Mates of power latin squares*

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Abstract

We study the number of mates of latin squares which are powers of cyclic squares. For $k > 1$ the cyclic square C_k is the Cayley table of Z_k. The power square C_k^n is obtained by taking a repeated Kronecker product of C_k with itself.

We first consider the square C_2^3 which is known to have $70,272 \cdot 8!$ mates. We obtain results on the structure of these mates, and obtain a combinatorial enumeration of a family of $6,144 \cdot 8!$ mates of C_2^3.

We then consider the power squares C_k^n for $k > 2$ and $n > 1$. For each of these squares, we enumerate a family of mates of a particular form. This gives an asymptotic lower bound for the number of mates that a latin square can have in terms of its size.

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Contents

1 Introduction

In our research, we classify and enumerate mates of powers of cyclic latin squares. In particular, we study mates of C_2^3 and regular mates of C_k^n. We do so with an ultimate goal of determining how many mates a latin square of a given size can have.

We extend the work of Bryant, Figler, Garcia, Mummert, and Singh [2] from a past REU. They found that the maximum number of mates for a size 7 latin square is $635 \cdot 7!$ mates and the maximum number of mates for a size 8 latin square is exactly $70,272 \cdot 8!$ mates. Additionally, their work found a construction to build a single mate for C_2^n when $n \geq 2$.

We began our research by analyzing structural properties of mates of C_2^3, as described in Section 2. These structural results allow us to divide these mates into four distinct classes based on how many 4-element sets make up the 2×2 primary blocks of a given mate. A computer search verified that all mates of C_2^3 have either 2, 4, 8 or 16 4-element sets which make up their primary blocks.

In the case where a mate of C_2^3 has two 4-element sets, we prove there are six possible forms a mate can take. These forms are described in Section 3, where we obtain a bijection between each pair of these forms. We then develop a construction which allows us to combinatorially enumerate a family of $6,144 \cdot 8!$ mates of C_2^3. Future work on the same subject may extend this research to combinatorially enumerate the remaining mates of C_2^3.

In Section 4, we focus on mates of C_3^2. Using a computational search, we discovered that this square has $12,445,836 \cdot 9!$ mates. We obtain a construction that enumerates a family of $6^6 \cdot 9!$ mates of C_3^2. We then generalize this construction for mates of C_k^n where $k \geq 3$ and $n \geq 2$. This generalization allows us to enumerate a family of mates of C_k^n. It also gives an asymptotic lower bound on the number of mates for a latin square of a given size.

1.1 Background and Definitions

In this section, we summarize the definitions and theorems needed for our research. Laywine and Mullen [1] and Mullen and Mummert [3] give thorough descriptions of this background.

A *latin array* is an $m \times n$ array with $m \cdot n$ possible symbols where each symbol appears at most once in each row and each column. A *latin square* is a latin array of size $n \times n$ with n symbols in which each symbol appears once in each row and once in each column. A *cyclic latin square* of size n is formed by filling the first row of a square with symbols in any order. The next row is filled by shifting all of the symbols left one place and moving the first symbol to the end of the row. The subsequent rows are filled by continuing this pattern, with each row shifted one place to the left of the previous row and the first symbol moving to the end of the row. For the remainder of this report, we will let C_n denote the unique reduced cyclic square of size n.

$$\begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} \qquad \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{array} \qquad \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{array}$$

$$C_2 \qquad\qquad C_3 \qquad\qquad C_4$$

Two latin squares of the same size are *orthogonal mates* if every possible ordered pair of their symbols is present when the squares are superimposed. A latin square with symbols $1, 2, \ldots, n$ is *semireduced* if the first row is in the natural order $1, 2, \ldots, n$. In addition, a latin square is *reduced* if the first row and the first column are in this natural order.

Definition 1.1. A *power square* is obtained by taking a repeated Kronecker product of a fixed square with itself [2].

For example, C_2^3 is the 8×8 latin square obtained by taking the product of the cyclic square C_2 with itself two times. That is,

$$C_2^3 = C_2 \otimes C_2 \otimes C_2.$$

$$
\begin{array}{cc}
1 & 2 \\
2 & 1
\end{array}
\qquad
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 1 & 4 & 3 \\
3 & 4 & 1 & 2 \\
4 & 3 & 2 & 1
\end{array}
\qquad
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \\
3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \\
4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 \\
5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \\
6 & 5 & 8 & 7 & 2 & 1 & 4 & 3 \\
7 & 8 & 5 & 6 & 3 & 4 & 1 & 2 \\
8 & 7 & 6 & 5 & 4 & 3 & 2 & 1
\end{array}
$$

$$C_2 \qquad C_2^2 = C_2 \otimes C_2 \qquad C_2^3 = C_2 \otimes C_2 \otimes C_2$$

Definition 1.2. A reverse C_3 square, $\widehat{C}_3$, is obtained when the rows are shifted in the opposite direction of a normal C_3 square.

$$
\begin{array}{ccc}
1 & 2 & 3 \\
3 & 1 & 2 \\
2 & 3 & 1
\end{array}
\qquad
\begin{array}{ccc|ccc|ccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
3 & 1 & 2 & 6 & 4 & 5 & 9 & 7 & 8 \\
2 & 3 & 1 & 5 & 6 & 4 & 8 & 9 & 7 \\
\hline
7 & 8 & 9 & 1 & 2 & 3 & 4 & 5 & 6 \\
9 & 7 & 8 & 3 & 1 & 2 & 6 & 4 & 5 \\
8 & 9 & 7 & 2 & 3 & 1 & 5 & 6 & 4 \\
\hline
4 & 5 & 6 & 7 & 8 & 9 & 1 & 2 & 3 \\
6 & 4 & 5 & 9 & 7 & 8 & 3 & 1 & 2 \\
5 & 6 & 4 & 8 & 9 & 7 & 2 & 3 & 1
\end{array}
$$

$$\widehat{C}_3 \qquad \widehat{C}_3^2 = \widehat{C}_3 \otimes \widehat{C}_3$$

$\widehat{C}_3$ is the only semireduced mate of C_3. $\widehat{C}_3^2$ is a an example of a mate of C_3^2 in which each 3×3 primary block is a latin square, and there are only three symbol sets in the primary blocks

Definition 1.3. A latin square of even size can be divided into four *quadrants*, which are obtained by dividing a latin square in half once vertically and once horizontally.

$$
\begin{array}{cccc|cccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \\
3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \\
4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 \\
\hline
5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \\
6 & 5 & 8 & 7 & 2 & 1 & 4 & 3 \\
7 & 8 & 5 & 6 & 3 & 4 & 1 & 2 \\
8 & 7 & 6 & 5 & 4 & 3 & 2 & 1
\end{array}
$$

C_2^3 divided into four quadrants

Definition 1.4. A *primary block* is a $k \times k$ block of a latin square of size k^n obtained by repeatedly dividing the square into equal sections.

For example, a latin square of size 8 such as C_2^3 has four 4×4 primary blocks, each of which is made of four 2×2 primary blocks, giving us 16 total 2×2 primary blocks.

$$
\begin{array}{cccc|cccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \\
3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \\
4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 \\
\hline
5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \\
6 & 5 & 8 & 7 & 2 & 1 & 4 & 3 \\
7 & 8 & 5 & 6 & 3 & 4 & 1 & 2 \\
8 & 7 & 6 & 5 & 4 & 3 & 2 & 1
\end{array}
\qquad
\begin{array}{cc|cc|cc|cc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \\
\hline
3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \\
4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 \\
\hline
5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \\
6 & 5 & 8 & 7 & 2 & 1 & 4 & 3 \\
\hline
7 & 8 & 5 & 6 & 3 & 4 & 1 & 2 \\
8 & 7 & 6 & 5 & 4 & 3 & 2 & 1
\end{array}
$$

4 primary blocks of size 4×4 16 primary blocks of size 2×2

Definition 1.5. A *subsquare* is a latin square that is found within a larger latin square.

For example, the four circled symbols in the latin square below form a 2×2 subsquare.

$$
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \\
3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \\
4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 \\
5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \\
6 & 5 & 8 & 7 & 2 & 1 & 4 & 3 \\
7 & 8 & 5 & 6 & 3 & 4 & 1 & 2 \\
8 & 7 & 6 & 5 & 4 & 3 & 2 & 1
\end{array}
$$

Subsquares differ from primary blocks and quadrants in that primary blocks and quadrants must be contiguous. In a subsquare, other symbols may lie between elements of the subsquare within the original square.

Definition 1.6. The *complement* of a subsquare is made by removing all rows and all columns containing the original subsquare. The cells left over make up the complement. This is illustrated in Figure 1.

2 Mates of $(C_2)^3$

In this section we analyze structural properties of mates of C_2^3. This analysis yields two of the most important building blocks of our later results. First, every primary block of a mate of C_2^3 has four distinct symbols, as we prove in Theorem 2.4. This theorem is important because it allows us to define $\mathcal{S}(M)$, which is the set of all 4-element sets

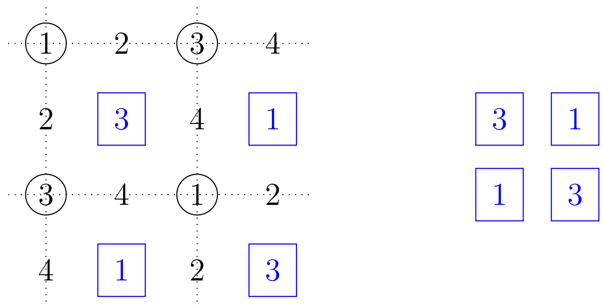


Figure 1: A subsquare and its complement

which make up the primary blocks of a mate M of C_2^3. This allows us to divide the mates of C_2^3 into four classes, based on whether $|\mathcal{S}(M)|$ is 2, 4, 8 or 16.

In this section, we also prove some structural properties which are not used further, but which we feel are interesting and could be of some significance. These properties describe the appearance of symbols in mates of C_2^3, the behavior of subsquares, and the construction of mates via transformations.

2.1 Components of Latin Squares

Theorem 2.1. *For any mate M of C_2^3 each symbol of M appears twice in each quadrant of M.*

Proof. Suppose M is a mate of C_2^3 and α is a symbol of M. We proceed by contradiction. Suppose α appears more than 2 times in some quadrant Q. Then there exists a $c > 0$ such that α appears $2 + c$ times in quadrant Q. We will compute how often α occurs in each of the other three quadrants.

Let V be the quadrant of M above or below Q. There are only 4 columns in V and Q and α appears once in each column because M is a latin square. Therefore, since α appears $2 + c$ times in Q, it appears $4 - (2 + c)$ times in V. That is, α appears $2 - c$ times in V.

Let R be the quadrant of M to the left or right of Q. There are only 4 rows in Q and R, and α appears once in each row since M is a latin square. Since α appears $2 + c$ times in Q, α appears $2 - c$ times in R.

Let D be the quadrant of M diagonally opposite of Q. Let Q' be the quadrant of C_2^3 corresponding to Q and let D' be the quadrant of C_2^3 corresponding to D. Then Q' and D' contain the same selection of 4 symbols which don't appear in the other 2 quadrants of C_2^3. If we superimpose D' and D, and also superimpose Q' and Q, we will get 4 ordered pairs overall whose second component is α. Since α appears

$2 + c$ times in Q and M is a mate of C_2^3, we will get $2 + c$ of those ordered pairs when Q and Q' are superimposed. Hence, we get $2 - c$ pairs when D and D' are superimposed. So, α must occur $2 - c$ times in D.

The total number of times α appears in M must be the sum of the number of times α appears in each of the quadrants. We have shown α appears $2 + c$ times in Q and $2 - c$ times in each of V, R, and D. Thus, the total number of times α appears is $(2+c)+3(2-c) = 8-2c$. Since $c > 0$, α appears less than 8 times in M. This is a contradiction because M is a latin square. Therefore, for any mate M of C_2^3, each symbol of M appears exactly twice in each quadrant of M. □

Theorem 2.2. *If $n \geq 2$, and M is a mate of C_2^n, each symbol of M appears 2^{n-2} times in each quadrant of M.*

Proof. Suppose M is a mate of C_2^n and α is a symbol of M. We proceed by contradiction. Suppose α appears more than 2^{n-2} times in some quadrant, Q. Then there exists a $c > 0$ such that α appears $2^{n-2} + c$ times in quadrant Q.

Let V be the quadrant of M above or below Q. There are only 2^{n-1} columns in V and Q and α appears once in each column because M is a latin square. Therefore, since α appears $2^{n-2} + c$ times in Q, it appears $2^{n-1} - (2^{n-2} + c)$ times in V. That is, α appears $2^{n-2} - c$ times in V.

Let R be the quadrant of M to the left or right of Q. There are only 2^{n-1} rows in Q and R, and α appears once in each row since M is a latin square. Since α appears $2^{n-2} + c$ times in Q, α appears $2^{n-2} - c$ times in R.

Let D be the quadrant of M diagonally opposite of Q. Let Q' be the quadrant of C_2^n corresponding to Q and let D' be the quadrant of C_2^n corresponding to D. Then Q' and D' contain the same selection of 2^{n-1} symbols of which don't appear in the other 2 quadrants of C_2^n. If we superimpose D' and D, and also superimpose Q' and Q, we will get 2^{n-1} ordered pairs overall whose second component is α. Since α appears $2^{n-2} + c$ times in Q and M is a mate of C_2^n, we will get $2^{n-2} + c$ of those ordered pairs when Q and Q' are superimposed. Hence, we get $2^{n-2} - c$ pairs when D and D' are superimposed. So, α must occur $2^{n-2} - c$ times in D.

The total number of times α appears in M must be the sum of the number of times α appears in each of the quadrants. We have shown α appears $2^{n-2} + c$ times in Q and $2^{n-2} - c$ times in each of V, R, and D. Thus, the total number of times α appears is $(2^{n-2} + c) + 3(2^{n-2} - c) = 2^n - 2c$. Since $c > 0$, α appears less than 2^n times in M. This is a contradiction because M is a latin square. Therefore, for any mate M of C_2^n, each symbol of M appears exactly 2^{n-2} times in each quadrant of M. □

Lemma 2.3. *For $n \geq 2$, every 2×2 primary block of C_2^n takes the form*

$$\begin{matrix} x & y \\ y & x \end{matrix}$$

for some distinct x and y.

Proof. Because C_2^n is a latin square, every 2×2 primary block of C_2^n is also a latin square. In particular, no symbol can be the same as any other symbol in its row or column because it is latin.

Because C_2^n is the Kronecker product of multiple C_2 squares, every primary block will take the form shown above due to the fact that the operation will make copies of C_2 throughout C_2^n. □

Theorem 2.4. *In a mate of C_2^n where $n \geq 2$, each 2×2 primary block takes the form*

$$\begin{matrix} a & b \\ c & d \end{matrix}$$

where a, b, c, and d are pairwise distinct.

Proof. Suppose there is a 2×2 primary block L in a mate of C_2^n of the form

$$\begin{matrix} a & b \\ c & d \end{matrix} .$$

In order to better analyze this structure, we superimpose L with the 2×2 primary block of C_2^n that we found in Lemma 2.3. This will result in a new latin square with the collection of ordered pairs

$$\begin{matrix} a, x & b, y \\ c, y & d, x \end{matrix} .$$

We know the symbols in each row and column in L need to be different in order for the square to be latin. Furthermore, if $d = a$, then an ordered pair would repeat and L wouldn't be a mate. The same problem would arise if $c = b$. Hence, each 2×2 primary block takes the form

$$\begin{matrix} a & b \\ c & d \end{matrix}$$

where a, b, c, and d are pairwise distinct. □

Theorem 2.5. *For $n > 1$, if S is a subsquare of half the size of C_2^n, the complement of S is also a subsquare with the same symbols.*

Proof. Suppose we have a subsquare S of C_2^n that is half the size of C_2^n. Let Σ be the set of symbols of S. Let S^c be the complement of S. We know S^c is a latin array and we want to prove it is a subsquare. We will show S^c is a subsquare with the same symbols as S. We do this by counting. Because S is size $n/2 \times n/2$, S^c will be of size $n/2 \times n/2$. So, there will be $n^2/4$ locations in S^c.

There are $n/2$ symbols in Σ and each symbol of Σ appears n times in C_2^n. Altogether the symbols of Σ appear $n^2/2$ total times in C_2^n. When we find S^c, $n/2$ of each symbol is removed. Thus, $n/2$ of each symbol remain. Therefore, each of the $n/2$ symbols from Σ appears $n/2$ times in S^c. So, we need $n^2/4$ locations in S^c to hold the symbols from Σ.

Because there are $n^2/4$ locations in S^c and we need $n^2/4$ locations in S^c to hold the symbols from Σ, then no other symbols can appear in S^c. Therefore, S^c is a subsquare with the same symbols as S. $\square$

Theorem 2.6. *If $n > 1$ and C_2^n has a subsquare S_1 of size 2^{n-1}, there are three other subsquares S_2, S_3, and S_4 of size 2^{n-1} such that S_1, S_2, S_3, and S_4 cover C_2^n.*

Proof. Suppose $n > 1$ and let C_2^n have a subsquare S_1 of size 2^{n-1}. By Theorem 2.5, we know there is exactly one more subsquare S_2 which has the same symbols as S_1. Because S_1 and S_2 are half the size of C_2^n, half the symbols in every row and column have been used after obtaining S_2. Thus, half of the symbols remain in every row and column. These symbols are disjoint from the symbols already used in S_1 and S_2. Hence, we have another subsquare S_3 of size 2^{n-1}. Again, by Theorem 2.5, we have another subsquare S_4 with the same symbols as S_3. Now every symbol in C_2^n has been used. Four subsquares of size 2^{n-1} now cover C_2^n.

$\square$

Theorem 2.7. *Suppose S is a 2×2 subsquare of C_2^2. For $n > 2$, if we have a mate M of C_2^n, and we pick the region R of M that corresponds to S, then each symbol from 1 to 2^n appears exactly 2^{n-2} times in R.*

Proof. Suppose we have a 2×2 subsquare S of C_2^2 where M is a mate of C_2^n and $n > 2$. Also assume that a region R corresponds to S. We proceed by contradiction.

Assume the symbol α appears $2^{n-2} + c$ times in R when $c > 0$. As a result, the other region in the same row as R is limited to having $2^{n-2} - c$ appearances of α. Additionally, the region in the same column as R is limited to $2^{n-2} - c$ appearances of α. The final region corresponds to the complementary region of R which is limited to $2^{n-2} - c$ appearances.

Thus, the total number of times α appears in all of C_2^n is $(2^{n-2} + c) + 3(2^{n-2} - c) = 2^n - 2c$. Since $c > 0$, α appears less than 2^n times in M. This is a contradiction because M is a latin square. Therefore, for any mate M of C_2^n, each symbol of M appears exactly 2^{n-2} times in each region of M. □

The following theorem is known in the literature. We will use it to establish a maximum size for subsquares of C_2^n and its mates.

Theorem 2.8. *If L is a latin square of size $2n \times 2n$ and S is a proper subsquare of L, then the size of S is no more than n.*

Proof. We proceed by contradiction. By permuting rows and columns, we can assume S is the top left $(n+c) \cdot (n+c)$ region of L where $0 < c < n$. So all of the $n+c$ symbols of S must appear once in each of the remaining $n-c$ rows. They must also appear in the $(n-c) \cdot (n-c)$ region diagonal from S.

So, we need $(n+c) \cdot (n-c)$ occurrences of symbols in $(n-c)^2$ locations. But if $(n-c) \cdot (n+c) = (n+c)^2$, then because $n \neq c$, we can divide by $(n-c)$ to obtain $(n+c) = (n-c)$ which gives $c = 0$. This is a contradiction. □

Theorem 2.9. *Suppose L is a latin square of size $2n \times 2n$. Let M be a mate of the square $C_2 \otimes L$. Then each symbol of M appears n times in each quadrant of M.*

Proof. Suppose L is a latin square of size $2n \times 2n$ and let M be a mate of the square $C_2 \otimes L$. We know $C_2 \otimes L$ will always be of size $4n \times 4n$ since C_2 is size 2×2 and L is size $2n \times 2n$. We proceed by contradiction. Suppose α appears more than n times in some quadrant, Q. Then there exists a c such that $c > 0$ and so α appears $n+c$ times in quadrant Q.

Let V be the quadrant of M above or below Q. There are only n columns in V and Q and α appears once in each column because M is a latin square. Because α appears $n+c$ times in Q, it appears $n-c$ times in V.

Let R be the quadrant of M to the left or right of Q. There are only n rows in Q and R, and α appears once in each row since M is a latin square. Because α appears $n+c$ times in Q, it appears $n-c$ times in R.

Let D be the quadrant of M diagonally opposite of Q. Let Q' be the quadrant of $C_2 \otimes L$ corresponding to Q and let D' be the quadrant of $C_2 \otimes L$ corresponding to D. Then Q' and D' contain the same selection of $n+c$ symbols of which don't appear in the other two quadrants of $C_2 \otimes L$. If we superimpose D' and D, and also superimpose Q' and Q, we will get n ordered pairs overall whose second component is α. Since α appears $n+c$ times in Q and M is a mate of $C_2 \otimes L$, we will get $n+c$ of these ordered pairs when Q and Q' are superimposed. Thus, we get $n-c$ pairs when D and D' are superimposed. So, α must occur $n-c$ times in D.

The total number of times α must appear in M is the sum of the number of times it appears in each quadrant of M. That is, the total number of times α appears in M is $(n+c)+3(n-c)=4n-2c$. Since $c>0$, α appears $4n-2$ times at the most. This is a contradiction because we need α to appear $4n$ times if α is to appear n times in each row of M. $\square$

2.2 Symbol Sets

We proved in Theorem 2.4 that if M is a mate of C_2^3, each 2×2 primary block of M contains 4 distinct symbols. This leads us to make the following definition, which allows us to track which sets of 4 symbols appear in a particular mate.

Definition 2.10. If M is a mate of C_2^3, then $\mathcal{S}(M)$ is the collection of 4-element sets of symbols that appear in the primary 2×2 blocks of M.

For example, consider the following two mates A and B of C_2^3, for which we have

$$\begin{aligned}\mathcal{S}(A) &= \{\{1,2,6,7\},\{3,4,5,8\},\{3,4,5,6\},\{1,2,7,8\}\}\\ \mathcal{S}(B) &= \{\{1,2,5,7\},\{3,4,6,8\},\{1,2,5,6\},\{3,4,7,8\},\\ &\qquad \{4,5,7,8\},\{1,2,3,6\},\{4,5,6,8\},\{1,2,3,7\}\}\end{aligned}$$

$$\begin{array}{cc|cc|cc|cc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8\\
6 & 7 & 8 & 5 & 4 & 3 & 2 & 1\\ \hline
4 & 5 & 2 & 1 & 6 & 7 & 8 & 3\\
3 & 6 & 7 & 8 & 1 & 2 & 5 & 4\\ \hline
7 & 8 & 5 & 6 & 3 & 4 & 1 & 2\\
2 & 1 & 4 & 3 & 8 & 5 & 6 & 7\\ \hline
8 & 3 & 6 & 7 & 2 & 1 & 4 & 5\\
5 & 4 & 1 & 2 & 7 & 8 & 3 & 6
\end{array}$$

Mate A of C_2^3 with $|\mathcal{S}(A)|=4$

$$\begin{array}{cc|cc|cc|cc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8\\
7 & 5 & 8 & 6 & 2 & 1 & 4 & 3\\ \hline
8 & 7 & 6 & 3 & 4 & 5 & 2 & 1\\
5 & 4 & 1 & 2 & 6 & 8 & 3 & 7\\ \hline
3 & 8 & 2 & 1 & 7 & 4 & 5 & 6\\
4 & 6 & 7 & 5 & 8 & 3 & 1 & 2\\ \hline
6 & 3 & 4 & 7 & 1 & 2 & 8 & 5\\
2 & 1 & 5 & 8 & 3 & 7 & 6 & 4
\end{array}$$

Mate B of C_2^3 with $|\mathcal{S}(B)|=8$

A computational search of all mates of C_2^3 shows that if M is a mate then $|\mathcal{S}(M)|$ is 2, 4, 8, or 16. In Table 1, the number of mates which possess each number of distinct primary block sets is shown. In this section, we prove results to explain this computational evidence.

Theorem 2.11. *For any mate of C_2^3, $|\mathcal{S}(M)|\neq 1$.*

Proof. Suppose M is some mate of C_2^3. Assume by contradiction that $\mathcal{S}(M)=1$. Then there is one set of four symbols which makes up each primary block of M. Without loss of generality, suppose this set is $A=\{a,b,c,d\}$. There are only two ways to arrange this set in a

$\lvert\mathcal{S}(M)\rvert$	*Frequency*
2	6,144
4	26,112
8	34,560
16	3,456

Table 1: Frequency Table for $\lvert\mathcal{S}(M)\rvert$

primary block in some 2×8 row or column so that the square would remain latin. Because M is a mate of C_2^3 we know it is of size 8×8. This means there are four 2×2 primary blocks in any 2×8 row or column. However, we can only have two primary blocks in any 2×8 row or column and still have a latin square. So, if we filled any 2×8 row or column with the set A, we would no longer have a latin square. This is a contradiction. Therefore, $\lvert\mathcal{S}(M)\rvert \neq 1$. □

Theorem 2.12. *For any mate of C_2^3, $\lvert\mathcal{S}(M)\rvert \neq 3$.*

Proof. Suppose M is some mate of C_2^3. Assume by contradiction that $\lvert\mathcal{S}(M)\rvert = 3$. Assume the three distinct sets of $\mathcal{S}(M)$ are A, B, and C. Then there are two cases in which any 2×8 row or column in M can be arranged. Since M is a mate of C_2^3 we know it is of size 8×8. This means there are four 2×2 primary blocks in any 2×8 row or column. In Case 1, the rows or columns are arranged with two of any two of A, B, or C. In Case 2, one of the sets are repeated in a row or column with all three sets appearing.

Case 1: Suppose every 2×8 row or column of M is arranged so that exactly two out of the three sets A, B, and C appear, with each of these two appearing twice. Without loss of generality, assume the top row is arranged in this fashion with A and B. Then in some other row, without loss of generality, this arrangement must occur with A and C. Because M is latin, A must contain half the symbols and B must contain the other half. That is, $A^c = B$. Similarly, $A^c = C$. Hence, $B = C$. Thus, there are only two distinct sets that make up the primary blocks of M. This is a contradiction as we assumed there are three distinct sets.

Case 2: Suppose there is a 2×8 column or row in which all three of the sets A, B, and C appear. Without loss of generality, assume the top 2×8 row of M is arranged in this fashion with A appearing twice. We know that eight distinct symbols appear in the first row. Four of these eight symbols must come from A and the other four from $B \cup C$. The symbols of B that do not appear in the top row must appear in the second row, and must do so in C because all eight symbols from A are accounted for. Similarly, the two symbols of C that do not appear in the top row must appear in B, where they must appear in the top row. This shows that $B = C$, which is a contradiction.

In both cases we arrived at a contradiction. Therefore, $|\mathcal{S}(M)| \neq 3$. $\square$

2.3 Constructing Mates Using Transformations

We wish to construct all the mates of C_2^3 where $|\mathcal{S}(M)| = 2$. To explain the construction, we begin with two 2×2 matrices, B_1 and B_2. When $|\mathcal{S}(M)| = 2$, there are a total of three possible choices for the elements within B_1.

$$\begin{array}{ccc} \begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix} & \begin{matrix} 1 & 2 \\ 5 & 6 \end{matrix} & \begin{matrix} 1 & 2 \\ 7 & 8 \end{matrix} \\ \text{Choice 1} & \text{Choice 2} & \text{Choice 3} \end{array}$$

Lemma 2.13. *When $|\mathcal{S}(M)| = 2$, the only possible sets that can fill the top left 2×2 primary block of a mate of C_2^3 are {1,2,3,4}, {1,2,5,6}, and {1,2,7,8}.*

Proof. Suppose we have a semireduced mate M of C_2^3 where $|\mathcal{S}(M)| = 2$. Since M is semireduced, the top row of symbols in M will be in natural order. This will lock in 2 symbols for every primary block in the top row of primary blocks in M. In particular, 1 and 2 will be in the top left primary block, putting 1 and 2 in a set together, call it $\mathcal{S}(A)$. By contradiction, suppose $\mathcal{S}(A)$ is completed with a pair of symbols that aren't 3-4, 5-6, or 7-8. Without loss of generality, suppose the pair 3-6 completes $\mathcal{S}(A)$ so we have $\mathcal{S}(A) = \{1, 2, 3, 6\}$. However, we know there is a set that contains the pair 3-4 since M is semireduced and contains 3-4 in one of its primary blocks. This would mean there are 2 sets which aren't distinct that make up the primary blocks of M. This is a contradiction because $|\mathcal{S}(M)| = 2$ and we must have two distinct sets. Therefore, $\mathcal{S}(A)$ must be completed with one of the pairs of symbols 3-4, 5-6, or 7-8. Thus, $\mathcal{S}(A)$ must be {1,2,3,4}, {1,2,5,6}, or {1,2,7,8}. $\square$

Our construction forms an 8×8 matrix by replacing each entry of a 4×4 latin array, F, with one of these 2×2 matrices in some arrangement. This array is formed by keeping track of which elements exist within each of the primary blocks in the mate regardless of order. In this array, each element appears twice in each row, twice in each column and twice on each diagonal. This type of pattern is uniquely determined from $\mathcal{S}(M)$ and from the assumption that the mate is semireduced.

$$\begin{matrix} B_1 & B_2 & B_1 & B_2 \\ B_2 & B_1 & B_2 & B_1 \\ B_2 & B_1 & B_2 & B_1 \\ B_1 & B_2 & B_1 & B_2 \end{matrix}$$

Array F

$$\begin{array}{cccc} B_1 & B_2 & B_1 & B_2 \\ B_2 & B_1 & B_2 & B_1 \\ B_2 & B_1 & B_2 & B_1 \\ B_1 & B_2 & B_1 & B_2 \end{array} \rightarrow \begin{array}{cccc} B_1 & B_2 & rB_1 & rB_2 \\ B_2 & cB_1 & B_2 & crB_1 \\ dB_2 & dB_1 & rdB_2 & rdB_1 \\ cdB_1 & rdB_2 & crdB_1 & dB_2 \end{array}$$

Figure 2: Transformations on F

$$\begin{array}{cc|cc|cc|cc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \\ \hline 3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \\ 4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 \\ \hline 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \\ 6 & 5 & 8 & 7 & 2 & 1 & 4 & 3 \\ \hline 7 & 8 & 5 & 6 & 3 & 4 & 1 & 2 \\ 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{array} \qquad \begin{array}{cc|cc|cc|cc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \\ \hline 7 & 8 & 2 & 1 & 3 & 4 & 6 & 5 \\ 3 & 4 & 6 & 5 & 7 & 8 & 2 & 1 \\ \hline 4 & 7 & 5 & 2 & 8 & 3 & 1 & 6 \\ 8 & 3 & 1 & 6 & 4 & 7 & 5 & 2 \\ \hline 2 & 5 & 8 & 3 & 6 & 1 & 4 & 7 \\ 6 & 1 & 4 & 7 & 2 & 5 & 8 & 3 \end{array}$$

C_2^3 Mate of C_2^3

Figure 3: Mate produced by Transformations

To help explain, we proceed by example. By selecting Choice 2 to be B_1, a natural B_2 follows.

$$\begin{array}{cc} 1 & 2 \\ 5 & 6 \end{array} \qquad \begin{array}{cc} 3 & 4 \\ 7 & 8 \end{array}$$

B_1 B_2

Through row, column, and diagonal switches, each set of these two matrices can be transformed to create a mate for C_2^3. To keep track of the matrix manipulations, rB is the matrix obtained by swapping the rows of matrix B and cB is the matrix obtained by swapping the columns of matrix B. In order to keep the diagonal switches consistent, dB is the matrix obtained by swapping the two elements in the right column within B. The transformation from the matrix F to the completed construction is shown in Figure 2. This process produces the mate shown in Figure 3.

There are a total of 24 possible transformations that can be made to a matrix A, eight of which are distinct. The following group presentation helps show that this is an 8-element subgroup of S_4 where e is the identity and the elements are not commutative.

$$r, c, d \left| \begin{array}{l} r^2 = c^2 = d^2 = e \\ dr = rd \\ dc = crd \\ cr = rc \end{array} \right.$$

As a result, large-element manipulations can be simplified using this identity property and the matrix manipulations can be limited to these eight distinct transformations. For example, through the associative property, the transformation $cdrdcrr = cd(rd)c(rr)$. Since rr is the identity and $rd{=}dr$, we can substitute back in and re-associate so that $cd(rd)c(rr) = c(dd)r(e)$. Because dd is also the identity, we can conclude that this original transformation is equivalent to cr. That being said, any string of transformations can be simplified to one of these eight possible manipulations within a primary 2×2 block:

$$\begin{array}{cccc} \begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix} & \begin{matrix} 3 & 4 \\ 1 & 2 \end{matrix} & \begin{matrix} 2 & 1 \\ 4 & 3 \end{matrix} & \begin{matrix} 1 & 4 \\ 3 & 2 \end{matrix} \\ B & rB & cB & dB \\ \\ \begin{matrix} 4 & 3 \\ 2 & 1 \end{matrix} & \begin{matrix} 4 & 1 \\ 2 & 3 \end{matrix} & \begin{matrix} 3 & 2 \\ 1 & 4 \end{matrix} & \begin{matrix} 2 & 3 \\ 4 & 1 \end{matrix} \\ rcB & cdB & rdB & crdB \end{array}.$$

3 Mates of C_2^3 with two symbol sets

In this section we show that, in the case where $|\mathcal{S}(M)| = 2$, there are six forms a mate of C_2^3 can take; see Lemma 3.2. We show that there is a bijection between the set of mates in any one of these forms and the set of mates in any other of these forms. The six forms are shown in Figure 3.

We then combinatorially enumerate the $6,144 \cdot 8!$ mates that have $|\mathcal{S}(M)| = 2$. To make this enumeration, we use "dot diagrams" to show how symbols in a mate of C_2^n can be arranged. There are 20 diagrams witth dot placements that can produce a mate. From the dot diagrams, we get equations describing the relationships between certain primary blocks. From these equations, we construct graphs which show the possible solutions to a dot diagram. In other words, the graphs and equations allow us to count how many mates can be made from a particular diagram.

We name the six forms $1a$, $1b$, $2a$, $2b$, $3a$, and $3b$. The forms $1a$ and $1b$ focus on the pattern where there are only row flips, the forms $2a$ and $2b$ focus only on column flips, and the forms $3a$ and $3b$ focus on diagonal changes. In a particular mate, for example, B_0 is the set that contains $\{1, 2, 3, 4\}$ in any order and B_1 is the set that contains $\{5, 6, 7, 8\}$ in any order.

Lemma 3.1. *Suppose L_1 is a latin array using cells marked B_1 in form 1a with the symbols $\{5, 6, 7, 8\}$ such that each 2×2 primary block contains all four symbols and no pair appears twice when L_1 is superimposed with C_2^3. Suppose L_0 is a latin array using the cells marked B_0*

$$\begin{array}{cc|cc} B_0 & B_0 & B_1 & B_1 \\ B_1 & B_1 & B_0 & B_0 \\ \hline B_0 & B_0 & B_1 & B_1 \\ B_1 & B_1 & B_0 & B_0 \end{array} \qquad \begin{array}{cc|cc} B_0 & B_1 & B_0 & B_1 \\ B_0 & B_1 & B_0 & B_1 \\ \hline B_1 & B_0 & B_1 & B_0 \\ B_1 & B_0 & B_1 & B_0 \end{array} \qquad \begin{array}{cc|cc} B_0 & B_1 & B_0 & B_1 \\ B_1 & B_0 & B_1 & B_0 \\ \hline B_1 & B_0 & B_1 & B_0 \\ B_0 & B_1 & B_0 & B_1 \end{array}$$

$$1a \qquad\qquad 2a \qquad\qquad 3a$$

$$\begin{array}{cc|cc} B_0 & B_0 & B_1 & B_1 \\ B_1 & B_1 & B_0 & B_0 \\ \hline B_1 & B_1 & B_0 & B_0 \\ B_0 & B_0 & B_1 & B_1 \end{array} \qquad \begin{array}{cc|cc} B_0 & B_1 & B_1 & B_0 \\ B_0 & B_1 & B_1 & B_0 \\ \hline B_1 & B_0 & B_0 & B_1 \\ B_1 & B_0 & B_0 & B_1 \end{array} \qquad \begin{array}{cc|cc} B_0 & B_1 & B_1 & B_0 \\ B_1 & B_0 & B_0 & B_1 \\ \hline B_0 & B_1 & B_1 & B_0 \\ B_1 & B_0 & B_0 & B_1 \end{array}$$

$$1b \qquad\qquad 2b \qquad\qquad 3b$$

Figure 4: Forms for mates of C_2^3 with $|\mathcal{S}(M)| = 2$

in form 1a with the symbols $\{1, 2, 3, 4\}$, such that each primary 2×2 block uses all four of those symbols and no pair appears twice when L_0 is superimposed with C_2^3. Then, if we put L_0 and L_1 together to make a latin square, the resulting square will be a mate of C_2^3.

Proof. Suppose L_1 is a latin array using cells marked B_1 in form 1a with the symbols $\{5, 6, 7, 8\}$ such that each 2×2 primary block contains all four symbols and no pair appears twice when L_1 is superimposed with C_2^3. Suppose L_0 is a latin array using the cells marked B_0 in form 1a with the symbols $\{1, 2, 3, 4\}$, such that each primary 2×2 block uses all four of those symbols and no pair appears twice when L_0 is superimposed with C_2^3.

Without loss of generality, let's begin with the latin array L_1 in the form 1a, call it H. Now if we put L_0 into H we will be adding two primary blocks to every row and column whose symbols are disjoint from the symbols of L_1. Adding two primary blocks who were already latin to every row and column of H who already had two primary blocks of which were latin will leave us with a latin square.

By our supposition, L_1 and L_0 have the property that every pair appears when superimposed with C_2^3. Then L_1 forms 32 pairs when superimposed with C_2^3 since it contains 32 symbols. Similarly, L_2 forms 32 pairs when superimposed with C_2^3. In total, we have 64 pairs which is what we need. Since we are combining the 32 pairs from L_1 and the 32 pairs from L_0 into the form 1a to form the square H, we will get all possible 64 pairs since the symbols of L_1 and L_0 are disjoint.

Therefore, the resulting square H is a mate of C_2^3. □

Lemma 3.2. *Forms 1a, 1b, 2a, 2b, 3a, and 3b are the only ways to make an array of size 4×4 with two symbols $\{B_0, B_1\}$ in which each symbol appears twice in each column, twice in each row, twice on each* *diagonal and twice on each 2×2 subsquare of C_2^2, and in which the top left cell is a B_0.*

Proof. Assume we wish to construct all the possible forms to make an array that is described in Lemma 3.2. We know the top left cell of any mate of C_2^3 where $|\mathcal{S}(M)| = 2$ will be made up of the set B_0. We also know that each symbol appears twice on each 2×2 subsquare of C_2^2, so once B_0 is placed in the top left cell, there are three possible positions for the other B_0 to take.

$$\begin{array}{ccc}
\begin{array}{cc|cc} B_0 & B_0 & & \\ & & & \\ \hline & & & \\ & & & \end{array} &
\begin{array}{cc|cc} B_0 & & & \\ B_0 & & & \\ \hline & & & \\ & & & \end{array} &
\begin{array}{cc|cc} B_0 & & & \\ & B_0 & & \\ \hline & & & \\ & & & \end{array} \\
1 & 2 & 3
\end{array}$$

To further complete these arrays, we know that wherever the B_0's go, the B_1's need to take the other positions in that top left 2×2 subsquare. Furthermore, each symbol needs to appear twice in each row and twice in each column. So, where there are already two B_0's in a row or column, we can complete it by inserting B_1's in the blank spaces. Similarly, where there are already two B_1's in a row or column, we can complete it by inserting B_0's in the blank spaces. We also know that two B_0's and two B_1's need to exist on each diagonal so where there are already two B_0's on the diagonal in form 3, we can complete it by inserting B_1's in the blank spaces.

$$\begin{array}{ccc}
\begin{array}{cc|cc} B_0 & B_0 & B_1 & B_1 \\ B_1 & B_1 & B_0 & B_0 \\ \hline & & & \\ & & & \end{array} &
\begin{array}{cc|cc} B_0 & B_1 & & \\ B_0 & B_1 & & \\ \hline B_1 & B_0 & & \\ B_1 & B_0 & & \end{array} &
\begin{array}{cc|cc} B_0 & B_1 & & \\ B_1 & B_0 & & \\ \hline & & B_1 & \\ & & & B_1 \end{array} \\
1 & 2 & 3
\end{array}$$

In order to further complete these forms, we are provided with more choices. In form 1, the top two rows were completely determined by the placement of B_0 in the first cell. However, the bottom half of the 4×4 array is not locked in. We are able to make choices in our placement of blocks. This exists in forms 2 and 3 as well. We know each 2×2 subsquare of C_2^2 contains two B_0's and two B_1's so we have two choices for each of the 3 forms in order to create these subsquares.

$$\begin{array}{ccc}
\begin{array}{cc|cc} B_0 & B_0 & B_1 & B_1 \\ B_1 & B_1 & B_0 & B_0 \\ \hline & B_0 & B_1 & \\ & & & \end{array} &
\begin{array}{cc|cc} B_0 & B_1 & B_0 & B_1 \\ B_0 & B_1 & B_0 & B_1 \\ \hline & B_0 & B_1 & \\ & & & \end{array} &
\begin{array}{cc|cc} B_0 & B_1 & B_0 & \\ B_1 & B_0 & B_1 & \\ \hline & & B_1 & B_0 \\ & & B_0 & B_1 \end{array} \\
1a & 2a & 3a
\end{array}$$

$$
\begin{array}{cc|cc}
B_0 & B_0 & B_1 & B_1 \\
B_1 & B_1 & B_0 & B_0 \\
\hline
 & B_1 & B_0 & \\
 & & & \\
\end{array}
\qquad
\begin{array}{cc|cc}
B_0 & B_1 & B_1 & B_0 \\
B_0 & B_1 & B_1 & B_0 \\
\hline
 & B_0 & B_0 & \\
 & & & \\
\end{array}
\qquad
\begin{array}{cc|cc}
B_0 & B_1 & B_1 & \\
B_1 & B_0 & B_0 & \\
\hline
 & & B_1 & B_0 \\
 & & B_0 & B_1 \\
\end{array}
$$

$1b$ $\qquad$ $2b$ $\qquad$ $3b$

Forms for mates of C_2^3

Once these subsquares are filled in for forms 1a, 1b, 2a, 2b, 3a and 3b, the rest of the square is uniquely determined. □

Lemma 3.3. *If we have any mate M of C_2^3 with $|S(M)| = 2$, its 2×2 primary blocks will satisfy the hypotheses of Lemma 3.2 and so it will be in form 1a, 1b, 2a, 2b, 3a, or 3b.*

Proof. Suppose we have a mate M of C_2^3 with $|S(M)| = 2$. Then there are two sets B_0 and B_1 which make up the 2×2 primary blocks of M. Let the primary blocks of C_2^3 be represented by A_1, A_2, A_3, and A_4 and be in form of

$$
\begin{array}{c|c|c|c}
A_1 & A_2 & A_3 & A_4 \\
\hline
A_2 & A_1 & A_4 & A_3 \\
\hline
A_3 & A_4 & A_1 & A_2 \\
\hline
A_4 & A_3 & A_2 & A_1
\end{array}.
$$

Because there only only two sets B_0 and B_1 which make up the primary blocks of M, neither B_0 and B_1 could appear more than twice in any row or column because M would no longer be latin. Thus, each of B_0 and B_1 appear twice in every row and column.

Also, no more than two of B_0 and B_1 could appear more than twice in a diagonal because we would get 12 ordered pairs with the same first symbol when M and C_2^3 are superimposed. We can only have 8 ordered pairs with the same symbol since there are only 8 symbols. Thus, there must be 2 of each B_0 and B_1 in each diagonal.

By Theorem 2.7, there are 2 symbols in any region of M that corresponds to any 2×2 subsquare of the representation of C_2^3 which is in the form of C_2^2. Hence, if B_0 and B_1 appeared more than twice in any region then a pair would be repeated which can't happen since M is a mate. Thus, there must be 2 of each B_0 and B_1 on each 2×2 subsquare of C_2^2. □

Theorem 3.4. *For any mate M of C_2^3 where M has form $1a$ or $1b$, the top half of M uniquely determines the bottom half of M.*

Proof. Suppose M is a mate of C_2^3 that is in form $1a$. Let α be the number of elements in the set B_0 and β be the number of elements in the set B_1. Therefore, $\alpha = 4$ and $\beta = 4$. We proceed by process of elimination.

Suppose we try to fill a cell c in the bottom half of M. We start with 8 symbols that could fill c. If we look at the location that corresponds to c in the mate form, it tells us which symbol set makes up the primary block that contains c. Without loss of generality, let's say this symbol set is B_0. This reduces our choices to 4 possible symbols because $\alpha = 4$. In all mates of form $1a$, the cells above c already contain two elements from B_0 and two elements from B_1. Therefore, because of the latin property, we must eliminate these two choices from our list of possible elements to fill c. As a result, $\alpha = 2$.

Next, we notice that the 2×2 primary block that contains c corresponds to a primary block in C_2^3 that contains some elements $\{x, y\}$. So if we look at what regions in the top half of M correspond to the $\{x, y\}$ primary blocks in C_2^3, we can eliminate more choices.

In every case, four of the eight possible ordered pairs appear in the top half of M, two of which are from B_0 and two from B_1. These ordered pairs look like (x, a), (y, b), (y, c), and (x, d) because of the forms described in Lemma 2.3 and Theorem 2.1. In every mate of this form, one of those elements will appear in the column meaning it was already eliminated from the possibilities. The other element from B_0 can be eliminated as a choice, leaving $\alpha = 1$. Therefore, the set B_0 only contains one specific possibility to fill the cell. $\square$

Theorem 3.4 does not hold for mates of C_2^3 where the mate is in form $2a$, $2b$, $3a$ or $3b$. The top half filled in is not enough to uniquely determine the rest of the square. Two mates of form $3a$ can have the same top half, but a unique bottom half.

$$\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
6 & 5 & 7 & 8 & 2 & 1 & 3 & 4 \\
4 & 3 & 6 & 5 & 8 & 7 & 2 & 1 \\
8 & 7 & 1 & 2 & 4 & 3 & 5 & 6 \\ \hline
7 & 8 & 2 & 1 & 3 & 4 & 6 & 5 \\
3 & 4 & 5 & 6 & 7 & 8 & 1 & 2 \\
5 & 6 & 8 & 7 & 1 & 2 & 4 & 3 \\
2 & 1 & 4 & 3 & 6 & 5 & 8 & 7
\end{array}
\qquad
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
6 & 5 & 7 & 8 & 2 & 1 & 3 & 4 \\
4 & 3 & 6 & 5 & 8 & 7 & 2 & 1 \\
8 & 7 & 1 & 2 & 4 & 3 & 5 & 6 \\ \hline
7 & 8 & 2 & 6 & 3 & 4 & 1 & 5 \\
3 & 4 & 5 & 1 & 7 & 8 & 6 & 2 \\
5 & 1 & 8 & 7 & 6 & 2 & 4 & 3 \\
2 & 6 & 4 & 3 & 1 & 5 & 8 & 7
\end{array}$$

Mate of C_2^3 in $3a$ — Different Mate of C_2^3 in $3a$

Conjecture 3.5. *For any mate M of C_2^3 where M has form $1b$, $2a$, $2b$, $3a$ or $3b$, the bottom right quadrant of M is uniquely determined by the remainder of M.*

3.1 Connections Between Forms

We want to combinatorially enumerate the mates of C_2^3 starting with the case where $|\mathcal{S}(M)| = 2$. There are six forms which occur in this case: $1a$, $1b$, $2a$, $2b$, $3a$, and $3b$, and each has the same number of mates. We show that there are 32 ways to fill in the primary blocks of a mate that are made up of the set B_1 and 32 ways to fill in the primary blocks made up of B_0. By Lemma 3.1, we know these regions are independent from each other. So, once we come up with an enumeration all 32 options for B_1, then we know we can combine them with the similar options B_0 to create a mate.

Some of the forms are linked by transpose which is a bijection between all mates in those forms. The transpose of form $1a$ is form $2a$ so if there are 32 mates in form $1a$, we know there are the same number of mates in form $2a$. Similarly, if we start with a mate in form $2a$ and take its transpose, we will get a mate of form $1a$. This means it is also surjective. In addition, the transpose of form $1b$ is form $2b$ and the transpose of form $3a$ is form $3b$.

$$\begin{array}{rl} 1a \rightarrow \text{transpose} \rightarrow & 2a \\ 1b \rightarrow \text{transpose} \rightarrow & 2b \\ 3a \rightarrow \text{transpose} \rightarrow & 3b \end{array}$$

We want to be able to find a bijection between all mates in form $1a$, $1b$ and $3a$. That way, we only need to prove one form has 32 mates in order to prove that all six forms have 32 mates. We also know that some of the forms are linked by row and column flips. So, if we take a mate in a given form and perform the same row/column flips on both C_2^3 and the mate, then it will still be a mate. Further, if we organize these flips in such a way that get us back to C_2^3 but produces a mate in a different form, then there is a bijection between the starting form and the ending form.

Lemma 3.6. *There is a bijection between the set of mates of C_2^3 of form $1a$ and the set of mates of C_2^3 of form $1b$.*

Proof. Say we are given a mate in form $1a$ and we want to find a series of flips that produces a mate in form $1b$ and brings C_2^3 back to its original position.

$$\begin{array}{cc|cc} C & C & D & D \\ D & D & C & C \\ \hline C & C & D & D \\ D & D & C & C \end{array} \qquad \begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ \hline 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{array}$$

Mate in form $1a$ Primary blocks of C_2^3

We know we can flip rows in both a mate and C_2^3 and get two squares that are mates. One of them may not be C_2^3 anymore, but at least this manipulation allows us to preserve the mate property. So, if we flip the first row of primary blocks of the mate with the second row of primary blocks of the mate and we flip the first row of of primary blocks of C_2^3

with the second row of primary blocks of C_2^3, we get two squares that are mates.

$$\begin{array}{cc|cc} D & D & C & C \\ C & C & D & D \\ \hline C & C & D & D \\ D & D & C & C \end{array} \qquad \begin{array}{cc|cc} 2 & 1 & 4 & 3 \\ 1 & 2 & 3 & 4 \\ \hline 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{array}$$

Square 1 Square 2

This row flip allowed our first square to take the form of $1b$ which is what we wanted. However, we also want the second square to be in the same form of C_2^3 so we can draw a connection between the mates of form $1a$ and mates of form $1b$. Because Square 1 is in form $1b$ but Square 2 is not C_2^3, our next step is to flip the first and second columns of Square 1 and to flip the first and second columns of Square 2.

$$\begin{array}{cc|cc} D & D & C & C \\ C & C & D & D \\ \hline C & C & D & D \\ D & D & C & C \end{array} \qquad \begin{array}{cc|cc} 1 & 2 & 4 & 3 \\ 2 & 1 & 3 & 4 \\ \hline 4 & 3 & 1 & 2 \\ 3 & 4 & 2 & 1 \end{array}$$

Square 1 Square 2

By permuting the symbols in Square 2 such that the 3's becomes 4's and the 4's become 3's, we can get Square 2 to be back to C_2^3 while preserving the matehood. Because this process always holds, there is a bijection between all mates in form $1a$ and all mates in form $1b$.

$$\begin{array}{cc|cc} D & D & C & C \\ C & C & D & D \\ \hline C & C & D & D \\ D & D & C & C \end{array} \qquad \begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ \hline 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{array}$$

Mate in form $1b$ Primary blocks of C_2^3

$\square$

Lemma 3.7. *There is a bijection between the set of mates of C_2^3 of form $3a$ and the set of mates of C_2^3 of form $1b$.*

Proof. We know there is a bijection between all mates of form $1a$ and all mates of form $1b$. So, if we can find a bijection between all mates of form $1b$ and all mates of form $3a$, there is a bijection between $1a$ and $3a$. Say we are given a mate in form $3a$ and we want to find a series of flips that produces a mate in form $1b$ and brings C_2^3 back to its original position.

$$\begin{array}{cc|cc} C & D & C & D \\ D & C & D & C \\ \hline D & C & D & C \\ C & D & C & D \end{array} \qquad \begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ \hline 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{array}$$

Mate in form $3a$ Primary blocks of C_2^3

We know we can flip rows in both a mate and C_2^3 and get two squares that are mates. So, if we flip the two middle columns of the mate and we flip the two middle columns of C_2^3, we get two squares that are mates.

$$\begin{array}{cc|cc} C & C & D & D \\ D & D & C & C \\ \hline D & D & C & C \\ C & C & D & D \end{array} \qquad \begin{array}{cc|cc} 1 & 3 & 2 & 4 \\ 2 & 4 & 1 & 3 \\ \hline 3 & 1 & 4 & 2 \\ 4 & 2 & 3 & 1 \end{array}$$

Square 1 Square 2

This row flip allowed our first square to take the form of $1b$ which is what we wanted. However, we also want the second square to be in the same form of C_2^3 so we can draw a connection between the mates of form $1a$ and mates of form $1b$. Because Square 1 is in form $1b$ but Square 2 is not C_2^3, our next step is to flip the middle two rows of Square 1 and to flip the middle two rows of Square 2.

$$\begin{array}{cc|cc} D & D & C & C \\ C & C & D & D \\ \hline C & C & D & D \\ D & D & C & C \end{array} \qquad \begin{array}{cc|cc} 1 & 3 & 2 & 4 \\ 3 & 1 & 4 & 2 \\ \hline 2 & 4 & 1 & 3 \\ 4 & 2 & 3 & 1 \end{array}$$

Square 1 Square 2

By permuting the symbols in Square 2 such that the 2's becomes 3's and the 3's become 2's, we can get Square 2 to be back to C_2^3 while preserving the matehood. Because this process always holds, there is a bijection between all mates in form $3a$ and all mates in form $1b$.

$$\begin{array}{cc|cc} D & D & C & C \\ C & C & D & D \\ \hline C & C & D & D \\ D & D & C & C \end{array} \qquad \begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ \hline 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{array}$$

Mate in form $1a$ Primary blocks of C_2^3

$\square$

Because there is a bijection between all of the forms, they all have the same number of mates. In addition, we can be given a mate of any form and the whole process can be run backwards which means it is surjective.

Theorem 3.8. *Each of the six forms has the same number of mates of* C_2^3 *as each of the other forms.*

The following diagram illustrates the bijections between the forms.

Form $1a \to$ Transpose $\to$ Form $2a$
$\downarrow$
Lemma 3.6
$\downarrow$
Form $1b \to$ Transpose $\to$ Form $2b$
$\downarrow$
Lemma 3.7
$\downarrow$
Form $3a \to$ Transpose $\to$ Form $3b$

3.2 Completing the Enumeration

We use dot diagrams to show how the symbols in a mate of C_2^3 can be placed and oriented. In addition, we used dot diagrams to prove that each of the forms have 32 possible mates. Like-shaped dots represent the two-element pairs {5,6} or {7,8}. Arrows from dot to dot always point from the "smaller" symbol to the "larger" symbols. Arrows that point down or right are considered positive and arrows that point up or left are considered negative. Because these squares are mates of C_2^3, certain blocks correspond to one another in order to maintain their mate property.

For example, the regions A and H correspond in form $1a$ because they both contain the symbol set {5,6} in C_2^3. In addition, certain blocks correspond to one another to maintain their latin property. For example, the arrows in A need to point in the opposite direction as the arrows in E because there would be two symbols in the same column if they did not. Therefore, A and E correspond in form $1a$. To help illustrate the relationships between the primary blocks within a certain form, we labeled each region with a letter.

		A	B
C	D		
		E	F
G	H		

Regions in form $1a$

We developed dot diagrams to find a general construction that shows how the symbols in a mate of $(C_2)^3$ can be placed and oriented. In form $1a$, we know the top row of the primary block A will be {5, 6} and the top row of the primary block B will be {7, 8} because the mate is semireduced. Suppose we are trying to create a dot diagram in form $1a$. The circle dots and arrows represent the positioning of the elements {5, 6} and the square dots and arrows represent the positioning of the elements {7, 8}. Because the arrow points from lowest to highest, the circle arrow in A will point right from 5 to 6 and the circle arrow in B will point right from 5 to 6.

Next, we know that there can't be three dots in one row or in one column because if a set of two elements needs to fill three cells, then one of them will have to get repeated at least once and that would make the mate not latin. Also, if two arrows are pointing in the same direction, then two of the same elements are in the same column and then the square wouldn't be latin. So, we have several choices for where our next set of circle dots could go. In E, the circle dots could either go horizontally on the top or horizontally on the bottom. Once this decision is made, the dot placement in F is determined because there has to be two dots in one row. So, if the dots in E are on the top, then the dots in F are on the bottom. If the dots in E are on the bottom then the dots in F are on the top. We know the dots in E cannot be vertical because if they were, there would be three dots in one column.

We have another decision for the placement of the dots in C. The dots can either be horizontal on the top, horizontal on the bottom, vertical on the right or vertical on the left. If the dots in C are horizontal, the dots in D are determined. If the dots in C are vertical, the dots in G are determined.

If the dots in C are horizontal, we know the dots in G will be horizontal too because there can't be three dots in one column. However, we can decide if we want the dots in G to be on the top or on the bottom. If the dots in C are vertical, we know the dots in D will be vertical too because there can't be three dots in one row. However, we can decide if we want the dots in D to be on the left or on the right. Overall, there are 24 possible dot diagrams in form $1a$. Some of these produce one mate, some produce more than one mate, and some produce no mates.

In Figure 5, the circle dots in E are horizontal on the bottom, the circle dots in C are vertical on the right and the circle dots in D are vertical on the right. This makes the circle dots in F horizontal on the top and the circle dots in G vertical on the left. We know the regions B and G correspond in C_2^3 because they both contain the elements $\{7, 8\}$ while also being in a region that corresponds to B_1 in the mate form. So, in the region B of the dot diagram, the circle arrow points right. In the same region of C_2^3, the same arrow would point from 7 to 8. This is representative of the ordered pairs (7,5) and (8,6) such that an opposite orientation would give the pairs (7,6) and (8,5). In these ordered pairs, the first element comes from C_2^3 and the second element comes from the corresponding element in the mate.

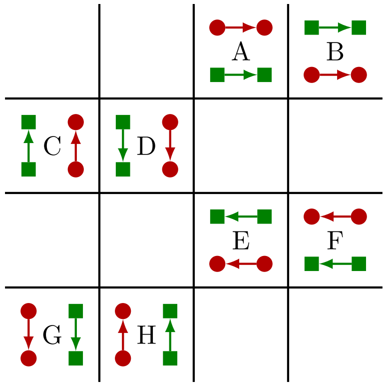
Because B and G correspond, whatever ordered pair is created in B, the opposite pairing should happen in G in order to create every possible ordered pair $(7,5)$, $(8,5)$, $(7,6)$, and $(8,6)$ for the circle arrows. So, the circle arrow is pointing right from 8 to 7 in the region B of C_2^3 creating the ordered pairs $(8,5)$ and $(7,6)$. In order to create the other two ordered pairs, the arrow should point from 7 to 8 in G of C_2^3. This is represented by a circle arrow pointing down in G which creates the rest of the ordered pairs $(7, 5)$ and $(8, 6)$. This same process can be repeated for each corresponding region.

Once the dot diagrams are obtained, we get 8 equations using the blocks that correspond to each other. As stated in Section 3.2, we defined arrows that point down or to the right as positive and arrows that point up or to the left as negative. If two arrows are both positive or both negative, then the primary blocks containing the arrows are equal to each other. If one arrow is positive and one is negative between two blocks that correspond, then one block is equal to the negative of the other. In Figure 5, we see that $A = -E$ because the circle arrow in A is pointing to the right and the circle arrow in E is pointing to the left to keep the square latin. This is also the case for regions B and F, C and D, and G and H. We also know that there is a relationship between the regions that contain the same elements in C_2^3. So, for a mate in form $1a$, we look for the equations between $A-H$, $B-G$, $C-F$, and $D-E$ because they correspond to each other due to matehood. We can only get equations from blocks which correspond to each other. Because the circle arrow in B is pointing to the right while the circle arrow in G is pointing to the down, we get $B = G$. For similar reasons, the other regions' equations are $A = -H$, $D = -E$, and $C = F$.

From the equations we can make a graph. The graph is formed by making vertices and labeling them with the names of the primary blocks. Edges are then drawn between vertices that correspond to each other. If one vertex is the negative of the vertex it is connected to, then a negative sign is put along the edge to show this relationship. There must be an even number of negatives along the edges in order for the cycle to get all the way around and back to where you started. In Figure 5, the graph makes an entire cycle because there are no choices that can be made to the orientation of the arrows once we know the square is semireduced. If a change is made by switching a negative to a positive, every other edge in the cycle would be changed as a result.

This isn't always the case, however. In Figure 6, there are two disjoint cycles because some regions are locked in due to the semireduced regions but others allow for manipulation. The region A will always point to the right because 5 and 6 are paired together while 7 and 8 are paired together. This makes the signs on the disjoint graph containing A locked in place, but the signs on the other disjoint graph can be switched to produce a different mate. The circle arrows can be flipped in the opposite direction which produces a different mate and the square arrows can be flipped in the opposite direction to produce a different mate. Because of this, the set of equations has four solutions.

There are some dot diagrams which provide no solution. In Figure 7, there is a contradiction between the latin property and the mate property. The dots in A and H are both horizontal, but there is no

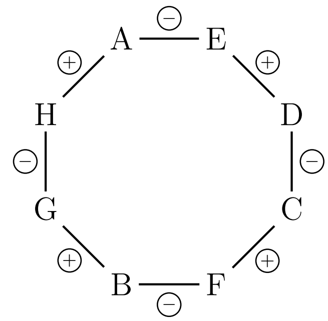


Dot diagram

$A = -E$
$B = -F$
$C = -D$
$G = -H$

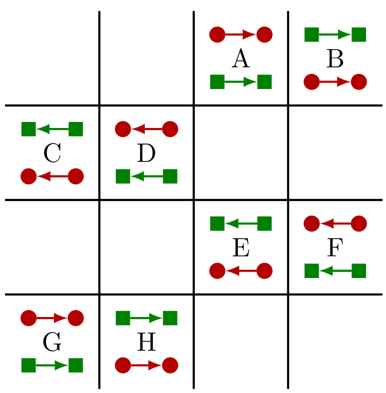
$A = -H$
$B = G$
$C = F$
$D = -E$

Equations



Dependence Graph

Figure 5: In this subcase of form 1a, the equations lead to a graph with a single 8-cycle. The orientation of A then determines the orientation of each of the other blocks, so there is exactly one solution.

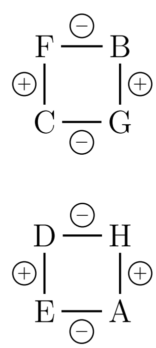


Dot diagram

$A = -E$
$B = -F$
$C = -G$
$D = -H$

$A = H$
$B = G$
$C = F$
$D = E$

Equations



Dependence Graph

Figure 6: In this subcase of form 1a, the equations lead to a graph with two 4-cycles. The orientation of A then determines the orientation of each of the other blocks in that 4-cycles. The orientation of block B then determines the orientation of the other four blocks. There are two different directions the circle arrows can point and two different directions the square arrows can point, resulting in four solutions

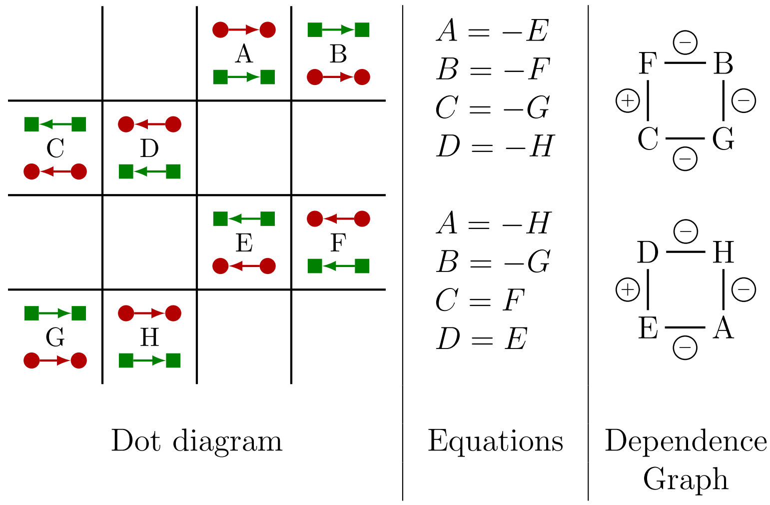


Figure 7: In this subcase of form 1a, the equations lead to a graph with two 4-cycles. The orientation of A then determines the orientation of each of the other blocks in that 4-cycle. The orientation of B then determines the orientation of the other blocks. These orientations create a pair of graphs with an odd number of negatives around the edges, resulting in no solution.

way to orient the arrows that creates both a latin square and a mate. Because the ordered pairs created in A are $(5,5)$ and $(6,6)$, then we need the corresponding region H to create the ordered pairs $(5,6)$ and $(6,5)$. In order to do this, the arrow needs to point to the left. Next, because the regions H and D correspond due to the latin property, their arrows go in opposite directions. Therefore, if the arrow in H is pointing right, the arrow in D points left. Next, the ordered pairs created in D correspond to the ordered pairs in E. So the arrow pointing left in D creates the ordered pairs $(1,6)$ and $(2,5)$ so the pairs in E should be $(1,5)$ and $(2,6)$. In order to do this, the circle arrow in E needs to point right. However, the arrows in A and E cannot point in the same direction because the square will not be latin. Therefore, in this case, the square cannot be both latin and a mate at the same time.

The equations made from this graph confirm that this has no solution. Because of the latin property, $A = -E$, $B = -F$, $C = -G$, and $D = -H$. The mate relationships give us the equations $A = -H$, $B = -G$, $C = F$, and $D = E$. When we string these equations together, we see that $A = -H = D = E = -A$. Clearly, $A \neq -A$ so this set of equations has no solution. We can also see from the graph that there is no solution because there needs to be an even number of negative signs along the edges. In the dependence graph section of the figure, there are three negative signs with one disjoint graph and three negative signs with the other. Therefore, there is no solution.

Using the same processes illustrated in Section 3.2, we constructed all of the possible dot diagrams for form $1a$ by hand and proved there are 32 possible mates. Because of the bijections described in Section 3.1, if there are 32 mates in form $1a$, there are 32 mates for all of the other forms. We have enumerated 32 mates for the B_0 fillings and 32 mates for the B_1 fillings in the mate form. Therefore, there are $(32)^2 \cdot 6 = 6,144$ mates of C_2^3 in which $|\mathcal{S}(M)| = 2$.

4 Enumerating regular mates of C_k^n

In this section, we focus on mates of C_3^2 and, more generally, C_k^n for $k \geq 3$ and $n \geq 2$. We found that C_3^2 has $12,445,836 \cdot 9!$ mates through a computational search, but we are only able to enumerate some of these mates.

We first prove some properties of C_3^2 and its mates in which every 3×3 primary block is a latin square. We then use these properties to enumerate a family of $6^6 \cdot 9!$ mates of C_3^2.

We then generalize this work to enumerate certain mates of C_k^n for $n \geq 2$ and $k \geq 3$. This enumeration establishes an asymptotic lower bound for the number of mates that a latin square of a given size can have.

4.1 Mates of C_3^2

For mates of C_3^2 in which every 3×3 primary block is a latin square, we prove that the primary blocks will look like $\widehat{C}_3$, that the primary blocks are arranged in the form of $\widehat{C}_3$, and that the order of $S(M)$ equals 3. We then enumerate a family of $6^6 \cdot 9!$ mates in which every 3×3 primary block is a latin square by using these properties to prove the existence of a bijection between sets of symbol permutations and latin squares of size 9. Through a computational search, we discovered that C_3^2 has $12,445,836 \cdot 9!$ mates, so in total, we enumerate about 0.375% of the total mates of C_3^2.

Lemma 4.1. *Every 3×3 primary block of C_3^n is in the form*

$$\begin{matrix} x & y & z \\ y & z & x \\ z & x & y \end{matrix},$$

where the symbols x, y, and z are distinct.

Proof. C_3^n is the Kronecker product of multiple C_3 squares and C_3 follows the above form, so the operation will make copies of C_3 throughout C_3^n. As a result, every 3×3 primary block of C_3^n will follow this form. □

Lemma 4.2. *If a* 3×3 *primary block in a mate* M *of* C_3^n *is a latin square, then each primary block of* M *will be in the form* $\widehat{C}_3$*,*

$$\begin{matrix} a & b & c \\ c & a & b \\ b & c & a \end{matrix} \,,$$

where a, b, and c, are distinct.

Proof. Suppose M is a mate of C_3^n. Each 3×3 primary block of C_3^n takes the form of C_3, which only has one semireduced mate, $\widehat{C}_3$. Each primary block in M needs to be a mate of its corresponding primary block in C_3^n, because if not then M wouldn't actually be a mate. For the primary blocks in M to be mates of those in C_3^n, the primary blocks in M must be latin squares. Each one of M's primary blocks must follow the form $\widehat{C}_3$, because every mate of C_3 takes this form as well. $\square$

Lemma 4.3. *If* M *is a mate of* C_3^2 *in which each* 3×3 *primary block is a latin square, then* $|S(M)| = 3$*.*

Proof. Suppose M is a mate of C_3^2 in which each 3×3 primary block is a latin square. M will take the primary block structure

$$\begin{matrix} A & B & C \\ D & E & F \\ G & H & I \end{matrix} \,,$$

If a symbol appears in a primary block, then it must appear 3 times. Due to this, the symbol sets of the first three primary blocks of M will be $\{1, 2, 3\}$, $\{4, 5, 6\}$, and $\{7, 8, 9\}$ if M is semireduced. If we take the set any symbol from the set of $\{1, 2, 3\}$ in the primary block a and try to place it in d, we run into a latin issue and we can't place it in f because we get repeated ordered pairs. We then run into the same two problems in the third row with the blocks b and c, so we place the symbols in a. As a result, the symbols in the set $\{1, 2, 3\}$ never appear in any other set in M. If we run through the same process with the other two sets, we find that they also possess this property. Therefore for any mate of C_3^2, $|S(M)| = 3$. $\square$

Lemma 4.4. *If every* 3×3 *primary block in a mate of* C_3^2 *is a latin square, then the* 3×3 *primary blocks must be arranged in the form*

$$\begin{matrix} B_1 & B_2 & B_3 \\ B_3 & B_1 & B_2 \\ B_2 & B_3 & B_1 \end{matrix} \,,$$

where the primary blocks B_1*,* B_2*, and* B_3 *contain distinct sets of symbols.*

Proof. Suppose M is a mate of C_3^2 in which every 3×3 primary block is a latin square. Notice that the 3×3 primary blocks in C_3^2 are arranged in the form

$$\begin{matrix} A_1 & A_2 & A_3 \\ A_2 & A_3 & A_1 \\ A_3 & A_1 & A_2 \end{matrix} .$$

If we superimpose M with the C_3^2 primary block arrangement, we will get the collection of ordered pairs

$$\begin{matrix} B_1, A_1 & B_2, A_2 & B_3, A_3 \\ B_4, A_2 & B_5, A_3 & B_6, A_1 \\ B_7, A_3 & B_8, A_1 & B_9, A_2 \end{matrix} .$$

Given that the mate is latin, the first row will contain three distinct primary blocks B_1, B_2, and B_2. These primary blocks correspond to the three other distinct primary blocks A_1, A_2, and A_3 respectively from C_3^2. There are only three distinct primary blocks in C_3^2 due to Lemma 4.2 and only three distinct primary blocks in M due to Lemma 4.3. We want the primary blocks in the mate to be a latin square, so the remaining blocks $B_4, \ldots, B_9$ need to equal one of B_1, B_2, or B_3.

If we attempt to set $B_1 = B_4$, then the arrangement is no longer a latin square and if we try to set $B_1 = B_6$, then we get the repeated ordered pair (B_1, A_1) and the arrangement won't be a mate. $B_1 = B_5$ because no issues arise and there aren't any other remaining locations in the second row for B_1. We run into the same restrictions on the third row which forces the placement of B_1 into the bottom right location and we get the form

$$\begin{matrix} B_1 & B_2 & B_3 \\ B_4 & B_1 & B_6 \\ B_7 & B_8 & B_1 \end{matrix} ,$$

By repeating the same process with the other two symbols B_2 and B_3 for the second and third rows, the form

$$\begin{matrix} B_1 & B_2 & B_3 \\ B_3 & B_1 & B_2 \\ B_2 & B_3 & B_1 \end{matrix} ,$$

results as the 3×3 primary block arrangement for a mate of C_3^2. □

Lemma 4.5. *If every 3×3 primary block of a mate of C_3^2 is a latin square, and we permute the symbols within just one 3×3 primary block, the resulting latin square is another mate.*

Proof. Suppose M is a mate of C_3^2 in which every 3×3 primary block is a latin square. Let's say we pick a 3×3 primary block P. This primary block in M will take the form of $\widehat{C}_3$. If we permute the symbols of P, the resulting P' will be a mate of the corresponding primary block from C_3^2 due to the fact that if you permute the symbols of a mate, the resulting latin square will also be a mate, so P' is a mate of C_3.

Considering that P is a latin square, none of the symbols that are in P appear again in the rows or columns that contain P. So no matter which symbol permutation is performed on P, the resulting mate M' will always be a latin square. M' is indeed a mate of C_3^2 because the symbols that appear in P and P' match up with those in the corresponding primary block in C_3^2 which is due to the the primary block arrangement of M and the fact that $|S(M)| = 3$ in M. $\square$

Lemma 4.6. *There are no more than* 6^6 *semireduced mates of* C_3^2 *in which every* 3×3 *primary block is a latin square.*

Proof. For each 3×3 primary block in a mate M of C_3^2, there are 6 ways to permute the symbols and for each permutation, each new square will be a mate. This is because every 3×3 primary block in a mate of C_3^n takes the form of $\widehat{C}_3$ and there are only 6 ways to permute the symbols of a 3×3 latin square.

In order to keep the mates semireduced, the 3×3 primary blocks containing the first row can't have their symbols permuted. As a result, only 6 of the 3×3 primary blocks in M can have their symbols permuted to get different semireduced mates. Because there are 6 ways to permute each of the 6 remaining primary blocks, there are at most 6^6 different mates. $\square$

Recall that S_3 is the group of permutations of $\{1, 2, 3\}$.

Lemma 4.7. *We can enumerate a family of* 6^6 *semireduced mates of* C_3^2 *in which every* 3×3 *primary block is a latin square, using a map from* $S_3 \times S_3 \times S_3 \times S_3 \times S_3 \times S_3 = (S_3)^6$ *to latin squares of size 9.*

Proof. Suppose we have a map from $(S_3)^6$ to L, the set of latin squares of size 9,

$$\phi : (S_3)^6 \to L.$$

In addition, we choose the mate of C_3^2, $\widehat{C}_3^2$ which has the following 3×3 primary blocks:

$$\begin{array}{c|c|c} B_1 & B_2 & B_3 \\ \hline B_a & B_b & B_c \\ \hline B_d & B_e & B_f \end{array}.$$

The primary blocks that contain the first row are ignored, because we need to keep the square semireduced. The function ϕ takes $\widehat{C}_3^2$ and applies a $g \in (S_3)^6$ to its 3×3 primary blocks, which takes each symbol permutation, σ in $g = (\sigma_a, \ldots, \sigma_f)$ and applies it to the corresponding primary block. This produces a new mate of the form

$$\phi(g) = \begin{array}{c|c|c} B_1 & B_2 & B_3 \\ \hline \sigma_a(B_a) & \sigma_b(B_b) & \sigma_c(B_c) \\ \hline \sigma_d(B_d) & \sigma_e(B_e) & \sigma_f(B_f) \end{array}.$$

We know that you do in fact get a mate, because we're permuting the symbols in the 3×3 primary blocks of a mate of C_3^2 which satisfies Lemma 4.5.

Suppose that $\phi(g_0) = M$ and $\phi(g_1) = M$. Then we have

$$\underbrace{\begin{array}{c|c|c} B_1 & B_2 & B_3 \\ \hline \sigma_{a0}(B_a) & \sigma_{b0}(B_b) & \sigma_{c0}(B_c) \\ \hline \sigma_{d0}(B_d) & \sigma_{e0}(B_e) & \sigma_{f0}(B_f) \end{array}}_{\phi(g_0)} = \quad M \quad = \underbrace{\begin{array}{c|c|c} B_1 & B_2 & B_3 \\ \hline \sigma_{a1}(B_a) & \sigma_{b1}(B_b) & \sigma_{c1}(B_c) \\ \hline \sigma_{d1}(B_d) & \sigma_{e1}(B_e) & \sigma_{f1}(B_f) \end{array}}_{\phi(g_1)}.$$

Because the primary blocks are the same, we can say that $g_0 = g_1$. Thus the function ϕ is injective.

We know that this function produces latin squares that are mates and that it is injective, so if we take the cardinality of $(S_3)^6$, then we know that there are at least 6^6 semireduced mates in this family where each 3×3 primary block of every mate is a latin square. $\square$

Theorem 4.8. *There are exactly 6^6 semireduced mates of C_3^2 in which every primary block is a latin square and they are in bijection with $(S_3)^6$.*

Proof. From Lemma 4.7, we assume that M takes the form of $\widehat{C}_3^2$ in which a $g \in (S_3)^6$ has been applied to the primary blocks. To show that ϕ is surjective, we demonstrate how to obtain g from M.

If we semireduce the bottom 6 primary blocks of M by using symbol permutations we will get $\widehat{C}_3^2$,

$$\begin{array}{c|c|c} B_1 & B_2 & B_3 \\ \hline \sigma_a(\sigma_a(B_a)) & \sigma_b(\sigma_b(B_b)) & \sigma_c(\sigma_c(B_c)) \\ \hline \sigma_d(\sigma_d(B_d)) & \sigma_e(\sigma_e(B_e)) & \sigma_f(\sigma_f(B_f)) \end{array} = \begin{array}{c|c|c} B_1 & B_2 & B_3 \\ \hline B_a & B_b & B_c \\ \hline B_d & B_e & B_f \end{array}.$$

We can determine g by semireducing the 3×3 primary blocks of M, therefore we know that the function ϕ from Theorem 4.7 is surjective. ϕ is both injective and surjective, so it's bijective. $\square$

Corollary 4.9. *C_3^2 has exactly $46,656 \cdot 9!$ mates in which each primary 3×3 block is a latin square.*

Proof. By Theorem 4.8, there are $6^6 = 46,656$ semireduced mates of C_3^2 in which each primary 3×3 block is a latin square. Because each semireduced mate can be used to produce $n!$ distinct mates of the same square by permuting symbols, we get $46,656 \cdot 9!$ total mates of C_3^2. $\square$

The next corollary follows from the definitions and the proof of Theorem 4.8.

Corollary 4.10. *There is a free transitive action of $S_9 \times (S_3)^6$ on the mates of C_3^2 in which each 3×3 primary block is a latin square.*

4.2 Generalization

In this section, we classify the family of *regular* mates of C_k^n by generalizing Theorem 4.8 for $k \geq 3$ and $n \geq 2$. We prove that, for regular mates of C_k^n, the $k \times k$ primary blocks will each be mates of C_k, that the primary blocks are arranged in the form of a mate of C_k^{n-1}, and that $S(M) = k^{n-1}$. We obtain a combinatorial enumeration of the family of regular mates.

This enumeration gives us an asymptotic lower bound for the number of mates that a latin square of a given size can have. We end the section with a summary of the algorithm for constructing regular mates, and example showing how to bootstrap this algorithm to compute the number of regular mates of C_k^n.

Definition 4.11. A mate of C_k^n in which $n \geq 2$ and $k \geq 3$ is *regular* if every $k \times k$ primary block of M is a latin square and $|S(M)| = k^{n-1}$.

Definition 4.12. SRM(L) is the set of semireduced mates for a latin square, L.

Lemma 4.13. *Every $k \times k$ primary block of C_k^n will be in the form of C_k.*

Proof. Each $k \times k$ primary block of C_k^n takes the form of C_k because C_k^n is the result of a Kronecker product of multiple C_k squares. □

Lemma 4.14. *If a mate M of C_k^n is regular, then every $k \times k$ primary block will be in the form of a mate of C_k.*

Proof. Suppose M is a regular mate of C_k^n. Now suppose that some primary block of M does not take the form of a mate of C_k. If we superimpose M and C_k^n and we choose a single ordered pair, this same ordered pair will appear twice because the primary block in C_k^n and the one in M aren't mates. As a result, this ordered pair will appear twice in the larger superimposed squares. However, this is impossible because M and C_k^n are mates and an ordered pair can only appear once. Therefore, every $k \times k$ primary block of M take the form of a mate of C_k. □

Lemma 4.15. *If a mate of C_k^n is regular, then the $k \times k$ primary blocks will be arranged in the form of a mate of C_k^{n-1}.*

Proof. Suppose M is a regular mate of C_k^n and that the $k \times k$ primary blocks aren't arranged in the form of a mate of C_k^{n-1}. There are k^{n-1} distinct sets of symbols for the primary blocks in M and the primary blocks are latin squares because M is regular so the symbols in one set won't appear with different symbols in another set.

In order for M and C_k^n to be mates, each primary block must match once with each distinct primary block of C_k^n. When M and C_k^n are superimposed, their primary block arrangements aren't mates so there will be repeated matchings of the primary blocks in M and C_k^n, which will produce repeated ordered pairs. This can't happen since M and C_k^n are mates, so the primary blocks in M must be arranged in the form of a mate of C_k^{n-1}. □

Lemma 4.16. *If a mate of C_k^n is regular and we permute the symbols within just one $k \times k$ primary block, the resulting latin square is another mate.*

Proof. Suppose M is a regular mate of C_k^n. Let's say we pick a $k \times k$ primary block P. This primary block in M will take the form of a mate of C_k due to Lemma 4.14. If we permute the symbols of P, the resulting P' will be a mate of the corresponding primary block from C_k^n due to the fact that if you permute the symbols of a mate, the resulting latin square will also be a mate, so P' is a mate of C_k.

Considering that P is a latin square, none of the symbols that are in P appear again in the rows or columns that contain P. So no matter which symbol permutation is performed on P, the resulting mate M' will always be a latin square. M' is indeed a mate of C_k^n because the symbols that appear in P and P' match up with those in the corresponding primary block in C_k^n which is due to the the primary block arrangement of M and the fact that $|S(M)| = k^{n-1}$ in both M and M'. □

Lemma 4.17. *There are no more than*

$$B(n,k) = |\operatorname{SRM}(C_k^{n-1})| \cdot |\operatorname{SRM}(C_k)^{k^{n-1}}| \cdot (k!)^{F(n,k)} \cdot (k^n)!$$

regular mates of C_k^n. Here $F(n,k) = k^{2n-2} - k^{n-1}$ is the the number of primary blocks in the mate of C_k^n which don't contain the first row.

Proof. Suppose M is a $k^n \times k^n$ regular mate of C_k^n.

To determine the primary block arrangement of M, pick a semireduced mate S of C_k^{n-1}. This gives us $|\operatorname{SRM}(C_k^{n-1})|$ initial choices; the number of semireduced mates of C_k^{n-1}. Next, replace each entry S with a semireduced mate of C_k. These mates are the primary blocks of M and the set of those mates has the size $|\operatorname{SRM}(C_k)^{k^{n-1}}|$, which is the number of semireduced mates of C_k raised to the number of primary blocks in M.

For each primary block of M which doesn't contain the first row, permute the symbols of that block arbitrarily. This gives us $(k!)^{F(n,k)}$ choices where $F(n,k) = k^{2n-2} - k^{n-1}$ is the the number of primary blocks not in the first row in M and $k!$ is the number of available permutations for each primary block. By permuting all the symbols of M arbitrarily, we get $(k^n)!$ mates overall.

If we combine all four of these steps, we can conclude that there are no more than

$$B(n,k) = |\operatorname{SRM}(C_k^{n-1})| \cdot |\operatorname{SRM}(C_k)^{k^{n-1}}| \cdot (k!)^{F(n,k)} \cdot (k^n)!$$

regular mates for C_k^n. □

Lemma 4.18. *We can enumerate a family of*

$$B(n,k) = |\operatorname{SRM}(C_k^{n-1})| \cdot |\operatorname{SRM}(C_k)^{k^{n-1}}| \cdot (k!)^{F(n,k)} \cdot (k^n)!$$

regular mates of C_k^n.

Proof. Suppose we have a map, Φ from

$$\operatorname{SRM}(C_k^{n-1}) \times \operatorname{SRM}(C_k)^{k^{n-1}} \times (S_k)^{F(n,k)} \times (S_{k^n})$$

to L, the set of latin squares of size k^n. Suppose further that we start with a semireduced mate N of C_k^{n-1}, which comes from the set $\operatorname{SRM}(C_k^{n-1})$.

Next we replace each symbol of N with some semireduced mate of C_k. Each of these mates come from the set $\operatorname{SRM}(C_k)^{k^{n-1}}$ in which k^{n-1} is the number of symbols in N. These mates are the primary blocks of M, a regular mate of C_k^n.

Now choose a set of symbol permutations for each one of the primary blocks which do not contain the first row. The sets of symbol permutations can be represented by $(S_k)^{F(n,k)}$ where S_k is the symbol permutation and $F(n,k) = k^{2n-2} - k^{n-1}$ is the number of blocks which don't contain the first row. This results in a semireduced mate of C_k^n. By permuting the symbols of the entire mate, we can find the remaining mates of C_k^n which are not semireduced. This is represented by (S_{k^n}).

Suppose that $\Phi(S_0, s_0, G_0, g_0) = M$ and $\Phi(S_1, s_1, G_1, g_1) = M$, where $S \in \operatorname{SRM}(C_k^{n-1})$, $s \in \operatorname{SRM}(C_k)^{k^{n-1}}$, $G \in (S_k)^{F(n,k)}$, and $g \in (S_{k^n})$. Because they produce the same regular mate M, we can determine that $\Phi(S_0, s_0, G_0, g_0) = \Phi(S_1, s_1, G_1, g_1)$ and that $(S_0, s_0, G_0, g_0) = (S_1, s_1, G_1, g_1)$. Thus Φ is injective.

Because Φ is injective and produces mates which are latin squares, we can enumerate a family of

$$B(n,k) = |\operatorname{SRM}(C_k^{n-1})| \cdot |\operatorname{SRM}(C_k)^{k^{n-1}}| \cdot (k!)^{F(n,k)} \cdot (k^n)!$$

regular mates of C_k^n. □

Theorem 4.19. *There are exactly*

$$B(n,k) = |\operatorname{SRM}(C_k^{n-1})| \cdot |\operatorname{SRM}(C_k)^{k^{n-1}}| \cdot (k!)^{F(n,k)} \cdot (k^n)!$$

regular mates of C_k^n.

Proof. From Lemma 4.18, we assume that a regular mate M of C_k^n is the result of a $\Phi(S, s, G, g)$ where $S \in \mathrm{SRM}(C_k^{n-1})$, $s \in \mathrm{SRM}(C_k)^{k^{n-1}}$, $G \in (S_k)^{F(n,k)}$, and $g \in (S_{k^n})$. Below, we show that Φ is surjective by obtaining (S, s, G, g) from M.

If we first semireduce M by permuting all the symbols, we can find g due to the fact that the same g can be used to revert this new M' to its initial state, M. If we semireduce each of the primary blocks in M', we get the set of permutations G for the same reason and now we have M''.

By looking at each individual $k \times k$ primary block in M'', we can find the set of semireduced mates of C_k that make up these primary blocks, s. Then, we can look at the arrangement of the primary blocks in M'' and determine the semireduced mate of C_k^{n-1}, S.

Because we can obtain (S, s, G, g) simply from M, we know that the function Φ from Theorem 4.18 is surjective. The mapping Φ is both injective and surjective, so it is bijective and there are exactly

$$B(n, k) = |\,\mathrm{SRM}(C_k^{n-1})| \cdot |\,\mathrm{SRM}(C_k)^{k^{n-1}}| \cdot (k!)^{F(n,k)} \cdot (k^n)!$$

regular mates of C_k^n. $\square$

Corollary 4.20. *Cyclic squares of prime size always have mates, and thus powers of these squares have many mates.*

Corollary 4.21. *Cyclic squares of even size have no mates, and thus powers of these squares have no regular mates.*

The following algorithm is the summarized version of the process used in Theorem 4.19 to construct regular mates of C_k^n.

Algorithm for Regular Mates of C_k^n:

1. Choose a semireduced mate M of $(C_k)^{n-1}$.

2. Replace each entry i of M uniformly with some semireduced mate B_i of C_k.

3. For each primary block not in the first row, permute the k symbols within that block arbitrarily.

4. Permute the k^n symbols of the new mate arbitrarily. In this way, we can enumerate all the regular mates of $(C_k)^n$.

We can bootstrap this algorithm for C_k^{n+1} and for further cases to find an asymptotic lower bound for the number of mates of a latin square of a given size. Instead of using $\mathrm{SRM}(C_k^(n-1))$, we can use the number of regular mates, giving a lower bound on the overall number of mates of C_k^n.

Example 4.22. We can first obtain a lower bound for the number of semireduced mates of C_3^3:

$$\begin{aligned}\mathrm{SRM}(C_3^3) &\geq |\,\mathrm{SRM}(C_3^{3-1})| \cdot |\,\mathrm{SRM}(C_3)^{3^{3-1}}| \cdot (3!)^{F(3,3)} \\ &\geq 6^6 \cdot 1 \cdot (6)^{3^4-3^2} \\ &= 6^{78}.\end{aligned}$$

We can then obtain a lower bound for the number of semireduced mates of C_3^4 by bootstrapping using our bound for the number of semireduced mates of C_3^3:

$$\begin{aligned}\mathrm{SRM}(C_3^4) &\geq |\,\mathrm{SRM}(C_3^{4-1})| \cdot |\,\mathrm{SRM}(C_3)^{3^{4-1}}| \cdot (3!)^{F(4,3)} \\ &\geq 6^{78} \cdot 1 \cdot (3!)^{3^6-3^3} \\ &= 6^{780}.\end{aligned}$$

5 Questions

1. Can we enumerate the remaining mates of C_2^3 out of the 70,272?
2. If we take a 4×4 subsquare of C_2^3, are there always permutations of the rows and columns that move the subsquare to the first quadrant so that the resulting square is also obtained by permuting symbols of C_2^3?
3. Will there ever be an odd subsquare for C_2^n?
4. If you have one 2×2 subsquare, will there always be at least one other subsquare that contains the same symbols? The answer is yes, if we start with a power square of C_2.
5. Suppose that M is a mate of C_2^3. Then $\mathcal{S}(M)$ has 2, 4, 8, or 16 sets in it, according to our computational data. Why?
6. Can we make a mate with any of the choices from Section 2.3? If so, how many?
7. If we fill in the top half of an 8×8 latin square with a 4×8 latin array and its entries make up 32 ordered pairs when superimposed with C_2^3, can the square be completed into a mate of C_2^3?
8. How many subsquares of size 3 are in each of the mates of C_3^2?

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