Math 520: Final exam

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1. (10 marks) For the following problems,

   (a) Draw a direction field for the given differential equation. on other Page

   (b) Based on an inspection of the direction field, describe how solutions behave for large t. on other page.

   (c) Find the general solution of the given differential equation and use it to determine how solutions behave as $t \to \infty$.

   (i)
$$\frac{dy}{dt} + y = te^{-t^2}.$$

   (ii)
$$\frac{dy}{dt} - ty = t^2 \sin t.$$

PART C

$\boxed{i}$ $dy/dt + y = te^{-t^2} \to y' + y = te^{-t^2}$

$P(t) = 1$; $g(t) = te^{-t^2}$; $\mu(t) = e^t$

Thus $y = \dfrac{\int_0^t (s \cdot e^{-s^2+s})\,ds + C}{e^t}$

This integral can be solved by using the Gauss Error Function, but I don't know how to show that, so I am leaving it in integral form.

$\boxed{i}$ $\lim_{t\to\infty} \left(\dfrac{\int_0^t (s \cdot e^{-s^2+s})\,ds + C}{e^t} \right)$

$= \lim_{t\to\infty} \left(\dfrac{t \cdot e^{-t^2+t}}{e^t} \right)$

$= \dfrac{\lim_{t\to\infty} (t \cdot e^{-t^2+t})}{\lim_{t\to\infty} (e^t)} = \dfrac{0}{\infty} = 0$

$\boxed{ii}$ $dy/dt - ty = t^2 \sin t \to y' - ty = t^2 \sin t$

$P(t) = -t$; $g(t) = t^2 \sin t$; $\mu(t) = e^{-1/2 t^2} = e^{-t^2/2}$

$y(t) = \dfrac{\int_0^t (e^{-s^2/2} \cdot s^2 \sin(s))\,ds + C}{e^{-t^2/2}}$

$\int (t^2 e^{-t^2/2} \sin t)\,dt$, by integration by parts

$= -te^{-t^2/2} \sin(t) - \int (e^{-t^2/2}(-\sin t - t\cos t))\,dt$, by integration by parts

$= -te^{-t^2/2} \sin(t) - e^{-t^2/2} \cos(t)$

$= -e^{-t^2/2}(t \sin(t) + \cos(t)) \to y(t) = -t\sin(t) - \cos(t) + C$

$\boxed{ii}$ $\lim_{t\to\infty} (-t\sin(t) - \cos(t) + C)$

$\to$ Sine is a periodic function, and does not converge. Therefore the limit does not exist.

2. (10 marks) Consider a differential equation by

$$\frac{dy}{dt} = (1-y)(1-2y)y.$$

(a) Draw a direction field for the given differential equation. on other Page

(b) Based on an inspection of the direction field, describe how solutions behave for large t. on other Page

(c) Find the general solution of the given differential equation and use it to determine how solutions behave as $t \to \infty$.

PART C

$$dy/dt = (1-y)(1-2y)y \rightarrow \frac{1}{(1-y)(1-2y)y}\,dy = 1 \cdot dt \rightarrow \int \frac{1}{(1-y)(1-2y)y}\,dy = \int 1\,dt \rightarrow$$

Partial Fraction Decomposition $\rightarrow \int \left(\frac{-4}{2y-1} + \frac{1}{y} + \frac{1}{y-1} \right) dy = t + C$

$$\rightarrow -2\ln(|2y-1|) + \ln(|y|) + \ln(|y-1|) = t + C \rightarrow \cancel{(2y-1)^{-2} + (y-1) + y = e^{t+C}}$$

$$(2y-1)^{-2} + (y-1) + y = e^{t+c} \rightarrow \frac{2y-1}{(2y-1)^2} = C_1 e^{t} \rightarrow \text{Square both sides}$$

$$\rightarrow \frac{(2y-1)^2}{(2y-1)} = C_3 e^{2t} \rightarrow 2y-1 = C_3 e^{2t} \rightarrow y = \frac{Ce^{2t}+1}{2}$$

$$\lim_{t\to\infty} \left(\frac{Ce^{2t}+1}{2} \right) = C \cdot \lim_{t\to\infty} \left(\frac{e^{2t}}{2} \right) + \lim_{t\to\infty} (1/2) = C \cdot \lim_{t\to\infty} \left(\frac{e^{2t}}{2} \right) + 1/2$$

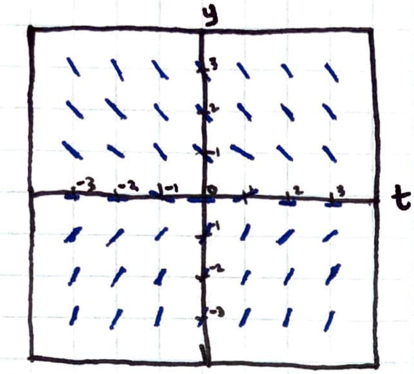
$$= \frac{C}{2} \left(\lim_{t\to\infty} (e^{2t}) \right) + 1/2 = \frac{C}{2}(\infty) + 1/2 = \infty$$

$$\therefore \lim_{t\to\infty} \left(\frac{Ce^{2t}+1}{2} \right) = \infty$$

MATH 520 FINAL EXAM - MATHEW HOUSER

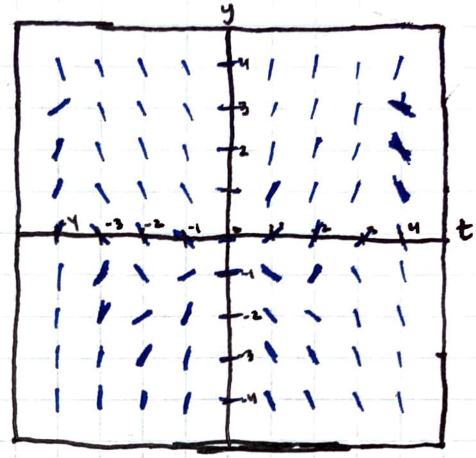
1. PART A

i.



$dy/dt + y = te^{-t^2}$

ii.



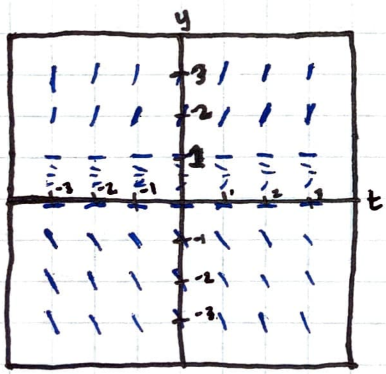
$dy/dt - ty = t^2 \sin(t)$

1. PART B

i. As $t \to \infty$; $y \to \emptyset$

ii. As $t \to \infty$, Any value 'y' can approach ∞ or $-\infty$, The limit does not exist

2. PART A



2. PART B

As $t \to \infty$;

$$\begin{cases} y > 1, & y \to \infty \\ 0 < y < 1, & y \to 1/2 \\ y < 0, & y \to -\infty \end{cases}$$

3. (20 marks) Find the general solution for the following differential equations.

(i)

$$(3xy - y^2)dx + x(x - y)dy = 0.$$

(ii)

$$(x + y)\sin y dx + (x \sin y + \cos y)dy = 0.$$

$\boxed{i}$ $3xy - y^2 + x(x-y)\,dy/dx = 0 \rightarrow x(3xy - y^2) + x^2(x-y)y' = 0$

Exact Equation

Then $\psi(x,y) = x^2(xy - \frac{y^2}{2}) + c_1$. Let $\psi(x,y) = c_2$.

$x^2(xy - y^2/2) + c_1 = c_2 \rightarrow x^2(xy - y^2/2) = c.$

$$x^3y - \frac{1}{2}x^2y^2 = c$$

$$\frac{1}{2}x^2y^2 - x^3y - c = 0$$

$$y = \frac{x^3 \pm \sqrt{x^6 + 2cx^2}}{x^2} =$$

Thus $y = \dfrac{x^2 \pm \sqrt{x^4 - c}}{x}$.

$\boxed{ii}$ $(x+y)\sin(y) + (x\sin(y) + \cos(y))y' = 0;$

$x + y + \dfrac{x\sin y + \cos y}{\sin y} = 0$. Then $\psi(x,y) = \ln(\sin(y)) + xy + \frac{x^2}{2} + c_1$

~~Let $\psi(x,y) = c_2$. Then~~

~~$c = \ln(\sin(y)) + xy + \frac{x^2}{2}$~~

4. (20 marks) Find general solutions of the following differential equations.

(i)

$$y'' + 4y' + 3y = t\sin t.$$

(ii)

$$y''' + 3y'' + 3y' + y = t.$$

(i) **Complementary Solution:** Assume $y = e^{rt}$, then $y'' + 4y' + 3y = r^2 + 4r + 3$. Thus $r_1 = -3$ and $r_2 = -1$. Thus the complementary solution is $y_c(t) = C_1 e^{-3t} + C_2 e^{-t}$.

Particular Solution: ~~Let $y'' + 4y' + 3y = t\sin t$, suppose $y(t) = At\sin(t) + B\cos(t)$. Then $y' = (At+B)\cos(t) + (A-Bt)\sin(t)$, $y'' = (-At-2B)\sin(t) + (2A-Bt)\cos(t)$. Then $y'' + 4y' + 3y =$~~

$$y_p(t) = \frac{t\sin(t)}{10} - \frac{t\cos(t)}{5} + \frac{\sin(t)}{25} + \frac{11\cos(t)}{50}$$

General Solution:

$$y(t) = C_1 e^{-3t} + C_2 e^{-t} + \frac{t\sin(t)}{10} - \frac{t\cos(t)}{5} + \frac{\sin(t)}{25} + \frac{11\cos(t)}{50}.$$

Complementary Solution: $y_c(t) = C_1 e^{-t} + C_2 t e^{-t} + C_3 t^2 e^{-t}$

Particular Solution: $y_p(t) = t - 3$

General Solution: $y = C_1 e^{-t} + C_2 t e^{-t} + C_3 t^2 e^{-t} + t - 3.$

5. (10 marks) Find general solutions by a power of series at $x_0 = 0$.

$$y'' - xy' - y = 0.$$

Let $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$

Then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$ $\qquad = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x)^n$

Substituting,

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n$$

a_2

Recursive Pattern,

Compute Coefficients

a_0 $\quad a_1$ $\quad a_3$ $\quad a_4$

$y = a_0$

6. (30 marks) Find the general solution of the given system of equations and describe the behavior of the solution as $t \to \infty$.

(i)

$$\dot{x} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} x.$$

(ii)

$$\dot{x} = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} x.$$

(iii)

$$\dot{x} = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} x.$$

$\boxed{i}$ $\begin{bmatrix} 3-r & -2 \\ 2 & -2-r \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \begin{vmatrix} 3-r & -2 \\ 2 & -2-r \end{vmatrix} = r^2 - r - 2 \to (r-2)(r+1) \to r_1 = 2,\ r_2 = -1$

For $r_1 = 2$, $\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to \begin{matrix} \xi_1 - 2\xi_2 = 0 \\ 2\xi_2 - 4\xi_2 = 0 \end{matrix} \to \xi_1 = 2\xi_2 \to \xi^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

For $r_2 = -1$, $\begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to \begin{matrix} 4\xi_1 - 2\xi_2 = 0 \\ 2\xi_1 - \xi_2 = 0 \end{matrix} \to 2\xi_1 = \xi_2 \to \xi^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Then $x^{(1)}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$ and $x^{(2)}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$ $\cancel{\to} 0$

so $x = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$

As $\cancel{t} \to \infty$, $x \to e^{3t}$

$\boxed{ii}$ $\begin{bmatrix} 3-r & -1 \\ -1 & 3-r \end{bmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \begin{vmatrix} 3-r & -1 \\ -1 & 3-r \end{vmatrix} = r^2 - 6r + 8 = (r-4)(r-2) \to r_1 = 4,\ r_2 = 2$

For $r_1 = 4$, $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to \begin{matrix} -\xi_1 - \xi_2 = 0 \\ -\xi_1 - \xi_2 = 0 \end{matrix} \to -\xi_1 = \xi_2 \to \xi^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

For $r_2 = 2$, $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to \begin{matrix} \xi_1 - \xi_2 = 0 \\ -\xi_1 + \xi_2 = 0 \end{matrix} \to \xi_1 = \xi_2 \to \xi^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

then $x^{(1)}(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t}$ and $x^{(2)}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$

so $x = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$

As $t \to \infty$, $x \to e^{2t}$

$\boxed{iii}$ $\begin{bmatrix} 1-r & -1 \\ 5 & -3-r \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \begin{vmatrix} 1-r & -1 \\ 5 & -3-r \end{vmatrix} = r^2 + 2r + 2 \to r_1 = -1 + i$ and $r_2 = -1 - i$

Then $\xi^{(1)} = \begin{pmatrix} 2+i \\ 5 \end{pmatrix}$ and $\xi^{(2)} = \begin{pmatrix} 2-i \\ 5 \end{pmatrix}$

$x = c_1 e^{-t}\left(\cos(t)\begin{pmatrix} 2 \\ 5 \end{pmatrix} - \sin(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + c_2 e^{-t}\left(\cos(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin(t)\begin{pmatrix} 2 \\ 5 \end{pmatrix}\right)$

As $t \to \infty$, $x \to 0$

real part is negative → Spiral Point