MATH 554 Homework #10 — Mathew Houser
§4 # 19, 21, 23, 28, 32, 33, 36

19. Show that $\cos x = x^3 + x^2 + 4x$ has exactly one root in $[0, \pi/2]$.
Define $f(x) = x^3 + x^2 + 4x - \cos x$ → continuous & Differentiable
$f(0) = -1 < 0 < \frac{\pi^3}{8} + \frac{\pi^2}{4} + \frac{4\pi}{2} = f(\pi/2)$; thus by Bolzano's Theorem, $\exists\, z \in (0, \pi/2)$ such that $f(z) = 0$. Furthermore $f'(x) = 3x^2 + 2x + 4 + \sin x > 0 \;\forall x \in [0, \pi/2]$, thus by Theorem 4.9 f is increasing and 1-1 on the interval $[0, \pi/2]$
Therefore $\cos x = x^3 + x^2 + 4x$ has exactly one root in $[0, \pi/2]$.

21. Let $f: [0,1] \to \mathbb{R}$ and $g: [0,1] \to \mathbb{R}$ be differentiable with $f(0) = g(0)$ and $f'(x) > g'(x) \;\forall x \in [0,1]$.
Prove that $f(x) > g(x) \;\forall x \in (0,1]$.
f and g are both continuous and differentiable
By the Cauchy Mean Value Theorem, $\exists\, c \in (0,1)$ such that
$\frac{f(x) - f(0)}{g(x) - g(0)} = \frac{f'(c)}{g'(c)}$. Since $f'(x) > g'(x) \;\forall x \in [0,1]$, we can say that $f'(c)/g'(c) > 1$.
It follows that
$\frac{f(x) - f(0)}{g(x) - g(0)} > 1 \implies f(x) - f(0) > g(x) - g(0)$. Then since $f(0) = g(0)$,
we conclude that $f(x) > g(x) \;\forall x \in (0, 1]$.

23. Use the Mean Value Theorem to prove that $\sqrt{1+h} < 1 + \frac{1}{2}h \;\forall h > 0$.
Let $f(x) = \sqrt{1+x}$ → continuous Differentiable. By the Mean Value Theorem, $\exists\, t \in (0, h)$ such that $f'(t) = \frac{f(h) - f(0)}{h - 0}$. Then,
$\frac{f(h) - f(0)}{h - 0} = \frac{\sqrt{1+h} - \sqrt{1}}{h - 0} = \frac{\sqrt{1+h} - 1}{h} = f'(t) = \frac{1}{2\sqrt{1+t}} < \frac{1}{2}$ (since $\sqrt{1+t} > 1$, $\frac{1}{\sqrt{1+t}} < 1 \to \frac{1}{2\sqrt{1+t}} < \frac{1}{2}$)
$\implies \sqrt{1+h} < \frac{1}{2}h + 1 \;\forall h > 0$.

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28. Prove that $f(x) = 2x^3 + 3x^2 - 36x + 5$ is 1-1 on the interval $[-1, 1]$. Is f increasing or decreasing?

f is obviously differentiable. Observe that

$f'(x) = 6x^2 + 6x - 36 = 6(x^2 + x - 6) = 6(x+3)(x-2)$. $\rightarrow$

$f'(x) \neq 0 \; \forall x \in [-1, 1]$. Thus f is 1-1 and monotone on $[-1, 1]$.

$f(-1) = 42$ and $f(1) = -26 \Rightarrow f$ is decreasing.

32. Use L'Hospital's Rule to find the limits

(a) $\lim_{x \to 1} \frac{\ln x}{x - 1} = \frac{0}{0} \xrightarrow[\text{L'Hospital}]{\text{Apply}} \lim_{x \to 1} \frac{\frac{1}{x}}{1} = 1$

(b) $\lim_{x \to 0} \frac{x}{e^x - 1} = \frac{0}{0} \xrightarrow[\text{L'Hospital}]{\text{Apply}} \lim_{x \to 0} \frac{1}{e^x} = 1$

(c) $\lim_{x \to 0} \frac{\sin x}{x} = \frac{0}{0} \xrightarrow[\text{L'Hospital}]{\text{Apply}} \lim_{x \to 0} \frac{\cos x}{1} = 1$

33. Use L'Hospital's Rule to find the limit:

$$\lim_{x \to 0} \frac{x^2 \sin x}{\sin x - x\cos x} = \frac{0}{0}$$

$$\lim_{x \to 0} \frac{x^2 \cos x + 2x \sin x}{x \sin x} = \frac{0}{0}$$

$$\lim_{x \to 0} \frac{4x\cos x + 2\sin x - x^2 \sin x}{x \cos x + \sin x} = \frac{0}{0}$$

$$\lim_{x \to 0} \frac{6\cos x - 6x\sin x - x^2 \cos x}{2\cos x - x\sin x} = \frac{6}{2} = 3.$$

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36. Use the Inverse-Function Theorem to derive the formula for the derivative of the inverse of ~~sin~~. $\sin x$ on the interval $[-\pi/2, \pi/2]$.
Consider $f(x) = \sin(x)$. f is obviously continuous and differentiable on $[-\pi/2, \pi/2]$ and the derivative $f'(x) = \cos(x) \neq 0$ $\forall x \in [-\pi/2, \pi/2]$. Thus f is 1-1 and the inverse exists. Now consider $g(x) = \arcsin(x)$. Then

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\cos(g(x))}$$

and since $\cos^2 x + \sin^2 x = 1$, it follows that

$$g'(x) = \frac{1}{\sqrt{1 - \sin^2(g(x))}} = \frac{1}{\sqrt{1 - (\sin(g(x)))^2}}$$

$$= \frac{1}{\sqrt{1 - (\sin(\arcsin(x)))^2}} = \frac{1}{\sqrt{1 - x^2}}$$