MATH 554 - Homework #6 - Mathew Houser

Chapter 2: 17, 19, 22, 23, 24, 25

17. Define $f:\mathbb{R}\to\mathbb{R}$ as follows:

$$f(x)=\begin{cases} x-\lfloor x\rfloor & \text{if } \lfloor x\rfloor \text{ is even} \\ x-\lfloor x+1\rfloor & \text{if } \lfloor x\rfloor \text{ is odd} \end{cases}$$

Determine where f has a limit and justify.

We will show that f has no limit at x_0 iff x_0 is an odd integer. Assume x_0 is an integer and consider the sequence $\{x_n\}_{n=1}^{\infty}=\{x_0+(-1)^n\frac{1}{n}\}_{n=1}^{\infty}$. Then

$$\{\lfloor x_n\rfloor\}_{n=1}^{\infty}=\begin{cases} x_0 & \text{if } n \text{ is even.} \\ x_0-1 & \text{if } n \text{ is odd.} \end{cases}$$

Assume x_0 is an odd integer. Then

$$\begin{cases} \text{if } n \text{ is even, } \lfloor x_n\rfloor = x_0 \to \lfloor x\rfloor \text{ is odd, } f(x)=x_n-x_0 \\ \text{if } n \text{ is odd, } \lfloor x_n\rfloor = x_0-1 \to \lfloor x\rfloor \text{ is even, } f(x)=x_n-x_0-1 \end{cases}$$

$\{f(x_n)\}_{n=1}^{\infty}$ diverges thus x_0 is not a limit of f.

Now assume x_0 is an even integer. Then

$$\begin{cases} \text{if } n \text{ is even, } \lfloor x_n\rfloor = x_0 \to \lfloor x\rfloor \text{ is even, } f(x)=x_n-x_0 \\ \text{if } n \text{ is odd, } \lfloor x_n\rfloor = x_0-1 \to \lfloor x\rfloor \text{ is odd, } f(x)=x_n-x_0 \end{cases}$$

$\{f(x_n)\}_{n=1}^{\infty}$ converges thus x_0 is a limit of f.

Lastly, assume x_0 is not an integer. Choose any arbitrary $\varepsilon>0$ and set $\delta=\min\{\varepsilon, x_0-\lfloor x_0\rfloor, x_0-\lfloor x_0\rfloor+1\}$. Then for all $0<|x-x_0|<\delta$, $x\in(\lfloor x_0\rfloor, \lfloor x_0\rfloor+1)$ thus $\lfloor x\rfloor=\lfloor x_0\rfloor$.

$$\begin{cases} \text{if } \lfloor x_0\rfloor \text{ is even, } |f(x)-f(x_0)|=|(x-\lfloor x\rfloor)-(x_0-\lfloor x_0\rfloor)|=|x-x_0|<\delta\le\varepsilon \\ \text{if } \lfloor x_0\rfloor \text{ is odd, } |f(x)-f(x_0)|=|(x-\lfloor x+1\rfloor)-(x_0-\lfloor x_0+1\rfloor)|=|x-x_0|<\delta\le\varepsilon. \end{cases}$$

Thus x_0 is a limit of f. Therefore f has a limit at x_0 iff x_0 is not an odd integer. ■

19. Define $f:(0,1)\to\mathbb{R}$ by $f(x)=\frac{\sqrt{9-x}-3}{x}$. Prove that f has a limit at 0 & find it

$$f(x)=\frac{\sqrt{9-x}-3}{x}\cdot\left(\frac{-\sqrt{9-x}-3}{-\sqrt{9-x}-3}\right)=\frac{-x}{x(3+\sqrt{9-x})}=\frac{-1}{3+\sqrt{9-x}}$$

$\lim_{x\to0}(-1)=-1$, $\lim_{x\to0}(3+\sqrt{9-x})=3+\sqrt{9}=6$.

Therefore by Theorem 2.4, $\lim_{x\to0}f(x)=-1/6$. ■

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22. Show by example that, even though f and g fail to have limits at x_0, it is possible for $f+g$ to have a limit at x_0. Give similar examples for fg and f/g.

$\lim_{x\to 0} \lfloor x \rfloor = \text{DNE}$; $\lim_{x\to 0} -\lfloor x \rfloor = \text{DNE}$; $\lim_{x\to 0} \lfloor x \rfloor - \lfloor x \rfloor = 0$. $(f+g)$

$\lim_{x\to 0} \frac{|x|}{x} = \text{DNE}$; $\lim_{x\to 0} \frac{-|x|}{x} = \text{DNE}$; $\lim_{x\to 0} \frac{|x|}{x} \cdot \frac{-|x|}{x} = -1$. (fg)

$\lim_{x\to 0} \frac{1}{x} = \infty$; $\lim_{x\to 0} \frac{-1}{x} = -\infty$; $\lim_{x\to 0} \frac{1/x}{-1/x} = -1$. (f/g)

23. Let $f:[\alpha,\beta]\to\mathbb{R}$ be decreasing and $\forall x\in(\alpha,\beta)$ define $U(x) = \inf\{f(y) : y < x\}$ and $L(x) = \sup\{f(y) : y > x\}$. Then f has a limit at $x_0 \in (\alpha,\beta)$ iff $U(x_0) = L(x_0)$, and in this case $\lim_{x\to x_0} f(x) = f(x_0) = U(x_0) = L(x_0)$.

$(\Rightarrow)$ Suppose f has a limit at $x_0 \in (\alpha,\beta)$, i.e. $\lim_{x\to x_0} f(x) = A$ for some $A\in\mathbb{R}$. Choose any arbitrary $\varepsilon > 0$. Then $\exists$ a $\delta > 0$ such that $\forall x \in [\alpha,\beta]$ and $0 < |x - x_0| < \delta$, $|f(x) - A| < \varepsilon$. $x_0 \in (\alpha,\beta)$ and (α,β) is an open set. Then $\exists\, x, y \in (\alpha,\beta)$ such that $x_0 - \delta < x < x_0 < y < x_0 + \delta$. Then by the definition of $U(x)$ and $L(x)$ it follows that:
$A - \varepsilon < f(y) \le L(x_0) \le f(x_0) \le U(x_0) \le f(x) < A + \varepsilon$. Thus $U(x_0) - L(x_0) < (A+\varepsilon) - (A-\varepsilon) = 2\varepsilon$. Since $\varepsilon > 0$ and arbitrary, we have $U(x_0) = L(x_0) = f(x_0)$. Moreover, since $A - \varepsilon < f(x_0) < A + \varepsilon$ $\forall \varepsilon > 0$, $f(x_0) = A$. Therefore $\lim_{x\to x_0} f(x) = f(x_0) = U(x_0) = L(x_0)$.

$(\Leftarrow)$ Suppose $U(x_0) = L(x_0)$. Then by the properties of decreasing functions $L(x_0) \le f(x_0) \le U(x_0)$, hence $L(x_0) = f(x_0) = U(x_0)$. Choose any $\varepsilon > 0$ arbitrarily. Clearly, $L(x_0) - \varepsilon$ is not an upperbound for $\{f(y) : y > x_0\}$ and $U(x_0) + \varepsilon$ is not a lower bound for $\{f(y) : y < x_0\}$. Then $\exists\, y_1, y_2 \in (\alpha,\beta)$ such that $\alpha \le y_1 \le x_0 \le y_2 \le \beta$ and $L(x_0) - \varepsilon < f(y_1)$ and $f(y_2) < U(x_0) + \varepsilon$.
Let $\delta = \min\{x_0 - y_1, y_2 - x_0\} \Rightarrow (x_0 - \delta, x_0 + \delta) \subseteq (y_1, y_2)$. Then for any $0 < |x - x_0| < \delta$ and $x \in (y_1, y_2)$ we have $f(x_0) - \varepsilon = L(x_0) - \varepsilon \le f(y_2) \le f(x) \le f(y_1) \le U(x_0) + \varepsilon = f(x_0) + \varepsilon \Rightarrow |f(x) - f(x_0)| < \varepsilon$. Thus $\lim_{x\to x_0} f(x) = f(x_0)$. ∎

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24. Let $f:[a,b]\to\mathbb{R}$ be monotone. Prove that f has a limit at both a and b.

WLOG assume f is increasing. Suppose f has a limit A at a, i.e $\lim_{x\to a} f(x) = A$ for some $A\in\mathbb{R}$. Choose any arbitrary $\varepsilon_1>0$. Then $\exists$ a $\delta_1>0$ such that $\forall x\in[a,b]$ and $0<|x-a|<\delta_1$, $|f(x)-A|<\varepsilon_1$. Then $\exists\, x,y\in[a,b]$ such that $a-\delta_1<x\le a\le y<a+\delta_1$. Then $A-\varepsilon_1<f(x)\le L(a)\le f(a)\le U(a)\le f(y)<A+\varepsilon_1$. Then we have $U(a)-L(a)<(A+\varepsilon_1)-(A-\varepsilon_1)=3\varepsilon_1$. Since this is true for any $\varepsilon>0$, $U(a)=L(a)=f(a)=A$

Suppose f has a limit B at b, i.e $\lim_{x\to b} f(x) = B$ for some $B\in\mathbb{R}$. Chose any arbitrary $\varepsilon_2>0$. Then $\exists$ a $\delta_2>0$ such that $\forall x\in[a,b]$ and $0<|x-b|<\delta_2$, $|f(x)-B|<\varepsilon_2$. Then $\exists\, x,y\in[a,b]$ such that $b-\delta_2<x\le b\le y\le b+\delta_2$ Then $B-\varepsilon_2<f(x)\le L(b)\le f(b)\le U(b)\le f(y)<B+\varepsilon_2$. Then we have $U(b)-L(b)<(B+\varepsilon_2)-(B-\varepsilon_2)=2\varepsilon$. Since this is true for any $\varepsilon>0$, $U(b)=L(b)=f(b)=B$. Therefore by Lemma 2.7 f has a limit at both a and b. ∎

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25. Suppose $f: [a,b] \to \mathbb{R}$ and define $g: [a,b] \to \mathbb{R}$ by $g(x) = \sup\{f(t) : a \le t \le x\}$. Prove that g has a limit at x_0 if f has a limit at x_0 and $\lim_{t \to x_0} f(t) = f(x_0)$.

Assume f has a limit at x_0 and $\lim_{t \to x_0} f(t) = f(x_0)$. Then $\forall\, \varepsilon > 0$, $\exists$ a $\delta > 0$ such that $\forall\, t \in [a,b]$ and $0 < |t - x_0| < \delta$, $|f(t) - f(x_0)| < \varepsilon/2$. Suppose $x_0 < x < x_0 + \delta$. If $g(x) = g(x_0)$, $|g(x) - g(x_0)| = 0 < \varepsilon$. If $g(x) \ne g(x_0)$, then $g(x) > g(x_0)$ and $g(x) = \sup\{f(t) : x_0 < t \le x\}$. $g(x) - \varepsilon/2$ is not an upper bound, so $\exists\, t \in (x_0, x)$ such that $f(t) > g(x) - \varepsilon/2$. $g(x_0) \ge f(x_0)$, so $|g(x) - g(x_0)| = g(x) - g(x_0) < f(t) + \varepsilon/2 - f(x_0) < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

The result is similar for $x_0 - \delta < x < x_0$. ∎