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Regular Mates of Power Squares

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Introduction

This project explores the question "How many mates can a latin square of a certain size have?" We extend the work of Bryant, Figler, Garcia, Mummert, and Singh[2] from a 2011–2012 Computational Science REU. They performed a computational search of squares of sizes 7 and 8, and found a construction to build a single mate for $(C_2)^n$ when $n \geq 2$.

We first focused on the mates of $(C_3)^2$. A computational search showed that this square has 12,445,836·9! mates. We obtained a combinatorial enumeration of the regular mates of this square. This set of $6^6 \cdot 9!$ mates consists of only about 0.375% of the total mates of $(C_3)^2$.

We generalized our construction to enumerate the regular mates of power squares $(C_k)^n$ where $k \geq 3$ and $n \geq 2$. This allows us to obtain a lower bound for the number of mates of a latin square of a given size. No nontrivial upper bound is known.

Goals

- Explore the question: How many mates can a latin square of a given size have?
- Find a construction that produces a large number of mates.
- Find a lower bound for the number of mates of a latin square of a given size.

Definitions

A *latin square* is an $m \times m$ array containing m distinct symbols, where each symbol appears once in each row and once in each column.

The latin square C_k is the *cyclic square* of size k, which is the Cayley table of $\mathbb{Z}_k$.

A *power square* is obtained by taking a repeated Kronecker product of a fixed square with itself. The square $(C_k)^n$ is the nth power of the square C_k.

Two latin squares of the same size are *orthogonal mates* if every possible ordered pair of their symbols is present when the squares are superimposed.

1	2	3
2	3	1
3	1	2

C_3

1	2	3
3	1	2
2	3	1

$\widehat{C}_3$

1,1	2,2	3,3
2,3	3,1	1,2
3,2	1,3	2,1

Superimposed

A latin square is *semireduced* if the m symbols in the first row are in the natural order $1, \ldots, m$. We work with semireduced mates because each one represents $m!$ distinct mates.

A *primary block* is a $k \times k$ block of a latin square of size k^n obtained by repeatedly dividing the square into equal sections.

$S(M)$ is the set of sets of symbols that appear in the $k \times k$ primary blocks of a mate M of $(C_k)^n$.

Mates of Power Squares

1	2	3	4	5	6	7	8	9
2	3	1	5	6	4	8	9	7
3	1	2	6	4	5	9	7	8
4	5	6	7	8	9	1	2	3
5	6	4	8	9	7	2	3	1
6	4	5	9	7	8	3	1	2
7	8	9	1	2	3	4	5	6
8	9	7	2	3	1	5	6	4
9	7	8	3	1	2	6	4	5

$(C_3)^2$

1	2	3	4	5	6	7	8	9
3	1	2	6	4	5	9	7	8
2	3	1	5	6	4	8	9	7
9	8	7	3	2	1	5	6	4
7	9	8	1	3	2	4	5	6
8	7	9	2	1	3	6	4	5
6	5	4	7	8	9	2	1	3
4	6	5	9	7	8	3	2	1
5	4	6	8	9	7	1	3	2

Mate M_1

1	2	3	4	5	6	7	8	9
3	1	2	6	4	5	9	7	8
6	3	4	5	2	1	8	9	7
9	8	7	1	6	3	4	5	2
7	9	8	3	1	2	6	4	5
8	7	9	2	3	4	5	6	1
4	5	6	9	7	8	1	2	3
2	4	5	8	9	7	3	1	6
5	6	1	7	8	9	2	3	4

Mate M_2

1	2	3	4	5	6	7	8	9
4	5	6	7	8	9	1	2	3
7	8	9	1	2	3	4	5	6
3	1	2	6	4	5	9	7	8
6	4	5	9	7	8	3	1	2
9	7	8	3	1	2	6	4	5
2	3	1	5	6	4	8	9	7
5	6	4	8	9	7	2	3	1
8	9	7	2	3	1	5	6	4

Mate M_3

A square L of size k^n is *regular* if each $k \times k$ primary block of L is a latin square and $|S(L)| = k^{n-1}$.

Both of the squares $(C_3)^2$ and M_1 above are regular squares. M_1, M_2, and M_3 are mates of $(C_3)^2$.

The squares M_2 and M_3 are not regular, but are semireduced.

- $|S(M_2)| = 3$, but not all the 3×3 primary blocks are latin.
- M_3 is not regular because $|S(M_3)| = 1$.

Construction of Mates of $(C_k)^n$

We developed an algorithm that enumerates the regular mates of $(C_k)^n$ when $n \geq 2$ and $k \geq 3$.

Step 1: Choose a semireduced mate M of $(C_k)^{n-1}$.

This mate determines the primary block arrangement of the mate of $(C_k)^n$. In the case $(C_3)^2$, we choose $\widehat{C}_3$ because it is the only semireduced mate of C_3.

Step 2: Replace each entry i of M uniformly with some semireduced mate B_i of C_k.

In the $(C_3)^2$ case, because $\widehat{C}_3$ is the only semireduced mate of C_3, each B_i will be $\widehat{C}_3$.

A	B	C
C	A	B
B	C	A

Step 1

$\Rightarrow$

B_1	B_2	B_3
B_a	B_b	B_c
B_d	B_e	B_f

Step 2

Step 3: For each primary block not in the first row, permute the k symbols within that block arbitrarily.

To keep the square semireduced, we cannot permute the symbols of blocks in the first row.

B_1	B_2	B_3
$\sigma_a(B_a)$	$\sigma_b(B_b)$	$\sigma_c(B_c)$
$\sigma_d(B_d)$	$\sigma_e(B_e)$	$\sigma_f(B_f)$

Using this process, we can enumerate every regular semireduced mate of $(C_k)^n$.

Step 4: Permute the k^n symbols of the new mate arbitrarily. In this way, we can enumerate all the regular mates of $(C_k)^n$.

General Result

We characterize and enumerate the regular mates of $(C_k)^n$.

Theorem. For $n \geq 3$ and $k \geq 2$, the number of regular mates of $(C_k)^n$ is

$$\left|\mathrm{SRM}((C_k)^{n-1})\right| \cdot \left|\mathrm{SRM}(C_k)\right|^{k^{n-1}} \cdot (k!)^{F(n,k)} \cdot (k^n)!$$

where

- $\mathrm{SRM}(L)$ is the set of semireduced mates of a latin square L.
- $F(n,k) = k^{2n-2} - k^{n-1}$ is the number of primary blocks in the mate of $(C_k)^n$ not in the first row.

This gives an asymptotic lower bound for the number of mates of a latin square whose size is a power of a prime.

Corollary. Cyclic squares of prime size always have mates, and thus powers of these squares have many regular mates.

Corollary. Cyclic squares of even size have no mates, and thus powers of these squares have no regular mates.

References

[1] Charles F. Laywine and Gary L. Mullen, *Discrete mathematics using Latin squares*, 1998.

[2] Megan Bryant, James Figler, Roger Garcia, Carl Mummert, and Yudhishtir Singh, *The number of mates of latin squares of sizes 7 and 8*, Congressus Numerantium (2013).

[3] Gary L. Mullen and Carl Mummert, *Finite fields and applications*, 2007.

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