Homework #3 — Mathew Houser

1.4 Find Upper & Lower bounds for the Sequence

$\left\{\frac{3n+7}{n}\right\}_{n=1}^{\infty} = \{10, 13/2, 16/3, 19/4, 21/5, \ldots\}$ $\lim_{n\to\infty}\left(\frac{3n+7}{n}\right) = 3$

Upperbound = 10, Lower bound = 3

1.6 Prove that the Sequence $\left\{\frac{3n}{2n+1}\right\}_{n=1}^{\infty}$ Converges!

Choose an arbitrary $\varepsilon > 0$. Set $N = \lfloor 3/4\varepsilon + 1/2 \rfloor + 1$

Then $\forall n \geq N$, we have

$$\left|\frac{3n}{2n+1} - \frac{3}{2}\right| = \left|\frac{-3}{4n-2}\right| < \frac{3}{4\lfloor 3/4\varepsilon + 1/2\rfloor - 2} = 3/3/\varepsilon = \varepsilon. \blacksquare$$

1.8 Suppose $\{a_n\}_{n=1}^{\infty}$ Converges to A, and define a new Sequence $\{b_n\}_{n=1}^{\infty}$ by $b_n = \frac{a_n + a_{n+1}}{2}$ $\forall n$. Prove that $\{b_n\}_{n=1}^{\infty}$ converges to A. Since $\{a_n\}_{n=1}^{\infty}$ converges to A, then by definition $\lim_{n\to\infty} a_n = A$. It is obvious that $\lim_{n\to\infty} a_{n+1} = A$. We then see that

$$\lim_{n\to\infty} \frac{a_n + a_{n+1}}{2} = \lim_{n\to\infty} \frac{a_n}{2} + \lim_{n\to\infty} \frac{a_{n+1}}{2} = \frac{A}{2} + \frac{A}{2} = A$$

Hence $\lim_{n\to\infty} \frac{a_n + a_{n+1}}{2} = A$ & the sequence $\{b_n\}_{n=1}^{\infty}$ Converges. $\blacksquare$

1.10 Prove that if $\{a_n\}_{n=1}^{\infty}$ Converges to A, then $\{|a_n|\}_{n=1}^{\infty}$ Converges to $|A|$. Let $\varepsilon > 0$. Since $\{a_n\}_{n=1}^{\infty}$ converses to A, there exists $N \in \mathbb{J}$ such that $\forall n \geq N$, $|a_n - A| < \varepsilon$.

Then $\forall n \geq N$ $||a_n| - |A|| \leq |a_n - A| < \varepsilon$. $\blacksquare$

Is the converse true? The converse is not true, counter example $\{(-1)^n\}$ ~~oscilates while~~ is divergent however $\{|(-1)^n|\}$ converses to 1. $\blacksquare$

1.15 Prove that if $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are Cauchy, then $\{a_n + b_n\}_{n=1}^{\infty}$ is also Cauchy.

Choose an arbitrary $\varepsilon > 0$. Since $\{a_n\}$ and $\{b_n\}$ are Cauchy, there exists some $N_1, N_2 \in \mathbb{J}$ such that $|a_m - a_n| < \varepsilon/2$ for $m,n > N_1$ and $|b_m - b_n| < \varepsilon/2$ for $m,n > N_2$. Choose $N = \max(N_1, N_2)$, then $|(a_m + b_m) - (a_n + b_n)| \leq |a_m - a_n| + |b_m - b_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ for $m,n > N$. Hence $\{a_n + b_n\}_{n=1}^{\infty}$ is Cauchy. ■

1.17 Prove that the sequence $\left\{\frac{2n+1}{n}\right\}_{n=1}^{\infty}$ is Cauchy.

$\lim_{n\to\infty}\left(\frac{2n+1}{n}\right) = 2$. Hence the sequence is convergent, thus by definition the sequence is also a Cauchy sequence. ■

1.21 Determine the accumulation points of the set $A = \{2^n + \frac{1}{k} : n \,\&\, k$ are positive integers$\}$

$\lim_{n\to\infty}(2^n + \frac{1}{k}) = \infty$ $\qquad \lim_{k\to\infty}(2^n + \frac{1}{k}) = 2^n$

The accumulation points are the set $\{2^n\}$.