Signal Processing

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Contents

1 Fourier

1.1 Fourier Series

A periodic function with period $2L$ and let $f(x)$ and $f'(x)$ be piecewise continuous on the interval $-L < x < L$

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right) = \sum_{n=-\infty}^{\infty} c_n e^{jn\pi x/L}$$

The coefficients are given by:

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \qquad n \geq 0$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \qquad n > 0$$

$$c_n = \frac{1}{2}(a_n - jb_n) \qquad n > 0$$

1.2 Fourier Transform

If $h(t)$ is a periodic function then the Fourier transform is given by:

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t}\, dt$$

Inverse Fourier tranformation of $H(\omega)$:

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{j\omega t}\, d\omega$$

1.3 Examples

1.3.1 Example 1: Fourier Series

Find the Fourier coefficients and Fourier Series for the square wave shown below:

$$f(t) = \begin{cases} 1 & \text{for } t - a > 0 \\ 0 & \text{for } t - a < 0 \end{cases}$$

and

$$f(x+2) = f(x)$$

Ans:

$$f(x) = \frac{1}{2} + \sum_{n=1,3,5,\ldots} \frac{2}{n\pi} \sin(n\pi x)$$

1.3.2 Example 2: Fourier Transform

The unit step function is defined as:

$$u(t-a) = \begin{cases} 1 & \text{for } t - a > 0 \\ 0 & \text{for } t - a < 0 \end{cases}$$

is used to define the rectangular pulse function:

$$x(t) = u(t-a) - u(t-b) \qquad \text{where } a < b$$

Ans:

$$X(\omega) = \frac{e^{-j\omega a} - e^{-j\omega b}}{j\omega}$$

1.3.3 Example 3: Inverse Fourier Transform

Consider the signal: $x(t) = \sin(@omega_0 t)$ where ω_0 is a constant. Find the Fourier Transform of this signal to find $X(\omega)$.

Ans:

$$X(\omega) = -j\pi(\delta(\omega - \omega_0) - \delta(\omega + \omega_0))$$

2 Laplace

Is a generalisation of the Fourier transform and defined as:

$$H(s) = \mathcal{L}\{h(t)\} = \int_{0^-}^{\infty} h(t)e^{-st}\, dt \qquad s \in \mathbb{C}$$

s is a complex number $s = \sigma + j\omega$ and is identical with Fourier transform, if s is set to $j\omega$.
Inverse Laplace tranformation:

$$f(t) = \frac{1}{2\pi j} \int_{\sigma_c - j\infty}^{\sigma_c + j\infty} F(s)e^{st}\, ds$$

2.1 Examples

2.1.1 Example 1: Laplace Transform

Using the Laplace transform, find the solution for the following equation:

$$\frac{\partial^2}{\partial t^2} y(t) + 2\frac{\partial}{\partial t} y(t) + 2y(t) = 0$$

with initial conditions $y(0) = 1$ and $Dy(0) = -1$

2.1.2 Example 2: Trabsfer Function

Consider a mass-spring-damper system with the following differential equation:

$$m\ddot{x} = -kx - b\dot{x} + f$$

Find the transfer function for the system with input f and output x.

Ans:

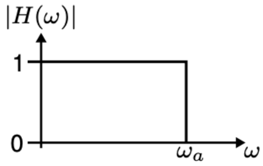
$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

3 Filter-Functions

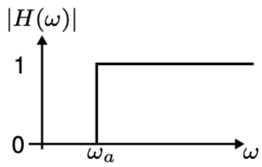
Pass band: Frequency range where the signal passes through the filter.
Stop band: Frequency range where the signal is attenuated by the filter.
Cut-off frequency: Frequency where the signal is attenuated by 3 dB.
Attenuation: The decrease in signal strength.
Ripple: The variation in the pass band.
Form factor: The ratio between the stop band and the pass band.
Group delay: The time delay of the filter.
Phase delay: The phase shift of the filter.

3.1 Electronic filters

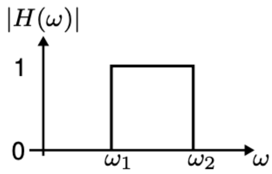
Low-pass filter: Ideally only allows low angle frequencies ($\omega < \omega_a$)



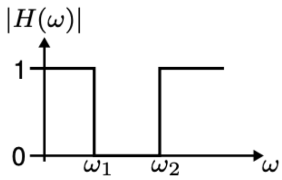
High-pass filter: Ideally only high angle frequencies ($\omega > \omega_a$)



Band-pass filter: Ideally only certain range of angle frequencies ($\omega_1 < \omega > \omega_2$)



Band-stop filter: Ideally filters out a certain range of angle frequencies ($\omega_1 > \omega < \omega_2$)



3.2 Filter specification

Specifications for a low-pass filter could be:

1. A cut-off frequency ω_a that specifies the upper limit of the passband.
2. A stopband frequency ω_s at which a stopband attenuation As is specified.
3. Information about permissible gain variation in the passband.

3.3 Group delay

Measures time for the filter and often depend on frequency ω.

$$T_g = -\frac{d\phi(\omega)}{d\omega}$$

where $\phi(\omega)$ is the angle of the filter.

The phase characteristic of a filter also affects the input-output behavior of a filter. If pulse overshoot or damped oscillation (ringing) is to be avoided, the filter must have a constant group delay, T_g.

3.4 Filter transfer functions

In practice, the ideal filters cannot be realized, but can be approximated using various filter function types.

These are all-pole filters (Only poles, no zeros) When selecting the filter function for a given application, we are interested in the following characteristics:

1. Constant gain in passband.
2. High attenuation after cut-off frequency.
3. Linear phase.

3.4.1 Butterworth

$$|H(j\omega)|^2 = \frac{1}{1 + (\frac{\omega}{\omega_a})^{2N}}$$

where ω_a is the cut-off frequency [rad/s]

All poles for $H(j\omega)$ lie in the left half-plane on a circle with radius ω_a and centre at the origin.

A Butterworth filter has the following characteristics 1. Optimal in terms of constant gain in the passband. 2. Has 3 dB attenuation at the cut-off frequency and then the filter gain drops rapidly by 20 dB/dec. (dec is when the power of 10 changes by 1) 3. The phase of the filter is not constant in the passband, which causes ringing at the step input. All pole pairs of a Butterworth filter have a natural natural natural frequency ω_n, which is equal to the cut-off frequency ω_a.

3.4.2 Chebyshev

$$|H(\omega)|^2 = \frac{1}{1 + e^2 T_N^2(\omega - \omega_a)}$$

where $T_N(\omega)$ is Chebyshev polynomial of degree N given by

$$T_N(\omega) = \begin{cases} \cos(N \cdot \cos^{-1}(\omega)) & : |\omega| \leq 0 \\ \cosh(N \cdot \cosh^{-1}(\omega)) & : |\omega| > 0 \end{cases}$$

The size of the passband ripple δ in dB is given by ϵ

$$\delta = 10 \cdot$$

A Chebyshev filter has the following characteristics 1. Has varying gain in the passband - the size of the passband ripple can be freely selected. 2. The gain decreases rapidly around the cut-off frequency. 3. The phase of the filter is not constant in the passband, which causes ringing at the step input.

The DC gain of a Chebyshev low-pass filter is not the maximum gain of the filter if the pole count is even.

3.4.3 Bessel

The transfer function for an Nth order Bessel filter is

$$H_N(s) = \frac{b_0}{s^N + b_{N-1}s^{N-1} + \cdots + b_1 s + b_0}$$

where

$$b_k = \frac{(2N-k)!}{2^{N-k}k!(N-k)!}$$

A Bessel filter has the following properties 1. Does not have ripple in the passband, but the amplitude is not as constant as a Butterworth filter. 2. Has attenuation that is very smooth. The amplitude characteristic of the filter is like that of the first order filter for the first 6 dB of attenuation regardless of the number of poles. 3. The phase is almost linear with frequency within the passband.

The filter's order number The order number of filters can be found by reading the amplitude characteristic of a given filter function and comparing it to the attenuation requirements at the stopband frequency.

$$Y = 20 \log \left(\frac{A}{A_{\text{ max}}} \right)$$

Frequency normalised (form factor)

$$X = \frac{\omega}{\omega_a}$$

Filter transformers Low pass to high pass transformation High-pass filters can be designed from normalised prototype low-pass filters based on the following specification: - The filter function (Bessel, Butterworth, Chebyshev) - The filter cut-off frequency ω_a - The filter stopband frequency ω_s - The filter stopband attenuation A_s at the stopband frequency ω_s

![[Pasted image 20230918163041.png—center—600]] A low-pass filter can be transformed into a high-pass filter by:

$$H_{\text{ hp }}(s) = H_{\text{ lp }}(\bar{s})\big|_{\bar{s}=\frac{1}{s}}$$

¿ [!example]- Low pass to high pass transformation ¿ 1. Find the stopband frequency for the normalized low-pass filter (used for the the design):

$$\frac{\omega_a}{\omega_s}$$

¿ 2. Filter order selection based on amplitude characteristics for lowpass filters. ¿ 3. The transfer function of the normalized low-pass filter is transformed to the transfer function of the normalized high-pass filter by replacing s with $\frac{1}{s}$ ¿ 4. The transfer function of the denormalized high-pass filter is found by replacing s with $\frac{s}{\omega_a}$

Low pass to bandpass transformation Bandpass filters can be designed from normalized prototype low-pass filters based on the following specification - The filter function (Bessel, Butterworth, Chebyshev) - The filter center frequency ω_s - The passband bandwidth $\Delta\omega_a$ - Stopband bandwidth $\Delta\omega_s$ - Filter stopband attenuation A_s

![[Pasted image 20230918163750.png—center—600]] A low-pass filter can be transformed into a band-pass filter by

$$H_{\text{ bp }}(s) = H_{\text{ lp }}(\bar{s})\big|_{\bar{s}=\frac{1}{W_a}\left(s+\frac{1}{s}\right)}$$

To design the filter, the bandpass filter is normalised, i.e. the normalised stopband width and the normalised passband width are calculated as:

$$W_a = \frac{\Delta f_a}{f_c}$$

$$W_s = \frac{\Delta f_s}{f_c}$$

and the form factor for bandpass filter:

$$F = \frac{\Delta f_s}{\Delta f_a} = \frac{W_s}{W_a}$$

¿ [!example] Low pass to bandpass transformation ¿ 1. To design the filter, the bandpass filter is normalized, i.e. the normalized stopband width and the normalized passband width are calculated. ¿ 2. The form factor is found. ¿ 3. Filter order selection based on amplitude characteristics for lowpass filters. ¿ 4. The normalised filter is found by replacing s with $\frac{1}{W_a}\left(s+\frac{1}{s}\right)$ in the low-pass filter. ¿ 5. The denormalized bandpass filter is found by replacing s in the normalized bandpass filter with $\frac{s}{\omega_c}$.

Low pass to bandstop transformation Bandstop filters can be designed from normalised prototype low-pass filters based on the following specification - Filter function (Bessel, Butterworth, Chebyshev) - Filter centre frequency ω_c - Passband bandwidth $\Delta\omega_c$ - Stopband bandwidth $\Delta\omega_s$ - Filter stopband attenuation A_s

![[Pasted image 20230918173146.png—center—600]] A low-pass filter can be transformed into a bandstop filter by

$$H_{\text{ bs }}(s) = H_{\text{ lp }}(\bar{s})\big|_{\bar{s}=\frac{W_a}{s+\frac{1}{s}}}$$

and the form factor for bandstop filter:

$$F = \frac{\Delta f_a}{\Delta f_s} = \frac{W_a}{W_s}$$

¿ [!example]- Low pass to bandstop transformation ¿ 1. To design the filter, the bandstop filter is normalized, i.e. the normalized stopband width and the normalized passband width are calculated. ¿ 2. The form factor is found. ¿ 3. Filter order selection based on amplitude characteristics for lowpass filters. ¿ 4. The normalised filter is found by replacing s with $\frac{W_a}{s+\frac{1}{s}}$ in the low-pass filter. ¿ 5. The denormalized bandpass filter is found by replacing s in the normalized bandpass filter with $\frac{s}{\omega_c}$.

4 Sampling-Reconstruction

Nyquist-Shannon Phrase:
A continuous-time signal $x(t)$ can only be correctly recovered from $x_s(t)$ if the sampling frequency is at least **twice** the highest frequency in the spectrum of $x(t)$.

5 FFT

6 Z-transformation

7 System-analysis

8 Digital-Realization-Structures

9 FIR-Filters

Finite Impulse Response filters where the impulse response of the system is of finite duration. A FIR filter with N samples has a impulse response:

$$h(n) = \begin{cases} a_n & \text{for } 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore the filter has N coefficients a_n for $n = 0, 1, \cdots, N-1$.

9.1 Linear phase

9.2 Frequency response

9.3 FIR-Filter design

If there is N samples in the impulse response, then:

$$M = \frac{N-1}{2}$$

And the transfer function of the filter is:

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{i=0}^{2M} a_i z^{-i}$$

Filter type	$\mathbf{c_0}$	$\mathbf{c_m = c_{-m}}$	a_i
Low-pass	$2Tf_a$	$\frac{1}{m\pi}\sin(2\pi mTf_a)$	c_{M-i}
High-pass	$1 - 2Tf_a$	$\frac{1}{m\pi}(\sin(m\pi) - \sin(2\pi mTf_a))$	c_{M-i}
Band-pass	$2T(f_{a_2} - f_{a_1})$	$\frac{1}{m\pi}\big(\sin(2\pi mTf_{a_2}) - \sin(2\pi mTf_{a_1})\big)$	c_{M-i}
Band-stop	$1 - 2T(f_{a_2} - f_{a_1})$	$\frac{1}{m\pi}(\sin(m\pi) + \sin(2\pi mTf_{a_1}) - \sin(2\pi mTf_{a_2}))$	c_{M-i}

Using the inverse z-transform, the impulse response is found:

$$y(n) = \sum_{i=0}^{2M} a_i x(n-i)$$

9.4 Window functions

Due to the ripples in the passband and stopband, the ideal filter is not realizable. Therefore, the ideal impulse response is multiplied by a window function $w(n)$ to obtain a realizable impulse response $h(n)$:

$$h(n) = h_\infty(n)w(n)$$

Following are some common window functions:

9.4.1 Rectangular window

$$w(n) = \begin{cases} 1 & \text{if } -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

Window	$\mathbf{B_n}$	$\mathbf{M_{min}}$	Min. stopband attenuation	Min. passband ripple
Rectangular	2	f_s/Δ_f	20 dB	1.5 dB
Barlett	4	$2f_s/\Delta_f$	25 dB	0.1 dB
Hamming	4	$2f_s/\Delta_f$	50 dB	0.05 dB
Hanning	4	$2f_s/\Delta_f$	45 dB	0.1 dB
Kasier ($\beta = \pi$)	2.8	$1.4f_s/\Delta_f$	40 dB	0.2 dB
Kasier ($\beta = 2\pi$)	4.4	$2.2f_s/\Delta_f$	65 dB	0.01 dB

9.4.2 Barlett window

$$w(n) = \begin{cases} 1 - \frac{|n|}{M} & \text{if } -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

9.4.3 Hamming and Hanning window

$$w(n) = \begin{cases} \alpha + (1-\alpha)\cos\left(\frac{n\pi}{M}\right) & \text{if } -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha = 0.54$ for Hamming and $\alpha = 0.5$ for Hanning.

9.4.4 Kaizer window

$$w(n) = \begin{cases} \frac{I_0\left(\beta\sqrt{1-\left(\frac{n}{M}\right)^2}\right)}{I_0(\beta)} & \text{if } -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

9.5 Examples

9.5.1 Low-pass filter

9.5.2 High-pass filter

9.5.3 Band-pass filter

9.5.4 Band-stop filter

10 IRR-Filters

10.1 Design of IRR-filters

Design of IRR-filters is done in the following steps:

1. The specifications of the IRR-filter.
2. The transfer function of the IRR-filter.
3. The optimal realization of the IRR-filter.
4. Program for signalprocessing or a circuit diagram for analog signalprocessing.

10.2 Matched z-transform

By using the matched z-transform, the poles and zeros of the IRR-filter are directly transferred to the z-plane. The transfer function of the IRR-filter is given by:

$$z = e^{sT}$$

10.2.1 1. Order IRR-filters

The transfer function for a first order system is given by:

$$H(s) = \frac{s + A_0}{s + B_0} = \frac{s - \sigma_1}{s - \sigma_2}$$

where $-A_0 = \sigma_1$ is a real zero and $-B_0 = \sigma_2$ is a real pole of the IRR-filter.

A digital first order transfer function is given by:

$$H(z) = \frac{z - e^{\sigma_1 T}}{z - e^{\sigma_2 T}} = \frac{1 - e^{\sigma_1 T} z^{-1}}{1 - e^{\sigma_2 T} z^{-1}}$$

A first order system without a zero is given by:

$$H(s) = \frac{\omega_a}{s + \omega_a}$$

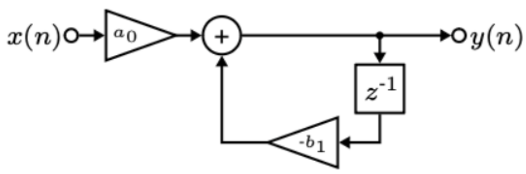
Using the matched z-transform, the digital transfer function is given by:

$$H(z) = \frac{1}{1 - e^{\sigma_1 T} z^{-1}}$$

where σ_1 is the pole of $H(s)$ and T is the sampling period.
The transfer function $H(z)$ does not have a DC-gain of 1. To get a DC-gain of 1, the transfer function is multiplied by a gain factor a_0:

$$H(z) = \frac{a_0}{1 + b_1 z^{-1}}$$

where $b_0 = -e^{\sigma_1 T}$ and $a_0 = 1 + b_1$.
The corresponding digital realization structure:



10.2.2 2. Order IRR-filters

The transfer function for a second order system with complex conjugate pole and zero pair is given by:

$$H(s) = \frac{s^2 + A_1 s + A_0}{s^2 + B_1 s + B_0}$$

which has zeros in $s_1 = \sigma_1 + j\omega_1, s_1^* = \sigma_1 - j\omega_1$ and poles in $s_2 = \sigma_2 + j\omega_2, s_2^* = \sigma_2 - j\omega_2$

Using the matched z-transform:

$$H(z) = \frac{(z - e^{\sigma_1 T} e^{j\omega_1 T})(z - e^{\sigma_2 T} e^{-j\omega_1 T})}{(z - e^{\sigma_2 T} e^{j\omega_2 T})(z - e^{\sigma_2 T} e^{-j\omega_2 T})}$$

10.3 Impulse invariance method

10.4 Examples

10.4.1 Example 1: Matched 1. order z-transform

Consider the following analog first order lowpass filter with cutoff frequency $\omega_a = 2\pi f_a$ where $f_a = 300\,\text{Hz}$:

$$H(s) = \frac{\omega_a}{s + \omega_a} = \frac{1885}{s + 1885}$$

which has a pole in $s_1 = \sigma_1 = -1885$.
Find the digital IRR-filter using the matched z-transform with sampling frequency $f_s = 16\,\text{kHz}$.

The coefficients of the digital IRR-filter are given by:

$$b_1 = -e^{\sigma_1 T} = -e^{-1885 \cdot \frac{1}{16000}} = -0.8889$$

$$a_0 = 1 + b_1 = 0.1111$$

The digital lowpass filter is given by:

$$H(z) = \frac{a_1}{1 + b_0 z^{-1}} = \frac{0.1111}{1 - 0.8888 z^{-1}}$$