Math 141: Midterm 1

NAME: Mathew Houser STUDENT ID: A43010232

1. (10 marks) For the following problems,

   (a) Draw a direction field for the given differential equation. Other Page

   (b) Based on an inspection of the direction field, describe how solutions behave for large t. other page

   (c) Find the general solution of the given differential equation and use it to determine how solutions behave as $t \to \infty$.

   (i)
$$\frac{dy}{dt} + 2ty = 2te^{-t^2}.$$

   (ii)
$$t\frac{dy}{dt} - y = t^2e^{-t}.$$

(i) Part C.

$dy/dt = 2te^{-t^2} - 2ty \rightarrow y' + 2ty = 2te^{-t^2}$

$P(t) = 2t$; $g(t) = 2te^{-t^2}$; $\mu(t) = e^{t^2}$

$y(t) = \int^t (2se^{-s^2})(e^{s^2})ds + c/e^{t^2}$.

$\int^t (2se^{-s^2})(e^{s^2})ds = \int^t (2s)ds = s^2|^t = t^2$.

Thus $y(t) = \frac{t^2+c}{e^{t^2}}$

L'Hopital:

$\lim_{t\to\infty} \left(\frac{t^2+c}{e^{t^2}}\right) = \lim_{t\to\infty} \left(\frac{1}{e^{t^2}}\right) = \frac{1}{\infty} = 0$

As $t \to \infty$, $y \to 0$.

(ii) Part C.

$dy/dt - 1/t\, y = te^{-t}$ $\quad \mu(t) = e^{\int^t p(s)ds} = e^{-\ln(t)} = t$

$P(t) = -1/t$; $g(t) = te^{-t}$; $\mu(t) = t$

$y(t) = \int^t (se^{-s})(s)\, ds + c/t$

$\int^t (s^2e^{-s})\, ds = +s^2e^{-s} + 2se^{-s} + 2e^{-s} = +(s^2+2s+2)e^{-s}|^t$

Thus $y(t) = \frac{+t^2e^{-t} + 2te^{-t} + 2e^{-t} + c}{-t}$

$\lim_{t\to\infty} (y(t)) = \frac{\lim_{t\to\infty} (+t^2e^{-t} + 2te^{-t} + 2e^{-t} + c)}{\lim_{t\to\infty} (-t)} = -\frac{c}{\infty} = 0$

As $t \to \infty$, $y \to 0$

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2. (10 marks) For the following problems,

   (a) Draw a direction field for the given differential equation. Other Page

   (b) Based on an inspection of the direction field, describe how solutions behave for large t. Other Page

   (c) Find the general solution of the given differential equation and use it to determine how solutions behave as $t \to \infty$.

(i)

$$\frac{dy}{dt} = (1 - \frac{y}{2})y.$$

(ii)

$$\frac{dy}{dt} = -(1 - \frac{y}{2})(1 - \frac{y}{3})y.$$

(i) Part C.

$\frac{dy}{dt} = (1-\frac{y}{2})y \rightarrow dy/dt = -1/2\,(y-2)y$

$\int \frac{-2}{(y-2)y}\,dy = \int 1\,dt$

$-\ln(|2/y - 1|) = t + C_0 \rightarrow \ln(|2/y - 1|) = -t + C_0$

$2/y - 1 = e^{-t+C_0} \rightarrow 2/y = e^{-t+C_0} + 1$

$y(t) = \frac{2}{C_1 e^{-t} + 1}$

$\lim_{t\to\infty} \left(\frac{2}{C_1 e^{-t}+1}\right) = \frac{\lim_{t\to\infty}(2)}{\lim_{t\to\infty}(C_1 e^{-t}+1)}$

$= \frac{2}{1} \rightarrow \lim_{t\to\infty}(y(t)) = 2$

As $t \to \infty$, $y \to 2$.

(ii) Part C.

$dy/dt = -1(1-y/2)(1-y/3)y$

$= (-1/2)(2-y)(1/3)(3-y)(y)$

$-\frac{1}{6}\int \frac{1}{(y-3)(y-2)(y)}\,dy = -\frac{1}{6}\int \frac{dy}{6y} - \frac{dy}{2(y-2)} + \frac{dy}{3(y-3)} = \int 1\,dt$

$-\ln(|y|) + 3\ln(|y-2|) - 2\ln(|y-3|) = t + c$

By graphing this function, it is clear that for all values of C, y approaches either Ø or 3 as t approaches infinity

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3. (20 marks) Solve the following differential equations with provided integrating factors.

(i)

$$x^2y^3 + x(1+y^2)\frac{dy}{dx} = 0, \quad \mu(x,y) = \frac{1}{xy^3}.$$

(ii)

$$ydx + (2x - ye^y)dy = 0, \quad \mu(x,y) = y.$$

(i) $\left(\frac{1}{xy^3}\right)(x^2y^3) + \left(\frac{1}{xy^3}\right)(x)(1+y^2)dy/dx = \left(\frac{1}{xy^3}\right)(0)$

$(x) + (1/y^3 + 1/y)\, dy/dx = 0$

$\psi_x(x,y) = x \Rightarrow \psi(x,y) = 1/2 \cdot x^2 + h(y) \Rightarrow \psi_y(x,y) = h'(y).$

$h'(y) = 1/y^3 + 1/y \Rightarrow h(y) = \ln(|y|) - 1/2y^2.$

Hence solutions of 3(i) are given implicitly by

$1/2 \cdot x^2 - 1/2y^2 + \ln(|y|) = C$

(ii) Dividing both sides by dx yields $y + (2x - ye^y)dy/dx = 0$. Then,

$(y)(y) + (y)(2x - ye^y)dy/dx = 0$

$(y^2) + (2xy - y^2e^y)dy/dx = 0$

$\psi_x(x,y) = y^2 \Rightarrow \psi(x,y) = xy^2 + h(y) \Rightarrow \psi_y(x,y) = 2xy + h'(y).$

$2xy + h'(y) = 2xy - y^2e^y \Rightarrow h'(y) = -y^2e^y \Rightarrow h(y) = -(y-1)^2 \cdot e^y$

Hence solutions of 3(ii) are given implicitly by

$xy^2 - (y-1)^2 e^y = C$

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4. (10 marks) Find an integrating factor and solve the following differential equation.

$$\frac{dy}{dx} = e^{2x} + y - 1.$$

$dy/dx = e^{2x} + y - 1$

$(e^{2x} + y - 1) + (-1)\, dy/dx = 0$

$M(x,y) = e^{2x} + y - 1$; $M_y(x,y) = 1$; $N(x,y) = -1$; $N_x(x,y) = 0$

$d\mu/dx = \frac{1-0}{-1}\mu = -\mu \Rightarrow \int \frac{1}{\mu}\, d\mu = \int -1\, dx \Rightarrow \ln(|\mu|) = -x \Rightarrow \mu(x) = e^{-x}$

$(e^{-x})(e^{2x} + y - 1) + (e^{-x})(-1)\, dy/dx = 0$

$(e^{x} + ye^{-x} - e^{-x}) + (-e^{-x})\, dy/dx = 0$

$\psi_x(x,y) = e^{x} + ye^{-x} - e^{-x} \Rightarrow \psi(x,y) = e^{x} - ye^{-x} + e^{-x} + h(y)$

$\psi_y(x,y) = -e^{-x} + h'(y).$

$-e^{-x} + h'(y) = -e^{-x} \Rightarrow h'(y) = 0 \Rightarrow h(y) = C_0$

Thus $e^{x} - ye^{-x} + e^{-x} = C_0$. Multiplying both sides by $\frac{1}{\mu(x)}$ yields

$e^{2x} - y + 1 = C_0 e^{x}.$

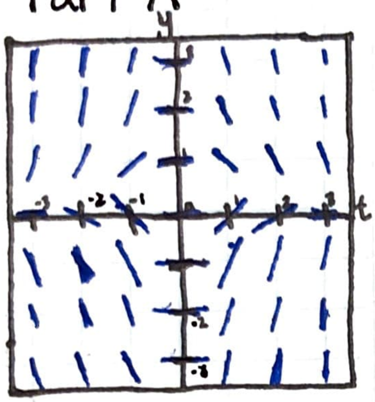
Therefore, $y(x) = e^{2x} + C_1 e^{x} + 1.$

Math 520 - Exam #1

Mathew Houser - Direction Fields

1. Part A

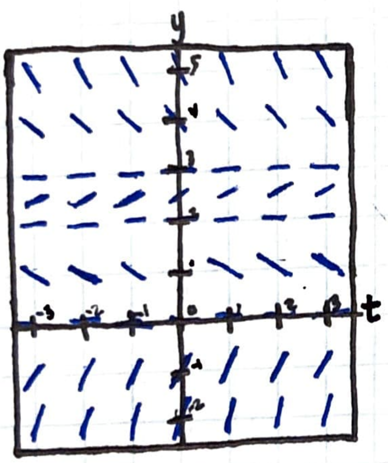
(i)



Part B

(i) As $t \to \infty$, $y \to 0$.

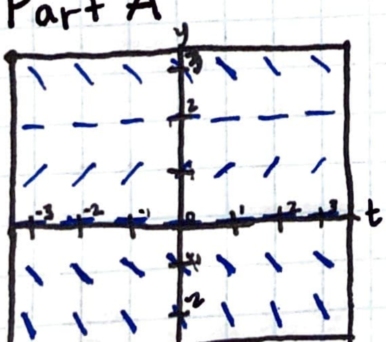
(ii)



(ii) As $t \to \infty$, $\begin{cases} y > 2, & y \to 3 \\ y < 2, & y \to 0 \end{cases}$

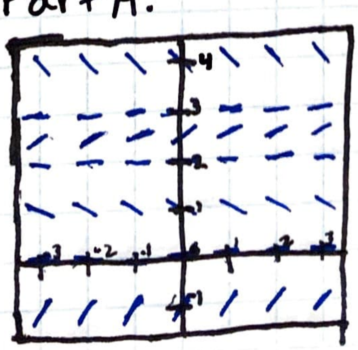
2. Part A

(i)



2. Part A.

(ii)



(i) Part B

As $t \to \infty$, $\begin{cases} y > 0, & y \to 2 \\ y < 0, & y \to \infty \end{cases}$

(ii) Part B

As $t \to \infty$, $\begin{cases} y > 2, & y \to 3 \\ y < 2, & y \to 0 \end{cases}$