Signal Processing

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1 Fourier

1.1 Fourier Series

A periodic function with period $2L$ and let $f(x)$ and $f'(x)$ be piecewise continuous on the interval $-L < x < L$

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right) = \sum_{n=-\infty}^{\infty} c_n e^{jn\pi x/L}$$

The coefficients are given by:

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \qquad n \geq 0$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \qquad n > 0$$

$$c_n = \frac{1}{2}(a_n - jb_n) \qquad n > 0$$

1.2 Fourier Transform

If $h(t)$ is a periodic function then the Fourier transform is given by:

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} \, dt$$

Inverse Fourier tranformation of $H(\omega)$:

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{j\omega t} \, d\omega$$

Signal	**Fourier Transform**
$\delta(t)$	1
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$
$\delta(t - t_0)$	$e^{-j\omega t_0}$
$\sin(\omega_0 t)$	$-j\pi(\delta(\omega - \omega_0) - \delta(\omega + \omega_0))$
$\cos(\omega_0 t)$	$\pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$
1	$2\pi\delta(\omega)$

1.3 Examples

1.3.1 Example 1: Fourier Series

Find the Fourier coefficients and Fourier Series for the square wave shown below:

$$f(x) = \begin{cases} 0 & \text{for } -1 \leq x \leq 0 \\ 1 & \text{for } 0 \leq x \leq 1 \end{cases}$$

and

$$f(x + 2) = f(x)$$

The fourier series is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

Find the L value:

$$2L = 2 \qquad \Rightarrow \qquad L = 1$$

Find a_0:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx \qquad n \geq 0$$

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\cos\left(\frac{0\pi x}{1}\right) dx = \int_{-1}^{1} f(x)dx = \int_{-1}^{0} 0dx + \int_{0}^{1} 1dx = 1$$

Find a_n:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx \qquad n \geq 0$$

$$= \frac{1}{1}\int_{-1}^{1} f(x)\cos\left(\frac{n\pi x}{1}\right) dx = \int_{-1}^{1} f(x)\cos(n\pi x)\, dx$$

$$= \int_{-1}^{0} 0\cos(n\pi x)\, dx + \int_{0}^{1} 1\cos(n\pi x)\, dx = 0 + \left[\frac{\sin(n\pi x)}{n\pi}\right]_0^1 = \frac{\sin(\pi n)}{\pi n}$$

For all n:

$$\frac{\sin(\pi n)}{\pi n} = 0$$

Find b_n:

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx \qquad n > 0$$

$$b_n = \int_{-1}^{1} f(x)\sin\left(\frac{n\pi x}{1}\right) dx = \int_{-1}^{0} 0\sin(n\pi x)\, dx + \int_{0}^{1} 1\sin(n\pi x)\, dx$$

$$= 0 + \left[\frac{-\cos(n\pi x)}{n\pi}\right]_0^1 = \frac{-\cos(n\pi 1)}{n\pi} - \frac{-\cos(n\pi 0)}{n\pi}$$

If n is even the function will cancel out, therefore $n = 1, 3, 5, \dots$ (odd):

$$= \frac{1}{n\pi} + \frac{1}{n\pi} = \frac{2}{n\pi}$$

Ans:

$$f(x) = \frac{1}{2} + \sum_{n=1,3,5,\dots} \frac{2}{n\pi}\sin(n\pi x)$$

1.3.2 Example 2: Fourier Transform

The unit step function is defined as:

$$u(t-a) = \begin{cases} 1 & \text{for } t-a>0 \\ 0 & \text{for } t-a<0 \end{cases}$$

is used to define the rectangular pulse function:

$$x(t) = u(t-a) - u(t-b) \qquad \text{where } a<b$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$X(\omega) = \int_{-\infty}^{a} 0e^{-j\omega t}\,dt + \int_{a}^{b} 1e^{-j\omega t}\,dt + \int_{b}^{\infty} 0e^{-j\omega t}\,dt$$

$$X(\omega) = 0 + \left[\frac{-e^{-j\omega t}}{j\omega}\right]_a^b + 0$$

Insert the limits:

$$X(\omega) = \frac{e^{-j\omega a} - e^{-j\omega b}}{j\omega}$$

2 Laplace transform

Is a generalisation of the Fourier transform and defined as:

$$H(s) = \mathcal{L}\{h(t)\} = \int_{0^-}^{\infty} h(t)e^{-st}\,dt \qquad s \in \mathbb{C}$$

s is a complex number $s = \sigma + j\omega$ and is identical with Fourier transform, if s is set to $j\omega$.
Inverse Laplace tranformation:

$$h(t) = \mathcal{L}^{-1}\{H(t)\} = \frac{1}{2\pi j}\int_{\sigma_c - j\infty}^{\sigma_c + j\infty} H(s)e^{st}\,ds$$

Signal	Laplace Transform
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$
$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$

2.1 General Formulas

$$\mathcal{L}\{f(t)\} = F(s)$$
$$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$$
$$\mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0) - f'(0)$$
$$\mathcal{L}\{f^n\} = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - \cdots - f^{n-1}(0)$$

2.2 Examples

2.2.1 Example 1: Laplace Transform

Using the Laplace transform, find the solution for the following equation:

$$\frac{\partial^2}{\partial t^2}y(t) + 2\frac{\partial}{\partial t}y(t) + 2y(t) = 0$$

with initial conditions $y(0) = 1$ and $y'(0) = -1$

Take laplace transform of the equation:

$$s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + 2(Y(s)) = 0$$
$$s^2Y(s) - s + 1 + 2sY(s) - 2 + 2Y(s) = 0$$

$$s^2Y(s) + 2sY(s) + 2Y(s) = 1 + s$$
$$(s^2 + 2s + 2)Y(s) = 1 + s$$
$$Y(s) = \frac{1+s}{(s^2+2s+2)} = \frac{1+s}{((s+1)^2+1)}$$

From table lookup:

$$\mathcal{L}\{e^{at}\cos(\omega t)\} = \frac{s-a}{(s-a)^2+\omega^2}$$
$$a = -1 \qquad \omega = 1$$
$$y(t) = e^{-t}\cos(t)$$

2.2.2 Example 2: Transfer Function

Consider a mass-spring-damper system with the following differential equation:

$$m\ddot{x} = -kx - b\dot{x} + f$$

Find the transfer function for the system with input f and output x.

$$ms^2X(s) = -kX(s) - bsX(s) + F(s)$$
$$ms^2X(s) + kX(s) + bsX(s) = F(s)$$
$$(ms^2 + k + bs)X(s) = F(s)$$

The transfer function is:

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2+k+bs}$$

2.2.3 Example 3: Differential equation

Consider:

$$y''(t) + y'(t) = 0.5t$$

where $y(0) = 0$ and $y'(0) = 0$ Use Laplace transform to solve the equation and find $y(t)$

From Laplace transform table:

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

Laplace transform of given differential equation:

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = 0.5\frac{1}{s^2}$$
$$(s^2+1)Y(s) = \frac{0.5}{s^2}$$
$$Y(s) = \frac{0.5}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}$$
$$0.5 = As(s^2+1) + B(s^2+1) + (Cs+D)s^2$$
$$0.5 = As^3 + As + Bs^2 + B + Cs^3 + Ds^2$$
$$0.5 = s^3(A+C) + s^2(B+D) + sA + B$$
$$B = 0.5$$

$$A = 0$$
$$B + D = 0 \qquad \Rightarrow \qquad D = -0.5$$
$$A + C = 0 \qquad \Rightarrow \qquad C = 0$$

Therefore the partial fractions are:

$$Y(s) = \frac{0}{s} + \frac{0.5}{s^2} + \frac{0s - 0.5}{s^2 + 1} = \frac{0.5}{s^2} + \frac{-0.5}{s^2 + 1}$$

From Laplace transform table:

$$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2 + a^2}$$

The inverse laplace transform of $Y(s)$:

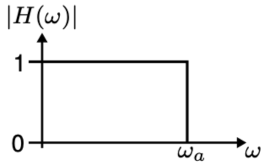
$$y(s) = 0.5t - 0.5\sin(t)$$

3 Filter-Functions

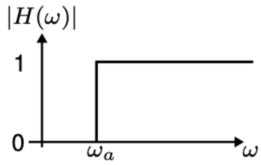
Pass band: Frequency range where the signal passes through the filter.
Stop band: Frequency range where the signal is attenuated by the filter.
Cut-off frequency: Frequency where the signal is attenuated by 3 dB.
Attenuation: The decrease in signal strength.
Ripple: The variation in the pass band.
Form factor: The ratio between the stop band and the pass band.
Group delay: The time delay of the filter.
Phase delay: The phase shift of the filter.

3.1 Electronic filters

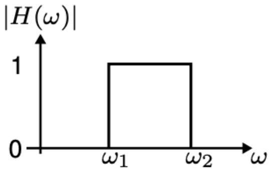
Low-pass filter: Ideally only allows low angle frequencies ($\omega < \omega_a$)



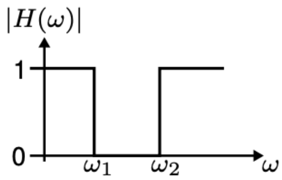
High-pass filter: Ideally only high angle frequencies ($\omega > \omega_a$)



Band-pass filter: Ideally only certain range of angle frequencies ($\omega_1 < \omega > \omega_2$)



Band-stop filter: Ideally filters out a certain range of angle frequencies ($\omega_1 > \omega < \omega_2$)



3.2 Filter specification

Specifications for a low-pass filter could be:

1. A cut-off frequency ω_a that specifies the upper limit of the passband.
2. A stopband frequency ω_s at which a stopband attenuation As is specified.
3. Information about permissible gain variation in the passband.

3.3 Group delay

Measures time for the filter and often depend on frequency ω.

$$T_g = -\frac{d\phi(\omega)}{d\omega}$$

where $\phi(\omega)$ is the angle of the filter.

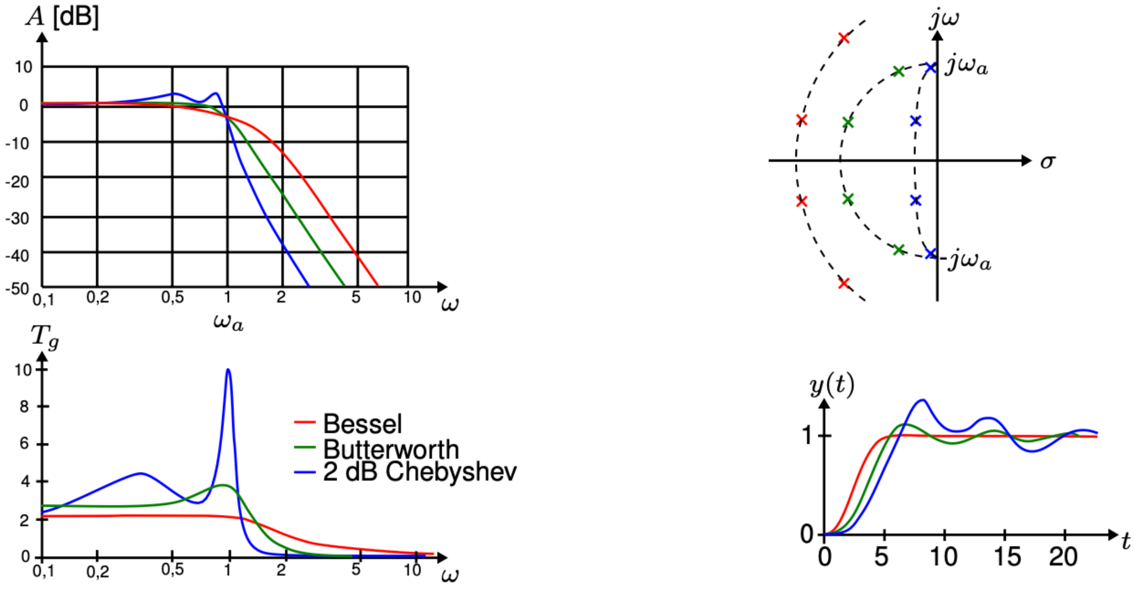
The phase characteristic of a filter also affects the input-output behavior of a filter. If pulse overshoot or damped oscillation (ringing) is to be avoided, the filter must have a constant group delay, T_g.

3.4 Filter transfer functions

In practice, the ideal filters cannot be realized, but can be approximated using various filter function types.

These are all-pole filters (Only poles, no zeros) When selecting the filter function for a given application, we are interested in the following characteristics:

1. Constant gain in passband.
2. High attenuation after cut-off frequency.
3. Linear phase.



3.4.1 Butterworth

$$|H(j\omega)|^2 = \frac{1}{1 + (\frac{\omega}{\omega_a})^{2N}}$$

where ω_a is the cut-off frequency [rad/s]

All poles for $H(j\omega)$ lie in the left half-plane on a circle with radius ω_a and centre at the origin.

A Butterworth filter has the following characteristics:

1. Optimal in terms of constant gain in the passband.
2. Has 3 dB attenuation at the cut-off frequency and then the filter gain drops rapidly by 20 dB/dec. (dec is when the power of 10 changes by 1)

3. The phase of the filter is not constant in the passband, which causes ringing at the step input.

All pole pairs of a Butterworth filter have a natural frequency ω_n, which is equal to the cut-off frequency ω_a.

Construction tables: Page 169 (Digital Signal Behandling: Erik Hüche)
Amplitude characteristics to determine order: Page 161 (Digital Signal Behandling: Erik Hüche)

3.4.2 Chebyshev

$$|H(\omega)|^2 = \frac{1}{1 + e^2 T_N^2(\omega - \omega_a)}$$

where $T_N(\omega)$ is Chebyshev polynomial of degree N given by

$$T_N(\omega) = \begin{cases} \cos(N \cdot \cos^{-1}(\omega)) & : |\omega| \leq 0 \\ \cosh(N \cdot \cosh^{-1}(\omega)) & : |\omega| > 0 \end{cases}$$

The size of the passband ripple δ in dB is given by ϵ

$$\delta = 10 \cdot \log(\epsilon^2 + 1)$$

A Chebyshev filter has the following characteristics

1. Has varying gain in the passband - the size of the passband ripple can be freely selected.
2. The gain decreases rapidly around the cut-off frequency.
3. The phase of the filter is not constant in the passband, which causes ringing at the step input.

The DC gain of a Chebyshev low-pass filter is not the maximum gain of the filter if the pole count is even.

Construction tables: Page 170 (Digital Signal Behandling: Erik Hüche)
Amplitude characteristics to determine order: Page 162 (Digital Signal Behandling: Erik Hüche)

3.4.3 Bessel

The transfer function for an Nth order Bessel filter is

$$H_N(s) = \frac{b_0}{s^N + b_{N-1}s^{N-1} + \cdots + b_1 s + b_0}$$

where

$$b_k = \frac{(2N-k)!}{2^{N-k}k!(N-k)!}$$

A Bessel filter has the following properties:

1. Does not have ripple in the passband, but the amplitude is not as constant as a Butterworth filter.
2. Has attenuation that is very smooth. The amplitude characteristic of the filter is like that of the first order filter for the first 6 dB of attenuation regardless of the number of poles.
3. The phase is almost linear with frequency within the passband.

Construction tables: Page 173 (Digital Signal Behandling: Erik Hüche)
Amplitude characteristics to determine order: Page 163 (Digital Signal Behandling: Erik Hüche)

3.4.4 The filter's order number

The order number of filters can be found by reading the amplitude characteristic of a given filter function and comparing it to the attenuation requirements at the stopband frequency.

$$Y = 20\log\left(\frac{A}{A_{\text{ max}}}\right)$$

Frequency normalised (form factor)

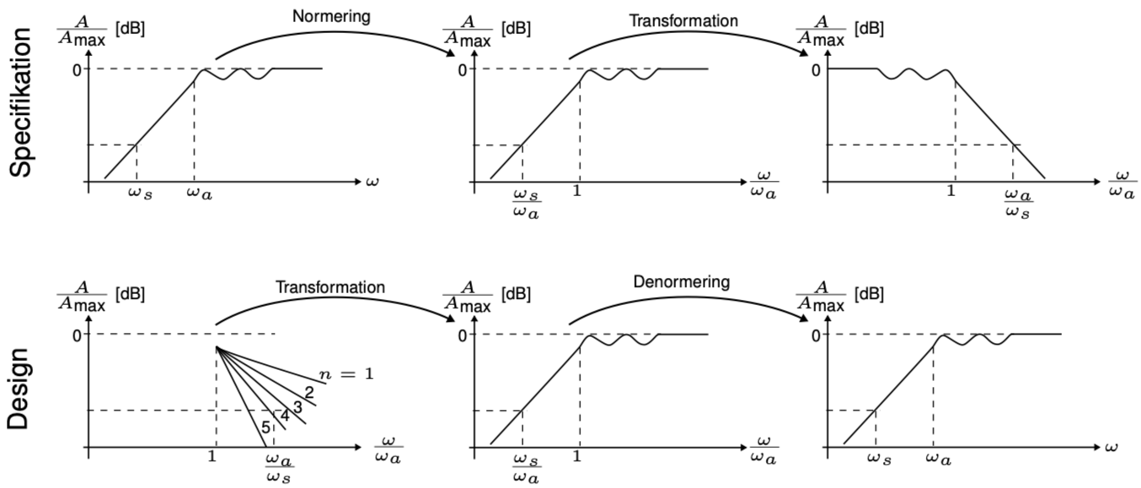
$$X = \frac{\omega}{\omega_a}$$

3.5 Filter transformers

3.5.1 Low pass to high pass transformation

High-pass filters can be designed from normalised prototype low-pass filters based on the following specification:

- The filter function (Bessel, Butterworth, Chebyshev)
- The filter cut-off frequency ω_a
- The filter stopband frequency ω_s
- The filter stopband attenuation A_s at the stopband frequency ω_s



A low-pass filter can be transformed into a high-pass filter by:

$$H_{\text{hp}}(s) = H_{\text{lp}}(\bar{s})|_{\bar{s}=\frac{1}{s}}$$

Process: Low pass to high pass transformation

1. Find the stopband frequency for the normalized low-pass filter (used for the design):

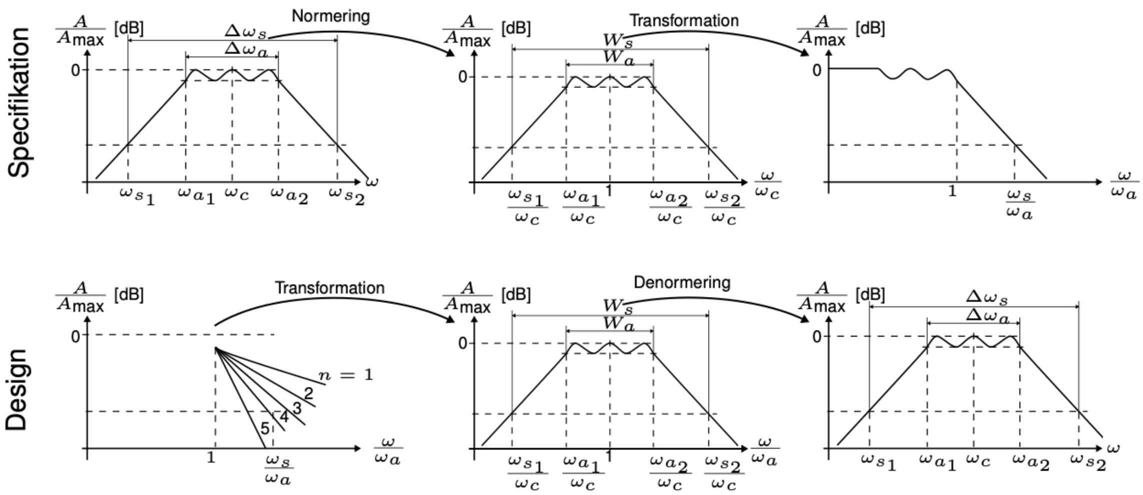
$$\frac{\omega_a}{\omega_s} = \frac{f_a}{f_s}$$

2. Filter order selection based on amplitude characteristics for lowpass filters.
3. The transfer function of the normalized low-pass filter is transformed to the transfer function of the normalized high-pass filter by replacing s with $\frac{1}{s}$
4. The transfer function of the denormalized high-pass filter is found by replacing s with $\frac{s}{\omega_a}$

3.5.2 Low pass to bandpass transformation

Bandpass filters can be designed from normalized prototype low-pass filters based on the following specification

- The filter function (Bessel, Butterworth, Chebyshev)
- The filter center frequency ω_s
- The passband bandwidth $\Delta\omega_a$
- Stopband bandwidth $\Delta\omega_s$
- Filter stopband attenuation A_s



A low-pass filter can be transformed into a bandpass filter by

$$H_{\mathrm{bp}}(s) = H_{\mathrm{lp}}(\bar{s})|_{\bar{s}=\frac{1}{W_a}\left(s+\frac{1}{s}\right)}$$

To design the filter, the bandpass filter is normalised, i.e. the normalised stopband width and the normalised passband width are calculated as:

$$W_a = \frac{\Delta f_a}{f_c}$$

$$W_s = \frac{\Delta f_s}{f_c}$$

$$\omega_c = \sqrt{\omega_{a_1} \cdot \omega_{a_1}}$$

If ω_c and $\Delta\omega_a$ is known:

$$\omega_{a_1} = \sqrt{\frac{(\Delta\omega_a)^2}{4} + \omega_c^2} - \frac{\Delta\omega_a}{2}$$

$$\omega_{a_2} = \sqrt{\frac{(\Delta\omega_a)^2}{4} + \omega_c^2} + \frac{\Delta\omega_a}{2}$$

These can also be used with ω_s

The form factor for bandpass filter:

$$F = \frac{\Delta f_s}{\Delta f_a} = \frac{W_s}{W_a}$$

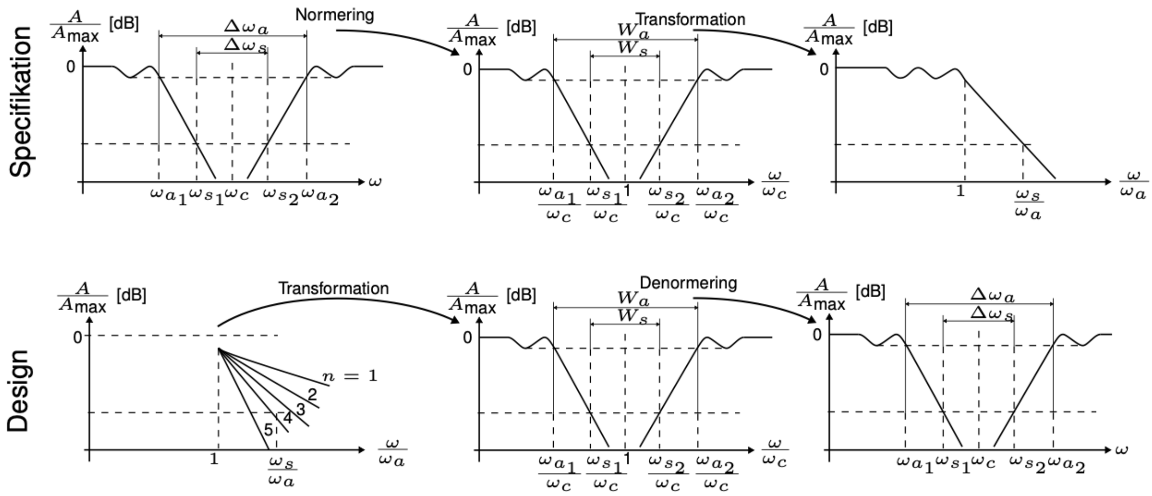
Process: Low pass to bandpass transformation

1. To design the filter, the bandpass filter is normalized, i.e. the normalized stopband width and the normalized passband width are calculated.
2. The form factor is found.
3. Filter order selection based on amplitude characteristics for lowpass filters.
4. The normalised filter is found by replacing s with $\frac{1}{W_a}\left(s+\frac{1}{s}\right)$ in the low-pass filter.
5. The denormalized bandpass filter is found by replacing s in the normalized bandpass filter with $\frac{s}{\omega_c}$.

3.5.3 Low pass to bandstop transformation

Bandstop filters can be designed from normalised prototype low-pass filters based on the following specification:

- Filter function (Bessel, Butterworth, Chebyshev)
- Filter centre frequency ω_c
- Passband bandwidth $\Delta\omega_c$
- Stopband bandwidth $\Delta\omega_s$
- Filter stopband attenuation A_s



A low-pass filter can be transformed into a bandstop filter by

$$H_{\text{bs}}(s) = H_{\text{lp}}(\bar{s})\big|_{\bar{s}=\frac{W_a}{s+\frac{1}{s}}}$$

and the form factor for bandstop filter:

$$F = \frac{\Delta f_a}{\Delta f_s} = \frac{W_a}{W_s}$$

Process: Low pass to bandstop transformation

1. To design the filter, the bandstop filter is normalized, i.e. the normalized stopband width and the normalized passband width are calculated.
2. The form factor is found.
3. Filter order selection based on amplitude characteristics for lowpass filters.

4. The normalised filter is found by replacing s with $\frac{W_a}{s+\frac{1}{s}}$ in the low-pass filter.
5. The denormalized bandpass filter is found by replacing s in the normalized bandpass filter with $\frac{s}{\omega_c}$.

3.6 Examples

3.6.1 Example 1: Butterworth High-pass filter

Design a Butterworth high-pass filer that have stopband attenuation of 50 dB and the following amplitude characteristics:

$$\omega_s = 1200 \qquad \omega_a = 6000$$

Stopband frequency for the low-pass filter:

$$\frac{\omega_a}{\omega_s} = \frac{6000}{1200} = 5$$

Using fig. 3.9 on Page 161 (Digital Signal Behandling: Erik Hüche):
The filter orden is 4.

Using table 3.1 on Page 169 (Digital Signal Behandling: Erik Hüche):
The normalized filter is:

$$H(s)_{lp} = \frac{1}{s^2 + 1.84776s + 1} \frac{1}{s^2 + 0.76537s + 1}$$

Transform to high by $s = 1/s$:

$$H(s)_{hp} = \frac{1}{(1/s)^2 + 1.84776/s + 1} \frac{1}{(1/s)^2 + 0.76537/s + 1}$$

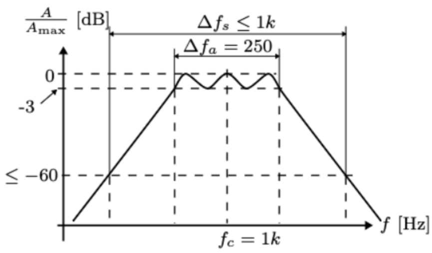
$$H(s)_{hp} = \frac{s^3}{1 + 1.84776 + 1} \frac{s^3}{(1 + 0.76537 + 1}$$

Denormalize $s = s/\omega_a$:

$$H(s)_{hp} = \frac{(s/6000)^3}{1 + 1.84776 + 1} \frac{(s/6000)^3}{(1 + 0.76537 + 1}$$

3.6.2 Example 2: Chebyshev Band-pass filter

Design a Chebyshev band-pass filter with the following amplitude characteristics:



Find W_a, W_s, F for the normalised bandpass filter:

$$F = \frac{\Delta f_s}{\Delta f_a} = \frac{W_s}{W_a} = \frac{1000}{250} = 4$$

The filter orden from fig 3.12 2 dB Chebyshev filter since 3 dB is not in the book:
4th order filter must be used.

Using table 3.1 on Page 169 (Digital Signal Behandling: Erik Hüche):
The normalized filter is:

4 Sampling-Reconstruction

Nyquist-Shannon Phrase:
A continuous-time signal $x(t)$ can only be correctly recovered from $x_s(t)$ if the sampling frequency is at least **twice** the highest frequency in the spectrum of $x(t)$.

4.1 Examples

4.1.1 Example 1: Reconstruction

Which of the following sample frequencies can be used to fully reconstruct the signal:

$$x(t) = \cos(6\pi \cdot t + 2) + \sin(5\pi \cdot t + 4)$$

1. $f_s = 1$ kHz
2. $f_s = 2$ kHz
3. $f_s = 3$ kHz
4. $f_s = 10$ kHz

Using Nyquist-Shannon Phrase, we know the samples frequency must be a least twice as high as the highest frequency in the signal.

The highest in $x(t)$ is $2\pi f_s = 6\pi \quad \Rightarrow \quad 2f_s \geq 6$ therefore 10 kHz can be used to fully reconstruct the signal.

5 FFT

5.1 DFT

The DTF is given by:

$$X(m) = \sum_{n=0}^{N-1} x(n) W_N^{mn} = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi mn}{N}}$$

for $m = 0, 1, \ldots, N-1$ and $W_N = e^{-j2\pi/N}$

The sequence can be found from the spectrum $X(m)$:

$$x(n) = \frac{1}{N} \sum_{m=0}^{N-1} X(m) W_N^{-mn} = \frac{1}{N} \sum_{m=0}^{N-1} X(m) e^{\frac{j2\pi mn}{N}}$$

for $m = 0, 1, \ldots, N-1$

When performing an N-point DFT, the distance between frequencies in the DFT will be $F = f_s/N$. If a higher resolution of the frequency spectrum is desired, zero filling can be used to increase the signal to L samples

5.2 Examples

5.2.1 Example 1: DFT of a sequence

Consider the sequence:

$$x(n) = \begin{cases} 10 & n = 0 \\ 0 & n = 1 \\ 0 & n = 2 \\ -10 & n = 3 \\ 0 & n = 4 \end{cases}$$

Calculate the 5-point discrete Fourier transform of the sequence. Find $X(1)$ and $X(2)$

$$X(m) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi mn}{N}}$$

Insert $N = 5$ and calculate the sum:

$$X(m) = \underbrace{10e^{\frac{-j2\pi m \cdot 0}{5}}}_{n=0} + \underbrace{0}_{n=1} + \underbrace{0}_{n=2} \underbrace{-10e^{\frac{-j2\pi m \cdot 3}{5}}}_{n=3} + \underbrace{0}_{n=4}$$

$$X(m) = 10 - e^{\frac{-j2\pi m \cdot 3}{5}}$$

Find $X(1)$ and $X(2)$:

$$X(1) = 10 - 10e^{\frac{-j2\pi \cdot 1 \cdot 3}{5}} = 10 - 10e^{\frac{-j6\pi \cdot}{5}}$$

$$X(2) = 10 - 10e^{\frac{-j2\pi \cdot 2 \cdot 3}{5}} = 10 - 10e^{\frac{-j12\pi}{5}}$$

6 Z-transformation

6.1 Z-transformation

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

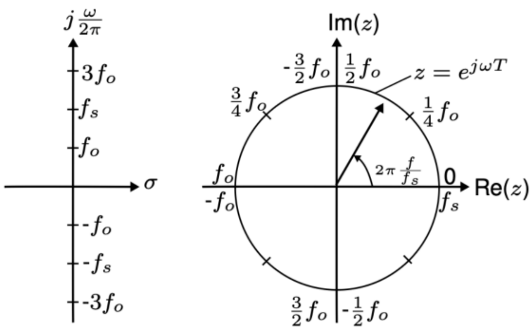
Notation:

$$X(z) = \mathcal{Z}\{x(n)\} \qquad x(n) = \mathcal{Z}^{-1}\{X(z)\}$$

6.1.1 s and z domain

$$X_s(s) = X(z) \qquad \text{when } z = e^{st} \quad s = \frac{1}{T}\ln(z)$$

where $s = \sigma + j\omega$



Here we see $\sigma < 0$ is mapped inside the unit circle $|z| < 1$.

6.1.2 Transformation rules/pairs

Rule	$x(n)$	$X(z)$
Z1	$ax_1(n) + bx_2(n)$	$aX_1(z) + bX_2(z)$
Z2	$x(n-m)$	$z^{-m}X(z)$
Z3	$x(n)a^{-n}$	$X(az)$
Z4	$x(n)s^{-bn}$	$X(e^{bT}z)$
Z5	$\sum_{m=0}^{n} x(m)h(n-m)$	$X(z)H(z)$

Pair	$x(n)$	$X(z)$
ZT1	$\delta(n)$	1
ZT2	$u(n)$	$\frac{z}{z-1}$
ZT3	n	$\frac{z}{(z-1)^2}$
ZT4	a^n	$\frac{z}{z-a}$
ZT5	$e^{s_0 nT}$	$\frac{z}{z-e^{s_0 T}}$
ZT6	$\sin(\omega_0 nT)$	$\frac{\sin(\omega_0 T)z}{z^2-2\cos(\omega_0 T)z+1}$
ZT7	$\cos(\omega_0 nT)$	$\frac{z^2-\cos(\omega_0 T)z}{z^2-2\cos(\omega_0 T)z+1}$

If $|x| < 1$:

$$\sum_{i=0}^{\infty} = \frac{1}{1-x}$$

6.2 Differential equations

A N'th order differential equation, that describes a causal system:

$$y(n) + b_1 y(n-1) + \cdots + b_N y(n-N) = a_0 x(n) + a_1 x(n-1) + \cdots + a_N x(n-N)$$

where $x(n-i)$ is the time-delayed input sequence, $y(n-i)$ is the time-delayed output sequence and a_i, b_i are real koefficients.

This can also be written as:

$$y(n) = \sum_{i=0}^{N} a_i x(n-i) - \sum_{i=1}^{N} b_i y(n-i)$$

6.3 Transfer Functions

Discrete-time systems (like continuous-time systems) can be described by transfer functions, defined as:

$$H(z) = \frac{Y(z)}{X(z)}$$

$H(z)$ is the transfer function.
$X(z)$ is the input sequence.
$Y(z)$ is the output sequence.

A transfer function is found by Z-transformation of a differential equation:

$$y(n) + \sum_{i=1}^{N} b_i y(n-i) = \sum_{i=0}^{N} a_i x(n-i)$$

By Z-transformation using Z2:

$$Y(z) + Y(z) \sum_{i=1}^{N} b_i z^{-i} = X(z) \sum_{i=0}^{N} a_i z^{-1}$$

The transfer function becomes:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{N} a_i z^{-i}}{1 + \sum_{i=1}^{N} b_i z^{-i}} = \frac{a_0 z^N + a_1 z^{N-1} + a_2 z^{N-2} + \cdots + a_N}{z^N + b_1 z^{N-1} + b_2 z^{N-2} + \cdots \cdot + b_N}$$

6.3.1 Poles and Zeros

$$H(z) = \frac{P(z)}{Q(z)}$$

Poles ($H(z) = \infty$) of z when $Q(z) = 0$
Zeros ($H(z) = 0$) of z when $P(z) = 0$

Therefore a transfer function can be written as:

$$H(z) = a_0 \frac{(z-z_1)(z-z_2)\cdots(z-z_N)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$$

6.4 Inverse Z-transformation

The inverse transform is used to determine the output response $y(n)$ of a time-discrete system for a given input stimulus $x(n)$. This analysis is performed according to the following procedure:

1. The transfer function $H(z)$ of the system is set up with positive powers of z.
2. The input sequence $x(n)$ is z-transformed. (Use table lookup)
3. The output response in z domain is calculated $Y(z) = H(z)X(z)$.
4. The output sequence $y(n)$ is calculated by inverse z-transformation of $Y(z)$.

6.4.1 Partial fractions

1. Setup a expression for $Y(z)$ with positive powers of z in factorized form:

$$Y(z) = \frac{T(z)}{N(z)} = \frac{T(z)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$$

where $p_1, p_2, \ldots, p_N$ are roots of the denominator polynomial of $Y(z)$

2. Divide $Y(z)$ with z so that the denominator's ordinal number is greater than the numerator's ordinal number. This expression resolves into fractions:

$$\frac{Y(z)}{z} = \frac{T(z)}{zN(z)} = \frac{k_1}{z-p_1} + \frac{k_2}{z-p_2} + \cdots + \frac{k_N}{z-p_N}$$

3. The coefficients are calculated as:

$$k_i = (z-p_i)\frac{Y(z)}{z}|_{z=p_i}$$

4. Write $\frac{Y(z)}{z}$ in partial fraction resolved form and multiply by z

6.5 Examples

6.5.1 Example 1: Z-transformation

6.5.2 Example 2: Inverse Z-transformation

Given a transfer function:

$$H(z) = \frac{1}{1-0.5z^{-1}} = \frac{z}{z-0.5}$$

Find the output sequence when the input sequence is the unit step sequence $u(n)$. $y(n) = 0 \quad n < 0$

First find the z-transformation of $u(n)$ (ZT2):

$$X(z) = \frac{z}{z-1}$$

$$Y(z) = X(z) \cdot H(z) = \frac{z}{z-1} \cdot \frac{z}{z-0.5} = \frac{z^2}{(z-1)(z-0.5)}$$

The two poles are:

$$p_1 = 1 \qquad p_2 = 0.5$$

Find the partial fractions:

$$\frac{Y(z)}{z} = \frac{z}{(z-1)(z-0.5)} = \frac{k_1}{z-1} + \frac{k_2}{z-0.5}$$

Using:

$$k_i = (z - p_i)\frac{Y(z)}{z}|_{z=p_i}$$

We get:

$$k_1 = (z-1)\frac{z}{(z-1)(z-0.5)}|_{z=1} = \frac{1}{1-0.5} = 2$$

$$k_2 = (z-0.5)\frac{z}{(z-1)(z-0.5)}|_{z=0.5} = \frac{0.5}{0.5-1} = -1$$

$$\frac{Y(z)}{z} = \frac{2}{z-1} - \frac{1}{z-0.5}$$

$$Y(z) = \frac{2z}{z-1} - \frac{z}{z-0.5}$$

Find the inverse z-transformation using ZT2 and ZT4:

$$y(n) = 2u(n) - 0.5^n = 2 - 0.5^n$$

6.5.3 Example 3: Inverse Z-transformation

Given a transfer function:

$$H(z) = \frac{1+z^{-1}}{1-z^{-1}+0.8125z^{-2}}$$

Find the output sequence when the input sequence is the unit step sequence $u(n)$. $y(n) = 0 \quad n < 0$

Positive exponents of z:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^2+z}{z^2-z+0.8125}$$

$$Y(z) = \frac{z^2+z}{z^2-z+0.8125} \cdot \frac{z}{z-1}$$

Find the poles:

$$p_1 = 1 \qquad p_2 = 0.5+0.75j \qquad p_2^* = 0.5-0.75j$$

Setup partial fraction:

$$\frac{Y(z)}{z} = \frac{k_1}{z-1}\frac{k_2}{z-(0.5+0.75j)}\frac{k_2^*}{z-(0.5-0.75j)}$$

$$\frac{Y(z)}{z} = \frac{z^2+z}{(z-1)(z-(0.5+0.75j))(z-(0.5-0.75j))}$$

Using:

$$k_i = (z - p_i)\frac{Y(z)}{z}|_{z=p_i}$$

We get:

$$k_1 = (z-1)\frac{z^2+z}{(z-1)(z-(0.5+0.75j))(z-(0.5-0.75j))}|_{z=1} = 2.46154$$

$$k_2 = (z-(0.5+0.75j))\frac{z^2+z}{(z-1)(z-(0.5+0.75j))(z-(0.5-0.75j))}|_{z=(0.5+0.75j)}$$
$$= -0.730769 - 0.846154j$$

$$k_2^* = (z-1)\frac{z^2+z}{(z-1)(z-(0.5+0.75j))(z-(0.5-0.75j))}|_{z=1} = -0.730769 + 0.846154j$$

Insert:

$$\frac{Y(z)}{z} = \frac{2.46154}{z-1} \frac{-0.730769 - 0.846154j}{z-(0.5+0.75j)} \frac{-0.730769 + 0.846154j}{z-(0.5-0.75j)}$$

$$Y(z) = \frac{2.46154z}{z-1} \frac{z(-0.730769 - 0.846154j)}{z-(0.5+0.75j)} \frac{z(-0.730769 + 0.846154j)}{z-(0.5-0.75j)}$$

Find the inverse z-transformation using ZT2 and ZT4:

$$y(n) = 2.46144 + (-0.730769 - 0.846154j)(0.5+0.75j)^n + (-0.730769 + 0.846154j)(0.5-0.75j)^n$$

6.5.4 Example 4: Differential equation

Find the transfer function of:

$$2y(n) + 3y(n-2) = 3x(n) + 2x(n-2)$$

$$2Y(z) + 3Y(z)z^{-2} = 3X(z) + 2X(z)z^{-2}$$
$$Y(z)(2+3z^{-2}) = X(z)(3+2z^{-2})$$
$$\frac{Y(z)}{X(z)} = \frac{3+2z^{-2}}{2+3z^{-2}} = \frac{3z^2+2}{2z^2+3}$$

6.5.5 Example 5: Differential equation

Find the transfer function from the following differential equation. It must have positive exponents.

$$y_k + 5y_{k-1} + 2y_{k-2} = x_k - x_{k-1}$$

$$Y(z) + 5Y(z)z^{-1} + 2Y(z)z^{-2} = X(z) - X(z)z^{-1}$$
$$Y(z)(1+5z^{-1}+2z^{-2}) = X(z)(1-z^{-1})$$
$$\frac{Y(z)}{X(z)} = \frac{1-z^{-1}}{1+5z^{-1}+2z^{-2}} \frac{z^2}{z^2} = \frac{z^2-z}{z^2+5z+2}$$

6.5.6 Example 6: Poles and Zeros

How many poles and zeros does the transfer function have:

$$G(z) = \frac{z^2-3}{z^3+2z}$$

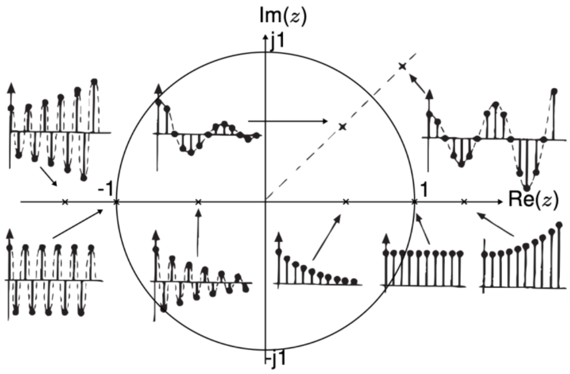
There are as many poles and zeroes as the ordinal number:
2 Zeros
3 Poles

7 System-analysis

7.1 Impulse response

The impulse response of a discrete-time system is called $h(n)$, and is identical to the output sequence of the system when the input sequence is a unit sample $\delta(n)$

The impulse response sequence $h(n)$ of a system is found by inverse z-transformation of the system's transfer function $H(z)$.



7.2 Stability

Stable system: A system is stable if its impulse response $h(n)$ goes to zero when n goes to infinity.

$$|h(n)| \to 0 \quad \text{when } n \to \infty$$

If all poles of the transfer function $H(z)$ are inside the unit circle.

$$|p_i| \leq 1 \quad \text{for } i = 1, 2, \ldots, N$$

Marginally stable system: A system is marginally stable if its impulse response $h(n)$ approaches a constant value different from zero or oscillates with constant amplitude and frequency as it approaches infinity.

If one pole of the transfer function $H(z)$ are on the unit circle and the rest are inside.

$$|p_i| \leq 1 \quad \text{for } i = 1, 2, \ldots, N$$

$$|p_j| = 1 \quad \text{for } j \in \{1, 2, \ldots, N\}$$

Unstable system: A system is unstable if its impulse response $h(n)$ grows indefinitely as it approaches infinity.

$$|h(n)| \to \infty \quad \text{when } n \to \infty$$

If one pole of the transfer function $H(z)$ are outside the unit circle.

$$|p_j| > 1 \quad \text{for } j \in \{1, 2, \ldots, N\}$$

7.3 Frequency response analysis

A frequency response analysis gives the response of a system at a sinusoidal input sequence.

$$y(n) = AM(\omega)\cos(\omega t + \varphi(\omega))$$

where

$$M(\omega) = |H(j\omega)| \qquad \varphi(\omega) = \angle H(j\omega)$$

$$H(j\omega) = H(z)|_{z=e^{j\omega T}} \qquad H(j\omega) = |H(\omega)|\angle\varphi(\omega)$$

7.4 Examples

7.4.1 Example 1: Stability

Is the following transfer function stable?

$$G(z) = \frac{(z+0.2)(z-0.2)}{(z+0.5)(z-1.1)}$$

Looking at the poles we see:

$$p_1 = -0.5 \qquad p_2 = 1.1$$

Since $|p_2| > 1$ the transfer function is not stable.

7.4.2 Example 2: Stability

Is the following transfer function stable?

$$G(z) = \frac{z+2}{(z-0.9)(z+0.2)}$$

Looking at the poles we see:

$$p_1 = 0.9 \qquad p_2 = -0.2$$

Since $|p_1| < 1$ and $|p_2| < 1$ the transfer function is stable.

7.4.3 Example 3: Impluse Response

Find the Impluse Response Sequence of the following discrete transfer function:

$$G(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 0.75z^{-1} + 0.125z^{-2}}$$

The input impluse $x(n) = \delta(n)$:

$$G(z) = \frac{Y(z)}{X(z)} = 1 \cdot \frac{z^2}{z^2 - 0.75z + 0.125}$$

Factor:

$$G(z) = \frac{z^2}{(z-0.5)(z-0.25)}$$

$$G(z) = \frac{z}{z-0.5} \cdot \frac{z}{z-0.25}$$

Inverse z-transformation:

$$g(n) = \mathcal{Z}^{-1}\left\{\frac{z}{z-0.5}\right\} \cdot \mathcal{Z}^{-1}\left\{\frac{z}{z-0.25}\right\}$$

Using table lookup (ZT4):

$$a^n = \mathcal{Z}^{-1}\left\{\frac{z}{z-a}\right\}$$

$$g(n) = 0.5^n \cdot 0.25^n$$

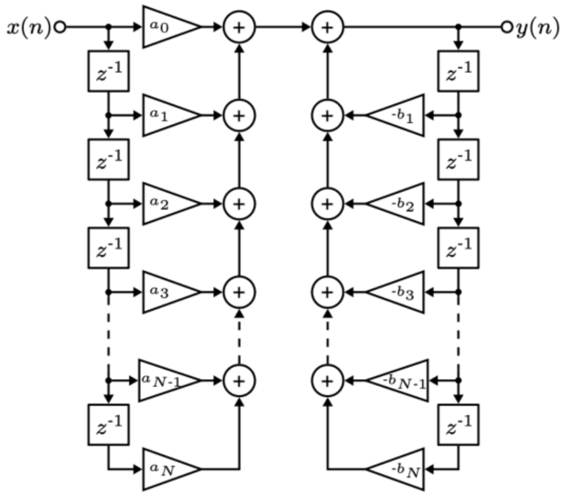
7.4.4 Example 3: Frequency Response

8 Digital-Realization-Structures

8.1 Direct realization structures

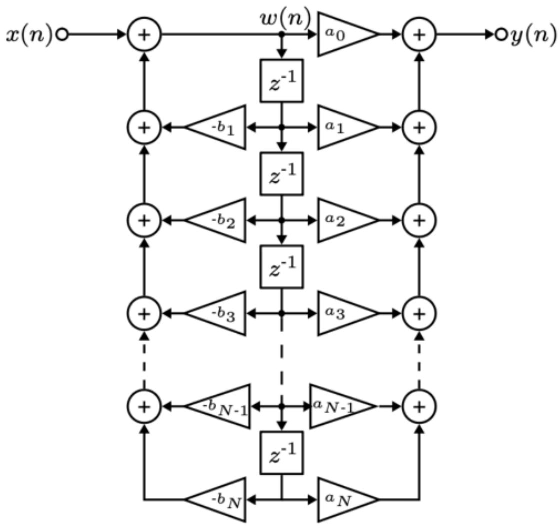
8.1.1 Type 1

$$y(n) = \sum_{i=0}^{N} a_i x(n-i) - \sum_{i=1}^{N} b_i y(n-i)$$



8.1.2 Type 2

$$y(n) = \sum_{i=0}^{N} a_i x(n-i) - \sum_{i=1}^{N} b_i y(n-i)$$



8.2 Cascade and Parallel Realization

A higher-order system can be implemented as a cascade of M sections given as follows:

$$H(z) = H_1(z) \cdot H_2(z) \cdot H_3(z) \cdot \cdots \cdot H_M(z) \cdot$$

A higher-order system can be implemented as a parallel realization of transfer functions that are a 1st-order or 2nd-order transfer function. These transfer functions are found by partial fraction resolution of $H(z)$.

$$H(z) = C + H_1(z) + H_2(z) + \cdots + H_M(z)$$

9 IRR-Filters

Infinite Impulse Response filters where the impulse response of the system is of infinite duration.

9.1 Design of IRR-filters

Design of IRR-filters is done in the following steps:

1. The specifications of the IRR-filter.
2. The transfer function of the IRR-filter.
3. The optimal realization of the IRR-filter.
4. Program for signalprocessing or a circuit diagram for analog signalprocessing.

9.2 Matched z-transform

By using the matched z-transform, the poles and zeros of the IRR-filter are directly transferred to the z-plane. The transfer function of the IRR-filter is given by:

$$z = e^{sT}$$

Following procedure is used for the matched z-transform:

1. Determine the frequency-normalized and factorized transfer function $H(s)$ of the analog prototype filter.
2. Determine the analog frequency-normalized poles and zero points.
3. Determine the denormalized poles and zero points.
4. Determine the coefficients of the digital transfer function.
5. Implement the transfer function as a cascade structure.

9.2.1 1. Order Matched z-transformation

The transfer function for a first order system is given by:

$$H(s) = \frac{s + A_0}{s + B_0} = \frac{s - \sigma_1}{s - \sigma_2}$$

where $-A_0 = \sigma_1$ is a real zero and $-B_0 = \sigma_2$ is a real pole of the IRR-filter.

A digital first order transfer function is given by:

$$H(z) = \frac{z - e^{\sigma_1 T}}{z - e^{\sigma_2 T}} = \frac{1 - e^{\sigma_1 T} z^{-1}}{1 - e^{\sigma_2 T} z^{-1}}$$

A first order system without a zero is given by:

$$H(s) = \frac{\omega_a}{s + \omega_a}$$

Using the matched z-transform, the digital transfer function is given by:

$$H(z) = \frac{1}{1 - e^{\sigma_1 T} z^{-1}}$$

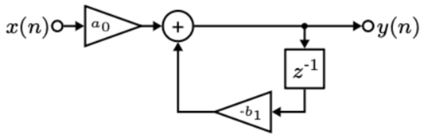
where σ_1 is the pole of $H(s)$ and T is the sampling period.
The transfer function $H(z)$ does not have a DC-gain of 1. To get a DC-gain of 1, the transfer function is multiplied by a gain factor a_0:

$$H(z) = \frac{a_0}{1 + b_1 z^{-1}}$$

where

$$b_0 = -e^{\sigma_1 T} \qquad a_0 = 1 + b_1$$

The corresponding digital realization structure:



9.2.2 2. Order Matched z-transformation

The transfer function for a second order system with complex conjugate pole and zero pair is given by:

$$H(s) = \frac{s^2 + A_1 s + A_0}{s^2 + B_1 s + B_0}$$

which has zeros in $s_1 = \sigma_1 + j\omega_1, s_1^* = \sigma_1 - j\omega_1$ and poles in $s_2 = \sigma_2 + j\omega_2, s_2^* = \sigma_2 - j\omega_2$

Using the matched z-transform:

$$H(z) = \frac{(z - e^{\sigma_1 T} e^{j\omega_1 T})(z - e^{\sigma_2 T} e^{-j\omega_1 T})}{(z - e^{\sigma_2 T} e^{j\omega_2 T})(z - e^{\sigma_2 T} e^{-j\omega_2 T})}$$

Using Euler's identity:

$$H(z) = \frac{z^2 - (2e^{\sigma_1 T}\cos(\omega_1 T))z + e^{2\sigma_1 T}}{z^2 - (2e^{\sigma_2 T}\cos(\omega_2 T))z + e^{2\sigma_2 T}} = \frac{1 - (2e^{\sigma_1 T}\cos(\omega_1 T))z^{-1} + e^{2\sigma_1 T} z^{-2}}{1 - (2e^{\sigma_2 T}\cos(\omega_2 T))z^{-1} + e^{2\sigma_2 T} z^{-2}}$$

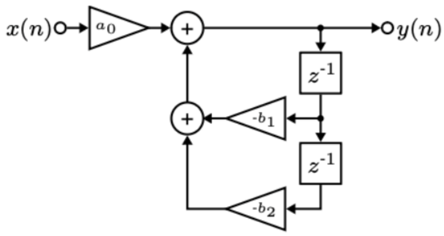
A second order low-pass filter is given by:

$$H(z) = \frac{a_0}{1 + b_1 z^{-1} + b_2 z^{-2}}$$

where

$$a_0 = 1 + b_1 + b_2 \qquad b_1 = -2e^{\sigma_1 T}\cos(\omega_1 T) \qquad b_2 = e^{2\sigma_2 T}$$

The corresponding digital realization structure:



9.3 Impulse invariance method

The following procedure is used to design digital IIR filters using the impulse invariant z transform.

1. Determine the frequency-normalized transfer function $H(s)$ of the analog prototype filter.

2. Partial fractionally resolve $H(s)$ into 1st and 2nd order transfer functions (maximum number of 2nd order transfer functions).
3. Denormalize the coefficients k_i and the poles $\sigma_i + j\omega_i$ by multiplication with the cutoff frequency or center frequency.
4. Determine the coefficients of the digital transfer function.
5. Implement the transfer function as a parallel structure.

To find impulse invariant z-transformation, the impulse response is z-transformed:

$$H(z) = T \cdot \mathcal{Z}[h(n)]$$

Given a N'th order filter:

$$H(s) = \frac{\sum_{i=0}^{M} A_i s^i}{\sum_{i=0}^{N} B_i s^i} \qquad M \leq N$$

where s_i is the i pole of the analog filter.

Using partial fractions, it can be written as:

$$H(s) = \sum_{i=1}^{N} \frac{k_i}{s - s_i}$$

And the z-transformation is:

$$H(z) = T \cdot \sum_{i=1}^{N} \frac{k_i}{1 - e^{s_i T} z^{-1}}$$

9.4 1st Order System

$$H(s) = \frac{A_0}{s + B_0} = \frac{-\sigma_i}{s - \sigma_i}$$

Using the impulse invariant z-transformation:

$$H(z) = \frac{a_0}{1 + b_1 z^{-1}}$$

where

$$a_0 = -\sigma_i T \qquad b_1 = -e^{\sigma_i T}$$

9.5 2nd Order System

$$H(s) = \frac{A_0}{s^2 + B_1 s + B_0}$$

Find the partial fractions:

$$H(s) = \frac{k_i}{s - s_i} + \frac{k_i^*}{s - s_i^*}$$

where

$$s_i = \sigma_i + j\omega_i \qquad k_i = \alpha_i + j\beta_i$$

Using the impulse invariant z-transformation:

$$H(z) = \frac{a_0 + a_1 z^{-1}}{1 + b_1 z^{-1} + b_2 z^{-2}}$$

where

$$a_0 = -2\alpha_i T \qquad a_1 = -2e^{\sigma_i T}(\alpha_i \cos(\omega_i T) - \beta_i \sin(\omega_i T))$$
$$b_1 = -(2e^{\sigma_i T} \cos(\omega_i T)) \qquad b_2 = e^{2\sigma_i T}$$

9.6 Bilinear z transformation

The following is the procedure for the design of digital filters using bilinearz transformation.

1. The prewarping constant is determined as

$$C = \cot\left(\frac{\omega_i T}{2}\right)$$

   where $i = a$ for low pass or high pass filter design and $i = c$ for band pass or band stop filter design.

2. The order number of the filter is determined based on the prewarped stopband frequency.
3. The frequency-normalized and factorized analog transfer functionH(s)is set up.
4. The coefficients of the digital transfer function are calculated.
5. The filter is implemented as a cascaded realization structure.

9.6.1 Warping and Pre-warping

$$\Omega = \frac{2}{T}\tan\left(\frac{\omega T}{2}\right) \qquad \text{[rad/s]}$$

$$\omega = \frac{2}{T}\tan^{-1}\left(\frac{\Omega T}{2}\right) \qquad \text{[rad/s]}$$

$$\Omega_a = C\tan\left(\frac{\omega_a T}{2}\right) \qquad \text{[rad/s]}$$

Normalized stop-band frequency:

$$\frac{\Omega_i}{\Omega_s} = \frac{1}{\Omega_s}$$

where $i = a$ for low and high pass and $i = c$ for band-pass and band-stop.

9.6.2 1st Order Bilinear z transformation

First order system:

$$H(s) = \frac{A_1 s + A_0}{B_1 s + B_0}$$

Using bilinear z transformation:

$$H(z) = \frac{a_0 + a_1 z^{-1}}{1 + b_1 z^{-1}}$$

where

$$a_0 = \frac{A_0 + A_1 C}{B_0 + B_1 C} \qquad a_1 = \frac{A_0 - A_1 C}{B_0 + B_1 C} \qquad b_1 = \frac{B_0 - B_1 C}{B_0 + B_1 C}$$

9.6.3 2nd Order Bilinear z transformation

A second order system:

$$H(s) = \frac{A_2 s^2 + A_1 s + A_0}{B_2 s^2 + B_1 s + B_0}$$

Using bilinear z transformation:

$$H(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2}}{1 + b_1 z^{-1} + b_2 z^{-2}}$$

where

$$a_0 = \frac{A_0 + A_1C + A_2C^2}{B_0 + B_0C + B_2C^2} \qquad a_1 = \frac{2(A_0 - A_2C^2)}{B_0 + B_1C + B_2C^2} \qquad a_2 = \frac{A_0 - A_1C + A_2C^2}{B_0 + B_1C + B_2C^2}$$

$$b_1 = \frac{2(B_0 - B_2C^2)}{B_0 + B_1C + B_2C^2} \qquad b_2 = \frac{B_0 - B_1C + B_2C^2}{B_0 + B_1C + B_2C^2}$$

9.7 Examples

9.7.1 Example 1: Matched 1st order z-transform

Consider the following analog first order lowpass filter with cutoff frequency $\omega_a = 2\pi f_a$ where $f_a = 300\,\text{Hz}$:

$$H(s) = \frac{\omega_a}{s + \omega_a} = \frac{1885}{s + 1885}$$

which has a pole in $s_1 = \sigma_1 = -1885$.
Find the digital IRR-filter using the matched z-transform with sampling frequency $f_s = 16\,\text{kHz}$.

The coefficients of the digital IRR-filter are given by:

$$b_1 = -e^{\sigma_1 T} = -e^{-1885 \cdot \frac{1}{16000}} = -0.8889$$

$$a_0 = 1 + b_1 = 0.1111$$

The digital lowpass filter is given by:

$$H(z) = \frac{a_1}{1 + b_0 z^{-1}} = \frac{0.1111}{1 - 0.8888z^{-1}}$$

9.7.2 Example 2: Matched 3rd order matched z-transform

Consider the following Butterworth 3rd order low-pass filter (frequency-normalized filter) and design an equivalent digital low-pass filter with cut-off frequency at 1 kHz and sample frequency at 8 kHz.

$$\tilde{H}(s) = \frac{1}{s+1}\frac{1}{s^2 + s + 1}$$

Using the matched z-transform

---

1. Determine the frequency-normalized and factorized transfer function $H(s)$ of the analog prototype filter.
Is in this case it's already given.

2. Determine the analog frequency-normalized poles and zero points.
There is no zeros in the transfer function, so only the poles are determined:

$$s_1 = -1 \qquad s_2 = s_2^* = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$$

3. Determine the denormalized poles and zero points.
Only the poles are denormalized, since there are no zeros:

$$\omega_a = 2\pi f_a = 2\pi \cdot 1000 = 6283.2$$

$$\sigma_1 = -1 \cdot \omega_a = -6283.2$$

$$\sigma_2 \pm j\omega_2 = s_2 \cdot \omega_a = -3141.6 \pm j5434.4$$

4. Determine the coefficients of the digital transfer function.

$$H(s) = H_1(s)H_2(s) = \frac{1}{s+1}\frac{1}{s^2+s+1}$$

$$H(z) = H_1(z)H_2(z)$$

Using the matched z-transform:

$$H_1(z) = \frac{1}{1-e^{\sigma_1 T}z^{-1}} = \frac{a_{0_1}}{1+b_{1_1}z^{-1}}$$

$$b_{1_1} = -e^{\sigma_1 T} = -e^{-6283.2\cdot\frac{1}{8000}} = -0.4559 \qquad a_{0_1} = 1 + b_{1_1} = 0.5441$$

$$H_1(z) = \frac{0.5441}{1-0.4559z^{-1}}$$

and

$$H_2(z) = \frac{1}{(z-e^{\sigma_2 T}e^{j\omega_2 T})(z-e^{\sigma_2 T}e^{-j\omega_2 T})} = \frac{a_{0_2}}{1+b_{1_2}z^{-1}+b_{2_2}z^{-2}}$$

$$b_{1_2} = -2e^{\sigma_2 T}\cos(\omega_2 T) = -2e^{-3141.6\cdot\frac{1}{8000}}\cos(5434.4\cdot\frac{1}{8000}) = -1.0507$$

$$b_{2_2} = e^{2\sigma_2 T} = e^{2\cdot -3141.6\cdot\frac{1}{8000}} = 0.4559$$

$$a_{0_2} = 1 + b_{1_2} + b_{2_2} = 1 - 1.0507 + 0.4559 = 0.4052$$

$$H_2(z) = \frac{0.4559}{1-1.0507z^{-1}+0.4559z^{-2}}$$

Combining the two transfer functions:

$$H(z) = H_1(z)H_2(z) = \frac{0.5441}{1-0.4559z^{-1}}\frac{0.4052}{1-1.0507z^{-1}+0.4559z^{-2}}$$

9.7.3 Example 3: Matched 3. order impulse invariance z-transform

Consider the following Butterworth 3rd order low-pass filter (frequency-normalized filter) and design an equivalent digital low-pass filter with cut-off frequency at 1 kHz and sample frequency at 8 kHz.

$$\tilde{H}(s) = \frac{1}{s+1}\frac{1}{s^2+s+1}$$

Using the impulse invariance method

---

1. Determine the frequency-normalized and factorized transfer function $H(s)$ of the analog prototype filter.
Is in this case it's already given.

2. Partial fractionally resolve $H(s)$ into 1st and 2nd order transfer functions (maximum number of 2nd order transfer functions).
The poles are given by:

$$s_1 = -1 \qquad s_2 = s_2^* = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$$

The transfer function is given by:

$$H(s) = \frac{k_1}{s-(s_1)} + \frac{k_2}{s-(s_2)} + \frac{k_2^*}{s-(s_2^*)} = \frac{k_1}{s-(-1)} + \frac{k_2}{s-(-\frac{1}{2}+j\frac{\sqrt{3}}{2})} + \frac{k_2^*}{s-(-\frac{1}{2}-j\frac{\sqrt{3}}{2})}$$

3. Denormalize k_i and the poles

4. Find the coefficients of the digital transfer function

9.7.4 Example 4: Bilinear z-transform

An analog 5th order Butterworth low-pass filter $H(s)$ has a -3 dB cut-off frequency $f_3 = 3$ kHz and -30 dB stopband frequency $f_{30} = 6$ kHz. The filter is digitized by bilinear z-transformation with a sample rate of 16 kHz.

$$N\left[\frac{2\tan^{-1}\left(\frac{3000}{16000}\frac{2\pi}{2}\right)}{\frac{1}{16000}}\right]$$

10 FIR-Filters

Finite Impulse Response filters where the impulse response of the system is of finite duration. A FIR filter with N samples has a impulse response:

$$h(n) = \begin{cases} a_n & \text{for } 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore the filter has N coefficients a_n for $n = 0, 1, \cdots, N-1$.

10.1 Linear phase

10.2 Frequency response

10.3 FIR-Filter design

If there is N samples in the impulse response, then:

$$M = \frac{N-1}{2}$$

And the transfer function of the filter is:

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{i=0}^{2M} a_i z^{-i}$$

Filter type	$\mathbf{c_0}$	$\mathbf{c_m = c_{-m}}$	a_i
Low-pass	$2Tf_a$	$\frac{1}{m\pi}\sin(2\pi mTf_a)$	c_{M-i}
High-pass	$1 - 2Tf_a$	$\frac{1}{m\pi}(\sin(m\pi) - \sin(2\pi mTf_a))$	c_{M-i}
Band-pass	$2T(f_{a_2} - f_{a_1})$	$\frac{1}{m\pi}\big(\sin(2\pi mTf_{a_2}) - \sin(2\pi mTf_{a_1})\big)$	c_{M-i}
Band-stop	$1 - 2T(f_{a_2} - f_{a_1})$	$\frac{1}{m\pi}(\sin(m\pi) + \sin(2\pi mTf_{a_1}) - \sin(2\pi mTf_{a_2}))$	c_{M-i}

Using the inverse z-transform, the impulse response is found:

$$y(n) = \sum_{i=0}^{2M} a_i x(n-i)$$

10.4 Window functions

Due to the ripples in the passband and stopband, the ideal filter is not realizable. Therefore, the ideal impulse response is multiplied by a window function $w(n)$ to obtain a realizable impulse response $h(n)$:

$$h(n) = h_\infty(n)w(n)$$

Following are some common window functions:

Window	$\mathbf{B_n}$	$\mathbf{M_{min}}$	Min. stopband attenuation	Min. passband ripple
Rectangular	2	f_s/Δ_f	20 dB	1.5 dB
Barlett	4	$2f_s/\Delta_f$	25 dB	0.1 dB
Hamming	4	$2f_s/\Delta_f$	50 dB	0.05 dB
Hanning	4	$2f_s/\Delta_f$	45 dB	0.1 dB
Kasier ($\beta = \pi$)	2.8	$1.4f_s/\Delta_f$	40 dB	0.2 dB
Kasier ($\beta = 2\pi$)	4.4	$2.2f_s/\Delta_f$	65 dB	0.01 dB

The new fouirer coefficients c'_m if $-M \leq m \leq M$:

$$c'_m = c_m w_m$$

Therefore

$$a_i = c'_{M-i}$$

10.4.1 Rectangular window

$$w(n) = \begin{cases} 1 & \text{if } -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

10.4.2 Barlett window

$$w(n) = \begin{cases} 1 - \frac{|n|}{M} & \text{if } -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

10.4.3 Hamming and Hanning window

$$w(n) = \begin{cases} \alpha + (1-\alpha)\cos\left(\frac{n\pi}{M}\right) & \text{if } -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha = 0.54$ for Hamming and $\alpha = 0.5$ for Hanning.

10.4.4 Kaizer window

$$w(n) = \begin{cases} \frac{I_0\left(\beta\sqrt{1-\left(\frac{n}{M}\right)^2}\right)}{I_0(\beta)} & \text{if } -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

10.5 Examples

10.5.1 Example 1: High-pass filter

A FIR high-pass filter with cutoff frequency $f_a = 1\,\text{kHz}$, a trasisition band of $\Delta f \leq 0.5\,\text{kHz}$, maximum stopband attenuation of $H_s \leq -50\,\text{dB}$ and sample frequency $f_s = 5\,\text{kHz}$ is designed.

Which window function should be used?
To satisfy the stopband attenuation, the Hamming or the Kaiser window should be used, as they both have 50 dB or higher stopband attenuation. The Hamming window is chosen, which has a $B_n = 4$ and minimum stopband attenuation of 50 dB.

What is the minimum filter order?
The filter order is given by $2M$, where M is:

$$M = \frac{B_n f_s}{2\Delta f} = \frac{4 \cdot 5000}{2 \cdot 500} = 20$$

Therefore the minimum filter order is $2M = 40$.

What is the coefficients for the filter?

$$c_0 = 1 - 2Tf_a \qquad c_m = \frac{1}{m\pi}(\sin(m\pi) - \sin(2\pi m T f_a))$$

And the Hamming window ($\alpha = 0.54$):

$$w_m = \alpha + (1-\alpha)\cos\left(\frac{m\pi}{M}\right)$$

Calculating the coefficients for the filter without window:

$$\begin{aligned}
c_0 &= 1 - 2 \cdot \frac{1}{5000} \cdot 1000 = 0.6 \\
c_1 = c_{-1} &= \frac{1}{1 \cdot \pi}\left(\sin(1 \cdot \pi) - \sin(2\pi \cdot 1 \cdot \frac{1}{5000} \cdot 1000)\right) = -0.3027 \\
c_2 = c_{-2} &= \frac{1}{2 \cdot \pi}\left(\sin(2 \cdot \pi) - \sin(2\pi \cdot 2 \cdot \frac{1}{5000} \cdot 1000)\right) = -0.0935 \\
c_3 = c_{-3} &= \frac{1}{3 \cdot \pi}\left(\sin(3 \cdot \pi) - \sin(2\pi \cdot 3 \cdot \frac{1}{5000} \cdot 1000)\right) = 0.0624
\end{aligned}$$

And so on... to $c_M = c_{20}$

Calculating the coefficients for the window:

$$w_0 = 0.54 + (1-0.54)\cos\left(\frac{0 \cdot \pi}{20}\right) = 1$$

$$w_1 = w_{-1} = 0.54 + (1-0.54)\cos\left(\frac{1\pi}{20}\right) = 0.9943$$

$$w_2 = w_{-2} = 0.54 + (1-0.54)\cos\left(\frac{2\pi}{20}\right) = 0.9775$$

$$w_3 = w_{-3} = 0.54 + (1-0.54)\cos\left(\frac{3\pi}{20}\right) = 0.9499$$

And so on... to $w_M = w_{20}$

Calculate the final coefficients $a_i = c_{M-i} w_{M-i}$:

$$a_{20} = c_{20-20} w_{20-20} = c_0 w_0 = 0.6 \cdot 1 = 0.6$$

$$a_{19} = c_1 w_1 = -0.3027 \cdot 0.9943 = -0.3010$$

$$a_{18} = c_2 w_2 = -0.0935 \cdot 0.9775 = -0.0914$$

$$a_{17} = c_3 w_3 = 0.0624 \cdot 0.9499 = 0.0592$$

And so on... from $a_0 \rightarrow a_{2M} = a_{40}$

10.5.2 Example 2: Band-stop filter

Find the filter coefficients for a FIR band-stop filter without a window. The filter has cutoff frequencies: $f_{a_1} = 1$ kHz $f_{a_2} = 2$ kHz and a sample frequency of $f_s = 10$ kHz. The filter shall have 5 samples.

For a band-stop filter:

$$c_0 = 1 - 2T(f_{a_2} - f_{a_1}) \qquad c_m = \frac{1}{m\pi}(\sin(m\pi) + \sin(2\pi mTf_{a_1}) - \sin(2\pi mTf_{a_2}))$$

Calculate the coefficients:

$$T = \frac{1}{f_s} = 0,0001$$

$$M = \frac{N-1}{2} = 2$$

$$c_0 = 1 - 2T(f_{a_2} - f_{a_1}) = 1 - 2(0,0001)(2000 - 1000) = 1 - 0.2 = \boxed{0.8}$$

$$\begin{aligned} c_m &= \frac{1}{m\pi}(\sin(m\pi) + \sin(2\pi m(0,0001)1000) - \sin(2\pi m(0.0001)2000)) \\ &= \frac{1}{m\pi}(\sin(m\pi) + \sin(2\pi m(0.1)) - \sin(2\pi m(0.2))) \end{aligned}$$

$$\begin{aligned} c_1 = c_{-1} &= \frac{1}{1\pi}(\sin(1\pi) + \sin(2\pi 1(0.1)) - \sin(2\pi 1(0.2))) \\ &= \frac{1}{\pi}(\sin(0.2\pi) - \sin(0.4\pi) = \boxed{-0.1156} \end{aligned}$$

$$\begin{aligned} c_2 = c_{-2} &= \frac{1}{2\pi}(\sin(2\pi) + \sin(2\pi 2(0.1)) - \sin(2\pi 2(0.2))) \\ &= \frac{1}{2\pi}(\sin(0.4\pi) - \sin(0.8\pi) = \boxed{0.0578} \end{aligned}$$

The coefficients are ($a_i = c_{M-i}$):

$$a_0 = 0.0578$$

$$a_1 = -0.1156$$

$$a_2 = 0.8$$

$$a_3 = -0.1156$$

$$a_4 = 0.0578$$