Compressive Sensing

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1 Introduction

After the famous Shanon theorem about the information entropy, the introduction of Compressive Sensing is likewise a major breakthrough in the signal processing community.

Compressive Sensing (CS) is basically used for the acquisition of signals which are either sparse or compressible. Sparsity is the inherent property of those signals for which, all information present in the signal can be represented only with the help of few important components, as compared to the total length of the signal. Similarly, if the sorted components of a signal decay rapidly obeying the power law, then these signals are classified as compressible signals.

A signal can have compressible representation either in original domain or in some transform domais likewise Fourier transform, cosine transform, wavelet transform, etc. A hardly any examples of signals having sparse representation in certain domain are: natural images which have sparse representation in wavelet domain, speech signal can be represented by fewer components using Fourier transform, using Radon transform it is possible obtain a better model for medical images.

The CS measurements are non-adaptive, i.e., it is not possible learning using previous measurements. The resulted fewer compressive measurements can be comfortably stored or transmitted. This fact gives an impression of compressing the signal at the time of acquisition only and it won the name Compressive Sensing. CS allows the faithful reconstruction of the original singal back from fewer random measurements by using some non-linear reconstruction methods. For the reason of all these features, Cs finds its applications especially in the areas where:

- the number of sensors are limited duw to high cost, e.g., non-visible wavelengths;
- taking measurements is too expensive, e.g., high speed A/D converters, imaging via neutro scattering;
- sensing is time consuming, e.g., medical imaging;
- sensing is power constrained, etc.

The Compressive Sensing method is split in two mainly processes: Acquisiton and Reconstruction. And each of them has its model, and these models will be presented below.

2 Acquisition Model

Compressive Sensing works by taking few group of random measurements which are non-adaptive. The CS acquisition model can be described mathematically by

$$y = \varphi x$$

where, $x \in R^n$ or C^n is an input signal of length n, $\varphi \in R^{m \times n}$ or $C^{m \times n}$ is an $m \times n$ random measurement matrix and $y \in R^m$ or C^m is the measurement vector of length m. The input signal and the random measurement matrix are multiplied together to generate compressive measurements. Here, the number of measurements extract is inferior than the length of the input signal, i.e., $m \ll n$.

The size of measurement matrix and consequently measurements number is proportional to the sparsity of input signal. To additionally reduce the measurements number which are necessary for perfect reconstruction, the measurement matrix must be disconnected with basis in which signal has sparse representation.

3 Reconstruction Model

The inputs to the reconstruction algorithm are the vector of measurement y and reconstruction matrix Θ, where $\Theta = \varphi \times \psi \in R^{m \times n}$ or $C^{m \times n}$ and ψ is the signal x sparsifying basis. The signal x can be represented as a columns linear combination of ψ or the basis vectors as

$$x = \sum_{i=1}^{n} s_i \psi_i = \psi s$$

where, $s \in R^n$ is the sparse coefficient vector of length n, having a couple nonzero entries. Through the solution of Acquisition Model equation, it is possible to recover back the original signal from the compressive measurements. The Acquisition Model equation is an underspecific system of linear equations and have innite number of possible solutions.

In these cases, the only solution can be obtained by creating the reconstruction problem as an ℓ_0-optimization problem given by the next equation. The ℓ_0-optimization problem look for a solution having minimum ℓ_0-norm subject to the specified constraints. This is equivalent to trying all the possibilities to find the expected solution.

$$\hat{s} = \arg\min_{\mathbf{s}} \|s\|_0, \text{ subject to } \Theta s = y$$

when $\hat{s}$ is the estimate of s and $\|s\|_0$ denotes the ℓ_0-norm of s. Although ℓ_0 is not a proper norm, it is a pseudonorm or quasinorm, which represents the nonzero elements number of a vector. Searching for a solution of the earlier equation by trying all possible combinations is huge computationally exercise for a medium sized problem. Additionally, ℓ_0-minimization problem has been defined as NP-hard.

Alternates have been proposed in literature, which are the capability to obtain a similar solution of the ℓ_0-minimization for the above problem, in near polynomial time. One of the options is using a convex optimization and looking for a solution that has a minimum ℓ_1-norm, as specified below. This is considered as a practical solution for the reason that solvers available from linear programming can be used for solving the ℓ_1-minimization problems in approximately polynomial time.

$$\hat{s} = \arg\min_{\mathbf{s}} \|s\|_1, \text{ subject to } \Theta s = y$$

where $\|s\|_1$ denotes the ℓ_1-norm of s, which indicates the absolute sum of a vector elements. The generalized expression of a norm is provided by the equation below, from which definition of ℓ_1 and other important norms can be obtained wherever demanded.

$$\ell_P : \|x\|_P = \sqrt[P]{\sum_i \|x_i\|^P}$$

The output of Compressive Sensing reconstruction algorithm is an estimate of sparse representation of x, i.e., $\hat{s}$. The estimate of x, i.e., $\hat{x}$ can be acquired from $\hat{s}$ by using its inverse transform.

4 The Future of Compressive Sensing

The Compressive Sensing has gained an enormous acceptance in a shorter time span, as a sampling method for sampling the signals at their information rate. CS utilises the advantage of underlying signal compressibility to simultaneously sample and compress the signal. And CS has also a strong mathematical foundation that contributed to its development and acceptance. These aspects can guarantee a strong interest in this area.

The introduction of CS in many areas of signal processing has revolutionized them. Some of the massive contributions are faster Magnetic Resonance Imaging, high quality image and video acquisition using single pixel camera, acquisition of Ultra-wide band signals while drastically reducing the power consumption.

5 References

- A Systematic Review of Compressive Sensing: Concepts, Implementations and Applications, Rani, Dhok & Deshmukh, IEEE, 2018