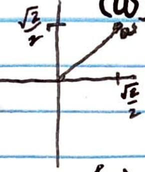
Math 532 Midterm 2

1. Let $w = 1+i$ and $z = -\sqrt{3} - i$



(w) $|w| = \sqrt{1^2+1^2} = \sqrt{2}$

$1+i = \sqrt{2}\, e^{i\theta} = \sqrt{2}\cos\theta + i\sqrt{2}\sin\theta$

$1 = \sqrt{2}\cos\theta$ $\quad 1 = \sqrt{2}\sin\theta$

$\cos\theta = \sqrt{2}/2$ $\quad \sin\theta = \sqrt{2}/2 \rightarrow \theta = \pi/4$

(z) $|z| = \sqrt{3+1} = 2$

$-\sqrt{3} - i = 2e^{i\theta} = 2\cos\theta + i\,2\sin\theta$

$\cos\theta = \sqrt{3}/2$ $\quad \sin\theta = -1/2 \rightarrow \theta = \mp\pi/6$

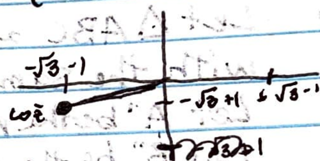
($w\bar{z}$) $\bar{z} = -\sqrt{3} + i$ $\quad \arg(\bar{z}) = 5\pi/6$

$w\bar{z} = (1+i)(-\sqrt{3}+i) = (-\sqrt{3}-1) + i(-\sqrt{3}+1)$

$|w\bar{z}| = \sqrt{(-\sqrt{3}-1)^2 + (-\sqrt{3}+1)^2}$

$= \sqrt{3+1+2\sqrt{3}+3+1-2\sqrt{3}}$

$= \sqrt{8} = 2\sqrt{2}$



$\arg(w\bar{z}) = \arg(w) + \arg(\bar{z})$

$= \pi/4 + 5\pi/6 \rightarrow \theta = 13\pi/12$

(w/z) $1/z = \frac{1}{-\sqrt{3}-i} \cdot \frac{-\sqrt{3}+i}{-\sqrt{3}+i} = \frac{-\sqrt{3}+i}{-2} = \sqrt{3}/2 - 1/2\, i$ $\quad |\bar{z}| = 1$

$w/z = (1+i)(\sqrt{3}/2 - 1/2\, i) = (\sqrt{3}/2 + 1/2) + i(\sqrt{3}/2 - 1/2)$

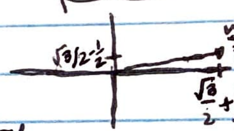
$|w/z| = \sqrt{(\sqrt{3}/2+1/2)^2 + (\sqrt{3}/2 - 1/2^2)}$

$= \sqrt{3/4 + \sqrt{3}/2 + 1/4 + 3/4 - \sqrt{3}/2 + 1/4} = \sqrt{1+1} = \sqrt{2}$

$1/z = \sqrt{3}/2 - 1/2\, i = e^{i\theta} = \cos\varphi + i\sin\varphi$

$\cos\varphi = \sqrt{3}/2$ $\quad \sin\varphi = -1/2 \rightarrow \varphi = \arg(1/z) = 11\pi/6$

$\arg(w/z) = \pi/4 + 11\pi/6 = 25\pi/12 = \pi/12 \rightarrow \theta = \pi/12$



2. Let A, B, C, D be the verticies of an arbritrary quadrangle.

Let P be the mid point of AB $= \frac{1}{2}(A+B)$

~~Let R be the mid point of BC = 1/2(B+C)~~

Let Q be the mid point of BC $= 1/2(B+C)$

Let R be the mid point of CD $= 1/2(C+D)$

Let S be the mid point of AD $= 1/2(A+D)$

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Then,

$\overrightarrow{PQ} = \frac{1}{2}(B+C) - \frac{1}{2}(A+B) = \frac{1}{2}(C-A)$

$\overrightarrow{QR} = \frac{1}{2}(C+D) - \frac{1}{2}(B+C) = \frac{1}{2}(D-B)$

$\overrightarrow{RS} = \frac{1}{2}(A+D) - \frac{1}{2}(C+D) = \frac{1}{2}(A-C)$

$\overrightarrow{SP} = \frac{1}{2}(A+B) - \frac{1}{2}(A+D) = \frac{1}{2}(B-D)$

It is clear that $\overline{PQ} = \overline{RS}$ and $\overline{QR} = \overline{SP}$.
Since the opposing sides of Quadrangle PQRS are congruent, Quadrangle PQRS is by definition a parallelogram. ∎

3. Let ΔABC and $\Delta A'B'C'$ be equilateral triangles with the same orientation. Then
Let A'' be the midpoint of AA'; $A'' = \frac{1}{2}(A+A')$
Let B'' be the midpoint of BB'; $B'' = \frac{1}{2}(B+B')$
Let C'' be the midpoint of CC'; $C'' = \frac{1}{2}(C+C')$, then

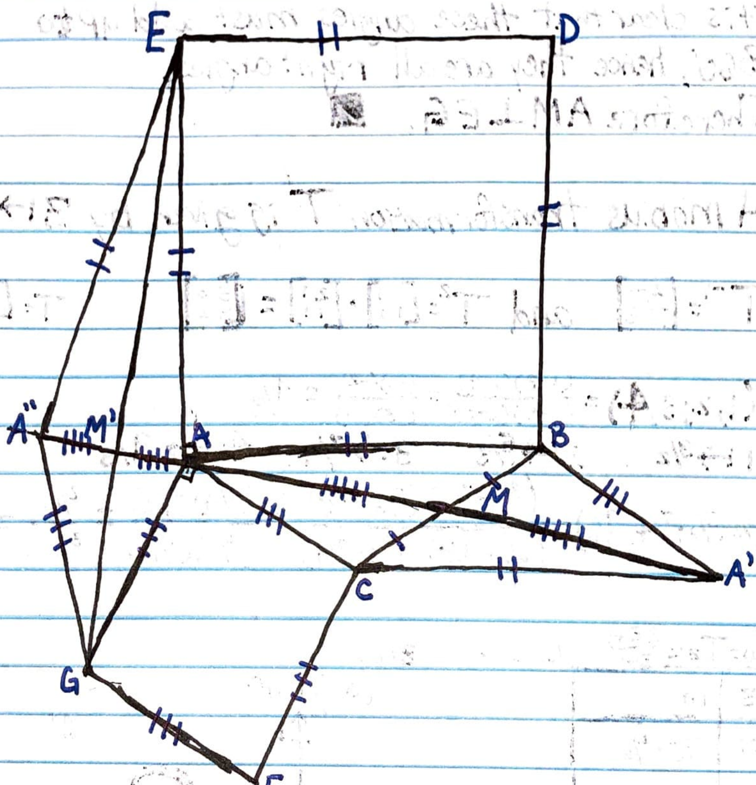
$A''B'' = 1/2(B+B') - 1/2(A+A')$
$= 1/2(B-A) + 1/2(B'-A')$
$= \frac{1}{2}(AB + A'B')$

$B''C'' = 1/2(C+C') - 1/2(B+B')$
$= 1/2(C-B) + 1/2(C'-B')$
$= \frac{1}{2}(BC + B'C')$

$A''C'' = 1/2(C+C') + 1/2(A+A')$
$= 1/2(C-A) + 1/2(C'-A')$
$= 1/2(AC + A'C')$

Since ΔABC and $\Delta A'B'C'$ are equilateral, $AB = BC = AC$ and $A'B' = B'C' = A'C'$
thus $A''B'' = B''C'' = A''C''$, and therefor $\Delta A''B''C''$ is equilateral by definition. ∎

4. Given a triangle $\triangle ABC$, draw external squares $ABDE$ and $ACFG$ on the sides AB and AC respectively. Let M be the mid point of the third side BC.



Proof: Doube the median AM to a Point A'. This creates $\triangle ABA'$ and $\triangle ACA'$. It is obvious that $\triangle ABA' \cong \triangle ACA'$. Then we see that $\measuredangle EAG = 180° - \measuredangle BAM - \measuredangle MAC$. By CPCTC, $\measuredangle MAC \cong \measuredangle BA'M$. Thus $\measuredangle EAG \cong \measuredangle ABA'$. Then by the Side Angle Side Postulate, $\triangle EAG \cong \triangle ABA' \cong \triangle ACA'$. Then $\overline{EG} = \overline{AA'} = 2\overline{AM}$. □

Extend AM to a point M' on the line $\overline{EG}$, then AM' to a point A''. This creates $\triangle EAG$ and $\triangle EA''G$, which are obviously congruent. (Cont...)

$\measuredangle EM'A$ and $\measuredangle GM'A''$ are vertical Angles, hence they are congruent. Similarly $\measuredangle EM'A'' \cong \measuredangle GM'A$. By CPCTC, $\measuredangle EM'A \cong \measuredangle EM'A''$ and similarly $\measuredangle GM'A \cong \measuredangle GM'A''$. Thus $\measuredangle EM'A \cong \measuredangle EM'A'' \cong \measuredangle GM'A \cong \measuredangle GM'A''$. Additionally, it is clear that these angles must add up to $360°$, hence they are all right angles. Therefore $AM \perp EG$. ∎

5. A möbius transformation T is given by $z \mapsto \frac{2z+1}{z+1}$.

(a) $T^{-1} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$ and $T^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$ $\qquad T = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

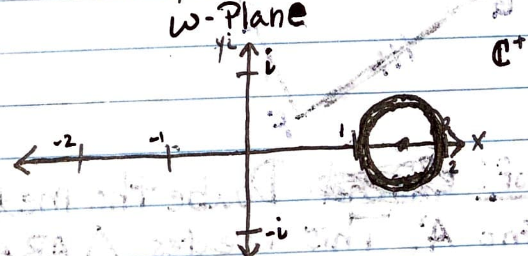
(b) $(1,2;3,4) = \frac{(1-3)(2-4)}{(1-4)(2-3)} = \frac{(-2)(-2)}{(-3)(-1)} = 4/3$

$1 \mapsto 3/2 \quad 2 \mapsto 5/3 \quad 3 \mapsto 7/4 \quad 4 \mapsto 9/5$

$(3/2, 5/3; 7/4, 9/5) = \frac{(3/2 - 7/4)(5/3 - 9/5)}{(3/2 - 9/5)(5/3 - 7/4)} = \frac{(-1/4)(-2/15)}{(-3/10)(-1/12)} = \frac{2/60}{3/120} = 4/3$

(c) $w = Tz = \frac{2z+1}{z+1}$. Line $x=0$: $z = iy$

z	w
$-2i$	$9/5 - 2/5\,i$
$-i$	$3/2 - 1/2\,i$
0	1
i	$3/2 + 1/2\,i$
$2i$	$9/5 + 2/5\,i$



$(x - 3/2)^2 + (y)^2 = 1/4$