MATH 554 Homework #9 — Mathew Houser

Chapter 4 # 3, 6, 11, 14, 15, 16

3. Use the definition to find the derivative of $f(x)=\sqrt{x}$, for $x>0$. Is f differentiable at zero? Explain.

Let $f:(0,\infty)\to\mathbb{R}$ be defined by $f(x)=\sqrt{x}$.
Let $x_0 \in (0,\infty)$ be an accumulation point of $(0,\infty)$, then for each $x\in(0,\infty)$, with $x \neq x_0$ we have

$$T(x) = \frac{f(x)-f(x_0)}{x-x_0} = \frac{\sqrt{x}-\sqrt{x_0}}{x-x_0} \cdot \frac{(\sqrt{x}+\sqrt{x_0})}{(\sqrt{x}+\sqrt{x_0})}$$

$$= \frac{(x-x_0)}{(x-x_0)(\sqrt{x}+\sqrt{x_0})} = \frac{1}{\sqrt{x}+\sqrt{x_0}} \quad \text{and}$$

$$\lim_{x\to x_0} \frac{1}{\sqrt{x}+\sqrt{x_0}} = \frac{1}{2\sqrt{x_0}}.$$ If $x_0=0$ then the limit is undefined hence f is not differentiable at zero. ■

6. Suppose $f:(a,b)\to\mathbb{R}$ is differentiable at $x\in(a,b)$. Prove that $\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}$ exists and equals $f'(x)$. Give an example of a function where this limit exists, but the function is not differentiable.

$$\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h} = \lim_{h\to 0}\frac{f(x+h)-f(x)+f(x)-f(x+h)}{2(x+h-x)}$$

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{2(x+h-x)} + \lim_{h\to 0}\frac{f(x)-f(x-h)}{2(x-(x-h))}$$

$$= \tfrac{1}{2}f'(x) + \tfrac{1}{2}f'(x) = f'(x).$$

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11. Prove $f:(0,1)\to\mathbb{R}$ defined by $f(x)=\sqrt{2x^2-3x+6}$ is differentiable on $(0,1)$ and compute the derivative.

Let $f(x)=\sqrt{2x^2-3x+6}$. Then (Power Rule & Chain Rule)

$f'(x)=\frac{1}{2}(2x^2-3x+6)^{-1/2}\cdot(4x-3)$ hence

$f'(x)=\dfrac{4x-3}{2\sqrt{2x^2-3x+6}}$. which is defined $\forall x\in(0,1)$ thus f is differentiable on $(0,1)$. $\blacksquare$

14. Suppose $f:\mathbb{R}\to\mathbb{R}$ is differentiable and define $g(x)=x^2 f(x^3)$. Show that g is differentiable and compute g'.

(Product Rule)

g is composed of differentiable functions, therefore

$g'(x)= x^2(3x^2)+(x^3)(2x)=3x^4+2x^4=5x^4$. $\blacksquare$

15. Define $f(x)=\sqrt{x+\sqrt{x+\sqrt{x}}}$ for $x\geq 0$. Determine where f is differentiable and compute the derivative.

$f(x)=\sqrt{x+\sqrt{x+\sqrt{x}}}=(x+(x+(x)^{1/2})^{1/2})^{1/2}$. Then (Power Rule & Chain Rule)

$f'(x)=\frac{1}{2}(x+(x+(x)^{1/2})^{1/2})^{-1/2}\cdot(1+\frac{1}{2}(x+(x)^{1/2})^{-1/2}\cdot(1+\frac{1}{2}(x)^{-1/2})\cdot(1)$

$$f'(x)=\left(\frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\right)\left(1+\frac{1}{2\sqrt{x+\sqrt{x}}}\right)\left(1+\frac{1}{2\sqrt{x}}\right)$$

which is defined $\forall x>0$. Hence f is differentiable on $(0,\infty)$ $\blacksquare$

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16. Define $f:[0,2] \to \mathbb{R}$ by $f(x) = \sqrt{2x - x^2}$.
Show that f satisfies the conditions of Rolle's Theorem and find c such that $f'(c) = 0$.

Let $g(x) = 2x - x^2$. $g(x)$ is a polynomial so it is continuous $\forall x \in \mathbb{R}$. Furthermore $g(x) \geq 0$ $\forall x \in [0,2]$. As we showed in chapter 2.3, $\lim_{x \to x_0} \sqrt{g(x)} = \sqrt{\lim_{x \to x_0} g(x)} = \lim_{x \to x_0} f(x)$. Thus f is continuous on $[0,2]$ and differentiable. $f(0) = 0 = f(2)$. Also,
$f'(x) = \frac{2-2x}{2\sqrt{2x-x^2}} = 0 \Rightarrow 2 = 2x \Rightarrow x = 1$. Then
$f'(1) = \frac{2-2}{2\sqrt{2-1}} = 0$. ∎