MATH 554 Homework #5 — Mathew Houser

Chapter 2: 2, 5, 7, 9, 10, 11, 13

2. Define $f:(-2,0)\to\mathbb{R}$ by $f(x)=\frac{2x^2+3x-2}{x+2}$.
Prove that f has a limit at -2 and find it.
Choose any arbitrary $\varepsilon>0$, and set $\delta=\varepsilon/2$.
If $0<|x+2|<\delta$ and $x\in(-2,0)$, then it follows that
$|f(x)+5|=|2x-1+5|=|2x+4|=2|x+2|<2\delta=\varepsilon$
Thus f has a limit $L=-5$ at $x_0=-2$. ■

5. Suppose $f:D\to\mathbb{R}$ with x_0 an accumulation point of D. Assume L_1 and L_2 are limits of f at x_0. Prove that $L_1=L_2$.
Choose any arbitrary $\varepsilon>0$. Then there is a $\delta_1,\delta_2>0$ such that $\forall x\in D$ $0<|x-x_0|<\delta_1$ implies that $|f(x)-L_1|<\varepsilon/2$, and $0<|x-x_0|<\delta_2$ implies that $|f(x)-L_2|<\varepsilon/2$. Let $\delta=\min(\delta_1,\delta_2)>0$.
Then there is $x\in D$ such that $0<|x-x_0|<\delta$. It follows that $|L_1-L_2|\leq|L_1-f(x)|+|f(x)-L_2|<\varepsilon/2+\varepsilon/2<\varepsilon$
Since ε is arbitrary and $\varepsilon>0$, $L_1=L_2$. ■

7. Define $f:(0,1)\to\mathbb{R}$ by $f(x)=x\cos(1/x)$.
Does f have a limit at 0? Justify.
yes, f has a limit $L=0$ at $x_0=0$.
Let $f(x)=x\cos(1/x)$, $g(x)=-|x|$, and $h(x)=|x|$.
Since $-1\leq\cos(1/x)\leq1$ $\forall x\neq0$, it follows that $g(x)\leq f(x)\leq h(x)$ $\forall x\neq0$. Now, observe that $\lim_{x\to0}g(x)=\lim_{x\to0}h(x)=0$. The Squeeze Theorem can now be applied to conclude that
$\lim_{x\to0}f(x)=\lim_{x\to0}x\cos(\frac{1}{x})=0$. ■

9. Define $f:(-1,1)\to\mathbb{R}$ by $f(x)=\frac{x+1}{x^2-1}$.
Does f have a limit at $x_0=1$? Justify.
No, f does not have a limit at $x_0=1$. First, observe that $f(x)=\frac{x+1}{x^2-1}=\frac{x+1}{(x+1)(x-1)}=\frac{1}{x-1}$. Now, Let L be any real number, and Choose $\varepsilon>0$ Such that $L+\varepsilon>0$. Suppose $0<x<\frac{1}{L+\varepsilon}$, then $L+\varepsilon<\frac{1}{x}<\frac{1}{x-1}=f(x)$, hence $|f(x)-L|>\varepsilon$. Thus it is impossible to find a $\delta>0$ that fulfills the requirements of the definition, i.e, L is not a limit of f at 1. Since L is any real number, we conclude that f does not have a limit at $x_0=1$. $\blacksquare$

10. Consider $f:(0,2)\to\mathbb{R}$ defined by $f(x)=x^x$.
Assume that f has a limit at 0 and find that limit.
$\lim\limits_{x\to 0} x^x = e^{\ln x^x} = e^{x\ln x} = e^{\frac{\ln x}{x^{-1}}}$ ($=\frac{-\infty}{\infty}$ $\Rightarrow$ Apply L'Hopital's Rule)
$= e^{\frac{x^{-1}}{-x^{-2}}} = e^{-x} = e^0 = 1$. Thus $\lim\limits_{x\to 0} x^x = 1$. $\blacksquare$

11. Suppose f, g, and $h: D\to\mathbb{R}$ where x_0 is an accumulation point of D, $f(x)\le g(x)\le h(x)$ $\forall x\in D$, and that f and h have limits at x_0 with $\lim\limits_{x\to x_0} f(x)=\lim\limits_{x\to x_0} h(x)$.
Prove that g has a limit at x_0 and $\lim\limits_{x\to x_0} f(x)=\lim\limits_{x\to x_0} g(x)=\lim\limits_{x\to x_0} h(x)$.
Let $L=\lim\limits_{x\to x_0} f(x)=\lim\limits_{x\to x_0} h(x)$. Choose any $\varepsilon>0$. then, $\exists\delta_1>0$ such that $0<|x-x_0|<\delta_1 \Rightarrow |f(x)-L|<\varepsilon$, hence $0<|x-x_0|<\delta_1 \Rightarrow -\varepsilon<f(x)<\varepsilon$. Also observe that $\exists\delta_2>0$ such that $0<|x-x_0|<\delta_2 \Rightarrow |h(x)-L|<\varepsilon$, hence $0<|x-x_0|<\delta_2 \Rightarrow -\varepsilon<h(x)<\varepsilon$. Given that $f(x)\le g(x)\le h(x)$ it is clear that $f(x)-L\le g(x)-L\le h(x)-L$. Let $\delta=\min(\delta_1,\delta_2)$. Then, for all $|x-x_0|<\delta$, $-\varepsilon<f(x)-L\le g(x)-L\le h(x)-L<\varepsilon$, thus
$-\varepsilon<g(x)-L<\varepsilon$, Therefore $\lim\limits_{x\to x_0} g(x)=L$ and $\lim\limits_{x\to x_0} f(x)=\lim\limits_{x\to x_0} g(x)=\lim\limits_{x\to x_0} h(x)=L$. $\blacksquare$

13. Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x - \lfloor x \rfloor$. Determine those points at which f has a limit, and justify.

$f(x) = x - \lfloor x \rfloor$ has a limit at x_0 iff x_0 is not an integer.

Assume x_0 is an integer and consider the sequence

$$\left\{ x_0 + (-1)^n \frac{1}{n} \right\}_{n=1}^{\infty} \Rightarrow \begin{array}{ll} x_0 - 1 < x_n < x_0 & \text{if } n \text{ is odd} \\ x_0 < x_n < x_0 + 1 & \text{if } n \text{ is even} \end{array}$$

Let $g: \mathbb{R} \to \mathbb{R}$, $g(x) = \lfloor x \rfloor$. Then for

$$\{g(x_n)\}_{n=1}^{\infty} = \begin{cases} x_0 - 1 & \text{if } n \text{ is odd} \\ x_0 & \text{if } n \text{ is even} \end{cases}$$

$f(x) = x - \lfloor x \rfloor = x_n - g(x_n)$. Then

$$\{f(x_n)\}_{n=1}^{\infty} = \begin{cases} x_n - x_0 + 1 & \text{if } n \text{ is odd} \\ x_n - x & \text{if } n \text{ is even} \end{cases}$$

Hence $f(x)$ diverges, so f doesn't have a limit at x_0 if x_0 is an integer.

~~x_0 is an integer Now Assume x_0 is~~

Now assume x_0 is not an integer and let δ be the distance from x_0 to the nearest integer. Now $\delta > 0$, and if $0 < |x - x_0| < \delta$, then $\lfloor x \rfloor = \lfloor x_0 \rfloor$; hence $|g(x) - \lfloor x_0 \rfloor| = 0 < \varepsilon \ \forall \varepsilon > 0$. Thus $g(x)$ has a limit $\lfloor x_0 \rfloor$ at x_0. Since $f(x) = x - g(x)$, ($\lim_{x \to x_0} x = x_0$ is obvious), $f(x)$ has a limit $x_0 - \lfloor x_0 \rfloor$ at x_0 if x_0 is not an integer. ∎