MATH 554 Homework #8 – Mathew Houser

Chapter 3 # 22, 27, 35, 37, 39, 41, 43

22. Define $f:(2,7)\to\mathbb{R}$ by $f(x)=x^3-x+1$
Show that f is uniformly continuous on $(2,7)$ without using Theorem 3.8.

Choose any arbritrary $\varepsilon>0$. Then we have

$$\begin{aligned}|f(x)-f(y)| &= |(x^3-x+1)-(y^3-y+1)|\\ &= |(x^3-y^3)-(x-y)|\\ &= |(x-y)(x^2+xy+y^2)-(x-y)|\\ &= |(x-y)(x^2+xy+y^2-1)|\end{aligned}$$

Although 7 is not in the domain of f, we can use that number to bound $|x^2+xy+y^2-1|$, that is
$|x^2+xy+y^2-1| < 7^2+(7)(7)+7^2-1 = 146$

Choose $\delta = \varepsilon/146$. Then $\forall x,y\in(2,7)$, if $|x-y|<\delta$, then
$|f(x)-f(y)| = |(x-y)(x^2+xy+y^2-1)| < 146\delta = \varepsilon$.

Therefore f is uniformly continuous on $(2,7)$. ■

27. Prove that every set of the form $\{x: a<x<b\}$ is open, and that every set of the form $\{x: a\le x\le b\}$ is closed.

Let $E\subseteq\mathbb{R}$ be any set of the form $\{x: a<x<b\}$. Then $\forall n\in J$, the set $(a-\frac{1}{n}, a+\frac{1}{n})$ is a neighborhood of a containing infinitely many points in E. Similarly, the set $(b-\frac{1}{n}, b+\frac{1}{n})$ is a neighborhood of b containing infinitely many points in E. Thus a and b are both accumulation points of E, however $a,b\notin E$, therefore E is open by definition.

Let $E\subseteq\mathbb{R}$ be any set of the form $\{x: a\le x\le b\}$. Then $\forall n\in J$ the sets $(a-\frac{1}{n}, a+\frac{1}{n})$ and $(b-\frac{1}{n}, b+\frac{1}{n})$ are neighborhoods of a and b respectively both containing infinitely many points in $\mathbb{R}\setminus E$, hence a and b are accumulation points of $\mathbb{R}\setminus E$ however $a,b\notin\mathbb{R}\setminus E$, hence the set $\mathbb{R}\setminus E$ is open and by Theorem 3.6, the set E is closed. ■

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35. Let E be compact and nonempty. Prove that E is bounded and that $\sup(E)$ and $\inf(E)$ both belong to E.

Let E be compact and nonempty. Then by Theorem 3.7 E is also closed and bounded, hence $\sup(E)$ and $\inf(E)$ both exist. Let $a = \sup(E)$. Then there is a sequence of points in E $\{x_n\}_{n=1}^{\infty}$ converging to a. Since E is closed, $a \in E$, hence $\sup(E) \in E$. Similarly, Let $b = \inf(E)$. Then there is a sequence of points in E $\{y_n\}_{n=1}^{\infty}$ converging to b. Since E is closed, $b \in E$ hence $\inf(E) \in E$. ■

37. Let $f: [a,b] \to \mathbb{R}$ have a limit at each $x \in [a,b]$. Prove that f is bounded.

Suppose $f: [a,b] \to \mathbb{R}$ has a limit at each $x \in [a,b]$. Then by Theorem 3.1, f is continuous at each $x \in [a,b]$. From exercise 27, we know that $[a,b]$ is a closed set. In the proof of Theorem 3.7, we showed that any closed interval is compact hence $[a,b]$ is closed and bounded. Then by Theorem 3.8, f is uniformly continuous. Choose $\varepsilon^* = 1$. Then $\exists\, \delta > 0$ such that $\forall\, x, y \in [a,b]$, if $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon^* = 1$. Since $[a,b]$ is bounded, $\exists \{x_1, x_2, \cdots, x_n\} \subseteq [a,b]$ such that $E \subseteq \bigcup_{i=1}^{n} (x_i - \delta, x_i + \delta)$, and consequently

$f([a,b]) \subseteq \bigcup_{i=1}^{n} (f(x_i) - 1, f(x_i) + 1)$, therefore f is bounded. ■

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39. Suppose that $f: \mathbb{R} \to \mathbb{R}$ is continuous and has the property that for each $\varepsilon > 0$, there is $M > 0$ such that if $|x| \geq M$, then $|f(x)| < \varepsilon$. Show that f is uniformly continuous.

Choose any $\varepsilon > 0$ arbitrarily. Then $\exists\, M > 0$ such that if $|x| \geq M$, then $|f(x)| < \varepsilon/2$.
f is continuous on the closed interval $[-M, M]$ so by Theorem 3.8, f is uniformly continuous on $[-M, M]$.
Hence $\exists\, \delta > 0$ such that $\forall\, x, y \in [-M, M]$, if $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon/2$. Given any $x, y \in \mathbb{R}$ such that $|x-y| < \delta$, there are 4 cases.

(i) Suppose $x, y \in [-M, M]$. Then
$$|f(x) - f(y)| < \varepsilon/2 < \varepsilon.$$

(ii) Suppose $x \in [-M, M]$ and $y \in \mathbb{R} \setminus [-M, M]$
then
$$|f(x) - f(y)| \leq |f(x) - f(M)| + |f(M) - f(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

(iii) Suppose $x \in \mathbb{R} \setminus [-M, M]$ and $y \in [-M, M]$
then
$$|f(x) - f(y)| \leq |f(x) - f(M)| + |f(M) - f(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

(iv) Suppose $x, y \in \mathbb{R} \setminus [-M, M]$. Then
$$|f(x) - f(y)| \leq |f(x)| + |f(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus $\forall\, x, y \in \mathbb{R}$, $\exists\, \delta > 0$ such that if $|x-y| < \delta$, $|f(x) - f(y)| < \varepsilon$. Since ε is arbitrary, f is uniformly continuous on $\mathbb{R}$. ■

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41. Find an interval of length one that contains a root of the equation $xe^x = 1$
Let $f(x) = xe^x - 1$. Then
$f(0) = -1 < 0 < f(1) = e - 1$.
Therefore by Bolzano's Theorem, there is a root in the interval $(0, 1)$. ∎

43. Suppose $f : [a,b] \to \mathbb{R}$ is continuous and $f(b) \leq y \leq f(a)$. Prove that $\exists\, c \in [a,b]$ such that $f(c) = y$.
Define $g : [a,b] \to \mathbb{R}$ by $g(x) = f(x) - y$.
If $f(b) \leq x \leq y$, then $g(x) < 0$.
If $y \leq x \leq f(a)$, then $g(x) > 0$.
By the modified Bolzano's Theorem,
$\exists\, c \in [a,b]$ such that $g(c) = f(c) - y = 0$
$\Rightarrow f(c) = y$. ∎