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MATH 574 - Homework #5 - Due September 29th, 2021

6.5.58. A. $(7)(6)(5)(4)(3) = 2{,}520$ ways

B. Since $5<7$, there is only one way

C. $\binom{7}{5}\frac{7!}{5!2!} = 21$ ways

D. Since $5<7$, there is only one way

8.1.12. A. $a_n = a_{n-1} + a_{n-2} + a_{n-3}$

B. The initial Conditions are:

$a_1 = 1$, Since there is only one way $\{(1)\}$ to climb one stair,
$a_2 = 2$, Since there are 2 ways $\{(2)(1,1)\}$ to climb 2 stairs, and
$a_3 = 4$, Since there are 4 ways $\{(3)(2,1)(1,2)(1,1,1)\}$ to climb 3 stairs.

C. $a_8 = a_7 + a_6 + a_5 = 44 + 24 + 13 = 81$ ways to climb 8 stairs.

8.1.28 PROOF (by Induction): $\forall n \geq 5$, $n \in \mathbb{Z}$, the Fibonacci Numbers satisfy the recurrence relation

$$f_n = 5f_{n-4} + 3f_{n-5} \quad (\star),$$

Given initial Conditions $f_0 = 0$, $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, and $f_4 = 3$.

(Basis Step): Let $n=5$, then $f_5 = 5f_1 + 3f_0 = 5(1) + 3(0) = 5$, which is true, thus $(\star)$ holds true for $n=5$.

(Induction Step): Assume $(\star)$ holds true for $n=k$, ~~[illegible]~~ that is, $f_k = 5f_{k-4} + 3f_{k-5}$. Now suppose $n=k+1$. Since by definition $f_{n+1} = f_n + f_{n-1}$, it follows that
$f_{k+1} = 5f_{k-3} + 3f_{k-4} = 3f_{k-2} + 2f_{k-3} = 2f_{k-1} + f_{k-2} = f_k + f_{k-1}$, which is true, thus $(\star)$ holds true for $n=k+1$. There fore by the Principle of Mathematical Induction, $(\star)$ holds true $\forall n \geq 5$, $n \in \mathbb{Z}$. ■

From the Recurrence Relation Above, it follows that

PROOF (by Induction): $\forall n > 0$, $n \in \mathbb{Z}$; $5 \mid f_{5n}$ $(*)$.

(Basis Step): Let $n=1$, then $f_5 = 5f_1 + 3f_0 = 5(1) + 3(0) = 5$, thus $5 \mid f_5$, ~~[illegible]~~ so $(*)$ holds true for $n=1$.

(Induction Step): Assume $(*)$ holds true for $n=k$, that is
$f_{5k} = 5f_{5k-4} + 3f_{5k-5} = 5a$, for some $a \in \mathbb{Z}$.

Now Suppose $n=k+1$, then

$$f_{5(k+1)} = 5f_{5(k+1)-4} + 3f_{5(k+1)-5} = 5f_{5k+1} + 3f_{5k}$$

Since $f_{5k+1} \in \mathbb{Z}$, and $5 \mid 5$, it follows that $5 \mid 5f_{5k+1}$.

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Since $3 \in \mathbb{Z}$, and by hypothesis $5 \mid f_{5k}$, it follows that $5 \mid 3f_{5k}$.
$5 \mid 5f_{5k+1}$, and $5 \mid 3f_{5k}$, so $5 \mid (5f_{5k+1} + 3f_{5k})$.
Then $5 \mid f_{5(k+1)}$, and thus (✱) holds true for $n = k+1$.
Therefore, by the principle of mathematical Induction, $5 \mid f_{5n}$, $\forall n \in \mathbb{Z}, n > 0$. ■

Supplemental A. PROOF (by Induction): $\forall n > 1$, $GCD(f_n, f_{n-1}) = 1$ (▲).
(Basis Step): Let $n = 2$, then $gcd(f_2, f_1) = gcd(1, 1) = 1$.
Thus (▲) holds true for $n = 2$.
(Induction Step): Assume (▲) is true for $n = k$, that is, $gcd(f_k, f_{k-1}) = 1$. Now suppose $n = k+1$, then, by the Euclidean Algorithm,
$gcd(f_{k+1}, f_k) = gcd(f_k, f_{k-1}) = 1$, by hypothesis.
Therefore, by the principle of mathematical Induction, $gcd(f_n, f_{n-1}) = 1$, $\forall n > 1$. ■

Supplemental B. PROOF (by Induction): $\forall n > 0$, $f_1 + f_3 + f_5 + \cdots + f_{2n-1} = f_{2n}$ (◘).
(Basis Step): Let $n = 1$, then $f_1 = 1 = f_2$.
Thus (◘) holds true for $n = 1$.
(Induction Step): Assume (◘) holds true for $n = k$, that is
$f_1 + f_3 + f_5 + \cdots + f_{2k-1} = f_{2k}$. Suppose now that $n = k+1$, then
$f_1 + f_3 + f_5 + \cdots + f_{2k-1} + f_{2k+1} = f_{2k} + f_{2k+1} = f_{2k+2} = f_{2(k+1)}$.
Thus (◘) holds true for $n = k+1$. Therefore by the principle of mathematical Induction, $f_1 + f_3 + f_5 + \cdots + f_{2n-1} = f_{2n}$, $\forall n > 0$. ■

Supplemental C. PROOF (by Induction): $\forall n > 0$, $f_1^2 + f_2^2 + f_3^2 + \cdots + f_n^2 = (f_n)(f_{n+1})$ (●).
(Basis Step): Let $n = 1$, then $f_1^2 = 1 = (f_1)(f_2)$,
thus (●) holds true for $n = 1$.
(Induction Step): Assume (●) holds true for $n = k$, that is,
$f_1^2 + f_2^2 + f_3^2 + \cdots + f_k^2 = (f_k)(f_{k+1})$. Now suppose $n = k+1$, then
$f_1^2 + f_2^2 + f_3^2 + \cdots + f_k^2 + f_{k+1}^2 = (f_k)(f_{k+1}) + (f_{k+1})^2 = (f_{k+1})(f_k + f_{k+1}) = (f_{k+1})(f_{k+2})$.
Thus (●) holds true for $n = k+1$. Therefore by the principle of mathematical Induction, $f_1^2 + f_2^2 + f_3^2 + \cdots + f_n^2 = (f_n)(f_{n+1})$, $\forall n > 0$. ■