Math 520: Midterm 2

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1. (10 marks) Find the general solution of the given differential equation.

$$y'' + y = t(1 + 2\sin t).$$

$y'' + y = t(1+2\sin(t)) = t + 2t\sin(t)$

Complementary Solution: $y''+y=0$. Assume $y = e^{rt}$, then $y''+y = r^2+1$, and thus $r_1 = i$ and $r_2 = -i$. ($\lambda = \emptyset$, $\mu = 1$). So $y_c(t) = C_1 \cdot \cos(t) + C_2 \cdot \sin(t)$.

Particular Solution: $y''+y = t + 2t\cdot\sin(t)$. $g(t)$ is the sum of 2 terms, so we split into 2 equations.

Let $y''+y = t$. Suppose $y(t) = At$, then $y'(t) = A$, $y''(t) = 0$. Then $At = t$, so $A = 1$, and $Y_{P_1}(t) = t$.

Let $y''+y = 2t\cdot\sin(t)$. Suppose $y(t) = At\cdot\sin(t) + Bt\cdot\cos(t)$. Then,

$y'(t) = (At+B)\cdot\cos(t) + (A-Bt)\cdot\sin(t)$, and

$y''(t) = (-At-2B)\cdot\sin(t) + (2A-Bt)\cdot\cos(t)$. Then by substituting

$y''+y = (-At-2B)\cdot\sin(t) + (2A-Bt)\cdot\cos(t) + At\cdot\sin(t) + Bt\cdot\cos(t) = 2t\cdot\sin(t)$

$\Rightarrow -2B\cdot\sin(t) + 2A\cdot\cos(t) = 2t\cdot\sin(t)$, So $B = -1$, and $A = 0$. Thus $Y_{P_2}(t) = -t\cdot\cos(t)$.

$Y_P = Y_{P_1} + Y_{P_2} = t - t\cdot\cos(t)$.

General Solution: $y(t) = y_c(t) + Y_P(t)$, so

$y(t) = C_1\cdot\cos(t) + C_2\cdot\sin(t) + t - t\cdot\cos(t)$.

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2. (30 marks) Find general solutions of the following differential equations.

(i) $y(t) = C_1 e^{-5t} + C_2 e^{-t}$

$$y'' + 6y' + 5y = 0.$$

(ii) $y(t) = C_1 e^{-2t} + C_2 t e^{-2t}$

$$y'' + 4y' + 4y = 0.$$

(iii) $y(t) = C_1 e^{-t} \cdot \cos(t) + C_2 e^{-t} \cdot \sin(t)$

$$y'' + 2y' + 2y = 0.$$

$\boxed{i}$ $y'' + 6y' + 5y = 0$. Assume $y = e^{rt}$, then $r^2 + 6r + 5 = (r+5)(r+1)$, so the possible values of r are $r_1 = -5$ and $r_2 = -1$. Thus the general solution is $y(t) = C_1 e^{-5t} + C_2 e^{-t}$.

$\boxed{ii}$ $y'' + 4y' + 4y = 0$. Assume $y = e^{rt}$, then $r^2 + 4r + 4 = (r+2)(r+2)$, so the possible values of r are $r_1 = -2$ and $r_2 = -2$. $r_1 = r_2$, so the general solution is $y(t) = C_1 e^{-2t} + C_2 t e^{-2t}$.

$\boxed{iii}$ $y'' + 2y' + 2y = 0$. Assume $y = e^{rt}$, then $r^2 + 2r + 2$, so the possible values of r are $r_1 = -1 + i$ and $r_2 = -1 - i$. r_1 and r_2 are complex conjugates where $\lambda = -1$ and $\mu = 1$, so the general solution is $y(t) = C_1 e^{-t} \cdot \cos(t) + C_2 e^{-t} \cdot \sin(t)$.

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3. (10 marks) Find the solution of the following initial value differential equation.

$$y'' + 2y' + y = t^2, \quad y(0) = 1, \quad y'(0) = 1.$$

Complementary Solution: $y'' + 2y' + y = 0$. Assume $y = e^{rt}$. Then $r^2 + 2r + 1 = (r+1)(r+1)$, so the possible values of r are $r_1 = -1$ and $r_2 = -1$. $r_1 = r_2$, So the complementary solution is $y_c(t) = c_1 e^{-t} + c_2 t e^{-t}$.

Particular Solution: Let $y'' + 2y' + y = t^2$. Suppose $y(t) = At^2 + Bt + C$. Then $y' = 2At + B$, and $y'' = 2A$. By subsitution, $2A + 4At + 2B + At^2 + Bt + C = t^2$, and so $A = 1$, $B = -4$, and $C = 6$. Therefore the particular solution is $y_p(t) = t^2 - 4t + 6$.

General Solution: $y(t) = y_c(t) + y_p(t)$, so $y(t) = c_1 e^{-t} + c_2 t e^{-t} + t^2 - 4t + 6$.

Also, $y'(t) = -c_1 e^{-t} + c_2 e^{-t} - c_2 t e^{-t} + 2t - 4$.

Initial Values: Substituting the first initial value $y(0) = 1$ yields $c_1 + 6 = 1$, which implies that $c_1 = -5$. Substituting the second initial value $y'(0) = 1$ yields $-c_1 + c_2 - 4 = 1$, which implies that $c_2 = 0$.

Solution: There fore by substitution, the solution of the initial value differential Equation

is $y(t) = -5e^{-t} + t^2 - 4t + 6$.