Multi Variable Calculus

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Last updated: December 23, 2023

Contents

1 Fourier

1.1 Fourier Series

A periodic function with period $2L$ and let $f(x)$ and $f'(x)$ be piecewise continuous on the interval $-L < x < L$

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right) = \sum_{n=-\infty}^{\infty} c_n e^{jn\pi x/L}$$

The coefficients are given by:

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \qquad n \geq 0$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \qquad n > 0$$

$$c_n = \frac{1}{2}(a_n - jb_n) \qquad n > 0$$

1.2 Fourier Transform

If $h(t)$ is a periodic function then the Fourier transform is given by:

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t}\, dt$$

Inverse Fourier tranformation of $H(\omega)$:

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{j\omega t}\, d\omega$$

1.3 Examples

1.3.1 Example 1

2 Laplace

Is a generalisation of the Fourier transform and defined as:

$$H(s) = \mathcal{L}\{h(t)\} = \int_{0^-}^{\infty} h(t)e^{-st}\,dt \qquad s \in \mathbb{C}$$

s is a complex number $s = \sigma + j\omega$ and is identical with Fourier transform, if s is set to $j\omega$.
Inverse Laplace tranformation:

$$f(t) = \frac{1}{2\pi j}\int_{\sigma_c - j\infty}^{\sigma_c + j\infty} F(s)e^{st}\,ds$$

2.1 Examples

2.1.1 Example 1: Laplace Transform

Using the Laplace transform, find the solution for the following equation:

$$\frac{\partial^2}{\partial t^2}y(t) + 2\frac{\partial}{\partial t}y(t) + 2y(t) = 0$$

with initial conditions $y(0) = 1$ and $Dy(0) = -1$

3 Several-Variables

4 Double-Integrals

4.1 Riemann Sum

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta A_i$$

$$I = \iint_D f(x,y)dA \qquad \text{where } D \text{ is a region in } \mathbb{R}^2 \text{ and } dA \text{ is } dxdy$$

4.2 Double Integrals over General domains

If $f(x,y)$ is defined and bounded on domain D, then $\hat{f}(x,y)$ is zero outside D.

$$\iint_D f(x,y)\,dA = \iint_R \hat{f}(x,y)\,dA$$

4.3 Iteration of Double Integrals

If $f(x,y)$ is continuous on the bounded y-simple domain D given by $a \leq x \leq b$ and $c(x) \leq y \leq d(x)$, then:

$$\iint f(x,y)\,dA = \int_a^b dx \int_{c(x)}^{d(x)} f(x,y)dy$$

If $f(x,y)$ is continuous on the bounded x-simple domain D given by $c \leq x \leq d$ and $a(x) \leq y \leq b(x)$, then:

$$\iint f(x,y)\,dA = \int_c^d dx \int_{a(x)}^{b(x)} f(x,y)dy$$

4.4 Double Integrals in Polar Coordinates

$$dA = dxdy = r\ drd\theta$$

$$\begin{array}{ll} x = r\cos(\theta) & r^2 = x^2 + y^2 \\ y = r\sin(\theta) & \tan(\theta) = \frac{y}{x} \end{array}$$

4.5 Change of Variables in Double Integrals

If x and y are given as a function of u and v:

$$\begin{array}{l} x = x(u,v) \\ y = y(u,v) \end{array}$$

These can be transformed or mapped from points (u,v) in the uv-plane to points (x,y) in the xy-plane.
The inverse transformation is given by:

$$\begin{array}{l} u = u(x,y) \\ v = v(x,y) \end{array}$$

Scaled area element:

$$dA = dxdy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

where the Jacobian is:

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Let $x(u, v)$ and $y(u, v)$ be a one-to-one transformation from a domain S in the uv-plane onto a domain D xy-plane.
Suppose, that function x and y, and first partial derivatives with respect to u and v are continuous in S. If $f(x, y)$ is integrable on D, then $g(u, v) = f(x(u, v), y(u, v))$ is integrable on S and:

$$\iint_D f(x, y)dA = \iint_S g(u, v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv$$

4.6 Examples

4.6.1 Example 1: Change of Variables

Evaluate the double integral:

$$\iint (x - 3y)dA$$

where R is triangular region with vertices $(0, 0)$, $(2, 1)$, and $(1, 2)$ using the transformation:

$$\begin{aligned} x &= 2u + v \\ y &= u + 2v \end{aligned}$$

4.6.2 Example 2: Riemann Sum

4.6.3 Example 3: By iteration

4.6.4 Example 4: Double integral

Find the area within the region:

$$\iint (\sin x + \cos y)dA \qquad R : \left\{ x, y \,\middle|\, \begin{array}{l} 0 \le x \le \pi/2 \\ 0 \le y \le \pi/2 \end{array} \right.$$

4.6.5 Example 5: Defining limits of integration

5 Tripple-Integrals

5.1 Examples

5.1.1 Example 1: Tripple Integral

Find $\iiint (x^2 + y^2 + z^2)dV$, where the region is bounded by $z = c\sqrt{(x^2 + y^2)}$ and $x^2 + y^2 + z^2 = a^2$

6 Fields-Curve

6.1 Curve & Parameterization

Representation of a curve in 3 space by using its position vector is given as:

$$r = r(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \qquad \text{where } a \leq t \leq b$$

6.2 Vector Fields

$$\mathbf{F}(x, y, z) = \underbrace{f_x(x, y, z)}_{\text{Scaler function}} \mathbf{i} + \underbrace{f_y(x, y, z)}_{\text{Scaler function}} \mathbf{j} + \underbrace{f_z(x, y, z)}_{\text{Scaler function}} \mathbf{k}$$

$$\frac{\partial f}{\partial x} = f_1 = f_x \qquad \tfrac{\partial f}{\partial y} = f_2 = f_y \qquad \frac{\partial f}{\partial z} = f_3 = f_z$$

Position vector:

$$r = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Unit vector with magnitude 1:

$$r = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

6.2.1 Scalar field

$$f(x, y, z) = f_1(x, y, z) = f_2(x, y, z) = f_3(x, y, z)$$

The gradient of a scalar field is a vector field:

$$\nabla f = \text{grad } f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$

6.2.2 Field lines

6.2.3 Convervation field

If $\mathbf{F}(x, y, z) = \nabla\phi(x, y, z)$ in a domain D, then $\mathbf{F}$ is a conservative vector field in D and function ϕ is the potential function.

$$\mathbf{F}(x, y, z) = \nabla\phi(x, y, z) = \phi_x(x, y, z)\mathbf{i} + \phi_y(x, y, z)\mathbf{j} + \phi_z(x, y, z)\mathbf{k}$$

If the vetor field is conservative, then all the following equations are true:

$$\begin{aligned} \frac{\partial f_x}{\partial y} &= \frac{\partial f_y}{\partial x} \\ \frac{\partial f_x}{\partial z} &= \frac{\partial f_z}{\partial x} \\ \frac{\partial f_y}{\partial z} &= \frac{\partial f_z}{\partial y} \end{aligned}$$

6.2.4 Vector field in Polar Coordinates

$$\mathbf{F} = f(r, \theta) = f_r(r, \theta)\hat{\mathbf{r}} + f_\theta(r, \theta)\hat{\theta}$$

where:

$$\hat{\mathbf{r}} = \cos(\theta)i + \sin(\theta)j$$

$$\hat{\theta} = -\sin(\theta)i + \cos(\theta)j$$

6.3 Line Integral

$$f(x,y)ds = \text{Area (tiny point)}$$

$$\text{Length of } \mathcal{C} = \int_{\mathcal{C}} f(x,y,z)\,ds = \int_a^b f(r(t)) \left|\frac{dr}{dt}\right| \,dt$$

6.3.1 Line integral of a vector field

$$W = \int_{\mathcal{C}} F.\hat{T}\,ds = \int F\,dr = \int_{\mathcal{C}} f_1(x,y,z)dx + f_2(x,y,z)dy + f_3(x,y,z)dz$$

6.4 Examples

6.4.1 Example 1: Conservative vector field and potential

Determine whether the given vector field is conservative, and find a potential function if it is:

$$\mathbf{F}(x,y,z) = (2xy - z^2)\mathbf{i} + (2yz + x^2)\mathbf{j} - (2zx - y^2)\mathbf{k}$$

6.4.2 Example 2: Line integral

Evaluate $\oint x^2y^2\ dx + x^3y\ dy$ counterclockwise around the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$

6.4.3 Example 3: Line integral

Evalute the line integral for $f(x,y) = x^2y^2$ along a straight line from origin to the point $(2,1)$

Ans: $(5\sqrt{5})/3$

6.4.4 Example 4: Line integral vector field

Evaluate the line integral for $\mathbf{F}(x,y) = y^2\mathbf{i} + 2xy\mathbf{j}$ from $(0,0)$ to $(1,1)$
Along the line $y = x$
Allong the line $y = x^2$

6.4.5 Example 5: Gradient

6.4.6 Example 6: Parametrize a curve

Use $t = y$ to parametrize the part of the line of intersection of the two planes:
Plane 1: $y = 2x - 4$
Plane 2: $z = 3x + 1$ from $(2,0,7)$ to $(3,2,10)$

Ans: $r(t) = \left(\frac{1}{2}(t+4)\right)\mathbf{i} + (t)\mathbf{j} + \left(\frac{3}{2}t + 7\right)\mathbf{k}$

7 Theorems

7.1 Green's Theorem

$$\oint_C f_1(x,y)dx + f_2(x,y)dy = \iint_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dA$$

7.2 Stokes' Theorem

$$\oint_C F \cdot dr = \iint_S \text{curl } F \cdot \widehat{N} dS$$

7.3 Divergence Theorem

$$\nabla \cdot F(x,y,z) = \text{div } F = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$F(x,y,z) = f_1(x,y,z)i + f_2(x,y,z)j + f_3(x,y,z)k$$

$$\oiint_s (F \cdot \widehat{N}) dS = \iiint_V \text{div } F dV$$

More variants:

$$\iiint_D \text{curl } F dV = -\oiint_s (F \times \widehat{N}) dS$$

$$\iiint_D \text{grad } \phi \, dV = \oiint_s \phi \, dS$$

7.4 Examples

7.4.1 Example 1: Div and Curl

Calculate the divergence and curl of the following vector field:

$$\mathbf{F} = \cos x \, \mathbf{i} - \sin y \, \mathbf{j} + z \, \mathbf{k}.$$

Divergence: Curl:

7.4.2 Example 2: Green's Theorem

Using Green's Theorem evaluate $\oint_e (x^2 y) \, dx + (xy^2) dy$, clockwise boundary of the region:

$$0 \leq y \leq \sqrt{9 - x^2}$$

7.4.3 Example 3: Stokes' Theorem

7.4.4 Example 4: Divergence Theorem

Use the Divergence Theorem to calculate the flux of the given vector field out of the sphere s with equation $x^2 + y^2 + z^2 = a^2$, where $a > 0$ and

$$\mathbf{F} = (x^2 + y^2)\mathbf{i} + (y^2 - z^2)\mathbf{j} + z\mathbf{k}$$

8 PDE

Partial Differential Equations are equations with multiple variables and derivatives. They are used to model many physical phenomena, such as heat, sound, and light. The totality of solutions to a PDE is called its general solution, and there can be a lot.

8.1 Classification of PDEs

General representation of a PDE:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

Conditions:

Linear: A, B, C, D, E, F are constants
Quasi-linear: A, B, C are constants
Fully non-linear: A, B, C, D, E, F are functions of u and its partial derivatives

8.2 Characteristics of PDEs

$B^2 - 4AC > 0$	2 real roots	2 characteristics	**Hyperbolic PDE**
$B^2 - 4AC = 0$	1 real roots	1 characteristics	**Parabolic PDE**
$B^2 - 4AC < 0$	0 real roots	0 characteristics	**Elliptic PDE**

Tyoes of varius PDEs:

Wave Equation: Hyperbolic PDE

Heat Equation: Parabolic PDE

Laplace Equation: Elliptic PDE

8.3 Initial and Boundary Conditions

8.4 Wave Equation (1D)

One dimensional wave equation is given by:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{where } c^2 = \left[\frac{T(x,t)}{\mu_x}\right] \tag{1}$$

Steps to solve:

1. Method of Separation of Variables $\quad u(x,t) = X(x)T(t)$
2. Satify the Boundary Conditions test
3. Fourier Series Validation

8.4.1 D'Alembert's Solution of the Wave Equation

His solution is given by eq. (1) but extended to two variables:

$$v = (x - ct) \qquad\qquad w = (x + ct) \tag{2}$$

I.e. $u(v, w)$. Partial derivatives from chain rule:

$$u_x = u_v \cdot v_x + u_w \cdot w_x = u_v + u_w$$

For double derivatives:

$$u_{xx} = (u_v + u_w)_x = (u_v + u_w)_v v_x + (u_v + u_w)_w w_x = u_{vv} + 2u_{vw} + u_{ww}$$

With respect to t:

$$u_{tt} = c^2 u_{xx} = c^2(u_{vv} + 2u_{vw} + u_{ww})$$

From eq. (1) and eq. (2):

$$u_{vw} = \frac{\partial^2 u}{\partial w \partial v} = 0$$

This can be solved by integrating with respect to v and w:

$$\frac{\partial u}{\partial v} = h(v) \text{ and } u = \int h(v)\ dv + \psi(w)$$

Here, $h(v)$ and $\psi(w)$ are arbitrary functions of v and w, respectively. The solution in term for x:

$$u = \phi(v) + \psi(w)$$

This is d'Alembert's solution, which is the general solution to the wave equation.

$$u(x,t) = \phi(x + ct) + \psi(x - ct)$$

This solution satisfies the wave equation and the initial conditions:

8.5 Heat Equation (1D)

$$\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2} \tag{3}$$

Conditions:

- PDE is linear and homogeneous.
- Boundary conditions are linear and homogeneous.The two for $u(x,t)$ is $u(0,t) = 0$ and $u(L,t) = 0$ for all $t > 0$.
- One initial condition at time ($t = 0$): $u(x,0) = f(x)$.

Solve the

Solution:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda_n^2 t}$$

where

$$\lambda_n = \frac{cn\pi}{L}$$

and

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx \qquad \text{for } n = 1, 2, 3 \ldots$$

8.6 Examples