Homework #2 - Mathew Houser

Chapter 0; #24, 29, 31, 32, 40, 42, 43.

24. Define $f: \mathbb{J} \to \mathbb{J}$ by
$f(1)=1$, $f(2)=2$; $f(3)=3$, and
$f(n)=f(n-1)+f(n-2)+f(n-3)$ for $n \geq 4$.
Prove that $f(n) \leq 2^n$ $\forall n \in \mathbb{J}$.

Let $P(n)$ denote the statement of $f(n) \leq 2^n$ $\forall n \in \mathbb{J}$.

Base Case

First we will show that $P(1)$, $P(2)$, $P(3)$ and $P(4)$ are true. For $n=1$, $f(1)=1$ and $2^1=2$, since $1<2$, $P(1)$ is true. For $n=2$, $f(2)=2$ and $2^2=4$. Since $2<4$, $P(2)$ is true. For $n=3$, $f(3)=3$ and $2^3=8$. Since $3<8$, $P(3)$ is true. For $n=4$, $f(4)=f(3)+f(2)+f(1)=3+2+1=6$, and $2^4=16$. Since $6<16$, $P(4)$ is true.

Inductive Hypothesis

Assume now that $P(K)$ is true, forsome $K \in \mathbb{J}$. That is, $f(K) \leq 2^K$. We want to show that $P(K+1)$ is true.

Induction step.

For $n=K+1$ we see that $f(K+1)=f(K)+f(K-1)+f(K-2)$. We know that $f(K)=f(K-1)+f(K-2)+f(K-3)$ is at most 2^K. Substituting 2^K for $f(K)$ leaves $f(K+1) \leq 2^K + f(K-1) + f(K-2)$.
Similarly we see from $P(K)$ that $f(K-1)+f(K-2)$ is at most $2^K - f(K-3)$.
Substituting $2^K - f(K-3)$ leaves us with
$f(K+1) \leq 2^K + 2^K - f(K-3) = 2^{K+1} - f(K-3)$
Hence $f(K+1)$ is at most $2^{K+1} - f(K-3)$
if $f(K-3)=0$ then $f(K+1)=2^{K+1}$,
If $f(K-3)>0$, Then $f(K+1)<2^{K+1}$. Thus $P(K+1)$ is true, and by the principle of mathematical Induction, $P(n)$, $f(n) \leq 2^n$ $\forall n \in \mathbb{J}$, is true. ∎

29. Define $f(n)$ as follows: for $n \in \mathbb{Z}$, $n \geq 0$, $f(0)=7$, $f(1)=4$, and for $n \geq 2$, $f(n) = 6 \cdot f(n-2) - f(n-1)$. Prove that $f(n) = 5 \cdot 2^n + 2(-3)^n \ \forall n \in \mathbb{Z}, n \geq 0$.

Let $P(n)$ be the statement $f(n) = 5 \cdot 2^n + 2(-3)^n \ \forall n \in \mathbb{Z}, n \geq 0$.

Base case

First we will show that ~~P(1), P(2) and P(0) are~~ $P(0)$, $P(1)$, and $P(2)$ are true. If $n=0$, $f(0)=7$, and $f(0) = 5 \cdot 2^0 + 2(-3)^0 = 5+2 = 7$. $7=7$, so $P(0)$ is true. If $n=1$, $f(1)=4$, and $f(1) = 5 \cdot 2^1 + 2 \cdot (-3)^1 = 10 - 6 = 4$. $4=4$ so $P(1)$ is true. If $n=2$, $f(2) = 6 \cdot f(0) - f(1)$ $= (6)(7) - (4) = 42 - 4 = 38$. Also $f(2) = 5 \cdot 2^2 + 2(-3)^2 = 20 + 18 = 38$. $38 = 38$ so $P(2)$ is true.

Inductive Hypothesis

Assume now that $P(k)$ is true, for some $k \in \mathbb{Z}$, $k \geq 2$. that is, $6 \cdot f(k-2) - f(k-1) = 5 \cdot 2^k + 2(-3)^k$.

Induction step.

We want to show that $P(k+1)$ is true. We see that $f(k+1) = 6 \cdot f(k-1) - f(k)$

$= 6 \cdot f(k-1) - 5 \cdot 2^k - 2(-3)^k$

$= 6 \cdot (5 \cdot 2^{k-1} + 2(-3)^{k-1}) - 5 \cdot 2^k - 2(-3)^k$

$= (30)(2)^{k-1} + (12)(-3)^{k-1} - 5 \cdot 2^k - 2(-3)^k$

$= (30)(2)^{k-1} + (12)(-3)^{k-1} - 10 \cdot 2^{k-1} + 6(-3)^{k-1}$

$= (20)(2)^{k-1} + (18)(-3)^{k-1}$

$= (10)(2)^k + (-6)(-3)^k$

$= 5(2)^{k+1} + 2(-3)^{k+1}$

Thus $P(k+1)$ is true, and by the principle of Mathematical induction, $P(n)$, $f(n) = 5 \cdot 2^n + 2(-3)^n$ is true $\forall n \in \mathbb{Z}, n \geq 0$. ■

Homework #2 - Mathew House

31. Find a 1-1 function f from $\mathbb{J}$ onto S where S is the set of all odd integers.
Let $f:\mathbb{J}\to S$ be defined by

$$f(n)=\begin{cases}\frac{n-1}{2}, & \text{when } n \text{ is odd}\\ -n/2, & \text{when } n \text{ is even.}\end{cases}$$ ∎

32. Let P_n be the set of all polynomials of degree n with integer coefficients. Prove that P_n is countable. The map
~~The map~~
$a_0+a_1x+\cdots+a_nx^n \mapsto (a_0, a_1, \cdots, a_n)$
is an injection from P_n to $\mathbb{Z}^{n+1}$, where $\mathbb{Z}^{n+1}$ is the finite Cartesian Product of the countable set $\mathbb{Z}$. The product of countable sets is also countable, so P_n is countable. ∎

40. If $x\geq 0$ and $y\geq 0$, Prove that $\sqrt{xy}\leq\frac{x+y}{2}$. (Hint: $(\sqrt{x}-\sqrt{y})^2\geq 0$).
We know that $(\sqrt{x}-\sqrt{y})^2\geq 0$. By expanding this, it follows that:

$$(\sqrt{x}-\sqrt{y})(\sqrt{x}-\sqrt{y})\geq 0$$

$$x-2\sqrt{xy}+y\geq 0$$

$$x+y\geq 2\sqrt{xy}$$

$$\sqrt{xy}\leq\frac{x+y}{2}.$$ ∎

42. If $x, y, a,$ and $b>0$, and $\frac{x}{y}<\frac{a}{b}$, Prove that $\frac{x}{y}<\frac{x+a}{y+b}<\frac{a}{b}$. Given $\frac{x}{y}<\frac{a}{b}\Rightarrow bx<ay$. Starting on the left, we will show that $\frac{x}{y}<\frac{x+a}{y+b}$ by first finding a common denominator

$$\frac{x}{y}<\frac{x+a}{y+b}\Rightarrow\frac{xy+bx}{y^2+by}<\frac{xy+ay}{y^2+by}\Rightarrow xy+bx<xy+ay$$

$$\Rightarrow bx<ay$$

which was shown to be true above.

thus, $\frac{x}{y} < \frac{x+a}{y+b}$. Similarly, we will show that $\frac{x+a}{y+b} < \frac{a}{b}$ by finding a common denominator.

$$\frac{x+a}{y+b} < \frac{a}{b} \Rightarrow \frac{bx+ab}{yb+b^2} < \frac{ay+ab}{yb+b^2} \Rightarrow bx+ab < ay+ab$$

$\Rightarrow bx < ay$, as shown above.

There fore, $\frac{x}{y} < \frac{x+a}{y+b} < \frac{a}{b}$. ■

43. Let $A = \{r : r \text{ is a rational number and } r^2 < 2\}$. Prove that A has no largest member. Hint: If $r^2 < 2$ and $r > 0$, choose a rational number δ such that $0 < \delta < 1$ and $\delta < \frac{2-r^2}{2r+1}$. Show that $(r+\delta)^2 < 2$.

Let r be a rational number such that $r^2 < 2$. Consider a rational number $p = (r+\delta)$, where δ is a positive real number less than one satisfying $\delta < \frac{2-r^2}{2r+1}$. It is obvious that $p > r$. We Observe that

$$\delta < \frac{2-r^2}{2r+1} \Rightarrow 2\delta r + \delta < 2 - r^2$$

$$\Rightarrow r^2 + 2r\delta < 2 - \delta$$

$\Rightarrow r^2 + 2r\delta + \delta^2 < 2 - \delta + \delta^2$. Since $0 < \delta < 1$, $\delta^2 < \delta \Rightarrow -\delta + \delta^2 < 0 \Rightarrow 2 - \delta + \delta^2 < 2$

$(r+\delta)^2 < 2 \Rightarrow p^2 < 2$. Hence the set A has no maximum. ■