$$S^{(2)} = \frac{-i}{2} \sum_{m_2=0}^{\infty} (-iH_0)^{m_2} X \int_0^{\delta t} dt_M \dots \int_0^{t_{M-m_2+1}} dt_{M-m_2}\, e^{i\omega_2[2] t_{M-m_2}}\, S^{(1)}(g(\vec{\omega}_2), t_{M-m_2})$$

$$= \frac{-i}{2} \sum_{m_2=0}^{\infty} (-iH_0)^{m_2} X \int_0^{\delta t} dt_M \dots \int_0^{t_{M-m_2+1}} dt_{M-m_2}\, e^{i\omega_2[2] t_{M-m_2}} \left(\frac{-i t_{M-m_2}}{2} \sum_{k^{(1)},k^{(0)}} \frac{i}{\lambda_{k^{(1)}} t_{M-m_2} - (\lambda_{k^{(0)}} - \omega_2[1]) t_{M-m_2}} \left(e^{-i\lambda_{k^{(1)}} t_{M-m_2}} - e^{-i(\lambda_{k^{(0)}} - \omega_2[1]) t_{M-m_2}} \right) \right) |k^{(1)}\rangle\langle k^{(1)}| X |k^{(0)}\rangle\langle k^{(0)}|$$

$$= \left(\frac{-i}{2}\right)^2 \sum_{k^{(1)},k^{(0)}} \sum_{m_2=0}^{\infty} (-iH_0)^{m_2} X \int_0^{\delta t} dt_M \dots \int_0^{t_{M-m_2+1}} dt_{M-m_2}\, e^{i\omega_2[2] t_{M-m_2}} \left(\frac{i}{\lambda_{k^{(1)}} - (\lambda_{k^{(0)}} - \omega_2[1])} \left(e^{-i\lambda_{k^{(1)}} t_{M-m_2}} - e^{-i(\lambda_{k^{(0)}} - \omega_2[1]) t_{M-m_2}} \right) \right) |k^{(1)}\rangle\langle k^{(1)}| X |k^{(0)}\rangle\langle k^{(0)}|$$

$$= \left(\frac{-i}{2}\right)^2 \sum_{k^{(1)},k^{(0)}} \sum_{m_2=0}^{\infty} (-iH_0)^{m_2} X \int_0^{\delta t} dt_M \dots \int_0^{t_{M-m_2+1}} dt_{M-m_2} \left(\frac{i}{\lambda_{k^{(1)}} - (\lambda_{k^{(0)}} - \omega_2[1])} \left(e^{-i(\lambda_{k^{(1)}} - \omega_2[2]) t_{M-m_2}} - e^{-i(\lambda_{k^{(0)}} - \omega_2[1] - \omega_2[2]) t_{M-m_2}} \right) \right) |k^{(1)}\rangle\langle k^{(1)}| X |k^{(0)}\rangle\langle k^{(0)}|$$

$$= \left(\frac{-i}{2}\right)^2 \left(\frac{i}{\lambda_{k^{(1)}} - (\lambda_{k^{(0)}} - \omega_2[1])}\right) \sum_{k^{(1)},k^{(0)}} \sum_{m_2=0}^{\infty} (-iH_0)^{m_2} X \int_0^{\delta t} dt_M \dots \int_0^{t_{M-m_2+1}} dt_{M-m_2} \left(e^{-i(\lambda_{k^{(1)}} - \omega_2[2]) t_{M-m_2}} - e^{-i(\lambda_{k^{(0)}} - \omega_2[1] - \omega_2[2]) t_{M-m_2}} \right) |k^{(1)}\rangle\langle k^{(1)}| X |k^{(0)}\rangle\langle k^{(0)}|$$

$$= \left(\frac{-i}{2}\right)^2 \left(\frac{i}{\lambda_{k^{(1)}} - (\lambda_{k^{(0)}} - \omega_2[1])}\right) \sum_{k^{(1)},k^{(0)}} \sum_{m_2=0}^{\infty} (-iH_0)^{m_2} X \int_0^{\delta t} dt_{M-m_2} \left(e^{-i(\lambda_{k^{(1)}} - \omega_2[2]) t_{M-m_2}} - e^{-i(\lambda_{k^{(0)}} - \omega_2[1] - \omega_2[2]) t_{M-m_2}} \right) \int_{t_{M-m_2}}^{\delta t} dt_{M-m_2+1} \dots \int_{t_{M-1}}^{\delta t} dt_M \; |k^{(1)}\rangle\langle k^{(1)}| X |k^{(0)}\rangle\langle k^{(0)}| X \dots$$

$$= \left(\frac{-i}{2}\right)^2 \left(\frac{i}{\lambda_{k^{(1)}} - (\lambda_{k^{(0)}} - \omega_2[1])}\right) \sum_{k^{(1)},k^{(0)}} \int_0^{\delta t} dt_{M-m_2} \left(e^{-i(\lambda_{k^{(1)}} - \omega_2[2]) t_{M-m_2}} - e^{-i(\lambda_{k^{(0)}} - \omega_2[1] - \omega_2[2]) t_{M-m_2}} \right) \sum_{m_2=0}^{\infty} \int_{t_{M-m_2}}^{\delta t} dt_{M-m_2+1} \dots \int_{t_{M-1}}^{\delta t} dt_M \; |k^{(1)}\rangle\langle k^{(1)}| X |k^{(0)}\rangle\langle k^{(0)}| X \dots (-iH_0)^{m_2} X$$

$$= \left(\frac{-i}{2}\right)^2 \left(\frac{i}{\lambda_{k^{(1)}} - (\lambda_{k^{(0)}} - \omega_2[1])}\right) \sum_{k^{(1)},k^{(0)}} \int_0^{\delta t} dt_{M-m_2} \left(e^{-i(\lambda_{k^{(1)}} - \omega_2[2]) t_{M-m_2}} - e^{-i(\lambda_{k^{(0)}} - \omega_2[1] - \omega_2[2]) t_{M-m_2}} \right) \sum_{k^{(2)}} e^{-i\lambda_{k^{(2)}}(\delta t - t_{M-m_2})} \; |k^{(2)}\rangle\langle k^{(2)}| X |k^{(1)}\rangle\langle k^{(1)}| X |k^{(0)}\rangle\langle k^{(0)}| X |k^{(2)}\rangle\langle k^{(2)}|$$

$$= \left(\frac{-i}{2}\right)^2 \left(\frac{i}{\lambda_{k^{(1)}} - (\lambda_{k^{(0)}} - \omega_2[1])}\right) \sum_{k^{(2)},k^{(1)},k^{(0)}} \int_0^{\delta t} dt_{M-m_2} \left(e^{i(\lambda_{k^{(2)}} - \lambda_{k^{(1)}} + \omega_2[2]) t_{M-m_2}} - e^{i(\lambda_{k^{(2)}} - \lambda_{k^{(0)}} + \omega_2[1] + \omega_2[2]) t_{M-m_2}} \right) e^{-i\lambda_{k^{(2)}}\delta t} \; |k^{(2)}\rangle\langle k^{(2)}| X |k^{(1)}\rangle\langle k^{(1)}| X |k^{(0)}\rangle\langle k^{(0)}| X |k^{(2)}\rangle\langle k^{(2)}|$$

$$= \left(\frac{-i}{2}\right)^2 \left(\frac{i}{\lambda_{k^{(1)}} - (\lambda_{k^{(0)}} - \omega_2[1])}\right) \sum_{k^{(2)},k^{(1)},k^{(0)}} \left(\int_0^{\delta t} dt_{M-m_2} e^{i(\lambda_{k^{(2)}} - \lambda_{k^{(1)}} + \omega_2[2]) t_{M-m_2}} - \int_0^{\delta t} dt_{M-m_2} e^{i(\lambda_{k^{(2)}} - \lambda_{k^{(0)}} + \omega_2[1] + \omega_2[2]) t_{M-m_2}} \right) e^{-i\lambda_{k^{(2)}}\delta t} \; |k^{(2)}\rangle\langle k^{(2)}| X |k^{(1)}\rangle\langle k^{(1)}| X |k^{(0)}\rangle\langle k^{(0)}| X |k^{(2)}\rangle\langle k^{(2)}|$$

$$= \left(\frac{-i}{2}\right)^2 \left(\frac{i}{\lambda_{k^{(1)}} - (\lambda_{k^{(0)}} - \omega_2[1])}\right) \sum_{k^{(2)},k^{(1)},k^{(0)}} \left(\frac{e^{i(\lambda_{k^{(2)}} - \lambda_{k^{(1)}} + \omega_2[2]) t_{M-m_2}}}{i(\lambda_{k^{(2)}} - \lambda_{k^{(1)}} + \omega_2[2])} \Big|_0^{\delta t} - \frac{e^{i(\lambda_{k^{(2)}} - \lambda_{k^{(0)}} + \omega_2[1] + \omega_2[2]) t_{M-m_2}}}{i(\lambda_{k^{(2)}} - \lambda_{k^{(0)}} + \omega_2[1] + \omega_2[2])} \Big|_0^{\delta t} \right) e^{-i\lambda_{k^{(2)}}\delta t} \; |k^{(2)}\rangle\langle k^{(2)}| X |k^{(1)}\rangle\langle k^{(1)}| X |k^{(0)}\rangle\langle k^{(0)}| X |k^{(2)}\rangle\langle k^{(2)}|$$

$$= \left(\frac{-i}{2}\right)^2 \left(\frac{i}{\lambda_{k^{(1)}} - (\lambda_{k^{(0)}} - \omega_2[1])}\right) \sum_{k^{(2)},k^{(1)},k^{(0)}} \left(\frac{(e^{i(\lambda_{k^{(2)}} - \lambda_{k^{(1)}} + \omega_2[2])\delta t} - 1)}{i(\lambda_{k^{(2)}} - \lambda_{k^{(1)}} + \omega_2[2])} - \frac{(e^{i(\lambda_{k^{(2)}} - \lambda_{k^{(0)}} + \omega_2[1] + \omega_2[2])\delta t} - 1)}{i(\lambda_{k^{(2)}} - \lambda_{k^{(0)}} + \omega_2[1] + \omega_2[2])} \right) e^{-i\lambda_{k^{(2)}}\delta t} \; |k^{(2)}\rangle\langle k^{(2)}| X |k^{(1)}\rangle\langle k^{(1)}| X |k^{(0)}\rangle\langle k^{(0)}| X |k^{(2)}\rangle\langle k^{(2)}|$$

$$= \left(\frac{-i}{2}\right)^2 \left(\frac{i}{\lambda_{k^{(1)}} - (\lambda_{k^{(0)}} - \omega_2[1])}\right) \sum_{k^{(2)},k^{(1)},k^{(0)}} \left(\frac{i(e^{-i\lambda_{k^{(2)}}\delta t} - e^{-i(\lambda_{k^{(0)}} - \omega_2[1] - \omega_2[2])\delta t})}{\lambda_{k^{(2)}} - (\lambda_{k^{(0)}} - \omega_2[1])} - \frac{i(e^{-i\lambda_{k^{(2)}}\delta t} - e^{-i(\lambda_{k^{(0)}} - \omega_2[1] - \omega_2[2])\delta t})}{\lambda_{k^{(2)}} - (\lambda_{k^{(0)}} - \omega_2[1] - \omega_2[2])} \right) |k^{(2)}\rangle\langle k^{(2)}| X |k^{(1)}\rangle\langle k^{(1)}| X |k^{(0)}\rangle\langle k^{(0)}| X |k^{(2)}\rangle\langle k^{(2)}|$$

$$= \left(\frac{-i}{2}\right)^2 \left(\frac{i}{\lambda_{k^{(1)}} - (\lambda_{k^{(0)}} - \omega_2[1])}\right) \frac{\delta t}{\delta t} \sum_{k^{(2)},k^{(1)},k^{(0)}} \left(\frac{i(e^{-i\lambda_{k^{(2)}}\delta t} - e^{-i(\lambda_{k^{(0)}} - \omega_2[1] - \omega_2[2])\delta t})}{\lambda_{k^{(2)}} - (\lambda_{k^{(0)}} - \omega_2[1])} - \frac{i(e^{-i\lambda_{k^{(2)}}\delta t} - e^{-i(\lambda_{k^{(0)}} - \omega_2[1] - \omega_2[2])\delta t})}{\lambda_{k^{(2)}} - (\lambda_{k^{(0)}} - \omega_2[1] - \omega_2[2])} \right) \frac{\delta t}{\delta t} \; |k^{(2)}\rangle\langle k^{(2)}| X |k^{(1)}\rangle\langle k^{(1)}| X |k^{(0)}\rangle\langle k^{(0)}| X |k^{(2)}\rangle\langle k^{(2)}|$$

$$= \left(\frac{-i\delta t}{2}\right)^2 \left(\frac{i}{\lambda_{k^{(1)}}\delta t - (\lambda_{k^{(0)}} - \omega_2[1])\delta t}\right) \sum_{k^{(2)},k^{(1)},k^{(0)}} \left(f(\lambda_{k^{(2)}}\delta t, (\lambda_{k^{(1)}} - \omega_2[2])\delta t) - f(\lambda_{k^{(2)}}\delta t, (\lambda_{k^{(0)}} - \omega_2[1] - \omega_2[2])\delta t) \right) \langle k^{(2)}|X|k^{(1)}\rangle\langle k^{(1)}|X|k^{(0)}\rangle \; |k^{(2)}\rangle\langle k^{(0)}|$$

$$= \left(\frac{-i\delta t}{2}\right)^2 \sum_{k_n} f(\vec{\lambda}_2(\vec{k}_2)\delta t - \vec{c}(\vec{\omega}_2)\delta t) \; \langle k^{(2)}|X|k^{(1)}\rangle\langle k^{(1)}|X|k^{(0)}\rangle \; |k^{(2)}\rangle\langle k^{(0)}|$$

où : $\vec{\lambda}_2(\vec{k}_2) = (\lambda_{k^{(0)}}, \lambda_{k^{(1)}}, \lambda_{k^{(2)}}) \Rightarrow \vec{\lambda}_2(\vec{k}_2)\delta t = (\lambda_{k^{(0)}}, \lambda_{k^{(1)}}, \lambda_{k^{(2)}})\delta t$

$\vec{c}(\vec{\omega}_2) = (\omega_2[2] + \omega_2[1],\ \omega_2[2],\ 0) \Rightarrow \vec{c}(\vec{\omega}_2)\delta t = (\omega_2[2] + \omega_2[1],\ \omega_2[2],\ 0)\delta t$

$\vec{\lambda}_2(\vec{k}_2)\delta t - \vec{c}(\vec{\omega}_2)\delta t = (\lambda_{k^{(0)}} - \omega_2[2] - \omega_2[1],\ \lambda_{k^{(1)}} - \omega_2[2],\ \lambda_{k^{(2)}})\delta t$

On vérifie : $f(\vec{\lambda}_2(\vec{k}_2)\delta t - \vec{c}(\vec{\omega}_2)\delta t) = i\, \dfrac{f((\lambda_{k^{(1)}} - \omega_2[2])\delta t, \lambda_{k^{(2)}}\delta t) - f((\lambda_{k^{(0)}} - \omega_2[2] - \omega_2[1])\delta t, \lambda_{k^{(2)}}\delta t)}{(\lambda_{k^{(1)}} - \omega_2[2])\delta t - (\lambda_{k^{(0)}} - \omega_2[2] - \omega_2[1])\delta t}$

$$= \frac{i}{\lambda_{k^{(1)}}\delta t - (\lambda_{k^{(0)}} - \omega_2[1])\delta t} \left(\frac{i}{\lambda_{k^{(2)}}\delta t - (\lambda_{k^{(0)}} - \omega_2[1])\delta t} \left(e^{-i\lambda_{k^{(2)}}\delta t} - e^{-i(\lambda_{k^{(0)}} - \omega_2[2])\delta t} \right) - \frac{i}{\lambda_{k^{(2)}}\delta t - (\lambda_{k^{(0)}} - \omega_2[1] - \omega_2[2])\delta t} \left(e^{-i\lambda_{k^{(2)}}\delta t} - e^{-i(\lambda_{k^{(0)}} - \omega_2[2] - \omega_2[1])\delta t} \right) \right)$$

Ce qui est comme en haut.