Mathew Houser
Page one

MATH 532 — Homework 1 — Due September 3rd, 2021

1a. $(1-i)(2-i)(3-i) = (2-i-2i+i^2)(3-i)$
$= (1-3i)(3-i) = 3-i-9i+3i^2 = 0-10i$.

1b. $(\sqrt{3}+i)^6 = (\sqrt{3}+i)(\sqrt{3}+i)(\sqrt{3}+i)^4 = (3+\sqrt{3}i+\sqrt{3}i+i^2)(\sqrt{3}+i)^4$
$= (2+2\sqrt{3}i)(2+2\sqrt{3}i)(\sqrt{3}+i)^2 = (4+4\sqrt{3}i+4\sqrt{3}i+12i^2)(\sqrt{3}+i)^2$
$= (-8+8\sqrt{3}i)(2+2\sqrt{3}i) = (-16-16\sqrt{3}i+16\sqrt{3}i+48i^2) = -64$.

1c. $\dfrac{4+3i}{3-4i} = \dfrac{4+3i}{3-4i} \times \dfrac{3+4i}{3+4i} = \dfrac{12+16i+9i+12i^2}{9+12i-12i-16i^2} = \dfrac{0+25i}{25} = i$.

1d. Substituting $z=4+3i$; $\dfrac{5-z}{5+z} = \dfrac{5-(4+3i)}{5+(4-3i)} = \dfrac{5-4-3i}{5+4-3i} = \dfrac{1-3i}{9+3i} =$

$\dfrac{1-3i}{9+3i} \times \dfrac{9-3i}{9-3i} = \dfrac{9-3i-27i+9i^2}{81-27i+27i-9i^2} = \dfrac{0-30i}{90-0i} = \dfrac{-30i}{90} = -\dfrac{1}{3}i$.

4. Let $z = \frac{1+it}{1-it}$. If $z=-1$, then $-1 = \frac{1+it}{1-it} \Rightarrow -1+it = 1+it$
$\Rightarrow -2=0$, thus $z \neq -1$. For $z \neq 1$, and $t \in \mathbb{R}$,
$|z| = \frac{|1+it|}{|1-it|} = \frac{1+t^2}{1+(-t)^2} = \frac{1+t^2}{1+t^2} = 1$. Therefore any complex number
$z \neq -1$, with $|z|=1$ and $t \in \mathbb{R}$ can be expressed as $z = \frac{1+it}{1-it}$. ■

5. Let $z = a+bi$ $(a, b \in \mathbb{R})$. Then $z^2 = (a+bi)(a+bi) = a^2+2abi-b^2$.
$z^4 = (a^2+2abi-b^2)(a^2+2abi-b^2) = a^4+2a^3bi-a^2b^2+2a^3bi$
$+4a^2b^2i^2-2ab^3i-a^2b^2-2ab^3i+b^4 \rightarrow$
$z^4 = (a^4-6a^2b^2+b^4)+(4a^3b-4ab^3)i$

a. If z^4 is real, then $4a^3b-4ab^3=0 \rightarrow 4a^3b = 4ab^3 \rightarrow$
$a^2=b^2$. Thus z^4 is real if $a=b$ or $a=-b$.

b. If z^4 is purely imaginary, then $a^4-6a^2b^2+b^4=0$.
Then by the Quadratic Formula, $a^2 = (3 \pm 2\sqrt{2})b^2$.
Suppose $a^2 = (3+2\sqrt{2})b^2$. Then $a = \pm\sqrt{3+2\sqrt{2}} \cdot \sqrt{b^2}$
$= \pm b \cdot \sqrt{(\sqrt{2})^2+2\sqrt{2}+1} = \pm b \cdot \sqrt{(\sqrt{2}+1)^2} = \pm b \cdot (\sqrt{2}+1)$
Suppose $a^2 = (3-2\sqrt{2})b^2$. Then similarly,
$a = \pm b \cdot (\sqrt{2}-1)$. Thus z^4 is purely imaginary
when $a = \sqrt{2}b+b$, $\sqrt{2}b-b$, $b-\sqrt{2}b$, or $-\sqrt{2}b-b$. ■

6a. $|3+2i| = \sqrt{9+4} = \sqrt{13}$

6b. $|-1+i\sqrt{3}| = \sqrt{1+3} = 2$

6c. $|-i(1+i)(2-3i)(4+3i)| = |11-23i| = \sqrt{650} = 5\sqrt{26}$

Mathew Houser

MATH 532 — Homework 1 — Due September 3^{rd}, 2021

6d. $\left|\frac{(3-i)(-1+2i)}{2-3i}\right| = \left|\frac{-1+7i}{2-3i}\right| = \left|\frac{-23}{13}+\frac{11i}{13}\right| = \sqrt{\left(\frac{23}{13}\right)^2+\left(\frac{11}{13}\right)^2} = \sqrt{\frac{50}{13}}$.

7a. Let $z=a+ib$, $w=c+id$ $(a,b,c,d \in \mathbb{R})$

$|zw| = \sqrt{(ac-bd)^2+(ad+bc)^2} = \sqrt{a^2c^2+b^2d^2+a^2d^2+b^2c^2}$

$|z||w| = (\sqrt{a^2+b^2})(\sqrt{c^2+d^2}) = \sqrt{(a^2+b^2)(c^2+d^2)} = \sqrt{a^2c^2+b^2d^2+a^2d^2+b^2c^2}$

$|z\bar{w}| = \sqrt{(ac+bd)^2+(bc-ad)^2} = \sqrt{a^2c^2+b^2d^2+a^2d^2+b^2c^2}$

$\therefore\ |zw| = |z||w| = |z\bar{w}| = \sqrt{a^2c^2+b^2d^2+a^2d^2+b^2c^2}$

7b. Through trial and error: $(p,q,r) = (8,9,12)$, $(7,11,13)$ and $(11,13,17)$ ---etc.

7c. Let $S = \{p \in \mathbb{N};\ p = m^2+n^2 \text{ for some } m,n \in \mathbb{N}\}$, and let $p = m^2+n^2$ and $q = x^2+y^2$, where $p,q \in S$, and $m,n,x,y \in \mathbb{N}$.
Then $pq = (m^2+n^2)(x^2+y^2) = m^2x^2+m^2y^2+n^2x^2+n^2y^2$.
From 7a we know $m^2x^2+m^2y^2+n^2x^2+n^2y^2 = (mx-ny)^2+(nx+my)^2$.
Since $\mathbb{N}$ is closed under multiplication and addition, $(mx-ny) \in \mathbb{N}$ and $(nx+my) \in \mathbb{N}$. Thus, $pq \in S$;
Therefore the set "S" is closed under multiplication. ∎

8a. Let $\alpha, \beta \in \mathbb{C}$. Then $|\alpha+\beta|^2 = (\alpha+\beta)(\overline{\alpha+\beta}) = (\alpha+\beta)(\bar{\alpha}+\bar{\beta})$
$= \alpha\bar{\alpha}+\alpha\bar{\beta}+\beta\bar{\alpha}+\beta\bar{\beta} = |\alpha|^2+|\beta|^2+\alpha\bar{\beta}+\bar{\alpha}\beta$. Also,
$|\alpha-\beta|^2 = (\alpha-\beta)(\overline{\alpha-\beta}) = (\alpha-\beta)(\bar{\alpha}-\bar{\beta})$
$= \alpha\bar{\alpha}+\beta\bar{\beta}-\alpha\bar{\beta}-\bar{\alpha}\beta = |\alpha|^2+|\beta|^2-\alpha\bar{\beta}-\bar{\alpha}\beta$. Then

$|\alpha+\beta|^2+|\alpha-\beta|^2 = |\alpha|^2+|\beta|^2+|\alpha|^2+|\beta|^2+\bar{\alpha}\beta-\bar{\alpha}\beta+\alpha\bar{\beta}-\alpha\bar{\beta}$.
$= 2|\alpha|^2+2|\beta|^2 = 2(|\alpha|^2+|\beta|^2)$. ∎

8b. Suppose now that $|\alpha| = |\beta|$. Then for any $\gamma \in \mathbb{C}$,
$|\alpha+\gamma|^2+|\alpha-\gamma|^2 = 2(|\alpha|^2+|\gamma|^2)$, Substituting $|\alpha| = |\beta|$,
$= 2(|\beta|^2+|\gamma|^2) = |\beta+\gamma|^2+|\beta-\gamma|^2$. ∎

8c. The sum of the squares of the lengths of the four sides of a parallelogram = The sum of the squares of the lengths of the 2 diagonals. Part (b) shows that this law holds regardless of orientation.

21. Let $A=3+i$, $B=1-2i$, $C=-2+4i$. There are 3 solutions for finding vertex "D":

$D_1 = A+B-C = (3+i)+(1-2i)-(-2+4i) = 6-5i$

$D_2 = B+C-A = (1-2i)+(-2+4i)-(3+i) = -4+i$

$D_3 = A+C-B = (3+i)+(-2+4i)-(1-2i) = 7i$

Mathew Houser

MATH 532 — Homework 1 — Due September 3^{rd}, 2021

25a. Let $\alpha+\beta+\gamma=0$, $\alpha z_1+\beta z_2+\gamma z_3=0$, where α,β,γ are not all zero. without loss of Generality, Let $\alpha\neq 0$, then since $\alpha+\beta+\gamma=0$, at least one of $\beta,\gamma\neq 0$. Then without loss of Generality, let $\beta\neq 0$. Then, $\gamma=-(\alpha+\beta) \rightarrow \alpha z_1+\beta z_2=(\alpha+\beta)z_3 \rightarrow$ $z_1+\frac{\beta}{\alpha}z_2=\frac{\alpha+\beta}{\alpha}z_3$. Add $-(1+\frac{\beta}{\alpha})z_1$ to both sides. Then, $\beta/\alpha(z_2-z_1)=\frac{\alpha+\beta}{\alpha}(z_3-z_1)$. Multiply both sides by $\frac{\alpha}{\beta}$, then $z_2-z_1=\frac{\alpha+\beta}{\beta}(z_3-z_1)$. Thus $z_2-z_1=b(z_3-z_1)$, for some $b\in\mathbb{R}$. There fore $z_1,z_2,z_3\in\mathbb{C}$ are collinear.

Now suppose z_1,z_2,z_3 are collinear, then there exists some $b\in\mathbb{R}$ such that $z_2-z_1=b(z_3-z_1)$ $\rightarrow z_2-z_1-bz_3+bz_1=0 \rightarrow (b-1)z_1+z_2-bz_3=0$ This condition is fulfilled when $\alpha=b-1$, $\beta=1$, $\gamma=-b$. ∎

25b. Yes, this can be extended for many collinear points.

36. For any complex number $a\neq 0$, $a,-a,1/\bar{a},-1/\bar{a},0$ are collinear if for some $j,k,l,m,n\in\mathbb{R}$, $j(a)+k(-a)+l(1/\bar{a})+m(-1/\bar{a})+n(0)=0$. This is true for all $n\in\mathbb{R}$, $a\in\mathbb{C}$ when $j=1$, $k=-1$, $l=|a|^2$, and $m=-|a|^2$. ∎

37a. $z=1+i$. $r=\sqrt{1+1}=\sqrt{2}$ $\theta=\tan^{-1}(1)=\pi/4$.
$z=1+i=\sqrt{2}e^{i\pi/4}$

37b. $z=4-3i$. $r=\sqrt{16+9}=5$ $\theta=\tan^{-1}(-3/4)$
$z=4-3i=5e^{i\tan^{-1}(-3/4)}$

37c. $z=1+w$, where $w^2+w+1=0$. By Quadratic Formula, $w=-1/2\pm(\sqrt{3}/2)i$. 2 Cases:
If $w=-1/2+i\sqrt{3}/2$, then
$z=1+(-1/2)+(\sqrt{3}/2)i=1/2+(\sqrt{3}/2)i$, and
$r=\sqrt{1/4+3/4}=1$; $\theta=\tan^{-1}((\sqrt{3}/2)/1/2)=\tan^{-1}(\sqrt{3})=\pi/3$
If $w=-1/2-i(\sqrt{3}/2)$, then $z=1-1/2-i\sqrt{3}/2=1/2-i\sqrt{3}/2$
$r=\sqrt{1/4+3/4}=1$ $\theta=\tan^{-1}(-\frac{\sqrt{3}}{2}/1/2)=\tan^{-1}(-\sqrt{3})=-\pi/3$
so $z=1+w=e^{\pm(\pi/3)i}$

MATH 532 — Homework 1 — Due September 3rd, 2021

37d. $z = \frac{1}{w}$, where $w^2 + w + 1 = 0$. By Quadratic Formula, $w = -1/2 \pm \sqrt{3}\, i/2$, so $\frac{1}{w} = -2 \pm \frac{2\sqrt{3}}{3}$. Then, $r = \sqrt{4 + 4/3} = 4\sqrt{3}/3$. Then θ has 2 cases
Let $z = -2 + \frac{2\sqrt{3}}{3}$, then $\theta = \tan^{-1}(\frac{2\sqrt{3}}{3}/-2) = \tan^{-1}(\sqrt{3}/3) = -\pi/6$
Let $z = -2 - \frac{2\sqrt{3}}{3}$, then $\theta = \tan^{-1}(\frac{-2\sqrt{3}}{3}/-2) = \tan^{-1}(\sqrt{3}/3) = \pi/6$.
So, $z = 1/w = \frac{4\sqrt{3}}{3} e^{\pm(\pi/6)i}$.

---

3. $e^{i\theta} = \cos(\theta) + i\sin(\theta)$
$e^{in\theta} = (\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$
$e^{i3\theta} = \cos(3\theta) + i\sin(3\theta) = (\cos\theta + i\sin\theta)^3$
$= \cos^3\theta + 2i\cos^2\theta\sin\theta - \cos\theta\sin^2\theta + i\cos^2\theta\sin - 2\cos\theta\sin^2\theta - i\sin^3\theta$
$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$. Then

★ $\cos(3\theta) = \cos^3\theta - 3\cos\theta\sin^2\theta = 4\cos^3\theta - 3\cos\theta$.

★ $\sin(3\theta) = 3\cos^2\theta\sin\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta$.

$e^{i4\theta} = \cos(4\theta) + i\sin(4\theta) = (\cos\theta + i\sin\theta)^4$
$= (4\cos^3\theta - 3\cos\theta + 3i\sin\theta - 4i\sin^3\theta)(\cos\theta + i\sin\theta)$
$= 4\cos^4\theta - 3\cos^2\theta + 3i\cos\theta\sin\theta - 4i\cos\theta\sin^3\theta$
$+ 4i\cos^3\theta\sin\theta - 3i\cos\theta\sin\theta - 3\sin^2\theta + 4\sin^4\theta$
$= (\cos^4\theta - 3\cos^2\theta - 3\sin^2\theta + 4\sin^4\theta) +$
$i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$.
$\cos(4\theta) = 4\cos^4\theta + 4\sin^4\theta - 3 = 4\cos^4\theta + 4(\cos^4\theta - 2\cos^2\theta + 1) - 3 \rightarrow$

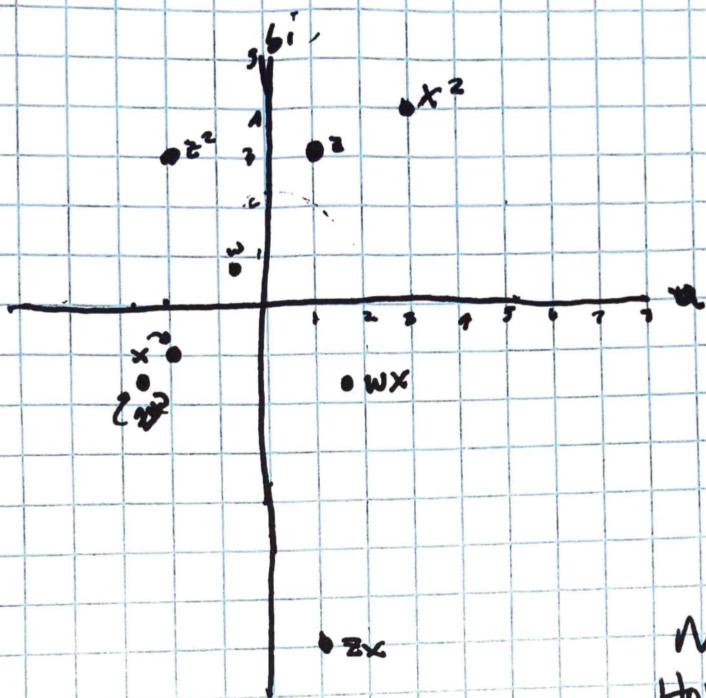
★ $\cos(4\theta) = 8\cos^4\theta - 8\cos^2\theta + 1$

★ $\sin(4\theta) = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta = (4\cos\theta\sin\theta)(\cos^2\theta - \sin^2\theta)$
$= (4\cos\theta\sin\theta)(1 - 2\sin^2\theta) = 4\sin\theta\cos\theta - 8\sin^3\cos\theta$.

4a. $z = 1+3i = \sqrt{10}\, e^{i\tan^{-1}(3)}$
$w = e^{3\pi/4\, i} = \cos(3\pi/4) + i\sin(3\pi/4) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$
$x = -2 - i = \sqrt{5}\, e^{i\tan^{-1}(1/2)}$

4c. $z^2 = -2 + 3i$
$zw = \cos(3\pi/4) - 3\sin(3\pi/4) + 3i\cos(3\pi/4) + i\sin(3\pi/4) = -2\sqrt{2} - \sqrt{2}\,i$
$wx = -2\cos(3\pi/4) + \sin(3\pi/4) - i\cos(3\pi/4) - 2i\sin(3\pi/4) = \frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i$
$zx = 1 - 7i$
$x^2 = 3 + 4i$

4. b, c



MATH 532
Homework 1
MATHEW HOUSER

4.d. $|z| = \sqrt{1+9} = \sqrt{10}$

$|w| = \sqrt{\frac{2}{4} + \frac{2}{4}} = 1$

$|x| = \sqrt{4+1} = \sqrt{5}$

4e. From the Graph we can tell that zx has the largest Length.