Multi Variable Calculus

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1 Fourier

2 Laplace

3 Several-Variables

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6 Theorems

7 PDE

Partial Differential Equations are equations with multiple variables and derivatives. They are used to model many physical phenomena, such as heat, sound, and light. The totality of solutions to a PDE is called its general solution, and there can be a lot.

7.1 Classification of PDEs

General representation of a PDE:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

Conditions:

Linear: A, B, C, D, E, F are constants
Quasi-linear: A, B, C are constants
Fully non-linear: A, B, C, D, E, F are functions of u and its partial derivatives

7.2 Initial and Boundary Conditions

7.3 Wave Equation (1D)

One dimensional wave equation is given by:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{where } c^2 = \left[\frac{T(x,t)}{\mu_x}\right] \tag{1}$$

Steps to solve:

1. Method of Separation of Variables $\quad u(x,t) = X(x)T(t)$
2. Satify the Boundary Conditions test
3. Fourier Series Validation

7.3.1 D'Alembert's Solution of the Wave Equation

His solution is given by eq. (1) but extended to two variables:

$$v = (x - ct) \qquad\qquad w = (x + ct) \tag{2}$$

I.e. $u(v, w)$. Partial derivatives from chain rule:

$$u_x = u_v \cdot v_x + u_w \cdot w_x = u_v + u_w$$

For double derivatives:

$$u_{xx} = (u_v + u_w)_x = (u_v + u_w)_v v_x + (u_v + u_w)_w w_x = u_{vv} + 2u_{vw} + u_{ww}$$

With respect to t:

$$u_{tt} = c^2 u_{xx} = c^2(u_{vv} + 2u_{vw} + u_{ww})$$

From eq. (1) and eq. (2):

$$u_{vw} = \frac{\partial^2 u}{\partial w \partial v} = 0$$

This can be solved by integrating with respect to v and w:

$$\frac{\partial u}{\partial v} = h(v) \text{ and } u = \int h(v)\ dv + \psi(w)$$

Here, $h(v)$ and $\psi(w)$ are arbitrary functions of v and w, respectively. The solution in term for x:

$$u = \phi(v) + \psi(w)$$

This is d'Alembert's solution, which is the general solution to the wave equation.

$$u(x,t) = \phi(x + ct) + \psi(x - ct)$$

This solution satisfies the wave equation and the initial conditions:

7.4 Heat Equation (1D)

7.5 Example exercises