

# Multi Variable Calculus

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# 1 Fourier

## 1.1 Fourier Series

A periodic function with period  $2L$  and let  $f(x)$  and  $f'(x)$  be piecewise continuous on the interval  $-L < x < L$

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right) = \sum_{n=-\infty}^{\infty} c_n e^{jn\pi x/L}$$

The coefficients are given by:

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx \quad n \geq 0$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad n > 0$$

$$c_n = \frac{1}{2}(a_n - jb_n) \quad n > 0$$

## 1.2 Fourier Transform

If  $h(t)$  is a periodic function then the Fourier transform is given by:

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

Inverse Fourier transformation of  $H(\omega)$ :

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{j\omega t} d\omega$$

## 1.3 Examples

### 1.3.1 Example 1: Fourier Series

Find the Fourier coefficients and Fourier Series for the square wave shown below:

$$f(t) = \begin{cases} 1 & \text{for } t - a > 0 \\ 0 & \text{for } t - a < 0 \end{cases}$$

and

$$f(x+2) = f(x)$$

Ans:

$$f(x) = \frac{1}{2} + \sum_{n=1,3,5,\dots} \frac{2}{n\pi} \sin(n\pi x)$$

### 1.3.2 Example 2: Fourier Transform

The unit step function is defined as:

$$u(t-a) = \begin{cases} 1 & \text{for } t-a > 0 \\ 0 & \text{for } t-a < 0 \end{cases}$$

is used to define the rectangular pulse function:

$$x(t) = u(t-a) - u(t-b) \quad \text{where } a < b$$

Ans:

$$X(\omega) = \frac{e^{-j\omega a} - e^{-j\omega b}}{j\omega}$$

### 1.3.3 Example 3: Inverse Fourier Transform

Consider the signal:  $x(t) = \sin(\omega_0 t)$  where  $\omega_0$  is a constant. Find the Fourier Transform of this signal to find  $X(\omega)$ .

Ans:

$$X(\omega) = -j\pi(\delta(\omega - \omega_0) - \delta(\omega + \omega_0))$$

## 2 Laplace

Is a generalisation of the Fourier transform and defined as:

$$H(s) = \mathcal{L}\{h(t)\} = \int_{0^-}^{\infty} h(t)e^{-st} dt \quad s \in \mathbb{C}$$

$s$  is a complex number  $s = \sigma + j\omega$  and is identical with Fourier transform, if  $s$  is set to  $j\omega$ .  
Inverse Laplace transformation:

$$f(t) = \frac{1}{2\pi j} \int_{\sigma_c - j\infty}^{\sigma_c + j\infty} F(s)e^{st} ds$$

### 2.1 Examples

#### 2.1.1 Example 1: Laplace Transform

Using the Laplace transform, find the solution for the following equation:

$$\frac{\partial^2}{\partial t^2}y(t) + 2\frac{\partial}{\partial t}y(t) + 2y(t) = 0$$

with initial conditions  $y(0) = 1$  and  $Dy(0) = -1$

#### 2.1.2 Example 2: Transfer Function

Consider a mass-spring-damper system with the following differential equation:

$$m\ddot{x} = -kx - b\dot{x} + f$$

Find the transfer function for the system with input  $f$  and output  $x$ .

Ans:

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

### 3 Several-Variables

#### 3.1 Examples

##### 3.1.1 Example 1: Gradient

Find the rate of change of  $f(x, y) = y^4 + 2xy^3 + x^2y^2$  at  $(0, 1)$  in each of the following directions:

1.  $\mathbf{i} + 2\mathbf{j}$
2.  $\mathbf{j} - 2\mathbf{i}$
3.  $3\mathbf{i}$
4.  $\mathbf{i} + \mathbf{j}$

##### 3.1.2 Example 2: Jacobian

Find the Jacobian  $Df(1, 0)$  for the transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^3$  given by:

$$\mathbf{f}(x, y) = (xe^y + \cos(\pi y), x^2, x - e^y)$$

##### 3.1.3 Example 3: Chain Rule

If  $z = \sin(x^2y)$  where  $x = st^2$  and  $y = s^2 + \frac{1}{t}$ ,  $y = e^{-t}$ , find  $\frac{\partial z}{\partial s}$  and  $\frac{\partial z}{\partial t}$  using chain rule.

##### 3.1.4 Example 4: Substitution

If  $z = \sin(x^2y)$  where  $x = st^2$  and  $y = s^2 + \frac{1}{t}$ ,  $y = e^{-t}$ , find  $\frac{\partial z}{\partial s}$  and  $\frac{\partial z}{\partial t}$  using substitution.

##### 3.1.5 Example 5: Partial Differentiation

Calculate  $f_{223}(x, y, z)$ ,  $f_{232}(x, y, z)$ , and  $f_{322}(x, y, z)$  of the following function:

$$f(x, y, z) = e^{x-2y+3z}$$

## 4 Double-Integrals

### 4.1 Riemann Sum

$$\sum_{i=1}^n f(x_i^*, y_i^*) \Delta A_i$$

$$I = \iint_D f(x, y) dA \quad \text{where } D \text{ is a region in } \mathbb{R}^2 \text{ and } dA \text{ is } dxdy$$

### 4.2 Double Integrals over General domains

If  $f(x, y)$  is defined and bounded on domain  $D$ , then  $\hat{f}(x, y)$  is zero outside  $D$ .

$$\iint_D f(x, y) dA = \iint_R \hat{f}(x, y) dA$$

### 4.3 Iteration of Double Integrals

If  $f(x, y)$  is continuous on the bounded y-simple domain  $D$  given by  $a \leq x \leq b$  and  $c(x) \leq y \leq d(x)$ , then:

$$\iint f(x, y) dA = \int_a^b dx \int_{c(x)}^{d(x)} f(x, y) dy$$

If  $f(x, y)$  is continuous on the bounded x-simple domain  $D$  given by  $c \leq x \leq d$  and  $a(x) \leq y \leq b(x)$ , then:

$$\iint f(x, y) dA = \int_c^d dx \int_{a(x)}^{b(x)} f(x, y) dy$$

### 4.4 Double Integrals in Polar Coordinates

$$dA = dxdy = r drd\theta$$

$$\begin{aligned} x &= r \cos(\theta) & r^2 &= x^2 + y^2 \\ y &= r \sin(\theta) & \tan(\theta) &= \frac{y}{x} \end{aligned}$$

### 4.5 Change of Variables in Double Integrals

If  $x$  and  $y$  are given as a function of  $u$  and  $v$ :

$$\begin{aligned} x &= x(u, v) \\ y &= y(u, v) \end{aligned}$$

These can be transformed or mapped from points  $(u, v)$  in the  $uv$ -plane to points  $(x, y)$  in the  $xy$ -plane.

The inverse transformation is given by:

$$\begin{aligned} u &= u(x, y) \\ v &= v(x, y) \end{aligned}$$

Scaled area element:

$$dA = dxdy = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

where the Jacobian is:

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$

Let  $x(u, v)$  and  $y(u, v)$  be a one-to-one transformation from a domain  $S$  in the  $uv$ -plane onto a domain  $D$  in the  $xy$ -plane.

Suppose, that function  $x$  and  $y$ , and first partial derivatives with respect to  $u$  and  $v$  are continuous in  $S$ . If  $f(x, y)$  is integrable on  $D$ , then  $g(u, v) = f(x(u, v), y(u, v))$  is integrable on  $S$  and:

$$\iint_D f(x, y) dA = \iint_S g(u, v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

## 4.6 Examples

### 4.6.1 Example 1: Change of Variables

Evaluate the double integral:

$$\iint (x - 3y) dA$$

where  $R$  is triangular region with vertices  $(0, 0)$ ,  $(2, 1)$ , and  $(1, 2)$  using the transformation:

$$\begin{aligned} x &= 2u + v \\ y &= u + 2v \end{aligned}$$

### 4.6.2 Example 2: Double integral

Evaluate the double integral by iteration

$$\iint_R (x^2 + y^2) dA$$

where  $R$  is the rectangle  $0 \leq x \leq a$ ,  $0 \leq y \leq b$

### 4.6.3 Example 3: By iteration

Evaluate the double integral by iteration:

$$\iint_D x \cos y dA$$

where  $D$  is the finite region in the first quadrant bounded by the coordinate axes and the curve  $y = 1 - x^2$ .

Ans:

$$\frac{1 - \cos(1)}{2}$$

### 4.6.4 Example 4: Double integral

Find the area within the region:

$$\iint (\sin x + \cos y) dA \quad R : \left\{ x, y \mid \begin{array}{l} 0 \leq x \leq \pi/2 \\ 0 \leq y \leq \pi/2 \end{array} \right.$$

### 4.6.5 Example 5: Defining limits of integration



## 5 Tripple-Integrals

### 5.1 Examples

#### 5.1.1 Example 1: Tripple Integral

Find  $\iiint (x^2 + y^2 + z^2) dV$ , where the region is bounded by  $z = c\sqrt{x^2 + y^2}$  and  $x^2 + y^2 + z^2 = a^2$

## 6 Fields-Curve

### 6.1 Curve & Parameterization

Representation of a curve in 3 space by using its position vector is given as:

$$r = r(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \quad \text{where } a \leq t \leq b$$

### 6.2 Vector Fields

$$\mathbf{F}(x, y, z) = \underbrace{f_x(x, y, z)}_{\text{Scaler function}} \mathbf{i} + \underbrace{f_y(x, y, z)}_{\text{Scaler function}} \mathbf{j} + \underbrace{f_z(x, y, z)}_{\text{Scaler function}} \mathbf{k}$$

$$\frac{\partial f}{\partial x} = f_1 = f_x \quad \frac{\partial f}{\partial y} = f_2 = f_y \quad \frac{\partial f}{\partial z} = f_3 = f_z$$

Position vector:

$$r = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Unit vector with magnitude 1:

$$r = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

#### 6.2.1 Scalar field

$$f(x, y, z) = f_1(x, y, z) = f_2(x, y, z) = f_3(x, y, z)$$

The gradient of a scalar field is a vector field:

$$\nabla f = \text{grad } f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$

#### 6.2.2 Field lines

#### 6.2.3 Conervation field

If  $\mathbf{F}(x, y, z) = \nabla\phi(x, y, z)$  in a domain  $D$ , then  $\mathbf{F}$  is a conservative vector field in  $D$  and function  $\phi$  is the potential function.

$$\mathbf{F}(x, y, z) = \nabla\phi(x, y, z) = \phi_x(x, y, z)\mathbf{i} + \phi_y(x, y, z)\mathbf{j} + \phi_z(x, y, z)\mathbf{k}$$

If the vector field is conservative, then all the following equations are true:

$$\begin{aligned} \frac{\partial f_x}{\partial y} &= \frac{\partial f_y}{\partial x} \\ \frac{\partial f_x}{\partial z} &= \frac{\partial f_z}{\partial x} \\ \frac{\partial f_y}{\partial z} &= \frac{\partial f_z}{\partial y} \end{aligned}$$

#### 6.2.4 Vector field in Polar Coordinates

$$\mathbf{F} = f(r, \theta) = f_r(r, \theta)\hat{\mathbf{r}} + f_\theta(r, \theta)\hat{\boldsymbol{\theta}}$$

where:

$$\begin{aligned} \hat{\mathbf{r}} &= \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j} \\ \hat{\boldsymbol{\theta}} &= -\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j} \end{aligned}$$

## 6.3 Line Integral

$$f(x, y)ds = \text{Area (tiny point)}$$

$$\text{Length of } \mathcal{C} = \int_{\mathcal{C}} f(x, y, z) ds = \int_a^b f(r(t)) \left| \frac{dr}{dt} \right| dt$$

### 6.3.1 Line integral of a vector field

$$W = \int_{\mathcal{C}} \mathbf{F} \cdot \hat{T} ds = \int F dr = \int_{\mathcal{C}} f_1(x, y, z)dx + f_2(x, y, z)dy + f_3(x, y, z)dz$$

## 6.4 Examples

### 6.4.1 Example 1: Conservative vector field and potential

Determine whether the given vector field is conservative, and find a potential function if it is:

$$\mathbf{F}(x, y, z) = (2xy - z^2)\mathbf{i} + (2yz + x^2)\mathbf{j} - (2zx - y^2)\mathbf{k}$$

### 6.4.2 Example 2: Line integral

Evaluate  $\oint x^2y^2 dx + x^3y dy$  counterclockwise around the square with vertices  $(0, 0)$ ,  $(1, 0)$ ,  $(1, 1)$ , and  $(0, 1)$

### 6.4.3 Example 3: Line integral

Evaluate the line integral for  $f(x, y) = x^2y^2$  along a straight line from origin to the point  $(2, 1)$

Ans:  $(5\sqrt{5})/3$

### 6.4.4 Example 4: Line integral vector field

Evaluate the line integral for  $\mathbf{F}(x, y) = y^2\mathbf{i} + 2xy\mathbf{j}$  from  $(0, 0)$  to  $(1, 1)$

Along the line  $y = x$

Along the line  $y = x^2$

### 6.4.5 Example 5: Gradient

### 6.4.6 Example 6: Parametrize a curve

Use  $t = y$  to parametrize the part of the line of intersection of the two planes:

Plane 1:  $y = 2x - 4$

Plane 2:  $z = 3x + 1$  from  $(2, 0, 7)$  to  $(3, 2, 10)$

Ans:  $r(t) = \left(\frac{1}{2}(t+4)\right)\mathbf{i} + (t)\mathbf{j} + \left(\frac{3}{2}t+7\right)\mathbf{k}$

## 7 Theorems

### 7.1 Green's Theorem

$$\oint_C f_1(x, y)dx + f_2(x, y)dy = \iint_R \left( \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dA$$

### 7.2 Stokes' Theorem

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot \hat{\mathbf{N}} dS$$

### 7.3 Divergence Theorem

$$\nabla \cdot \mathbf{F}(x, y, z) = \text{div } \mathbf{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\mathbf{F}(x, y, z) = f_1(x, y, z)\mathbf{i} + f_2(x, y, z)\mathbf{j} + f_3(x, y, z)\mathbf{k}$$

$$\oint_s (\mathbf{F} \cdot \hat{\mathbf{N}}) dS = \iiint_V \text{div } \mathbf{F} dV$$

More variants:

$$\iiint_D \text{curl } \mathbf{F} dV = - \oint_s (\mathbf{F} \times \hat{\mathbf{N}}) dS$$

$$\iiint_D \text{grad } \phi dV = \oint_s \phi dS$$

### 7.4 Examples

#### 7.4.1 Example 1: Div and Curl

Calculate the divergence and curl of the following vector field:

$$\mathbf{F} = \cos x \mathbf{i} - \sin y \mathbf{j} + z \mathbf{k}.$$

**Divergence: Curl:**

#### 7.4.2 Example 2: Green's Theorem

Using Green's Theorem evaluate  $\oint_c (x^2y) dx + (xy^2) dy$ , clockwise boundary of the region:

$$0 \leq y \leq \sqrt{9 - x^2}$$

#### 7.4.3 Example 3: Stokes' Theorem

#### 7.4.4 Example 4: Divergence Theorem

Use the Divergence Theorem to calculate the flux of the given vector field out of the sphere  $s$  with equation  $x^2 + y^2 + z^2 = a^2$ , where  $a > 0$  and

$$\mathbf{F} = (x^2 + y^2)\mathbf{i} + (y^2 - z^2)\mathbf{j} + z\mathbf{k}$$

## 8 PDE

Partial Differential Equations are equations with multiple variables and derivatives. They are used to model many physical phenomena, such as heat, sound, and light. The totality of solutions to a PDE is called its general solution, and there can be a lot.

### 8.1 Classification of PDEs

General representation of a PDE:

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = G$$
$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

Conditions:

**Linear:**  $A, B, C, D, E, F$  are constants

**Quasi-linear:**  $A, B, C$  are constants

**Fully non-linear:**  $A, B, C, D, E, F$  are functions of  $u$  and its partial derivatives

### 8.2 Characteristics of PDEs

$B^2 - 4AC > 0$	2 real roots	2 characteristics	<b>Hyperbolic PDE</b>
$B^2 - 4AC = 0$	1 real roots	1 characteristics	<b>Parabolic PDE</b>
$B^2 - 4AC < 0$	0 real roots	0 characteristics	<b>Elliptic PDE</b>

Types of various PDEs:

**Wave Equation:** Hyperbolic PDE

**Heat Equation:** Parabolic PDE

**Laplace Equation:** Elliptic PDE

### 8.3 Initial and Boundary Conditions

### 8.4 Wave Equation (1D)

One dimensional wave equation is given by:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where } c^2 = \left[ \frac{T(x,t)}{\mu_x} \right] \quad (1)$$

Steps to solve:

1. Method of Separation of Variables  $u(x, t) = X(x)T(t)$
2. Satisfy the Boundary Conditions test
3. Fourier Series Validation

#### 8.4.1 D'Alembert's Solution of the Wave Equation

His solution is given by eq. (1) but extended to two variables:

$$v = (x - ct) \quad w = (x + ct) \quad (2)$$

I.e.  $u(v, w)$ . Partial derivatives from chain rule:

$$u_x = u_v \cdot v_x + u_w \cdot w_x = u_v + u_w$$

For double derivatives:

$$u_{xx} = (u_v + u_w)_x = (u_v + u_w)_v v_x + (u_v + u_w)_w w_x = u_{vv} + 2u_{vw} + u_{ww}$$

With respect to  $t$ :

$$u_{tt} = c^2 u_{xx} = c^2 (u_{vv} + 2u_{vw} + u_{ww})$$

From eq. (1) and eq. (2):

$$u_{vw} = \frac{\partial^2 u}{\partial w \partial v} = 0$$

This can be solved by integrating with respect to  $v$  and  $w$ :

$$\frac{\partial u}{\partial v} = h(v) \text{ and } u = \int h(v) dv + \psi(w)$$

Here,  $h(v)$  and  $\psi(w)$  are arbitrary functions of  $v$  and  $w$ , respectively. The solution in term for  $x$ :

$$u = \phi(v) + \psi(w)$$

This is d'Alembert's solution, which is the general solution to the wave equation.

$$u(x, t) = \phi(x + ct) + \psi(x - ct)$$

This solution satisfies the wave equation and the initial conditions:

## 8.5 Heat Equation (1D)

$$\frac{\partial u}{\partial t} = c \frac{\partial^2 u}{\partial x^2} \quad (3)$$

Conditions:

- PDE is linear and homogeneous.
- Boundary conditions are linear and homogeneous. The two for  $u(x, t)$  is  $u(0, t) = 0$  and  $u(L, t) = 0$  for all  $t > 0$ .
- One initial condition at time ( $t = 0$ ):  $u(x, 0) = f(x)$ .

Solve the

Solution:

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda_n^2 t}$$

where

$$\lambda_n = \frac{cn\pi}{L}$$

and

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad \text{for } n = 1, 2, 3 \dots$$

## 8.6 Examples