Multi Variable Calculus

Mathias Balling Christiansen

Last updated: January 3, 2024

Contents

1 Fourier

1.1 Fourier Series

A periodic function with period $2L$ and let $f(x)$ and $f'(x)$ be piecewise continuous on the interval $-L < x < L$

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right) = \sum_{n=-\infty}^{\infty} c_n e^{jn\pi x/L}$$

The coefficients are given by:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx \qquad n \geq 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx \qquad n > 0$$

$$c_n = \frac{1}{2}(a_n - jb_n) \qquad n > 0$$

1.2 Fourier Transform

If $h(t)$ is a periodic function then the Fourier transform is given by:

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\, dt$$

Inverse Fourier tranformation of $H(\omega)$:

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{j\omega t}\, d\omega$$

Signal	**Fourier Transform**
$\delta(t)$	1
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$
$\delta(t - t_0)$	$e^{-j\omega t_0}$
$\sin(\omega_0 t)$	$-j\pi(\delta(\omega - \omega_0) - \delta(\omega + \omega_0))$
$\cos(\omega_0 t)$	$\pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$
1	$2\pi\delta(\omega)$

1.3 Examples

1.3.1 Example 1: Fourier Series

Find the Fourier coefficients and Fourier Series for the square wave shown below:

$$f(x) = \begin{cases} 0 & \text{for } -1 \leq x \leq 0 \\ 1 & \text{for } 0 \leq x \leq 1 \end{cases}$$

and

$$f(x+2) = f(x)$$

The fourier series is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

Find the L value:

$$2L = 2 \qquad \Rightarrow \qquad L = 1$$

Find a_0:

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \qquad n \geq 0$$

$$a_0 = \frac{1}{1} \int_{-1}^{1} f(x) \cos\left(\frac{0\pi x}{1}\right) dx = \int_{-1}^{1} f(x) dx = \int_{-1}^{0} 0 dx + \int_{0}^{1} 1 dx = 1$$

Find a_n:

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \qquad n \geq 0$$

$$= \frac{1}{1} \int_{-1}^{1} f(x) \cos\left(\frac{n\pi x}{1}\right) dx = \int_{-1}^{1} f(x) \cos(n\pi x)\, dx$$

$$= \int_{-1}^{0} 0 \cos(n\pi x)\, dx + \int_{0}^{1} 1 \cos(n\pi x)\, dx = 0 + \left[\frac{\sin(n\pi x)}{n\pi}\right]_0^1 = \frac{\sin(\pi n)}{\pi n}$$

For all n:

$$\frac{\sin(\pi n)}{\pi n} = 0$$

Find b_n:

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \qquad n > 0$$

$$b_n = \int_{-1}^{1} f(x) \sin\left(\frac{n\pi x}{1}\right) dx = \int_{-1}^{0} 0 \sin(n\pi x)\, dx + \int_{0}^{1} 1 \sin(n\pi x)\, dx$$

$$= 0 + \left[\frac{-\cos(n\pi x)}{n\pi}\right]_0^1 = \frac{-\cos(n\pi 1)}{n\pi} - \frac{-\cos(n\pi 0)}{n\pi}$$

If n is even the function will cancel out, therefore $n = 1, 3, 5, \ldots$ (odd):

$$= \frac{1}{n\pi} + \frac{1}{n\pi} = \frac{2}{n\pi}$$

Ans:

$$f(x) = \frac{1}{2} + \sum_{n=1,3,5,\ldots} \frac{2}{n\pi} \sin(n\pi x)$$

1.3.2 Example 2: Fourier Transform

The unit step function is defined as:

$$u(t-a) = \begin{cases} 1 & \text{for } t-a>0 \\ 0 & \text{for } t-a<0 \end{cases}$$

is used to define the rectangular pulse function:

$$x(t) = u(t-a) - u(t-b) \qquad \text{where } a < b$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$X(\omega) = \int_{-\infty}^{a} 0e^{-j\omega t}\,dt + \int_{a}^{b} 1e^{-j\omega t}\,dt + \int_{b}^{\infty} 0e^{-j\omega t}\,dt$$

$$X(\omega) = 0 + \left[\frac{-e^{-j\omega t}}{j\omega}\right]_a^b + 0$$

Insert the limits:

$$X(\omega) = \frac{e^{-j\omega a} - e^{-j\omega b}}{j\omega}$$

2 Laplace transform

Is a generalisation of the Fourier transform and defined as:

$$H(s) = \mathcal{L}\{h(t)\} = \int_{0^-}^{\infty} h(t)e^{-st}\,dt \qquad s \in \mathbb{C}$$

s is a complex number $s = \sigma + j\omega$ and is identical with Fourier transform, if s is set to $j\omega$.
Inverse Laplace tranformation:

$$h(t) = \mathcal{L}^{-1}\{H(t)\} = \frac{1}{2\pi j}\int_{\sigma_c - j\infty}^{\sigma_c + j\infty} H(s)e^{st}\,ds$$

Signal	Laplace Transform
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$
$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$

2.1 General Formulas

$$\mathcal{L}\{f(t)\} = F(s)$$
$$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$$
$$\mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0) - f'(0)$$
$$\mathcal{L}\{f^n\} = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - \cdots - f^{n-1}(0)$$

2.2 Examples

2.2.1 Example 1: Laplace Transform

Using the Laplace transform, find the solution for the following equation:

$$\frac{\partial^2}{\partial t^2}y(t) + 2\frac{\partial}{\partial t}y(t) + 2y(t) = 0$$

with initial conditions $y(0) = 1$ and $y'(0) = -1$

Take laplace transform of the equation:

$$s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + 2(Y(s)) = 0$$

$$s^2Y(s) - s + 1 + 2sY(s) - 2 + 2Y(s) = 0$$

$$s^2Y(s) + 2sY(s) + 2Y(s) = 1 + s$$

$$(s^2 + 2s + 2)Y(s) = 1 + s$$

$$Y(s) = \frac{1+s}{(s^2+2s+2)} = \frac{1+s}{((s+1)^2+1)}$$

From table lookup:

$$\mathcal{L}\{e^{at}\cos(\omega t)\} = \frac{s-a}{(s-a)^2+\omega^2}$$

$$a = -1 \qquad \omega = 1$$

$$y(t) = e^{-t}\cos(t)$$

2.2.2 Example 2: Transfer Function

Consider a mass-spring-damper system with the following differential equation:

$$m\ddot{x} = -kx - b\dot{x} + f$$

Find the transfer function for the system with input f and output x.

$$ms^2X(s) = -kX(s) - bsX(s) + F(s)$$

$$ms^2X(s) + kX(s) + bsX(s) = F(s)$$

$$(ms^2 + k + bs)X(s) = F(s)$$

The transfer function is:

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2+k+bs}$$

2.2.3 Example 3: Differential equation

Consider:

$$y''(t) + y'(t) = 0.5t$$

where $y(0) = 0$ and $y'(0) = 0$ Use Laplace transform to solve the equation and find $y(t)$

From Laplace transform table:

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

Laplace transform of given differential equation:

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = 0.5\frac{1}{s^2}$$

$$(s^2+1)Y(s) = \frac{0.5}{s^2}$$

$$Y(s) = \frac{0.5}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}$$

$$0.5 = As(s^2+1) + B(s^2+1) + (Cs+D)s^2$$

$$0.5 = As^3 + As + Bs^2 + B + Cs^3 + Ds^2$$

$$0.5 = s^3(A+C) + s^2(B+D) + sA + B$$

$$B = 0.5$$

$$A = 0$$

$$B + D = 0 \qquad \Rightarrow \qquad D = -0.5$$

$$A + C = 0 \qquad \Rightarrow \qquad C = 0$$

Therefore the partial fractions are:

$$Y(s) = \frac{0}{s} + \frac{0.5}{s^2} + \frac{0s - 0.5}{s^2 + 1} = \frac{0.5}{s^2} + \frac{-0.5}{s^2 + 1}$$

From Laplace transform table:

$$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2 + a^2}$$

The inverse laplace transform of $Y(s)$:

$$y(s) = 0.5t - 0.5\sin(t)$$

3 Several-Variables

3.1 Examples

3.1.1 Example 1: Gradient

Find the rate of change of $f(x, y) = y^4 + 2xy^3 + x^2y^2$ at $(0, 1)$ in each of the following directions:

1. $\mathbf{i} + 2\mathbf{j}$
2. $\mathbf{j} - 2\mathbf{i}$
3. $3\mathbf{i}$
4. $\mathbf{i} + \mathbf{j}$

3.1.2 Example 2: Jacobian

Find the Jacobian $Df(1, 0)$ for the transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ given by:

$$\mathbf{f}(x, y) = (xe^y + \cos(\pi y), x^2, x - e^y)$$

3.1.3 Example 3: Chain Rule

If $z = \sin(x^2y)$ where $x = st^2$ and $y = s^2 + \frac{1}{t}$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ using chain rule.

For:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial x} = 2xy\cos(x^2y)$$

$$\frac{\partial x}{\partial s} = t^2$$

$$\frac{\partial z}{\partial y} = x^2\cos(x^2y)$$

$$\frac{\partial y}{\partial s} = 2s$$

Find:

$$\frac{\partial z}{\partial s} = (2xy\cos(x^2y))t^2 + (x^2\cos(x^2y))2s$$

Replace x and y with $x = st^2$ and $y = s^2 + \frac{1}{t}$:

$$\frac{\partial z}{\partial s} = (4s^3t^4 + 2st^3)\cos(s^4t^4 + s^2t^3)$$

Same principal for:

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

3.1.4 Example 4: Substitution

If $z = \sin(x^2 y)$ where $x = st^2$ and $y = s^2 + \frac{1}{t}$ find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ using substitution.

$$z = \sin\left((st^2)^2 s^2 + \frac{1}{t}\right) = \sin\left(s^2 t^4 (s^2 + \frac{1}{t})\right) = s^4 t^4 + s^2 t^3$$

$$\frac{\partial z}{\partial s} = \sin(s^4 t^4 + s^2 t^3) = (4s^3 t^4 + 2st^3)\cos(s^4 t^4 + s^2 t^3)$$

$$\frac{\partial z}{\partial t} = \sin(s^4 t^4 + s^2 t^3) = (s^4 4t^3 + s^2 3t^2)\cos(s^4 t^4 + s^2 t^3)$$

3.1.5 Example 5: Partial Differentiation

Calculate $f_{223}(x, y, z)$, $f_{232}(x, y, z)$, and $f_{322}(x, y, z)$ of the following function:

$$f(x, y, z) = e^{x-2y+3z}$$

Normal Differentiation. For $f_{223}(x, y, z)$, start with y, then y again, and then z:

$$\frac{\partial}{\partial z}\frac{\partial}{\partial y}\frac{\partial f(x, y, z)}{\partial y}$$

4 Double-Integrals

4.1 Riemann Sum

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta A_i$$

$$I = \iint_D f(x,y)dA \qquad \text{where } D \text{ is a region in } \mathbb{R}^2 \text{ and } dA \text{ is } dxdy$$

4.2 Double Integrals over General domains

If $f(x,y)$ is defined and bounded on domain D, then $\hat{f}(x,y)$ is zero outside D.

$$\iint_D f(x,y)\,dA = \iint_R \hat{f}(x,y)\,dA$$

4.3 Iteration of Double Integrals

If $f(x,y)$ is continuous on the bounded y-simple domain D given by $a \leq x \leq b$ and $c(x) \leq y \leq d(x)$, then:

$$\iint f(x,y)\,dA = \int_a^b dx \int_{c(x)}^{d(x)} f(x,y)dy$$

If $f(x,y)$ is continuous on the bounded x-simple domain D given by $c \leq x \leq d$ and $a(x) \leq y \leq b(x)$, then:

$$\iint f(x,y)\,dA = \int_c^d dx \int_{a(x)}^{b(x)} f(x,y)dy$$

4.4 Double Integrals in Polar Coordinates

$$dA = dxdy = r\ drd\theta$$

$$\begin{aligned} x &= r\cos(\theta) \qquad & r^2 &= x^2 + y^2 \\ y &= r\sin(\theta) \qquad & \tan(\theta) &= \tfrac{y}{x} \end{aligned}$$

4.4.1 Limits for Polar Coordinates

r is the radius from origin to the point

$$r \geq 0$$

θ is the angle in the positive direction of the xy-plane

$$0 \leq \theta \leq 2\pi$$

4.5 Change of Variables in Double Integrals

If x and y are given as a function of u and v:

$$x = x(u,v)$$
$$y = y(u,v)$$

These can be transformed or mapped from points (u, v) in the uv-plane to points (x, y) in the xy-plane.
The inverse transformation is given by:

$$u = u(x, y)$$
$$v = v(x, y)$$

Scaled area element:

$$dA = dxdy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

where the Jacobian is:

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Let $x(u, v)$ and $y(u, v)$ be a one-to-one transformation from a domain S in the uv-plane onto a domain D xy-plane.
Suppose, that function x and y, and first partial derivatives with respect to u and v are continuous in S. If $f(x, y)$ is integrable on D, then $g(u, v) = f(x(u, v), y(u, v))$ is integrable on S and:

$$\iint_D f(x,y)dA = \iint_S g(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

4.6 Examples

4.6.1 Example 1: Change of Variables

Evaluate the double integral:

$$\iint (x - 3y)dA$$

where R is triangular region with vertices $(0, 0)$, $(2, 1)$, and $(1, 2)$ using the transformation:

$$x = 2u + v$$
$$y = u + 2v$$

As x and y are dependent on u and v the transformation of dA is given by

$$dA = dxdy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

Using:

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

the Jacobian can be calculated:

$$\frac{\partial x}{\partial u} = 2 \quad \frac{\partial y}{\partial v} = 2$$
$$\frac{\partial x}{\partial v} = 1 \quad \frac{\partial y}{\partial u} = 1$$
$$2 \cdot 2 - 1 \cdot 1 = 3$$

Find the boundaries for the double integral
$y_1 : (0, 0) \rightarrow (2, 1)$ is $y = \frac{1}{2}x$
$y_2 : (0, 0) \rightarrow (1, 2)$ is $y = 2x$
$y_3 : (1, 2) \rightarrow (2, 1)$ is $y = 3 - x$

Replace x and y with their transformation:
$u + 2v = \frac{2u+v}{2} \qquad v = 0$
$u + 2v = 4u + 2v \qquad u = 0$
$u + 2v = 3 - 2u - v \qquad u = 1 - v$

Therefore $0 \leq u \leq 1 - v$ and $0 \leq v \leq 1$
Now transform the original function $x - 3y$:

$$\begin{aligned} x - 3y &= 2u + v - 3(u + 2v) \\ &= 2u - 3u + v - 6v \\ &= -u - 5v \end{aligned}$$

$$\begin{aligned} \int_0^1 \int_0^{1-v} (-u - 5v) \cdot 3 \; dudv &= -3 \int_0^1 \left[\frac{u^2}{2} + 5uv \right]_0^{1-v} dv \\ &= -\frac{3}{2} \int_0^1 \left(\frac{27v^2}{2} - 12v \right) dv \\ &= \left[\frac{9v^3}{2} - 6v^2 - \frac{3v}{2} \right]_0^1 \\ &= \boxed{-3} \end{aligned}$$

4.6.2 Example 2: Double integral

Evaluate the double integral by iteration

$$\iint_R (x^2 + y^2) \, dA$$

where R is the rectangle $0 \leq x \leq a$, $0 \leq y \leq b$

Insert the limits and solve the integral:

$$\begin{aligned} \int_0^b \int_0^a (x^2 + y^2) dxdy &= \int_0^b \left[\frac{x^3}{3} + xy^2 \right]_0^a dy \\ &= \int_0^b \left(\frac{a^3}{3} + ay^2 \right) dy \\ &= \boxed{\frac{a^3 b}{3} + \frac{ab^3}{3}} \end{aligned}$$

4.6.3 Example 3: By iteration

Evaluate the double integral by iteration:

$$\iint_D x \cos y \, dA$$

where D is the finite region in the first quadrant bounded by the coordinate axes and the curve $y = 1 - x^2$.

Given the region the minimum for x and y must be 0 and for x the maximum is 1:

$$
\begin{aligned}
\int_0^1 \int_0^{1-x^2} (x\cos(y))dydx &= \int_0^1 x\sin\left(1-x^2\right)dx \\
&= \left[\frac{1}{2}\sin(1)\sin\left(x^2\right) + \frac{1}{2}\cos(1)\cos\left(x^2\right)\right]_0^1 \\
&= \boxed{\sin^2\left(\frac{1}{2}\right)}
\end{aligned}
$$

4.6.4 Example 4: By iteration

Evaluate the double integral by iteration

$$\iint_R xy^2 \; dA$$

where R is the finite region in the first quadrant bounded by the curves $y = x^2$ and $x = y^2$

$$x = \sqrt{y} \qquad x = y^2$$

Since it is bounded in the first quadrant here intercepts in $(0,0)$ and $(1,1)$: $0 \leq y \leq 1$
In this region $x = \sqrt{y} \geq x = y^2$
Solve the integral:

$$\int_0^1 \int_{y^2}^{\sqrt{y}} xy^2 dxdy$$

$$= \int_0^1 \left[\frac{x^2y^2}{2}\right]_{y^2}^{\sqrt{y}} dy = \frac{1}{2}\int_0^1 (\sqrt{y}^2y^2) - ((y^2)^2y^2)dy$$

$$= \frac{1}{2}\int_0^1 y^3 - y^6 dy = \left[\frac{y^4}{4} - \frac{y^7}{7}\right]_0^1$$

$$= \frac{1}{2}\left(\frac{1}{4} - \frac{1}{7}\right) = \frac{1}{2} \cdot \frac{7-4}{28} = \frac{3}{56}$$

5 Tripple-Integrals

5.1 Tripple Integrals over General domains

$$\iiint_D f(x,y,z)dxdxdz$$

5.2 Change of Variables

$$dV = dx\ dy\ dz = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\ dv\ dw$$

$$\iiint f(x,y,z)\, dx\ dy\ dz = \iiint g(u,v,w) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\ dv\ dw$$

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \frac{\partial x}{\partial u}\left(\frac{\partial y}{\partial v}\frac{\partial z}{\partial w} - \frac{\partial y}{\partial w}\frac{\partial z}{\partial v}\right) - \frac{\partial x}{\partial v}\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial w} - \frac{\partial y}{\partial w}\frac{\partial z}{\partial u}\right) + \frac{\partial x}{\partial w}\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)$$

5.3 Cylyndrical Coordinates

$$\begin{aligned} x &= r\cos(\theta) \quad & r^2 &= x^2 + y^2 \\ y &= r\sin(\theta) \quad & \tan\theta &= (\tfrac{y}{x}) \\ z &= z \end{aligned}$$

From the Jacobian using change of variables:

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = r$$

$$dV = dx\ dy\ dz = r\ dr\ d\theta\ dz$$

5.3.1 Limits for Cylyndrical Coordinates

r is the radius from origin to the point

$$r \geq 0$$

θ is the angle in the positive direction of the x-axis

$$0 \leq \theta \leq 2\pi$$

5.4 Spherical Coordinates

$$\begin{aligned} x &= \rho\sin(\phi)\cos(\theta) \quad & \rho^2 &= x^2 + y^2 + z^2 \\ y &= \rho\sin(\phi)\sin(\theta) \quad & \cos(\phi) &= \tfrac{z}{\rho} \\ z &= \rho\cos(\phi) \quad & \tan\theta &= \tfrac{y}{x} \end{aligned}$$

$$dxdydz = dV = \rho^2\sin(\phi)d\rho d\phi d\theta$$

5.4.1 Limits for Spherical Coordinates

ρ is the distance from the origin O to the point P

$$\rho \geq 0$$

ϕ is the angle by the radial line OP to the positive direction of the z-axis

$$0 \leq \phi \leq \pi$$

θ is the angle in the positive direction of the x-axis to the point P in the xy-plane

$$0 \leq \theta \leq 2\pi$$

5.5 Examples

5.5.1 Example 1: Tripple Integral

Find $\iiint(x^2+y^2+z^2)dV$, where the region is bounded by $z = c\sqrt{(x^2+y^2)}$ and $x^2+y^2+z^2 = a^2$

Using Cylyndrical Coordinates:

$$z = c\sqrt{r^2} = cr \qquad r^2+z^2 = a^2 \quad \Rightarrow \quad z = \sqrt{a^2-r^2}$$

Therefore the region is bounded by:

$$0 \leq r \leq a \qquad 0 \leq \theta \leq 2\pi \qquad cr \leq z \leq \sqrt{a^2-r^2}$$

Convert the function:

$$x^2+y^2+z^2 \quad \Rightarrow \quad r^2+z^2$$

$$\int_0^{2\pi}\int_0^a\int_{cr}^{\sqrt{a^2-r^2}} (x^2+r^2)dzdrd\theta$$

5.5.2 Example 2: Sphereical Coordinates

Find the volume of:

$$\iiint \sqrt{x^2+y^2+z^2}dxdydz$$

where the region is bounded by $x^2+y^2+z^2 \leq 1$ in spherical domain.

Transform the region:

$$\rho^2 = x^2+y^2+z^2 \leq 1 \qquad \text{Therefore: } 0 \leq \rho \leq 1$$

For θ and ϕ we have:

$$0 \leq \theta \leq 2\pi \qquad 0 \leq \phi \leq \pi$$

and for dV:

$$dV = dxdydz = \rho^2 \sin\phi d\rho d\phi d\theta$$

Transform the function:

$$\sqrt{x^2+y^2+z^2} = \sqrt{\rho^2} = \rho$$

Solve the integral:

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho\cdot\rho^2\sin(\phi)d\rho d\phi d\theta &= \int_0^{2\pi}\int_0^{\pi}\left[\frac{1}{4}p^4\sin(\phi)\right]_0^1 d\phi d\theta \\
&= \int_0^{2\pi}\int_0^{\pi}\frac{\sin(\phi)}{4}d\phi d\theta \\
&= \int_0^{2\pi}\left[-\frac{\cos(\phi)}{4}\right]_0^{\pi} d\theta \\
&= \frac{1}{2}\int_0^{2\pi} d\theta \\
&= \frac{1}{2}\cdot 2\pi \\
&= \boxed{\pi}
\end{aligned}$$

5.5.3 Example 3: Jacobian Transformation

Find the Jacobian using change of variables from uv-space to xy-space when:

$$x = 2u + w \qquad y = 2u - 2v \qquad z = u + v^2 - 2w^2$$

The Jacobian is:

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\begin{aligned}
\begin{vmatrix} 2 & 0 & 1 \\ 2u & -2v & 0 \\ 1 & 2v & -4w \end{vmatrix} &= 2(-2v\cdot -4w - 0\cdot 2v) - 0 + 1(2u\cdot 2v - 2v\cdot 1) \\
&= 16vw + 4uv - 2v
\end{aligned}$$

5.5.4 Example 4: Triple Integral

Find the volume of solid bounded by:

$$x^2 + y^2 + z^2 = 9 \qquad x^2 + y^2 = 8z$$

Using cylyndrical coordinates:

$$r^2 + z^2 = 9 \qquad r^2 = 8z$$

This gives two bounds for z:

$$z = \sqrt{9 - r^2} \qquad z = \frac{r^2}{8}$$

Bounds for r is the intersection:

$$\sqrt{9 - r^2} = z = \frac{r^2}{8} \quad \Rightarrow \quad r = 2\sqrt{2}$$

From the r bounds we see:

$$z = \underbrace{\sqrt{9-r^2}}_{\text{Upper bound}} \qquad z = \underbrace{\frac{r^2}{8}}_{\text{Lower bound}}$$

And for θ:

$$0 \leq \theta \leq 2\pi$$

Setup the integral:

$$\int_0^{2\pi} \int_0^{2\sqrt{2}} \int_{r^2/8}^{\sqrt{9-r^2}} r \ dz \ dr \ d\theta$$

$$= \int_0^{2\pi} \int_0^{2\sqrt{2}} [rz]_{r^2/8}^{\sqrt{9-r^2}} \ drd\theta$$

$$= \int_0^{2\pi} \int_0^{2\sqrt{2}} r\sqrt{9-r^2} - r^3/8drd\theta$$

$$= \int_0^{2\pi} \left(\int_0^{2\sqrt{2}} r\sqrt{9-r^2}dr - \int_0^{2\sqrt{2}} r^3/8dr \right) d\theta$$

Using u substitution:

$$\int_0^{2\sqrt{2}} r\sqrt{u}dr \qquad u = 9 - r^2 \quad \Rightarrow \quad \frac{du}{dr} = -2r \quad \Rightarrow \quad dr = \frac{du}{-2r}$$

New limits:

$$r = 0: \quad u = 9 - 0^2 = 9 \qquad\qquad r = 2\sqrt{2}: \quad u = 9 - (2\sqrt{2})^2 = 1$$

$$\int_9^1 r\sqrt{u}\frac{du}{-2r} = \frac{-1}{2}\int_9^1 \sqrt{u}du = \frac{-1}{2}\left[\frac{2u^{3/2}}{3}\right]_9^1 = \frac{26}{3}$$

Solve the other integral:

$$= \int_0^{2\sqrt{2}} r^3/8dr = \left[r^4/32\right]_0^{2\sqrt{2}} = 2$$

Insert results:

$$= \int_0^{2\pi} \frac{26}{3} - 2d\theta = \boxed{\frac{40}{3}\pi}$$

6 Fields-Curve

6.1 Curve & Parameterization

Representation of a curve in 3 space by using its position vector is given as:

$$r = r(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \qquad \text{where } a \leq t \leq b$$

6.2 Vector Fields

$$\mathbf{F}(x, y, z) = \underbrace{f_1(x, y, z)}_{\text{Scaler function}}\ \mathbf{i} + \underbrace{f_2(x, y, z)}_{\text{Scaler function}}\ \mathbf{j} + \underbrace{f_3(x, y, z)}_{\text{Scaler function}}\ \mathbf{k}$$

$$\frac{\partial f}{\partial x} = f_1 = f_x \qquad \frac{\partial f}{\partial y} = f_2 = f_y \qquad \frac{\partial f}{\partial z} = f_3 = f_z$$

Position vector:

$$r = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Unit vector with magnitude 1:

$$r = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

6.2.1 Scalar field

$$F(x, y, z) = f_1(x, y, z) + f_2(x, y, z) + f_3(x, y, z)$$

The gradient of a scalar field is a vector field:

$$\nabla f = \text{grad } f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$

6.2.2 Field lines

$$\frac{dx}{f_1(x, y, z)} = \frac{dy}{f_2(x, y, z)} = \frac{dz}{f_3(x, y, z)}$$

6.2.3 Convervation field

If $\mathbf{F}(x, y, z) = \nabla\phi(x, y, z)$ in a 3d domain D, then $\mathbf{F}$ is a conservative vector field in D and function ϕ is the potential function.

$$\mathbf{F}(x, y, z) = \nabla\phi(x, y, z) = \phi_x(x, y, z)\mathbf{i} + \phi_y(x, y, z)\mathbf{j} + \phi_z(x, y, z)\mathbf{k}$$

If the vetor field is conservative, then all the following equations are true:

$$\begin{aligned} \frac{\partial f_1}{\partial y} &= \frac{\partial f_2}{\partial x} \\ \frac{\partial f_1}{\partial z} &= \frac{\partial f_3}{\partial x} \\ \frac{\partial f_2}{\partial z} &= \frac{\partial f_3}{\partial y} \end{aligned}$$

If $\mathbf{F}(x, y) = \nabla\phi(x, y)$ in a 2d domain D, then $\mathbf{F}$ is a conservative vector field in D and function ϕ is the potential function.

$$\frac{\partial f_1}{\partial y} = \frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial f_2}{\partial x}$$

6.2.4 Vector field in Polar Coordinates

$$\mathbf{F} = f(r,\theta) = f_r(r,\theta)\hat{\mathbf{r}} + f_\theta(r,\theta)\hat{\theta}$$

where:

$$\hat{\mathbf{r}} = \cos(\theta)i + \sin(\theta)j$$
$$\hat{\theta} = -\sin(\theta)i + \cos(\theta)j$$

6.3 Line Integral

$$f(x,y)ds = \text{Area (tiny point)}$$
$$\text{Length of } \mathcal{C} = \int_{\mathcal{C}} f(x,y,z)\,ds = \int_a^b f(r(t)) \left|\frac{dr}{dt}\right| dt$$

6.3.1 Line integral of a vector field

$$W = \int_{\mathcal{C}} F.\hat{T}\,ds = \int F\,dr = \int_{\mathcal{C}} f_1(x,y,z)dx + f_2(x,y,z)dy + f_3(x,y,z)dz$$

6.4 Examples

6.4.1 Example 1: Conservative vector field and potential

Determine whether the given vector field is conservative, and find a potential function if it is:

$$\mathbf{F}(x,y,z) = (2xy - z^2)\mathbf{i} + (2yz + x^2)\mathbf{j} - (2zx - y^2)\mathbf{k}$$

The field is convervative if:

$$\begin{aligned} \frac{\partial f_1}{\partial y} &= \frac{\partial f_2}{\partial x} \\ \frac{\partial f_1}{\partial z} &= \frac{\partial f_3}{\partial x} \\ \frac{\partial f_2}{\partial z} &= \frac{\partial f_3}{\partial y} \end{aligned}$$

$$\frac{\partial(2xy - z^2)}{\partial y} = 2x = \frac{\partial(2yz + x^2)}{\partial x}$$
$$\frac{\partial(2xy - z^2)}{\partial z} = -2z = \frac{\partial(-2zx + y^2)}{\partial x}$$
$$\frac{\partial(2yz + x^2)}{\partial z} = 2y = \frac{\partial(-2zx + y^2)}{\partial y}$$

All equations are satisfied! The field is convervative.

Find the potential function $\phi(x,y,z)$:

$$f_1 = \frac{\partial \phi}{\partial x} \quad \Rightarrow \quad \phi = \int f_1 dx = x^2 y - z^2 x + c(y,z)$$

$$\frac{\partial \phi}{\partial y} = x^2 + \frac{\partial c(y,z)}{\partial y}$$

$$f_2 = \frac{\partial \phi}{\partial y} = x^2 + \frac{\partial c(y,z)}{\partial y} = 2yz + x^2 \quad \Rightarrow \quad \frac{\partial c(y,z)}{\partial y} = 2yz$$

This means c is a function of y and z and can be found by taking the anit-derivative

$$c(y,z) = y^2 z + c(z)$$

Insert $c(y,z)$:

$$\phi = x^2 y - z^2 x + y^2 z + c(z)$$

$$f_3 = \frac{\partial \phi}{\partial z} = -2zx + y^2 + \frac{\partial c(z)}{\partial z} = -2zx + y^2 \quad \Rightarrow \quad \frac{\partial c(z)}{\partial z} = 0$$

A scalar potential function of F:

$$\phi(x,y,z) = x^2 y - z^2 x + y^2 z$$

6.4.2 Example 2: Line integral

Evaluate $\oint x^2 y^2 \; dx + x^3 y \; dy$ counterclockwise around the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$

Find the parameterization for each of the lines:

c_1: (0,0) to (1,0)

$$x(t) = t \qquad y(t) = 0 \qquad 0 \le t \le 1$$

$$\frac{dx}{dt} = 1 \qquad \Rightarrow \qquad dx = dt$$

$$\frac{dy}{dt} = 0 \qquad \Rightarrow \qquad dy = 0$$

$$\int_0^1 (t^2 0^2 dt + t^3 \cdot 0 \cdot 0) = \boxed{0}$$

c_2: (1,0) to (1,1)

$$x(t) = 1 \qquad y(t) = t \qquad 0 \le t \le 1$$

$$\frac{dx}{dt} = 0 \qquad \Rightarrow \qquad dx = 0$$

$$\frac{dy}{dt} = 1 \qquad \Rightarrow \qquad dy = dt$$

$$\int_0^1 (1^2 t^2 \cdot 0 + 1^3 \cdot t dt) = \left[\frac{t^2}{2}\right]_0^1 = \boxed{\frac{1}{2}}$$

c_3: (1,1) to (0,1)

$$x(t) = 1 - t \qquad y(t) = 1 \qquad 0 \le t \le 1$$

$$\frac{dx}{dt} = -1 \qquad \Rightarrow \qquad dx = -dt$$

$$\frac{dy}{dt} = 0 \qquad \Rightarrow \qquad dy = 0$$

$$\int_0^1 ((1-t)^2 1^2 (-dt) + (1-t)^3 \cdot 1 \cdot 0) = -\int_0^1 (1-t)^3 dt = \left[\frac{(1-t)^3}{3}\right]_0^1 = \boxed{-\frac{1}{3}}$$

c_4: (0,1) to (0,0)

$$x(t) = 0 \qquad y(t) = 1 - t \qquad 0 \le t \le 1$$

$$\frac{dx}{dt} = 0 \qquad \Rightarrow \qquad dx = 0$$

$$\frac{dy}{dt} = -1 \qquad \Rightarrow \qquad dy = -dt$$

$$\int_0^1 (0^2(1-t)^2 0 + 0^3 \cdot (1-t)(-dt)) = \boxed{0}$$

Therefore

$$\oint x^2y^2\ dx + x^3y\ dy = 0 + \frac{1}{2} - \frac{1}{3} + 0 = \frac{1}{6}$$

6.4.3 Example 3: Line integral

Evalute the line integral for $f(x,y) = x^2y^2$ along a straight line from origin to the point $(2,1)$

The parmeterization of arc length over t:

$$x = f(t) = t \qquad \frac{df(t)}{dt} = 1$$

$$y = g(t) = 2t \qquad \frac{dg(t)}{dt} = 2$$

Setup integral with bounds: $0 \leq t \leq 1$

$$\int_0^1 t^2(2t)^2\sqrt{f'(t)^2 + g'(t)^2}dt = \int_0^1 5t^2\sqrt{5}dt$$

$$= \left[\frac{5t^3\sqrt{5}}{3}\right]_0^1 = \frac{5\sqrt{5}}{3}$$

6.4.4 Example 4: Line integral vector field

Evaluate the line integral of the tangential compnent of the given vector field along the given curve:

$$F(x,y) = xy\mathbf{i} - x^2\mathbf{j}$$

For a vector field:

$$W = \int F\,dr$$

Along the line $y = x^2$: Parametrize x and y:

$$x(t) = t \qquad y(t) = t^2 \qquad r(t) = t\mathbf{i} + t^2\mathbf{j}$$

$$\frac{dr}{dt} = \mathbf{i} + 2t\mathbf{j} \quad \Rightarrow \quad dr = (\mathbf{i} + 2t\mathbf{j})dt$$

Setup integral with bounds: $0 \leq t \leq 1$

$$\int_0^1 (t^3\mathbf{i} - t^2\mathbf{j})(\mathbf{i} + 2t\mathbf{j})dt = \int_0^1 t^3 - 2t^3 dt = \int_0^1 -t^3 dt$$

$$= \left[-\frac{t^4}{4}\right]_0^1 = -\frac{1}{4}$$

6.4.5 Example 5: Line integral over specified curve

Evaluate the given line integral over the specified curve $\mathcal{C}$

$$\int_{\mathcal{C}}(x+y)ds \qquad \mathbf{r} = at\mathbf{i} + bt\mathbf{j} + ct\mathbf{k} \qquad 0 \leq t \leq m$$

$$\int_{\mathcal{C}} f(x,y,z)ds = \int_a^b f(r(t)) \left|\frac{dr}{dt}\right| dt = \sqrt{a^2+b^2+c^2}dt$$

$$ds = \left|\frac{dr}{dt}\right| dt = |a\mathbf{i} + b\mathbf{j} + c\mathbf{k}|dt$$

$$f(r(t)) = at + bt$$

Solve the integral:

$$\int_0^m (at+bt)\sqrt{a^2+b^2+c^2}dt = \sqrt{a^2+b^2+c^2}\int_0^m (a+b)tdt$$

$$= \sqrt{a^2+b^2+c^2}\left[\frac{(a+b)t^2}{2}\right]_0^m = \frac{\sqrt{a^2+b^2+c^2}(a+b)m^2}{2}$$

6.4.6 Example 6: Parametrize a curve

Use $t = y$ to parametrize the part of the line of intersection of the two planes:
Plane 1: $y = 2x - 4$
Plane 2: $z = 3x + 1$ from $(2,0,7)$ to $(3,2,10)$

Find parameterization for x with $y = t$:

$$t = 2x - 4 \qquad \Rightarrow \qquad x(t) = \frac{t+4}{2}$$

Find parameterization for z using $x(t)$:

$$z(t) = 3\left(\frac{t+4}{2}\right) + 1 \qquad \Rightarrow \qquad \left(\frac{3t+12}{2}\right) + 1 \qquad \Rightarrow \qquad \frac{3t}{2} + 7$$

The parameterization is given by:

$$r(t) = \left(\frac{1}{2}(t+4)\right)\mathbf{i} + (t)\mathbf{j} + \left(\frac{3}{2}t + 7\right)\mathbf{k}$$

7 Surface-Integrals

7.1 Parametric Surface

For curve parametrization:

$$r = r(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \qquad \text{where } a \leq t \leq b$$

For surface parametrization:

$$r = r(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k} \qquad \text{where } a \leq u \leq b, \quad c \leq v \leq d$$

7.2 Surface Area

For a surface the area is given by:

$$\iint_S f(x,y,z)dS$$

$$dS = \left|\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right| dudv = \sqrt{\left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2} dudv$$

For a parametrized surface S given by $r = r(u,v)$, where (u,v) is in the domain D in the uv-plane, the surface area is given by:

$$\iint_S f\ dS = \iint_D f(r(u,v)) \left|\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right| dudv$$

$$= \iint_b f(x(u,v), y(u,v), z(u,v)) \sqrt{\left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2} du\,dv$$

For a surface S given by $z = g(x,y)$, where (x,y) is in the domain D in the xy-plane, the surface area is given by:

$$\iint_S f(x,y,z)dS = \iint_D f(x,y,z(x,y)) \sqrt{1 + \left(\frac{\partial g(x,y)}{\partial x}\right)^2 + \left(\frac{\partial g(x,y)}{\partial y}\right)^2} dxdy$$

The projection of normal vector onto the xy-plane is given by:

$$\cos(\gamma) = \frac{1}{\sqrt{1 + \left(\frac{\partial g(x,y)}{\partial x}\right)^2 + \left(\frac{\partial g(x,y)}{\partial y}\right)^2}} \qquad \text{hence } dS = \frac{1}{\cos(\gamma)} dxdy$$

7.3 Oriented Surface

- A smooth surface S in 3-space is said to be orientable if there exists a unit vector field $\widehat{N}(P)$.
- $\widehat{N}(P)$ defined on S that varies continuously as P ranges over S and that is everywhere normal to S.
- Any such vector field $\widehat{N}(P)$ determines an orientation of S.
- The oriented surface must have two sides.
- $\widehat{N}(P)$ can have only one value at each point P with two sides.

7.4 Flux

$$\mathbf{F} = f_1(x,y,z)\mathbf{i} + f_2(x,y,z)\mathbf{j} + f_3(x,y,z)\mathbf{k}$$

Given any continuous vector field $\mathbf{F}$, flux of $\mathbf{F}$ across the orientable surface S is integral of the normal component of $\mathbf{F}$ over S

$$\iint_S \mathbf{F} \cdot dS = \iint_S (\mathbf{F} \cdot \widehat{\mathbf{N}}) dS$$

If the surface is closed, then the flux is given by:

$$\oiint_S \mathbf{F} \cdot dS = \oiint_S (\mathbf{F} \cdot \widehat{\mathbf{N}}) dS$$

If S is a parametrized surface given by $r = r(u,v)$, where (u,v) is in the domain D in the uv-plane, then the flux is given by:

$$\iint_S \mathbf{F} \cdot dS = \iint_B \mathbf{F} \cdot \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right) du\, dv$$

$$= \iint_D \left(f_1 \frac{\partial(y,z)}{\partial(u,v)} + f_2 \frac{\partial(z,x)}{\partial(u,v)} + f_3 \frac{\partial(x,y)}{\partial(u,v)} \right) du\ dv$$

For a surface S given by $z = g(x,y)$, where (x,y) is in the domain D in the xy-plane, the flux is given by:

$$\iint_S \mathbf{F} \cdot dS = \iint_D \left(-f_1 \frac{\partial z}{\partial x} - f_2 \frac{\partial z}{\partial y} + f_3 \right) dx\, dy$$

7.5 Examples

7.5.1 Example 1: Surface area

Find $\iint_S x\ dS$ over the part of the parabolic cylinder $z = x^2/2$ that lies inside the first octant part of the cylinder $x^2 + y^2 = 1$.

Since $z = g(x,y)$ is a function of x and y:

$$\iint_S f(x,y,z) dS = \iint_D f(x,y,z(x,y)) \sqrt{1 + \left(\frac{\partial g(x,y)}{\partial x}\right)^2 + \left(\frac{\partial g(x,y)}{\partial y}\right)^2} dxdy$$

Find the length:

$$\frac{\partial}{\partial x}(x^2/2) = x$$

$$\frac{\partial}{\partial y}(x^2/2) = 0$$

$$\sqrt{1 + \left(\frac{\partial g(x,y)}{\partial x}\right)^2 + \left(\frac{\partial g(x,y)}{\partial y}\right)^2} = \sqrt{1 + (2x)^2}$$

Using $r^2 = x^2 + y^2$ we know that x and y must between 1 and 0:

$$y = \sqrt{1 - x^2}$$

Setup integral:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} x\sqrt{1 + x^2} dydx$$

$$= \int_0^1 x\sqrt{1+x^2}\sqrt{1-x^2}dx = \int_0^1 x\sqrt{1-x^4}dx$$

Using table lookup:

$$\left[\frac{1}{4}x^2\sqrt{1-x^4} - \frac{1}{4}\tan^{-1}\left(\frac{\sqrt{1-x^4}}{x^2}\right)\right]_0^1 = \frac{\pi}{8}$$

7.5.2 Example 2: Flux

Find the flux of $F = x\mathbf{i} + x\mathbf{j} + \mathbf{k}$ upward through the part of the surface $z = x^2 - y^2$ inside the cylinder $x^2 + y^2 = a^2$

For a surface S given by $z = g(x, y)$, where (x, y) is in the domain D in the xy-plane, the flux is given by:

$$\iint_S \mathbf{F} \cdot dS = \iint_D \left(-f_1\frac{\partial z}{\partial x} - f_2\frac{\partial z}{\partial y} + f_3\right) dx\,dy$$

$$\frac{\partial z}{\partial x} = 2x \qquad \frac{\partial z}{\partial y} = -2y$$

Setup integral:

$$\iint (-x(2x) - x(-2y) + 1)dxdy = \iint (-2x^2 + 2yx + 1)dxdy$$

Using $r^2 = x^2 + y^2$ we know the radius is a:

$$0 \le r \le a \qquad 0 \le \theta \le 2\pi$$

$$dxdy = rdrd\theta$$

$$x = r\cos(\theta)$$

$$y = r\sin(\theta)$$

$$\int_0^{2\pi}\int_0^a (-2(r\cos(\theta))^2 + 2(r\sin(\theta))(r\cos(\theta)) + 1)rdrd\theta$$

$$= \int_0^{2\pi} -\frac{1}{2}a^4\cos^2(\theta) + \frac{1}{2}a^4\sin(\theta)\cos(\theta) + \frac{a^2}{2}d\theta$$

$$= -\frac{1}{2}\pi a^2\left(a^2 - 2\right)$$

7.5.3 Example 3: Flux (Parametrized Surface)

Find the flux of $F = 2x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ upward through the surface $r = u^2v\mathbf{i} + uv^2\mathbf{j} + v^3\mathbf{k}$ where $(0 \le u \le 1, 0 \le v \le 1)$

If S is a parametrized surface given by $r = r(u, v)$, where (u, v) is in the domain D in the uv-plane, then the flux is given by:

$$\iint_S \mathbf{F} \cdot dS = \iint_B \mathbf{F} \cdot \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right) du\,dv$$

$$= \iint_D \left(f_1\frac{\partial(y,z)}{\partial(u,v)} + f_2\frac{\partial(z,x)}{\partial(u,v)} + f_3\frac{\partial(x,y)}{\partial(u,v)}\right) du\ dv$$

$$F(r(u,v)) = 2(u^2v)\mathbf{i} + uv^2\mathbf{j} + v^3\mathbf{k}$$

$$\frac{\partial r}{\partial u} = 2uv\mathbf{i} + v^2\mathbf{j} \qquad \frac{\partial r}{\partial v} = u^2\mathbf{i} + 2uv\mathbf{j} + 3v^2\mathbf{k}$$

$$\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2uv & v^2 & 0 \\ u^2 & 2uv & 3v^2 \end{vmatrix} = (v^2 \cdot 3v^2 - 0 \cdot 2uv)\mathbf{i} - (2uv \cdot 3v^2 - u^2 \cdot 0)\mathbf{j} + (2uv \cdot 2uv - v^2 \cdot u^2)\mathbf{k}$$

$$= 3v^4\mathbf{i} - 6uv^3\mathbf{j} + 3u^2v^2\mathbf{k}$$

Setup integral:

$$\int_0^1 \int_0^1 (2(u^2v)\mathbf{i} + uv^2\mathbf{j} + v^3\mathbf{k}) \cdot \left(3v^4\mathbf{i} - 6uv^3\mathbf{j} + 3u^2v^2\mathbf{k}\right)\, du\, dv$$

$$\int_0^1 \int_0^1 (6u^2v^5 - 6u^2v^5 + 3u^2v^5)\, du\, dv = \int_0^1 \int_0^1 3u^2v^5\, du\, dv$$

$$= \int_0^1 \left[u^3v^5\right]_0^1\, dv = \int_0^1 v^5\, dv$$

$$= \left[\frac{v^6}{6}\right]_0^1 = \boxed{\frac{1}{6}}$$

8 Theorems

8.1 Differential Operators

8.1.1 Gradient

The gradient of a scalar field is a vector field that points in the direction of the steepest increase of the scalar field.

$$\text{grad } f(x,y,z) = \nabla f(x,y,z) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$
$$\mathbf{F}(x,y,z) = f_x(x,y,z)\mathbf{i} + f_y(x,y,z)\mathbf{j} + f_z(x,y,z)\mathbf{k}$$

8.1.2 Divergence

The divergence of a velocity field represents the net flow of fluid out of a small volume in a scalar field.

$$\text{div } \mathbf{F}(x,y,z) = \nabla \cdot \mathbf{F}(x,y,z) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

8.1.3 Curl

The curl or field circulation of the electric field gives the rate of change of the magnetic field.

$$\text{curl } \mathbf{F}(x,y,z) = \nabla \times \mathbf{F}(x,y,z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$
$$= \left(\frac{\partial f_1}{\partial y} - \frac{\partial f_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f_3}{\partial z} - \frac{\partial f_1}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\mathbf{k}$$

8.2 Green's Theorem

Let R be a regular, closed region in the xy-plane whose boundary, C, consists of one or more piecewise smooth, simple closed curves that are positively oriented (counterclock vise) with respect to R.

$$\oint_C f_1(x,y)dx + f_2(x,y)dy = \iint_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dA$$

8.3 Stokes' Theorem

Let S be a piecewise smooth, oriented surface in 3-space, having unit normal field $\widehat{N}$ and boundary C consisting of one or more piecewise smooth, closed curves with orientation inherited from S.

$$\oint_C F \cdot dr = \iint_S \text{curl } F \cdot \widehat{N} dS$$

8.4 Divergence Theorem

Let S, be a closed piecewise smooth surface, which is the boundary of V with normal $\widehat{N}$ pointing outwards.

$$\oiint_s (F \cdot \widehat{N}) dS = \iiint_V \text{div}\, F dV$$

More variants:

$$\iiint_D \text{curl } F dV = -\oiint_s (F \times \widehat{N}) dS$$
$$\iiint_D \text{grad } \phi\, dV = \oiint_s \phi\, dS$$

8.5 Examples

8.5.1 Example 1: Div and Curl

Calculate the divergence and curl of the following vector field:

$$\mathbf{F} = \cos x\,\mathbf{i} - \sin y\,\mathbf{j} + z\,\mathbf{k}.$$

Divergence:

$$\text{div }\mathbf{F}(x,y,z) = \nabla \cdot \mathbf{F}(x,y,z) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$
$$\text{div}F = -\sin(x) - \cos(y) + 1$$

Curl:

$$\text{curl}F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$
$$= \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)\mathbf{k}$$
$$= (0-0)i + (0-0)j + (0+0)k$$

8.5.2 Example 2: Green's Theorem

Using Green's Theorem evaluate $\oint_e (x^2 y)\,dx + (xy^2)dy$, clockwise bounded of the region:

$$0 \le y \le \sqrt{9 - x^2}$$

$$\oint_C f_1(x,y)dx + f_2(x,y)dy = \iint_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dA$$
$$f_1(x,y) = x^2 y \qquad f_2 = xy^2$$
$$\frac{\partial f_1}{\partial y} = x^2 \qquad \frac{\partial f_2}{\partial x} = y^2$$
$$\iint_R \left(y^2 - x^2\right) dA$$

But since it is clockwise:

$$-\iint_R \left(y^2 - x^2\right) dA = \iint_R \left(x^2 - y^2\right) dA$$

Using polar coordinates:

$$y^2 = 9 - x^2 \quad \Rightarrow \quad r = 3$$

Since $y \ge 0$:

$$0 \le \theta \le \pi$$

Convert the function to polar:

$$x^2 = (r\cos(\theta))^2 \qquad y^2 = (r\sin(\theta))^2$$

$$\int_0^\pi \int_0^3 (r^2\cos^2(\theta) - r^2\sin^2(\theta))rdrd\theta = \int_0^\pi \int_0^3 r^3(\cos^2(\theta) - \sin^2(\theta))drd\theta$$

$$= \int_0^\pi \left[\frac{1}{4}r^4(\cos^2(\theta) - \sin^2(\theta))\right]_0^3 d\theta$$

$$= \frac{81}{4}\int_0^\pi (\cos^2(\theta) - \sin^2(\theta))d\theta = \frac{81}{4}\int_0^\pi \frac{1+\cos(2\theta)}{2} - \frac{1-\cos(2\theta)}{2}d\theta$$

$$\frac{81}{4}\int_0^\pi \frac{1+\cos(2\theta)}{2} - \frac{1-\cos(2\theta)}{2}d\theta = \frac{81}{4}\int_0^\pi \cos(2\theta)d\theta$$

$$= \frac{81}{4}\left[\frac{\sin(2\theta)}{2}\right]_0^\pi = 0 - 0 = 0$$

8.5.3 Example 3: Stokes' Theorem

Evaluate $\oint F \cdot dr$, where $F = -y^3i + x^3j - z^3k$ and C is the curve of intersection of the cylinder $x^2 + y^2 \leq 1$ and the plane $2x + 2y + z = 3$ oriented to have a counterclockwise projection onto the xy-plane.

8.5.4 Example 4: Divergence Theorem

Use the Divergence Theorem to calculate the flux of the given vector field out of the sphere s with equation $x^2 + y^2 + z^2 = a^2$, where $a > 0$ and

$$\mathbf{F} = (x^2 + y^2)\mathbf{i} + (y^2 - z^2)\mathbf{j} + z\mathbf{k}$$

9 PDE

Partial Differential Equations are equations with multiple variables and derivatives. They are used to model many physical phenomena, such as heat, sound, and light. The totality of solutions to a PDE is called its general solution, and there can be a lot.

9.1 Classification of PDEs

General representation of a PDE:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

Conditions:

Linear: A, B, C, D, E, F are only function of x,y variables, not u.

Quasi-linear: A, B, C, D, E, F may be function of (x, y, u, u_x, u_y)

Fully non-linear: A, B, C, D, E, F may be function of $(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy})$

9.2 Characteristics of PDEs

$B^2 - 4AC > 0$	2 real roots	2 characteristics	**Hyperbolic PDE**
$B^2 - 4AC = 0$	1 real roots	1 characteristics	**Parabolic PDE**
$B^2 - 4AC < 0$	0 real roots	0 characteristics	**Elliptic PDE**

Tyoes of varius PDEs:

Wave Equation: Hyperbolic PDE

Heat Equation: Parabolic PDE

Laplace Equation: Elliptic PDE

9.3 Important Second-Order PDEs

$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$	One-dimensional wave equation
$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$	One-dimensional heat equation
$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$	Two-dimensional Laplace equation
$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$	Two-dimensional Poisson equation
$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$	Two-dimensional wave equation
$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$	Three-dimensional Laplace equation

9.4 Initial and Boundary Conditions

9.5 Wave Equation (1D)

One dimensional wave equation is given by:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{where } c^2 = \left[\frac{T(x,t)}{\mu_x}\right] \tag{1}$$

With two boundary conditions $x = 0$ and $x = L$:

$$u(0,t) = 0 \qquad u(L,t) = 0 \qquad \text{For all } t \geq 0$$

And two initial conditions, initial displacement and initial velocity at time $t = 0$:

$$u(x,0) = f(x) \qquad \frac{\partial u}{\partial t}(x,0) = g(x) \qquad \text{For all } 0 \leq x \leq L$$

Steps to solve:

1. Method of Separation of Variables $\quad u(x,t) = X(x)T(t)$
2. Satisfy the Boundary Conditions test
3. Fourier Series Validation

9.5.1 D'Alembert's Solution of the Wave Equation

His solution is given by eq. (1) but extended to two variables:

$$v = (x - ct) \qquad w = (x + ct) \tag{2}$$

I.e. $u(v,w)$. Partial derivatives from chain rule:

$$u_x = u_v \cdot v_x + u_w \cdot w_x = u_v + u_w$$

For double derivatives:

$$u_{xx} = (u_v + u_w)_x = (u_v + u_w)_v v_x + (u_v + u_w)_w w_x = u_{vv} + 2u_{vw} + u_{ww}$$

With respect to t:

$$u_{tt} = c^2 u_{xx} = c^2(u_{vv} + 2u_{vw} + u_{ww})$$

From eq. (1) and eq. (2):

$$u_{vw} = \frac{\partial^2 u}{\partial w \partial v} = 0$$

This can be solved by integrating with respect to v and w:

$$\frac{\partial u}{\partial v} = h(v) \text{ and } u = \int h(v)\, dv + \psi(w)$$

Here, $h(v)$ and $\psi(w)$ are arbitrary functions of v and w, respectively. The solution in term for x:

$$u = \phi(v) + \psi(w)$$

This is d'Alembert's solution, which is the general solution to the wave equation.

$$u(x,t) = \phi(x + ct) + \psi(x - ct)$$

This solution satisfies the wave equation and the initial conditions:

9.6 Heat Equation (1D)

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{3}$$

Conditions:

- PDE is linear and homogeneous.
- Boundary conditions are linear and homogeneous.The two for $u(x,t)$ is $u(0,t) = 0$ and $u(L,t) = 0$ for all $t > 0$.
- One initial condition at time ($t = 0$): $u(x,0) = f(x)$.

Solve the

Solution:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda_n^2 t}$$

where

$$\lambda_n = \frac{cn\pi}{L}$$

and

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx \qquad \text{for } n = 1,2,3\ldots$$

9.7 Examples

9.7.1 Example 1: Type, Normal Form, and solve

Find the type, transform to normal form, and solve.

$$u_{xy} - u_{yy} = 0$$

Find A, B, C:

$$Au_{xx} + 2Bu_{xy} + Cuyy = f(x, y, u, u_x, u_y)$$

$$A = 0 \qquad 2B = 1 \Rightarrow B = \frac{1}{2} \qquad C = -1$$

Find the type:

$$B^2 - 4AC = \left(\frac{1}{2}\right)^2 - 4 \cdot 0 \cdot -1 = \frac{1}{4}$$

Since $B^2 - 4AC > 0$ the PDE is hyperbolic.

Transform to normal form:

$$Ay'' - 2By' + C = 0 \quad \Rightarrow \quad -y' - 1 = 0 \quad \Rightarrow \quad y' = -1$$

$$y = \int \frac{dy}{dx} = \int -1dx = -x + c_1$$

$$c_1 = x + y$$

Transform the variables:
Only one constant, therefore $v = x$

$$v = x \qquad v_x = 1 \qquad v_y = 0$$

$$w = x + y \qquad w_x = 1 \qquad w_y = 1$$

$$u_x = u_v v_x + u_w w_x = u_v \cdot 1 + u_w \cdot 1 = u_v + u_w$$

$$u_{xy} = (u_v + u_w)_v v_y + (u_v + u_w)_w w_y = u_{xy} = (u_v + u_w)_v \cdot 0 + (u_v + u_w)_w \cdot 1 = \boxed{u_{vw} + u_{ww}}$$

$$u_y = u_v u_y + u_w w_y = u_v \cdot 0 + u_w \cdot 1 = u_w$$

$$u_{yy} = (u_w)_v v_y + (u_w)_w w_y = (u_w)_v \cdot 0 + (u_w)_w \cdot 1 = \boxed{u_{ww}}$$

The normal form is:

$$u_{vw} + u_{ww} - u_{ww} = 0 \quad \Rightarrow \quad \boxed{u_{vw} = 0}$$

Solve:

$$u_{vw} = 0 \quad \Rightarrow \quad u_v = h(v)$$

$$u = g(w) + \int h(v)$$

$$u(v, w) = g(w) + f(v)$$

Insert x, y:

$$u(x, y) = g(x + y) + f(x)$$

Where f and g er orbitrary functions.

9.7.2 Example 2: Type, Normal Form, and solve

Find the type, transform to normal form, and solve.

$$u_{xy} - u_{yy} = 0$$
