has nonzero entries, then the first nonzero entry is a 1, called the leading 1 in this row.
- If a column contains a leading 1, then all the other entries in that column are 0.
- If a row contains a leading 1, then each row above it contains a leading 1 further to the left. That means, rows of 0's, if any, appear at the bottom of the matrix.

Definition. The number of leading 1's in the rref of a matrix A is called the rank of A.

Inverse Matrix

Definition. The inverse of A is A^{-1} only when

$$AA^{-1} = A^{-1}A = I.$$

In this case, we call matrix A invertible.

Theorems.

- If matrix A is invertible, then $|A| \neq 0$.
- If $|A| \neq 0$, then A is invertible and $A^{-1} = \frac{1}{|A|}A^*$.

For now, you only need to know

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Exercise

Exercise 1.01 Find rref of matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Exercise

Exercise 1.01 Find rref of matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Solution 1.01

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \operatorname{rref} A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \operatorname{rank} A = 2.$$

Thank you!

¿ Q & A ?