Algebraic Geometry

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Abstract

Algebraic Geometry Reading Notes

Contents

1 Category theory

1.1 Categories and functors

universal property : Given two sets M and N, a product is a set P, along with maps $\mu : P \to M$ and $\nu : P \to N$, such that for any set P' with maps $\mu' : P' \to M$ and $\nu' : P' \to N$, these maps must factor uniquely through P

$$\begin{tikzcd}
P' \arrow[dr, dashed, "\exists!"] \arrow[drr, "\nu'"] \arrow[ddr, "\mu'"'] & & \\
& P \arrow[r, "\nu"'] \arrow[d, "\mu"] & N \\
& M &
\end{tikzcd}$$

product : a product is a diagram

$$\begin{tikzcd}
P \arrow[r, "\nu"] \arrow[d, "\mu"'] & N \\
M &
\end{tikzcd}$$

category $\mathscr{C}$: a category consists of a collection of **objects**, and for each pair of objects, a set of **morphisms** (or **arrows**) between them

Morphisms are often informally called **maps**

obj($\mathscr{C}$) : The collection of objects of a category $\mathscr{C}$ is often denoted obj($\mathscr{C}$), but we will usually denote the collection also by $\mathscr{C}$

$\boldsymbol{Mor(A, B)}$: If $A, B \in \mathscr{C}$, then the set of morphisms from A to B is denoted $Mor(A, B)$

A morphism is often written $f : A \to B$, and A is said to be the **source** of f, and B the **target** of f

$Mor(B, C) \times Mor(A, B) \to Mor(A, C)$, and if $f \in Mor(A, B)$ and $g \in Mor(B, C)$, then their composition is denoted $g \circ f$. Composition is associative: if $f \in Mor(A, B), g \in Mor(B, C)$, and $h \in Mor(C, D)$, then $h \circ (g \circ f) = (h \circ g) \circ f$

identity morphism : For each object $A \in \mathscr{C}$, there is always an identity morphism $id_A : A \to A$, such that when you (left- or right-)compose a morphism with the identity, you get the same morphism. More precisely, for any morphisms $f : A \to B$ and $g : B \to C$, $id_B \circ f = f$ and $g \circ id_B = g$

isomorphism : a notion of isomorphism between two objects of a category (a morphism $f : A \to B$ such that there exists some — necessarily unique — morphism $g : B \to A$, where $f \circ g$ and $g \circ f$ are the identity on B and A respectively)

automorphism : an isomorphism of the object with itself

Example 1.1 *the category of sets, denoted* ***Sets****. The objects are sets, and the morphisms are maps of sets*

Example 1.2 *the category* $\boldsymbol{Vec_k}$ *of vector spaces over a given field* k*. The objects are* k*-vector spaces, and the morphisms are linear transformations*

Example 1.3 *A category in which each morphism is an isomorphism is called a* ***groupoid***

(a) A perverse definition of a group is: a groupoid with one object. Make sense of this

(b) Describe a groupoid that is not a group

automorphism group : If A is an object in a category $\mathscr{C}$, then the invertible elements of $Mor(A, A)$ form a group (called the automorphism group of A, denoted $\boldsymbol{Aut(A)}$)

the automorphism groups of the objects in

Example 1.1 : $\{bijections\ between\ A\}$

Example 1.2 : $\{linear\ isomorphism\ on\ A\}$

Two isomorphic objects have isomorphic automorphism groups, but these groups are not canonically isomorphic : if X is a topological space, then the fundamental groupoid is the category where the objects are points of X, and the morphisms $x \to y$ are paths from x to y, up to homotopy. Then the automorphism group of x_0 is the (pointed) fundamental group $\pi_1(X, x_0)$. In the case where X is connected, and $\pi_1(X)$ is not abelian, this illustrates the fact that for a connected groupoid, $\{x\}$ and $\{y\}$ have isomorphic automorphism groups, but $\pi_1(X, x)$ and $\pi_1(X, y)$ are not canonically isomorphic

Example 1.4 *abelian groups: The abelian groups, along with group homomorphisms, form a category* $\boldsymbol{Ab}$

Example 1.5 *Modules over a ring. If* A *is a ring, then the* A*-modules form a category* $\boldsymbol{Mod_A}$

Taking $A = k$*, we obtain Example 1.2; taking* $A = \mathbb{Z}$*, we obtain Example 1.4*

Example 1.6 *rings. There is a category* ***Rings****, where the objects are rings, and the morphisms are maps of rings in the usual sense*

Example 1.7 *topological spaces. The topological spaces, along with continuous maps, form a category* $\boldsymbol{Top}$*. The isomorphisms are homeomorphisms.*

Example 1.8 *partially ordered sets. A* ***partially ordered set****, (or* ***poset****), is a set* S *along with a binary relation* $\geq$ *on* S *satisfying:*

(i) $x \geq x$ *(reflexivity)*

(ii) $x \geq y$ *and* $y \geq z$ *imply* $x \geq z$ *(transitivity)*

(iii) if $x \geq y$ *and* $y \geq x$ *then* $x = y$ *(antisymmetry)*

A partially ordered set $(S, \geq)$ *can be interpreted as a category whose objects are the elements of* S*, and with a single morphism from* x *to* y *if and only if* $x \geq y$ *(and no morphism otherwise).*

Example 1.9 *the category of subsets of a set, and the category of open subsets of a topological space. If* X *is a set, then the subsets form a partially ordered set, where the order is given by inclusion. Informally, if* $U \subset V$*, then we have exactly one morphism* $U \to V$ *in the category (and otherwise* *none). Similarly, if X is a topological space, then the open sets form a partially ordered set, where the order is given by inclusion.*

subcategory : a subcategory $\mathscr{A}$ of a category $\mathscr{B}$ has as its objects some of the objects of $\mathscr{B}$, and some of the morphisms, such that the morphisms of $\mathscr{A}$ include the identity morphisms of the objects of $\mathscr{A}$, and are closed under composition

covariant functor : a covariant functor F from a category $\mathscr{A}$ to a category $\mathscr{B}$, denoted $F : \mathscr{A} \to \mathscr{B}$, is the following data. It is a map of objects $F : obj(\mathscr{A}) \to obj(\mathscr{B})$, and for each $A_1, A_2 \in A$, and morphism $m : A_1 \to A_2$, a morphism $F(m) : F(A_1) \to F(A_2)$ in $\mathscr{B}$

We require that F preserves identity morphisms (for $A \in \mathscr{A}, F(id_A) = id_{F(A)}$), and that F preserves composition $(F(m_2 \circ m_1) = F(m_2) \circ F(m_1))$

A trivial example is the **identity functor** $id : \mathscr{A} \to \mathscr{A}$

forgetful functor : Consider the functor from the category of vector spaces (over a field k) Vec_k to $Sets$, that associates to each vector space its underlying set. The functor sends a linear transformation to its underlying map of sets. This is an example of a forgetful functor, where some additional structure is forgotten

Another example of a forgetful functor is $Mod_A \to Ab$ from A-modules to abelian groups, remembering only the abelian group structure of the A-module

Example 1.10 *Topological examples. (Examples of covariant functors)*

fundamental group functor $\boldsymbol{\pi_1}$, which sends a topological space X with choice of a point $x_0 \in X$ to a group $\pi_1(X, x_0)$

The ith homology functor $\boldsymbol{Top \to Ab}$, which sends a topological space X to its ith homology group $H_i(X, \mathbb{Z})$

The covariance corresponds to the fact that a (continuous) morphism of pointed topological spaces $\phi : X \to Y$ with $\phi(x_0) = y_0$ induces a map of fundamental groups $\pi_1(X, x_0) \to \pi_1(Y, y_0)$, and similarly for homology groups

Example 1.11 *Suppose A is an object in a category $\mathscr{C}$. Then there is a functor $\boldsymbol{h^A : \mathscr{C} \to Sets}$ sending $B \in \mathscr{C}$ to $Mor(A, B)$, and sending $f : B_1 \to B_2$ to $Mor(A, B_1) \to Mor(A, B_2)$ described by*

$$[g : A \to B_1] \mapsto [f \circ g : A \to B_1 \to B_2]$$

composition : If $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{C}$ are covariant functors, then we define a functor $G \circ F : \mathscr{A} \to \mathscr{C}$ (the composition of G and F) in the obvious way

A covariant functor $F : \mathscr{A} \to \mathscr{B}$ is **faithful** if for all $A, A' \in \mathscr{A}$, the map $Mor_{\mathscr{A}}(A, A') \to Mor_{\mathscr{B}}(F(A), F(A'))$ is injective, and **full** if it is surjective. A functor that is full and faithful is **fully faithful**

A subcategory $i : \mathscr{A} \to \mathscr{B}$ is a full subcategory if i is full. (Inclusions are always faithful, so there is no need for the phrase "faithful subcategory".) Thus a subcategory $\mathscr{A}'$ of $\mathscr{A}$ is full if and only if for all $A, B \in obj(\mathscr{A}')$, $Mor_{\mathscr{A}'}(A, B) = Mor_{\mathscr{A}}(A, B)$

Example 1.12 *the forgetful functor $Vec_k \to Sets$ is faithful, but not full; and if A is a ring, the category of finitely generated A-modules is a full subcategory of the category Mod_A of A-modules*

contravariant functor : A contravariant functor is defined in the same way as a covariant functor, except the arrows switch directions: in the above language, $F(A_1 \to A_2)$ is now an arrow from $F(A_2)$ to $F(A_1)$. (Thus $F(m_2 \circ m_1) = F(m_1) \circ F(m_2)$, not $F(m_2) \circ F(m_1)$.)

It is wise to state whether a functor is covariant or contravariant, unless the context makes it very clear. If it is not stated (and the context does not make it clear), the functor is often assumed to be covariant

opposite category : $\mathscr{C}^{opp}$ is the same category as $\mathscr{C}$ except that the arrows go in the opposite direction. Here $\mathscr{C}^{opp}$ is said to be the opposite category to $\mathscr{C}$

Sometimes people describe a contravariant functor $\mathscr{C} \to \mathscr{D}$ as a covariant functor $\mathscr{C}^{opp} \to \mathscr{D}$

Example 1.13 *Linear algebra example 1.2*

If Vec_k is the category of k-vector spaces, then taking duals gives a contravariant functor $()^\vee : Vec_k \to Vec_k$. Indeed, to each linear transformation $f : V \to W$, we have a dual transformation $f^\vee : W^\vee \to V^\vee$, and $(f \circ g)^\vee = g^\vee \circ f^\vee$

Example 1.14 *Topological example 1.10*

The ith cohomology functor $H_i(\cdot, \mathbb{Z}) : Top \to Ab$ is a contravariant functor

Example 1.15 *There is a contravariant functor $Top \to Rings$ taking a topological space X to the ring of real-valued continuous functions on X. A morphism of topological spaces $X \to Y$ (a continuous map) induces the pullback map from functions on Y to functions on X*

Example 1.16 *the* ***functor of points*** *1.11*

Suppose A is an object of a category $\mathscr{C}$. Then there is a contravariant functor $h_A : \mathscr{C} \to Sets$ sending $B \in \mathscr{C}$ to $Mor(B, A)$, and sending the morphism $f : B_1 \to B_2$ to the morphism $Mor(B_2, A) \to Mor(B_1, A)$ via

$$[g : B_2 \to A] \mapsto [g \circ f : B_1 \to B_2 \to A]$$

Examples 1.13 and 1.15 may be interpreted as special cases

natural transformation of covariant functors $F \to G$: Suppose F and G are two covariant functors from $\mathscr{A}$ to $\mathscr{B}$. A natural transformation of covariant functors $F \to G$ is the data of a morphism $\mathfrak{m}_A : F(A) \to G(A)$ for each $A \in \mathscr{A}$ such that for each $f : A \to A'$ in $\mathscr{A}$, the diagram commutes:

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\mathfrak{m}_A"'] & F(A') \arrow[d, "\mathfrak{m}_{A'}"] \\ G(A) \arrow[r, "G(f)"'] & G(A') \end{tikzcd}$$

natural isomorphism : A natural isomorphism of functors is a natural transformation such that each m_A is an isomorphism. (We make analogous definitions when F and G are both contravariant.)

equivalence of categories : The data of functors $F : \mathscr{A} \to \mathscr{B}$ and $F' : \mathscr{B} \to \mathscr{A}$ such that $F \circ F'$ is naturally isomorphic to the identity functor $id_{\mathscr{B}}$ on $\mathscr{B}$ and $F' \circ F$ is naturally isomorphic to $id_{\mathscr{A}}$ is said to be an equivalence of categories

Example 1.17 *Let $f.d.Vec_k$ be the category of finite-dimensional vector spaces over k*

Let $(\cdot)^{\vee\vee} : f.d.Vec_k \to f.d.Vec_k$ be the double dual functor from the category of finite-dimensional vector spaces over k to itself. Then $(\cdot)^{\vee\vee}$ is naturally isomorphic to the identity functor on $f.d.Vec_k$. (Without the finite-dimensionality hypothesis, we only get a natural transformation of functors from id to $(\cdot)^{\vee\vee}$)

Let $\mathscr{V}$ be the category whose objects are the k-vector spaces k^n for each $n \geq 0$ (there is one vector space for each n), and whose morphisms are linear transformations. The objects of $\mathscr{V}$ can be thought of as vector spaces with bases, and the morphisms as matrices. There is an obvious functor $\mathscr{V} \to f.d.Vec_k$, as each k^n is a finite-dimensional vector space

Example 1.18 *$\mathscr{V} \to f.d.Vec_k$ gives an equivalence of categories, by describing an "inverse" functor*

1.2 Universal properties determine an object up to unique isomorphism

Products were defined by a universal property

initial objects : An object of a category $\mathscr{C}$ is an initial object if it has precisely one map to every object

final objects : It is a final object if it has precisely one map from every object

zero objects : It is a zero object if it is both an initial object and a final object

Example 1.19 *Any two initial objects are uniquely isomorphic. Any two final objects are uniquely isomorphic*

This (partially) justifies the phrase "the initial object" rather than "an initial object", and similarly for "the final object" and "the zero object"

(Convention: we often say "the", not "a", for anything defined up to unique isomorphism)

Example 1.20 *The initial and final objects in Sets, Rings, and Top (if they exist)?*

Localization of rings and modules

$A \to S^{-1}A$ satisfies the following universal property:

$S^{-1}A$ is initial among A-algebras B where every element of S is sent to an invertible element in B. (Recall: the data of "an A-algebra B" and "a ring map $A \to B$" are the same)

Translation: any map $A \to B$ where every element of S is sent to an invertible element must factor uniquely through $A \to S^{-1}A$

Another translation: a ring map out of $S^{-1}A$ is the same thing as a ring map from A that sends every element of S to an invertible element

Category theoretic description : If A is a ring and S is a subset, consider all A-algebras B, so that, under the canonical homomorphism $A \to B$, every element of S is mapped to a unit. These algebras are the objects of a category, with A-algebra homomorphisms as morphisms. Then, the localization of A at S is the initial object of this category. (This is a more abstract way of expressing the universal property above.)

Furthermore, an $S^{-1}A$-module is the same thing as an A-module for which $s \times \cdot : M \to M$ is an A-module isomorphism for all $s \in S$

Let's get some practice with this by defining localizations of modules by universal property. Suppose M is an A-module. We define the A-module map $\phi : M \to S^{-1}M$ as being initial among A-module maps $M \to N$ such that elements of S are invertible in N ($s \times \cdot : N \to N$ is an isomorphism for all $s \in S$). More precisely, any such map $\alpha : M \to N$ factors uniquely through ϕ:

$$
\begin{tikzcd}
M \arrow[r, "\phi"] \arrow[dr, "\alpha"'] & S^{-1}M \arrow[d, dashed, "\exists!"] \\
& N
\end{tikzcd}
$$

(Translation: $M \to S^{-1}M$ is universal (initial) among A-module maps from M to modules that are actually $S^{-1}A$-modules)

Tensor products

Another important example of a universal property construction is the notion of a tensor product of A-modules

$$
\begin{aligned}
\otimes_A : obj(Mod_A) \times obj(Mod_A) &\to obj(Mod_A) \\
(M, N) &\to M \otimes_A N
\end{aligned}
$$

Proposition 1.21 *$(\cdot) \otimes_A N$ gives a covariant functor $Mod_A \to Mod_A$. Then $(\cdot) \otimes_A N$ is a **right-exact** functor, i.e., if $M' \to M \to M'' \to 0$ is an exact sequence of A-modules, then the induced sequence is also exact:*

$$M' \otimes_A N \to M \otimes_A N \to M'' \otimes_A N \to 0$$

Category theoretic description : a tensor product of M and N is an A-module T along with an A-bilinear map $t : M \times N \to T$, such that given any A-bilinear map $t' : M \times N \to T'$, there is a unique A-linear map $f : T \to T'$ such that $t' = f \circ t$

$$
\begin{tikzcd}
M \times N \arrow[r, "t"] \arrow[dr, "t'"'] & T = M \otimes_A N \arrow[d, dashed, "\exists!\ f"] \\
& T'
\end{tikzcd}
$$

Remark 1.22 *(a) If M is an A-module and $A \to B$ is a morphism of rings, give $B \otimes_A M$ the structure of a B-module. This describes a functor $Mod_A \to Mod_B$*
(b) If further $A \to C$ is another morphism of rings, show that $B \otimes_A C$ has a natural structure of a ring. Hint: multiplication will be given by $(b_1 \otimes c_1)(b_2 \otimes c_2) = (b_1 b_2) \otimes (c_1 c_2)$

Remark 1.23 *If S is a multiplicative subset of A and M is an A-module, there is a natural isomorphism $(S^{-1}A) \otimes_A M \cong S^{-1}M$ (as $S^{-1}A$-modules and as A-modules)*

Remark 1.24 *tensor products commute with arbitrary direct sums: if M and $\{N_i\}_{i \in I}$ are all A-modules, describe an isomorphism*

$$M \otimes (\oplus_{i \in I} N_i) \cong \oplus_{i \in I} (M \otimes N_i)$$

Fibered products

Suppose we have morphisms $\alpha : X \to Z$ and $\beta : Y \to Z$ (in any category). Then the fibered product (or fibred product) is an object $X \times_Z Y$ along with morphisms $pr_X : X \times_Z Y \to X$ and $pr_Y : X \times_Z Y \to Y$, where the two compositions $\alpha \circ pr_X$, $\beta \circ pr_Y : X \times_Z Y \to Z$ agree, such that given any object W with maps to X and Y (whose compositions to Z agree), these maps factor through some unique $W \to X \times_Z Y$:

$$\begin{tikzcd}
W \arrow[dr, dotted, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_Z Y \arrow[r, "pr_Y"] \arrow[d, "pr_X"] & Y \arrow[d, "\beta"] \\
& X \arrow[r, "\alpha"] & Z
\end{tikzcd}$$

Depending on your religion, the diagram

$$\begin{tikzcd}
X \times_Z Y \arrow[r, "pr_Y"] \arrow[d, "pr_X"] & Y \arrow[d, "\beta"] \\
X \arrow[r, "\alpha"] & Z
\end{tikzcd}$$

is called a **fibered/pullback/Cartesian diagram/square** (six possibilities — even more are possible if you prefer "fibred" to "fibered")

Example 1.25 *In Sets,* $X \times_Z Y = \{(x, y) \in X \times Y : \alpha(x) = \beta(y)\}$.

Example 1.26 *If Z is the final object in a category $\mathscr{C}$, and $X, Y \in \mathscr{C}$, then $X \times_Z Y = X \times Y$: "the" fibered product over Z is uniquely isomorphic to "the" product. (Assume all relevant (fibered) products exist)*

Example 1.27 *If the two squares in the following commutative diagram are Cartesian diagrams, show that the "outside rectangle" (involving U, V, Y, and Z) is also a Cartesian diagram*

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & V \arrow[d] \\
W \arrow[r] \arrow[d] & X \arrow[d] \\
Y \arrow[r] & Z
\end{tikzcd}$$

Example 1.28 *Given morphisms $X_1 \to Y, X_2 \to Y$, and $Y \to Z$, then there is a natural morphism $X_1 \times_Y X_2 \to X_1 \times_Z X_2$*

Remark 1.29 ***The magic diagram***

Suppose we are given morphisms $X_1, X_2 \to Y$ and $Y \to Z$. Then the following diagram is a Cartesian square :

$$\begin{tikzcd} X_1 \times_Y X_2 \arrow[r] \arrow[d] & X_1 \times_Z X_2 \arrow[d] \\ Y \arrow[r] & Y \times_Z Y \end{tikzcd}$$

(Assume all relevant (fibered) products exist)

(Hint : if $Y \times_Z Y$ exists, then $X_1 \times_Z X_2 \to X_2 \to Y$ and $X_1 \times_Z X_2 \to X_1 \to Y$ are equal)

Coproducts

Define coproduct in a category by reversing all the arrows in the definition of product.

For $\mathscr{C}$ a category and ,$X, Y \in Obj(\mathscr{C})$ two objects, their coproduct is an object $X \amalg Y$ in $\mathscr{C}$ equipped with two morphisms

$$\begin{tikzcd} & Y \arrow[d, "i_Y"'] \\ X \arrow[r, "i_X"] & X \coprod Y \end{tikzcd}$$

such that this is universal with this property, meaning such that for any other object with maps like this

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Q \end{tikzcd}$$

there exists a unique morphism $(f, g) : X \coprod Y \to Q$ such that we have the following commuting diagram:

$$\begin{tikzcd} & Y \arrow[d, "i_Y"'] \arrow[ddr, "g"] & \\ X \arrow[r, "i_X"] \arrow[drr, "f"'] & X \coprod Y \arrow[dr, dotted, "\exists!(f,g)"] & \\ & & Q \end{tikzcd}$$

This morphism (f, g) are called copairing of f and g. The morphism $X \to X \coprod Y$ and $Y \to X \coprod Y$ coprojections or sometimes "injections" or "inclusions", although in general they may not be monomorphisms

fibered coproduct

Define fibered coproduct in a category by reversing all the arrows in the definition of fibered product.

$$\begin{tikzcd}
W & & \\
& X \coprod_Z Y \arrow[ul, dotted, "\exists!"'] & Y \arrow[l, "g"] \arrow[ull, "\nu"'] \\
& X \arrow[u, "f"] \arrow[uul, "\mu"] & Z \arrow[l, "\alpha"] \arrow[u, "\beta"']
\end{tikzcd}$$

Example 1.30 *Coproduct for Sets is disjoint union*

Remark 1.31 *Suppose $A \to B$ and $A \to C$ are two ring morphisms, so in particular B and C are A-modules. Recall that $B \otimes_A C$ has a ring structure. Then there is a natural morphism $B \to B \otimes_A C$ given by $b \mapsto b \otimes 1$. (This is not necessarily an inclusion) Similarly, there is a natural morphism $C \to B \otimes_A C$. This gives a fibered coproduct on rings, i.e., that*

$$\begin{tikzcd}
B \otimes_A C & C \arrow[l] \\
B \arrow[u] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

satisfies the universal property of fibered coproduct

monomorphism : a morphism $\pi : X \to Y$ is a monomorphism if any two morphisms $\mu_1 : Z \to X$ and $\mu_2 : Z \to X$ such that $\pi \circ \mu_1 = \pi \circ \mu_2$ must satisfy $\mu_1 = \mu_2$. In other words, there is at most one way of filling in the dotted arrow so that the diagram

$$\begin{tikzcd}
Z \arrow[d, dotted, "\leq 1"'] \arrow[dr] & \\
X \arrow[r, "\pi"'] & Y
\end{tikzcd}$$

commutes — for any object Z, the natural map $Mor(Z, X) \to Mor(Z, Y)$ is an injection

Remark 1.32 *The composition of two monomorphisms is a monomorphism*

Remark 1.33 *A morphism $\pi : X \to Y$ is a monomorphism $\Leftrightarrow$ the fibered product $X \times_Y X$ exists, and the induced morphism $X \to X \times_Y X$ is an isomorphism*

Remark 1.34 *If $Y \to Z$ is a monomorphism, then the morphism $X_1 \times_Y X_2 \to X_1 \times_Z X_2$ is an isomorphism*

epimorphism : a morphism $\pi : Y \to X$ is a monomorphism if any two morphisms $\mu_1 : X \to Z$ and $\mu_2 : X \to Z$ such that $\mu_1 \circ \pi = \mu_2 \circ \pi$ must satisfy $\mu_1 = \mu_2$. In other words, there is at most one way of filling in the dotted arrow so that the diagram

$$\begin{tikzcd} Z & \\ X \arrow[u, dotted, "\leq 1"] & Y \arrow[l, "\pi"] \arrow[ul] \end{tikzcd}$$

commutes — for any object Z, the natural map $Mor(X, Z) \to Mor(Y, Z)$ is an injection

Representable functors and Yoneda's Lemma Suppose A is an object of category $\mathscr{C}$. For any object $C \in \mathscr{C}$, we have a set of morphisms $Mor(C, A)$. If we have a morphism $f : B \to C$, we get a map of sets

$$Mor(C, A) \to Mor(B, A)$$

by composition: given a map from C to A, we get a map from B to A by precomposing with $f : B \to C$. Hence this gives a contravariant functor $h_A : \mathscr{C} \to Sets$. Yoneda's Lemma states that the functor h_A determines A up to unique isomorphism. More precisely:

Lemma 1.35 *(a) Suppose you have two objects A and A' in a category $\mathscr{C}$, and morphisms*

$$i_C : Mor(C, A) \to Mor(C, A')$$

that commute with the maps

$$Mor(C, A) \to Mor(B, A)$$

for all the (B, C, f) that $f : B \to C$

Then the i_C (as C ranges over the objects of C) are induced from a unique morphism $g : A \to A'$. More precisely, there is a unique morphism $g : A \to A'$ such that for all $C \in \mathscr{C}$, i_C is $u \mapsto g \circ u$
(b) If furthermore the i_C are all bijections, then the resulting g is an isomorphism

(Hint : looking for an element $Mor(A, A')$. So just plug in $C = A$ to the first equation, and see where the identity goes

There is an analogous statement with the arrows reversed, where instead of maps into A, you think of maps from A. The role of the contravariant functor h_A is played by the covariant functor h^A. The proof is the same (with the arrows reversed)

Lemma 1.36 *(a) Suppose A and B are objects in a category $\mathscr{C}$. There is a bijection between the natural transformations $h_A \to h_B$ of covariant functors $\mathscr{C} \to Sets$ and the morphisms $B \to A$*
(b) Suppose A and B are objects in a category $\mathscr{C}$. There is a bijection between the natural transformations $h_A \to h_B$ of contravariant functors $\mathscr{C} \to Sets$ and the morphisms $A \to B$

Representable functors *: a contravariant functor F from $\mathscr{C}$ to Sets is said to be* ***representable*** *if there is a natural isomorphism :*

$$\xi : F \cong h_A$$

Thus the representing object A is determined up to unique isomorphism by the pair (F, ξ). There is a similar definition for covariant functors. (The element $\xi^{-1}(id_A) \in F(A)$ is often called the "universal object"; do you see why?)

(c) ***Yoneda's Lemma****. Suppose F is a covariant functor $\mathscr{C} \to Sets$, and $A \in \mathscr{C}$. There is a bijection between the natural transformations $h^A \to F$ and $F(A)$*

1.3 Limits and colimits

Limits

small category : we say that a category is a small category if the objects and the morphisms are sets

index category : Suppose $\mathscr{I}$ is any small category, and $\mathscr{C}$ is any category. Then a functor $F : \mathscr{I} \to \mathscr{C}$ (i.e., with an object $A_i \in \mathscr{C}$ for each element $i \in \mathscr{I}$, and appropriate commuting morphisms dictated by $\mathscr{I}$) is said to be a **diagram indexed by** $\mathscr{I}$. We call $\mathscr{I}$ an index category

Example 1.37 *if $\square$ is the category in the figure below, and $\mathscr{A}$ is a category, then a functor $\square \to \mathscr{A}$ is precisely the data of a commuting square in $\mathscr{A}$*

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}$$

limit of the diagram (inverse limit / projective limit) : the limit of the diagram is an object $\varprojlim_{\mathscr{I}} A_i$ of $\mathscr{C}$ along with morphisms $f_j : \varprojlim_{\mathscr{I}} A_i \to A_j$ for each $j \in \mathscr{I}$, such that if $m : j \to k$ is a morphism in $\mathscr{I}$, then

$$\begin{tikzcd}
\varprojlim_{\mathscr{I}} A_i \arrow[d, "f_j"] \arrow[dr, "f_k"] & \\
A_j \arrow[r, "F(m)"'] & A_k
\end{tikzcd}$$

commutes, and this object and maps to each A_i are universal (final) with respect to this property

More precisely, given any other object W along with maps $g_i : W \to A_i$ commuting with the $F(m)$ (if $m : j \to k$ is a morphism in I , then $g_k = F(m) \circ g_j$), then there is a unique map

$$g : W \to \varprojlim_{\mathscr{I}} A_i$$

so that $g_i = f_i \circ g$ for all i.

Example 1.38 *fibered product*

If $\mathscr{I}$ is the partially ordered set:

$$\begin{tikzcd}
 & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}$$

we obtain the fibered product

Example 1.39 *product*

If $\mathscr{I}$ is $\bullet \quad \bullet$, we obtain the product

*If $\mathscr{I}$ is a set (i.e., the only morphisms are the identity maps), then the limit is called the **product** of the A_i, and is denoted $\prod_i A_i$. The special case where $\mathscr{I}$ has two elements is the example of the previous paragraph*

Example 1.40 *If $\mathscr{I}$ has an initial object e, then A_e is the limit, and in particular the limit always exists*

Example 1.41 ***p-adic integers***

a p-adic integer is an element of the inverse limit in the category of the rings :

$$\begin{tikzcd}
 & \mathbb{Z}_p \arrow[d] \arrow[dr] \arrow[drr] & & \\
\cdots \arrow[r] & \mathbb{Z}/p^3 \arrow[r] & \mathbb{Z}/p^2 \arrow[r] & \mathbb{Z}/p
\end{tikzcd}$$

Remark 1.42 *In the category Sets*

$$\left\{(a_i)_{i\in\mathscr{I}} \in \prod_i A_i : F(m)(a_j) = a_k \; for \; all \; m \in Mor_{\mathscr{I}}(j,k) \in Mor(\mathscr{I})\right\}$$

along with the obvious projection maps to each A_i, is the limit $\lim\limits_{\overleftarrow{\mathscr{I}}} A_i$

This clearly also works in the category Mod_A of A-modules (in particular Vec_k and Ab), as well as *Rings*

From this point of view, $2 + 3p + 2p^2 + \cdots \in \mathbb{Z}_p$ can be understood as the sequence $(2, 2 + 3p, 2 + 3p + 2p^2, \cdots)$

colimits (direct limit / inductive limit / injective limit)

the colimit of the diagram is an object $\lim\limits_{\overrightarrow{\mathscr{I}}} A_i$ of $\mathscr{C}$ along with morphisms $f_j : A_j \to \lim\limits_{\overrightarrow{\mathscr{I}}} A_i$ for each $j \in \mathscr{I}$, such that if $m : j \to k$ is a morphism in $\mathscr{I}$, then

$$\begin{tikzcd}
\lim\limits_{\overrightarrow{\mathscr{I}}} A_i & \\
A_j \arrow[u, "f_j"] \arrow[r, "F(m)"'] & A_k \arrow[ul, "f_k"']
\end{tikzcd}$$

commutes, and this object and maps to each A_i are universal (final) with respect to this property

More precisely, given any other object W along with maps $g_i : A_i \to W$ commuting with the $F(m)$ (if $m : j \to k$ is a morphism in $\mathscr{I}$, then $g_j = g_k \circ F(m)$), then there is a unique map $g : \lim\limits_{\overrightarrow{\mathscr{I}}} A_i \to W$ so that $g_i = g \circ f_i$ for all i

Example 1.43 *The set $5^{-\infty}\mathbb{Z}$ of rational numbers whose denominators are powers of 5 is a colimit $\lim\limits_{\overrightarrow{\mathscr{I}}} 5^{-i}\mathbb{Z}$. More precisely, $5^{-\infty}\mathbb{Z}$ is the colimit of the diagram*

$$\mathbb{Z} \longrightarrow 5^{-1}\mathbb{Z} \longrightarrow 5^{-2}\mathbb{Z} \longrightarrow \cdots$$

Example 1.44 *coproduct*

If $\mathscr{I}$ is $\bullet \quad \bullet$, we obtain the coproduct

If $\mathscr{I}$ is a set (i.e., the only morphisms are the identity maps), then the colimit is called the ***coproduct*** *of the A_i, and is denoted $\coprod_i A_i$. The special case where $\mathscr{I}$ has two elements is the example of the previous paragraph*

Example 1.45 $\mathbb{Q} = \varinjlim_{\mathscr{I}} \frac{1}{n}\mathbb{Z}$

Example 1.46 *Define $U \to V$ iff $V \subset U$, then some subsets of a given set as a colimit. (Dually, the intersection can be interpreted as a limit.) The objects of the category in question are the subsets of the given set.*

filtered (directed partially ordered set / filtered index category) :

a nonempty partially ordered set $(S, \geq)$ is filtered (or is said to be a filtered set) if for each $x, y \in S$, there is a z such that $x \geq z$ and $y \geq z$. More generally, a nonempty category $\mathscr{I}$ is filtered if:

(i) for each $x, y \in \mathscr{I}$, there is $az \in \mathscr{I}$ and arrows $x \to z$ and $y \to z$, and

(ii) for every two arrows $u : x \to y$ and $v : x \to y$, there is an arrow $w : y \to z$ such that $w \circ u = w \circ v$

Remark 1.47 *Suppose $\mathscr{I}$ is filtered. (We will almost exclusively use the case where $\mathscr{I}$ is a filtered set.) Recall the symbol $\coprod$ for disjoint union of sets. Then any diagram in Sets indexed by $\mathscr{I}$ has the following, with the obvious maps to it, as a colimit:*

$$\left\{(a_i, i) \in \coprod_{i \in \mathscr{I}} A_i\right\} / ((a_i, i) \sim (a_j, j) \text{ if and only if there are } f : A_i \to A_k \text{ and } g : A_j \to A_k \text{ in the diagram for which } f(a_i) = g(a_j) \text{ in } A_k)$$

Example 1.48 *The colimit $\varinjlim M_i$ in the category of A-modules Mod_A can be described as follows. The set underlying $\varinjlim M_i$ is defined as in Remark 1.47. To add the elements $m_i \in M_i$ and $m_j \in M_j$, choose an $l \in \mathscr{I}$ with arrows $u : i \to l$ and $v : j \to l$, and then define the sum of m_i and m_j to be $F(u)(m_i) + F(v)(m_j) \in M_l$. The element $m_i \in M_i$ is 0 if and only if there is some arrow $u : i \to k$ for which $F(u)(m_i) = 0$, i.e., if it becomes 0 "later in the diagram". Last, multiplication by an element of A is defined in the obvious way*

The A-module described above is indeed the colimit

Example 1.49 *Direct limits exist in the category (Set) and (Ab) of sets and abelian groups. More precisely:*

(1) Let $(\{A_i\}_{i \in I}, \{\rho_{ij}\}_{i,j \in I})$ be a direct system in (Set). Then $\varinjlim_{i \in I} A_i = \coprod_{i \in I} A_i / \sim$, where $a_i \sim a_j$ if and only if there exists $k \in I$ such that $i, j \leq k$ and $\rho_{ik}(a_i) = \rho_{jk}(a_j)$, together with the maps $\rho_i : A_i \to \coprod_{i \in I} A_i / \sim$ induced by the inclusions is a direct limit of $(\{A_i\}_{i \in I}, \{\rho_{ij}\}_{i,j \in I})$.

(2) Let $(\{A_i\}_{i\in I}, \{\rho_{ij}\}_{i,j\in I})$ *be a direct system in* (Ab)*. Then* $\varinjlim_{i\in I} A_i = \bigoplus_{i\in I} A_i/N$ *where* N *is the subgroup generated by elements of the form* $a_i - \rho_{ij}(a_i)$ *together with the maps* $\rho_i : A_i \to \bigoplus_{i\in I} A_i/N$ *induced by the inclusions is a direct limit of* $(\{A_i\}_{i\in I}, \{\rho_{ij}\}_{i,j\in I})$*.*

(3) Let $(\{A_i\}_{i\in I}, \{\rho_{ij}\}_{i,j\in I})$ *be a direct system in* (Ab)*. Forgetting the group structure, it becomes a direct system in* (Set)*. The induced map* $\prod_{i\in I} A_i/\sim \to \bigoplus_{i\in I} A_i/N$ *is a bijection that identifies the* f_i *in (1) and (2).*

Similarly, if $(\{A_i\}_{i\in I}, \{\rho_{ij}\}_{i,j\in I})$ *is a direct system of modules or rings, then the direct limit exists and the underlying set is the direct limit of the underlying sets.*

Example 1.50 *Suppose* S *is a multiplicative set of integral domain* A*, then* $S^{-1}A = \varinjlim \frac{1}{s}A$ *where the limit is over* $s \in S$*, and in the category of* A*-modules*

Example 1.51 *COLIMITS OF A-MODULES WITHOUT THE FILTERED CONDITION*

Suppose you are given a diagram of A-modules indexed by $\mathscr{I} : F : \mathscr{I} \to Mod_A$*, where we let* $M_i := F(i)$*. Show that the colimit is* $\oplus_{i\in\mathscr{I}}$ *Mi modulo the relations* $m_i - F(n)(m_i)$ *for every n:* $i \to j$ *in* $\mathscr{I}$ *(i.e., for every arrow in the diagram). (Some what more precisely: "modulo" means "quotiented by the submodule generated by")*

1.4 Adjoints

adjoints :

Two covariant functors $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ are adjoint if there is a natural bijection for all $A \in \mathscr{A}$ and $B \in \mathscr{B}$

$$\tau_{AB} : Mor_{\mathscr{B}}(F(A), B) \to Mor_{\mathscr{A}}(A, G(B))$$

We say that (F, G) form an **adjoint pair**, and that F is **left-adjoint** to G (and G is **right-adjoint** to F). We say F is a **left adjoint** (and G is a **right adjoint**)

By "natural" we mean the following. For all $f : A \to A'$ in $\mathscr{A}$, we require

$$\begin{tikzcd}
Mor_{\mathscr{B}}\ (F(A'), B) \arrow[r, "Ff^*"] \arrow[d, "\tau_{A'B}"'] & Mor_{\mathscr{B}}\ (F(A), B) \arrow[d, "\tau_{AB}"] \\
Mor_{\mathscr{A}}\ (A', G(B)) \arrow[r, "f^*"'] & Mor_{\mathscr{A}}\ (A, G(B))
\end{tikzcd}$$

to commute, and for all $g : B \to B'$ in $\mathscr{B}$ we want a similar commutative diagram to commute

$$\begin{tikzcd}
Mor_{\mathscr{B}}\ (F(A), B) \arrow[r, "g^*"] \arrow[d, "\tau_{AB}"'] & Mor_{\mathscr{B}}\ (F(A), B') \arrow[d, "\tau_{AB'}"] \\
Mor_{\mathscr{A}}\ (A, G(B)) \arrow[r, "Gg^*"'] & Mor_{\mathscr{A}}\ (A, G(B'))
\end{tikzcd}$$

(Here f^* is the map induced by $f : A \to A'$, and Ff^* is the map induced by $Ff : F(A) \to F(A')$)

In fact, we have :

$$\begin{tikzcd}
Mor_{\mathscr{B}}\ (F(A'), B) \arrow[r, "Ff^*"] \arrow[d, "\tau_{A'B}"'] & Mor_{\mathscr{B}}\ (F(A), B) \arrow[r, "g^*"] \arrow[d, "\tau_{AB}"] & Mor_{\mathscr{B}}\ (F(A), B') \arrow[d, "\tau_{AB'}"] \\
Mor_{\mathscr{A}}\ (A', G(B)) \arrow[r, "f^*"'] & Mor_{\mathscr{A}}\ (A, G(B)) \arrow[r, "Gg^*"'] & Mor_{\mathscr{A}}\ (A, G(B'))
\end{tikzcd}$$

Remark 1.52 *The map τ_{AB} has the following properties :*

For each A there is a map $\eta_A : A \to GF(A)$ so that for any $g : F(A) \to B$, the corresponding $\tau_{AB}(g) : A \to G(B)$ is given by the composition

$$A \xrightarrow{\eta_A} GF(A) \xrightarrow{Gg} G(B)$$

Similarly, there is a map $\epsilon_B : FG(B) \to B$ for each B so that for any $f : A \to G(B)$, the corresponding map $\tau_{AB}^{-1}(f) : F(A) \to B$ is given by the composition

$$F(A) \xrightarrow{Ff} FG(B) \xrightarrow{\epsilon_B} B$$

Proof:

$$
\begin{tikzcd}
Mor_{\mathscr{B}}\,(F(A),F(A)) \arrow[r, "g^*"] \arrow[d, "\tau_{A,F(A)}"'] & Mor_{\mathscr{B}}\,(F(A),B) \arrow[d, "\tau_{AB}"] \\
Mor_{\mathscr{A}}\,(A,GF(A)) \arrow[r, "Gg^*"'] & Mor_{\mathscr{A}}\,(A,G(B))
\end{tikzcd}
$$

So $\eta_A = \tau_{A,F(A)}(1_{F(A)})$ $\square$

Example 1.53 *Suppose M, N, and P are A-modules (where A is a ring). There is a bijection*

$$Hom_A(M \otimes_A N, P) \leftrightarrow Hom_A(M, Hom_A(N,P))$$

(Hint : try to use the universal property of $\otimes$)

$(\cdot) \otimes_A N$ *and* $Hom_A(N,\cdot)$ *are adjoint functors*

Example 1.54 *Suppose $B \to A$ is a morphism of rings. If M is an A-module, We can create a B-module M_B by considering it as a B-module. This gives a functor $\cdot B : Mod_A \to Mod_B$. Then this functor is right-adjoint to $\cdot \otimes_B A$. In other words, there is a bijection*

$$Hom_A(N \otimes_B A, M) \cong Hom_B(N, M_B)$$

Example 1.55 *For those familiar with representation theory:*

Frobenius reciprocity may be understood in terms of adjoints. Suppose V is a finite-dimensional representation of a finite group G, and W is a representation of a subgroup $H < G$. Then induction and restriction are an adjoint pair (Ind_H^G, Res_H^G) between the category of G-modules and the category of H-modules

Example 1.56 *The loop space is dual to the suspension of the same space; this duality is sometimes called Eckmann–Hilton duality. The basic observation is that $[\sum Z, X] \cong [Z, \Omega X]$*

Example 1.57 ***groupification of abelian semigroups***

Here is another motivating example: getting an abelian group from an abelian semigroup. (An abelian semigroup is just like an abelian group, except we don't require an identity or an inverse. Morphisms of abelian semigroups are maps of sets preserving the binary operation. One example is the non-negative integers $\mathbb{N}$ under addition. Another is the positive integers $\mathbb{N}^*$ under multiplication. You may enjoy groupifying both.) From an abelian semigroup, you can create an abelian group. Here is a formalization of that notion.

A groupification of a semigroup S is a map of abelian semigroups $\pi : S \to G$ such that G is an abelian group, and any map of abelian semigroups from S to an abelian group G' factors uniquely through G:

$$\begin{tikzcd} S \arrow[r, "\pi"] \arrow[dr] & G \arrow[d, dotted, "\exists!"] \\ & G' \end{tikzcd}$$

Construct the "groupification functor" H from the category of nonempty abelian semigroups to the category of abelian groups.

Let F be the forgetful functor from the category of abelian groups Ab to the category of abelian semigroups. Then H is left-adjoint to F.

Example 1.58 *$S^{-1}A$-modules are a fully faithful subcategory of the category of A-modules (via the obvious inclusion $Mod_{S^{-1}A} \hookrightarrow Mod_A$). Then $Mod_A \to Mod_{S^{-1}A}$ can be interpreted as an adjoint to the forgetful functor $Mod_{S^{-1}A} \to ModA$*

Remark 1.59 *Here is a table of most of the adjoints that will come up for us.*

situation	**category**$\mathscr{A}$	**category**$\mathscr{B}$	**left adjoint** $F : \mathscr{A} \to \mathscr{B}$	**right adjoint** $G : \mathscr{B} \to \mathscr{A}$
A-modules			$(\cdot) \otimes_A N$	$Hom_A(N, \cdot)$
ring maps $B \to A$	Mod_B	Mod_A	$(\cdot)\otimes_B$ (extension of scalars)	$M \to M_B$ (restriction of scalars)
(pre)sheaves on a topological space X	presheaves on X	sheaves on X	sheafification	forgetful
(semi)groups	semigroups	groups	groupification	forgetful
sheaves, $\pi : X \to Y$	sheaves on Y	sheaves on X	π^{-1}	π_*
sheaves of abelian groups or $\mathscr{O}$-modules, open embeddings $\pi : U \hookrightarrow Y$	sheaves on U	sheaves on Y	$\pi_!$	π^{-1}
ring maps $B \to A$	Mod_A	Mod_B	$M \mapsto M_B$ (restriction of scalars)	$N \mapsto Hom_B(A, N)$
quasicoherent sheaves, affine $\pi : X \to Y$	$QCoh_X$	$QCoh_Y$	π_*	$\pi^!_{sh}$

1.5 Abelian categories

preadditive category : a category $\mathscr{A}$ is called preadditive if each morphism set $Mor_{\mathscr{A}}(x, y)$ is endowed with the structure of an abelian group such that the compositions

$$Mor(x, y) \times Mor(y, z) \to Mor(x, z)$$

are bilinear

additive functor : a functor $F : \mathscr{A} \to \mathscr{B}$ of preadditive categories is called additive if and only if $F : Mor(x, y) \to Mor(F(x), F(y))$ is a homomorphism of abelian groups for all $x, y \in Ob(\mathscr{A})$

In particular for every x, y there exists at least one morphism $x \to y$, namely the zero map.

Let 0 be the zero map from x to y, by bilinearity, for an arbitrary object of $\mathscr{A}$: z, $f \in Mor(y, z), g \in Mor(z, x)$, we have $f \circ 0$ =the zero map from x to z, $0 \circ g$ =the zero map from z to y. (Hint : $f \circ 0 = f \circ (0 + 0)$)

Lemma 1.60 *Let $\mathscr{A}$ be a preadditive category. Let x be an object of $\mathscr{A}$. The following are equivalent:*

(1) x is an initial object,

(2) x is a final object, and

(3) $id_x = 0$ in $Mor_{\mathscr{A}}(x, x)$.

Furthermore, if such an object 0 exists, then a morphism $\alpha : x \to y$ factors through 0 if and only if $\alpha = 0$.

Proof: First assume that x is either (1) initial or (2) final. In both cases, it follows that $Mor(x, x)$ is a trivial abelian group containing id_x, thus $id_x = 0$ in $Mor(x, x)$, which shows that each of (1) and (2) implies (3).

Now assume that $id_x = 0$ in $Mor(x, x)$. Let y be an arbitrary object of $\mathscr{A}$ and let $f \in Mor(x, y)$. Denote $C : Mor(x, x) \times Mor(x, y) \to Mor(x, y)$ the composition map. Then $f = C(0, f)$ and since C is bilinear we have $C(0, f) = 0$. Thus $f = 0$. Hence x is initial in $\mathscr{A}$. A similar argument for $f \in Mor(y, x)$ can be used to show that x is also final. Thus (3) implies both (1) and (2). $\square$

Lemma 1.61 *Let $\mathscr{A}$ be a preadditive category. Let $x, y \in Ob(\mathscr{A})$. If the product $x \times y$ exists, then so does the coproduct $x \coprod y$. If the coproduct $x \coprod y$ exists, then so does the product $x \times y$. In this case also $x \coprod y \cong x \times y$.*

Proof: Suppose that $z = x \times y$ with projections $p : z \to x$ and $q : z \to y$. Denote $i : x \to z$ the morphism corresponding to $(1, 0)$. Denote $j : y \to z$ the morphism corresponding to $(0, 1)$. Thus we have the commutative diagram

$$\begin{tikzcd}
x \arrow[rr, "1"] \arrow[dr, "i"'] & & x \arrow[dl, "p"] \\
& z & \\
y \arrow[ur, "j"] \arrow[rr, "1"'] & & y \arrow[ul, "q"']
\end{tikzcd}$$

where the diagonal compositions are zero. It follows that $i \circ p + j \circ q : z \to z$ is the identity since it is a morphism which upon composing with p gives p and upon composing with q gives q. Suppose given morphisms $a : x \to w$ and $b : y \to w$. Then we can form the map $a \circ p + b \circ q : z \to w$. In this way we get a bijection $Mor(z, w) = Mor(x, w) \times Mor(y, w)$ which show that $z = x \coprod y$. $\square$

direct sum $x \oplus y$: given a pair of objects x, y in a preadditive category $\mathscr{A}$, the direct sum $x \oplus y$ of x and y is the direct product $x \times y$ endowed with the morphisms i, j, p, q as in the Lemma above.

Remark 1.62 *Note that the proof of Lemma above shows that given p and q the morphisms i, j are uniquely determined by the rules $p \circ i = id_x, q \circ j = id_y, p \circ j = 0, q \circ i = 0$. Moreover, we automatically have $i \circ p + j \circ q = id_{x \oplus y}$. Similarly, given i, j the morphisms p and q are uniquely determined. Finally, given objects x, y, z and morphisms $i : x \to z, j : y \to z, p : z \to x$ and $q : z \to y$ such that $p \circ i = id_x, q \circ j = id_y, p \circ j = 0, q \circ i = 0$ and $i \circ p + j \circ q = id_z$, then z is the direct sum of x and y with the four morphisms equal to i, j, p, q.*

Lemma 1.63 *Let $\mathscr{A}, \mathscr{B}$ be preadditive categories. Let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor. Then F transforms direct sums to direct sums and zero to zero.*

Proof: Suppose F is additive. A direct sum z of x and y is characterized by having morphisms $i : x \to z, j : y \to z, p : z \to x$ and $q : z \to y$ such that $p \circ i = id_x, q \circ j = id_y, p \circ j = 0, q \circ i = 0$ and $i \circ p + j \circ q = id_z$, according to Remark above. Clearly $F(x), F(y), F(z)$ and the morphisms $F(i), F(j), F(p), F(q)$ satisfy exactly the same relations (by additivity) and we see that $F(z)$ is a direct sum of $F(x)$ and $F(y)$. Hence, F transforms direct sums to direct sums. $\square$

additive category : a category $\mathscr{C}$ is said to be additive if it satisfies the following properties.

Ad1. For each $A, B \in \mathscr{C}$, $Mor(A, B)$ is an abelian group, such that composition of morphisms distributes over addition (bilinear)

Ad2. $\mathscr{C}$ has a zero object, denoted 0. (This is an object that is simultaneously an initial object and a final object)

Ad3. It has products of two objects (a product $A \times B$ for any pair of objects), and hence by induction, products of any finite number of objects

In an additive category, the morphisms are often called **homomorphisms**, and Mor is denoted by Hom. In fact, this notation Hom is a good indication that you're working in an additive category

The 0-morphism in the abelian group $Hom(A, B)$ is the composition $A \to 0 \to B$. (We also remark that the notion of 0-morphism thus makes sense in any category with a 0-object)

Let $\mathscr{A}$ be a preadditive category. Let $f : x \to y$ be a morphism

kernel : a kernel of f is a morphism $i : z \to x$ such that (a) $f \circ i = 0$ and (b) for any $i' : z' \to x$ such that $f \circ i' = 0$ there exists a unique morphism $g : z' \to z$ such that $i' = i \circ g$.

$$\begin{tikzcd}
z' \arrow[d, dotted, "\exists!"'] \arrow[dr, "i'"'] \arrow[drr, "0"] & & \\
z \arrow[r, "i"'] \arrow[rr, bend right, "0"'] & x \arrow[r] & y
\end{tikzcd}$$

The kernel is written $kerf \to x$ The kernel of $f : x \to y$ is the limit of the diagram

$$\begin{tikzcd}
& 0 \arrow[d] \\
x \arrow[r, "f"] & y
\end{tikzcd}$$

cokernel : a cokernel of f is a morphism $p : y \to z$ such that (a) $p \circ f = 0$ and (b) for any $p' : y \to z'$ such that $p' \circ f = 0$ there exists a unique morphism $g : z \to z'$ such that $p' = g \circ p$.

$$\begin{tikzcd}
& & z' \\
x \arrow[r] \arrow[urr, "0"] \arrow[rr, bend right, "0"'] & y \arrow[r, "p"'] \arrow[ur, "p'"] & z \arrow[u, dotted, "\exists!"']
\end{tikzcd}$$

The cokernel of $f : x \to y$ is the colimit of the diagram

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d] & y \\
0 &
\end{tikzcd}$$

coimage : if a kernel of f exists, then a coimage of f is a cokernel for the morphism $ker(f) \to x$. If a kernel and coimage exist then we denote this $x \to Coim(f)$.

image : if a cokernel of f exists, then the image of f is a kernel of the morphism $y \to coker(f)$. If a cokernel and image of f exist then we denote this $Im(f) \to y$.

Lemma 1.64 *Let $\mathscr{C}$ be a preadditive category. Let $f : x \to y$ be a morphism in $\mathscr{C}$.*

(1) If a kernel of f exists, then this kernel is a monomorphism.

(2) If a cokernel of f exists, then this cokernel is an epimorphism.

(3) If a kernel and coimage of f exist, then the coimage is an epimorphism.

(4) If a cokernel and image of f exist, then the image is a monomorphism.

Lemma 1.65 *Let $f : x \to y$ be a morphism in a preadditive category such that the kernel, cokernel, image and coimage all exist. Then f can be factored uniquely as*

$$x \to Coim(f) \to Im(f) \to y$$

If $i : A \to B$ is a monomorphism, then we say that A is a **subobject** of B, where the map i is implicit. There is also the notion of **quotient object**, defined dually to subobject

abelian category : an abelian category is an additive category satisfying three additional properties :

(1) Every map has a kernel and cokernel

(2) Every monomorphism is the kernel of its cokernel

(3) Every epimorphism is the cokernel of its kernel

quotient : the cokernel of a monomorphism is called the quotient. The quotient of a monomorphism $A \to B$ is often denoted B/A (with the map from B implicit)

Theorem 1.66 *Freyd-Mitchell Embedding Theorem:*

If $\mathscr{C}$ is an abelian category such that $Hom(X,Y)$ is a set for all $X, Y \in \mathscr{C}$, then there is a ring A and an exact, fully faithful functor from $\mathscr{C}$ into Mod_A, which embeds $\mathscr{C}$ as a full subcategory

In the sense that $Hom_{\mathscr{A}}(X,Y) \cong Hom_A(M,N)$

(Unfortunately, the ring A need not be commutative)

The upshot is that to prove something about a diagram in some abelian category, we may assume that it is a diagram of modules over some ring, and we may then "diagram-chase" elements. Moreover, any fact about kernels, cokernels, and so on that holds in Mod_A holds in any abelian category)

Complexes, exactness, and homology (In this entire discussion, we assume we are working in an abelian category.) We say a sequence :

$$\cdots \to A \xrightarrow{f} B \xrightarrow{g} C \to \cdots \tag{1}$$

is a **complex at** B if $g \circ f = 0$, and is **exact at** B if $ker g = im f$. (More specifically, g has a kernel that is an image of f. Exactness at B implies being a complex at B)

Lemma 1.67 *The above sequence 1 is exact at B if and only if $Coker(f) = Coim(g)$*

A sequence is a **complex** (resp. **exact**) if it is a complex (resp. exact) at each (internal) term.

A **short exact sequence** is an exact sequence with five terms, the first and last of which are zeros – in other words, an exact sequence of the form :

$$0 \to A \to B \to C \to 0$$

Lemma 1.68 *Fix an abelian category $\mathscr{A}$. In this category,*

(i) $0 \to A \xrightarrow{f} B$ is exact if and only if $A \xrightarrow{f} B$ is a monomorphism. In this sense, $f = Im(f)$

(ii) $A \xrightarrow{f} B \to 0$ is exact if and only if $A \xrightarrow{f} B$ is an epimorphism. In this sense, $f = Coim(f)$

(iii) $0 \to A \to B \to 0$ is exact if and only if $A \to B$ is an isomorphism

(iv) $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ *is exact if and only if* f *is a kernel of* g

(V) $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *s exact if and only if* g *is a cokernel of* f

Proof: (i) The cokernel of $0 \to A$ is the identity map $A \xrightarrow{1_A} A$, which has as kernel $0 \to A$. So the image of $0 \to A$ is $0 \to A$. Thus $0 \to A \to B$ is exact if and only if $0 \to A$ is the kernel of $A \to B$.

So we are showing that being a monomorphism is the same as having kernel $0 \to A$. Suppose first that $A \xrightarrow{f} B$ has kernel $0 \to A$. Let g, h be two morphisms $Z \to A$ so that $f \circ g = f \circ h$. Then, by linearity, $f \circ (g - h) = 0$. By the universal property of kernels, $g - h$ factors through $0 \to A$, so $g - h$ must be the zero morphism. This implies $g = h$, so the defining property of monomorphisms is verified.

Conversely, suppose $A \xrightarrow{f} B$ is a monomorphism. Suppose $g : Z \to A$ is a morphism so that $f \circ g = 0$. Note that there is another map with this property, namely the zero morphism, $0 : Z \to A$. Since f is a monomorphism, $g = 0$. It follows that g factors uniquely through $0 \to A$. Since g was arbitrary, this verifies that $0 \to A$ has the universal property of the kernel.

(Note: So far, we have only used that $\mathscr{A}$ is additive and kernels and cokernels exist.)

(ii) Duality.

(iii) Suppose first that $A \to B$ is an isomorphism. Then $A \to B$ is both monic and epic, so the sequence is exact by parts (i) and (ii).

Suppose conversely that $0 \to A \to B \to 0$ is exact. Then $B \to 0$ is the cokernel of $A \to B$ and $0 \to A$ is the kernel of $A \to B$. This makes the image of $A \to B$ the morphism $id_B : B \to B$ and the coimage of $A \to B$ the morphism $id_A : A \to A$.

From (i) and (ii), we can get f is both a monomorphism and an epimorphism. By definition of abelian category, f is the kernel of its cokernel. So the image of $A \to B$ is $f : A \to B$. So f is an isomorphism.

□

homology : If (1) is a complex at B, then its homology at B (often denoted by H) is $ker g/im f$. (More precisely, there is some monomorphism $im f \to ker g$, and that H is the cokernel of this monomorphism.) Therefore, (1) is exact at B if and only if its homology at B is 0. We say that elements of $ker g$ are the **cycles**, and elements of $im f$ are the **boundaries** (so homology is "cycles mod boundaries").

If the complex is indexed in decreasing order, the indices are often written as subscripts, and H_i is the homology at $A_{i+1} \to A_i \to A_{i-1}$. If the complex is indexed in increasing order, the indices are often written as superscripts, and the homology H^i at $A^{i-1} \to A^i \to A^{i+1}$ is often called **cohomology**.

An exact sequence

$$A^\bullet \ : \ \cdots \to A^{i-1} \xrightarrow{f^{i-1}} A^i \xrightarrow{f^i} A^{i+1} \xrightarrow{f^{i+1}} \cdots \tag{2}$$

can be "factored" into short exact sequences

$$0 \to ker\ f^i \to A^i \to ker\ f^{i+1} \to 0$$

More generally, if (1) is assumed only to be a complex, then it can be "factored" into short exact sequences.

$$
\begin{aligned}
&0 \to ker\ f^i \to A^i \to im\ f^i \to 0 \\
&0 \to im\ f^{i-1} \to ker\ f^i \to H^i(A^\bullet) \to 0
\end{aligned} \tag{3}
$$

We also have :

$$
\begin{aligned}
&0 \to im\ f^i \to A^{i+1} \to coker\ f^i \to 0 \\
&0 \to H^i(A^\bullet) \to coker\ f^{i-1} \to im\ f^i \to 0
\end{aligned} \tag{4}
$$

(These are somehow dual to (3))

Remark 1.69 *Suppose*

$$0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} 0$$

is a complex of finite-dimensional k-vector spaces (often called $A^\bullet$ for short).

Define $h^i(A^\bullet) := dim\ H^i(A^\bullet)$.

Then $\sum(-1)^i dim A^i = \sum(-1)^i h^i(A^\bullet)$. In particular, if $A^\bullet$ is exact, then $\sum(-1)^i dim\ A^i = 0$.

Remark 1.70 *Suppose $\mathscr{C}$ is an abelian category. Define the category $Com_{\mathscr{C}}$ of complexes as follows. The objects are infinite complexes*

$$A^\bullet\ :\ \ \cdots \to A^{i-1} \xrightarrow{f^{i-1}} A^i \xrightarrow{f^i} A^{i+1} \xrightarrow{f^{i+1}} \cdots \tag{5}$$

in $\mathscr{C}$, and the morphisms $A^\bullet \to B^\bullet$ are commuting diagrams

$$
\begin{array}{ccccccc}
\cdots \to A^{i-1} & \xrightarrow{f^{i-1}} & A^i & \xrightarrow{f^i} & A^{i+1} & \xrightarrow{f^{i+1}} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\cdots \to B^{i-1} & \xrightarrow{g^{i-1}} & B^i & \xrightarrow{g^i} & B^{i+1} & \xrightarrow{g^{i+1}} & \cdots
\end{array} \tag{6}
$$

Then $Com_{\mathscr{C}}$ is an abelian category.

(Remark for experts: Essentially the same argument shows that the $\mathscr{C}^{\mathscr{I}}$ is an abelian category for any small category $\mathscr{I}$ and any abelian category $\mathscr{C}$. This immediately implies that the category of presheaves on a topological space X with values in an abelian category $\mathscr{C}$ is automatically an abelian category)

Remark 1.71 *(6) induces a map of homology $H^i(A^\bullet) \to H^i(B^\bullet)$*

Furthermore, H^i is a covariant functor $Com_{\mathscr{C}} \to \mathscr{C}$

Theorem 1.72 ***(Long exact sequence).*** *– A short exact sequence of complexes*

$$
\begin{tikzcd}
0^\bullet \arrow[d] & : & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
A^\bullet \arrow[d] & : & \cdots \arrow[r] & A^{i-1} \arrow[r] \arrow[d] & A^{i} \arrow[r] \arrow[d] & A^{i+1} \arrow[r] \arrow[d] & \cdots \\
B^\bullet \arrow[d] & : & \cdots \arrow[r] & B^{i-1} \arrow[r] \arrow[d] & B^{i} \arrow[r] \arrow[d] & B^{i+1} \arrow[r] \arrow[d] & \cdots \\
C^\bullet \arrow[d] & : & \cdots \arrow[r] & C^{i-1} \arrow[r] \arrow[d] & C^{i} \arrow[r] \arrow[d] & C^{i+1} \arrow[r] \arrow[d] & \cdots \\
0^\bullet & : & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}
$$

induces a ***long exact sequence in cohomology***

$$
\begin{aligned}
\cdots \quad &\to H^{i-1}(C^\bullet) \to \\
H^i(A^\bullet) \to H^i(B^\bullet) &\to H^i(C^\bullet) \to \\
H^{i+1}(A^\bullet) \to \quad &\cdots
\end{aligned}
\tag{7}
$$

Exactness of functors If $F : \mathscr{A} \to \mathscr{B}$ is a covariant additive functor from one abelian category to another, we say that F is **right-exact** if the exactness of

$$A' \to A \to A'' \to 0$$

in $\mathscr{A}$ implies that

$$F(A') \to F(A) \to F(A'') \to 0$$

is also exact.

Dually, we say that F is **left-exact** if the exactness of

$$0 \to A' \to A \to A'' \qquad \textit{implies}$$

$$0 \to F(A') \to F(A) \to F(A'') \qquad \textit{is exact}$$

A contravariant functor is **left-exact** if the exactness of

$$A' \to A \to A'' \to 0 \qquad \textit{implies}$$

$$0 \to F(A'') \to F(A) \to F(A') \qquad \textit{is exact}$$

Dually, we say that F is **right-exact** if the exactness of

$$0 \to A' \to A \to A'' \qquad \textit{implies}$$

$$F(A'') \to F(A) \to F(A') \to 0 \qquad \textit{is exact}$$

A covariant or contravariant functor is **exact** if it is both left-exact and right-exact.

Remark 1.73 *Suppose F is an exact functor. Then applying F to an exact sequence preserves exactness*

For example, if F is covariant, and $A' \to A \to A''$ is exact, then $FA' \to FA \to FA''$ is exact

Proof: Need to proof : If $F : \mathscr{C} \to \mathscr{D}$ is an exact functor between abelian categories and

$$\ldots \to X_n \xrightarrow{f_n} X_{n+1} \xrightarrow{f_{n+1}} X_{n+2} \to \ldots$$

is an exact sequence in $\mathscr{C}$, then its image

$$\ldots \to F\left(X_n\right) \xrightarrow{F(f_n)} F\left(X_{n+1}\right) \xrightarrow{F(f_{n+1})} F\left(X_{n+2}\right) \to \ldots$$

is exact.

We want to prove exactness at X_{n+1}. We can construct a commutative diagram:

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \arrow[dr] & & 0 & & \\
& \mathsf{Ker}\, f_n \arrow[dr] & & & & \mathsf{Im}\, f_{n+1} \arrow[ur] \arrow[dr] & & & \\
& & X_n \arrow[rr, "f_n"] \arrow[dr] & & X_{n+1} \arrow[rr, "f_{n+1}"] \arrow[ur] & & X_{n+2} \arrow[dr] & & \\
& & & \mathsf{Im}\, f_n \arrow[ur] \arrow[dr] & & & & \mathsf{Coker}\, f_{n+1} \arrow[dr] & \\
& & 0 \arrow[ur] & & 0 & & & & 0.
\end{tikzcd}$$

The diagonals are exact. (Proving this is a straightforward exercise.) The image of this diagram in $\mathscr{D}$:

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \arrow[dr] & & 0 & & \\
& F(\mathsf{Ker}\, f_n) \arrow[dr] & & & & F(\mathsf{Im}\, f_{n+1}) \arrow[ur] \arrow[dr] & & & \\
& & F(X_n) \arrow[rr, "F(f_n)"] \arrow[dr] & & F(X_{n+1}) \arrow[rr, "F(f_{n+1})"] \arrow[ur] & & F(X_{n+2}) \arrow[dr] & & \\
& & & F(\mathsf{Im}\, f_n) \arrow[ur] \arrow[dr] & & & & F(\mathsf{Coker}\, f_{n+1}) \arrow[dr] & \\
& & 0 \arrow[ur] & & 0 & & & & 0
\end{tikzcd}$$

also has exact diagonals. (Here we used the fact that $F(0) \cong 0$)

We can compute

$$\operatorname{Im} F\left(f_n\right) = \operatorname{Im}\left(F\left(X_n\right) \to F\left(\operatorname{Im} f_n\right) \to F\left(X_{n+1}\right)\right) = \operatorname{Im}\left(F\left(\operatorname{Im} f_n\right) \to F\left(X_{n+1}\right)\right)$$

since $F(X_n) \to F(\mathrm{Im}\, f_n)$ is epic. Also

$$\mathrm{Im}\,(F(\mathrm{Im}\, f_n) \to F(X_{n+1})) = \mathrm{Ker}\,(F(X_{n+1}) \to F(\mathrm{Im}\, f_{n+1})) = \mathrm{Ker}\,(F(X_{n+1}) \to F(\mathrm{Im}\, f_{n+1}) \to F(X_{n+2}))$$

since $F(\mathrm{Im}\, g) \to F(X_{n+2})$ is monic. Thus, $\mathrm{Im}\, F(f_n) = \mathrm{Ker}\, F(f_{n+1})$. □

Remark 1.74 *Suppose A is a ring, $S \subset A$ is a multiplicative subset, and M is an A-module.*

(a) Localization of A-modules $Mod_A \to Mod_{S^{-1}A}$ is an exact covariant functor

(b) $(\cdot) \otimes_A M$ is a right-exact covariant functor $Mod_A \to Mod_A$. (This is a repeat of Exercise 1.3.H.)

(c) $Hom(M, \cdot)$ is a left-exact covariant functor $Mod_A \to Mod_A$

If $\mathscr{C}$ is any abelian category, and $C \in \mathscr{C}$, then $Hom(C, \cdot)$ is a left-exact covariant functor $\mathscr{C} \to Ab$

(d) $Hom(\cdot, M)$ is a left-exact contravariant functor $Mod_A \to Mod_A$

If $\mathscr{C}$ is any abelian category, and $C \in \mathscr{C}$, then $Hom(\cdot, C)$ is a left-exact contravariant functor $\mathscr{C} \to Ab$

Example 1.75 *Suppose M is a **finitely presented** A**-module**: M has a finite number of generators, and with these generators it has a finite number of relations; or usefully equivalently, fits in an exact sequence*

$$A^{\oplus q} \to A^{\oplus q} \to M \to 0$$

We can use this exact sequence and the left-exactness of Hom to describe an isomorphism

$$S^{-1}Hom_A(M, N) \cong Hom_{S^{-1}A}(S^{-1}M, S^{-1}N)$$

Proof: From

$$A^{\oplus q} \to A^{\oplus q} \to M \to 0$$

We get

$$0 \to Hom_A(M, N) \to Hom_A(A^{\oplus q}, N) \to Hom_A(A^{\oplus q}, N)$$

$$0 \to S^{-1}Hom_A(M, N) \to S^{-1}Hom_A(A^{\oplus q}, N) \to S^{-1}Hom_A(A^{\oplus q}, N)$$

and

$$S^{-1}A^{\oplus q} \to S^{-1}A^{\oplus q} \to S^{-1}M \to 0$$

$$0 \to Hom_{S^{-1}A}(S^{-1}M, S^{-1}N) \to Hom_{S^{-1}A}(S^{-1}A^{\oplus q}, S^{-1}N) \to Hom_{S^{-1}A}(S^{-1}A^{\oplus q}, S^{-1}N)$$

So we just need to consider $M = A^{\oplus k}$

We know

$$Hom_A(A^{\oplus k}, N) \cong (Hom_A(A, N))^{\oplus k},\ Hom_A(A, N) \cong N$$

So

$$S^{-1}Hom_A(A^{\oplus k}, N) \cong S^{-1}(N^{\oplus k}) \cong Hom_{S^{-1}A}(S^{-1}A^{\oplus k}, S^{-1}N)$$

$$\frac{(n_1, \cdots, n_k)}{s} \mapsto (\frac{(a_1, \cdots, a_k)}{s} \mapsto \frac{a_1 n_1 + \cdots + a_k n_k}{s})$$

□

Example 1.76 *Hom doesn't always commute with localization :*

In the language of Example1.75, take $A = N = \mathbb{Z}$, $M = \mathbb{Q}$, and $S = \mathbb{Z} \setminus \{0\}$

$$S^{-1}Hom_A(M, N) = S^{-1}0 = 0, Hom_{S^{-1}A}(S^{-1}M, \ \ S^{-1}N) = Hom_{\mathbb{Q}}(\mathbb{Q}, \mathbb{Q}) = \mathbb{Q}$$

Remark 1.77 ***Interaction of homology and (right/left-)exact functors***

Suppose $F : \mathscr{A} \to \mathscr{B}$ is a covariant functor of abelian categories, and $C^\bullet$ is a complex in $\mathscr{A}$

(a) (F right-exact yields $\mathrm{FH}^\bullet \longrightarrow \mathrm{H}^\bullet\mathrm{F}$) If F is right-exact, describe a natural morphism $\mathrm{FH}^\bullet \to \mathrm{H}^\bullet\mathrm{F}$. (More precisely, for each i, the left side is F applied to the cohomology at piece i of $\mathrm{C}^\bullet$, while the right side is the cohomology at piece i of $\mathrm{FC}^\bullet$.)

(b) (F left-exact yields $\mathrm{FH}^\bullet \longleftarrow \mathrm{H}^\bullet\mathrm{F}$) If F is left-exact, describe a natural morphism $\mathrm{H}^\bullet\mathrm{F} \to \mathrm{FH}^\bullet$.

(c) (F exact yields $\mathrm{FH}^\bullet \overset{\sim}{\longleftrightarrow} \mathrm{H}^\bullet\mathrm{F}$) If F is exact, show that the morphisms of (a) and (b) are inverses and thus isomorphisms.

Hint for (a): use $\mathrm{C}^i \xrightarrow{d^i} \mathrm{C}^{i+1} \longrightarrow \mathrm{coker}\ d^i \longrightarrow 0$ to give an isomorphism $F\,coker\,d^i \cong\ coker\,F\,d^i$. Then use the first line of (4) to give a epimorphism $\mathrm{F}\,\mathrm{im}\mathrm{d}^i \longrightarrow \mathrm{im}\,\mathrm{Fd}^i$. Then use the second line of (4) to give the desired map $\mathrm{FH}^i\mathrm{C}^\bullet \longrightarrow \mathrm{H}^i\mathrm{FC}^\bullet$. While you are at it, you may as well describe a map for the fourth member of the quartet {coker, im, H, ker}: $\mathrm{F}\,\mathrm{kerd}^{\mathrm{i}} \longrightarrow \ker \mathrm{Fd}^i$

Remark 1.78 *If this makes your head spin, you may prefer to think of it in the following specific case, where both $\mathscr{A}$ and $\mathscr{B}$ are the category of A-modules, and F is $(\cdot) \otimes \mathrm{N}$ for some fixed N-module. Your argument in this case will translate without change to yield a solution to Remark 1.74(a) and (c) in general. If $\otimes N$ is exact, then N is called a flat A-module.*

For example, localization is exact (Remark 1.74), so $S^{-1}A$ is a flat A-algebra for all multiplicative sets S ($S^{-1}M \cong M \otimes_A S^{-1}A$). Thus taking cohomology of a complex of A-modules commutes with localization

Remark 1.79 ***Interaction of adjoints, (co)limits, and (left- and right-) exactness***

A surprising number of arguments boil down to the statement:

Limits commute with limits and right adjoints. In particular, in an abelian category, because kernels are limits, both right adjoints and limits are left-exact.

as well as its dual:

Colimits commute with colimits and left adjoints. In particular, because cokernels are colimits, both left adjoints and colimits are right-exact.

The latter has a useful extension:

In Mod_A, colimits over filtered index categories are exact.

Remark 1.80 ⋆⋆ *Caution. It is not true that in abelian categories in general, colimits over filtered index categories are exact.*

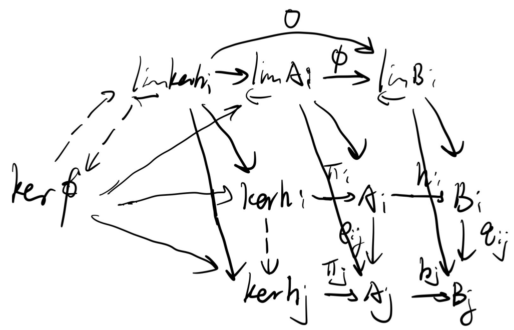
Fix a prime p. In the category Ab of abelian groups, for each positive integer n, we have an exact sequence $\mathbb{Z} \to \mathbb{Z}/(p^n) \to 0$. Taking the limit over all n in the obvious way, we obtain $\mathbb{Z} \to \mathbb{Z}_p \to 0$, which is certainly not exact.)

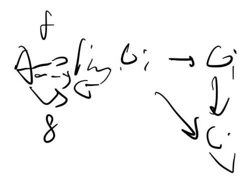
Theorem 1.81 ***(kernels commute with limits)***

Suppose $\mathscr{C}$ is an abelian category, and a: $\mathscr{I} \to \mathscr{C}$ and b: $\mathscr{I} \to \mathscr{C}$ are two diagrams in $\mathscr{C}$ indexed by $\mathscr{I}$. For convenience, let $A_i = a(i)$ and $B_i = b(i)$ be the objects in those two diagrams. Let $h_i : A_i \to B_i$ be maps commuting with the maps in the diagram.

(Translation: h is a natural transformation of functors $a \to b$) Then the $\ker h_i$ form another diagram in $\mathscr{C}$ indexed by $\mathscr{I}$. Describe a canonical isomorphism $\varprojlim \ker h_i \cong \ker\left(\varprojlim A_i \to \varprojlim B_i\right)$, assuming the limits exist.

Proof:





$\square$

Theorem 1.82 ***(Limits commute with limits)***

Let $F : I \times J \to C$ be a functor. If $\varprojlim_j F(i,j)$ exists for all $i \in I$ then we find that $\varprojlim_i \varprojlim_j F(i,j)$ exists iff $\varprojlim_{(i,j)} F(i,j)$ exists, in which case they are canonically isomorphic.

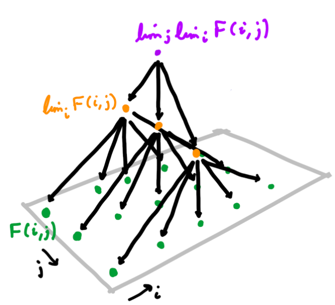
In particular

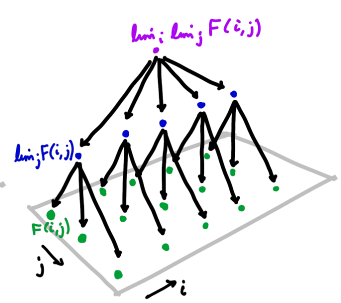
$$\varprojlim_i \varprojlim_j F(i,j) \cong \varprojlim_j \varprojlim_i F(i,j)$$

if both sides exist.

I should briefly clarify what $\varprojlim_i \varprojlim_j F(i,j)$ actually means. A morphism $f : i \to i'$ induces a natural transformation $F(i,-) \Rightarrow F(i',-)$, which induces a morphism $\varprojlim_j F(f,j) : \varprojlim_j F(i,j) \to \varprojlim_j F(i',j)$. So the limits_j *$\varprojlim_j F(i,j)$ in fact assemble into a functor $I \to C$, and $\varprojlim_i \varprojlim_j F(i,j)$ is the limit of this functor.*

Proof:





$\square$

Theorem 1.83 ***(right adjoints commute with limits)***

Suppose (F: $\mathscr{C} \to \mathscr{D}, \mathrm{G} : \mathscr{D} \to \mathscr{C}$*) is a pair of adjoint functors. If* $\mathrm{A} = \varprojlim_i \mathrm{A}_i$ *is a limit in* $\mathscr{D}$ *of a diagram indexed by* $\mathscr{I}$*, then* $\mathrm{GA} = \varprojlim_{\mathfrak{G}} \mathrm{GA}_i$ *(with the corresponding maps* $\mathrm{GA} \to \mathrm{GA}_i$ *) is a limit in* $\mathscr{C}$*.*

Proof: We must show that $\mathrm{GA} \to \mathrm{GA}_i$ satisfies the universal property of limits. Suppose we have maps $W \to \mathrm{GA}_i$ commuting with the maps of $\mathscr{I}$. We wish to show that there exists a unique $W \to GA$ extending the $W \to GA_i$. By adjointness of F and G, we can restate this as: Suppose we have maps $\mathrm{FW} \to A_i$ commuting with the maps of $\mathscr{I}$. We wish to show that there exists a unique $\mathrm{FW} \to \mathrm{A}$ extending the $\mathrm{FW} \to \mathrm{A}_i$. But this is precisely the universal property of the limit. □

Corollary 1.84 *If F and G are additive functors between abelian categories, and (F, G) is an adjoint pair, then (as kernels are limits and cokernels are colimits) G is left-exact and F is right-exact*

Example 1.85 *In Mod_A, colimits over filtered index categories are exact. (Your argument will apply without change to any abelian category whose objects can be interpreted as "sets with additional structure".) Right-exactness follows from the above discussion, so the issue is left-exactness.*

(Possible hint: After you show that localization is exact, or stalkification is exact, in a hands-on way, you will be easily able to prove this)

Example 1.86 *Filtered colimits commute with homology in* Mod_A

Hint: use the FHHF Theorem, and the previous Exercise

Remark 1.87 *Suppose*

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_{n+1} \arrow[r] \arrow[d] & B_{n+1} \arrow[r] \arrow[d] & C_{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_0 \arrow[r] \arrow[d] & B_0 \arrow[r] \arrow[d] & C_0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

is an inverse system of exact sequences of modules over a ring, such that the maps $A_{n+1} \to A_n$ are surjective. (We say: "transition maps of the left term are surjective".)

Then the limit

$$0 \longrightarrow \varprojlim A_n \longrightarrow \varprojlim B_n \longrightarrow \varprojlim C_n \longrightarrow 0$$

is also exact. (You will need to define the maps in)

Remark 1.88 *Based on these ideas, you may suspect that rightexact functors always commute with colimits. The fact that tensor product commutes with infinite direct sums may reinforce this idea. Unfortunately, it is not true - "double dual"* $\cdot^{\vee\vee} : Vec_k \to Vec_k$ *is covariant and right exact (why is it right-exact?), but does not commute with infinite direct sums, as* $\prod_{i=1}^{\infty}(k^{\vee\vee})$ *is not isomorphic to* $(\prod_{i=1}^{\infty} k)^{\vee\vee}$.

2 Sheaves

2.1 Sheaf and presheaf

Def 2.1 ***Presheaves***

Presheaves are a way of keeping track of algebraic data on a topological space. More precisely:

Let X be a topological space and $\mathcal{C}$ a category. A presheaf $\mathcal{F}$ of $\mathcal{C}$ on X consists of

(1) For all $U \subseteq X$ open, an object $\mathcal{F}(U)$ in $\mathcal{C}$.

(2) For all $V \subseteq U \subseteq X$ open, a morphism $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ in $\mathcal{C}$ such that

(i) For all $U \subseteq X$ open, $\rho_{UU} : \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity.

(ii) For all $W \subseteq V \subseteq U \subseteq X$ open, we have $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$, that is, the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\rho_{UV}"] \arrow[rr, bend left, "\rho_{UW}"] & \mathcal{F}(V) \arrow[r, "\rho_{VW}"] & \mathcal{F}(W) \end{tikzcd}$$

*Elements $s \in \mathcal{F}(U)$ are called **sections** of $\mathcal{F}$ over U. The object $\mathcal{F}(U)$ is called space of sections of $\mathcal{F}$ over U.*

*- Elements $s \in \mathcal{F}(X)$ are called **global sections** of $\mathcal{F}$. The object $\mathcal{F}(X)$ is called space of global sections of $\mathcal{F}$.*

- Alternative notations for $\mathcal{F}(U)$ are $\Gamma(U, \mathcal{F})$ and $H^0(U, \mathcal{F})$.

*- The morphisms ρ_{UV} are called **restriction maps**. For $V \subseteq U$ open and $s \in \mathcal{F}(U)$, we will sometimes write $s|_V$ for $\rho_{UV}(s)$ and call $s|_V$ the **restriction** of s (from U to V).*

Example 2.2 *Let X be a topological space.*

*(1) For any object A in a category $\mathcal{C}$, the **constant presheaf** with value A is the presheaf A' with $\underline{A}'(U) = A$ and $\rho_{UV} = \mathrm{id}_A$ for all $V \subseteq U \subseteq X$ open.*

*(2) The **presheaf of continuous functions** on X is defined by setting $\mathcal{C}^0(U) := C^0(U) := \{f : U \to \mathbb{R} \mid f$ continuous $\}$ and $\rho_{UV} : \mathcal{C}^0(U) \to \mathcal{C}^0(V)$, $f \mapsto f|_V$. This can be considered as a presheaf of sets, abelian groups, or even rings.*

(3) More generally, if $\pi : Y \to X$ is a continuous map of topological spaces, we can look at the presheaf (of sets) of continuous sections of π defined as $\mathcal{F}_\pi(U) := \{s : U \to Y \mid s$ continuous, $\pi \circ s = \mathrm{id}_U\}$ with ρ_{UV} the obvious restriction maps.

Remark 2.3 *Let X be a topological space and let Ouv_X be the category of open subsets of X, that is, the category with:*

- Objects: $U \subseteq X$ open

- Morphisms:

$$\mathrm{Hom}(U, V) = \begin{cases} \emptyset \text{ if } U \not\subseteq V \\ \iota \text{ if } U \subseteq V \text{ where } \iota \text{ denotes the inclusion} \end{cases}$$

Then, a presheaf on X with values in a category $\mathcal{C}$ is the same as a contravariant functor from Ouv_X *to* $\mathcal{C}$.

Def 2.4 *The* ***germ*** *of a differentiable function. Before we do, we first give another definition, that of the germ of a differentiable function at a point $p \in X$. Intuitively, it is a "shred" of a differentiable function at p. Germs are objects of the form $\{(f,\ open\ U) : p \in U, f \in \mathscr{O}(U)\}$ modulo the relation that $(f,U) \sim (g,V)$ if there is some open set $W \subset U,V$ containing p where $f|_W = g|_W$ (i.e., $res_{U,W} f = res_{V,W} g$). In other words, two functions that are the same in an open neighborhood of p (but may differ elsewhere) have the same germ. We call this set of germs the* ***stalk*** *at p, and denote it $\mathscr{O}_p$. Notice that the stalk is a ring: you can add two germs, and get another germ: if you have a function f defined on U, and a function g defined on V, then $f+g$ is defined on $U \cap V$. Moreover, $f+g$ is well-defined: if $\tilde{f}$ has the same germ as f, meaning that there is some open set W containing p on which they agree, and $\tilde{g}$ has the same germ as g, meaning they agree on some open W' containing p, then $\tilde{f}+\tilde{g}$ is the same function as $f+g$ on $U \cap V \cap W \cap W'$.*

Notice also that if $\mathrm{p} \in \mathrm{U}$*, you get a map* $\mathscr{O}(\mathrm{U}) \to \mathscr{O}_{\mathrm{p}}$. *Experts may already see that we are talking about germs as colimits.*

We can see that $\mathscr{O}_{\mathrm{p}}$ *is a local ring as follows. Consider those germs vanishing at p, which we denote* $\mathfrak{m}_{\mathfrak{p}} \subset \mathscr{O}_{\mathfrak{p}}$. *They certainly form an ideal:* $\mathfrak{m}_{\mathfrak{p}}$ *is closed under addition, and when you multiply something vanishing at p by any function, the result also vanishes at p. We check that this ideal is maximal by showing that the quotient ring is a field:*

$$0 \longrightarrow \mathfrak{m}_{\mathfrak{p}} := \textit{ ideal of germs vanishing at } \mathrm{p} \longrightarrow \mathscr{O}_{\mathfrak{p}} \overset{f \mapsto f(p)}{\longrightarrow} \mathbb{R} \longrightarrow 0$$

Def 2.5 ***germ, stalk***

Define the stalk of a presheaf $\mathcal{F}$ at a point p to be the set of germs of $\mathcal{F}$ at p, denoted $\mathcal{F}_p$, as in the example above. Germs correspond to sections over some open set containing p, and two of these sections are considered the same if they agree on some smaller open set. More precisely: the stalk is

$$\{(\textit{ open } \mathrm{U}, \mathrm{f}) : \mathrm{p} \in \mathrm{U}, \mathrm{f} \in \mathcal{F}(\mathrm{U})\}$$

modulo the relation that $(U,f) \sim (V,g)$ if there is some open set $W \subset U,V$ where $p \in W$ and $\mathrm{res}_{U,W} f = res_{\ V,W} g$. *Write $\overline{(U,f)}$ for the equivalence class of (U,f) in the stalk*

A useful equivalent definition of a ***stalk*** *is as a colimit of all* $\mathcal{F}(\mathrm{U})$ *over all open sets* U *containing* p *:*

$$\mathcal{F}_{\mathrm{p}} = \varinjlim_{p \in U} \mathcal{F}(\mathrm{U}).$$

(The index category is a filtered set (given any two such open sets, there is a third such set contained in both))

Gien $s \in \mathcal{F}(U)$ and $x \in U \subseteq X$, then we define the ***the germ of*** *s* ***at*** *x to be the image of s in $\mathcal{F}_p$: $s_x = \overline{(U,s)} \in \mathcal{F}_x$ If $\mathcal{B}$ is a basis for topology on X then we can rewrite the stalk as*

$$\mathcal{F}_x \cong \varinjlim_{x \in U, U \in \mathcal{B}} \mathcal{F}(U),$$

Def 2.6 ***Sheaf***

A presheaf is a sheaf if it satisfies two more axioms. Notice that these axioms use the additional information of when some open sets cover another.

Identity axiom*. If* $\{\mathrm{U}_i\}_{i\in I}$ *is an open cover of* U*, and* $f_1, f_2 \in \mathcal{F}(U)$*, and* $res_{U,U_i} f_1 = res_{U,U_i} f_2$ *for all* i*, then* $f_1 = f_2$*.*

(A presheaf satisfying the identity axiom is called a separated presheaf, but we will not use that notation in any essential way.)

Gluability axiom*. If* $\{\mathrm{U}_i\}_{\mathrm{i}\in\mathrm{I}}$ *is an open cover of* U*, then given* $\mathrm{f}_i \in \mathcal{F}(\mathrm{U}_i)$ *for all* i*, such that* $res_{U_i,U_i\cap U_j} f_i = res_{U_j,U_i\cap U_j} f_j$ *for all* i, j*, then there is some* $f \in \mathcal{F}(U)$ *such that* $res_{U,U_i} f = f_i$ *for all* i*.*

(In this case, identity means there is at most one way to glue, and gluability means that there is at least one way to glue)

The stalk of a sheaf at a point is just its stalk as a presheaf - the same definition applies - and similarly for the germs of a section of a sheaf.

Remark 2.7 *Interpretation in terms of the equaliser exact sequence. The two axioms for a presheaf to be a sheaf can be interpreted as "exactness" of the "equalizer exact sequence":*

$$\cdot \longrightarrow \mathcal{F}(\mathrm{U}) \longrightarrow \prod \mathcal{F}(\mathrm{U_i}) \rightrightarrows \prod \mathcal{F}(\mathrm{U_i} \cap \mathrm{U_j})$$

Identity is exactness at $\mathcal{F}(\mathrm{U})$*, and gluability is exactness at* $\prod \mathcal{F}(\mathrm{U_i})$*.*

Example 2.8 *Let* X *be a topological space.*

(1) (An additional axiom sometimes included is that $\mathcal{F}(\varnothing)$ *is a one-element set, and in general, for a sheaf with values in a category,* $\mathcal{F}(\varnothing)$ *is required to be the final object in the category. This actually follows from the above definitions, assuming that the empty product is appropriately defined as the final object.)*

If $\mathcal{F}$ *is a sheaf of sets (or abelian groups, rings, etc.), then, by Identity axiom of Def2.6 applied to* $I = \emptyset$*, the underlying set of* $\mathcal{F}(\emptyset)$ *has at most one element, and by Gluability axiom, it has at least one element.*

If U *and* V *are disjoint, then* $\mathcal{F}(\mathrm{U} \cup \mathrm{V}) = \mathcal{F}(\mathrm{U}) \times \mathcal{F}(\mathrm{V})$*. Here we use the fact that* $\mathcal{F}(\varnothing)$ *is the final object.*

(2) The presheaves $\mathcal{C}^0$ *(and* $\mathcal{C}^\infty$ *if* $X = \mathbb{R}^n$ *) on* X *are sheaves, since:*

(i) Two continuous functions f, g *that coincide on an open cover of* $U \subseteq X$ *coincide at every point, hence they are equal.*

(ii) Given $U \subseteq X$ *open, a cover* $U = \cup_{i\in I} U_i$*, and continuous functions* $f_i : U_i \to \mathbb{R}$ *that agree on* $U_i \cap U_j$*, the function*

$$f : U \to \mathbb{R}$$

$$x \mapsto f_i(x) \text{ if } x \in U_i$$

is well-defined. By construction, $f|_{U_i} = f_i$ and f is continuous (resp. differentiable), since continuity (resp. differentiability) at any point can be checked on an open neighborhood.

*(3)**constant sheaf***

let $\mathcal{F}(\mathrm{U})$ be the maps to S that are locally constant, i.e., for any point p in U, there is an open neighborhood of p where the function is constant. Show that this is a sheaf. (A better description is this: endow S with the discrete topology, and let $\mathcal{F}(\mathrm{U})$ be the continuous maps $\mathrm{U} \to \mathrm{S}$.) This is called the constant sheaf (associated to S); (do not confuse it with the constant presheaf)

We denote this sheaf $\underline{S}$

*(4)**morphisms glue***

Suppose Y is a topological space. Show that "continuous maps to Y " form a sheaf of sets on X. More precisely, to each open set U of X, we associate the set of continuous maps of U to Y. Show that this forms a sheaf. ((2), with $Y = \mathbb{R}$, and (3), with $Y = S$ with the discrete topology, are both special cases)

Example 2.9 *Important Example: **Restriction of a sheaf***

Suppose $\mathcal{F}$ is a sheaf on X, and U is an open subset of X. Define the restriction of $\mathcal{F}$ to U, denoted $\mathcal{F}|_{\mathrm{U}}$, to be the collection $\mathcal{F}|_{\mathrm{U}}(\mathrm{V}) = \mathcal{F}(\mathrm{V})$ for all open subsets $V \subset \mathrm{U}$. Clearly this is a sheaf on U.

Example 2.10 *Important Example: the **skyscraper sheaf***

Suppose X is a topological space, with $p \in X$, and S is a set. Let $i_p : p \to X$ be the inclusion. Then $i_{p,*}S$ defined by

$$i_{p,*}S(U) = \begin{cases} S & \text{if } p \in U, \text{ and} \\ \{e\} & \text{if } p \notin U \end{cases}$$

forms a sheaf. Here $\{e\}$ is any one-element set. This is called a skyscraper sheaf.

(Mild caution: this informal picture suggests that the only nontrivial stalk of a skyscraper sheaf is at p, which isn't the case)

There is an analogous definition for sheaves of abelian groups, except $i_{p,*}(S)(U) = \{0\}$ if $p \notin U$; and for sheaves with values in a category more generally, $i_{p,*}S(U)$ should be a final object

Theorem 2.11 *Let A be a ring, and let the basis $\mathcal{B}$ of $Spec\,A$ be the one made of principle open subsets, then $\mathscr{O}_{Spec\,A}$ is a sheaf on $\mathcal{B}$.*

We have a sheaf of rings on $Spec\,A$ which on principal open subsets $D(f)$ is simply

$$\mathcal{O}_{Spec\,A}(D(f)) = \mathcal{O}(D(f)) = A[f^{-1}].$$

Remark 2.12 $\star$ ***The space of sections (espace étalé) of a (pre)sheaf.***

Suppose $\mathcal{F}$ is a presheaf (e.g., a sheaf) on a topological space X. Construct a topological space F along with a continuous map $\pi : \mathrm{F} \to \mathrm{X}$ as follows: as a set, F is the disjoint union of all the stalks of $\mathcal{F}$. This naturally gives a map of sets $\pi : F \to X$. Topologize F as follows. Each s in $\mathcal{F}(\mathrm{U})$ determines *a subset $\{(x, s_x) : x \in U\}$ of F. The topology on F is the weakest topology such that these subsets are open. (These subsets form a base of the topology. For each $y \in F$, there is an open neighborhood V of y and an open neighborhood* U *of* π(y) *such that* $\pi|_V$ *is a homeomorphism from* V *to* U.*) The topological space F could be thought of as the space of sections of $\mathcal{F}$ (and in French is called the espace étalé of $\mathcal{F}$)*

Def 2.13 ***The pushforward sheaf/direct image sheaf***

Suppose $\pi : X \to Y$ is a continuous map, and $\mathcal{F}$ is a presheaf on X. Then define $\pi_*\mathcal{F}$ by $\pi_*\mathcal{F}(V) = \mathcal{F}\left(\pi^{-1}(V)\right)$, where V is an open subset of Y. Show that $\pi_*\mathcal{F}$ is a presheaf on Y, *and is a sheaf if $\mathcal{F}$ is. This is called the **pushforward** or **direct image** of $\mathcal{F}$. More precisely, $\pi_*\mathcal{F}$ is called the **pushforward of** $\mathcal{F}$ **by** π.*

Once we realize that sheaves form a category, we will see that the pushforward is a functor from sheaves on X to sheaves on Y (see next subsection).

pushforward induces maps of stalks

Suppose $\pi : X \to Y$ is a continuous map, and $\mathcal{F}$ is a sheaf of sets (or rings or A-modules) on X. If $\pi(p) = q$, we can describe a natural morphism of stalks $(\pi_*\mathcal{F})_q \to \mathcal{F}_p$. (Use the explicit definition of stalk using representatives, or the universal property)

(If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheavesy then for each $x \in X$ we have a well-defined induced map $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ defined by $\overline{(U, s)} \to \overline{(U, \phi_U(s))}$. The well-definedness of this map comes from the naturality properties of morphisms of presheaves)

Example 2.14 *As the notation suggests, the skyscraper sheaf can be interpreted as the pushforward of the constant sheaf $\underline{S}$ on a one-point space p, under the inclusion morphism $i_p : \{p\} \to X$.*

2.2 Morphisms of presheaves and sheaves

Def 2.15 *Let X be a topological space. A* ***morphism*** $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$ ***of presheaves of*** $\mathcal{C}$ ***on*** X *consists of a collection of morphisms $\varphi(U) : \mathcal{F}_1(U) \to \mathcal{F}_2(U)$ in $\mathcal{C}$ for every $U \subseteq X$ open compatible with restrictions, i.e. such that for all $V \subseteq U \subseteq X$ open, the following diagram commutes:*

$$\begin{tikzcd} \mathcal{F}_1(U) \arrow[r, "\varphi(U)"] \arrow[d, "\rho_{1,UV}"'] & \mathcal{F}_2(U) \arrow[d, "\rho_{2,UV}"] \\ \mathcal{F}_1(V) \arrow[r, "\varphi(V)"'] & \mathcal{F}_2(V) \end{tikzcd}$$

A morphism $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$ of sheaves $\mathcal{F}_i$ is a morphism $\mathcal{F}_1 \to \mathcal{F}_2$ between the $\mathcal{F}_i$ considered as presheaves.

Example 2.16 *Let X be a topological space.*

(1) Every morphism $f : A \to B$ of objects in $\mathcal{C}$ yields a morphism of presheaves $\varphi : \underline{A}' \to \underline{B}'$ by setting $\varphi(U) := f$.

(2) If $X = \mathbb{R}^n$, the association $\mathcal{C}^\infty(U) := C^\infty(U)$ with the usual restriction maps defines a presheaf and the inclusions $C^\infty(U) \subseteq C^0(U)$ define a morphism of presheaves $\mathcal{C}^\infty \to \mathcal{C}^0$.

(3) If $Y_2 \overset{\pi_2}{\to} Y_1 \overset{\pi_1}{\to} X$ are continuous maps of topological spaces, then there is an associated morphism $\mathcal{F}_{\pi_1 \circ \pi_2} \to \mathcal{F}_{\pi_1}$ of the presheaves of continuous sections.

Remark 2.17 *Note that there is a forgetful functor $\iota : \mathrm{Sh}_{\mathcal{C}}(X) \to \mathrm{PSh}_{\mathcal{C}}(X)$ that simply forgets the sheaf axioms. By our definition of morphism of sheaves, this functor is fully faithful.*

2.3 Properties determined at the level of stalks, and sheafification

Proposition 2.18 *Let X be a space with presheaves $\mathcal{F}$ and $\mathcal{G}$, and let $\phi, \psi : \mathcal{F} \to \mathcal{G}$ be morphisms of presheaves.*

1. If $\mathcal{F}$ is a sheaf and $U \subseteq X$ is open, then

$$\begin{aligned} \rho_U : \mathcal{F}(U) &\longrightarrow \textstyle\prod_{x \in U} \mathcal{F}_x \\ s &\longmapsto (s_x)_{x \in U} \end{aligned}$$

is injective

2. If $\mathcal{F}$ is a sheaf, then ϕ_U is injective for all open $U \subset X$ if and only if $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$, i.e. sections of sheaves are deterrmined by their germs.

3. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then ϕ_U is bijective for all open $U \subset X$ if and only if $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is bijective for all $x \in X$

4.If $\mathcal{G}$ is a sheaf, then $\phi = \psi$ if and only if $\varphi_x = \psi_x$ for all $x \in X$

Warning: surjectivity of the stalk function is a big deal. We can describe the failure in an equivalence of surjectivity in terms of homological algebra, and this is where we get sheaf cohomology from

Def 2.19 *Let X be a topological space, and let $\mathcal{F}$ and $\mathcal{G}$ be sheaves, and let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves.*

*1. The maps ϕ is called **injective** (resp.**bijective**) if for all $x \in X$ $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective (resp. bijective). This is eguivalent to ϕ_U being injective (resp.bijective) for all open subsets $U \subseteq X$*

*2. The map ϕ is called **surjective** if for all $x \in X$ $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective. This is not equivalent to ϕ_U being surjective for all $U \subseteq X$*

Remark 2.20 *surjectivity as sheaves does not imply surjectivity as presheaves :*

$f : \mathbb{R} \to S^1, r \mapsto e^{2\pi i r}$

define $\forall U \in Ouv_{S^1}, \mathcal{E}(U) = \{g : U \to \mathbb{R} | g \text{ continous and } f \circ g = id_U\}$, *also consider constant sheaf* $\forall U \in Ouv_{S^1}, \mathcal{C}(U) = \{id : U \to U\}$

define the map $\mathcal{E} \to \mathcal{C}$ *takes* $g \in \mathcal{E}(U) \mapsto f \circ g$

the suriective map of sheaves $(\mathcal{E}_x = \{g_n | g_n(x) = x + 2\pi i n\} \to \mathcal{C}_x = \{id_x\})$, *but* $\mathcal{E}(S^1) = \emptyset$ *while* $\mathcal{C}(S^1) = \{*\}$

Proposition 2.21 *Let X be a topological space and $\mathcal{F}$ be a presheaf on X, then there exists a sheaf $\widetilde{\mathcal{F}}$ with a morphism of presheaves $\iota_{\widetilde{\mathcal{F}}} : \mathcal{F} \to \widetilde{\mathcal{F}}$ such that for every morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$ where $\mathcal{G}$ is a sheaf, there exists a unique morphism $\overline{\phi} : \widetilde{\mathcal{F}} \to \mathcal{G}$ such that $\phi = \overline{\phi} \circ \iota_{\widetilde{\mathcal{F}}}$. The following also holds:*

1. The map $\iota_{\widetilde{\mathcal{F}}}$ induces bijections on stalks

2. The pair $(\widetilde{\mathcal{F}}, \iota_{\widetilde{\mathcal{F}}})$ is unique up to unique isomorphism

3. The pair $(\widetilde{\mathcal{F}}, \iota_{\widetilde{\mathcal{F}}})$ is natural in the presheaf variable $\mathcal{F}$ and morphisms of presheaves

4. The assignment $\mathcal{F} \mapsto \widetilde{\mathcal{F}}$ and $\phi \mapsto \overline{\phi}$ is a functor, left adjoint to the inclusion functor from the category of sheaves into the category of presheaves

$$Hom_{Psh}(\mathcal{F}, Forget(\mathcal{G})) \cong Hom_{Shv}(\widetilde{\mathcal{F}}, \mathcal{G})$$

Proof: For some open set $U \subseteq X$ we define

$$\widetilde{\mathcal{F}}(U) = \left\{ (s_x)_{x\in U} \in \prod_{x\in U} \mathcal{F}_x \right\} \forall x \in U, \exists V \in \mathrm{Ouv}(U) \text{ such that } x \in V, t \in \mathcal{F}(V) : \forall y \in V, s_y = t_y\}.$$

$\square$

Proposition 2.22 *Let A be a ring, $X = \operatorname{Spec} A$ and $x \in X$, then*

$$\mathcal{O}_{X,x} \cong A_{\mathfrak{p}_x}$$

Proof: Consider the basis for X of principal open subsets, so $\mathcal{B} = \{D(f) \subseteq X | f \in A\}$ Then we can rewrite the definition of the stalk of the structure sheaf at a point $x \in X$ as

$$\mathcal{O}_X|_x = \mathrm{colim}_{x\in U, U=D(f)} \mathcal{O}_X(U) = \mathrm{colim}_{x\in U\in\mathcal{B}} A[f^{-1}] =: B.$$

that $B \cong A_{\mathfrak{p}_x}$, The above colimit is taken over the structure maps $\frac{a}{f^n} \longmapsto \frac{a}{f^n} = \frac{ag^n}{(fg)^n}$ $\square$

2.4 Sheaves of abelian groups, and $\mathscr{O}_X$-modules, form abelian categories

Proposition 2.23 *Let I be a filtered partially ordered set. Then for each I-indexed inductive system*

$$0 \longrightarrow A_i \xrightarrow{\alpha_i} B_i \xrightarrow{\beta_i} C_i \longrightarrow 0$$

of short exact sequences of abelian groups, the sequence

$$0 \longrightarrow \varinjlim_{i \in I} A_i \longrightarrow \varinjlim_{i \in I} B_i \longrightarrow \varinjlim_{i \in I} C_i \longrightarrow 0$$

of colimits is again exact

Theorem 2.24 *Let X be a topological space*

The category $Sh_{ab}(X)$ of sheaves of abelian groups on X is an abelian category with all colimits and limits

Proposition 2.25 *A morphism $f : \mathcal{F} \to \mathcal{G}$ of sheaves of abelian groups is surjective on each stalk if and only if f is an epimorphism in $Sh_{ab}(X)$*

Example 2.26 *Let $X = S^1 \subset R^2$ with upper and lower hemisphere $i_+ : D_+ \to S^1, i_- : D_- \to S^1$. Set $i : D_+ \cap D_- \to S^1$, and $F := i_{-,*}\underline{\mathbb{Z}} \oplus i_{+,*}\underline{\mathbb{Z}}$, $G := i_*(\underline{\mathbb{Z}})$. We can construct an epimorphism $\mathcal{F} \to \mathcal{G}$ such that $\mathcal{F}(X) \to \mathcal{G}(X)$ is not surjective. Here, $\underline{\mathbb{Z}} = C(-, \underline{\mathbb{Z}})$ denotes the constant sheaf with value on $\mathbb{Z}$ on the respective topological spaces*

2.5 The inverse image sheaf

Def 2.27 *Let $f : X \to Y$ be a continuous map of topological spaces, and let $\mathcal{F}$ be a presheaf on X and $\mathcal{G}$ be a presheaf on Y*

1. *The* ***pushforward*** $\boldsymbol{f_*\mathcal{F}}$ *is the presheaf on Y defined by $(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$ for $V \in Ouv(Y)$*
2. *The* ***pullback*** $\boldsymbol{f^+G\ (f^*G)}$ *is the presheaf on X defined by $(f^+\mathcal{G})(U) = colim_{f(U)\subset V, V \in Ouv(Y)}\mathcal{G}(V)$ for an open set $U \subset X$*

Proposition 2.28 *We have the following natural correspondence :*

$$\boldsymbol{PreSh(X)(f^+\mathcal{G}, \mathcal{F}) \cong PreSh(Y)(\mathcal{G}, f_*\mathcal{F})}$$

We say that f^+ is left adjoint to f_*, it's right adjoint

Recall that the fact we have an adjoint pair of functors tells us a lot of information, like left adjoint preserve all colimits and right adjoints preserve all limits, and stuff like that

Proposition 2.29 *Let $x \in X$, then we have a natural identification $(f^+\mathcal{G})_x = \mathcal{G}_{f(x)}$*

Def 2.30 *Let $f : X \to Y$ be a continuous map of topological spaces, and let $\mathcal{G}$ be a sheaf on Y*

$\boldsymbol{f^{-1}\mathcal{G} = \widetilde{f^+\mathcal{G}}}$ *is the sheafification of $f^+\mathcal{G}$*

Proposition 2.31 *Let $f : X \to Y$ be a continuous map of topological spaces, and let $\mathcal{F}$ be a sheaf on X and $\mathcal{G}$ be a sheaf on Y*

1. *$f_*\mathcal{F}$ is a sheaf on Y*
2. *We have the following adjunction :* $\boldsymbol{Sh(X)(f^{-1}G, F) \cong Sh(Y)(G, f_*F)}$
3. *For all $x \in X$ we have $(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}$*

Corollary 2.32 *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces, then*

$$Hom\left((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)\right) \cong \left\{f : X \to Y \text{ and } f^{\#} : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X\right\} \cong \left\{f : X \to Y \text{ and } f^{\flat} : \mathcal{O}_Y \to f_*\mathcal{O}_X\right\}$$

2.6 Recovering sheaves from a "sheaf on a base"

Theorem 2.33 *Let X be a topological space and let $\mathcal{B}$ be a basis of the topology of X, stable under finite intersections. Let $Sh_{\mathcal{B}}(X)$ be the category of sheaves on the basis B as defined in the lecture. The functors*

$$Sh(X) \longrightarrow Sh_{\mathcal{B}}(X), \qquad (\mathcal{F} : O_{UV}(X)^{op} \to Sets) \longmapsto (\mathcal{F}|_{\mathcal{B}^{op}} : \mathcal{B}^{op} \to Sets)$$

and

$$Sh_{\mathcal{B}}(X) \longrightarrow Sh(X), \qquad \mathcal{F} \longmapsto (U \mapsto \lim_{\substack{\leftarrow \\ V \subseteq U, V \in \mathcal{B}}} \mathcal{F}(V))$$

are inverse equivalences of categories

3 Scheme

3.1 Locally Ringed Spaces and Schemes

Def 3.1 *A ring A is a* ***local ring*** *if it has a unique maximal deal $\mathfrak{m}_A$*

In this case. note that all the elements of $A \setminus \mathfrak{m}_A$ are invertible. Conversely, if A is a ring and $I \subseteq A$ is an ideal such that all elements of $A \backslash I$ are invertible, then A is a local ring with $\mathfrak{m}_A = I$

Def 3.2 *A spectral space X is called* ***local*** *if it has a unique closed point*

Lemma 3.3 *A ring A is a local ring if and only if $\operatorname{Spec} A$ is a local spectral space*

Proof: The closed points of an affine scheme Spec A are exactly the maximal ideals $\square$

Def 3.4 *A morphism $\phi : A \to B$ of local rings is a* ***local morphism*** *if $\phi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$ (so it preserves the local structure)*

A map $f : X \to Y$ of local spectral spaces is called ***local*** *if it maps the closed point of X to the closed point of Y*

Example 3.5 *The map $\mathbb{Z}_{(p)} \to \mathbb{Q}$ is not a local map*

Lemma 3.6 *A morphism of local rings $\phi : A \to B$ is local if and only if $\phi : \operatorname{Spec} B \to \operatorname{Spec} A$ is local.*

Def 3.7 *Given a ring A and a prime ideal $\mathfrak{p} \subseteq A$, then the* ***localisation of*** *A* ***at*** *$\mathfrak{p}$ is*

$$A_{\mathfrak{p}} = A[(A \setminus \mathfrak{p})^{-1}].$$

Given a spectral space X and $x \in X$. then the localisation of X at x is

$$X_x = \bigcap_{x \in U} U$$

Proposition 3.8 *Given a spectral space X, a point $x \in X$, a ring A and prime ideal $\mathfrak{p} \subseteq A$*

1. *The ring $A_{\mathfrak{p}}$ is a local ring with marimal ideal $\mathfrak{p}A_{\mathfrak{p}}$*
2. *The spectral space X_x is a local spectral space*
3. *If $X = \operatorname{Spec} A$, then we have a map $A \to A_{\mathfrak{p}}$ and $\operatorname{Spec} A_{\mathfrak{p}} \cong X_x$ where x corresponds to $\mathfrak{p}$ in $X = \operatorname{Spec} A$*

Proof: For part 3, we rewrite $A_{\mathfrak{p}}$ as the following filtered colimit

$$\operatorname{colim}_{f \notin \mathfrak{p}} A[f^{-1}].$$

This implies that Spec $A_{\mathfrak{p}}$ can be rewritten as the following cofiltered limit

$$\operatorname{Spec} A_{\mathfrak{p}} = \lim_{f \notin \mathfrak{p}} \operatorname{Spec} A[f^{-1}] = \lim_{f \notin \mathfrak{p}} D(f) = \bigcap_{f \notin \mathfrak{p}} D(f) = X_x$$

$\square$

Def 3.9 *1. A **ringed space** is a pair* $(X, \mathcal{O}_X)$ *of a topological space* X *and a sheaf of rings* $\mathcal{O}_X$

*2. A **locally ring space** is a ring space* $(X, \mathcal{O}_X)$ *, such that for all* $x \in X$*, the stalk* $\mathcal{O}_{X,x}$ *at* x *is a local ring*

3. If A *is a ring, then **Spec A** is the locally ringed space* $(Spec\, A, \mathcal{O}_{\mathrm{Spec}\, A})$

Def 3.10 *Let* $(X, \mathcal{O}_X)$ *and* $(Y, \mathcal{O}_Y)$ *be ringed spaces, then a **map of ringed spaces** is a continuous map of underlying topological spaces* $f : X \to Y$ *plus a map* $f^{\#}_{V\to U} : O_Y(V) \to O_X(U)$ *for all open* $U \subset X, V \subset Y$ *and* $f(U) \subset V$*, such that for all open set containments* $U' \subset U \subset X$ *and* $V' \subset V \subset Y$ *, and* $f(U') \subset V'$ *and* $f(U) \subset V$*, we have the following commutative diagram:*

$$\begin{tikzcd}
\mathcal{O}_Y(V) \arrow[r, "f^{\#}_{V\to U}"] \arrow[d, "res^V_{V'}"'] & \mathcal{O}_X(U) \arrow[d, "res^U_{U'}"] \\
\mathcal{O}_Y(V') \arrow[r, "f^{\#}_{V'\to U'}"'] & \mathcal{O}_X(U')
\end{tikzcd}$$

i.e.$f^{\#} : f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ *is a map of* Shv_{Ring}

Proposition 3.11 *Let* $(X, \mathcal{O}_X)$ *and* $(Y, \mathcal{O}_Y)$ *be ringed spaces, use prop 2.31:*

$$Hom((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)) \cong \{f : X \to Y \, and \, f^{\#} : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X\} \cong \{f : X \to Y \, and \, f^{\flat} : \mathcal{O}_Y \to f_*\mathcal{O}_X\}$$

Remark 3.12 *Let be* $f : X \to Y$ *a map of schemes. We say* $f^{\flat}$ *is abjoint with* $f^{\#}$*.*

The relation is between the map of Y*-sheaves* $\mathcal{O}_Y \to f_*\mathcal{O}_X$ *being surjective and the adjoint map of* X*-sheaves* $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ *being surjective :*

The first is asking that for all y *in* Y *that the stalk* $\mathcal{O}_{Y,y} \to (f_*\mathcal{O}_X)_y$ *map is surjective. While the second is asking that for all* x *in* X *that the stalk map* $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ *is surjective.*

Let me give counterexamples in one direction. Consider the map $f : \mathbb{P}^1_k \to Spec\, k$ *with* $X = \mathbb{P}^1_k$ *and* $Y = Spec\, k$*. Then this is an example where* $O_Y \to f_*\mathcal{O}_X$ *is surjective but* $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ *is not surjective.*

Let me give a counterexample in the other direction. Let X *be a disjoint union of two copies of* $Spec\, k$*, and* Y *just* $Spec\, k$*. Then* $f^{-1}\mathcal{O}_Y = \mathcal{O}_X$ *in particular the map is surjective. But the map* $\mathcal{O}_Y = k \to k^2 = f_*\mathcal{O}_X$ *is not surjective.*

Def 3.13 *Let* $(\phi =)f : B \to A$ *be a map of rings, and let* $X = \mathrm{Spec}\, A$ *and* $Y = \mathrm{Spec}\, B$ *then spaces, and* $\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *is a morphism of ringed spaces given by* $\phi : X \to Y$ *on the level of topological spaces, and* $\phi^{\flat} : \mathcal{O}_Y \to \phi_*\mathcal{O}_X$ *defined on a basis of principal opens by the nataral map for all* $s \in B$ *:*

$$\begin{tikzcd}
\mathcal{O}_Y(D(s)) \arrow[r] \arrow[d, "="'] & \mathcal{O}_X(\phi^{-1}D(s)) \arrow[d, "="] \\
B[s^{-1}] \arrow[r, "f[s^{-1}]"] & A[f(s)^{-1}]
\end{tikzcd}$$

where we can identify $\mathcal{O}_X(\phi^{-1}(D(s))) = \mathcal{O}_X(D(f(s))$.

Remark 3.14 *(Warning!). The functor from the category of rings to the category of ringed spaces is not fully faithful yet, because there are maps between affine schemes (now consider only as ringed spaces) that are not yet induced by maps of rings.*

Take p some prime, then there is a morphism of ringed spaces $f_{\mathrm{p}} : (\operatorname{Spec}\mathbb{Q}, \mathcal{O}_{\operatorname{Spec}\mathbb{Q}}) \to (\operatorname{Spec}\mathbb{Z}, \mathcal{O}_{\operatorname{Spec}\mathbb{Z}})$ which is defined by sending the point $\operatorname{Spec}\mathbb{Q} = *$ to $p \in \operatorname{Spec}\mathbb{Z}$. Indeed, take $f^{\#} : f_{\mathrm{p}}^{-1}\mathcal{O}_{\operatorname{Spec}\mathbb{Z}} = \mathbb{Z}_{(p)} \to \mathcal{O}_{\operatorname{Spec}\mathbb{Q}}$ to be the natural map, then this does not come from a ring map $\mathbb{Z} \to \mathbb{Q}$

Def 3.15 *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be locally ringed spaces, then a* ***map of locally ringed spaces*** *is a map of ringed spaces s.t. for all $x \in X$, the map $f_x^{\#} : (f^{-1}\mathcal{O}_Y)_x = (\mathcal{O}_Y)_{f(x)} \to (\mathcal{O}_X)_x$ is a local map*

Def 3.16 *An* ***affine scheme*** *is a locally ringed space $(X, \mathcal{O}_X)$ such that $(X, \mathcal{O}_X) \cong (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some ring A.*

A ***morphism of affine schemes*** *is exactly a morphism of locally ringed spaces.*

A ***scheme*** *is a locally ringed space $(X, \mathcal{O}_X)$ such that there exists a covering $X = \bigcup_{i\in\mathcal{I}} U_i$ of X by open subsets such that each $\left(U_i, \mathcal{O}_{X|U_i}\right)$ is isomorphic to an affine scheme.*

A ***morphism of schemes*** *is simply a morphisrm of locally riniged spaces*

Proposition 3.17 *If $f : B \to A$ is a map of rings inducing a map $(g, g^{\#}) : (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \to (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B})$, then it is a morphism of locally ringed spaces. For all $x = \mathfrak{p} \in \operatorname{Spec} A$, we have $\mathfrak{p} \subseteq A$, and $f^{-1}(\mathfrak{p}) = \mathfrak{q} \subseteq B$ and we have $g_x^{\#} : \mathcal{O}_{Y,g(x)} = B_{\mathfrak{q}} \to A_{\mathfrak{p}} = \mathcal{O}_{X,x}$ is a local map. On spectral spaces we have $x \mapsto y = f^{-1}(\mathfrak{p}) \in \operatorname{Spec} B$, which on localised spectral spaces is a map $\operatorname{Spec} A_{\mathfrak{p}} = X_x \to Y_y = \operatorname{Spec} B_{\mathfrak{q}}$*

Theorem 3.18 *The controvariant functor which sends $A \mapsto \left(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}\right)$ is an equivalence of categories between the category of rings and the category of affine schemes*

Theorem 3.19 *The functor $\operatorname{Spec} : \{Ring\}^{op} \to \{locally\ ringed\ spaces\}$ is fully faithful onto its image, which we'll call the category of affine schemes*

3.2 Affine Schemes with Strucutre Sheaf are Rings

3.2.1 $Hom\,((X,\mathcal{O}_X),(Y,\mathcal{O}_Y)) \cong Hom_{Rings}(B,\Gamma(X,\mathcal{O}_X))$

How can we think of elements of A as of functions of $Spec\,A$? Strictly speaking, we cannot. However, given $f \in A$ and $x \in Spec\,A$, we get an element $f(x) \in \kappa(x)$, where $f(x)$ is the residue class in $\kappa(x) := A_{\mathfrak{p}_x}/\mathfrak{p}_x A_{\mathfrak{p}_x}$ of the image of f in the localization $A_{\mathfrak{p}_x}$. This is completely analogous to the situation with prevarieties. However, we do not get a function with a well defined target, but rather a collection of values $f(x)$ in different targets $\kappa(x), x \in Spec\,A$. Nevertheless, this is a useful point of view. For instance, we can interpret $D(f)$ as the set of points where $f(x) = 0$, i. e. where the function f does not vanish

Let $(X,\mathcal{O}_X)$ be a locally ringed space and $x \in X$. We call the stalk $\mathcal{O}_{X,x}$ the local ring of X in x, denote by $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{X,x}$, and by $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ the residue field. If U is an open neighborhood of x and $f \in \mathcal{O}_X(U)$, we denote by $f(x) \in \kappa(x)$ the image of f under the canonical homomorphisms $\mathcal{O}_X(U) \to \mathcal{O}_{X,x} \to \kappa(x)$

Def 3.20 *Let $(X,\mathcal{O}_X)$ be a locally ringed space, and $f \in \mathcal{O}_X(X)$*

Define $\boldsymbol{X_f := \{x \in X; f(x) \neq 0\} = \{x \in X; f_x \notin \mathfrak{m}_x\}}$

Remark : $X_f = D(f)$ when X is an affine scheme, since $f_{\mathfrak{p}_x} \notin \mathfrak{p}_x A_{\mathfrak{p}_x} \Leftrightarrow f \notin \mathfrak{p}_x$

Proposition 3.21 *If $U = Spec\,B$ is an open affine subscheme of X, and if $f|_U \in B = \Gamma(U, \mathcal{O}_X|_U)$ is the restriction of f, Then $U \cap X_f = D(f|_U)$. Conclude that X_f is an open subset of X*

Proof: Recall:$\mathcal{F}_x = (\mathcal{F}|_U)_x$, so $\mathcal{O}_{X,x} = \mathcal{O}_{U,x}$

$D(f|_U) = \{x \in U, f|_U \notin x\} = \{\mathfrak{p}_x \in U, (f|_U)_{\mathfrak{p}_x} \notin \mathfrak{p}_x B_{\mathfrak{p}_x}\} = \{x \in U, f_x \notin \mathfrak{m}_x\} = U \cap X_f$ □

Remark 3.22 $(f, f^{\#}) : (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$

Then $f^{-1}(Y_s) = \{x \in X | f^{\#}(s)_x \notin \mathfrak{m}_x \subseteq \mathcal{O}_{X,x}\} = X_{f^{\#}(s)}$ is right when $(Y,\mathcal{O}_Y) = Spec\,B$. And it is this true for general schemes (even for locally ringed spaces!)

Proof: To prove it for schemes you can work locally and reduce this way to the case of affine schemes.

To prove it for locally ringed spaces take p in X. Since f is a morphism of locally ringed spaces the pullback of sections induces a local morphism of local rings $\mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$. So we get an injection between the residue fields $\kappa(f(p)) \to \kappa(p)$ mapping $s(f(p))$ to $(f^*s)(p)$. This shows that $s(f(p)) = 0$ iff $(f^*s)(p) = 0$ and therefore $D(f^*s) = f^{-1}(D(s))$.

Note that this is a trivial statement if the structure sheaves of the ringed spaces is their sheaf of continuous functions and the pullback of sections is given by precomposition with f. In that case $s(y)$ is really just the evaluation of the function at the point y. Therefore $(f^*s)(p) = (s \circ f)(p) = s(f(p))$ □

Def 3.23 *notation $\Gamma(X,\mathcal{F})$ and $\mathcal{F}(X)$ for the global sections of a sheaf $\mathcal{F}$ over the topological space X*

We treat $\Gamma(X,-) : Shv(X) \to Sets$ *as a functor*

We consider $\boldsymbol{\Gamma(X,\mathcal{O}_X)}$ *the ring of global sections*

Remark 3.24 $\Gamma(-,\mathcal{O}_-) : \{\textit{locally ringed spaces}\} \to \{Ring\}^{op}$ *a functor*

If $(g,g^{\#}) : (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$

$g^{\#} : g^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ *has adjoint* $g^{\flat} : \mathcal{O}_Y \to g_*\mathcal{O}_X$

$$\Gamma(g,g^{\#}) : \Gamma(Y,\mathcal{O}_Y) \xrightarrow{g^{\flat}} \Gamma(Y,g_*\mathcal{O}_X) = \mathcal{O}_X(g^{-1}(Y)) = \mathcal{O}_X(X) = \Gamma(X,\mathcal{O}_X)$$

Theorem 3.25 *Let* $(X,\mathcal{O}_X)$ *be a locally ringed space and let* $(Y,\mathcal{O}_Y)$ *be an affine scheme, say* $Y = \operatorname{Spec} B$*, then the map*

$$\boldsymbol{\Gamma(-,-) : Hom\,((X,\mathcal{O}_X),(Y,\mathcal{O}_Y)) \longrightarrow Hom_{Rings}(B,\Gamma(X,\mathcal{O}_X))}$$

is an isomorphism

(See images 5-9 for proof)

3.2.2 Open immersion

Def 3.26 ***open immersion***

A morphism of ringed spaces $(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *is an* ***open immersion*** *if* $f : X \to Y$ *is an open embedding and* $f^{\#} : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ *is an isomorphism or equivalently, there exists an open subset* $U \subset Y$ *such that* $(X, \mathcal{O}_X) \cong (U, \mathcal{O}_Y|_U)$*, , and this isomorphism factors* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *through the inclusion* $(U, \mathcal{O}_Y|_U) \to (Y, \mathcal{O}_Y)$

Proposition 3.27 *Let* $(Y, \mathcal{O}_Y)$ *be a scheme and* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *be an open immersion, then* $(X, \mathcal{O}_X)$ *is also a scheme*

Proof: Let $(Y, \mathcal{O}_Y)$ be a scheme, and U be an open subset of Y, then we'll show $(U, \mathcal{O}_Y|_U)$ is a scheme. In the proposition. $(U, \mathcal{O}_Y|_U)$ is the image of $(X, \mathcal{O}_X)$. Now let $\{V_i\}$ be a collection of open sets such that $Y = \bigcup V_i$ and $V_i \cong \operatorname{Spec} B_i$, then for all $x \in U$ we can choose i such that $x \in V_i$, so $V_i \cap U \subseteq V_i$ is an open neighbourhood of x. This implies that there exists $f \in B_i$ such that $x \in D_{V_i}(f) \subseteq V_i \cap U$. , so call $U_x = D_{V_i}(f) \subseteq V_i$ Then we have

$$\Gamma(U_x, \mathcal{O}_Y|_{U_x}) = \Gamma(U_x, (\mathcal{O}_Y|_{V_i})|_{U_x}) = B_i[f^{-1}],$$

which is an affine scheme, hence as x varies, all the U_x cover U

□

Proposition 3.28 ***Morphisms of ringed spaces glue***

Suppose $(X, \mathcal{O}_X)$ *and* $(Y, \mathcal{O}_Y)$ *are ringed spaces,* $X = \cup_i U_i$ *is an open cover of* X*, and we have morphisms of ringed spaces* $\pi_i \colon U_i \to Y$ *that "agree on the overlaps",i.e.* $\pi_i|_{U_i \cap U_j} = \pi_j|_{U_i \cap U_j}$*. Then there is a unique morphism of ringed spaces* $\pi \colon X \to Y$ *such that* $\pi|_{U_i} = \pi_i$

Example 3.29 *Let* $U_i, i = 1, 2,$ *be two schemes. Let* $V_i \subseteq U_i$ *be open subschemes and let* $\varphi : V_1 \xrightarrow{\sim} V_2$ *be an isomorphism*

1) There exists a scheme X *with open subschemes* $W_i \subseteq X$ *and isomorphisms* $\alpha_i : U_i \xrightarrow{\sim} W_i, i = 1, 2,$ *such that* $\alpha_i^{-1}(W_1 \cap W_2) = V_i$ *and* $\varphi = \alpha_2^{-1} \circ \alpha_1|V_1$

2) Let A *be a ring and let* $X_{\pm}$ *be the scheme obtained by glueing* $U_1 = U_2 = Spec(A[T])$ *along the isomorphism* $\varphi_{\pm} : Spec(A[T, T^{-1}]) \to Spec(A[T, T^{-1}]), T \mapsto T^{\pm 1}$

$X_{\pm}$ *are not affine schemes and that* X_+ *is not isomorphic to* X_-

Remark: X_+ *is called the affine line over* A *with doubled origin,* X_- *is called the projective line over* A

Proposition 3.30 *Let* $(Y, \mathcal{O}_Y)$ *be a locally ringed space. Show that* $(Z, \mathcal{O}_Z) \mapsto Z$ *induces a bijection between open subsets of* Y *and equivalence classes of open immersions* $(Z, \mathcal{O}_Z) \to (Y, \mathcal{O}_Y)$ *of locally ringed spaces. Here, open immersions are equivalent if they are isomorphic as locally ringed spaces over* $(Y, \mathcal{O}_Y)$

Theorem 3.31 *Let k be an algebraically closed field. For a (classical) quasi-projective variety $X \subseteq \mathbb{P}^n_k(k)$ let $\mathcal{O}$ be its sheaf of regular functions $U \mapsto \mathcal{O}(U)$. Let $\pi : X \to X^{sob}$ be the soberification (Sheet 3, Exercise 4).*

Then $X^{sch} := (X^{sob}, \pi_* \mathcal{O})$ is a scheme over $Spec(k)$ and that $X \mapsto X^{sch}$ is a fully faithful functor from the category of quasi-projective varieties to the category of schemes over $Spec(k)$

Hint: Reduce to the case of affine algebraic sets by glueing morphisms of locally ringed spaces

3.3 Scheme Valued Points

Def 3.32 *Given a category $\mathcal{C}$ and an object $S \in \mathcal{C}$, then the category $\boldsymbol{\mathcal{C}/S}$ is defined with objects the pairs $A \to S$, and morphisms $f : A \to B$ with the following commutative diagram:*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] & & B \arrow[dl] \\ & S & \end{tikzcd}$$

Fix a scheme S and $X \to S \in Sch/S$, and define for any $Z \in Sch$ (the category of schemes) the ***Z-valued point of X/S*** *as*

$$\boldsymbol{X_S(Z) = Hom_{Sch/S}(Z, X)}$$

We are being a little vague above, we really mean $X_S(Z)$ to be all morphisms of schemes $f : Z \to X$ such that the following diagram commutes:

$$\begin{tikzcd} Z \arrow[rr, "f"] \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd}$$

Since $Spec\,\mathbb{Z}$ is final in the category of schemes, if we take $S = Spec\,\mathbb{Z}$ then we will write $\boldsymbol{X(Z)}$, since the commutative diagram above becomes irrelevant. If $Y = Spec\,B$ or $S = Spec\,C$ we might also write $X_S(Y) = X_S(B) = X_C(Y) = X_C(B)$. This explains the notation for ***k-valued points*** *as* $\boldsymbol{X(k)}$ *(i.e. $X_{Spec\,\mathbb{Z}}(Spec\,k)$)*

A-valued points of an affine scheme $Spec\,\mathbb{Z}[X_1, \cdots, X_n]/(f_1, \cdots, f_m)$ (where $f_i \in \mathbb{Z}[X_1, \cdots, X_n]$ are relations) are precisely the solutions to the equations $f_1(x_1, \cdots, x_n) = \cdots = f_m(x_1, \cdots, x_n) = 0$ in the ring A. For example, the rational solutions to $x^2 + y^2 = 16$ are precisely the $\mathbb{Q}$-valued points of $Spec\,\mathbb{Z}[x, y]/(X^2 + X^2 = 16)$.

$$\begin{aligned} X(A) &= Hom_{Sch}(Spec\,A, Spec\,\mathbb{Z}[X_1, \cdots, X_n]/(f_1, \cdots, f_m)) \\ &\cong Hom_{Ring}(\mathbb{Z}[X_1, \cdots, X_n]/(f_1, \cdots, f_m), A) \\ &\cong \{x = (x_1, \cdots, x_n) \in A^n | f_1(x) = \cdots = f_m(x) = 0\} \end{aligned}$$

Given a base scheme S, and any $X \to S \in Sch/S$, we have a functor $\boldsymbol{X_S(-) : (Sch/S)^{op} \to Sets}$ defined by $Z \mapsto X_S(Z)$. We can use the Yoneda lemma to say something concrete about this

Proposition 3.33 *Let $\mathcal{C}$ be a category. For $X \in \mathcal{C}$ let $h_X := \mathrm{Hom}_{\mathcal{C}}(-, X)\,;\, \mathcal{C}^{\mathrm{op}} \to Sets$ be the associated functor. Let $F\colon \mathcal{C}^{\mathrm{op}} \to Sets$ be an arbitrary functor*

Recall the Yoneda lemma i.e. the map

$$\mathrm{Hom}(h_X, F) \to F(X),\ \eta \mapsto \eta_X(\mathrm{Id}_X)$$

is a bijection, natural in X and F

Let S be a scheme and let $X \to S, Y \to S$ be two schemes over S. Let $\mathcal{C} = \mathrm{Sch}/S$ be the category of schemes over S and let $\mathcal{D} \subset \mathcal{C}$ be the full subcategory consisting of objects $Z \to S \in \mathcal{C}$ with Z affine. Let $\mathrm{Hom}_S(X, Y)$ be the set of morphisms $f\colon X \to Y$ of schemes over S. Then there are bijections

$$\mathbf{Hom}_{\boldsymbol{S}}(\boldsymbol{X}, \boldsymbol{Y}) \cong \mathbf{Hom}(\boldsymbol{h_X}, \boldsymbol{h_Y}) \cong \mathbf{Hom}(\boldsymbol{h_{X|\mathcal{D}}}, \boldsymbol{h_{Y|\mathcal{D}}})$$

where $F|_{\mathcal{D}}$ denotes the restriction of a functor $F\colon C^{op} \to Sets$ to $\mathcal{D}^{op}$

Corollary 3.34 *Let $\mathrm{Sch}^{aff}/S.$ be the full subcategory of Sch/S of affine schemes over S, then the functor $Sch/S \to Fun\left(\left(\mathrm{Sch}^{aff}/S\right)^{op}, \mathrm{Sets}\right)$ is also fully faithful*

In particular, if we take $S = Spec\,\mathbb{Z}$ and use our equivalence of the category of affine schemes with the category of rings, we find that a scheme is equivalent to giving a functor from the category of rings to the category of sets, $X \mapsto (R \mapsto X(R))$.

Corollary 3.35 *The functor*

$$\boldsymbol{\Phi}\colon \mathbf{\{schemes\}} \to \mathbf{Fun(Rings, Sets)},\ \boldsymbol{X} \mapsto \mathbf{Hom}_{\boldsymbol{lrs}}(\mathbf{Spec}(-), \boldsymbol{X}) = (\boldsymbol{R} \mapsto \mathbf{Hom}_{\boldsymbol{lrs}}(\mathbf{Spec}(\boldsymbol{R}), \boldsymbol{X}))$$

is fully faithful

Example 3.36 *The functors $R \mapsto F_n(R) := x \in R | x^n = 1$ for $n \geq 1$, and $R \mapsto G(R) := \{(x, y) \in R^2 | R^2 \xrightarrow{(x,y)} R \text{ surjective}\}$ lie in the essential image of Φ*

3.4 Fiber products

Proposition 3.37 *Coproducts exist in Sch, if I is an index set, $(X_i)_{i\in I}$ is a family of schemes, then $\amalg_i X_i$ is the coproduct*

For example : $Spec(\Pi_{i=1}^n R_i) = \amalg_{i=1}^n Spec\, R_i$, but $Spec(\Pi_{i=1}^\infty R_i)$(quasicompact)$\neq \amalg_{i=1}^\infty Spec\, R_i$(not quasicompact)

Recall in a category $\mathcal{C}$ with finite limits, we can take the pullback (fiber product) of two maps $X \to S$ and $Y \to S$ to obtain an objects $X \times_S Y$ which is universal in some sense.

The universal property states that the Z-valued points of $X \times_S Y$ are canonically in bijection with the fiber product of the Z-valued points of X and Y over S in the category of sets. In symbols this reads,

$$(X \times_S Y)(Z) \cong X(Z) \times_{S(Z)} Y(Z),$$

Theorem 3.38 *Fiber products exist in Sch*

Proof: [Hartshone theorem 3.3]

Key: a) $Spec\, A \times_{Spec\, B} Spec\, C \cong Spec\,(A \otimes_B C)$

$$\begin{aligned}
Hom((X, \mathcal{O}_X), Spec\, A \times_{Spec\, B} Spec\, C) &= Hom((X, \mathcal{O}_X), Spec\, A) \times_{Hom((X,\mathcal{O}_X), Spec\, B} Hom((X, \mathcal{O}_X), Spec\, C) \\
&\cong Hom_{Rings}(A, \Gamma(X, \mathcal{O}_X)) \times_{Hom_{Rings}(B, \Gamma(X, \mathcal{O}_X))} Hom_{Rings}(C, \Gamma(X, \mathcal{O}_X)) \\
&\cong Hom_{Rings}(A \otimes_B C, \Gamma(X, \mathcal{O}_X)) \\
&\cong Hom((X, \mathcal{O}_X), Spec\,(A \otimes_B C))
\end{aligned}$$

□

Proposition 3.39 *1) Let $X \xrightarrow{f} S$. $Y \xrightarrow{g} S$ be two morphisms of locally ringed spaces. Let $X = \bigcup_i U_i$, $Y = \bigcup_j V_j$, $S = \bigcup S_{i,j}$ be open coverings. We view $U_i, V_j, S_{i,j}$ as locally ringed spaces via the restriction of the structure sheaves on X, Y, S. Using the universal property of fiber products (which exist in locally ringed spaces), we can show that for all i, j the map $U_i \times_{S_{i,j}} V_j arrow X \times_S Y$ identifies the source as an open subspace of the latter and*

$$\bigcup_{i,j} U_i \times_{S_{i,j}} V_j = X \times_S Y$$

2) Assume that X, Y, S are schemes. Then the natural map $|X \times_S Y| \to |X| \times_{|S|} |Y$ is surjective, but not injective in general

Proposition 3.40 *Let $f\colon X \to S, g\colon S' \to S$ be morphisms of schemes. Let $f'\colon X' := X \times_S S' \to S'$ be the projection.*

1) If f is an open (resp. closed) immersion, then f' is an open (resp. closed) immersion

2) Assume $S' = \mathrm{Spec}(k(s)) \to S$ is the canonical morphism for some $s \in S$. Then $|X'| \to |X| \times_{|S|} \{s\}$ is a homeomorphism

3.5 Reduced scheme

Def 3.41 *A scheme X is called **reduced** if for all $U \subset X$ open, the ring $\mathcal{O}_X(U)$ is a reduced ring (so $f^m = 0$ if and only if $f = 0$*

Proposition 3.42 *1. An affine scheme $X = \operatorname{Spec} A$ is reduced if and only if A is reduced*

2. A scheme X is reduced if and only if for all open affines $U = \operatorname{Spec} A \subseteq X$, the ring A is reduced if and only if X admits a cover by reduced affine schemes

Proof: 1. If A is reduced then $A[f^{-1}]$ is reduced for all $f \in A$, and if $U \subset X$ is an open subset, then A there exists $f_i \in A$ such that $U = \bigcup_i D(f_i)$. We have an injection

$$\mathcal{O}_X(U) \hookrightarrow \prod_i \mathcal{O}_X(D(f_i)) = \prod_i A[f_i^{-1}],$$

since $\mathcal{O}_X$ is a sheaf, and since the product of reduced rings is reduced, we see that $\mathcal{O}_X(U)$ is reduced. Conversely, if $X = \operatorname{Spec} A$ is reduced then in particular $\mathcal{O}_X(X) = A$ is reduced

2. First assume that X is reduced, ther $\mathcal{O}_X(U) = A$ is reduced if $U = \operatorname{Spec} A$. It is clear that all $\mathcal{O}_X(U_\alpha)$ are reduced for an affine cover $\{U_\alpha\}$ of X if $\mathcal{O}_X(U)$ is reduced for all affine opens U. Finally, if $U \subseteq U_\alpha$ are reduced for an affine cover $\{U_\alpha\}$ of X where $O_X(U_\alpha)$ is reduced, then clearly $\mathcal{O}_X(U)$ is reduced, but for any general $U \subset X$ the injection,

$$\mathcal{O}_X(U) \hookrightarrow \prod_\alpha \mathcal{O}_X(U \cap U_\alpha),$$

again shows us that $\mathcal{O}_X(U)$ is reduced

□

Proposition 3.43 *Let X be a scheme. The following are equivalent :*

1) The scheme X is reduced

2) There exists an affine open covering $X = \cup U_i$ such that each $\Gamma(U_i, \mathcal{O}_X)$ is reduced

3) For every affine open $U \subseteq X$ the ring $\mathcal{O}_X(U)$ is reduced

4) Every local ring $\mathcal{O}_{X,x}$ is reduced

Just as there is a canonical way to obtain a reduced ring from any commutative ring A (just take $A_{red} = A/N$ where N is the ideal of nilpotents of A), there is a canonical way to reduce a scheme

Proposition 3.44 *1. Given a scheme $X = (|X|, \mathcal{O}_X)$, then the scheme $\boldsymbol{X_{red} = (|X|, \mathcal{O}_{X_{red}})}$ is a reduced scheme where $\boldsymbol{\mathcal{O}_{X_{red}}}$ is defined as the sheafification of the presheaf $\mathcal{O}^0_{X_{red}}$, which is defined on some open subset $U \subseteq X$ as $\mathcal{O}^0_{X_{red}}(U) = (\mathcal{O}_X(U))_{red}$*

2. If $X = \operatorname{Spec} A$ is an affine scheme, then

$$X_{red} = \operatorname{Spec} A_{red}$$

3. For any reduced scheme Y, we have a bijection

$$Hom(Y, X_{red}) \longrightarrow Hom(Y, X).$$

This means that X_{red} has the same (although dual) universal property with respect to X, that A_{red} does with respect to A

Proof: 2. We know that $|\operatorname{Spec} A|$ and $|Spec\, A_{red}|$ are homeomorphic as topological spaces, so for any $f \in A$, we have $(A[f^{-1}])_{red} = A_{red}[\overline{f}^{-1}]$ This means that

$$\mathcal{O}^0_{(\operatorname{Spec} A)_{red}}(D(f)) = \mathcal{O}_{\operatorname{Spec} A_{red}}(D(f))$$

which implies $(\mathcal{O}_{\operatorname{Spec} A})_{red} = \mathcal{O}_{\operatorname{Spec} A_{red}}$

1. If $U = \operatorname{Spec} A \subseteq X$ is an open affine then

$$(|U|, \mathcal{O}_{X_{red}}|_U) = (|U|, \mathcal{O}_{U_{red}}) = \operatorname{Spec} A_{red}$$

where the last equality come from part 2. This implies that Spec A_{red} is an open affine of X_{red} and choosing an open affine cover of X will give us a reduced open affine cover of X_{red}

3. Let Y be a reduced scheme, then for all map $f : Y \to X$, the map of sheaves $f^\flat : \mathcal{O}_X \to f_*\mathcal{O}_Y$ factors uniquely through $\mathcal{O}_{X_{red}}$, so for all open subsets U of X we have the following diagram :

$$\begin{tikzcd}
\mathcal{O}_X(U) \arrow[r, "f^\flat(U)"] \arrow[d] & (f_*\mathcal{O}_Y)(U) = \mathcal{O}_Y(f^{-1}(U)) \\
\mathcal{O}_{X_{red}}(U) \arrow[r] & \mathcal{O}^0_{X_{red}}(U) = (\mathcal{O}_X(U))_{red} \arrow[u, "\tilde{f}(U)"']
\end{tikzcd}$$

These factorisations $\tilde{f}(U)$ glue to a unique map $Y \to X_{red}$, a map of schemes, using the universal property of sheafification. $\square$

Proposition 3.45 *Given a scheme X, then there is a bijection between closed reduced subschemes of X and closed subsets of $|X|$*

Proof: Assume that $X = Spec\, A$ then reduced closed subschemes are simply $Spec\, A/I$ where A/I is reduced, which are the same as ideals $I \subsetneq A$ where I is a radical ideal, which by definition give us our closed subsets $V(I)$. In general we just simply glue. We do observe though that the localisation of a reduced subscheme is still reduced $\square$

3.6 Closed immersion

Def 3.46 *A map $f : Y \to X$ of schemes is called a* ***closed immersion*** *if*

1) the induced map on topological spaces $|f| : |Y| \to |X|$ is a closed immersion (a homeomophism onto a closed subset)

2) $f^\flat : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is a surjective map of sheaves

Proposition 3.47 *Let $f : Y \to X$ be a map of schemes, then the following are equivalent :*

1) f is a closed immersion

2) For all open subsets $U \subseteq X$ with $U = \operatorname{Spec} A$, $f^{-1}(U) = \operatorname{Spec} B \subseteq Y$ is open affine, and $A \to B$ is surjective

3) There exists an open cover of X by affine schemes which satisfy the property of part 2)

(i.e. $\exists$ open cover $X = \cup_i U_i$ s.t. $U_i = Spec\, A_i$ with $f^{-1}(U_i) = Spec\, B_i$ and $A_i \to B_i$ is surjective)

Proof: clear 2)⇒3)⇒1) □

Remark 3.48 *In particular, for $X = \operatorname{Spec} A$ affine, then closed immersions are in bijection with surjections $A \to B$*

$$\{closed\ immersion\ Y \to X\} \cong \{surjection\ A \to B\}$$

3.7 $\mathcal{O}_X$-module

Def 3.49 *Given a ring A and an A-module M. Then we define a presheaf $\widetilde{M}$ on $X = \operatorname{Spec} A$, defined on the basis of principal opens by*

$$\widetilde{M}(D(f)) = M[f^{-1}], f \in A$$

We exactly know this defines a sheaf on this basis of principal opens which we can extend uniquely to a sheaf $\widetilde{M}$ on all of X. The expected thing happens on stalks of $\widetilde{M}$ too

Proposition 3.50 *Let $x \in X = \operatorname{Spec} A$, and Let $\mathfrak{p} \subseteq A$ be the corresponding prime ideal in A, then $\widetilde{M}_x = M_{\mathfrak{p}}$*

Proof:

$$\widetilde{M}_x = \operatorname{colim}_{D(f) \ni x} M[f^{-1}] = \operatorname{colim}_{f \notin \mathfrak{p}} M[f^{-1}] = M_{\mathfrak{p}}.$$

□

Def 3.51 *Let $(X, \mathcal{O}_X)$ be a ringed space, then a sheaf of $\mathcal{O}_X$-modules is a sheaf of abelian groups $\mathcal{M}$ together with a map $\mathcal{O}_X \times \mathcal{M} \to \mathcal{M}$ of sheaves such that $\mathcal{M}(U)$ is an $\mathcal{O}_X(U)$-module for all open subsets $U \subseteq X$*

The fact that we ask the action map $\mathcal{O}_X \times \mathcal{M} \to \mathcal{M}$ to be a map of sheaves assures us that our restriction maps respect this module structure

Just as we have modules over a ring, we have $\mathcal{O}_X$-modules over a sheaf of rings $\mathcal{O}_X$. There is only one possible definition that could go with the name $\mathcal{O}_X$-module – a sheaf of abelian groups $\mathcal{F}$ with the following additional structure

For each $U, \mathcal{M}(U)$ is an $\mathcal{O}_X(U)$-module. Furthermore, this structure should behave well with respect to restriction maps: if $U \subset V$, then :

$$\begin{tikzcd}
\mathcal{O}_X(V) \times \mathcal{M}(V) \arrow[r, "action"] \arrow[d, "res_{V,U} \times res_{V,U}"'] & \mathcal{M}(V) \arrow[d, "res_{V,U}"] \\
\mathcal{O}_X(U) \times \mathcal{M}(U) \arrow[r, "action"] & \mathcal{M}(U)
\end{tikzcd}$$

commutes

Recall that the notion of A-module generalizes the notion of abelian group, because an abelian group is the same thing as a $\mathbb{Z}$-module. Similarly, the notion of $\mathcal{O}_X$-module generalizes the notion of sheaf of abelian groups, because the latter is the same thingasa $\underline{\mathbb{Z}}$- module, where $\underline{\mathbb{Z}}$ is the constant sheaf associated to $\mathbb{Z}$. Hence when we are proving things about $\mathcal{O}_X$-modules, we are also proving things about sheaves of abelian groups

Ex 3.52 *If $(X, \mathcal{O}_X)$ is a ringed space, and $\mathcal{F}$ is an $\mathcal{O}_X$-module, describe how for each $p \in X, \mathcal{F}_p$ is an $\mathcal{O}_{X,p}$-module.*

Remark 3.53 ***vector bundles***

The motivating example of $\mathcal{O}_X$-modules is the sheaf of sections of a vector bundle. If $(X, \mathcal{O}_X)$ is a differentiable manifold (so $\mathcal{O}_X$ is the sheaf of differentiable functions, and $\pi : V \to X$ is a vector bundle over X, then the sheaf of differentiable sections $\sigma\colon X \to V$ is an $\mathcal{O}_X$-module. Indeed, given a section s of π over an open subset $U \subset X$, and a function f on U. we can multiply s by f to get a new section fs of π over U. Moreover, if U' is a smaller subset, then we could multiply f by s and then restrict to U', or we could restrict both f and s to U' and then multiply, and we would get the same answer

Proposition 3.54 *Given an A -module M and $X = \operatorname{Spec} A$, then $\widetilde{M}$ is a sheaf of $\mathcal{O}_X$-modules*

Proof: Taken $U = D(f) \subseteq X$ and $f \in A$ we have to give an action map $\mathcal{O}_X(U) \times \widetilde{M}(U) \to \widetilde{M}(U)$, but this can simply be the $A[f^{-1}]$-module structure,

$$A[f^{-1}] \times M[f^{-1}] \longrightarrow M[f^{-1}].$$

This clearly commutes with restrction maps etc □

We now have the techincal theorem which drives the types of results we desire

Theorem 3.55 *1) Let $X = Spec\, A$, M be an A-module and $\mathcal{N}$ an sheaf of $\mathcal{O}_X$-modules, then*

$$Hom_{\mathcal{O}_X}(\widetilde{M}, \mathcal{N}) \to Hom_A(M, \mathcal{N}(X)),$$

is a bijection.

2) The functor from A-modules to sheaves of $\mathcal{O}_X$-modules defined by $M \mapsto \widetilde{M}$ is fully faithful

3) Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_X$-modules. and assume there exists a cover of X by open affines $D(f_i)$ $f_i \in A$ such that $\mathcal{M}|_{D(f_i)} \cong \widetilde{M_i}$ for some $A[f_i^{-1}]$-module $\mathcal{M}_i$, then there exists an A-modules M such that $\mathcal{M} \cong \widetilde{M}$

Necessarily we have $M = \mathcal{M}(X)$ as $M = \widetilde{M}(X)$ but we still don't really know M exists yet. Assuming this theorem is true just for a little bit.

3.8 Quasi-coherent sheaf

3.8.1 Quasi-coherent sheaves

Def 3.56 ***quasi-coherent sheaf***

Let X be some scheme, then a ***quasi-coherent sheaf*** *on X is a sheaf of $\mathcal{O}_X$-modules $\mathcal{M}$ such that there exists a covering $\{U_i = \operatorname{Spec} A_i\}$ of X by open affines, and A_i-modules M_i such that $\mathcal{M}|_{U_i} \cong \widetilde{M_i}$*

A morphism of quasi-coherent sheaves is simply a morphism of $\mathcal{O}_X$-modules

Corollary 3.57 *Given an affine scheme $X = \operatorname{Spec} A$, then we have an equivalence of categories between A-modules and quasi-coherent sheaves on X, given by $M \mapsto \widetilde{M}$ with inverse $\mathcal{M} \mapsto \mathcal{M}(X)$*

Def 3.58 *A space X is called* ***quasi-compact*** *if every open cover has a finite subcover, and* ***quasi-separated*** *if given two quasi-compact open subsets U and V, then the intersection $U \cap V$ is also a quasi-compact open subset.*

A map of schemes is called ***quasi-compact*** *(resp.* ***quasi-separated****) if the inverse images of quasi-compact (resp. quasi-separated) open subsets of X are quasi-compact (resp. quasi-separated)*

Theorem 3.59 *X scheme, $\mathcal{M} \in Mod_{\mathcal{O}_X}$, TFAE :*

1) $\mathcal{M}$ qcoh (quasi-coherent)

2) $\exists$ open sets $X = \cup U_i$ with $\mathcal{M}|_{U_i}$ qcoh

3) $\exists$ open sets $X = \cup U_i$, $U_i = Spec\, A_i$ with $\mathcal{M}|_{U_i} = \widetilde{M_i}$ for some A_i-module M_i

Theorem 3.60 *Let $f : Y \to X$ be a map of schemes which is quasi-compact and quasi-separated, and let $\mathcal{M}$ be a quasi-coherent sheaf on Y, then $f_*\mathcal{M}$ is a quasi-coherent sheaf on X with its natural $\mathcal{O}_X$-structure*

Theorem 3.61 ***Gluing Modules***

The functor $M \to M_i = M[f_i^{-1}]$ from the category of A-modules to the category of collections of $A[f^{-1}]$-modules M_i and isomorphisms $\alpha_{ij} : M_i[f_j^{-1}] \to M_j[f_i^{-1}]$ which satisfy the cocycle condition, is an equivalence of categories.

Proof: We have equivalences of categories:

$$\begin{aligned}\{A - modules\} &\cong \{quasi - coherent\ sheaves\ on\ Spec\, A\} \\ &\cong \{quasi - coherent\ sheaves\ on\ D(f_i) + gluing\ data\} \\ &\cong \{collections(M_i, \alpha_{ij})\ as\ above\}\end{aligned}$$

□

Proposition 3.62 *1) Any direct sum of quasi-coherent sheaves is quasi-coherent*

2) Any colimit of quasi-coherent sheaves is quasi-coherent

3) The kernel and cokernel of a morphism of quasi-coherent sheaves is quasi-coherent. Given a short exact sequence of $\mathcal{O}_X$-modules $0 \to F_1 \to F_2 \to F_3 \to 0$ if two out of three are quasi-coherent so is the third

5) Given a morphism of schemes $f : Y \to X$ the pullback of a quasi-coherent $\mathcal{O}_X$-module is a quasi-coherent $\mathcal{O}_Y$-module. See Prop 3.106

6) Given two quasi-coherent $\mathcal{O}_X$-modules the tensor product is quasi-coherent, see The Stacks project : Modules, Lemma 17.16.6

7) Given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ the tensor, symmetric and exterior algebras on $\mathcal{F}$ are quasi-coherent, see Stacks project : Modules, Lemma 17.21.6

8) Given two quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$ such that $\mathcal{F}$ is of finite presentation, then the internal hom $Hom_{\mathcal{O}_X}(\mathcal{F}, vG)$ is quasi-coherent, see Prop 3.11.2

3.8.2 Quasi-coherent sheaves & closed immersions

$$A\ ring \xrightarrow{geometrize} (X := Spec\, A, \mathcal{O}_X)\ \ locally\ ringed\ space$$

$$\Gamma(Y, \mathcal{O}_Y) = \mathcal{O}_Y(Y) \longleftarrow Y$$

$$ring\ of\ global\ function$$

Intuition : $A = \mathbb{Z}[x_i, i \in I]/(f_j, j \in J) \to X$: vanishing locus of polynomials $f_j, j \in J$

Upshot can try to "geometrize" algebra and vice versa "algebrize" geometry

Purpose of this subsubsection : "Geometrize" sujections of rings $A \to B = A/I$ or equivalently , ideals $I \subseteq A$ and "Geometrize" modules over A

Recall :

A map $f : Y \to X$ of schemes is called a closed immersion if

1) the induced map on topological spaces $|f| : |Y| \to |X|$ is a closed immersion (a homeomophism onto a closed subset)

2) $f^{\flat} : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is a surjective map of sheaves

We know

$$(f_*\mathcal{O}_Y)_x = \begin{cases} \mathcal{O}_{Y,y} & if\ x = f(y) \\ 0 & else \end{cases}$$

Thus 2) $\Leftrightarrow \forall y \in Y\ f^{\flat}_y : \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is a surjective map

Lemma 3.63 *$\varphi : A \to B$ surjection of rings $\Rightarrow f = Spec\,\varphi : Spec\,B \to Spec\,A$ closed immersion*

Given a map $f : Y \to X$ can considers $\mathcal{I} := ker(\mathcal{O}_X \to f_*\mathcal{O}_Y)$. This is not merely a sheaf of abelian group

Let $(f, f^{\flat})$ be a map of ringed spaces $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$

If $\mathcal{N}$ is an $\mathcal{O}_Y$ -module, then the pushforward $f_*\mathcal{N}$ is the $\mathcal{O}_X$ -module with the structure morphism

$$\mathcal{O}_X \times f_*\mathcal{N} \xrightarrow{f^{\flat} \times id_{f_*\mathcal{N}}} f_*\mathcal{O}_Y \times f_*\mathcal{N} = f_*(\mathcal{O}_Y \times \mathcal{N}) \xrightarrow{f_*(-)} f_*\mathcal{N}$$

If $Y = \{x\}$ for some $x \in X$ closed, $\mathcal{O}_Y(Y) := \mathcal{O}_{X,x}$ and $g : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ given by :

$$|g|(x) = x$$

$$g_U^\flat : \mathcal{O}_X(U) \to g_*\mathcal{O}_Y(U) = \begin{cases} \mathcal{O}_{X,x} & if\ x \in U \\ 0 & else \end{cases}$$

canonical map

get injection $\mathcal{I} := ker(\mathcal{O}_X \to g_*\mathcal{O}_Y) \subseteq \mathcal{O}_X$ of $\mathcal{O}_X$-module (an "ideal" of $\mathcal{O}_X$)

However $Y, \mathcal{O}_Y$ is a scheme iff $\# Spec\, \mathcal{O}_{X,x} = 1$

Thus

$$\{closed\ immersions : Y \to X\}/isom\ over\ X \subsetneqq \{ideal\ \mathcal{I} \subseteq \mathcal{O}_X\}$$

Aim:

$$\{closed\ immersions : Y \to X\}/isom\ over\ X \xleftrightarrow{1:1} \{quasi-coherant\ ideals\ \mathcal{I} \subseteq \mathcal{O}_X\}$$

Def 3.64 *A closed embedding $f : Y \to X$ determines an ideal sheaf on X, as the kernel $\mathcal{I}_{Y/X}$ of the map of $\mathcal{O}_X$-modules :$\mathcal{O}_X \to f_*\mathcal{O}_X$*

*An **ideal sheaf** on X is what it sounds like: it is a sheaf of ideals. It is a sub $\mathcal{O}_X$-module $\mathcal{I}$ of $\mathcal{O}_X$. On each open subset, it gives an ideal $\mathcal{I}(U)$ of the ring $\mathcal{O}_X(U)$. We thus have an exact sequence (of $\mathcal{O}_X$-modules)*

$$0 \to \mathcal{I}_{Y/X} \to \mathcal{O}_X \to f_*\mathcal{O}_Y \to 0$$

(On $Spec\, B$, the epimorphism $\mathcal{O}_X \to f_*\mathcal{O}_Y$ is the surjection $B \to A$)

Thus for each affine open subset $Spec\, B \hookrightarrow X$, we have an ideal $\mathcal{I}(Spec\, B) = I(B) \subseteq B$ and we can recover Y from this information: the $I(B)$ (as $Spec\, B \hookrightarrow X$ varies over the affine open subsets) defines an $\mathcal{O}$-module on the base, hence an $\mathcal{O}_X$-module on X, and the cokernel of $\mathcal{I} \to \mathcal{O}_X$ is $\pi_*\mathcal{O}_Y$. It will be useful to understand when the information of the $I(B)$ (for all affine opens $Spec\, B \hookrightarrow X$) actually determines a closed subscheme

Remark 3.65 *Suppose $\mathcal{I}_{Y/X}$ is a sheaf of ideals corresponding to a closed embedding $Y \to X$. Suppose $Spec\, B \hookrightarrow X$ is an affine open subscheme, and $f \in B$*

Then the natural map $\mathcal{I}(Spec\, B)_f = I(B)_f \to I(B_f) = \mathcal{I}(Spec\, B_f)$ is an isomorphism

Remark 3.66 *Suppose X is a scheme, and for each affine open subset $Spec\, B$ of X, $I(B) \subseteq B$ is an ideal. Suppose further that for each affine open subset $Spec\, B \hookrightarrow X$ and each $f \in B$, restriction of functions from $B \to B_f$ induces an isomorphism $I(B_f) \cong I(B)_f$*

Then these data arise from a (unique) closed subscheme $Y \to X$ by the above construction. In other words, the closed embeddings $Spec\, B/I \to Spec\, B$ glue together in a well-defined manner to obtain a closed embedding $Y \to X$

Def 3.67 ***the vanishing scheme***

(1) Suppose X is a scheme, and $s \in \Gamma(X, \mathcal{O}_X)$. we can define the ***closed subscheme cut out by*** $\boldsymbol{s}$*. We call this the vanishing scheme* $\boldsymbol{V(s)}$ *of s, as it is the scheme-theoretic version of our earlier (set-theoretical) version of $V(s)$:*

on affine open $Spec\, B$, we just take $Spec\, B/(s_B)$, where s_B is the restriction of s to $Spec\, B$. The previous note tells us this yields a well-defined closed subscheme

(2) If u is an invertible function, then $V(s) = V(su)$

(3) If S is a set of functions, we can define $V(S)$

Proposition 3.68 *An $\mathcal{O}_X$-module $\mathcal{M}$ is quasicoherent if and only if for each such distinguished $Spec\, A_f \hookrightarrow Spec\, A \subseteq X$*

$\alpha : \mathcal{M}(Spec\, A)_f = \Gamma(Spec\, A, \mathcal{M})_f \to \Gamma(Spec\, A_f, \mathcal{M}) = \mathcal{M}(Spec\, A_f)$ is an isomorphism

Proof: Using Theorem 3.59 □

Now we get : a sheaf of ideals is quasicoherent if and only if it comes from a closed subscheme :

Theorem 3.69

$$\{closed\ immersions : Y \to X\}/isom\ over\ X \xleftrightarrow{1:1} \{quasi-coherant\ ideals\ \mathcal{I} \subseteq \mathcal{O}_X\}$$

We call

$$0 \to \mathcal{I}_{Y/X} \to \mathcal{O}_X \to f_*\mathcal{O}_Y \to 0$$

the **closed subscheme exact sequence** corresponding to $Y \hookrightarrow X$

Theorem 3.70 *Let X be a scheme. Let $i : Z \to X$ be a closed immersion of locally ringed spaces.*

1) The locally ringed space Z is a scheme,

2) the kernel $\mathcal{I}$ of the map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is a quasi-coherent sheaf of ideals,

3) for any affine open $U = Spec(R)$ of X the morphism $i^{-1}(U) \to U$ can be identified with $Spec(R/I) \to Spec(R)$ for some ideal $I \subseteq R$, and

4) we have $\mathcal{I}|_U = \widetilde{I}$.

In particular, any sheaf of ideals locally generated by sections is a quasi-coherent sheaf of ideals (and vice versa), and any closed subspace of X is a scheme

3.8.3 ⋆ Schematic image

See Algebraic Geometry I Schemes With Examples and Exercises, Ulrich Görtz and Torsten Wedhorn

Lemma 3.71 *Let $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be a morphism of ringed spaces. Assume i is a homeomorphism onto a closed subset of X and $i^\# : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. Denote $\mathcal{I} \subseteq \mathcal{O}_X$ the kernel of $i^\#$. The functor*

$$i_* : Mod(\mathcal{O}_Z) \to Mod(\mathcal{O}_X)$$

is exact, fully faithful, with essential image those $\mathcal{O}_X$-modules $\mathcal{G}$ such that $\mathcal{I}\mathcal{G} = 0$

Lemma 3.72 *Let $i : Z \to X$ be a closed immersion of schemes. Let $\mathcal{I} \subseteq \mathcal{O}_X$ be the quasi-coherent sheaf of ideals cutting out Z*

1) For any $\mathcal{O}_X$-module F the adjunction map $\mathcal{F} \to i_*i^*\mathcal{F}$ induces an isomorphism $\mathcal{F}/\mathcal{I}\mathcal{F} \cong i_*i^*\mathcal{F}$

2) The functor i^* is a left inverse to i_*, i.e., for any $\mathcal{O}_Z$-module $\mathcal{G}$ the adjunction map $i^*i_*\mathcal{G} \to \mathcal{G}$ is an isomorphism

3) The functor

$$i_* : QCoh(\mathcal{O}_Z) \to QCoh(\mathcal{O}_X)$$

is exact, fully faithful, with essential image those quasi-coherent $\mathcal{O}_X$-modules $\mathcal{G}$ such that $\mathcal{I}\mathcal{G} = 0$

Proof: By Modules, see Lemma 3.71 □

Lemma 3.73 *Let X be a scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{G} \subseteq F$ be a $\mathcal{O}_X$-submodule. There exists a unique quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{G}' \subseteq \mathcal{G}$ with the following property: For every quasi-coherent $\mathcal{O}_X$-module $\mathcal{H}$ the map*

$$Hom_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G}') \to Hom_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G})$$

is bijective. In particular $\mathcal{G}'$ is the largest quasi-coherent $\mathcal{O}_X$-submodule of $\mathcal{F}$ contained in $\mathcal{G}$

Proof: Let $\mathcal{G}_a, a \in A$ be the set of quasi-coherent $\mathcal{O}_X$-submodules contained in $\mathcal{G}$. Then the image $\mathcal{G}'$ of

$$\bigoplus_{a \in A} \mathcal{G}_a \to \mathcal{F}$$

is quasi-coherent as the image of a map of quasi-coherent sheaves on X is quasi-coherent and since a direct sum of quasi-coherent sheaves is quasi-coherent, see Prop 3.62. The module $\mathcal{G}'$ is contained in $\mathcal{G}$. Hence this is the largest quasi-coherent $\mathcal{O}_X$-module contained in $\mathcal{G}$.

To prove the formula, let $\mathcal{H}$ be a quasi-coherent $\mathcal{O}_X$-module and let $\alpha : \mathcal{H} \to \mathcal{G}$ be an $\mathcal{O}_X$-module map. The image of the composition $\mathcal{H} \to \mathcal{G} \to \mathcal{F}$ is quasi-coherent as the image of a map of quasi-coherent sheaves. Hence it is contained in $\mathcal{G}'$. Hence α factors through $\mathcal{G}'$ as desired □

Def 3.74 *Let X be a scheme. Let $Z, Y \subseteq X$ be closed subschemes corresponding to quasi-coherent ideal sheaves $\mathcal{I}, \mathcal{J} \subseteq \mathcal{O}_X$. The* ***scheme theoretic intersection*** *of Z and Y is the closed subscheme of X cut out by $\mathcal{I} + \mathcal{J}$. The* ***scheme theoretic union*** *of Z and Y is the closed subscheme of X cut out by $\mathcal{I} \cap \mathcal{J}$*

Proposition 3.75 *Let X be a scheme. Let $Z, Y \subseteq X$ be closed subschemes. Let $Z \cap Y$ be the scheme theoretic intersection of Z and Y. Then $Z \cap Y \to Z$ and $Z \cap Y \to Y$ are closed immersions and*

$$\begin{tikzcd} Z \cap Y \arrow[r] \arrow[d] & Z \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

is a cartesian diagram of schemes, i.e., $Z \cap Y = Z \times_X Y$

Proposition 3.76 *Let S be a scheme. Let $X, Y \subseteq S$ be closed subschemes. Let $X \cup Y$ be the scheme theoretic union of X and Y. Let $X \cap Y$ be the scheme theoretic intersection of X and Y. Then $X \to X \cup Y$ and $Y \to X \cup Y$ are closed immersions, there is a short exact sequence*

$$0 \to \mathcal{O}_{X\cup Y} \to \mathcal{O}_X \times \mathcal{O}_Y \to \mathcal{O}_{X\cap Y} \to 0$$

of $\mathcal{O}_S$-modules, and the diagram

$$\begin{tikzcd} X \cap Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X \cup Y \end{tikzcd}$$

is cocartesian in the category of schemes, i.e., $X \cup Y = X \sqcup_{X\cap Y} Y$

Lemma 3.77 *Let $f : X \to Y$ be a morphism of schemes. There exists a closed subscheme $Z \subseteq Y$ such that f factors through Z and such that for any other closed subscheme $Z' \subseteq Y$ such that f factors through Z' we have $Z \subseteq Z'$*

Proof: Let $\mathcal{I} = Ker(\mathcal{O}_Y \to f_*\mathcal{O}_X)$. If $\mathcal{I}$ is quasi-coherent then we just take Z to be the closed subscheme determined by $\mathcal{I}$

It requires us to show that there exists a largest quasi-coherent sheaf of ideals $\mathcal{I}'$ contained in $\mathcal{I}$. This follows from Lemma 3.73 $\square$

Def 3.78 *Let $f : X \to Y$ be a morphism of schemes. The* ***scheme theoretic image*** *(****schematic image****) of f is the smallest closed subscheme $Im(f) = Z \subseteq Y$ through which f factors*

Proposition 3.79 *Let $f : X \to Y$ be a morphism of schemes. Let $Z \subseteq Y$ be the scheme theoretic image of f. If f is quasi-compact then*

1) the sheaf of ideals $\mathcal{I} = Ker(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ is quasi-coherent,

2) the scheme theoretic image Z is the closed subscheme determined by $\mathcal{I}$,

3) for any open $U \subseteq Y$ the scheme theoretic image of $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is equal to $Z \cap U$, and

4) the image $f(X) \subseteq Z$ is a dense subset of Z, in other words the morphism $X \to Z$ is dominant

Proof: 1) : It suffices to prove that $\mathcal{I}$ is quasi-coherent. Since the property of being quasi-coherent is local we may assume Y is affine. As f is quasi-compact, we can find a finite affine open covering $X = \cup_{i=1}^n U_i$. Denote f' the composition

$$X' = \sqcup U_i \to X \to Y$$

Then $f_*\mathcal{O}_X$ is a subsheaf of $f'_*\mathcal{O}_{X'}$, and hence $\mathcal{I} = Ker(\mathcal{O}_Y \to f'_*\mathcal{O}_{X'})$. By Theorem 3.60 the sheaf $f'_*\mathcal{O}_{X'}$ is quasi-coherent on Y, so the ker is quasi-coherent

$\square$

3.9 Projective Space

This subsection we are going to talk about a scheme $\mathbb{P}^n$ which severly generalises topological projective space $\mathbb{RP}^n$ and $\mathbb{CP}^n$

Example 3.80 *Define the nth complex projective space $\mathbb{CP}^n$ as follows,*

$$\mathbb{CP}^n = \mathbb{C}^{n+1} - 0/(x_0, \ldots, x_n) \sim \lambda(x_0, \ldots, x_n), \forall \lambda \in \mathbb{C}^\times$$

We denote points in $\mathbb{CP}^n$ by homogenous coordinates, $(x_0 : \cdots : x_n)$ where $x_i \in \mathbb{C}$ and not all zero. Notice that each x_i is not well-defined, because we have this equivalence relation by a non-zero is by a non-zero complex number, however the ratios $\frac{x_i}{x_j}$ are well-defined whenever $x_j \neq 0$. The standard cover for $\mathbb{CP}^n$ is by $(n+1)$-many copies of $\mathbb{C}^n$ (hence $\mathbb{CP}^n$ is an n-dimensional complex manifold) defined by

$$\begin{aligned} U_i = \{(x_0 : \cdots : x_n) \in \mathbb{CP}^n \mid x_i \neq 0\} &\longrightarrow \mathbb{C}^n \\ (x_0 : \cdots : x_n) &\mapsto (\frac{x_0}{x_i}, \cdots, \frac{x_n}{x_i}) \end{aligned}$$

For all $i \neq j$, we have $U_i \cap U_j \cong \mathbb{C}^{n-1} \times \mathbb{C}^\times$ which we define as $\{(X_{i,k})_{k=0,\ldots,n,k\neq i} | X_{i,j} \neq 0\}$ We will use these types of sets when we start talking about projective space as schemes. Another way to construct $\mathbb{CP}^n$ would be to glue together all of these U_i along these $U_i \cap U_j = U_{i,j}$

Example 3.81 *More generally, for any field k we can define*

$$\mathbb{P}^n(k) = k^{n+1} - \{0\} / k^\times.$$

Our goal now is to construct a scheme $\mathbb{P}^n$ such that the k -valued points of $\mathbb{P}^n$ are given exactly by $\mathbb{P}^n(k)$.

There are three ways to go about doing this. There are three ways to go about doing this. We are going to do this super explicitly. We could also generalise the functor *Spec* to a functor called ***Proj***, and then define $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{Z}} = \mathrm{Proj}(\mathbb{Z}[x_0, \ldots, x_n])$

Another thing we could do is the functor of points approach, i.e. write down $R \to \mathbb{P}^n(R)$. We are going and show this is in the essential image of the fully faithful functor $\mathrm{Sch} \to \mathrm{Fun}(\mathrm{Ring}, \mathrm{Set})$ to do this explicitly, so don't worry

Caution! It will usually not be the case that $\mathbb{P}^n(R) = R^{n+1} - 0 / R^\times$, for a general ring R

Def 3.82 *(Construction of $\mathbb{P}^n$)*

For any $i = 0.....,n$, let

$$U_i = \operatorname{Spec} \mathbb{Z}[(X_{j/i})_{j=0,\ldots,n,i\neq j}] \cong \mathbb{A}^n_{\mathbb{Z}}.$$

This of $X_{i,j} = X_{j/i}$ as being the fraction "$\frac{X_j}{X_i}$". For each $i \neq j$ we have

$$U_{i,j} = D(X_{j/i}) \subseteq U_i,$$

so $U_{i,j} = D(X_{j/i}) \cong \operatorname{Spec}\mathbb{Z}[(X_{k/i})_{k\neq i,j}, (X_{j/i})^{\pm 1}]$ *We have an isomorphism between* $U_{j/i}$ *and* $U_{j/i}$ *denoted as*

$$\begin{aligned}
&\alpha_{i,j} : U_{i,j} \to U_{j,i} \\
&X_{k/i}, k \neq i \mapsto X_{k/j} \cdot X_{i/j}^{-1} \\
&X_{j/i} \mapsto X_{i/j}^{-1}
\end{aligned}$$

The inverse of this map is simply $X_{k/j}, k \neq j \mapsto X_{k/i} \cdot X_{j/i}^{-1}$

There is a lemma which tells us we can glue schemes together, so long as the pieces slot together coherently

Lemma 3.83 *Let* $\mathcal{I}$ *be a set and* U_i *for all* $i \in \mathcal{I}$ *be schemes. For* i, j *we have* $U_{i,j} \subseteq U_i = U_{i,i}$ *is an open subscheme and the isomorphisms* $\alpha_{i,j} : U_{i,j} \to U_{j,i}$ *which satisfy the cocycle condition, so* $\alpha_{i,k} = \alpha_{j,k} \circ \alpha_{i,j}$ *for all* $i, j, k \in \mathcal{I}$ *on* $U_{i,j,k} := U_{i,j} \cap U_{j,k}$. *Then we have a scheme*

$$X = \bigcup_{i \in \mathcal{I}} U_i$$

i.e. X *admits an open covering* $X = \cup V_i$ *with* $\beta_i : V_i \cong U_i$ *such that* $\beta_i : V_i \cap V_j \cong U_{i,j}$ *and* $\beta_j : V_i \cap V_j \cong U_{j,i}$, *and* $\alpha_{i,j} = \beta_j \circ \beta_i^{-1}$

Noticing that in our case we have $\alpha_{i,k} = \alpha_{j,k} \circ \alpha_{i,j}$, we apply this gluing lemma and obtain the scheme,

$$\mathbb{P}^n = \bigcup_{i=0}^{n} U_i = \bigcup_{i=0}^{n} \operatorname{Spec}\mathbb{Z}[(X_{j/i})_{j=0,\ldots,n, i\neq j}] = \bigcup_{i=0}^{n} \mathbb{A}^n_{\mathbb{Z}}$$

In particular, for all fields k we have

$$\mathbb{P}^n(k) = \bigcup_{i=0}^{n} U_i(k) = \bigcup_{i=0}^{n} k^n = \mathbb{P}^n(k).$$

Although this line seems tautological, the first $\mathbb{P}^n(k)$ is the k-valued points of a scheme and the latter is our old definition of projective space of a field

Def 3.84 *Let* $m \in \mathbb{Z}$, *we let* $\mathcal{O}_{\mathbb{P}^n}(m)(U_i) = \mathcal{O}(U_i)$ *(the def of* f^* *see 3.104)*

$$\begin{array}{lccc}
\alpha^m_{i,j} : & \mathcal{O}_{\mathbb{P}^n}(m)(U_{i,j}) & \xrightarrow{\sim} & \alpha^*_{j,i}\mathcal{O}_{\mathbb{P}^n}(m)(U_{j,i}) \\
& 1 & \longmapsto & (X_{i/j})^m
\end{array}$$

(notice $(X_{i/j})^m$ *is a unit in* $\mathcal{O}_{\mathbb{P}^n}(m)(U_{j,i})$*)*

Check the cocycle condition:

$$1 \xmapsto{\alpha^m_{i,j}} X^m_{i/j} \xmapsto{\alpha^m_{j,k}} X^m_{i/k}$$

$$1 \qquad X^m_{i/j} \qquad (X_{i/k} \cdot X^{-1}_{j/k})^m X^m_{j/k} = X^m_{i/k}$$

Now we get a sheaf of $\mathcal{O}_{\mathbb{P}^n}$*-module :* $\boldsymbol{(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m))}$

Proposition 3.85 $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = \{$*degree m homogenous polynomials in $n+1$-variables* $\}$

Proof: We only proof the case $m \geq 0$:

$$\mathbb{P}^n_A = \cup_{i=0}^n U_i, \quad U_i = \mathrm{Spec}A[x_{j/i}, j = 0, \dots, n, j \neq i]$$
$$U_{i,j} = D(x_{j/i}) \subseteq U_i, \quad U_{i,j} = \mathrm{Spec}A[x_{k/i}, k \neq i, j, x_{j/i}, (x_{j/i})^{-1}].$$

The glueing map is denoted by $\alpha_{i,j} : U_{i,j} \to U_{j,i}$, and it's induced by

$$\alpha'_{i,j} : A[x_{k/j}, k \neq i, j, x_{i/j}, (x_{i/j})^{-1}] \to A[x_{k/i}, k \neq i, j, x_{j/i}, (x_{j/i})^{-1}]$$
$$x_{k/j} \mapsto x_{k/i} \cdot (x_{j/i})^{-1}, \quad x_{i/j} \mapsto (x_{j/i})^{-1}.$$

The sheaf $\mathcal{O}_{\mathbb{P}^n_A}(m)$ is defined by letting $\mathcal{O}_{\mathbb{P}^n_A}(m)(U_i) = \mathcal{O}_{U_i}$ as $\mathcal{O}_{U_i}$-module, and glueing the sheaf by using the following map:

$$\alpha^m_{i,j} : \mathcal{O}_{\mathbb{P}^n}(m)(U_{i,j}) \to \alpha^*_{i,j}\mathcal{O}_{\mathbb{P}^n}(m)(U_{j,i}), \quad 1 \mapsto (x_{i/j})^m,$$

Remark: Since $U_{j,i}$ is affine, by Prop 3.103 in notes,

$$\alpha^*_{i,j}\mathcal{O}_{\mathbb{P}^n}(m)|_{U_{j,i}} \cong (\mathcal{O}_{\mathbb{P}^n}(m)(U_{j,i})) \widetilde{\otimes_{\mathcal{O}_{\mathbb{P}^n(m)(U_{j,i}))}}} \mathcal{O}_{\mathbb{P}^n}(m)(U_{i,j}))$$

And from Theorem 3.55 in notes, to give a morphism between two quasi-coherent sheaves, we only need to define on their global sections. The $\mathcal{O}_{U_i}$-module structure on $\alpha^*_{i,j}\mathcal{O}_{\mathbb{P}^n}(m)(U_{j,i})$ implies:

$$x_{k/i} \cdot 1 \mapsto (x_{i/j})^m \otimes (x_{k/i}) = x^m_{i/j} \cdot x_{k/j} \cdot x^{-1}_{i/j}$$

where the second equality follows from the definition of $\alpha'_{i,j}$.

The map $\Phi : A[x_0, \dots, x_n]_m \to \Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m))$ is defined as follows: given a homogeneous polynomial F of degree m, $F \mapsto (f_i)_{i=0}^n$, where $f_i \in \mathcal{O}_{\mathbb{P}^n_A}(U_i) = A[x_{j/i, j \neq i, j=0,\dots,n}]$ s.t. $\beta_i(f_i) = \frac{F}{x_i^m}$, and β_i is defined by

$$\beta_i : A[x_{j/i}, j \neq i, j = 0, \dots, n] \to A[x_0, \dots, x_n, x_i^{-1}], \quad x_{j/i} \mapsto \frac{x_j}{x_i}.$$

It's easy to check that β_i is injective. Since F is homogeneous of degree m and β_i is injective, f_i exists and is unique. Then we prove that Φ is well-defined: from the remark above, we have

$$\alpha^m_{i,j}(f_i) = f_i(x_{0/j} \cdot (x_{i/j})^{-1}, \dots, x_{i-1/j} \cdot (x_{i/j})^{-1}, x_{i+1/j} \cdot (x_{i/j})^{-1} \dots, x_{n/j} \cdot (x_{i/j})^{-1}) \cdot (x_{i/j})^m$$

Remark: $x_{j/i} \mapsto (x_{i/j})^{-1}$.

This implies

$$\begin{aligned}
\beta_j(\alpha^m_{i,j}(f_i)) &= f_i(\frac{x_0}{x_j} \cdot \frac{x_j}{x_i}, \dots, \frac{x_n}{x_j} \cdot \frac{x_j}{x_i}) \cdot \frac{x_i^m}{x_j^m} \\
&= f_i(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}) \cdot \frac{x_i^m}{x_j^m} \\
&= \frac{F(x_0, \dots, x_n)}{x_i^m} \cdot \frac{x_i^m}{x_j^m} = \frac{F(x_0, \dots, x_n)}{x_j^m} = \beta_j(f_j).
\end{aligned}$$

By injectivity of β_j, we have $\alpha^m_{i,j}(f_i) = f_j$. So $(f_i)^n_{i=0}$ defines a section in $\Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m))$.

Φ is injective: this follows from the injectivity of β_i.

To prove that Φ is surjective, we first prove that for a section $(f_i)^n_{i=0}$ in $\Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m))$, we must have deg $f_i \leq m$. Suppose there is a f_i s.t. deg$f_i = n > m$. WLOG we can suppose $f_i = g_i + h_i$, where g_i is homogeneous of deg n, and deg $h_i < n$. Then from the definition of $\alpha_{i,j}$, we get

$$\begin{aligned}\alpha^m_{i,j}(f_i) &= g_i(x_{0/j} \cdot (x_{i/j})^{-1}, \dots, x_{n/j} \cdot (x_{i/j})^{-1}) \cdot (x_{i/j})^m \\ &\quad + h_i(x_{0/j} \cdot (x_{i/j})^{-1}, \dots, x_{n/j} \cdot (x_{i/j})^{-1}) \cdot (x_{i/j})^m \\ = g_i(x_{0/j}, \dots, x_{n/j}) \cdot (x_{i/j})^{m-n} &+ h_i(x_{0/j} \cdot (x_{i/j})^{-1}, \dots, x_{n/j} \cdot (x_{i/j})^{-1}) \cdot (x_{i/j})^m\end{aligned}$$

which can't be a restriction of an element in $\mathcal{O}_{U_j}$ since $n > m$. Contradiction.

Then we can prove that Φ is surjective. Given $(f_i)^n_{i=0}, f_i \in \mathcal{O}_{U_i}, \alpha^m_{i,j}(f_i) = f_j, \text{deg} f \leq m$. Take $F = x_0^m \beta_0(f_0)$. Because deg $f_0 \leq m$, F is a well-defined homogeneous polynomial of degree m. And for any j,

$$\frac{F}{x_j^m} = \frac{x_0^m}{x_j^m}\beta_0(f_0) = \frac{x_0^m}{x_j^m}\beta_0(\alpha_{j,0}(f_j)) = \frac{x_0^m}{x_j^m} f_j(\frac{x_0}{x_j}, \dots, \frac{x_n}{x_j}) \cdot \frac{x_j^m}{x_0^m} = f_j(\frac{x_0}{x_j}, \dots, \frac{x_n}{x_j}) = \beta_j(f_j).$$

So $\Phi(F) = (f_i)^n_{i=0}$. $\square$

3.10 Vector Bundle

3.10.1 Line bundle

Def 3.86 *An A-module M is **invertible** if the endofunctor $M \otimes_A$ - on the category of A-modules is an equivalence of categories*

This is equivalent to the existence of an A-module M' such that $M \otimes_A M' \cong A$ (this is simple to prove, one should try it)

*A module M is called **finite locally free** if there exists $f_1, \cdots, f_n \in A$ generating the unit ideal such that $M[f_i^{-1}]$ is a free $A[f_i^{-1}]$-module of finite rank*

Theorem 3.87 *1) If M is invertible, then M is a direct summand of a finite free A-module*

2) A module M is finite locally free if and only if it is flat and finitely presented

3) A module M is invertible if and only if it is locally free of rank 1

(i.e. exist a cover of $Spec\, A$ by $D(f_j)$ such that $M[f_j^{-1}] \cong A[f_j^{-1}]$ is locally free of rank 1)

Theorem 3.88 *For all rings A, there is a natural (functorial in A) bijection :*

$\mathbb{P}^n(A)$ $(Hom_{Sch/S}(Spec\, A, \mathbb{P}^n)) \leftrightarrow$ the set of all surjections: $\{A^{n+1} \twoheadrightarrow M\}/\sim$

where M is an invertible A-module, modulo the equivalence relation that $p : A^{n+1} \to M$ is equivalent to $p' : A^{n+1} \to M'$ if and only if there exists an isomorphism $\alpha : M \to M'$ such that $p' = \alpha \circ p$

We can extend that result to arbitrary schemes

Def 3.89 ***line bundle***

*Let X be a scheme, a **line bundle** (also be called **invertible $\mathcal{O}_X$-module**) of X is an $\mathcal{O}_X$-module that is locally isomorphic to $\mathcal{O}_X$*

*More generally, a **vector bundle** V is a sheaf of $\mathcal{O}_X$-modules that is locally free of finite rank, i.e. there exists a cover $X = \cup_i U_i$ such that $V|_{U_i} \cong \mathcal{O}_{U_i}^{n_i}$, for some $n_i \geq 0$. V is a **rank n vector bundle** if it is locally isomorphism to $\mathcal{O}_X^n = \bigoplus_{i=1}^n \mathcal{O}_X$ (all $n_i = n$)*

Def 3.90 *Let $\mathcal{L}$ be a line bundle over X and $\sigma \in \Gamma(X, \mathcal{L})$ a global section. We define $V(\sigma) \subseteq X$ closed subscheme as follows:*

On small enough affine $Spec\, A = U \subseteq X$ such that $\mathcal{L}|_U \xrightarrow[t]{\sim} \widetilde{A}$

We let $V(\sigma)_U = Spec(A/t(\sigma))$, $V(\sigma)_U \hookrightarrow U$

The ideal $< t(\sigma) > \subseteq A$ does not depend on t. If $t_i : \mathcal{L} \xrightarrow{\sim} \widetilde{A}$ are two trivializations, then there is a unit $u \in A^\times$ such that $t_2 = t_1 u$

Def 3.91 *If $I \subseteq \Gamma(X, \mathcal{L})$, we let $V(I)$ be given by $Spec(A/t(I))$*

Proposition 3.92

$$\Gamma(X,\mathcal{L}) \xrightarrow{\sim} Hom_{\mathcal{O}_X}(\mathcal{O}_X,\mathcal{L})$$

$$s \longmapsto [a \mapsto a\cdot s, \forall a \in \mathcal{O}_X(U)]$$

$$\Gamma(f)(1) \in \Gamma(X,\mathcal{L}) \longleftarrow f$$

Def 3.93 *Define $F : Sch^{op} \to Sets$*

$F(X) = \{(\mathcal{L}, s_0, \cdots, s_n) | \mathcal{L}$ is line bundle, $s_0, \cdots, s_n \in \Gamma(X,\mathcal{L})$, s.t.$V(\{s_0,\cdots,s_n\}) = \emptyset\}/\sim$

$(\mathcal{L}', s_0', \cdots, s_n') \sim (\mathcal{L}', s_0', \cdots, s_n')$ if there is a isomorphism of $\mathcal{O}_X$-module $\alpha : \mathcal{L} \xrightarrow{\sim} \mathcal{L}'$ with $\alpha(s_i) = s_i'$

Example 3.94 *Fix k a field (PID), consider $F(k)$:*

$\mathcal{L} \cong \mathcal{O}_k, s_0, \cdots, s_n \in \Gamma(X,\mathcal{O}_X) = k$, $V(s_0,\cdots,s_n) = \emptyset \Leftrightarrow$ at least one $s_i \neq 0$. And $(k_0,\cdots,k_n) \sim (k_0',\cdots,k_n')$ means $(k_0,\cdots,k_n)/k^\times$, same as $k^{n+1}/\sim$, there is a automorphism of $\mathcal{O}_k$ sending $k_i \to k_i'$

So $F(k) = \mathbb{P}^n(k)$

Proposition 3.95 *$F(X) \cong \{\mathcal{O}_X^{n+1} \xrightarrow{s_0,s_1,\cdots s_n} \mathcal{L} \mid surjection$, with $\mathcal{L}$ an line bundle of $X\}/\sim$*

Proof: We can define this map on every small enough open affine set U:

$\mathcal{O}_X^{n+1}(U) \xrightarrow{s_0|_U, s_1|_U, \cdots s_n|_U} \mathcal{L}(U)$

surjection on every stalk $\Leftrightarrow V(\{s_0,\cdots,s_n\})_U = \emptyset$, $\forall U \Leftrightarrow V(\{s_0,\cdots,s_n\}) = \emptyset$

□

Def 3.96 *If $Y \xrightarrow{f} X$, $F(X) = \{(\mathcal{L}, s_0, \cdots, s_n) | \mathcal{L}$ is line bundle, $s_0, \cdots, s_n \in \Gamma(X,\mathcal{L})$, s.t.$V(\{s_0,\cdots,s_n\}) = \emptyset\}/\sim$*

Define $F(X) \xrightarrow{F(f)} F(Y)$ as follows :

$F(f)(\mathcal{L}, s_0, \cdots, s_n) = (f^*\mathcal{L}, f^*s_0, \cdots, f^*s_n)$

$\Gamma(X,\mathcal{L}) = \Gamma(X, f_*f^*\mathcal{L}) \to \Gamma(Y, f^*\mathcal{L})$

$s_i \mapsto f^*s_i$

global section : $s_i : \mathcal{O}_X \to \mathcal{L}$

$f^*s_i : f^*\mathcal{O}_X \to f^*\mathcal{L}$

An A-valued point of $\mathbb{P}^n$ is the data of a line bundle together with $n+1$ section (subject to a vanishing locus condition). If $y_0, y_1, ..., y_n$ denote the global sections A-linear combinations of the y_i make sense as global sections of this line bundle and in particular it makes sense to impose the condition that a specific A-linear combination vanishes.

Theorem 3.97 *F is representable by $\mathbb{P}^n_{\mathbb{Z}}$. ($F(X) \cong Hom(X, \mathbb{P}^n_{\mathbb{Z}})$)*

Recall: We say a contravariant functor from $\mathcal{C}$ to Sets is represented by Y if it is naturally isomorphic to the functor h_Y ($h_Y : \mathcal{C} \to Sets$ defined by $h_Y(X) = Mor(X,Y)$). We say it is ***representable*** *if it is represented by some Y*

Sometimes we also refer to $F(X)$ as $\mathbb{P}^n_{\mathbb{Z}}(X)$

(See images 5-7 for proof)

Def 3.98 *For A any ring, define $\mathbb{P}^n_A = \mathbb{P}^n_{\mathbb{Z}} \times_{Spec\,\mathbb{Z}} Spec\,A$*

3.10.2 Picard group

Def 3.99 *Let's fix X a scheme*

If $\mathcal{M},\mathcal{N}$ are $\mathcal{O}_X$-modules, we let $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N} \in \mathcal{O}_X$-module

$(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N})^{Pre} : Ouv_X \to Groups$

$\boldsymbol{(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N})^{Pre}(U) = \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{N}(U)}$

$\boldsymbol{\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}} = sheafification(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N})^{Pre} = \boldsymbol{\mathcal{M}\, \widetilde{\otimes_{\mathcal{O}_X}}\, \mathcal{N}}$

Def 3.100 *Define $\underline{Hom}_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}) \in \mathcal{O}_X$-module :*

$\boldsymbol{\underline{Hom}_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})(U) = Hom_{\mathcal{O}_X|_U}(\mathcal{M}|_U,\mathcal{N}|_U)} \subseteq Hom_{\mathcal{O}_X(U)}(\mathcal{M}(U),\mathcal{N}(U))$

Caution : The rule $U \mapsto Hom_{\mathcal{O}_X(U)}(\mathcal{M}(U),\mathcal{N}(U))$ is not a presheaf

Proposition 3.101

$$\boldsymbol{\underline{Hom}_{\mathcal{O}_X}(\mathcal{O}_X,\mathcal{N}) \cong \mathcal{N},\ \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{N} \cong \mathcal{N}}$$

Proof:

$$\begin{array}{ccc} \underline{Hom}_{\mathcal{O}_X}(\mathcal{O}_X,\mathcal{N})(U) = Hom_{\mathcal{O}_X|_U}(\mathcal{O}_X|_U,\mathcal{N}|_U) & \xrightarrow{\sim} & \mathcal{N}(U) \\ f & \longmapsto & f(U) \\ [a \mapsto a \cdot n] & \longleftarrow\!\shortmid & n \end{array}$$

As presheaves :

$$\begin{array}{ccc} \mathcal{O}_X(U) \otimes_{\mathcal{O}_X(U)} \mathcal{N}(U) & \xrightarrow{\sim} & \mathcal{N}(U) \\ a \otimes n & \longmapsto & a \cdot n \end{array}$$

□

Proposition 3.102 *Let $\mathcal{L},\mathcal{M},\mathcal{N}$ be $\mathcal{O}_X$-modules,then*

$$\boldsymbol{Hom_{\mathcal{O}_X}(\mathcal{L},\underline{Hom}(\mathcal{M},\mathcal{N})) \cong Hom_{\mathcal{O}_X}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M},\mathcal{N})}$$

$$\boldsymbol{\underline{Hom}(\mathcal{M},-) \perp (-) \otimes_{\mathcal{O}_X} \mathcal{M}}$$

Proposition 3.103 *1) If $\mathcal{M}$ and $\mathcal{N}$ are qcoh of $\mathcal{O}_X$-modules, then $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ is also qcho*

2) If $X = Spec\,A$, then $\boldsymbol{\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} = M\, \widetilde{\otimes_A}\, N}$

Proof: 1) follows from 2), now we proof 2): $f \in A$

$$
\begin{aligned}
(\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N})^{Pre}(D(f)) &= \widetilde{M}(D(f)) \otimes_{\mathcal{O}_X(D(f))} \widetilde{N}(D(f)) \\
&= \widetilde{M}[f^{-1}] \otimes_{A[f^{-1}]} \widetilde{N}[f^{-1}] \\
&\cong M \otimes_A N[f^{-1}] \\
&= \widetilde{M \otimes_A} N(D(f))
\end{aligned}
$$

□

Caution : $\underline{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ might not be quasicoherent even if $\mathcal{F}$ and $\mathcal{G}$ are

Def 3.104 *Let $(f, f^\flat)$ be a map of ringed spaces $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$*

1) If $\mathcal{N}$ is an $\mathcal{O}_Y$ -module, then the pushforward $f_*\mathcal{N}$ is the $\mathcal{O}_X$ -module with the structure morphism

$$\mathcal{O}_X \times f_*\mathcal{N} \xrightarrow{f^\flat \times id_{f_*\mathcal{N}}} f_*\mathcal{O}_Y \times f_*\mathcal{N} = f_*(\mathcal{O}_Y \times \mathcal{N}) \xrightarrow{f_*(-)} f_*\mathcal{N}$$

2) If $\mathcal{M}$ is an $\mathcal{O}_X$ -module, then $f^{-1}\mathcal{M}$ is a sheaf of $f^{-1}\mathcal{O}_X$ -modules via

$$f^{-1}\mathcal{O}_X \times f^{-1}\mathcal{M} = f^{-1}(\mathcal{O}_X \times \mathcal{M}) \longrightarrow f^{-1}(\mathcal{M}).$$

We now define the pallback $f^*\mathcal{M}$ as the $\mathcal{O}_Y$ -module

$$\boldsymbol{f^*\mathcal{M} = f^{-1}\mathcal{M} \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_Y}$$

Proposition 3.105 *There is an adjunction with left adjoint f^* and right adjoint f_*. In other words, for each $\mathcal{O}_X$-module $\mathcal{M}$, and each $\mathcal{O}_Y$-module $\mathcal{N}$. we have the following natural identification :*

$$\mathbf{Hom}_{\mathcal{O}_Y}(\boldsymbol{f^*\mathcal{M}, \mathcal{N}}) \cong \mathbf{Hom}_{\mathcal{O}_X}(\boldsymbol{\mathcal{M}, f_*\mathcal{N}})$$

Proof: Sketch of the Proof :

We already have the adjunction

$$\mathrm{Hom}(f^{-1}\mathcal{M}, \mathcal{N}) \cong \mathrm{Hom}(\mathcal{M}, f_*\mathcal{N}) \supseteq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, f_*\mathcal{N}),$$

and a subset on the right hand side. There is a subset on the left hand side corresponding to the $\mathcal{O}_X$-linear maps $\mathcal{M} \to f_*\mathcal{N}$, and that is $\mathrm{Hom}_{f^{-1}\mathcal{O}_X}\left(f^{-1}\mathcal{M}, \mathcal{N}\right)$ so we (have to check we) have an adjunction between these subsets,

$$\mathrm{Hom}_{f^{-1}\mathcal{O}_X}(f^{-1}\mathcal{M}, \mathcal{N}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, f_*\mathcal{N}).$$

We can now use a change of rings isomorphism to change the left hand side to

$$\mathrm{Hom}_{f^{-1}\mathcal{O}_X}\left(f^{-1}\mathcal{M}, \mathcal{N}\right) \cong \mathrm{Hom}_{\mathcal{O}_Y}\left(f^{-1}\mathcal{M} \otimes_{f^{-1}\mathcal{O}_X} O_Y, \mathcal{N}\right) \cong \mathrm{Hom}_{\mathcal{O}_Y}\left(f^*\mathcal{M}, \mathcal{N}\right).$$

□

Proposition 3.106 *1) Let $f : Y \to X$ be (any!) map of schemes, and $\mathcal{M}$ a quasi-coherent $\mathcal{O}_X$-module, then $f^*\mathcal{M}$ is a quasi-coherent $\mathcal{O}_Y$-module*

2) If $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$, then $\mathcal{M} \cong \widetilde{M}$ for some A-module M, and then we have $\boldsymbol{f^*\mathcal{M} \cong M \widetilde{\otimes_A} B}$

Proof: Proof For part 1, we notice that we can cover Y by open affines $V = \operatorname{Spec} B \subseteq Y$ mapping into an open affine $U = \operatorname{Spec} A \subseteq X$, then let $g : V \to U$ be the restriction of f to V, then $(f^*\mathcal{M})|_V = g^*(\mathcal{M}|_V)$ To check the quasi-coherentness of $f^*\mathcal{M}$, it suffices to check the quasi-coherentness of $g^*(\mathcal{M}|_V)$. This implies that we can replace Y by $Spec\, B$ and X by $Spec\, A$, thus we only have to prove part 2. In that case $\mathcal{M} = \widetilde{M}$ and for all $\mathcal{O}_Y$-modules $\mathcal{N}$ we have the following series of isomorphisms :

$$\operatorname{Hom}_{\mathcal{O}_Y}(f^*\mathcal{M}, \mathcal{N}) \cong \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{M}, f_*\mathcal{N}) \cong \operatorname{Hom}_A(M, f_*\mathcal{N}(X)) = \operatorname{Hom}_A(M, \mathcal{N}(Y)$$

$$\cong \operatorname{Hom}_B(M \otimes_A B, \mathcal{N}(Y)) \cong \operatorname{Hom}_{\mathcal{O}_Y}(\widetilde{M \otimes_A} B, \mathcal{N}).$$

The Yoneda lemma now tells us that $f^*\mathcal{M} \cong \widetilde{M \otimes_A} B$

$\square$

Def 3.107 *X a scheme. The definition of X_s makes sense when s is a global section of any $\mathcal{O}_X$-module $\mathcal{M}$:*

Let $s \in \mathcal{M}(X)$,

denote $i_x : Spec\, \kappa(x) \hookrightarrow X, \forall x \in X$

$$\Gamma(X, \mathcal{M} \to (i_x)_*(i_x)^*\mathcal{M}) : \mathcal{M}(X) \xrightarrow{ev_x} (i_x)^*\mathcal{M}(Spec\, \kappa(x)) = \mathcal{M}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x) = \mathcal{M}_x / \mathfrak{m}_x \mathcal{M}_x$$

define $\boldsymbol{X_s = \{p \text{ in } X : s(p) \neq 0 \text{ in } \mathcal{M}_p \otimes \kappa(p)\}}$ *(i.e. $D(s) = \{x \in X, ev_x(s) \neq 0\} = \{x \in X, s_x \notin \mathfrak{m}_x\mathcal{M}_x\} = \{x \in X, s_x \in \mathcal{M}_x$ is a generator$\}$), where $\kappa(p)$ is the residue field at p, $\mathcal{M}_p$ is the stalk of $\mathcal{M}$ at p and $s(p)$ is the image of s under the map $\mathcal{M}(X) \to \mathcal{M}_p \to \mathcal{M}_p \otimes_{\mathcal{O}_{X,p}} \kappa(p) = \mathcal{M}_p/\mathfrak{m}_p\mathcal{M}_p$.*

But for X_s to be an open subset of X (and therefore interesting to consider) we need $\mathcal{M}$ to be locally free. To prove this work locally and reduce to the case where $\mathcal{M}$ is the structure sheaf.

Remark 3.108 *Remark 3.22 is true even when f is a morphism of locally ringed spaces and s is a global section of a locally free sheaf :*

$(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, f is a morphism of locally ringed spaces and s is a global section of a locally free sheaf

Then $f^{-1}(Y_s) = X_{f^*(s)}$

Proposition 3.109 *If W is a vector bundle, then $\underline{Hom}_{\mathcal{O}_X}(W, \mathcal{O}_X)$ is a vectoe bundle*

Proof: If $W|_U \cong \mathcal{O}_X^n|_U$, then :

$$\begin{aligned}\underline{Hom}_{\mathcal{O}_X}(W, \mathcal{O}_X)|_U &\cong \underline{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^n|_U, \mathcal{O}_X|_U) \\ &= \underline{Hom}_{\mathcal{O}_X}(\mathcal{O}_U, \mathcal{O}_U)^n \\ &= \mathcal{O}_U^n \qquad [f \mapsto (f(e_1, e_2, \cdots, e_n))]\end{aligned}$$

□

Def 3.110 *Define* ***dual vector bundle*** *:* $W^\vee = \underline{Hom}_{\mathcal{O}_X}(W, \mathcal{O}_X)$

Proposition 3.111 *1) For vector bundle W, we have canonical evaluation pairings :*

$$e_V : W \otimes_{\mathcal{O}_X} W^\vee \to \mathcal{O}_X$$

2) When $W = \mathcal{L}$ is a line bundle, $e_V : \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee \to \mathcal{O}_X$ is an isomorphism

Proof: 1)

$$\begin{aligned}W(U) \otimes_{\mathcal{O}_X(U)} Hom(W|_U, \mathcal{O}_U) &\to \mathcal{O}_U \\ w \otimes f &\mapsto f(U)\end{aligned}$$

2) locally

$$\begin{aligned}\mathcal{O}_X \otimes_{\mathcal{O}_X} Hom(\mathcal{O}_X, \mathcal{O}_X) &\to \mathcal{O}_X \\ 1 \otimes id &\mapsto id(1)\end{aligned}$$

□

Def 3.112 ***Picard group***

Let X be a scheme, we let $Pic(X) = \{line\ bundles\}/\cong$

$$\begin{aligned}Pic(X) \times Pic(X) &\to Pic(X) \\ (\mathcal{L}_1, \mathcal{L}_2) &\mapsto \mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2 \\ (-)^{-1} : Pic(X) &\to Pic(X) \\ \mathcal{L} &\mapsto \mathcal{L}^\vee\end{aligned}$$

This is a group

Proposition 3.113 $\mathbb{Z} \cong Pic(\mathbb{P}^n_{\mathbb{Z}})$ $m \mapsto O_{\mathbb{P}^n}(m)$

$$\mathcal{O}_{\mathbb{P}^n}(m) = \begin{cases} \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus m} & m \geq 1 \\ \mathcal{O}_{\mathbb{P}^n} & m = 0 \\ \mathcal{O}_{\mathbb{P}^n}(-m)^\vee & m \leq -1 \;\; (\mathcal{O}(-1) = \mathcal{O}(1)^\vee) \end{cases}$$

Def 3.114 *Fix X a scheme and W a vector bundle over X*

$\mathbb{V}(W) : Sch/X \to Sets$*, then defines a functor $\mathbb{V}(W)$ on all schemes $f : Y \to X$ over X by*

$$\mathbb{V}(W)(Y) := (f^*W)(Y) = \Gamma(Y, f^*W)$$

Theorem 3.115 *This functor $\mathbb{V}(W)$ is representable by a scheme over X, also denoted by $\mathbb{V}(W)$ over X, such that there is a cover $X = \bigcup_i U_i$ such that*

$$\boldsymbol{\mathbb{V}(W) \times_X U_i = \mathbb{V}(W)|_{U_i} \cong U_i \times_{Spec\,\mathbb{Z}} \mathbb{A}^n_{\mathbb{Z}}}$$

Proof: (sketchy) It's enough to prove this result locally on X by general gluing lemmas. So assume that X is affine, $X = \operatorname{Spec} A$ so that $W|_{Spec\,A} \cong \mathcal{O}^n_{Spec\,A}$

$$\mathbb{V}(W)(Y) = f^*W(Y) = f^*(\mathcal{O}^n_X)(Y) \cong \mathcal{O}_Y(Y)^n \cong \operatorname{Hom}_{Ring}(\mathbb{Z}[T_1, \cdots, T_n], \mathcal{O}_Y(Y)).$$

We then use the universal property of fibre products to obtain :

$$\mathbb{V}(W)(Y) = \mathcal{O}_Y(Y)^n = \operatorname{Hom}_{Ring}(\mathbb{Z}[T_1, \cdots, T_n], \mathcal{O}_Y(Y)) \cong \operatorname{Hom}_{\mathrm{Sch}}(Y, \mathbb{A}^n) \cong \operatorname{Hom}_{\mathrm{Sch}\,/X}(Y, \mathbb{A}^n \times X)$$

In this way we can see that $\mathbb{V}(W)$ is represented by $X \times \mathbb{A}^n$, locally $\square$

Proposition 3.116 *If X is a scheme and W vector bundle of X, then*

$$\mathbb{V}(W)(Y) = \{(f, \gamma) | f : Y \to X, \gamma \in \Gamma(Y, f^*W)\}$$

Remark 3.117 *We can now recount the definition of $V(s)$:*

Take X scheme, E a vector bundle over X, $s \in \Gamma(X, E)$,

$$Hom_{Sch/X}(Y, \mathbb{V}(E)) = \left\{\begin{tikzcd} Y \arrow[rr, dotted] \arrow[dr] & & \mathbb{V}(E) \arrow[dl] \\ & X & \end{tikzcd}\right\} = \Gamma(Y, f^*E)$$

In particular, $Hom_{Sch/X}(X, \mathbb{V}(E)) = \Gamma(X, E)$

Define : let $s \in \Gamma(X, E)$, define $V(s)$ as the fiber product of

$$\begin{tikzcd} V(s) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "s"] \\ X \arrow[r, "0"'] & \mathbb{V}(E) \end{tikzcd}$$

Some facts abput $V(s)$:

· $\forall Y \xrightarrow{f} X$ *a X-scheme, f factors over $V(s) \Leftrightarrow f^*s \in \Gamma(Y, f^*E)$ is zero, in other words*

$$Hom_{Sch/X}(Y, V(s)) = \begin{cases} \emptyset & if\ f^*s \neq 0 \\ * & if\ f^*s = 0 \end{cases}$$

· *If $X = Spec\,A$ and $E = \widetilde{A^n}, s = (f_1, \cdots, f_n) \in A^n$, then $V(s) = Spec(A/(f_1, \cdots, f_n))$*

· *The underlying set of $V(s)$ is*

$$\{x \in X\ :\ s \in ker(E(X) \xrightarrow{ev_x} E_x \otimes k(x))\}$$

$$s \longmapsto s_x \otimes 1 =: s(x)$$

3.10.3 ⋆ Blowing Up

Def 3.118 ***Universal Property of Blowing Up***

Proposition 3.119 ***(Universal Property of Blowing Up)***

3.11 Internal Hom Sheaves

3.11.1 Zariski Sheaf

Recall: Prop 3.28 (Gluing of morphisms)

Let X,Y be locally ringed spaces. For evevy open subset $U \subseteq X$ let $\mathrm{Hom}(U, Y)$ be the set of morphisms $(U, \mathcal{O}_{X|U}) \to (Y, \mathcal{O}_Y)$ of locally ringed spaces. Then $U \mapsto \mathrm{Hom}(U, Y)$ is a sheaf of sets on X

In other words: If $X = \bigcup_i U_i$ is an open covering, then a family of morphisrms $U_i \to Y$ glues to a morphism $X \to Y$ if and only if the morphisms coincide on intersections $U_i \cap U_j$, and the resulting morphisrm $X \to Y$ is uniquely deterrmined

Def 3.120 *A contravariant functor $F : \mathrm{Sch}_S \to \mathrm{Set}$ is a **Zariski sheaf** (sheaf for the Zariski topology) if it satisfies the following condition:*

For every $X \in \mathrm{Obj}(\mathrm{Sch}_S)$ y and every open cover $\{U_i\}$ of X. the natural map

$$\{\eta \in F(X)\} \to \{\{\eta_i \in F(U_i)\} : \forall i, j, \eta_i|_{U_i \cap U_j} = \eta_j|_{U_i \cap U_i}\}$$

is a bijection

i.e. $F(X) \xrightarrow{\sim} Eq(\Pi_{i \in I} F(U_i) \rightrightarrows \Pi_{i,j \in I} F(U_i \cap U_j))$

Corollary 3.121 *Every representable functor $F : (Sch_S)^{op} \to (Sets)$ is a Zariski sheaf*

Theorem 3.122 *A Zariski sheaf is representable if and only if it admits an open cover by representable Zariski sheaves*

3.11.2 Internal Hom Sheaves

Back to $Hom_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) = Hom_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U) \to Hom_{\mathcal{O}_X(U)}(\mathcal{F}(U), \mathcal{G}(U)) \quad (\mathcal{O}_U = \mathcal{O}_X|_U)$

Example 3.123 *Suppose that $\mathcal{F}, \mathcal{G} \in \mathcal{O}_X$-modules are quasicoherent and $Spec\, A = U \subseteq X$, then*

$$\boldsymbol{Hom_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) = Hom_{\mathcal{O}_X(U)}(\mathcal{F}(U), \mathcal{G}(U))}$$

Example 3.124 *In general, even if $\mathcal{F}, \mathcal{G} \in Qcoh(X)$, $Hom_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ might not be quasicoherent. A counter example :*

$$X = Spec\,\mathbb{Z}, \quad Hom_{\mathcal{O}_X}(\oplus_{i=1}^{\infty}\mathcal{O}_X, \mathcal{O}_X)(D(p)) = Hom_{\mathbb{Z}[p^{-1}]}(\oplus_{i=1}^{\infty}\mathbb{Z}[\frac{1}{p}], \mathbb{Z}[\frac{1}{p}]) = \Pi_{i=1}^{\infty}\mathbb{Z}[\frac{1}{p}]$$

$$Hom_{\mathcal{O}_X}(\oplus_{i=1}^{\infty}\mathcal{O}_X, \mathcal{O}_X)(X) = \Pi_{i=1}^{\infty}\mathbb{Z}$$

$$but \quad \Pi_{i=1}^{\infty}\mathbb{Z}[\frac{1}{p}] \neq (\Pi_{i=1}^{\infty}\mathbb{Z})[\frac{1}{p}] \quad (that\ means\ M|_{D(f)} \neq \mathcal{M}(X)_f)$$

Def 3.125 *Let $X \in Sch, \mathcal{F} \in Qcoh(X)$. We say $\mathcal{F}$ is*

*1) **Finitely presented (f.p)***

*2) **Finitely generated (f.g)***

if there is an open cover $\cup_{i\in I} U_i \in X$ *s.t.* $U_i = Spec\, A_i$ *and* $\mathcal{F}(U_i)$ *is a*

1) Finitely presented A_i*-module*

2) Finitely generated A_i*-module*

Proposition 3.126 *If* $\mathcal{F} \in Qcoh(X)$ *is f.p (f.g) and* $Spec\, A = U \subseteq X$*, then* $\mathcal{F}(U)$ *is a f.p (f.g) A-module*

Proposition 3.127 *If* $X \in Sch, \mathcal{F}, \mathcal{G} \in Qcoh(X)$ *and* $\mathcal{F}$ *is f.p, then* $Hom_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ *is quasicoherent*

Proof: Use Ex 1.75 :

Suppose M is a finitely presented A-module, then :

$$S^{-1}Hom_A(M,N) \cong Hom_{S^{-1}A}(S^{-1}M, S^{-1}N)$$

□

3.11.3 Properties of morphisms of schemes

Def 3.128 *Let* P *be a property of morphisms of schemes, we say that* P *is*

1) ***presented under composition (P ∈ COMP)*** *if:*

$$(X \xrightarrow{f} Y) \in P \text{ and } (Y \xrightarrow{g} Z) \in P \Rightarrow (X \xrightarrow{g\circ f} Z) \in P$$

2) ***presented under base change (P ∈ BC)*** *if:*

$$(X \xrightarrow{f} Y) \in P \text{ and } (S \xrightarrow{g} Y) \Rightarrow (X \times_Y S \to S) \in P$$

3) ***local on the target (P ∈ LOCT)*** *if:*

$$\text{whenever } \cup_{i\in I} U_i \to Y \text{ is an open cover, and} (X \times_Y U_i \xrightarrow{f_i} U_i) \in P, \forall i \Rightarrow (X \xrightarrow{f} Y) \in P$$

4) ***local on source (P ∈ LOCS)*** *if:*

$$\text{whenever } \cup_{i\in I} V_i \to X \text{ is an open cover, and} (V_i \xrightarrow{f|_{V_i}} Y) \in P, \forall i \Rightarrow (X \xrightarrow{f} Y) \in P$$

Example 3.129 *If* $P_1 = \{open\ immersion\}$, $P_2 = \{closed\ immersion\}$

Then $P_1, P_2 \in COMP, BC, LOCT$*, but* $P_1, P_2 \notin LOCS$

3.12 Hausdorff Property for Schemes

A topological space X is Hausdorff if and only if the diagonal $\triangle \subseteq X \times X$ is a closed subset. The analogue in algebraic geometry is, given a scheme X over a base scheme S, to consider the diagonal morphism

$$\triangle_{X/S} : X \to X \times_S X$$

This is the unique morphism of schemes such that $pr_1 \circ \triangle_{X/S} = id_X$ and $pr_2 \circ \triangle_{X/S} = id_X$ (it exists in any category with fibre products)

3.12.1 Separated

Def 3.130 *A top space T is* ***Hausdorff*** *if and only if the diagonal $\triangle \subseteq T \times T$ is closed*

Def 3.131 *A morphism of schemes $f : X \to S$ is called* ***separated*** *if the diagonal $\triangle_{X/S} : X \to X \times_S X; x \mapsto (x, x)$ is a closed immersion*

$$\begin{tikzcd}
X \arrow[dr, dashed, "\triangle_{X/S}"] \arrow[drr] \arrow[ddr] & & \\
& X \times_S X \arrow[r] \arrow[d] & X \arrow[d] \\
& X \arrow[r] & S
\end{tikzcd}$$

A scheme X is called ***separated*** *if $X \to Spec\mathbb{Z}$ is separated*

Def 3.132 *A morphism $i : X \to Y$ is called a* ***locally closed immersion*** *if*

1. $|i| : |X| \to |Y|$ is a locally closed immersion, i.e $|X|$ is open in its closure : exists an open set $|V| \subseteq |Y|$ such that $|X| \subseteq |V|$ is closed

2. $i^{\#} : i^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is surjective

Proposition 3.133 *Equal to $i : X \to Y$ can factored into*

$$X \xrightarrow{\rho_1} Z \xrightarrow{\rho_2} Y$$

where ρ_1 is a closed embedding and ρ_2 is an open embedding

Proof: Both open and closed immersions are locally closed immerions, and composites of locally closed immersions are locally closed immersions, so we have one direction.

Conversely, take ρ_1 to be a locally closed immersion, then $|X| \subseteq |Y|$ is a locally closed map of spaces, as there exists an open set $|V| \subseteq |Y|$ such that $|X| \subseteq |V|$ is closed. Then there exists a unique open subscheme $V \subseteq Y$ with underlying space $|V|$, and $X \to V$ is a closed immersion $\square$

Example 3.134 *The morphism $Spec\, k[t, t^{-1}] \to Spec\, k[x, y]$ given by $(x, y) \to (t, 0)$ is a locally closed embedding*

Remark 3.135 *Suppose $V \to X$ is a morphism. Consider three conditions:*

(i) V is the intersection of an open subscheme of X and a closed subscheme of X

(ii) V is an open subscheme of a closed subscheme of X, i.e., it factors into an open embedding followed by a closed embedding

(iii) V is a closed subscheme of an open subscheme of X, i.e., V is a locally closed embedding

(i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii). But (iii) does not always imply (i) and (ii)

Proposition 3.136 *Let $f : X \to S$ be an morphism of schemes, then the diagonal map $\triangle_{X/S} : X \to X \times_S X$ is a locally closed immersion*

If X and S are affine, then this is in fact just a closed immersion

Proof: First assume that X and S are affine, so let $X = Spec\, A$ and $S = Spec\, R$, then we have, $X \times_S X \cong Spec\,(A \otimes_R A)$, and the map $\triangle_{X/S} : X \to X \times_S X$ corresponds to the multiplication map $A \otimes_R A \to A$, which is surjective. Hence $\triangle_{X/S}$ is a closed immersion in this case.

In general, for any $x \in X$, we choose some open affine U with $x \in U = Spec\, A \subseteq X$ mapping to $Spec\, R \subseteq S$. Then again we have, $W = Spec\, A \times_{Spec\, R} Spec\, A \cong Spec\,(A \otimes_R A)$, which is an open neighbourhood of $\triangle(U)$

$$\begin{tikzcd} U \arrow[r, "closed"] & \triangle(U) \arrow[rr, bend left, "open"] & \subset W \arrow[r, hook] & X \times X \end{tikzcd}$$

$U \xrightarrow{closed} \triangle(U)$ is closed by the affine case, hence $\triangle_{X/S}$ is a locally closed immersion

(Recall closed immersion and open immersion $\in LOCS$)

$\square$

Corollary 3.137 *If X is an affine scheme over an affine scheme S, X is separated inside of Sch/S*

Corollary 3.138 *The scheme over S, $f : X \to S$ is separated if and only if $|\triangle_{X/S}|(|X|) \subseteq |X \times_S X|$ is a closed subspace*

Proposition 3.139 *Consider the following commutative diagram of schemes:*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

1. *If f and g are separated, then so is h*
2. *If h is separated, then so is f*

3.12.2 Diagonal of scheme morphisms

Def 3.140 *Let $u : X \to S$ be an S-object. The morphism*

$$\triangle_{X/S} := \triangle_u := (id_X, id_X)_S : X \to X \times_S X$$

*is called the **diagonal (morphism)** of X over S*

(2) Let $f : X \to Y$ be a morphism of S-objects. The morphism

$$\Gamma_f := (id_X, f)_S : X \to X \times_S Y$$

is called the ***graph (morphism)*** *of f*

(3) Let $f, g : X \to Y$ be two S-morphisms. An S-object K together with an S-morphism $i : K \to X$ is called (difference) kernel of f and g if for all S-objects T the map $i(T)$ yields a bijection

$$K_S(T) \xrightarrow{\sim} \{x \in X_S(T); f(T)(x) = g(T)(x)\}$$

We denote the ***kernel of f and g*** *by $Ker(f,g)_S$ or simply $Ker(f,g)$ and call the morphism $i : Ker(f,g) \to X$ the* ***canonical morphism***

Proposition 3.141 *Let $u : X \to S, v : Y \to S$ be S-objects, let $p : X \times_S Y \to X$ and $q : X \times_S Y \to Y$ be the projections, and $f, g : X \to Y$ two S-morphisms*

$$\begin{tikzcd} X \times_S Y \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "v"] \\ X \arrow[r, "u"'] & S \end{tikzcd}$$

(1)

$$\triangle_{X/S} = \Gamma_{id_X} \qquad \Gamma_f = (can : Ker(X \times_S Y \underset{f \circ p}{\overset{g}{\rightrightarrows}} Y) \to X \times_S Y)$$

(2) All rectangles of the following diagram are cartesian :

$$\begin{tikzcd} Ker(f,g) \arrow[r, "can"] \arrow[d, "can"'] \arrow[dr, phantom, "\square"] & X \arrow[r, "f"] \arrow[d, "\Gamma_f"] \arrow[dr, phantom, "\square"] & Y \arrow[d, "\triangle_{Y/S}"] \\ X \arrow[r, "\Gamma_g"'] & X \times_S Y \arrow[r, "f \times_S id_Y"'] & Y \times_S Y \end{tikzcd}$$

(3) Let $s : S \to X$ be a section of f (i.e., $f \circ s = id_S$). The following diagram is cartesian :

$$\begin{tikzcd} S \arrow[r, "s"] \arrow[d, "s"'] & X \arrow[d, "\Gamma_{s \circ f}"] \\ X \arrow[r, "\triangle_{X/S}"'] & X \times_S X \end{tikzcd}$$

Proof: Using the Yoneda lemma it suffices to treat the case that $\mathcal{C}$ is the category of sets

$\square$

3.12.3 $\star$ Scheme theoretic closure and density

Def 3.142 *Let X be a scheme. Let $U \subseteq X$ be an open subscheme*

The scheme theoretic image of the morphism $U \subseteq X$ is called the ***scheme theoretic closure*** *of U in X*

We say U is ***scheme theoretically dense*** *in X if for every open $V \subseteq X$ the scheme theoretic closure of $U \cap V$ in V is equal to V*

Example 3.143 *Here is an example where scheme theoretic closure being X does not imply dense for the underlying topological spaces :*

Let k be a field. Set $A = k[x, z_1, z_2, \cdots]/(x^n z_n)$ Set $I = (z_1, z_2, \cdots) \subseteq A$. Consider the affine scheme $X = Spec(A)$ and the open subscheme $U = X \setminus V(I)$. Since $A \to \pi_n A_{z_n}$ is injective we see that the scheme theoretic closure of U is X. Consider the morphism $X \to Spec(k[x])$. This morphism is surjective (set all $z_n = 0$ to see this). But the restriction of this morphism to U is not surjective because it maps to the point $x = 0$. Hence U cannot be topologically dense in X

Lemma 3.144 *Let X be a scheme. Let $U \subseteq X$ be an open subscheme. If the inclusion morphism $U \to X$ is quasi-compact, then U is scheme theoretically dense in X if and only if the scheme theoretic closure of U in X is X*

Proof: Follows from Prop 3.79 part (3) □

Example 3.145 *Let A be a ring and $X = Spec(A)$. Let $f_1, \cdots, f_n \in A$ and let $U = D(f_1) \cup \cdots \cup D(f_n)$. Let $I = Ker(A \to \pi A_{f_i})$. Then the scheme theoretic closure of U in X is the closed subscheme $Spec(A/I)$ of X. Note that $U \to X$ is quasi-compact. Hence we see U is scheme theoretically dense in X if and only if $I = 0$*

Proposition 3.146 *Let $j : U \to X$ be an open immersion of schemes. Then U is scheme theoretically dense in X if and only if $\mathcal{O}_X \to j_* \mathcal{O}_U$ is injective*

Corollary 3.147 *Let X be a scheme. If U, V are scheme theoretically dense open subschemes of X, then so is UV*

Proof: Let $W \subseteq X$ be any open. Consider the map $\mathcal{O}_X(W) \to \mathcal{O}_X(W \cap V) \to \mathcal{O}_X(W \cap V \cap U)$. Both maps are injective. Hence the composite is injective. Hence $U \cap V$ is scheme theoretically dense in X □

Lemma 3.148 *Let $h : Z \to X$ be an immersion. If h is quasi-compact or Z is reduced, then we can factor $h = i \circ j$ with $j : Z \to \overline{Z}$ an open immersion and $i : Z \to X$ a closed immersion*

Proposition 3.149 *Let $h : Z \to X$ be an immersion. Assume either h is quasi-compact or Z is reduced. Let $\overline{Z} \subseteq X$ be the scheme theoretic image of h*

Then the morphism $Z \to \overline{Z}$ is an open immersion which identifies Z with a scheme theoretically dense open subscheme of $\overline{Z}$. Moreover, Z is topologically dense in $\overline{Z}$

Corollary 3.150 *Let X be a reduced scheme and let $U \subseteq X$ be an open subscheme. Then the following are equivalent*

1) U is topologically dense in X

2) The scheme theoretic closure of U in X is X

3) U is scheme theoretically dense in X

Proof: This follows from the fact that a closed subscheme Z of X whose underlying topological space equals X must be equal to X as a scheme $\square$

Corollary 3.151 *Let X be a scheme and let $U \subseteq X$ be a reduced open subscheme. Then the following are equivalent :*

1) The scheme theoretic closure of U in X is X

2) U is scheme theoretically dense in X

If this holds then X is a reduced scheme

Proposition 3.152 *Let X, Y be schemes over S. Let $a, b : X \to Y$ be morphisms of schemes over S. There exists a largest locally closed subscheme $Z \subseteq X$ such that $a|_Z = b|_Z$. In fact Z is the equalizer of (a, b)*

Moreover, if Y is separated over S, then Z is a closed subscheme

Proof: The equalizer of (a, b) is for categorical reasons the fibre product Z in the following diagram

$$\begin{tikzcd} Z = Y \times_{Y \times_S Y} X \arrow[r] \arrow[d] & X \arrow[d, "(a,b)"] \\ Y \arrow[r, "\triangle_{Y/S}"] & Y \times_S Y \end{tikzcd}$$

$\square$

Corollary 3.153 *Let S be a scheme. Let X, Y be schemes over S. Let $f, g : X \to Y$ be morphisms of schemes over S. Let $U \subseteq X$ be an open subscheme such that $f|_U = g|_U$. If the scheme theoretic closure of U in X is X and $Y \to S$ is separated, then $f = g$*

3.13 Finiteness Conditions

Proposition 3.154 ***affine communication***

Let P be some property of affine schemes, such that :

(i) If $Spec\,A$ has property P then for any $f \in A, D(f) = Spec\,A_f$ does too

(ii) If $(f_1, \cdots, f_n) = A$, and $Spec A_{f_i}$ has P for all i, then so does $Spec\,A$

Suppose that $X = \cup_{i\in I} Spec\,A_i$ where $Spec\,A_i$ has property P. Then every affine open subset of X, $Spec\,B$ has P too

(This property is also called affine-local)

Def 3.155 *A scheme X is* ***locally Noetherian*** *if it has a open cover by $X = \cup_{i\in I} Spec\,A_i$ where each A_i is a Noetherian ring*

Proposition 3.156 *If $Spec\,B = U \underset{open}{\subset} X$ affine open and X is locally Noetherian, then B is Noetherian*

Def 3.157 *A scheme X is* ***Noetherian*** *if it is locally Noetherian and quasicompact (or equivalently can be covered by finitely many such affine open sets)*

Def 3.158 *Let T be a topological space, then T is* ***Noetherian*** *if every decreasing sequence of closed subsets of T stabilises*

Remark 3.159 *If T is a Noetherian space, then T is quasi-compact*

Remark 3.160 *Any subspace A of a Noetherian space X is Noetherian, since closed subsets $Z_i \subseteq A$ come from closed subsets $\overline{Z_i} \subseteq X$, which stabilise by assumption*

Proposition 3.161 *Let X be a Notherian scheme, then $|X|$ is a Noetherian space*

Def 3.162 *Let X be a Noetherian scheme, A a* ***coherent*** *$\mathcal{O}_X$-module is a quasi-coherent $\mathcal{O}_X$-module of finite type*

Theorem 3.163 *Let X be a scheme*

1) The category $QCoh(X)$ of quasi-coherent modules on X is a full abelian subcategory of the category $\mathcal{O}_X$-Mod of $\mathcal{O}_X$-modules that is closed under extensions. The functor $QCoh(X) \to \mathcal{O}_X$-Mod is exact

2) Assume that X is Noetherian. The category $Coh(X)$ of coherent modules on X is a full abelian subcategory of the category $\mathcal{O}_X$-Mod of $\mathcal{O}_X$-modules that is closed under extensions. The functor $Coh(X) \to \mathcal{O}_X$-Mod is exact

Def 3.164 *Let $f : X \to Y$ be a map*

1) f is ***quasicompact*** *(qc) if for all $U \underset{open}{\subseteq} Y$ with U quasicompact, $f^{-1}(U) \subseteq X$ is quasicompact*

2) f is ***quasiseparated*** *(qs) if $\triangle_{X/Y} : X \to X \times_Y X$ is quasicompact*

Proposition 3.165 *Let $f : X \to S$ be a morphism of schemes. The following are equivalent :*

1) $f : X \to S$ is quasi-compact

2) The inverse image of every affine open is quasi-compact

3) There exists some affine open covering $S = \cup_{i\in I} U_i$ such that $f^{-1}(U_i)$ is quasi-compact for all i

Proposition 3.166 *Let $f : X \to S$ be a morphism of schemes. The following are equivalent:*

1) The morphism f is quasi-separated

2) For every pair of affine opens $U, V \subseteq X$ which map into a common affine open of S the intersection $U \cap V$ is a finite union of affine opens of X

3) There exists an affine open covering $S = \cup_{i\in I} U_i$ and for each i an affine open covering $f^{-1}U_i = \cup_{j\in I_i} V_j$ such that for each i and each pair $j, j' \in I_i$ the intersection $V_j \cap V_{j'}$ is a finite union of affine opens of X

Proof: Let us prove that (3) implies (1). By the covering $X \times_S X = \cup_i \cup_{j,j'} V_j \times_{U_i} V_{j'}$ is an affine open covering of $X \times_S X$. Moreover, $\triangle^{-1}_{X/S}(V_j \times_{U_i} V_{j'}) = V_j \cap V_{j'}$. Hence the implication follows from the Prop above

The implication (1) $\Rightarrow$ (2) follows from the fact that under the hypotheses of (2) the fibre product $U \times_S V$ is an affine open of $X \times_S X$. The implication (2) $\Rightarrow$ (3) is trivial

$\square$

Proposition 3.167 *Let $P_1 = \{quasicompact\ maps\}$, $P_2 = \{quasiseparated\ maps\}$*

Then $P_1, P_2 \in COMP, BC, LOCT$

Proof: For P_2 : $COMP$: $X \xrightarrow{f} Y \xrightarrow{g} Z$, $f, g \in P_2$

$$\begin{tikzcd}
X \arrow[r, "qc(hypo)"] \arrow[rr, bend left, "qc(qc\in COPM)"] & X \times_Y X \arrow[r, "qc(qc\in BC)"] \arrow[d] & X \times_Z X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \times X \arrow[d] \\
& Y \arrow[r, "qc(hypo)"'] & Y \times_Z Y \arrow[r] \arrow[d] & Y \times Y \arrow[d] \\
& & Z \arrow[r] & Z \times Z
\end{tikzcd}$$

BC : $[x \to Y] \in P_2$,

$$\begin{tikzcd}
V = X \times_Y U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & U \arrow[d] \\
X \arrow[r] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
V \arrow[rr, "qc(BC)"] \arrow[dr] & & V \times_U V \arrow[rr] \arrow[dr] \arrow[dd] & & V \arrow[dr] \arrow[dd] & \\
& X \arrow[rr, "qc(hypo)"'] & & X \times_Y X \arrow[rr] & & X \arrow[dd] \\
& & V \arrow[rr] \arrow[dr] & & U \arrow[dr] & \\
& & & X \arrow[rr] & & Y
\end{tikzcd}$$

$\square$

Remark 3.168 *X is qc as a scheme iff $X \to Spec\,\mathbb{Z}$ is qc*

X is qs as a scheme iff $X \to Spec\,\mathbb{Z}$ is qs

Proposition 3.169 *Let P be a property of maps with $P \in COMP, BC$. And the following commutative diagram of schemes:*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

Suppose $h \in P$ and that $[\triangle_{Y/Z} : Y \to Y \times_Z Y] \in P$, then $f \in P$

Proof: Graph cartesian diagram :

$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] \arrow[d] & X \times_Z Y \arrow[d] \\ Y \arrow[r, "\triangle_{Y/Z}"'] & Y \times_Z Y \end{tikzcd} \qquad \begin{tikzcd} X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d] & Y \arrow[d] \\ X \arrow[r, "h"'] & Z \end{tikzcd}$$

From $\triangle_{Y/Z}, h \in P$, and $P \in BC$, we know $\Gamma_f, \pi_Y \in P$. So $f = \pi_Y \circ \Gamma_f \in P$ since $P \in COMP$ $\square$

Corollary 3.170 *Set up as before and $X \xrightarrow{h} Z$ quasicompact and $Y \xrightarrow{g} Z$ quasiseparated $\Rightarrow X \xrightarrow{f} Y$ is quasicompact*

Def 3.171 *A map $f : X \to Y$ is **locally of finite type** (resp. **locally finite presentation**) if $\forall x \in X$, there is $X \in Spec\,A = U \underset{open}{\subseteq} X$ and $Spec\,B = V \underset{open}{\subseteq} Y$ with $f(U) \subseteq V$ and $B \to A$ finite type (resp. finite presentation)*

*A map is of **finite type** (resp. **finite presentation**) if it is locally of finite type (resp. finite presentation) and quasicompact*

i.e. a morphism $f : Y \to X$ of schemes is of finite type if f is quasicompact, and there is an open cover of Y by $Spec\,B_i$, such that $f|_{Spec\,B_i}$ factors over some Spec $A_i \subseteq X$, and B_i is a finitely generated A_i-algebra through the corresponding map of rings

Example 3.172 *1) $\bigcup\limits_{i=1}^{\infty} Spec\,\mathbb{Z} \to Spec\,\mathbb{Z}$ is locally of finite type*

2) $Spec\,k \hookrightarrow Spec\,k[x_1, \cdots, x_n]$ is of finite type, not of finite presentation

Proposition 3.173 *If $f : Y = Spec\,B \to X = Spec\,A$ is a morphism of finite type, then B is a finitely generated A-algebra*

Proposition 3.174 *$X \to Y$ finite type + Y Noetherian $\Rightarrow X$ is Noetherian and $X \to Y$ finite presentation*

Remark 3.175 *If k is an algebraically closed field, then the classical notation of varieties over k is essentially (up to being separated and irreducible as well) the same as a scheme of finite type over $Spec\,k$*

3.14 Dimension

Def 3.176 *Let X be an arbitrary topological space*

*(1) A point $x \in X$ is called **closed** if the set $\{x\}$ is closed*

*(2) We say that a point $\eta \in X$ is a **generic point** if $\overline{\{\eta\}} = X$*

*(3) Let x and x' be two points of X. We say that x is a **generization** of x' or that x' is a **specialization** of x if $x' \in \overline{\{x\}}$*

*(4) A point $x \in X$ is called a **maximal point** if its closure $\overline{\{x\}}$ is an irreducible component of X*

Example 3.177 *If $X = SpecA$ is the spectrum of a ring, the notions introduced above have the following algebraic meaning :*

(1) A point $x \in X$ is closed if and only if $\mathfrak{p}_x$ is a maximal ideal

(2) A point $\eta \in X$ is a generic point of X if and only if $\mathfrak{p}_\eta$ is the unique minimal prime ideal. This exists if and only if the nilradical of A is a prime ideal. Thus X is irreducible if and only if its nilradical is a prime ideal

(3) A point x is a generization of a point x' (in other words, x' is a specialization of x) if and only if $\mathfrak{p}_x \subseteq \mathfrak{p}_{x'}$

(4) A point $x \in X$ is a maximal point if and only if $\mathfrak{p}_x$ is a minimal prime ideal

Def 3.178 *Let X be a topological space. The dimension $dim\, X$ of X is the supremum of all lengths of chains $X_0 \supsetneqq X_1 \supsetneqq \cdots \supsetneqq X_n$ of irreducible closed subsets of X. (The length of a chain as above is n)*

If X is a scheme, then its dimension is by definition the dimension of the underlying topological space. A topological space X is called equidimensional (of dimension d), if all irreducible components of X have the same dimension (equal to d)

So the dimension is $-\infty$ (if and only if $X =$), a non-negative integer, or ∞

For an affine scheme $X = Spec\, A$, we have an inclusion reversing bijection between irreducible closed subsets of X and prime ideals of A. Thus we have $dim\, X = dim\, A$, where

$$dim\, A := sup\{n \in \mathbb{N}; \exists\, \mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \cdots \subsetneqq \mathfrak{p}_n \; chain\ of\ prime\ ideals\ of\ A\}$$

is called the Krull dimension or simply the dimension of the ring A

Proposition 3.179 *Let X be a topological space*

(1) Let Y be a subspace of X. Then $dim\, Y \leq dim\, X$. If X is irreducible, $dim\, X < \infty$, and $Y \subseteq X$ is a proper closed subset, then $dim\, Y < dim\, X$

(2) Let $X = \bigcup\limits_{\alpha} U_\alpha$ be an open covering. Then $dim\, X = \sup\limits_{\alpha} dim\, U_\alpha$

(3) Let I be the set of irreducible components of X. Then $dim\, X = \sup\limits_{Y \in I} dim\, Y$

From these definitions and properties, we now give the definition of the dimension of scheme :

Def 3.180 *Let T be a locally spectral space, then the **(Krull) dimension** of T is defined as the supremum minus one of the length of all chains of specialisations of points in T, i.e.*

$$\boldsymbol{dim\, T = sup_n\{x_0 > x_1 > x_2 > \cdots > x_n | x_i \in T, x_i \neq x_j, \forall i = j\}}$$

$x_1, x_2 \in T$, we say $x_1 > x_2$ if $x_2 \in \overline{\{x_1\}}$ and $x_1 \neq x_2$

Def 3.181 *If X is a scheme, define $\boldsymbol{dim\, X = dim\, |X|}$*

Proposition 3.182 *If $X = \bigcup_i U_i$ is an open cover of a scheme X, then $\boldsymbol{dim\, X = \sup_i dim\, U_i}$*

So $\boldsymbol{dim\, X = \sup_{x \in X} dim\, \mathcal{O}_{X,x}}$

Example 3.183 *1) Let k be an algebraically closed field, and let $X = \mathbb{A}^1_k$ then $|X|$ looks like k but with a generic point. The generic point specialises to all closed points. This gives us specialisations of length 1, which implies $dim\, X = 1$*

2) If k is an algebraically closed and $X = \mathbb{A}^2_k$ then the points of X are closed points, irreducible curves, and the generic point. We then have a specialisation of length 2, so at least $dim\, X \geq 2$. See that $dim\, X = 2$

Warning : there are Noetherian affine schemes of infinite dimension (Nagata)

Theorem 3.184 *Let A be a domain finitely generated over a field k ($A = k[x_1, \cdots, x_n]/I$)*

Let $n = tr.deg_k(Frac(A))$, then $dim\, Spec\, A = n$

Def 3.185 *A **prevariety** over k is a reduced, irreducible scheme of finite type over k*

*A **variety** over k is a reduced, irreducible, separated scheme of finite type over k*

Def 3.186 *A scheme X is **integral**, if $\mathcal{O}_X(U)$ is an integral domain for all $U \neq \emptyset$ open in X*

Proposition 3.187 *Let X be a scheme. The following are equivalent :*

1) The scheme X is irreducible.

2) There exists an affine open covering $X = \cup_{i\in I} U_i$ such that I is not empty, U_i is irreducible for all $i \in I$, and $U_i \cap U_j \neq \emptyset$ for all $i, j \in I$

3) The scheme X is nonempty and every nonempty affine open $U \subseteq X$ is irreducible

Proposition 3.188 *A scheme X is reduced and irreducible $\Leftrightarrow$ X is integral*

Proof: Assume first that for all nonempty $U = Spec\, A \subseteq X$ we have A is an integral domain. Then all such A are reduced, so A is a reduced domain. If X was not irreducible, then there would exist $U, V \subseteq X$, a pair of non-empty open subsets with $U \cap V = \emptyset$. Without loss of generality we can take U and V to be affine, but then $U \cup V = U \sqcup V$ is affine, with $\mathcal{O}_X(U \sqcup V) = \mathcal{O}_X(U) \times \mathcal{O}_X(V)$. This has zero divisors, which is a contradiction.

Conversely, assume that X is reduced and irreducible, and consider $Spec\, A \subseteq X$ a non-empty open subset. These hypotheses tell us that $Spec\, A$ is reduced and irreducible. Without loss of generality (again), we may take $X = Spec\, A$, so then X being reduced implies that A is a reduced ring. Now consider $f, g \in A$ with $fg = 0$, then $V(f) \cap V(g) = V(fg) = Spec\, A$, so $D(f) \cap D(g) = \emptyset$. The fact that X is irreducible implies that $D(f) = \emptyset$ or $D(g) = \emptyset$. Hence either f or g are nilpotent, but A is reduced, so $f = 0$ or $g = 0$ $\square$

3.15 $\star$ Geometric Properties of Schemes over Fields

See Red Book

Def 3.189 *Let P be one of the following properties of a scheme over a field:*

"irreducible", "connected", "reduced", or "integral"

We say that a k-scheme X possesses P geometrically if the K-scheme X_K possesses P for every field extension K of k

3.16 Affine Morphisms

3.16.1 Affine Morphisms

Def 3.190 *A morphism $f : X \to S$ of schemes is **affine** if there exists a cover of S by open affines $U_i = Spec\, A_i$ such that $f^{-1}(U_i)$ is an affine scheme in X for all i*

Proposition 3.191 *If $f : X \to S$ is affine, then for all $U \subseteq S$ open affine, the inverse image $f^{-1}(U)$ is affine in X*

Proof: First we let $S = SpecA$. We need to check that f is both quasi-compact and quasi-separated. This is true since both properties are $LOCT$ and affine schemes are quasi-compact and quasi-separated.

This implies that $f_*\mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_S$-algebra, and hence in this case $f_*\mathcal{O}_X = \widetilde{B}$ for some A-algebra B. We claim now that $X = Spec\, B$. Notice that,

$$B = \Gamma(S, f_*\mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)$$

so we have maps of schemes,

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr] & & Spec\, B \arrow[dl] \\ & Spec\, A & \end{tikzcd}$$

To check that φ is an isomorphism, it suffices to work locally on S, but locally this is true by assumption :

There is a cover $Spec\, A = \bigcup_{i=1}^{n} Spec\, A[f_i^{-1}] = \bigcup_{i=1}^{n} U_i$

$$\begin{tikzcd} Spec\, C_i = f^{-1}(U_i) \arrow[rr, "\varphi|_{f^{-1}(U_i)}"] \arrow[dr] & & Spec\, B[f_i^{-1}] \arrow[dl] \\ & Spec\, A[f_i^{-1}] & \end{tikzcd}$$

But $B[f_i^{-1}] = f_*\mathcal{O}_X(U_i) = \Gamma(U_i, \mathcal{O}_{U_i}) = C_i$, it is an isomorphism

□

Proposition 3.192 *The class of affine morphisms satisfies COMP, BC, PROD and LOCT*

Proposition 3.193 *An affine map $f : X \to S$ is separated*

Proposition 3.194 *Let $f : Y \to X$ be a morphism of schemes*

Assume $|f| : |Y| \to |X|$ is a topological immersion with closed image. Then that f is affine

(Hint: Show that each $x \in X$ has an open, affine neighborhood U_x such that $f^{-1}(U_x)$ is affine by using arguments as in sheet 8, exercise 4.i))

Proposition 3.195 *If X is separated and $Y = Spec\, B$, then any $f : Y \to X$ is affine*

Proof: Need to check $Spec\,A = U \subseteq (open)X$, then $f^{-1}(U)$ is affine

The diagram

$$\begin{tikzcd} f^{-1}(U) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & U \times Y \arrow[d] \\ X \arrow[r] & X \times X \end{tikzcd}$$

Since $U \times Y$ is affine and $f^{-1}(U) \to U \times Y$ is closed immersion, then $f^{-1}(U)$ is affine

□

Def 3.196 *Let $f : X \to Y$ be affine. Let $Spec\,A = U \subseteq (open)Y$ and $Spec\,B = f^{-1}(U)$*

1) We say f is ***integral*** *if for all such U, the ring map $A \to B$ is integral*

2) We say f is ***finite*** *if for all such U, the ring map $A \to B$ is finite*

Def 3.197 *$f : X \to S$ is* ***closed*** *if $|f| : |X| \to |S|$ is a closed map ($\forall Z \subseteq |X|$ closed subset $f(Z) \subseteq |S|$ is also closed)*

$f : X \to S$ is ***universally closed*** *if $\forall Y$ and maps*

$$\begin{tikzcd} X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & S \end{tikzcd}$$

then $X \times_S Y \to Y$ is also closed

Theorem 3.198 *Let $f : X \to S$ be a morphism of schemes. TFAE :*

1) f is integral

2) f is affine and universally closed

Proof: 1)→2):

By def it is affine. Suffices to prove integral ⇒ closed

WLOG $X = Spec\,R$ and $S = Spec\,T$ (we can check closed on affine cover). Since integral $\in$ $COMP$:

$$\begin{tikzcd} Spec\,R/I \arrow[r, hook] \arrow[rr, bend right, "integral"'] & Spec\,R \arrow[r] & Spec\,T \end{tikzcd}$$

Suffices to show $f(Spec\,R) \subseteq Spec\,T$ is closed. And this follows from going up of integral maps

2)→1): (affine universally closed ⇒ integral)

WLOG $f : Spec\,A \to Spec\,R$. We have $a \in A$ and consider $R[X] \to A$ $(x \mapsto a)$ and $I = ker(R[X] \to A)$

We need to prove $\exists x^n + r_{n-1}x^{n-1} + \cdots + r_0 \in I$

To be continued ...

□

Proposition 3.199 *Let $f : Y \to X$ be a morphism of schemes*

f is a universal homeomorphism if and only if it is surjective, universally injective and integral

(Hint : surjective is universally surjective)

3.16.2 Quasi-coherent $\mathcal{O}_X$-algebra

Theorem 3.200 *Moreover, for a fixed S, the functor from the category of affine morphisms into S to the opposite category of quasi-coherent $\mathcal{O}_S$-algebras, defined by $(f : Y \to S) \mapsto f_*\mathcal{O}_Y$ is an equivalence of categories*

The inverse functor is denoted as $\mathcal{A} \mapsto \underline{Spec}_{\mathcal{O}_S}\mathcal{A}$ (that means $\underline{Spec}_{\mathcal{O}_S}\mathcal{A}$ is a scheme affine/S)

Proof: If $S = Spec\, A$ is affine, then affine maps $Y \to S$ are mapped via an equivalence of categories to the A-algebra $\Gamma(Y, \mathcal{O}_Y)$, then we arrive at the chain of equivalences :

$$\{affine\ maps\ f : Y \to S\} \cong A - algebras \cong \{quasi - coherent\ \mathcal{O}_S - algebras\}$$

From this we obtain $\Gamma\widetilde{(Y, \mathcal{O}_Y)} = f_*\mathcal{O}_Y$, which comes from the fact that $f_*\mathcal{O}_Y$ is quasi-coherent. Hence the composite functor is an equivalence of categories. In general, the equivalence from affine maps $Y \to S$ to quasi-coherent $\mathcal{O}_S$-algebras via $Y \mapsto f_*\mathcal{O}_Y$ comes by general gluing lemmas □

Def 3.201 *$X \in Sch$ and $\mathcal{A}$ a quasicoherent $\mathcal{O}_X$-algebra*

Let (Not the same thing as the definition above) $\underline{Spec}_{\mathcal{O}_X}\mathcal{A} : Sch/X \to Sets$

$$\boldsymbol{\underline{Spec}_{\mathcal{O}_X}\mathcal{A}(Y \to X) = Hom_{\mathcal{O}_X - alg}(\mathcal{A}, f_*\mathcal{O}_Y)}$$

Proposition 3.202 *$\underline{Spec}_{\mathcal{O}_X}\mathcal{A}$ is representable by a scheme affine/X : $\underline{Spec}_{\mathcal{O}_X}\mathcal{A}$*

(since $Hom_{\mathcal{O}_x - alg}(Y, \underline{Spec}_{\mathcal{O}_X}\mathcal{A}) = Hom_{\mathcal{O}_x - alg}(\mathcal{A}, f_*\mathcal{O}_Y)$)

Moreover, all schemes affine over X are of this form

Proof: First we prove $\underline{Spec}_{\mathcal{O}_X}\mathcal{A}$ is a Zarsiki sheaf :

Let $\cup_{i\in I}U_i \to Y \xrightarrow{f} X$ Zarsiki cover of Y, $U_{ij} = U_i \cap U_j$, then :

$$\begin{array}{ccccc}
\underline{Spec}_{\mathcal{O}_X}\mathcal{A}(Y) & \longrightarrow & \prod_{i\in I}\underline{Spec}_{\mathcal{O}_X}\mathcal{A}(U_i) & \rightrightarrows & \prod_{i,j\in I}\underline{Spec}_{\mathcal{O}_X}\mathcal{A}(U_{ij}) \\
\big\downarrow{=} & & & & \\
Hom_{\mathcal{O}_X - alg}(\mathcal{A}, f_*\mathcal{O}_Y) & \longrightarrow & \prod_{i\in I} Hom_{\mathcal{O}_X - alg}(\mathcal{A}, f_*\mathcal{O}_{U_i}) & \rightrightarrows & \prod_{i,j\in I} Hom_{\mathcal{O}_X - alg}(\mathcal{A}, f_*\mathcal{O}_{U_{ij}})
\end{array}$$

Since $Hom_{\mathcal{O}_X - alg}(\mathcal{A}, -)$ is exact, and :

$$f_*\mathcal{O}_Y \xrightarrow{\sim} Eq(\prod_{i\in I} f_*\mathcal{O}_{U_i} \rightrightarrows \prod_{i,j\in I} f_*\mathcal{O}_{U_{ij}})$$

$\mathcal{O}_Y$ is a sheaf, so :

$$\begin{array}{ccc}
Hom_{\mathcal{O}_X - alg}(\mathcal{A}, f_*\mathcal{O}_Y) & \hookrightarrow & \Pi_{i\in I} Hom_{\mathcal{O}_X - alg}(\mathcal{A}, f_*\mathcal{O}_{U_i}) \\
\big\downarrow & & \big\downarrow \\
Hom_{\mathcal{O}_X}(\mathcal{A}, f_*\mathcal{O}_Y) & \hookrightarrow & \Pi_{i\in I} Hom_{\mathcal{O}_X}(\mathcal{A}, f_*\mathcal{O}_{U_i}))
\end{array}$$

Now we just need to proof $\underline{Spec}_{\mathcal{O}_X}\mathcal{A}$ is locally representable :

If $Spec\, B = U \subseteq X$, consider car seqence :

$$\begin{array}{ccc}
F & \longrightarrow & \underline{Spec}_{\mathcal{O}_X}\mathcal{A} \\
\big\downarrow & \lrcorner & \big\downarrow \\
U & \longrightarrow & X
\end{array}$$

Claim : $F = \underline{Spec}_{\mathcal{O}_X} \mathcal{A}(U)$

To be continued...

$\square$

Def 3.203 ***Tensor algebra constructions***

If M is an A-module, then the ***tensor algebra $T(M)$*** *is a noncommutative algebra, graded by $\mathbb{Z}_{\geq 0}$, defined as follows. $T_0(M) = A$, $T_n(M) = M \otimes_A \cdots \otimes_A M$ (where n terms appear in the product)*

$$T(M) := T_A(M) := \bigoplus_{n \geq 0} T_n(M)$$

where the product is given by

$$(m_1 \otimes \cdots \otimes m_n, m'_1 \otimes \cdots \otimes m'_{n'}) \mapsto m_1 \otimes \cdots \otimes m_n \otimes m'_1 \otimes \cdots \otimes m'_{n'}$$

The ***symmetric algebra $Sym(M)$*** *is a symmetric algebra, graded by $\mathbb{Z}_{\geq 0}$, defined as the quotient of $T(M)$ by the (two-sided) ideal generated by all elements of the form $x \otimes y - y \otimes x$ for all $x, y \in M$. Thus $Sym_n M$ is the quotient of $M \otimes \cdots \otimes M$ by the relations of the form $m_1 \otimes \cdots \otimes m_n - m'_1 \otimes \cdots \otimes m'_n$ where $m'_1, \cdots, m'_n$ is a rearrangement of $m_1, \cdots, m_n$. $Sym(M) = \bigoplus_{n \geq 0} Sym_n(M)$*

We have $Sym_0(M) = A$ and $Sym_1(M) = M$. The A-algebra $Sym(M)$ and the A-linear map $\iota : M = Sym_1(M) \hookrightarrow Sym(M)$ satisfy the following universal property :

For every commutative A-algebra B, composition with ι yields a bijection

$$Hom_{(A-Alg)}(Sym(M), B) \xrightarrow{\sim} Hom_{(A-Mod)}(M, B) \;\; \varphi \mapsto \varphi \circ \iota$$

If $\varphi : A \to B$ is a ring homomorphism, the isomorphism above implies that there is an isomorphism of B-algebras

$$Sym_A(M) \otimes_A B \xrightarrow{\sim} Sym_B(M \otimes_A B)$$

Also an isomorphism of graded A-algebras, functorial in A-modules M and M',

$$Sym(M \oplus M') \xrightarrow{\sim} Sym(M) \otimes_A Sym(M')$$

Def 3.204 *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{E}$ an $\mathcal{O}_X$-module. The sheaf associated to the presheaf $U \mapsto Sym_{\Gamma(U, \mathcal{O}_X)}(\Gamma(U, \mathcal{E}))$ on X is a commutative graded $\mathcal{O}_X$-algebra :*

$$\boldsymbol{Sym(\mathcal{E}) = \bigoplus_{n \geq 0} Sym_n(\mathcal{E})}$$

called the ***symmetric algebra of $\mathcal{E}$***

Proposition 3.205 *$\mathcal{E} \mapsto Sym(\mathcal{E})$ defines a functor from the category of $\mathcal{O}_X$-modules into the category of commutative $\mathcal{O}_X$-algebras which is left adjoint to the forgetful functor, that is, for every commutative $\mathcal{O}_X$-algebra $\mathcal{A}$ we have bijections which are functorial in $\mathcal{A}$ and in $\mathcal{E}$*

$$\boldsymbol{Hom_{\mathcal{O}_X-Alg}(Sym(\mathcal{E}), \mathcal{A}) \cong Hom_{\mathcal{O}_X-Mod}(\mathcal{E}, \mathcal{A})}$$

Remark 3.206 *If $\mathcal{E}$ is a vector bundle over X, $\mathbb{V}(\mathcal{E})(Y \to X) = \Gamma(Y, f^*\mathcal{E})$*

Claim : $\mathbb{V}\boldsymbol{(\mathcal{E})} \cong \underline{\boldsymbol{Spec}}_{\boldsymbol{\mathcal{O}_X}}\boldsymbol{(Sym(\mathcal{E}^\vee))}$

Proof:

$$\begin{aligned}
\mathbb{V}(\mathcal{E})[Y \xrightarrow{f} X] &= \Gamma(Y, f^*\mathcal{E}) \\
&\cong \Gamma(Y, \underline{Hom}(f^*\mathcal{E}^\vee, \mathcal{O}_Y)) \\
&\cong Hom_{\mathcal{O}_Y}(f^*\mathcal{E}^\vee, \mathcal{O}_Y) \\
&\cong Hom_{\mathcal{O}_X}(\mathcal{E}^\vee, f_*\mathcal{O}_Y) \\
&\cong Hom_{\mathcal{O}_X-alg}(Sym\mathcal{E}^\vee, f_*\mathcal{O}_Y) \\
&\cong \underline{Spec_{\mathcal{O}_X}}(Sym\mathcal{E}^\vee)[Y \to X]
\end{aligned}$$

$\square$

3.17 Proper Morphisms

Def 3.207 *A morphism $f : X \to S$ is **proper** if it is*

1) separated

2) finite type (locally of finite type + quasicompact)

3) universally closed

A scheme over S is proper if the structure map $X \to S$ is proper

Proposition 3.208 *Let $f : X \to Y$ be a map. TFAE:*

1) f is affine and proper

2) f is finite

Proof: 1) $\Rightarrow$ 2): affine + universally closed $\Rightarrow$ integral

integral + finite type $\Rightarrow$ finite

2) $\Rightarrow$ 1): finite $\Rightarrow$ integral, affine, separated, finite type

integral $\Rightarrow$ universally closed

$\square$

Example 3.209 *$\mathbb{A}^1_k \to Spec\,k$ is affine but not finite, so not proper*

$$\begin{tikzcd} V(xy-1) \subseteq \mathbb{A}^2_k \arrow[r, "p_1"] \arrow[d, "p_2"'] & \mathbb{A}^1_k \arrow[d] \\ \mathbb{A}^1_k \arrow[r] & Spec\,k \end{tikzcd}$$

$p_2(V(xy-1)) = \mathbb{A}^1_k \ \{0\}$

(In this place $V(xy-1) = Spec\,k[x,y]/(xy-1)$ is a scheme)

Proposition 3.210 *proper $\in LOCT, BC, COMP$*

Proof: separated and finite type $\in LOCT, BC, COMP$

closed $\in LOCT, COMP$

universally closed $\in BC$ (almost by def)

$$\begin{tikzcd} (X \times_Y Z) \times_Z W = X \times_Y W \arrow[r, "2"] \arrow[d] & W \arrow[d] \\ X \times_Y Z \arrow[r, "1"] \arrow[d] & Z \arrow[d] \\ X \arrow[r, "universally\ closed"'] & Y \end{tikzcd}$$

$1 + 2$ Cartesian $\Rightarrow$ extended square is also Cartesian $\square$

Proposition 3.211 *Let*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

be a commutable diagram

If g is separated and h is proper, then f is proper

Proof: Prop 3.169 tells we just need to prove $\triangle_g : Y \to Y \times_Z Y$ is proper, we know this map is a closed immersion since g is separated

Closed immersion is a finite map, so it is a proper map $\square$

Theorem 3.212 $\mathbb{P}^n_{\mathbb{Z}} \to Spec\,\mathbb{Z}$ *is proper*

We will prove it later. (useing Theorem 3.224)

Def 3.213 *We call a morphism $f : X \to S$ **locally quasi-projective** (resp. **locally projective**) if there exists an open affine covering $(U_\alpha)_\alpha$ of S and for all α a quasi-compact U_α-immersion (resp. a closed U_α-immersion) $f^{-1}(U_\alpha) \hookrightarrow \mathbb{P}^{n_\alpha}_{U_\alpha}$ for some integer $n_\alpha \geq 0$*

$$\begin{tikzcd} & \mathbb{P}^{n_\alpha}_{U_\alpha} \arrow[d, "\pi"] \\ f^{-1}(U_\alpha) \arrow[ur, hook, "\alpha"] \arrow[r, "f"'] & U_\alpha \end{tikzcd}$$

(where each α is a quasi-compact immersion (resp. a closed immersion), and π is first projective. $\mathbb{P}^{n_\alpha}_{U_\alpha} = U_i \times_{Spec\,\mathbb{Z}} \mathbb{P}^{n_\alpha}_{\mathbb{Z}}$)

Proposition 3.214 *locally projective $\Rightarrow$ proper*

Proof: proper $\in LOCT$, so WLOG $S = U_i$

From "$\mathbb{P}^n_{\mathbb{Z}} \to Spec\,\mathbb{Z}$ is proper" and "proper $\in BC$", we know $[\mathbb{P}^{n_\alpha}_{U_\alpha} \to U_\alpha]$ is proper

closed immersion is proper and proper $\in COMP$. So f is proper $\square$

Example 3.215 *Over $Spec\,k$, k is a field, locally projective means*

$$\begin{tikzcd} & \mathbb{P}^n_k \arrow[d, "\pi"] \\ Z \arrow[ur, hook, "s"] \arrow[r] & Spec\,k \end{tikzcd}$$

(s a closed immersion)

Theorem 3.216 ***Chow's lemma***

The Lemma of Chow says that every separated morphism of finite type (resp. every proper morphism) becomes quasi-projective (resp. projective) after modification with a birational projective morphism, more precisely:

Let S be a qcqs scheme and let $f : X \to S$ be a separated morphism of finite type (resp. proper morphism) such that X has only finitely many irreducible components (which is automatic if S is noetherian). Then there exists a commutative diagram

$$\begin{tikzcd} X \arrow[dr, "f"'] & & X' \arrow[ll, "\pi"'] \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

where g is quasi-projective (resp. projective) and where π is surjective, projective, and there exists a quasicompact open dense subscheme U of X such that $\pi^{-1}(U)$ is dense in X' and the restriction $\pi^{-1}(U) \to U$ of π is an isomorphism. Moreover:

(1) f is proper if and only if g is projective

(2) If X is reduced (resp. irreducible, resp. integral), then X' can be chosen to be reduced (resp. irreducible, resp. integral)

The definition of A will be given later, here we only need to use the special case of this lemma :

Corollary 3.217 *Let S be a Noetherian scheme. Let $f : X \to S$ be a separated morphism of finite type. Then there exist an $n \geq 0$ and a diagram*

$$\begin{tikzcd} X \arrow[dr, "f"'] & X' \arrow[l, "\pi"'] \arrow[r, hook, "i"] \arrow[d] & \mathbb{P}^n_S \arrow[dl] \\ & S & \end{tikzcd}$$

where $i : X' \to \mathbb{P}^n_S$ is an immersion, and $\pi : X' \to X$ is proper and surjective. Moreover, we may arrange it such that there exists a dense open subscheme $U \subset X$ such that $\pi^{-1}(U) \to U$ is an isomorphism

Remark 3.218 *In particular, if $S = Spec\, A$ (A is Noetherian), for example : $S = Spec\, k$*

Example 3.219 *If $X/Spec\, k$ is proper variety, then X' here is also proper and $i : X' \hookrightarrow \mathbb{P}^n_k$ is closed immersion*

(proper locally closed immersion $\Rightarrow$ closed immersion)

3.18 Valuative Criterion and Valuation Rings

3.18.1 Diagonal of scheme morphisms and separated morphisms

3.18.2 Valuative Criterion

Def 3.220 *A* ***valuation ring*** *is an integral domain V with fraction field K such that for all $x \in K^\times$, either $x \in V$ or $x^{-1} \in V$*

Theorem 3.221 *Let K be a field, and $A \subset K$ with $Frac A = K$ such that A is a local ring. Then the following are equivalent :*

1. *A is a valuation ring of K, so for all $x \in K - \{0\}$ either x or x^{-1} lies in A*
2. *There exists a valuation $v : K \to \Gamma \cup \{\infty\}$ such that A is $\{x \in K | v(x) \geq 0\}$*
3. *For all local rings $A \subseteq B \subseteq K$ where $A \hookrightarrow B$ is a local map, $A = B$*

Example 3.222 *Let k be a field, then $k[t]_{(t)} \subseteq k(t)$ is a valuation ring*

$$f \in k(t),\ f = t^n \cdot \frac{\Pi_{i=1}^n p_i(t)^{u_i}}{\Pi_{j=1}^m q_j(t)^{v_j}}$$

each p_i, q_j are irreducible polynomials and $p_i \neq 0, q_j \neq 0$

Theorem 3.223 ***Valuative Criterion for Separatedness***

A morphism $f : X \to S$ of schemes is separated if and only if f is quasi-separated and for any valuation ring V with fraction field $K = Frac(V)$, then any diagram of the following form,

$$\begin{tikzcd}
Spec\, K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
Spec\, V \arrow[r] \arrow[ur, dashed, "g", "\leq 1"'] & S
\end{tikzcd}$$

has at most one lift (the dotted arrow) such that the above diagram commutes

We will prove it later

Theorem 3.224 ***Valuation Criterion for Properness***

A morphism $f : X \to S$ of schemes is proper if and only if f is of finite type (which implies quasi-compact) and quasi-separated, and for any valuation ring V with fraction field K, then any diagram of the following form,

$$\begin{tikzcd}
Spec\, K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
Spec\, V \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

there exists exactly one lift such that the above diagram commutes

Example 3.225 *$G_m = Spec\, k[T, T^{-1}]$, generic point $\eta \in G_m$, then $k(\eta) = k(T)$ and $k[T]_{(T)}$ is a valuation ring of $k(\eta)$*

$$\begin{tikzcd}
Spec\, k(T) \arrow[r] \arrow[d] & G_m \arrow[d] \\
Spec\, k[T]_{(T)} \arrow[r] \arrow[ur, dashed] & Spec\, k
\end{tikzcd}$$

Example 3.226 *The line with two origins $\mathbb{A}^1_{0,0} \to Spec\, k$ does not satisfy valuation criterion :*

$K = k(t), V = k[t]_{(t)}$ with

$$\begin{tikzcd}
Spec\, k(t) \arrow[rr] \arrow[dd] & & \mathbb{A}^1_{0,0} \arrow[dd] \\
& \mathbb{A}^1_k \cup \mathbb{A}^1_k \arrow[ur, hook, shift left] \arrow[ur, hook, shift right] & \\
Spec\, k[t]_{(t)} \arrow[ur, bend left, "p_1"] \arrow[ur, bend right, "p_2"'] \arrow[rr] & & Spec\, k
\end{tikzcd}$$

3.18.3 Valuation Rings

Def 3.227 *A **totally order group** is an abelian group Γ with total order $\leq$ on Γ (so $x \leq y$ and $y \leq x$ implies that $x = y$), such that if $x \leq y$ and $x' \leq y'$ we have $x + x' \leq y + y'$*

Example 3.228 *1) $\mathbb{Z}$ with the usual order*

2) R with usual order

3) $R \oplus R$ with the lexicographical ordering, which we will write as $R \oplus R\epsilon$, to imply that ϵ is some infinitesimal, i.e. $\epsilon < r$ for all $r \in R_{>0}$

($a + b\epsilon \leq a' + b'\epsilon$ if $a \leq b$, or $a = a'$ and $b \leq b'$, i.e. the first summand is in control)

Def 3.229 *Given a ring R, then a **additive valuation** on R is a map $v : R \to G \cup \{\infty\}$ for some totally ordered group G, such that*

1. *$v(0) = \infty$ and $v(1) = 0_G$*
2. *$v(xy) = v(x) + v(y)$, with the convention that $a + \infty = \infty + a = \infty + \infty = \infty$*
3. *$v(x + y) \geq min(v(x), v(y))$*

If $G \cong \mathbb{Z}$, then v is a discrete valuation

Def 3.230 *Given a ring R, then a **multiplicative valuation** on R is a map $|\cdot| : R \to \Gamma \cup \{\ 0\}$ for some totally ordered group Γ, such that*

1. *$|0| = 0$ and $|1| = 1_\Gamma$*
2. *$|xy| = |x| \cdot |y|$ with the convention that $|x| \cdot 0 = 0 \cdot |x| = 0 \cdot 0 = 0$*
3. *$|x + y| \leq max(|x|, |y|)$*

Remark 3.231 *additive valuation = multiplicative valuation*

Example 3.232 *$v : R \to R \cup \{\infty\}$ corresponds to $|\cdot| : R \to R^{>0} \cup \{0\}, \forall r \in R, |r| = e^{-v(r)}$*

Def 3.233 *We say $|\cdot|$ is a **discrete valuation** if $\Gamma \cong \mathbb{Z}$ as an ordered group*

Example 3.234 *$\mathbb{Z} \subset \mathbb{R}$, the p-adic valuation*

$|\cdot|_p : \mathbb{Z} \to \mathbb{R}^{>0} \cup \{0\}$

Pick $0 < r < 1$ (traditionally $r = \frac{1}{p}$)

$$|n|_p = \begin{cases} r^\alpha & if\ n = p^\alpha q,\ (p, q) = 1 \\ 0 & if\ n = 0 \end{cases}$$

Example 3.235 *$R = k[T]$, the T-adic valuation*

$|\cdot|_T : R = k[T] \to \mathbb{R}^{>0} \cup \{0\}$

Pick $0 < r < 1$

$$|p(T)|_T = \begin{cases} r^{\alpha} & if\ p(T) = T^{\alpha}q(T),\ q(0) \neq 0 \\ 0 & if\ p(T) = 0 \end{cases}$$

Example 3.236 $\Gamma = \mathbb{R}^{>0} \oplus \epsilon^{\mathbb{Z}}$*, we think of ϵ as infinitesimally smaller than 1, meaning $\forall r \in \mathbb{R}^{>0}$ with $r < 1 \Rightarrow r < \epsilon$ but $\epsilon < 1$*

$|\cdot|_\epsilon : k[x,y] \to \Gamma \cup \{\infty\}$

Pick $r \in \mathbb{R}^{>0}$

If $p(x,y) = \sum_{i,j} a_{ij}x^i y^j$,

$$|p(x,y)| = \begin{cases} max_{i,j,a_{ij}\neq 0} r^i \epsilon^j & \\ 0 & if\ p(x,y) = 0 \end{cases}$$

examples: $|x+y| = max(r,\epsilon) = \epsilon$, $|xy^3 + x^2| = max(r\epsilon^3, r^2) = r\epsilon^3$

Proposition 3.237 *Let R be a ring and $|\cdot| : R \to \Gamma \cup \{0\}$ a multiplicative valuation the following hold :*

1) $\{r \in R|\ |r| = 0\}$ *is a prime ideal* $\mathfrak{p}_{|\cdot|}$

2) Let $K = Frac(R/\mathfrak{p}_{|\cdot|})$. We can extend $|\cdot|$ to a map $|\cdot|' : K \to \Gamma \cup \{0\}$ satisfying $|[r]|' = |r|$ if $r \in R, [r] \in Frac(R/\mathfrak{p}_{|\cdot|}$ and r represents $[r]$

Proposition 3.238 *Let K be a field, and $A \subset K$ with $FracA = K$ such that A is a local ring. Then the following are equivalent :*

1. *A is a valuation ring of K, so for all $x \in K - \{0\}$ either x or x^{-1} lies in A*
2. *There exists a valuation $|\cdot| : K \to \Gamma \cup \{0\}$ such that $A = \{x \in K||x| \leq 1\}$*
3. *For all local rings $A \subseteq B \subseteq K$ where $A \hookrightarrow B$ is a local map, $A = B$*

Now suppose $A = V$ is a valuation ring. Let $a, b \in K \setminus \{0\}, |a| \leq |b| \Leftrightarrow |\frac{a}{b} \geq_V 1 \Leftrightarrow \frac{a}{b} \in V$

$\Gamma = K^\times/V^\times$ becomes an ordered group, a partial order on $K^\times$ given by $a \leq b$ if $\frac{a}{b} \in V$. It is well defined on $K^\times/V^\times$: $\leq_V$

For valuation ring V. We let $\Gamma_V = (K^\times/V^\times, \leq_V)$

$|\cdot|_V : V \to \Gamma_V \cup \{0\}$ is the canonical one

Def 3.239 *If $K^\times/V^\times \cong \mathbb{Z}$. We say that V is a* ***DVR (discrete valuation ring)***

Remark 3.240 *A valuation $|\cdot| : R \to \Gamma \cup \{0\}$ with $|r| \leq 1, \forall r \in R$ of this form factor as follows :*

$$\begin{tikzcd} R \arrow[r, two heads] \arrow[d, "|\cdot|"'] & R/\mathfrak{p}_{|\cdot|} \arrow[r, hook] & V \arrow[d, "|\cdot|_V"] \\ \Gamma \cup \{0\} & & \Gamma_V \cup \{0\} \arrow[ll, hook'] \end{tikzcd}$$

with $Frac(R/\mathfrak{p}_{|\cdot|}) = Frac(V)$

Example 3.241 *$|\cdot|_\infty : k[T] \to \mathbb{R}^{>0} \cup \{0\}$ pick $r > 1$. $|p(T)|_\infty = r^d, d = deg(p(T))$*

Proposition 3.242 *Let V be a valuation ring and $|\cdot|_V : V \to \Gamma_V \cup \{0\}$ its canonical valuation, then:*

1) Principal ideals of V are in bijection with sets $S_{\leq r} = \{g \in \Gamma_V | g \leq r\}, I_{\leq r} = |\cdot|^{-1} S_{\leq r}$

2) Finitely generated ideals are principal (Bezout domain)

3) Ideals of V are bijection with sets $S \subseteq \Gamma_V \cup \{0\}$ satisfying if $r \in S$ and $r' \leq r$, then $r' \in S$. $I_S = |\cdot|^{-1} S$

4) The set of ideals of V is a total order under $\leq$

Remark 3.243 *Every valuation ring has a unique maximal ideal (it is a local ring)*

Corollary 3.244 *If V is a Noetherian valuation ring, then V is a PID and a DVR and $Spec\, V$ is 1-dimensional, $Spec\, V = \{(0), \mathfrak{m}\}$*

Theorem 3.245 *Let A be a local noetherian integral domain of $\dim Spec\, A = 1$, then $Spec\, A = \{(0), \mathfrak{m}\}$. More over A is normal if and only if $\mathfrak{m} \subseteq A$ is a principal ideal if and only if A is a discrete valuation ring*

Back to prove Theorem 3.212 :

Proof: Using Theorem 3.224, we just need to prove for all valuation rings V with $K = Frac(V)$:

$$\mathbb{P}^n(V) = Hom(Spec\, V, \mathbb{P}^n_{\mathbb{Z}}) \cong Hom(Spec\, k, \mathbb{P}^n_{\mathbb{Z}}) = \mathbb{P}^n(k)$$

Because V is a local, V has trivial line bundles. $\mathbb{P}^n(V) = (\mathbb{A}^{n+1} \setminus \{0\})(V)/V^\times$, $\mathbb{P}^n(K) = K^{n+1} \setminus \{0\}/K^\times$, so we need to prove :

$$\{(x_0, \cdots, x_n) \in V^n \ with \ x_i \in V^\times \ for \ some \ i\}/V^\times \xrightarrow{bijecive} (K^n \setminus \{0\})/K^\times$$

Given a tuple $(a_0, \cdots, a_n) \in K^{n+1} \setminus \{0\}$,there is unique $j \in \{0, \cdots, n\}, |a_i| \leq |a_j|, \forall i \in \{0, \cdots, n\}$ and $|a_i| < |a_j|, \forall i < j$, then $(\frac{a_0}{a_j}, \cdots, 1, \cdots, \frac{a_n}{a_j}) \in \mathbb{A}^{n+1} \setminus \{0\})(V)$ $\square$

3.19 Normalization

Def 3.246 *Let R be a domain, we say $Spec\, R$ is* ***normal*** *if R is integrally closed in its fraction field*

i.e. an integral domain A is normal if $x \in K$ satisfies $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ for some $a_i \in A$, then $x \in A$

Remark 3.247 *1) R integrally closed, then $S^{-1}R$ integrally closed*

2) If for all closed $\mathfrak{m} \in Spec\, R$, $R_{\mathfrak{m}}$ is integrally closed $\Rightarrow$ R is integrally closed

Def 3.248 *We say X a* ***integral scheme*** *if X is reduced and irreducible*

We say X ***normal*** *if $Spec\, \mathcal{O}_{X,x}$ is normal for all $x \in X$ (i.e. the local ring $\mathcal{O}_{X,x}$ is a normal integral domain)*

Example 3.249 *(Non-example)*

Let $A = k[x,y]/(y^2 - x^3)$, the cuspidal cubic. We claim that A is not normal. To see this, embed A into $k[t]$ by sending $x \mapsto t^2$ and $y \mapsto t^3$, then A is identified with $f = \sum_{i=0}^{n} r_i t^i$ with $r_i \in k$ and $r_1 = 0$ inside $k[t]$. In particular, A is an integral domain, since it injects into $k[t]$, and $t = y/x \in K = Frac(A) = Frac(k[t])$, and t satisfies the equation $t^2 - x = 0$. By $t \notin A$, so A is not normal

Proposition 3.250 *(1) If X is normal, then X is reduced*

(2) If X is integral, then TFAE :

a) X is normal

b) For all $U = Spec\, A \subseteq X$ an open affine subset, A is a normal integral domain

c) $\exists$ covering $X = \cup_i Spec\, A_i$, A_i is normal

Example 3.251 *If X is normal, but not irreducible, it can be that for some $U = Spec\, A \subseteq X$ we have that A is not an integral domain, for example $X = Spec\, k \sqcup Spec\, k = Spec\,(k \times k)$ for some field k. The definition of normality is designed to be a local property of X. One can show that if X is noetherian and normal, then $X = \sqcup_{i=1}^{n} X_i$ with X_i irreducible, normal and noetherian*

Def 3.252 *$f : Y \to X$ is* ***dominant*** *if $f(Y) \subseteq X$ is dense*

Def 3.253 *Let X be an integral scheme. Let $\mathcal{C}_X$ be category of $Y \to X$ such that the map $Y \to X$ is dominant and Y is irreducible and normal*

A ***normalization*** *is a final object in $\mathcal{C}_X$*

Proposition 3.254 *With set up as above, the normalization $\widetilde{X} \to X$ exists and it is integral over X, and $Frac(\widetilde{X}) = Frac(X)$*

Claim : $\widetilde{U} = Spec\, \widetilde{A} \to Spec\, A$ is the normalization of $U = Spec\, A$

If $Y \xrightarrow{f} U$ is dominant and Y is normal, we get maps

$$\begin{tikzcd} g : A \arrow[r, hook] \arrow[dr, hook] & \Gamma(Y, \mathcal{O}_Y) \arrow[r, hook] & Frac(Y) \\ & Frac(U) \arrow[ur] & \end{tikzcd}$$

then $A \hookrightarrow \widetilde{A} \subseteq \Gamma(Y, \mathcal{O}_Y)$ since $\Gamma(Y, \mathcal{O}_Y)$ is a normal ring

Example 3.255 *Let $X = Spec\, A$ with $A = k[x,y]/(y^2 - x^3)$ be our cuspidal cubic, then we have $(y/x)^2 = x$ inside $Frac(A) = K$. Let $t = y/x$, then we see that $t \in \widetilde{A} \subseteq K$, such that $x = t^2$ and $y = tx = t^3$, thus $\widetilde{A} \supseteq k[t] \supseteq A$, but $k[t]$ is normal so $\widetilde{A} = k[t]$. Let*

$$\widetilde{X} = Spec\, \widetilde{A} = \mathbb{A}^1_k \to Spec\, A = V(y^2 - x^3) \subseteq \mathbb{A}^2_k$$

be defined by $t \mapsto (t^2, t^3)$, then geometrically this map sends the vertical line $\mathbb{A}^1_k$ homeomorphically (not isomorphically as schemes) to the cuspidal cubic

3.20 Curve

Proposition 3.256 *$X \to Spec\, k$ morphism of finite type. TFAE :*

1) $dim\, X = 0$

2) $X = SpecA$, A finite dim k-algebra

3.20.1 smooth

Normal scheme is equivalent to a smooth scheme if the dimension of our scheme is less than or equal to 1

Proposition 3.257 *(Corollary 3.36). Let k be algebraically closed and let X be a k-scheme locally of finite type. Then*

$$\{x \in X; x\ closed\} = \{x \in X; k = \kappa(x)\} = X(k) := Hom_k(Spec\, k, X)$$

Def 3.258 *Let X be a scheme, and let $x \in X$. Then $\mathfrak{m}_x/\mathfrak{m}_x^2$ is a vector space over $\mathcal{O}_{X,x}/\mathfrak{m}_x = \kappa(x)$, so x is a k-rational point $x \in X(k)$ $(k = \kappa(x)?)$((considered as a closed point of X)), and the (Zariski, or absolute)* ***tangent space*** *of X in x is by definition the dual vector space*

$$T_x X = (\mathfrak{m}_x/\mathfrak{m}_x^2)^* = Hom_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k)$$

Remark 3.259 *Let X be a scheme and $x \in X$*

(1) Nakayama's lemma shows that if $\mathfrak{m}_x$ is finitely generated, $dim_{\kappa(x)} T_x X$ is the cardinality of a minimal generating set of $\mathfrak{m}_x$. In particular $dim_{\kappa(x)} T_x X$ is finite if X is locally noetherian

(2) If $U \subseteq X$ is an open neighborhood of x, we clearly have $T_x X = T_x U$

(3) The tangent space is functorial in (X, x) in the following sense. Let $f : X \to Y$ be a morphism of schemes and let $x \in X$ be a point such that $dim_{\kappa(f(x))} T_{f(x)} Y$ is finite. Then the local homomorphism $f_x^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ induces a $\kappa(x)$-linear map $\mathfrak{m}_{f(x)}/\mathfrak{m}_{f(x)}^2 \otimes_{\kappa(f(x))} \kappa(x) \to \mathfrak{m}_x/\mathfrak{m}_x^2$. If the extension $\kappa(x)/\kappa(f(x))$ is finite or $T_{f(x)} Y$ is a finite-dimensional $\kappa(f(x))$-vector space, then dualizing we obtain an induced map on tangent spaces

$$df_x : T_x X \to T_{f(x)} Y \otimes_{\kappa(f(x))} \kappa(x)$$

This construction is compatible with composition of morphisms in the obvious way

Proposition 3.260 *Let k be a field and let X be a normal scheme of finite type over k with $dimX \leq 1$*

Then every $x \in X(k)$ is a smooth point of X

Hint: Use that a 1-dimensional normal local noetherian domain is a discrete valuation ring

3.21 Permanence for properties of morphisms of schemes

See Red Book

affine. Closed immersions; (LOCT), (COMP), (BC), (CANC) if g is separated: Proposition 12.3; (IND): [EGAIV] (8.10.5); (DESC): Proposition 14.51.

bijective. (LOCT), (COMP); (DESC): Proposition 14.48.

closed. Closed immersions; (LOCT), (COMP); (DESC): Proposition 14.49.

closed immersion. (LOCT), (COMP): Remark 3.47; (BC): Proposition 4.32; (CANC) if g is separated: Remark 9.11; (IND): Proposition 10.75; (DESC): Proposition 14.51.

etale. Open immersions; (LOCS), (LOCT), (COMP), (BC), (CANC) if g is unramified: Volume II; (IND): [EGAIV] (17.7.8); (DESC): [EGAIV] (17.7.3). faithfully flat. (LOCT), (COMP), (BC): Remark 14.8; (CANC) if g is an open immersion; (DESC): Corollary 14.12. finite. Closed immersions; (LOCT), (COMP), (BC), (CANC) if g is separated, (IND): Proposition 12.11; (DESC): Proposition 14.51. finite locally free. (LOCT), (COMP), (BC), (IND), (DESC): follows from the permanence properties for “finite”, “flat” and “of finite presentation”. flat. Open immersions; (LOCS), (LOCT), (COMP), (BC): Proposition 14.3; (CANC) if g is unramified: Volume II; (IND): [EGAIV] (11.2.6); (DESC): Corollary 14.12. homeomorphism. (LOCT), (COMP); (DESC): Proposition 14.49. immersion. (LOCT), (COMP): Remark 3.47; (BC): Proposition 4.32; (CANC): Remark 9.11; (IND): Proposition 10.75; (DESC) if f is quasi-compact: Proposition 14.51. injective. Immersions; (LOCT), (COMP), (CANC); (DESC): Proposition 14.48. integral. Closed immersions; (LOCT), (COMP), (BC), (CANC) if g is separated, (IND): Proposition 12.11; (DESC): [EGAIV] (2.7.1). isomorphism. (LOCT), (COMP), (BC); (IND): Corollary 10.64; (DESC): Proposition 14.51. locally of finite type. Immersion; (LOCS), (LOCT), (COMP), (BC), (CANC): Proposition 10.7; (IND): by definition; (DESC): Proposition 14.51. locally of finite presentation. Open immersions; (LOCS), (LOCT), (COMP), (BC), (CANC) if g is locally of finite type: Proposition 10.35; (IND): by definition; (DESC): Proposition 14.51. monomorphism. Immersions; (LOCT), (COMP), (BC), (CANC); (IND): [EGAIV] (8.10.5); (DESC): Proposition 14.51. of finite type. Closed immersions, immersions of locally noetherian schemes; (LOCT), (COMP), (BC), (CANC) if g is quasi-separated (e.g., if Y is locally noetherian): Proposition 10.7; (IND): by definition; (DESC): Proposition 14.51. of finite presentation. Quasi-compact open immersions; (LOCT), (COMP), (BC), (CANC) if g is quasi-separated and locally of finite type: Proposition 10.35; (IND): by definition; (DESC): Proposition 14.51. open. Open immersions; (LOCS), (LOCT), (COMP); (DESC): Proposition 14.49. open immersion. (LOCT), (COMP): Remark 3.47; (BC): Proposition 4.32; (CANC) if g is unramified: Volume II; (IND): Proposition 10.75; (DESC): Proposition 14.51. projective. Closed immersions; (COMP) if Z is quasi-compact and quasi-separated, (BC), (CANC) if g is separated: [EGAII] (5.5.5); (IND): [EGAIV] (8.10.5). proper. Closed immersions; (LOCT), (COMP), (BC), (CANC) if g is separated: Proposition 12.58; (IND): [EGAIV] (8.10.5); (DESC): Proposition 14.51. purely inseparable. Immersions; (LOCT), (COMP), (BC), (CANC): [EGAInew] (3.7.6); (IND): [EGAIV] (8.10.5); (DESC): [EGAInew] (3.7.6)