Available online at www.sciencedirect.com

ScienceDirect

Applied Mathematical Modelling 31 (2007) 181–193

APPLIED MATHEMATICAL MODELLING

www.elsevier.com/locate/apm

A wavelet-based stochastic finite element method of thin plate bending

Jian-Gang Han [a], Wei-Xin Ren [b,*], Yih Huang [c]

[a] Department of Civil Engineering, Hainan University, Haiko, Hainan Province 570228, People's Republic of China
[b] Department of Civil Engineering, Central South University, Changsha, Hunan Province 410075, People's Republic of China
[c] School of Science, Xi'an University of Architecture & Technology, Xi'an, Shaanxi Province 710055, People's Republic of China

Received 1 May 2005; received in revised form 1 July 2005; accepted 31 August 2005
Available online 2 November 2005

Abstract

A wavelet-based stochastic finite element method is presented for the bending analysis of thin plates. The wavelet scaling functions of spline wavelets are selected to construct the displacement interpolation functions of a rectangular thin plate element and the displacement shape functions are expressed by the spline wavelets. A new wavelet-based finite element formulation of thin plate bending is developed by using the virtual work principle. A wavelet-based stochastic finite element method that combines the proposed wavelet-based finite element method with Monte Carlo method is further formulated. With the aid of the wavelet-based stochastic finite element method, the present paper can deal with the problem of thin plate response variability resulting from the spatial variability of the material properties when it is subjected to static loads of uncertain nature. Numerical examples of thin plate bending have demonstrated that the proposed wavelet-based stochastic finite element method can achieve a high numerical accuracy and converges fast.

Keywords: Wavelet function; Wavelet finite element; Spline; Thin plate; Bending

1. Introduction

The reliability of many engineering structures in the presence of uncertainty has been a crucial issue in their analysis and design. Primary systems related to civil engineering structures may be quite sensitive to small imperfections of pertinent design variables. These variables may include the quantities such as modulus of elasticity, Poisson's ratio, shear strength, applied loads, and a variety of other physical and mathematical parameters [1–4]. Several of these variables are inherently random and can be maximal appropriately modeled as random processes. Clearly, the complexity of civil engineering structures requires the use of versatile numerical algorithms, such as the finite element method, to obtain the mathematical approximation to their physical

* Corresponding author. Tel.: +86 731 2654349/591 87892454; fax: +86 591 83737442.
E-mail addresses: hanjg@fzu.edu.cn (J.-G. Han), renwx@mail.csu.edu.cn, ren@fzu.edu.cn (W.-X. Ren).
URL: http://bridge.fzu.edu.cn (W.-X. Ren).

0307-904X/$ - see front matter
doi:10.1016/j.apm.2005.08.020

behavior. A challenging task, however, is to accurately account for the randomness in a given problem while using some proven numerical algorithms.

Finite element method (FEM) based on the energy variational principle and discrete interpolation has been developed since the middle of twenty century, which makes a full use of advantages of conventional energy method and finite difference method. FEM is now a powerful tool in structural analysis and design [5–11]. In the development of FEM, the energy variational principle provides a theoretical foundation to develop varieties of finite element formulations. The most developed one, i.e. the finite element method provides the necessary modeling flexibility. Standard deterministic form of the finite element tool modified to the stochastic finite element method has been developed to analyze system stochasticity problems. Extensive reviews regarding this can be found [12–14].

Wavelet theory is a mathematical tool that was developed in recent decades. As a new branch of mathematics, it has gained more and more attentions in engineering fields, such as signal processing, image processing, pattern and phonetic recognition, quantum physics, earthquake reconnaissance, fluid mechanics, diagnosis and monitoring of machinery defect, etc. The wavelet analysis based on the wavelet transform has the extraordinary characteristics that combine the advantages of functional analysis, Fourier transform, spline analysis, harmonic analysis and numerical analysis [15–18]. It is convinced that the wavelet based methods are powerful in analyzing the field problems with changes in gradients and singularities due to the excellent multi-resolution properties of wavelet functions.

Several investigators have tried to integrate the wavelet analysis with the finite element concept to solve various filed problems. The wavelet-decomposed method was proposed to solve the Navier–Stokes equations [19]. Some investigators combined the Garlerkin method and wavelet analysis, which was called the wavelet Garlerkin method. This method had been applied to solve the Dirichlet problem [20]. Daubechies wavelet has been used to construct one-dimensional Daubechies wavelet beam element [21]. The numerical algorithm of Daubechies wavelet derivative with a higher order was presented, which was used to solve the differential equations of beam and plate [22,23]. However, the current wavelet-based finite elements were always constructed in the wavelet space where the wavelet coefficients were used as the parameters to be determined [24,25]. In this way, when a complex structural problem is analyzed, the interfaces between different elements and boundary conditions cannot be easily treated as the conventional displacement-based finite element methods do. This shortcoming limits a wide application of wavelet-based finite element methods. Recently, authors [26] proposed a multivariable wavelet-based finite element method with the help of the Hellinger–Reissner generalized variational principle. The formulation has been successfully applied to solve the thick plate problems.

The bending of a thin plate is a two-dimensional problem. Various formulations of robust thin plate bending elements have been proposed [27–31]. In this paper, a new wavelet-based stochastic finite element method is presented by using the spline wavelets. To overcome above shortcoming of wavelet-based elements and to have a wide range of applicability, the present spline wavelet finite element formulation of thin plate bending is constructed as the same way of conventional displacement-based finite element method. The spline wavelet functions are used as the displacement interpolation functions of thin plates and corresponding shape functions are derived. Combining the Monte Carlo method, a stochastic wavelet-based finite element formulation is to deal with the problem of the thin plate response variability resulting from the spatial variability of the material properties when it is subjected to static loads of uncertain nature. Numerical examples illustrate that current wavelet-based stochastic finite element method has a high analytical accuracy.

2. Spline wavelets and scaling functions

Wavelet analysis is fundamentally a different approach. The wavelet theory is based on the idea that any signal can be broken down a series of local basis functions called ‘‘wavelet’’. Any particular local feature of a signal can be analyzed according to the scale and translation characteristics of wavelets. Due to the compact support and orthogonality, the spline wavelet can describe the details of the field problem conveniently and accurately. More importance is that the spline wavelet has the explicit expressions which facilitate not only the theoretical formulation, but also numerical implementations with computers. To construct spline wavelet based finite elements, the properties of spline wavelets is briefly discussed as follows. More detailed description can be found in Ref. [17].

The m-order cardinal B-spline can be defined as follows:

$$N_m(x) = < N_{m-1} \ast N_1 > = \int_0^1 N_{m-1}(x-t)\,\mathrm{d}t \quad m \geqslant 2, \tag{1}$$

where N_1 is the characteristic function within an interval $[0,1]$ and N_m is the m-order cardinal spline wavelet function. Using N_1, the expressions of N_m can be easily determined. An advantage of definition N_m in Eq. (1) is that many important properties of N_m can be derived from. Among those are the nine properties listed in the following:

(1) For every $f \in \mathrm{C}$

$$\int_{-\infty}^{\infty} f(x)N_m(x)\,\mathrm{d}x = \int_0^1 \cdots \int_0^1 f(x_1 + \cdots + x_m)\,\mathrm{d}x_1 \cdots \mathrm{d}x_m.$$

(2) For every $g \in \mathrm{C}^m$

$$\int_{-\infty}^{\infty} g^{(m)}(x)N_m(x)\,\mathrm{d}x = \sum_{k=0}^{m} (-1)^{m-k} C_m^k g(k)$$

(3) $supp\ N_m = [0, m]$
(4) $N_m(x) > 0$, for $0 < x < m$
(5) $\sum\limits_{k=-\infty}^{\infty} N_m(x-k) = 1$, for all x
(6) $N_m'(x) = N_{m-1}(x) - N_{m-1}(x-1)$
(7) The cardinal B-spline N_m and N_{m-1} are related by the identity:

$$N_m(x) = \frac{x}{m-1} N_{m-1}(x) + \frac{m-x}{m-1} N_{m-1}(x-1)$$

(8) $N_m(x)$ is symmetric with respect to the center of its support, namely:

$$N_m\left(\frac{m}{2} + x\right) = N_m\left(\frac{m}{2} - x\right), \quad x \in R$$

(9) $N_m(x) \in \mathrm{C}^{m-2}(-\infty, +\infty)$.

Let H be the Hilbert space $L_2(R^2)$ of square integrable bivariate functions. A multi-resolution sequence $V = \{V_j\}_{j\in Z}$ consists of nested closed subspaces $V_j \subset H$ whose union is dense in H, i.e.

$$V_j \subset V_{j+1} \quad \mathrm{clos}_H\left(\bigcup_{j\in Z} V_j\right) = H. \tag{2}$$

A scaling function $\phi \in V_0$, with a non-vanishing integral, exists such that collection $\{\phi(x-k)|k \in Z^2\}$ is a Riesz basis of V_0. Since $\phi \in V_0 \subset V_1$, a sequence a_k exists such that the scaling function satisfies:

$$\phi(x) = 2\sum_k a_k \phi(2x - k). \tag{3}$$

This functional equation is often called the refinement function. It can be shown that the collection of functions $\phi_{j,\,k}(x) = 2^j\phi\,(2^jx - k)$ is also a Riesz basis of V_j. W_j is used to denote a space complementing V_j in V_{j+1}, i.e., a space that satisfies:

$$V_{j+1} = V_j \oplus W_j, \tag{4}$$

where the symbol $\oplus$ stands for a direct sum. In other words, each element of V_{j+1} can be expressed in a unique way as the sum of an element of W_j and an element of V_j. The space W_j is sometimes called the detail space because it contains the "detail" information needed to go from an approximation at resolution$j$ to an approximation at resolution $j+1$. Now, for fixed j, the wavelet functions $\{\psi_{jk}|k \in Z^2\}$ constitute the Riesz basis of the detail space W_j. They can be obtained by the dilatations and integer translations of a finite number of mother functions (mother wavelets). In fact, one has $\psi_{j,k}(x) = 2^j\psi(2^jx - k)$.

Eq. (4) implies that there exists a sequence b_k such that

$$\psi(x) = 2\sum_k b_k\phi(2x-k). \tag{5}$$

Letting $\phi_m(x) = N_m$, for each j, since $N_m(2^jx) \in V_j^m$ and $V_j^m \subset V_{j+1}^m$, the following equation can be obtained:

$$\phi_m(x) = \sum_{k=0}^{m} 2^{-m+1}C_m^k\phi_m(2x-k), \tag{6}$$

which is called the refinement function for the cardinal B-spline of order m.

The compactly supported wavelet ψ_m with a minimum support that corresponds to the m-order cardinal B-spline ϕ_m is unique, and is given by

$$\psi_m(x) = \sum_n q_n\phi_m(2x-n), \tag{7}$$

where

$$q_n = \frac{(-1)^n}{2^{m-1}}\sum_{k=0}^{m} C_k^m\phi_{2m}(n+1-k), \quad n = 0,\ldots,3m-2.$$

Since the differential and integral operation of a polynomial is easy and the local characteristic of almost all functions can be described, most of conventional finite element methods select the polynomials as displacement interpolation functions. The terms and order numbers of a polynomial must be considered based on element degrees of freedom and convergence of solving problem. At the same time geometry isotropy property also should be considered. In other words the displacement interpolation function should be independent on the orientation of local coordinate system. In order to satisfy the requirement of convergence, the displacement interpolation functions expressed by polynomials should include constant and linear terms.

In the spline wavelet finite element method, spline wavelet scaling functions are used to construct the element displacement interpolation functions. The wavelet-based displacement shape functions can be derived by

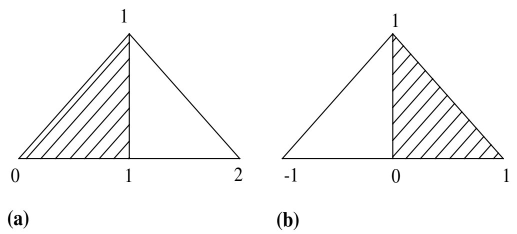

Fig. 1. The two-order scaling functions of spline wavelet.

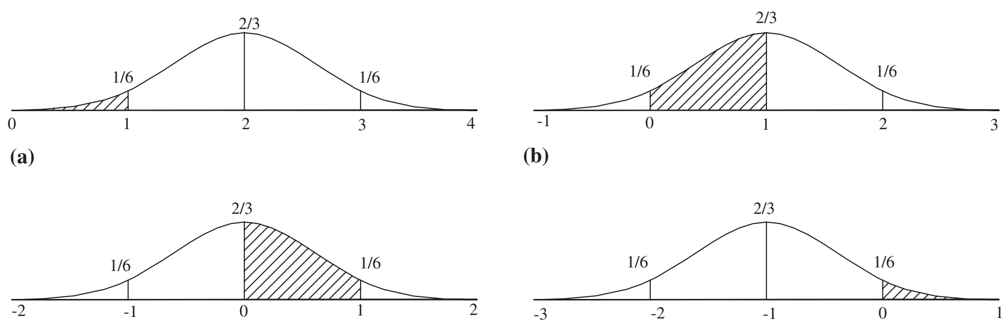

Fig. 2. The four-order scaling functions of spline wavelet.

the same as a conventional displacement-based finite element method. To construct the simple spline wavelets as the element displacement interpolation functions that satisfy the requirement of shape functions, two-order and four-order spline wavelet scaling functions are chosen. Two-order scaling functions $\phi_2(x)$ and $\phi_2(x+1)$ of spline wavelets are shown in Fig. 1, whereas four-order scaling functions $\phi_4(x)$, $\phi_4(x+1)$, $\phi_4(x+2)$ and $\phi_4(x+3)$ of spline wavelets are shown in Fig. 2.

3. Spline wavelet finite element formulation of thin plates

The rectangular and triangular elements of thin plates are the most commonly used elements in the thin plate analysis. In this section, the four-node rectangular element for thin plate bending are constructed by the spline wavelet scaling functions. The geometry of a four-node rectangular thin plate element for bending analysis is shown in Fig. 3. The both lengths of the plate are a and b, respectively. There is only one generic displacement that can fully represent the bending deformation of thin plates. It is the plate deflection w (translation in the z direction). Three degrees of freedom of each node are:

$$a_i = [\, w_i \quad \theta_{xi} \quad \theta_{yi} \,] = \left[w_i \quad \frac{\partial w_i}{\partial y} \quad -\frac{\partial w_i}{\partial x} \right]. \tag{8}$$

Note that the sign change for $\theta_{yi} = -\partial w_i/\partial x$ to correlate the slope with a positive nodal rotation. This type element of a thin plate has 12 degrees of freedom. The element displacement function is herein interpolatively represented by two-order and four-order spline wavelet scaling functions:

$$\begin{aligned} w = {} & a_1\phi_2(\xi)\phi_4(\eta) + a_2\phi_2(\xi)\phi_4(\eta+1) + a_3\phi_2(\xi+1)\phi_4(\eta) + a_4\phi_2(\xi+1)\phi_4(\eta+1) + a_5\phi_4(\xi)\phi_2(\eta) \\ & + a_6\phi_4(\xi)\phi_4(\xi)\phi_2(\eta+1) + a_7\phi_4(\xi+1)\phi_2(\eta) + a_8\phi_4(\xi+1)\cdot\phi_2(\eta+1) + a_9\phi_4(\xi+2)\phi_4(\eta+2) \\ & + a_{10}\phi_4(\xi+2)\cdot\phi_4(\eta+3) + a_{11}\phi_4(\xi+3)\phi_2(\eta+2) + a_{12}\phi_4(\xi+3)\phi_2(\eta+3), \end{aligned} \tag{9}$$

where $\phi_2(\xi)$ and $\phi_4(\xi)$ are respectively two-order and four-order spline wavelet scaling functions. a_i are the arbitrary parameters that can be determined from the node conditions of a element. If the origin of element local coordinates is set at the left corner node, the relationships between global and local coordinates are:

$$\begin{aligned} x &= x_1 + a\xi, \quad \xi = \frac{x - x_1}{a}, \\ y &= y_1 + b\eta, \quad \eta = \frac{y - y_1}{b}. \end{aligned} \tag{10}$$

The ranges of local coordinates are $0 \leqslant \xi \leqslant 1$ and $0 \leqslant \eta \leqslant 1$. The nodal displacement vector of a four-node rectangular thin plate element can be expressed as

$$a^e = [\, w_1 \quad \theta_{x1} \quad \theta_{y1} \quad w_2 \quad \theta_{x2} \quad \theta_{y2} \quad w_3 \quad \theta_{x3} \quad \theta_{y3} \quad w_4 \quad \theta_{x4} \quad \theta_{y4} \,]^{\mathrm{T}}. \tag{11}$$

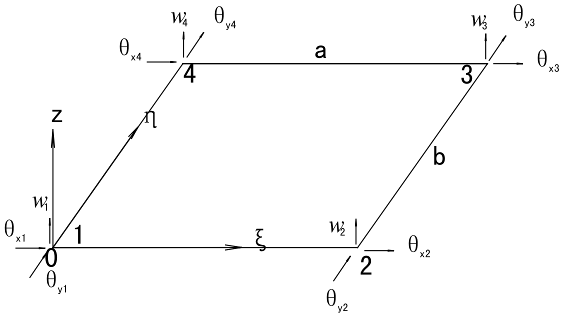

Fig. 3. Four-node rectangular thin plate bending element.

The node conditions of an element are as follow:

$$\xi = 0, \quad \eta = 0, \quad w = w_1, \quad w_{,y} = \theta_{x1}, \quad w_{,x} = -\theta_{y1}, \tag{12a}$$
$$\xi = 1, \quad \eta = 0, \quad w = w_2, \quad w_{,y} = \theta_{x2}, \quad w_{,x} = -\theta_{y2}, \tag{12b}$$
$$\xi = 1, \quad \eta = 1, \quad w = w_3, \quad w_{,y} = \theta_{x3}, \quad w_{,x} = -\theta_{y3}, \tag{12c}$$
$$\xi = 0, \quad \eta = 1, \quad w = w_4, \quad w_{,y} = \theta_{x4}, \quad w_{,x} = -\theta_{y4}. \tag{12d}$$

Substituting Eq. (12) into Eq. (9) produces:

$$w = Na^e, \tag{13}$$

where

$$N = [N_1 \quad N_2 \quad N_3 \quad N_4 \quad N_5 \quad N_6 \quad N_7 \quad N_8 \quad N_9 \quad N_{10} \quad N_{11} \quad N_{12}]$$

in which N_i is the spline wavelet-based displacement shape function in local coordinates. The element displacement field of thin plate bending is now represented by the shape functions and node displacements. The expressions of those spline wavelet shape functions are as follows:

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \\ N_6 \\ N_7 \\ N_8 \\ N_9 \\ N_{10} \\ N_{11} \\ N_{12} \end{Bmatrix} = \boldsymbol{A} \begin{Bmatrix} \phi_2(\xi)\phi_4(\eta) \\ \phi_2(\xi)\phi_4(\eta+1) \\ \phi_2(\xi+1)\phi_4(\eta) \\ \phi_2(\xi+1)\phi_4(\eta+1) \\ \phi_4(\xi)\phi_2(\eta) \\ \phi_4(\xi)\phi_2(\eta+1) \\ \phi_4(\xi+1)\phi_2(\eta) \\ \phi_4(\xi+1)\phi_2(\eta+1) \\ \phi_4(\xi+2)\phi_4(\eta+2) \\ \phi_4(\xi+2)\phi_4(\eta+3) \\ \phi_4(\xi+3)\phi_4(\eta+2) \\ \phi_4(\xi+3)\phi_4(\eta+3) \end{Bmatrix} \tag{14}$$

and

$$\boldsymbol{A} = \boldsymbol{C}_1 \cdot \boldsymbol{B}_1, \quad \boldsymbol{C}_1 = \mathrm{diag}(\boldsymbol{A}_1, \boldsymbol{A}_1, \boldsymbol{A}_1, \boldsymbol{A}_1),$$

$$\boldsymbol{B}_1 = \begin{bmatrix} -11/36 & 1/33 & 82/45 & -198/197 & -11/36 & 82/45 & 1/33 & -198/197 & 222/101 & 198/47 & 198/47 & -770/37 \\ 26/45 & 23/72 & 35/46 & -11/39 & -53/78 & -2/13 & -17/55 & -5/94 & 11/30 & -58/49 & 44/17 & -303/32 \\ 53/78 & 17/55 & 2/13 & 5/94 & -26/45 & -35/46 & -23/72 & 11/39 & -11/30 & -44/17 & 58/49 & 303/32 \\ -23/31 & -74/23 & -1/3 & -14/37 & 203/48 & 1/106 & 93/41 & 71/28 & 24/13 & -2562/197 & -615/62 & 1821/37 \\ -257/95 & -222/101 & -26/45 & -23/72 & 74/21 & 53/78 & 23/12 & 11/30 & 141/76 & -490/69 & -366/47 & 2160/73 \\ 13/50 & 1/9 & 5/34 & 1/51 & -14/71 & -71/30 & -2/17 & 35/54 & -7/31 & -19/17 & 18/71 & 159/52 \\ -102/43 & 23/16 & -14/27 & -6/25 & -102/43 & -14/27 & 23/16 & -6/25 & -199/44 & 2768/151 & 2768/151 & -3499/49 \\ 58/49 & -23/17 & -14/71 & -2/17 & 61/47 & 22/51 & 31/44 & 7/31 & 2/3 & -51/20 & -81/28 & 115/11 \\ -61/47 & -31/44 & -22/51 & -7/31 & -58/49 & 14/71 & 23/17 & 2/17 & -2/3 & 81/28 & 51/20 & -115/11 \\ 203/48 & 93/41 & 1/106 & 71/28 & -23/31 & -1/3 & -74/23 & -14/37 & 24/13 & -615/62 & -2562/197 & 1821/37 \\ 14/71 & 2/17 & 71/30 & -35/54 & -13/50 & -5/34 & -1/9 & -1/51 & 7/31 & -18/71 & 19/17 & -159/52 \\ -74/21 & -23/12 & -53/78 & -11/30 & 257/95 & 26/45 & 222/101 & 23/72 & -141/76 & 366/47 & 490/69 & -2160/73 \end{bmatrix}$$

$$\boldsymbol{A}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & a \end{bmatrix}.$$

The generalized strains or curvatures of thin plate bending can be obtained from plate strain–displacement relationship:

$$\boldsymbol{\Psi} = \boldsymbol{B}\boldsymbol{a}^e, \tag{15}$$

where $\boldsymbol{B}$ is the strain matrix of a four-node rectangular element for thin plate bending problem. They can be calculated from the shape functions as follows:

$$\boldsymbol{B} = -\begin{bmatrix} \frac{\partial^2 N_1}{\partial x^2} & \frac{\partial^2 N_2}{\partial x^2} & \frac{\partial^2 N_3}{\partial x^2} & \cdots & \frac{\partial^2 N_{12}}{\partial x^2} \\ \frac{\partial^2 N_1}{\partial y^2} & \frac{\partial^2 N_2}{\partial y^2} & \frac{\partial^2 N_3}{\partial y^2} & \cdots & \frac{\partial^2 N_{12}}{\partial y^2} \\ 2\frac{\partial^2 N_1}{\partial x \partial y} & 2\frac{\partial^2 N_2}{\partial x \partial y} & 2\frac{\partial^2 N_3}{\partial x \partial y} & \cdots & 2\frac{\partial^2 N_{12}}{\partial x \partial y} \end{bmatrix}.$$

Subsequently, the generalized stresses of thin plate bending can be determined from plate stress–strain relationship:

$$\boldsymbol{M} = \boldsymbol{D}_b \boldsymbol{B} \boldsymbol{a}^e \tag{16}$$

in which $\boldsymbol{D}_b$ is the matrix of the elastic matrix related with material constants

$$\boldsymbol{D}_b = D_b \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix},$$

where $D_b = \frac{Et^3}{12(1-\mu^2)}$ is the bending rigidity of a thin plate.

By using the virtual work principle, the spline wavelet finite element formulation of a four-node rectangular element for the bending of a thin plat can be obtained as follows:

$$\boldsymbol{K}^e \boldsymbol{a}^e = \boldsymbol{P}^e, \tag{17}$$

where $\boldsymbol{K}^e$ is the element stiffness matrix calculated from

$$\boldsymbol{K}^e = \int\int \boldsymbol{B}^{\mathrm{T}} \boldsymbol{D}_b \boldsymbol{B} \, \mathrm{d}x \, \mathrm{d}y$$

and $\boldsymbol{P}^e$ is the equivalent node force vector due to an uniformly distributed load $\boldsymbol{q}$:

$$\boldsymbol{P}^e = \int\int \boldsymbol{N}^{\mathrm{T}} \boldsymbol{q} \, \mathrm{d}x \, \mathrm{d}y.$$

If all element formulations are properly assembled, the global formulation of thin plate bending can be obtained and the structural analysis can be carried out by solving the global finite element equations.

4. Wavelet-based stochastic FEM

4.1. Monte Carlo simulation

The analytic method of stochastic problems can be divided into two types: statistical and non-statistical approximation method. These two methods have their advantages and disadvantages. The stochastic FEM in common use which developed on the base of the traditional FEM is a non-statistical numerical analytical method. The application of this method had been limited because of the following reasons: (1) the quantum of stochastic disturbance should be small; (2) the numerical results are an approximate solution and the accuracy is therefore limited; (3) the implement of solution is difficult. The Monte Carlo simulation-based stochastic FEM, however, is a statistical approximation method. When the distribution of stochastic variables is determinate, the solution of stochastic problems is accurate. The Monte Carlo simulation needs lots of sampling experiments, so the calculating work is tremendous. The appearance and rapid development of high speed electronic computers provide an extraordinary effective and cheapness tools for the Monte Carlo simulation of civil engineering structures.

The Monte Carlo method belongs to a stochastic numerical simulation method, which is also so called statistical experiment method. The method was developed on the base of the mathematical statistical principle.

The core idea is that the occurrence frequency of an event in abundance independently experiments approximates to its probability. It is considered as a relatively accurate method, so which is usually used to check the veracity of other methods. Monte Carlo method is heavily dependent on the fast, efficient production of streams of random numbers. Since physical processes, such as a white noise generated from electrical circuits, generally introduce new numbers much too slowly to be effective with today's digital computers, the random number sequences can be produced directly on the computer using software.

Let us consider an elementary event with a countable set of random outcomes, $A_1, A_2, \ldots, A_k$ (e.g. rolling a die). Suppose this event occurs repeatedly, say N times, with $N >> 1$, and we count how often the outcome A_k is observed (N_k). Then it makes sense to define the probability p_k for the outcome A_k or (it is assumed that all possible events have been enumerated)

$$p_k = \lim_{N\to\infty}(N_k/N), \quad \sum_k p_k = 1. \tag{18}$$

Obviously we have $0 \leqslant p_k \leqslant 1$ (if A_k never occurs, $p_k = 0$; if it is certain to occur, $p_k = 1$). In order to improve the accuracy, N should be enough bigger. Some investigators suggested that N should satisfy the following condition:

$$N \geqslant \frac{100}{P_\mathrm{f}}, \tag{19}$$

where P_f is the previously assumed failure probability. To improve the efficiency, some other conditions can be also implemented in the Monte Carlo simulation with reference to David and Kurt [32].

4.2. Spline wavelet-based stochastic FEM

The second moment method of stochastic finite element is a practical method using in reliability analysis of structures. But when it is used for the large-scale complex civil engineering structures, such as arch dam in water-power engineering, long-span bridges, high-rise building and offshore structures, the second moment method needs to process a lot of iterations. At the same time the limit state functions of such large-scale civil engineering structures have no clear explicit expressions. Practically, one of the most crude and simple methodologies in stochastic analysis is the Monte Carlo simulation, which is based on the fact that the stochastic governing equation can be approximated by a set of deterministic equations represented by the numerically generated, in compliance with the stochastic field under consideration, random samples. However, the Monte Carlo simulation cannot be directly used to calculate the failure probability of structures.

The original combination of Monte Carlo simulation and FEM can track back to 1970s. The Monte Carlo simulation-based stochastic FEM is built on the base of a number of determinacy finite element computing. It should not belong of a really stochastic finite element and the name of simulation finite element or statistical finite element would be more suitable. But the stochastic finite element is developed by the original and direct Monte Carlo simulation-based finite element in faith. Subsequently, the present wavelet-based stochastic FEM of thin plate bending is formulated by combining the above developed spline wavelet FEM and Monte Carlo simulation.

The first step of the wavelet-based stochastic FEM is to generate a series of random numbers in a uniform distribution for every random variable. This step can be easily carried out. The following step is to determine the random numbers which must fit the distribution law of each random variable by transforming. A group of random numbers corresponding to all variables is considered as once sampling. Then, the random numbers of stochastic variables are substituted into the derived spline wavelet-based finite element formulation and one group solutions can be obtained by solving the corresponding FEM equations. At last, the statistical analysis is implemented for these group solutions and the statistical quantities that describe the structural responses can be obtained. Using these statistical quantities, the corresponding reliability calculations can be performed. The flowchart of proposed wavelet-based stochastic finite element calculation with the Monte Carlo simulation is as shown in Fig. 4.

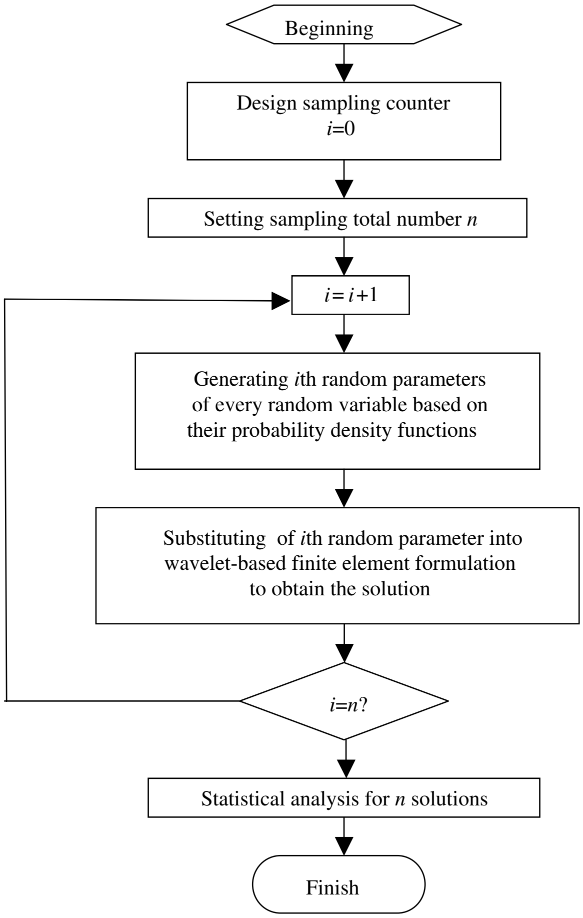

Fig. 4. Flowchart of wavelet-based stochastic finite element method incorporated with Monte Carlo simulation.

5. Numerical examples

The applications of the wavelet-based stochastic finite element method to the bending analysis of the rectangular thin plates with different boundary conditions are presented in this section to demonstrate the reliability and computation accuracy of the method. The first numerical example is aimed at verifying the developed spline wavelet-based FEM and the second example is to illustrate the feasibility of the proposed wavelet-based stochastic FEM. All programs are implemented in the Matlab environment [33].

5.1. Example 1-bending of square thin plates

A simply supported and a fixed thin plate under the uniform distribution loading q are considered in this numerical example. The thickness of the thin plate is t. The length is L, the bending rigidity is D_b, and the Possion ratio $\mu = 0.3$. The bending deflections and moments of the square thin plates ($t/l = 0.001$) have been calculated by using the developed spline wavelet-based thin plate element formulation. In Table 1, the calculated results are compared with the exact solutions that are obtained by directly solving the differential equations [27]. It can be seen that current results match the exact solutions very well. To investigate the effect of mesh density, the plate is divided into several meshes, say 6×6, 8×8 and 10×10. It is demonstrated that the proposed spline wavelet-based finite element formulation converges fast to exact solutions.

Table 1
Comparison of deflections and moments of simply and fixed supported square thin plates under uniformly distributed loading ($t/l = 0.001$)

Meshes	Center deflections ($l^4/100D_b$)		Bending moments ($ql^2/10$)	
	Simply supported	Fixed supported	Simply supported (plate center)	Fixed supported (mid point of one side)
Rectangular element 6 × 6	0.4041	0.1247	0.6146	−0.3558
Rectangular element 8 × 8	0.4058	0.1258	0.5129	−0.5016
Rectangular element 10 × 10	0.4060	0.1264	0.4801	−0.5124
Exact solution [27]	0.4062	0.1265	0.4789	−0.5133

5.2. Example 2-stochastic analysis of thin plate bending

A square thin plate with two edges free and two edges fixed subjected to a concentrated load as shown in Fig. 5 is considered in this numerical example. The length of this plate is 100 mm, the thickness is 1.0 mm, and the concentrated loading is 100.0 N. It is assumed that the materials parameters and applied load are sort of uncertain. The fabricated error of the plate is ±0.1 mm. The mean value of Young's modulus E is 210.0 N/mm^2, which follows a Gaussian distribution with a 5% standard deviation of the mean value. The random input variable density ρ is assumed as the uniform distribution with the lower limit of 4.0e−6 kg/mm^3 and the high limit of 12.0e−6 kg/mm^3. The applied load is assumed as a logarithmic normal distribution with a standard deviation of 10.0 N.

The stochastic bending analysis of this thin plate is carried out by using the wavelet-based stochastic FEM incorporated with the Monte Carlo simulation. Because the stresses and the deflections at the central point of the plate where the concentrated load is applied to are maximal, they are elected as the output variables. At first, when the maximum sampling number is 40, the numerical results obtained from the wavelet-based stochastic FEM demonstrate that the mean value and variance of the maximal plate deflection are 0.406 mm and 0.07201, respectively, whereas the mean value and variance of the maximal stress are 93.0 N/mm^2 and 11.6929,

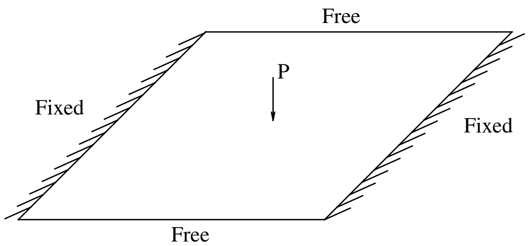

Fig. 5. A square thin plate with two edges free and two edges fixed.

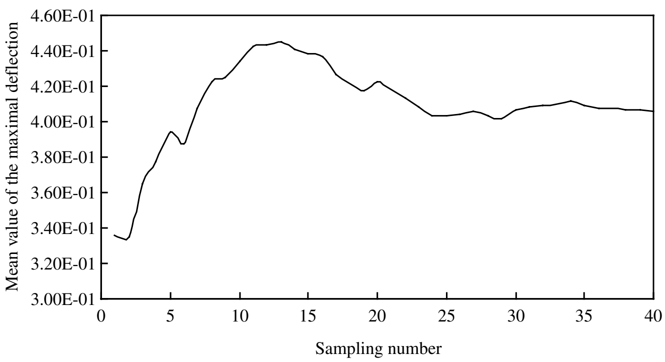

Fig. 6. Relationship between the mean value of maximal plate defection and sampling number.

respectively. The changes of the mean value and variance of the maximal plate stress and deflection with the sampling number are plotted in Figs. 6–9. It can be observed that the present wavelet-based stochastic FEM of thin plate bending converges fast. Further numerical results have shown that the mean value of the maximal plate deflection and stress are 0.409 mm and 93.6 N/mm^2, respectively when the maximum sampling number is increased to be 80. The discrepancy of the maximal stress and deflection are all less than 1%, so the maximum sampling number of 40 can satisfy the requirement of analyzing. Using the mean value and variance of output variables, the reliability calculations can be thereafter performed.

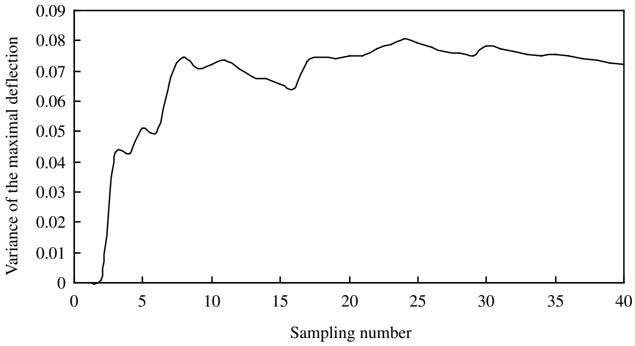

Fig. 7. Relationship between the variance of maximal plate deflection and sampling number.

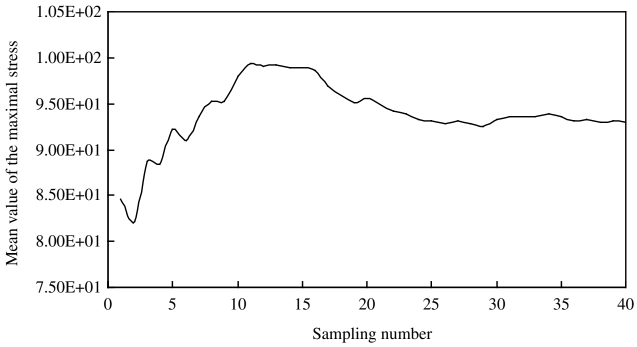

Fig. 8. Relationship between the mean value of maximal plate bending stress and sampling number.

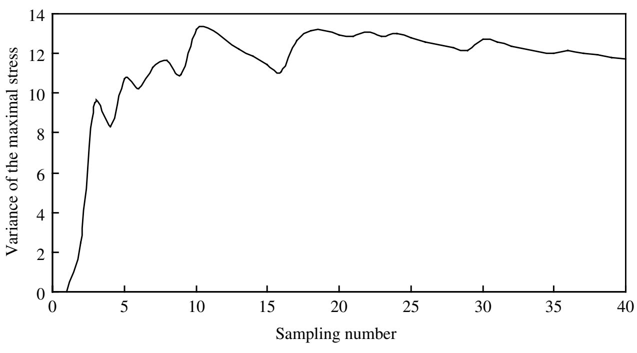

Fig. 9. Relationship between the variance of maximal plate bending stress and sampling number.

6. Conclusions

Based on the spline wavelet scaling functions that are used as the displacement interpolation function of thin plate bending, the spline wavelet-based finite element formulation is developed in the paper. Since the current wavelet-based finite element is constructed in the way of conventional displacement-based FEM, it has a wide range of applicability to deal with the versatile boundary conditions and interfaces between elements. The wavelet-based stochastic FEM of thin plate bending is further presented after the developed wavelet-based FEM is incorporated with the Monte Carlo simulation. The method can deal with the problem response variability resulting from the spatial variability of the material properties of a thin plate when it is subjected to static loads of uncertain nature. The numerical examples illustrate that the proposed wavelet-based FEM and the wavelet-based stochastic FEM can achieve a high analytical accuracy and converge fast in solving the stochastic problem of thin plate bending.

The wavelet-based methods are powerful in analyzing the field problems with changes in gradients and singularities due to the excellent multi-resolution properties of wavelet functions. It can be anticipated that the wavelet-based finite element methods would play an important role in analyzing the engineering problems with a local high gradient. Although the thin plate bending element is constructed in the paper, further extensive work is still required in solving the nonlinearity, high gradient, buckling and dynamic problems of structures in the future. The wavelet-based stochastic finite element method is also feasible to carry out the reliability analysis of large-scale civil engineering structures.

Acknowledgement

Support from the national Natural Science Foundation of China, grant number 59978038, is greatly acknowledged.

References

[1] E. Vanmarcke, Probabilistic modeling of soil profiles, J. Geotech. Engng. Div. ASCE 103 (11) (1977) 1227–1246.
[2] M. Shinozuka, Stochastic Mechanics, Vol. 1, Department of Civil Engineering, Columbia University, New York, NY, 1987.
[3] C.C. Chamis, Probabilistic structural analysis methods for space propulsion system components, Probab. Engng. Mech. 2 (2) (1987) 100–110.
[4] P.D. Spanos, G. Roger, Stochastic finite element expansion for random media, J. Engng. Mech. Div. ASCE 115 (5) (1989) 1035–1053.
[5] R.J. Melosh, Basis for derivation of matrices for the direct stiffness method, AIAA J. 1 (1963) 1631–1637.
[6] T.H.H. Pian, Derivation of element stiffness matrices by assumed stress distributions, AIAA J. 2 (1964) 1333–1336.
[7] Herrmann, Finite element bending analysis for plats, J. Engng. Mech. Div., ASCE 93 (1967) 13–26.
[8] K. Washizu, Variational Method in Elasticity and Plasticity, second ed., Pergamon Press, Oxford, 1975.
[9] H.C. Hu, Variational Principle and Application of Elastic Mechanics, Science Press, Beijing, 1981.
[10] W.J. William, R.J. Paul, Finite Element for Structural Analysis, Prentice-Hall, Inc., New Jersey, 1984.
[11] Z.M. Elias, Theory and Methods of Structural Analysis, John Wiley & Sons, 1986.
[12] E. Vanmarcke, M. Shinozuka, S. Nakagiri, G.I. Schueller, M. Grigoriu, Random fields and stochastic finite elements, Struct. Saf. 3 (3) (1986) 143–166.
[13] H. Benaroya, M. Rehak, Finite element methods in probabilistic structural analysis: a selective review, Appl. Mech. Rev. 41 (5) (1988) 201–213.
[14] G.I. Schueller, A state-of-the-art report on computational stochastic mechanics, Probab. Engng. Mech. 12 (4) (1997) 197–321.
[15] I. Daubechies, Ten lectures on wavelets. CBMS–NSF regional conference series in applied mathematics, Department of Mathematics, University of Lowell, MA., Society for Industrial and Applied Mathematics, Philadelphia, 1992.
[16] V.A. Barker, Some Computational Aspects of Wavelets, Informatics and Mathematical Modeling, Technical University of Denmark, Denmark, 2001.
[17] C.K. Chui, An Introduction to Wavelet, Academic Press, New York, 1992.
[18] C.S. Burrus, R.A. Gopinath, H. Guo, Introduction to Wavelets and Wavelet Transforms—A Primer, Prentice-Hall, Inc., Upper Saddle River, NJ, 1998.
[19] G.W. Wei, D.S. Zhang, S.C. Althorpe, D.J. Kouri, D.K. Hoffman, Wavelet-distributed approximating functional method for solving the Navier–Stokes equation, Comput. Phys. Commun. 115 (1998) 18–24.
[20] R.O. Wells, X. Zhou, Wavelet solutions for the Dirichlet problem, Numer. Math. 70 (1995) 379–396.
[21] J.X. Ma, J.J. Xue, S.J. Yang, Z.J. He, A study of the construction and application of a Daubechies wavelet-based beam element, Finite Elements Anal. Des. 39 (2003) 956–975.

[22] Y.H. Zhou, J.Z. Wang, X.J. Zheng, Application of wavelet Galerkin FEM to bending of beam and plate structure. Applied Mathematic and Mechanics, English Edition (from China), 19(8) (1998) 745–755.
[23] Y.H. Zhou, J.Z. Wang, X.J. Zheng, Application of wavelet Galerkin FEM to bending of plate structure, Acta Mech. Solida Sin. 12 (2) (1999) 136–143.
[24] J. Ko, A.J. Kurdilla, M.S. Pilant, A class of finite element methods based on orthonormal compactly supported wavelets, Comput. Mech. 16 (1995) 235–244.
[25] S.M. Luo, X.J. Zheng, The finite element method based on interpolating with wavelet basis function. Applied Mathematic and Mechanics, English Edition (from China), 21(1) (2000) 13–18.
[26] J.G. Han, W.X. Ren, Y. Huang, Multivariable wavelet based finite element method and its application to thick plates, Finite Elements Anal. Des. 41 (2005) 821–833.
[27] S.P. Timoshenko, D.H. Young, Theory of Structures, McGraw-Hill Company, Inc., New York, 1965.
[28] D.J. Allman, A quadrilateral finite element including vertex rotation for plan elasticity analysis, Int. J. Numer. Meth. Engng. 26 (1988) 717–730.
[29] O.C. Zienkiewiza, R.L. Taylor, fourth ed.The Finite Element Method, Vol. 2, McGraw-Hill, New York, 1991.
[30] R. Ayad, G. Dhatt, J.L. Batoz, A new hybrid-mixed variational approach for Reissner–Mindlin plates: the MISP model, Int. J. Numer. Meth. Engng. 42 (1998) 1149–1179.
[31] F. Kikuchi, K. Ishii, An improved 4-node quadrilateral plate bending element of Reissner–Mindlin type, Comput. Mech. 23 (1999) 240–249.
[32] P.L. David, B. Kurt, A Guide to Monte Carlo Simulations in Statistical Physics, Cambridge University Press, 2003.
[33] Mathworks Inc., Matlab release 5, Natick, MA, USA, 1996.