Multi Variable Calculus Exam

190483865

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Problem 1.1

$$e^{-j\omega a}$$

Problem 1.2

$$1 + t + \sin 2t + \cos(4t)$$

Problem 1.3

$$f(x, y, z) = 0$$

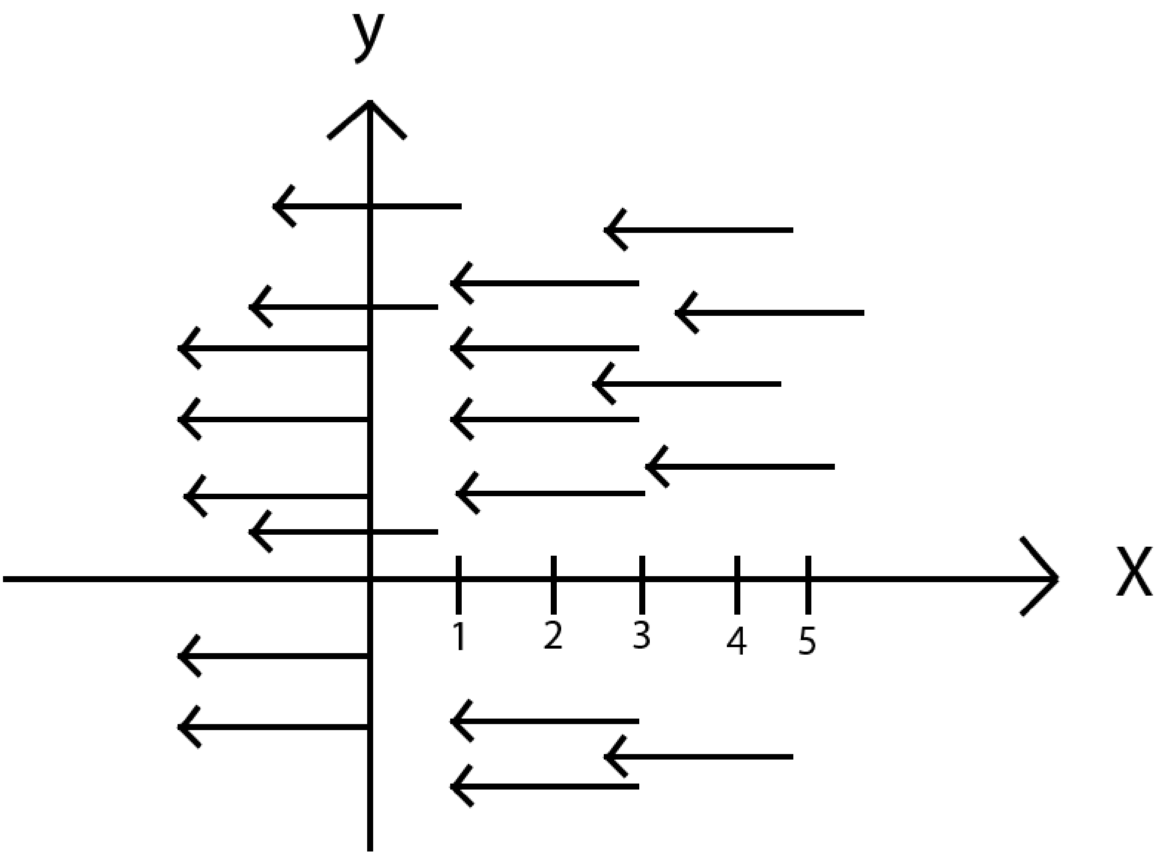
Problem 1.4

$$f = e^{3x+4y} \cos 5z$$
$$f_1 = 3e^{3x+4y} \cos(5z)$$
$$f_{12} = 12e^{3x+4y} \cos(5z)$$
$$f_{122} = 48e^{3x+4y} \cos(5z)$$

Problem 1.5

$$\lim_{(x,y)\to(2,-1)} = 2 \cdot -1 + 2^2 = -2$$

Problem 1.6



Problem 1.7

$$\text{div } F(x, y) = -\cos(x) + \cos(y)$$

Problem 1.8

b)

Problem 1.9

a)

Problem 1.10

c)

Problem 2.1

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}$$

Limits for integral:

$$-\frac{1}{2} \leq t \leq \frac{1}{1}$$

$$X(\omega) = \int_{-1/2}^{1/2} (1 - 2|t|)e^{-j\omega t}$$

$$= \left[\frac{e^{-it\omega}(-2 - i(2t-1)\omega)}{\omega^2}\right]_{-0.5}^{0.5}$$

$$= \left(\frac{-2e^{-i0.5\omega}}{\omega^2}\right) - \left(\frac{2i\omega e^{i0.5\omega}}{\omega^2}\right)$$

$$= \frac{-2e^{-i0.5\omega} - 2i\omega e^{i0.5\omega}}{\omega^2}$$

Problem 2.2

$$\oint_C F \cdot dr = \iint_S \text{curl } F \cdot \widehat{N} dS$$

$$\text{curl } F = (0-1)i + (0-1)j + (0-1)k = -i + -j + -k$$

$$\widehat{N} dS = (i + j + k)dxdy$$

$$\text{curl } F \cdot \widehat{N} dS = (-1 - 1 - 1)dxdy = -3dxdy$$

Using spherical coordinates:

$$x^2 + y^2 + z^2 = a^2 = \rho^2$$

Setup integral:

$$\int_0^{2\pi} \int_0^a -3rdrd\theta$$

$$-3 \int_0^{2\pi} \left[\frac{r^2}{2}\right]_0^a d\theta$$

$$\frac{-3a^2}{2} \int_0^{2\pi} d\theta$$

$$\frac{-3a^2}{2} [\theta]_0^{2\pi}$$

$$-3a^2\pi$$

Problem 3.1

Evaluate the double integral $\iint_R ydA$ where R is the region bounded by a shifted circle $x^2 + y^2 = 2x$ and a line $y = x$

Find limits (replace $x = y$):

$$x^2 + x^2 = 2x \qquad \Rightarrow \qquad 2x = 2 \qquad \Rightarrow \qquad x = \begin{cases} 0 \\ 1 \end{cases}$$

$$0 \le x \le 1$$

There y is also

$$0 \le y \le 1$$

$$\int_0^1 \int_0^1 ydxdy = \int_0^1 [yx]_0^1 \, dy$$
$$= \int_0^1 ydy = \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}$$

Problem 3.2

$$u_{xx} + 5u_{xy} + 4u_{yy} = 0$$

Find A,B and C:

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = f(x, y, u, u_x, u_y)$$
$$A = 1 \qquad 2B = 5 \quad \Rightarrow \quad B = 2.5 \qquad C = 4$$

Find the type:

$$AC - B^2 = 1 \cdot 4 - \left(\frac{5}{2}\right)^2 = \frac{16 - 25}{4} = -\frac{9}{4}$$

Since $AC - B^2 < 0$ the PDE is hyperbolic.

Transform to normal form:

$$Ay'' - 2By' + C = 0 \qquad \Rightarrow \qquad y'' - 5y' + 4 = 0 \qquad \Rightarrow \qquad y' = \begin{cases} 1 \\ 4 \end{cases}$$

Find the constants:

$$y' = 1 \quad \Rightarrow \quad y = x + c_1 \quad \Rightarrow \quad c_1 = y - x$$
$$y' = 4 \quad \Rightarrow \quad y = 4x + c_1 \quad \Rightarrow \quad c_2 = y - 4x$$

Transform the variables:

$$v = y - x \qquad \Rightarrow \qquad v_x = -1 \qquad \Rightarrow \qquad v_y = 1$$
$$w = y - 4x \qquad \Rightarrow \qquad w_x = -4 \qquad \Rightarrow \qquad w_y = 1$$

$$u_x = u_v v_x + u_w w_x = u_v \cdot -1 + u_w \cdot -4 = -u_v - 4u_w$$
$$u_{xx} = (u_x)_v v_x + (u_x)_w w_x = (-u_v - 4u_w)_v v_x + (-u_v - 4u_w)_w w_x$$

$$= (-u_{vv} - 4u_{wv}) \cdot -1 + (-u_{vw} - 4u_{ww}) \cdot -4 = \boxed{u_{vv} + 4u_{wv} + 4u_{vw} + 16u_{ww}}$$

$$u_{xy} = (u_x)_v v_y + (u_x)_w w_y = (-u_v - 4u_w)_v v_y + (-u_v - 4u_w)_w w_y$$
$$= (-u_v - 4u_w)_v + (-u_v - 4u_w)_w = -u_{vv} - 4u_{wv} - u_{vw} - 4u_{ww} =$$
$$= \boxed{-u_{vv} - 5u_{wv} - 4u_{ww}}$$

$$u_y = u_v v_y + u_w w_y = u_v \cdot 1 + u_w \cdot 1 = \boxed{u_v + u_w}$$
$$u_{yy} = (u_y)_v v_y + (u_y)_w w_y = (u_v + u_w)_v + (u_v + u_w)_w$$
$$= u_{vv} + u_{wv} + u_{vw} + u_{ww} = \boxed{u_{vv} + 2u_{vw} + u_{ww}}$$

Insert into equation:

$$u_{xx} + 5u_{xy} + 4u_{yy} = 0$$
$$u_{vv} + 4u_{wv} + 4u_{vw} + 16u_{ww} + 5(-u_{vv} - 5u_{wv} - 4u_{ww}) + 4(u_{vv} + 2u_{vw} + u_{ww}) = 0$$

Normal form:

$$-4u_{vw} = 0 \quad \Rightarrow \quad u_{vw} = 0$$

Solve:

$$u_{vw} = 0 \qquad \Rightarrow \qquad u_v = h(v)$$
$$u = g(w) + \int h(v)$$
$$u(v, w) = g(w) + f(v)$$

Insert x and y:

$$u(v, w) = g(y - 4x) + f(y - x)$$

Where g and f is arbitrary functions.