Math 554 Homework #4 — Mathew Houser

Chapter 1: # 26, 28, 32, 34, 37, 40, 45

26. Give an example in which $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ do not converge but $\{a_n + b_n\}_{n=1}^{\infty}$ converges.

$\{a_n\}_{n=1}^{\infty} = \{(-1)^n\}_{n=1}^{\infty} = \{-1, 1, -1, 1, -1, 1, \cdots\}$

$\{b_n\}_{n=1}^{\infty} = \{(-1)^{n+1}\}_{n=1}^{\infty} = \{1, -1, 1, -1, 1, -1, \cdots\}$

$\{a_n + b_n\}_{n=1}^{\infty} = \{(-1)^n + (-1)^{n+1}\}_{n=1}^{\infty} = \{0, 0, 0, 0, 0, 0, \cdots\}$.

28. If $\{a_n\}_{n=1}^{\infty}$ converges to a with $a_n \geq 0$ $\forall n$, show that $\{\sqrt{a_n}\}_{n=1}^{\infty}$ converges to $\sqrt{a}$.

Case 1: Suppose $a = 0$. Choose any arbitrary $\varepsilon > 0$, then $\exists N \in \mathbb{J}$ such that $|a_n - 0| < \varepsilon^2$ $\forall n \geq N$. Thus $|\sqrt{a_n} - 0| < \varepsilon$, hence $\{\sqrt{a_n}\}_{n=1}^{\infty}$ converges to $a = 0$.

Case 2: Suppose $a > 0$. Choose any arbitrary $\varepsilon > 0$. Then $\exists N \in \mathbb{J}$ such that $\forall n \geq N$, $|a_n - a| < \varepsilon\sqrt{a}$. Thus

$$|\sqrt{a_n} - \sqrt{a}| = \frac{|a_n - a|}{\sqrt{a_n} + \sqrt{a}} < \frac{|a_n - a|}{\sqrt{a}} < \frac{\varepsilon\sqrt{a}}{\sqrt{a}} = \varepsilon$$

Hence $\{\sqrt{a_n}\}_{n=1}^{\infty}$ converges to $\sqrt{a}$ $\forall n$. ■

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32. Find the limit of the sequences

(a) $\lim_{n\to\infty}\left(\frac{n^2+4n}{n^2-5}\right) = \frac{\lim_{n\to\infty}\left(\frac{n+4}{n}\right)}{\lim_{n\to\infty}\left(1-\frac{5}{n^2}\right)} = \frac{1}{1} = 1$

(b) $\lim_{n\to\infty}\left(\frac{\cos(n)}{n}\right) = 0$ (by the squeeze theorem)

(c) $\lim_{n\to\infty}\left(\frac{\sin(n^2)}{\sqrt{n}}\right) = 0$ (by Squeeze Theorem)

(d) $\lim_{n\to\infty}\left(\frac{n}{n^2-3}\right) = \frac{\lim_{n\to\infty}\left(\frac{1}{n}\right)}{\lim_{n\to\infty}\left(1-3/n^2\right)} = \frac{0}{1} = 0$

(e) $\lim_{n\to\infty}\left(\left(\sqrt{4-\frac{1}{n}}-2\right)n\right) = \lim_{n\to\infty}\left(\frac{-\frac{1}{n}}{\sqrt{4-\frac{1}{n}}+2}\cdot n\right) = \frac{\lim_{n\to\infty}(-1)}{\lim_{n\to\infty}\left(\sqrt{\frac{4n-1}{n}}+2\right)} = \frac{-1}{4}$

(f) $\lim_{n\to\infty}\left((-1)^n\frac{\sqrt{n}}{n+7}\right)$. $\lim_{n\to\infty}\left(\frac{\sqrt{n}}{n+7}\right) = \frac{\lim_{n\to\infty}\left(1/\sqrt{n}\right)}{\lim_{n\to\infty}\left(1+7/n\right)} = \frac{0}{1} = 0$

Since $\left\{\frac{\sqrt{n}}{n+7}\right\}_{n=1}^{\infty}$ converges to 0 and we know that $\{(-1)^n\}_{n=1}^{\infty}$ is bounded by $-1 \le a_n \le 1$, we apply theorem 1.13 to determine that $\lim_{n\to\infty}\left((-1)^n\frac{\sqrt{n}}{n+7}\right) = 0$. ∎

34. Find a convergent subsequence of the sequence $\left\{(-1)^n\left(1-\frac{1}{n}\right)\right\}_{n=1}^{\infty}$. Choose the sequence $\{n_k\}_{k=1}^{\infty}$ such that $n=2k$ for each $k \in \mathbb{J}$.

Then $\left\{(-1)^{2k}\left(1-\frac{1}{2k}\right)\right\}_{k=1}^{\infty}$ converges to 1. ∎

37. Prove that if $\{a_n\}_{n=1}^{\infty}$ is decreasing and bounded, then $\{a_n\}_{n=1}^{\infty}$ converges.

Suppose $\{a_n\}_{n=1}^{\infty}$ is a decreasing sequence. Then by definition $\{a_n\}_{n=1}^{\infty}$ is a monotone sequence. Since $\{a_n\}_{n=1}^{\infty}$ is monotone and bounded, it follows from theorem 1.16 that $\{a_n\}_{n=1}^{\infty}$ is convergent. ∎

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40. Show that the sequence defined by $a_1 = 6$ and $a_n = \sqrt{6 + a_{n-1}}$ $\forall n > 1$ is Convergent and find the Limit.

Suppose the sequence is convergent and Converges to L, that is $\lim_{n\to\infty}(a_n) = \lim_{n\to\infty}(a_{n-1}) = L$. Then $L = \sqrt{6+L} \to L^2 = 6 + L \to L^2 - L - 6 = 0$ $\to (L-3)(L+2) = 0$. Thus $\lim_{n\to\infty}(a_n) = \lim_{n\to\infty}(a_{n-1}) = L = 3$. Since we have shown that the limit exists we conclude that the sequence converges to 3. ■

45. Show that if x is any real number, then there is a sequence of rational numbers converging to x.

Let x be any real number. Then, there exists a sequence of rational numbers $\{q_n\}_{n=1}^{\infty}$ such that $x < q_n < x + 1/n$. We know that $\lim_{n\to\infty}(x + 1/n) = \lim_{n\to\infty}(x) = x$. Then by the Squeeze theorem it follows that $\lim_{n\to\infty}(q_n) = x$, hence the sequence $\{q_n\}_{n=1}^{\infty}$ converges to x. ■