MATH 554 — Homework #7 — Mathew Houser

§3: 2, 5, 7, 13, 15, 17

2. Define $f:[-4,0]\to\mathbb{R}$ by $f(x)=\frac{2x^2-18}{x+3}$ for $x\neq -3$ and $f(-3)=-12$. Show that f is continuous at -3.

Then $\forall x\in[-4,0]$, $x\neq -3$,

$$\lim_{x\to -3} f(x)=\lim_{x\to -3}\frac{2x^2-18}{x+3}=\lim_{x\to -3}\frac{2(x+3)(x-3)}{x+3}=\lim_{x\to -3} 2x-6$$

$$=2(-3)-6=-6-6=-12=f(-3).$$

Thus by Theorem 3.1, f is continuous at -3. ■

5. Define $f:(0,1)\to\mathbb{R}$ by $f(x)=\frac{1}{\sqrt{x}}-\sqrt{\frac{x+1}{x}}$.

Can one define $f(0)$ to make f continuous at 0? Explain.

Assume $x\neq 0$. Then $\forall x\in(0,1)$,

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}\left(\frac{1}{\sqrt{x}}\right)-\left(\lim_{x\to 0}\left(\frac{1}{\sqrt{x}}\right)\cdot\sqrt{\lim_{x\to 0}(x+1)}\right).$$

$\lim_{x\to 0}(x+1)=1$. $\sqrt{1}=1$. $\lim_{x\to 0}\left(\frac{1}{\sqrt{x}}\right)\cdot 1=\lim_{x\to 0}\left(\frac{1}{\sqrt{x}}\right)$.

Then $\lim_{x\to 0} f(x)=\lim_{x\to 0}\left(\frac{1}{\sqrt{x}}\right)-\lim_{x\to 0}\left(\frac{1}{\sqrt{x}}\right)=0$.

Now suppose $f(0)=0$. Then $\lim_{x\to 0} f(x)=0=f(0)$

Thus by Theorem 3.1, f is continuous at 0. ■

7. Suppose $f:\mathbb{R}\to\mathbb{R}$ is continuous and $f(r)=r^2$ for each rational number r. Determine $f(\sqrt{2})$ and Justify.

Consider the sequence of rational numbers $\{r_n\}_{n=1}^{\infty}$ where $r_n\in(\sqrt{2}-\frac{1}{n},\sqrt{2}+\frac{1}{n})$. Clearly $|r_n-\sqrt{2}|<\frac{1}{n}$, hence $\{r_n\}_{n=1}^{\infty}$ converges to $\sqrt{2}$ and $\lim_{n\to\infty} r_n=\sqrt{2}$.

Since f is continuous & $f(r_n)=r_n^2$, it follows that

$$f(\sqrt{2})=\lim_{n\to\infty} f(r_n)=\lim_{n\to\infty} r_n^2=\left(\lim_{n\to\infty} r_n\right)^2=(\sqrt{2})^2=2.$$

Thus $f(\sqrt{2})=2$. ■

MATH 554 - Homework #7 - Mathew Houser

§3: 2, 5, 7, 13, 15, 17

13. Let $f: D \to \mathbb{R}$ be continuous at $x_0 \in D$. Prove that there is $M > 0$ and a neighborhood Q of x_0 such that $|f(x)| \leq M \quad \forall x \in Q \cap D$.

Choose any $\varepsilon > 0$, and let $Q = (x_0 - \delta, x_0 + \delta)$ be a neighborhood of x_0. Then $\exists \delta > 0$ such that $\forall x \in Q \cap D$, $|f(x) - f(x_0)| < \varepsilon$. (Since f is continuous)

Let $M = |f(x_0)| + \varepsilon$. Clearly $M > 0$. Then it follows that

$|f(x) - f(x_0)| < \varepsilon \Rightarrow |f(x)| - |f(x_0)| \leq \varepsilon \Rightarrow |f(x)| \leq |f(x_0)| + \varepsilon = M$.

Therefore $\exists\ M > 0$ and a neighborhood Q of x_0 such that $|f(x)| \leq M \quad \forall x \in Q \cap D$. $\blacksquare$

15. Suppose $f, g: D \to \mathbb{R}$ are both continuous on D. Define $h: D \to \mathbb{R}$ by $h(x) = \max\{f(x), g(x)\}$. Show that h is continuous on D.

Choose $\varepsilon > 0$ arbitrarily. Let $\varepsilon^* =$

Then $\exists \delta_1 > 0$ s.t $\forall x \in D$ and $|x - x_0| < \delta_1$, $|f(x) - f(x_0)| < \varepsilon^*$, and $\exists \delta_2 > 0$ s.t. $\forall x \in D$ and $|x - x_0| < \delta_2$, $|g(x) - g(x_0)| < \varepsilon^*$.

Set $\delta = \min\{\delta_1, \delta_2\}$. Then $\forall x \in D$ and $|x - x_0| < \delta$,

$|h(x) - h(x_0)| = |\max\{f(x), g(x)\} - \max\{f(x_0), g(x_0)\}|$

$= \frac{1}{2}|f(x) + g(x) + |f(x) - g(x)| - f(x_0) - g(x_0) - |f(x_0) - g(x_0)|$

$= \frac{1}{2}|[f(x) - f(x_0)] + [g(x) - g(x_0)] + |f(x) - g(x)| - |f(x_0) - g(x_0)||$

$< \frac{1}{2}|\varepsilon^* + \varepsilon^* + |f(x) - g(x)| - |f(x_0) - g(x_0)|$

$< \frac{1}{2}|2\varepsilon^* + |f(x) - g(x) - f(x_0) + g(x_0)||$

$< \frac{1}{2}|2\varepsilon^* + |f(x) - f(x_0)| - |g(x) - g(x_0)||$

$< \frac{1}{2}|\varepsilon^* + \varepsilon^* + \varepsilon^* - \varepsilon^*| = \frac{1}{2}(2\varepsilon^*) = \varepsilon^* = \varepsilon$.

Thus h is continuous on D. $\blacksquare$

MATH 554 – Homework #7 – Mathew Hauser

§3: 2, 5, 7, 13, 15, 17

17. Suppose $f: D \to \mathbb{R}$ with $f(x) \geq 0 \ \forall x \in D$. Show that if f is continuous at x_0, then $\sqrt{f}$ is continuous at x_0.

Assume f is continuous at x_0.

Then f has a limit at x_0, and $\lim_{x \to x_0} f(x) = f(x_0)$.

Then $\lim_{x \to x_0} \sqrt{f(x)} = \sqrt{\lim_{x \to x_0} f(x)} = \sqrt{f(x_0)}$. Thus by Theorem 3.1, $\sqrt{f}$ is continuous at x_0. ■