

# Signal Processing

Mathias Balling Christiansen

Last updated: January 4, 2024

## Contents

<b>1</b>	<b>Fourier</b>	<b>3</b>
1.1	Fourier Series . . . . .	3
1.2	Fourier Transform . . . . .	3
1.3	Examples . . . . .	3
1.3.1	Example 1: Fourier Series . . . . .	3
1.3.2	Example 2: Fourier Transform . . . . .	5
<b>2</b>	<b>Laplace transform</b>	<b>6</b>
2.1	General Formulas . . . . .	6
2.2	Examples . . . . .	6
2.2.1	Example 1: Laplace Transform . . . . .	6
2.2.2	Example 2: Transfer Function . . . . .	7
2.2.3	Example 3: Differential equation . . . . .	7
<b>3</b>	<b>Filter-Functions</b>	<b>9</b>
3.1	Electronic filters . . . . .	9
3.2	Filter specification . . . . .	9
3.3	Group delay . . . . .	10
3.4	Filter transfer functions . . . . .	10
3.4.1	Butterworth . . . . .	10
3.4.2	Chebyshev . . . . .	11
3.4.3	Bessel . . . . .	11
3.4.4	The filter's order number . . . . .	11
3.5	Filter transformers . . . . .	12
3.5.1	Low pass to high pass transformation . . . . .	12
3.5.2	Low pass to bandpass transformation . . . . .	12
3.5.3	Low pass to bandstop transformation . . . . .	13
<b>4</b>	<b>Sampling-Reconstruction</b>	<b>15</b>
<b>5</b>	<b>FFT</b>	<b>16</b>
5.1	DFT . . . . .	16
5.2	Examples . . . . .	16
5.2.1	Example 1: DFT of a sequence . . . . .	16
<b>6</b>	<b>Z-transformation</b>	<b>17</b>
6.0.1	s and z domain . . . . .	17
6.1	Differential equations . . . . .	17
6.2	Transfer Functions . . . . .	17
6.3	Inverse Z-transformation . . . . .	17

6.4	Examples . . . . .	17
6.4.1	Example 1: Z-transformation . . . . .	17
6.4.2	Example 2: Inverse Z-transformation . . . . .	17
<b>7</b>	<b>System-analysis</b>	<b>18</b>
<b>8</b>	<b>Digital-Realization-Structures</b>	<b>19</b>
<b>9</b>	<b>IRR-Filters</b>	<b>20</b>
9.1	Design of IRR-filters . . . . .	20
9.2	Matched z-transform . . . . .	20
9.2.1	1. Order Matched z-transformation . . . . .	20
9.2.2	2. Order Matched z-transformation . . . . .	21
9.3	Impulse invariance method . . . . .	21
9.4	Bilinear z transformation . . . . .	22
9.4.1	Warping and Pre-warping . . . . .	22
9.4.2	1. Order Bilinear z transformation . . . . .	22
9.4.3	2. Order Bilinear z transformation . . . . .	22
9.5	Examples . . . . .	22
9.5.1	Example 1: Matched 1. order z-transform . . . . .	22
9.5.2	Example 2: Matched 3. order matched z-transform . . . . .	23
9.5.3	Example 2: Matched 3. order impulse invariance z-transform . . . . .	24
9.5.4	Example 3: Bilinear z-transform . . . . .	24
<b>10</b>	<b>FIR-Filters</b>	<b>25</b>
10.1	Linear phase . . . . .	25
10.2	Frequency response . . . . .	25
10.3	FIR-Filter design . . . . .	25
10.4	Window functions . . . . .	25
10.4.1	Rectangular window . . . . .	26
10.4.2	Barlett window . . . . .	26
10.4.3	Hamming and Hanning window . . . . .	26
10.4.4	Kaizer window . . . . .	26
10.5	Examples . . . . .	26
10.5.1	Low-pass filter . . . . .	26
10.5.2	High-pass filter . . . . .	26
10.5.3	Band-pass filter . . . . .	27
10.5.4	Band-stop filter . . . . .	27

# 1 Fourier

## 1.1 Fourier Series

A periodic function with period  $2L$  and let  $f(x)$  and  $f'(x)$  be piecewise continuous on the interval  $-L < x < L$

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right) = \sum_{n=-\infty}^{\infty} c_n e^{jn\pi x/L}$$

The coefficients are given by:

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx \quad n \geq 0$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad n > 0$$

$$c_n = \frac{1}{2}(a_n - jb_n) \quad n > 0$$

## 1.2 Fourier Transform

If  $h(t)$  is a periodic function then the Fourier transform is given by:

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

Inverse Fourier transformation of  $H(\omega)$ :

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{j\omega t} d\omega$$

Signal	Fourier Transform
$\delta(t)$	1
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$
$\delta(t - t_0)$	$e^{-j\omega t_0}$
$\sin(\omega_0 t)$	$-j\pi(\delta(\omega - \omega_0) - \delta(\omega + \omega_0))$
$\cos(\omega_0 t)$	$\pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$
1	$2\pi\delta(\omega)$

## 1.3 Examples

### 1.3.1 Example 1: Fourier Series

Find the Fourier coefficients and Fourier Series for the square wave shown below:

$$f(x) = \begin{cases} 0 & \text{for } -1 \leq x \leq 0 \\ 1 & \text{for } 0 \leq x \leq 1 \end{cases}$$

and

$$f(x+2) = f(x)$$

---

The fourier series is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

Find the  $L$  value:

$$2L = 2 \quad \Rightarrow \quad L = 1$$

Find  $a_0$ :

$$\begin{aligned} a_n &= \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx \quad n \geq 0 \\ a_0 &= \frac{1}{1} \int_{-1}^1 f(x) \cos\left(\frac{0\pi x}{1}\right) dx = \int_{-1}^1 f(x) dx = \int_{-1}^0 0 dx + \int_0^1 1 dx = 1 \end{aligned}$$

Find  $a_n$ :

$$\begin{aligned} a_n &= \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx \quad n \geq 0 \\ &= \frac{1}{1} \int_{-1}^1 f(x) \cos\left(\frac{n\pi x}{1}\right) dx = \int_{-1}^1 f(x) \cos(n\pi x) dx \\ &= \int_{-1}^0 0 \cos(n\pi x) dx + \int_0^1 1 \cos(n\pi x) dx = 0 + \left[ \frac{\sin(n\pi x)}{n\pi} \right]_0^1 = \frac{\sin(\pi n)}{\pi n} \end{aligned}$$

For all  $n$ :

$$\frac{\sin(\pi n)}{\pi n} = 0$$

Find  $b_n$ :

$$\begin{aligned} b_n &= \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad n > 0 \\ b_n &= \int_{-1}^1 f(x) \sin\left(\frac{n\pi x}{1}\right) dx = \int_{-1}^0 0 \sin(n\pi x) dx + \int_0^1 1 \sin(n\pi x) dx \\ &= 0 + \left[ \frac{-\cos(n\pi x)}{n\pi} \right]_0^1 = \frac{-\cos(n\pi 1)}{n\pi} - \frac{-\cos(n\pi 0)}{n\pi} \end{aligned}$$

If  $n$  is even the function will cancel out, therefore  $n = 1, 3, 5, \dots$  (odd):

$$= \frac{1}{n\pi} + \frac{1}{n\pi} = \frac{2}{n\pi}$$

Ans:

$$f(x) = \frac{1}{2} + \sum_{n=1,3,5,\dots} \frac{2}{n\pi} \sin(n\pi x)$$

### 1.3.2 Example 2: Fourier Transform

The unit step function is defined as:

$$u(t-a) = \begin{cases} 1 & \text{for } t-a > 0 \\ 0 & \text{for } t-a < 0 \end{cases}$$

is used to define the rectangular pulse function:

$$x(t) = u(t-a) - u(t-b) \quad \text{where } a < b$$

---

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

$$X(\omega) = \int_{-\infty}^a 0e^{-j\omega t} dt + \int_a^b 1e^{-j\omega t} dt + \int_b^{\infty} 0e^{-j\omega t} dt$$

$$X(\omega) = 0 + \left[ \frac{-e^{-j\omega t}}{j\omega} \right]_a^b + 0$$

Insert the limits:

$$X(\omega) = \frac{e^{-j\omega a} - e^{-j\omega b}}{j\omega}$$

## 2 Laplace transform

Is a generalisation of the Fourier transform and defined as:

$$H(s) = \mathcal{L}\{h(t)\} = \int_{0^-}^{\infty} h(t)e^{-st} dt \quad s \in \mathbb{C}$$

$s$  is a complex number  $s = \sigma + j\omega$  and is identical with Fourier transform, if  $s$  is set to  $j\omega$ .  
Inverse Laplace transformation:

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{1}{2\pi j} \int_{\sigma_c - j\infty}^{\sigma_c + j\infty} H(s)e^{st} ds$$

Signal	Laplace Transform
1	$\frac{1}{s}$
$t^n$	$\frac{n!}{s^{n+1}}$
$e^{at}$	$\frac{1}{s-a}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$e^{at} \cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$
$e^{at} \sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$

### 2.1 General Formulas

$$\mathcal{L}\{f(t)\} = F(s)$$

$$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$$

$$\mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0) - f'(0)$$

$$\mathcal{L}\{f^n\} = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - \dots - f^{n-1}(0)$$

### 2.2 Examples

#### 2.2.1 Example 1: Laplace Transform

Using the Laplace transform, find the solution for the following equation:

$$\frac{\partial^2}{\partial t^2}y(t) + 2\frac{\partial}{\partial t}y(t) + 2y(t) = 0$$

with initial conditions  $y(0) = 1$  and  $y'(0) = -1$

---

Take laplace transform of the equation:

$$s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + 2(Y(s)) = 0$$

$$s^2Y(s) - s + 1 + 2sY(s) - 2 + 2Y(s) = 0$$

$$\begin{aligned}
s^2Y(s) + 2sY(s) + 2Y(s) &= 1 + s \\
(s^2 + 2s + 2)Y(s) &= 1 + s \\
Y(s) &= \frac{1 + s}{(s^2 + 2s + 2)} = \frac{1 + s}{((s + 1)^2 + 1)}
\end{aligned}$$

From table lookup:

$$\begin{aligned}
\mathcal{L}\{e^{at} \cos(\omega t)\} &= \frac{s - a}{(s - a)^2 + \omega^2} \\
a &= -1 \quad \omega = 1 \\
y(t) &= e^{-t} \cos(t)
\end{aligned}$$

### 2.2.2 Example 2: Transfer Function

Consider a mass-spring-damper system with the following differential equation:

$$m\ddot{x} = -kx - b\dot{x} + f$$

Find the transfer function for the system with input  $f$  and output  $x$ .

---

$$\begin{aligned}
ms^2X(s) &= -kX(s) - bsX(s) + F(s) \\
ms^2X(s) + kX(s) + bsX(s) &= F(s) \\
(ms^2 + k + bs)X(s) &= F(s)
\end{aligned}$$

The transfer function is:

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + k + bs}$$

### 2.2.3 Example 3: Differential equation

Consider:

$$y''(t) + y'(t) = 0.5t$$

where  $y(0) = 0$  and  $y'(0) = 0$  Use Laplace transform to solve the equation and find  $y(t)$

---

From Laplace transform table:

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

Laplace transform of given differential equation:

$$\begin{aligned}
s^2Y(s) - sy(0) - y'(0) + Y(s) &= 0.5 \frac{1}{s^2} \\
(s^2 + 1)Y(s) &= \frac{0.5}{s^2} \\
Y(s) &= \frac{0.5}{s^2(s^2 + 1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 1} \\
0.5 &= As(s^2 + 1) + B(s^2 + 1) + (Cs + D)s^2 \\
0.5 &= As^3 + As + Bs^2 + B + Cs^3 + Ds^2 \\
0.5 &= s^3(A + C) + s^2(B + D) + sA + B \\
B &= 0.5
\end{aligned}$$

$$A = 0$$

$$B + D = 0 \quad \Rightarrow \quad D = -0.5$$

$$A + C = 0 \quad \Rightarrow \quad C = 0$$

Therefore the partial fractions are:

$$Y(s) = \frac{0}{s} + \frac{0.5}{s^2} + \frac{0s - 0.5}{s^2 + 1} = \frac{0.5}{s^2} + \frac{-0.5}{s^2 + 1}$$

From Laplace transform table:

$$\mathcal{L}\{\sin(at)\} = \frac{a}{s^2 + a^2}$$

The inverse laplace transform of  $Y(s)$ :

$$y(s) = 0.5t - 0.5 \sin(t)$$



### 3 Filter-Functions

**Pass band:** Frequency range where the signal passes through the filter.

**Stop band:** Frequency range where the signal is attenuated by the filter.

**Cut-off frequency:** Frequency where the signal is attenuated by 3 dB.

**Attenuation:** The decrease in signal strength.

**Ripple:** The variation in the pass band.

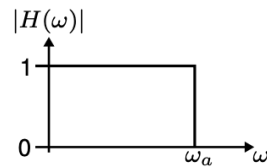
**Form factor:** The ratio between the stop band and the pass band.

**Group delay:** The time delay of the filter.

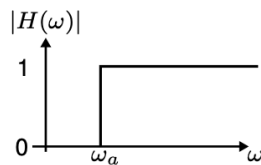
**Phase delay:** The phase shift of the filter.

#### 3.1 Electronic filters

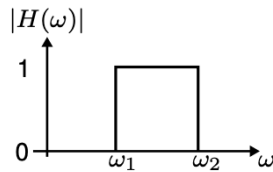
**Low-pass filter:** Ideally only allows low angle frequencies ( $\omega < \omega_a$ )



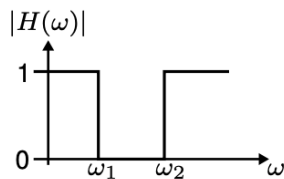
**High-pass filter:** Ideally only high angle frequencies ( $\omega > \omega_a$ )



**Band-pass filter:** Ideally only certain range of angle frequencies ( $\omega_1 < \omega < \omega_2$ )



**Band-stop filter:** Ideally filters out a certain range of angle frequencies ( $\omega_1 > \omega < \omega_2$ )



#### 3.2 Filter specification

Specifications for a low-pass filter could be:

1. A cut-off frequency  $\omega_a$  that specifies the upper limit of the passband.
2. A stopband frequency  $\omega_s$  at which a stopband attenuation  $A_s$  is specified.
3. Information about permissible gain variation in the passband.

### 3.3 Group delay

Measures time for the filter and often depend on frequency  $\omega$ .

$$T_g = -\frac{d\phi(\omega)}{d\omega}$$

where  $\phi(\omega)$  is the angle of the filter.

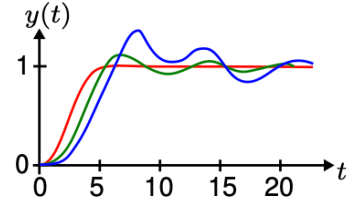
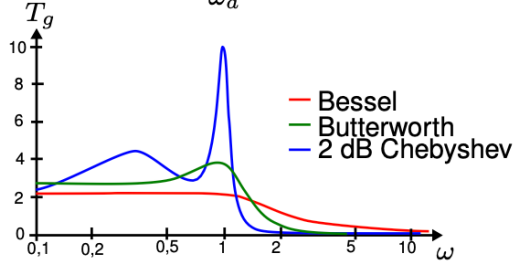
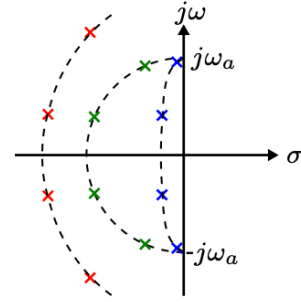
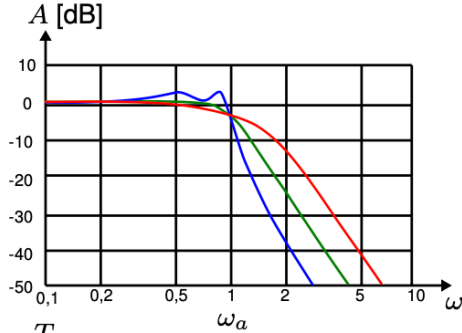
The phase characteristic of a filter also affects the input-output behavior of a filter. If pulse overshoot or damped oscillation (ringing) is to be avoided, the filter must have a constant group delay,  $T_g$ .

### 3.4 Filter transfer functions

In practice, the ideal filters cannot be realized, but can be approximated using various filter function types.

These are all-pole filters (Only poles, no zeros) When selecting the filter function for a given application, we are interested in the following characteristics:

1. Constant gain in passband.
2. High attenuation after cut-off frequency.
3. Linear phase.



#### 3.4.1 Butterworth

$$|H(j\omega)|^2 = \frac{1}{1 + (\frac{\omega}{\omega_a})^{2N}}$$

where  $\omega_a$  is the cut-off frequency [rad/s]

All poles for  $H(j\omega)$  lie in the left half-plane on a circle with radius  $\omega_a$  and centre at the origin.

A Butterworth filter has the following characteristics:

1. Optimal in terms of constant gain in the passband.
2. Has 3 dB attenuation at the cut-off frequency and then the filter gain drops rapidly by 20 dB/dec. (dec is when the power of 10 changes by 1)

3. The phase of the filter is not constant in the passband, which causes ringing at the step input.

All pole pairs of a Butterworth filter have a natural natural natural frequency  $\omega_n$ , which is equal to the cut-off frequency  $\omega_a$ .

### 3.4.2 Chebyshev

$$|H(\omega)|^2 = \frac{1}{1 + e^2 T_N^2(\omega - \omega_a)}$$

where  $T_N(\omega)$  is Chebyshev polynomial of degree N given by

$$T_N(\omega) = \begin{cases} \cos(N \cdot \cos^{-1}(\omega)) & : |\omega| \leq 1 \\ \cosh(N \cdot \cosh^{-1}(\omega)) & : |\omega| > 1 \end{cases}$$

The size of the passband ripple  $\delta$  in dB is given by  $\epsilon$

$$\delta = 10 \cdot$$

A Chebyshev filter has the following characteristics

1. Has varying gain in the passband - the size of the passband ripple can be freely selected.
2. The gain decreases rapidly around the cut-off frequency.
3. The phase of the filter is not constant in the passband, which causes ringing at the step input.

The DC gain of a Chebyshev low-pass filter is not the maximum gain of the filter if the pole count is even.

### 3.4.3 Bessel

The transfer function for an Nth order Bessel filter is

$$H_N(s) = \frac{b_0}{s^N + b_{N-1}s^{N-1} + \dots + b_1s + b_0}$$

where

$$b_k = \frac{(2N - k)!}{2^{N-k} k! (N - k)!}$$

A Bessel filter has the following properties:

1. Does not have ripple in the passband, but the amplitude is not as constant as a Butterworth filter.
2. Has attenuation that is very smooth. The amplitude characteristic of the filter is like that of the first order filter for the first 6 dB of attenuation regardless of the number of poles.
3. The phase is almost linear with frequency within the passband.

### 3.4.4 The filter's order number

The order number of filters can be found by reading the amplitude characteristic of a given filter function and comparing it to the attenuation requirements at the stopband frequency.

$$Y = 20 \log \left( \frac{A}{A_{\max}} \right)$$

Frequency normalised (form factor)

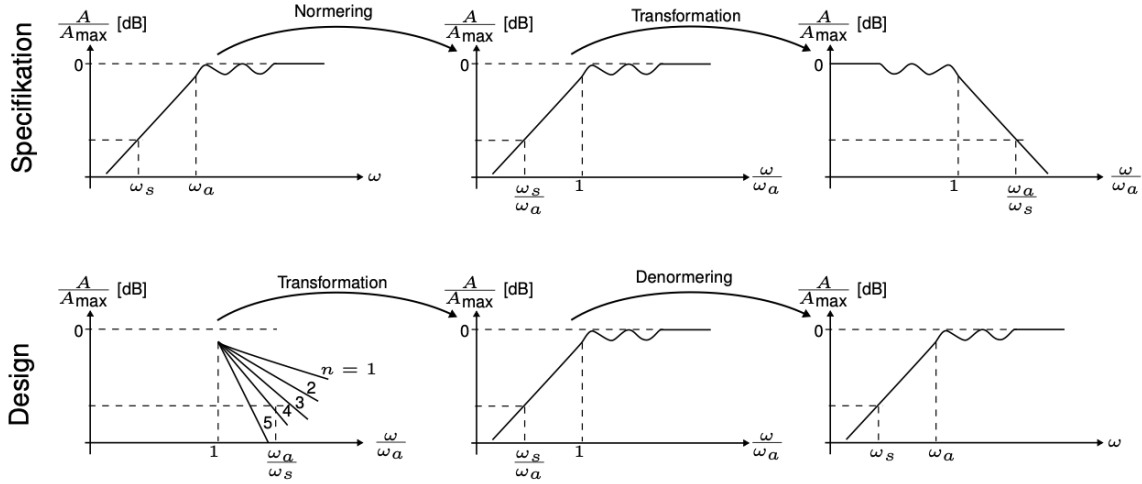
$$X = \frac{\omega}{\omega_a}$$

### 3.5 Filter transformers

#### 3.5.1 Low pass to high pass transformation

High-pass filters can be designed from normalised prototype low-pass filters based on the following specification:

- The filter function (Bessel, Butterworth, Chebyshev)
- The filter cut-off frequency  $\omega_a$
- The filter stopband frequency  $\omega_s$
- The filter stopband attenuation  $A_s$  at the stopband frequency  $\omega_s$



A low-pass filter can be transformed into a high-pass filter by:

$$H_{hp}(s) = H_{lp}(\bar{s})|_{\bar{s}=\frac{1}{s}}$$

Process: Low pass to high pass transformation

1. Find the stopband frequency for the normalized low-pass filter (used for the design):

$$\frac{\omega_a}{\omega_s}$$

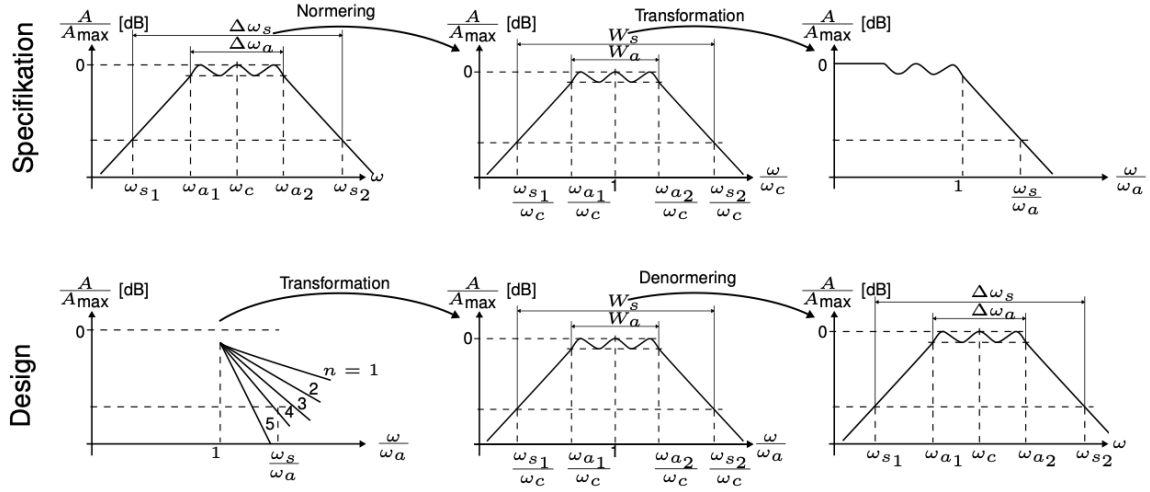
2. Filter order selection based on amplitude characteristics for lowpass filters.
3. The transfer function of the normalized low-pass filter is transformed to the transfer function of the normalized high-pass filter by replacing  $s$  with  $\frac{1}{s}$
4. The transfer function of the denormalized high-pass filter is found by replacing  $s$  with  $\frac{s}{\omega_a}$

#### 3.5.2 Low pass to bandpass transformation

Bandpass filters can be designed from normalized prototype low-pass filters based on the following specification

- The filter function (Bessel, Butterworth, Chebyshev)
- The filter center frequency  $\omega_s$
- The passband bandwidth  $\Delta\omega_a$
- Stopband bandwidth  $\Delta\omega_s$

- Filter stopband attenuation  $A_s$



A low-pass filter can be transformed into a bandpass filter by

$$H_{bp}(s) = H_{lp}(\bar{s})|_{\bar{s} = \frac{1}{W_a}(s + \frac{1}{s})}$$

To design the filter, the bandpass filter is normalised, i.e. the normalised stopband width and the normalised passband width are calculated as:

$$W_a = \frac{\Delta f_a}{f_c}$$

$$W_s = \frac{\Delta f_s}{f_c}$$

and the form factor for bandpass filter:

$$F = \frac{\Delta f_s}{\Delta f_a} = \frac{W_s}{W_a}$$

Process: Low pass to bandpass transformation

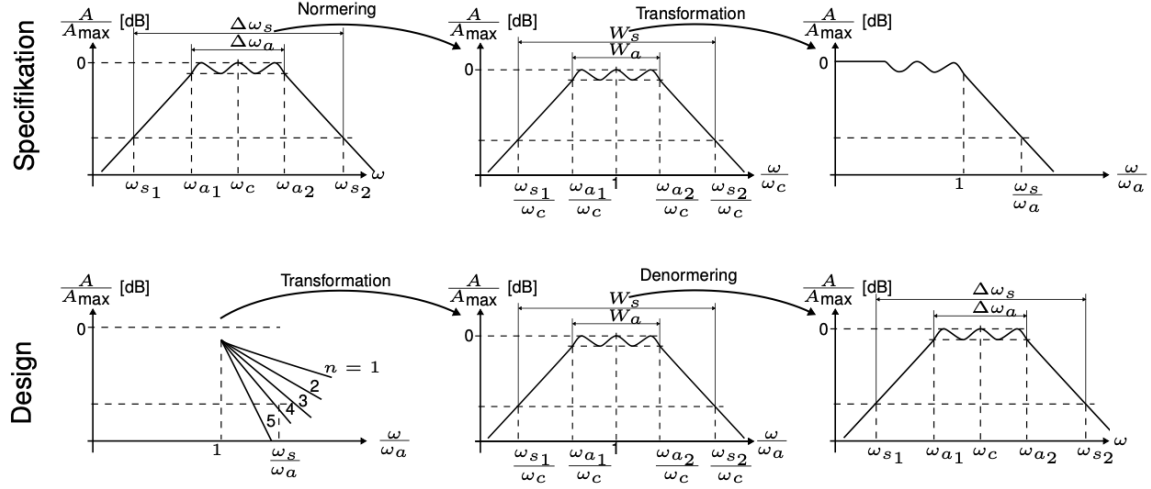
1. To design the filter, the bandpass filter is normalized, i.e. the normalized stopband width and the normalized passband width are calculated.
2. The form factor is found.
3. Filter order selection based on amplitude characteristics for lowpass filters.
4. The normalised filter is found by replacing  $s$  with  $\frac{1}{W_a}(s + \frac{1}{s})$  in the low-pass filter.
5. The denormalized bandpass filter is found by replacing  $s$  in the normalized bandpass filter with  $\frac{s}{\omega_c}$ .

### 3.5.3 Low pass to bandstop transformation

Bandstop filters can be designed from normalised prototype low-pass filters based on the following specification:

- Filter function (Bessel, Butterworth, Chebyshev)
- Filter centre frequency  $\omega_c$
- Passband bandwidth  $\Delta\omega_c$

- Stopband bandwidth  $\Delta\omega_s$
- Filter stopband attenuation  $A_s$



A low-pass filter can be transformed into a bandstop filter by

$$H_{bs}(s) = H_{lp}(\bar{s}) \Big|_{\bar{s} = \frac{\omega_a}{s + \frac{1}{s}}}$$

and the form factor for bandstop filter:

$$F = \frac{\Delta f_a}{\Delta f_s} = \frac{W_a}{W_s}$$

Process: Low pass to bandstop transformation

1. To design the filter, the bandstop filter is normalized, i.e. the normalized stopband width and the normalized passband width are calculated.
2. The form factor is found.
3. Filter order selection based on amplitude characteristics for lowpass filters.
4. The normalised filter is found by replacing  $s$  with  $\frac{W_a}{s + \frac{1}{s}}$  in the low-pass filter.
5. The denormalized bandpass filter is found by replacing  $s$  in the normalized bandpass filter with  $\frac{s}{\omega_c}$ .

## 4 Sampling-Reconstruction

### Nyquist-Shannon Phrase:

A continuous-time signal  $x(t)$  can only be correctly recovered from  $x_s(t)$  if the sampling frequency is at least **twice** the highest frequency in the spectrum of  $x(t)$ .

## 5 FFT

### 5.1 DFT

The DTF is given by:

$$X(m) = \sum_{n=0}^{N-1} x(n)W_N^{mn} = \sum_{n=0}^{N-1} x(n)e^{\frac{-j2\pi mn}{N}}$$

for  $m = 0, 1, \dots, N-1$  and  $W_N = e^{-j2\pi/N}$

The sequence can be found from the spectrum  $X(m)$ :

$$x(n) = \frac{1}{N} \sum_{m=0}^{N-1} X(m)W_N^{-mn} = \frac{1}{N} \sum_{m=0}^{N-1} X(m)e^{\frac{j2\pi mn}{N}}$$

for  $m = 0, 1, \dots, N-1$

### 5.2 Examples

#### 5.2.1 Example 1: DFT of a sequence

Consider the sequence:

$$x(n) = \begin{cases} 10 & n = 0 \\ 0 & n = 1 \\ 0 & n = 2 \\ -10 & n = 3 \\ 0 & n = 4 \end{cases}$$

Calculate the 5-point discrete Fourier transform of the sequence. Find  $X(1)$  and  $X(2)$

---

$$X(m) = \sum_{n=0}^{N-1} x(n)e^{\frac{-j2\pi mn}{N}}$$

Insert  $N = 5$  and calculate the sum:

$$X(m) = \underbrace{10e^{\frac{-j2\pi m \cdot 0}{5}}}_{n=0} + \underbrace{0}_{n=1} + \underbrace{0}_{n=2} + \underbrace{-10e^{\frac{-j2\pi m \cdot 3}{5}}}_{n=3} + \underbrace{0}_{n=4}$$
$$X(m) = 10 - e^{\frac{-j2\pi m \cdot 3}{5}}$$

Find  $X(1)$  and  $X(2)$ :

$$X(1) = 10 - 10e^{\frac{-j2\pi \cdot 1 \cdot 3}{5}} = 10 - 10e^{\frac{-j6\pi}{5}}$$
$$X(2) = 10 - 10e^{\frac{-j2\pi \cdot 2 \cdot 3}{5}} = 10 - 10e^{\frac{-j12\pi}{5}}$$



## 6 Z-transformation

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$$

Notation:

$$X(z) = \mathcal{Z}\{x(n)\} \quad x(n) = \mathcal{Z}^{-1}\{X(z)\}$$

### 6.0.1 s and z domain

### 6.1 Differential equations

### 6.2 Transfer Functions

$$H(z) = \frac{Y(z)}{X(z)}$$

$H(z)$  is the transfer function.

$X(z)$  is the input sequence.

$Y(z)$  is the output sequence.

### 6.3 Inverse Z-transformation

### 6.4 Examples

#### 6.4.1 Example 1: Z-transformation

#### 6.4.2 Example 2: Inverse Z-transformation

## 7 System-analysis

## 8 Digital-Realization-Structures

## 9 IRR-Filters

### 9.1 Design of IRR-filters

Design of IRR-filters is done in the following steps:

1. The specifications of the IRR-filter.
2. The transfer function of the IRR-filter.
3. The optimal realization of the IRR-filter.
4. Program for signalprocessing or a circuit diagram for analog signalprocessing.

### 9.2 Matched z-transform

By using the matched z-transform, the poles and zeros of the IRR-filter are directly transferred to the z-plane. The transfer function of the IRR-filter is given by:

$$z = e^{sT}$$

Following procedure is used for the matched z-transform:

1. Determine the frequency-normalized and factorized transfer function  $H(s)$  of the analog prototype filter.
2. Determine the analog frequency-normalized poles and zero points.
3. Determine the denormalized poles and zero points.
4. Determine the coefficients of the digital transfer function.
5. Implement the transfer function as a cascade structure.

#### 9.2.1 1. Order Matched z-transformation

The transfer function for a first order system is given by:

$$H(s) = \frac{s + A_0}{s + B_0} = \frac{s - \sigma_1}{s - \sigma_2}$$

where  $-A_0 = \sigma_1$  is a real zero and  $-B_0 = \sigma_2$  is a real pole of the IRR-filter.

A digital first order transfer function is given by:

$$H(z) = \frac{z - e^{\sigma_1 T}}{z - e^{\sigma_2 T}} = \frac{1 - e^{\sigma_1 T} z^{-1}}{1 - e^{\sigma_2 T} z^{-1}}$$

A first order system without a zero is given by:

$$H(s) = \frac{\omega_a}{s + \omega_a}$$

Using the matched z-transform, the digital transfer function is given by:

$$H(z) = \frac{1}{1 - e^{\sigma_1 T} z^{-1}}$$

where  $\sigma_1$  is the pole of  $H(s)$  and  $T$  is the sampling period.

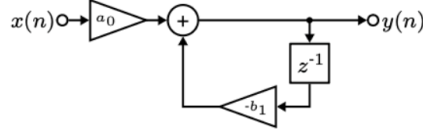
The transfer function  $H(z)$  does not have a DC-gain of 1. To get a DC-gain of 1, the transfer function is multiplied by a gain factor  $a_0$ :

$$H(z) = \frac{a_0}{1 + b_1 z^{-1}}$$

where

$$b_0 = -e^{\sigma_1 T} \quad a_0 = 1 + b_1$$

The corresponding digital realization structure:



### 9.2.2 2. Order Matched z-transformation

The transfer function for a second order system with complex conjugate pole and zero pair is given by:

$$H(s) = \frac{s^2 + A_1 s + A_0}{s^2 + B_1 s + B_0}$$

which has zeros in  $s_1 = \sigma_1 + j\omega_1, s_1^* = \sigma_1 - j\omega_1$  and poles in  $s_2 = \sigma_2 + j\omega_2, s_2^* = \sigma_2 - j\omega_2$

Using the matched z-transform:

$$H(z) = \frac{(z - e^{\sigma_1 T} e^{j\omega_1 T})(z - e^{\sigma_2 T} e^{-j\omega_1 T})}{(z - e^{\sigma_2 T} e^{j\omega_2 T})(z - e^{\sigma_2 T} e^{-j\omega_2 T})}$$

Using Euler's identity:

$$H(z) = \frac{z^2 - (2e^{\sigma_1 T} \cos(\omega_1 T))z + e^{2\sigma_1 T}}{z^2 - (2e^{\sigma_2 T} \cos(\omega_2 T))z + e^{2\sigma_2 T}} = \frac{1 - (2e^{\sigma_1 T} \cos(\omega_1 T))z^{-1} + e^{2\sigma_1 T} z^{-2}}{1 - (2e^{\sigma_2 T} \cos(\omega_2 T))z^{-1} + e^{2\sigma_2 T} z^{-2}}$$

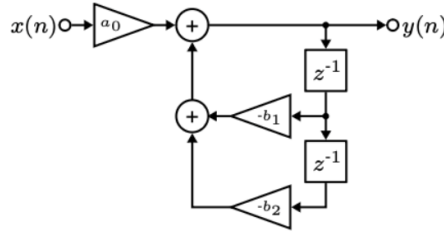
A second order low-pass filter is given by:

$$H(z) = \frac{a_0}{1 + b_1 z^{-1} + b_2 z^{-2}}$$

where

$$a_0 = 1 + b_1 + b_2 \quad b_1 = -2e^{\sigma_1 T} \cos(\omega_1 T) \quad b_2 = e^{2\sigma_2 T}$$

The corresponding digital realization structure:



## 9.3 Impulse invariance method

The following procedure is used to design digital IIR filters using the impulse invariantz transform.

1. Determine the frequency-normalized transfer function  $H(s)$  of the analog prototype filter.
2. Partial fractionally resolve  $H(s)$  into 1st and 2nd order transfer functions (maximum number of 2nd order transfer functions).
3. Denormalize the coefficients  $k_i$  and the poles  $\sigma_i + j\omega_i$  by multiplication with the cutoff frequency or center frequency.
4. Determine the coefficients of the digital transfer function.
5. Implement the transfer function as a parallel structure.

## 9.4 Bilinear z transformation

The following is the procedure for the design of digital filters using bilinearz transformation.

1. The prewarping constant is determined as

$$C = \cot \left( \frac{\omega_i T}{2} \right)$$

where  $i = a$  for low pass or high pass filter design and  $i = c$  for band pass or band stop filter design.

2. The order number of the filter is determined based on the prewarped stopband frequency.
3. The frequency-normalized and factorized analog transfer function  $H(s)$  is set up.
4. The coefficients of the digital transfer function are calculated.
5. The filter is implemented as a cascaded realization structure.

### 9.4.1 Warping and Pre-warping

#### 9.4.2 1. Order Bilinear z transformation

First order system:

$$H(s) = \frac{A_1 s + A_0}{B_1 s + B_0}$$

Using bilinear z transformation:

$$H(z) = \frac{a_0 + a_1 z^{-1}}{1 + b_1 z^{-1}}$$

where

$$a_0 = \frac{A_0 + A_1 C}{B_0 + B_1 C} \quad a_1 = \frac{A_0 - A_1 C}{B_0 + B_1 C} \quad b_1 = \frac{B_0 - B_1 C}{B_0 + B_1 C}$$

#### 9.4.3 2. Order Bilinear z transformation

A second order system:

$$H(s) = \frac{A_2 s^2 + A_1 s + A_0}{B_2 s^2 + B_1 s + B_0}$$

Using bilinear z transformation:

$$H(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2}}{1 + b_1 z^{-1} + b_2 z^{-2}}$$

where

$$a_0 = \frac{A_0 + A_1 C + A_2 C^2}{B_0 + B_1 C + B_2 C^2} \quad a_1 = \frac{2(A_0 - A_2 C^2)}{B_0 + B_1 C + B_2 C^2} \quad a_2 = \frac{A_0 - A_1 C + A_2 C^2}{B_0 + B_1 C + B_2 C^2}$$

$$b_1 = \frac{2(B_0 - B_2 C^2)}{B_0 + B_1 C + B_2 C^2} \quad b_2 = \frac{B_0 - B_1 C + B_2 C^2}{B_0 + B_1 C + B_2 C^2}$$

## 9.5 Examples

### 9.5.1 Example 1: Matched 1. order z-transform

Consider the following analog first order lowpass filter with cutoff frequency  $\omega_a = 2\pi f_a$  where  $f_a = 300$  Hz:

$$H(s) = \frac{\omega_a}{s + \omega_a} = \frac{1885}{s + 1885}$$

which has a pole in  $s_1 = \sigma_1 = -1885$ .

Find the digital IIR-filter using the matched z-transform with sampling frequency  $f_s = 16$  kHz.

The coefficients of the digital IIR-filter are given by:

$$b_1 = -e^{\sigma_1 T} = -e^{-1885 \cdot \frac{1}{16000}} = -0.8889$$

$$a_0 = 1 + b_1 = 0.1111$$

The digital lowpass filter is given by:

$$H(z) = \frac{a_1}{1 + b_0 z^{-1}} = \frac{0.1111}{1 - 0.8888 z^{-1}}$$

### 9.5.2 Example 2: Matched 3. order matched z-transform

Consider the following Butterworth 3rd order low-pass filter (frequency-normalized filter) and design an equivalent digital low-pass filter with cut-off frequency at 1 kHz and sample frequency at 8 kHz.

$$\tilde{H}(s) = \frac{1}{s+1} \frac{1}{s^2 + s + 1}$$

Using the matched z-transform

**1. Determine the frequency-normalized and factorized transfer function  $H(s)$  of the analog prototype filter.**

Is in this case it's already given.

**2. Determine the analog frequency-normalized poles and zero points.**

There is no zeros in the transfer function, so only the poles are determined:

$$s_1 = -1 \quad s_2 = s_2^* = -\frac{1}{2} + j \frac{\sqrt{3}}{2}$$

**3. Determine the denormalized poles and zero points.**

Only the poles are denormalized, since there are no zeros:

$$\omega_a = 2\pi f_a = 2\pi \cdot 1000 = 6283.2$$

$$\sigma_1 = -1 \cdot \omega_a = -6283.2$$

$$\sigma_2 \pm j\omega_2 = s_2 \cdot \omega_a = -3141.6 \pm j5434.4$$

**4. Determine the coefficients of the digital transfer function.**

$$H(s) = H_1(s)H_2(s) = \frac{1}{s+1} \frac{1}{s^2 + s + 1}$$

$$H(z) = H_1(z)H_2(z)$$

Using the matched z-transform:

$$H_1(z) = \frac{1}{1 - e^{\sigma_1 T} z^{-1}} = \frac{a_{0_1}}{1 + b_{1_1} z^{-1}}$$

$$b_{1_1} = -e^{\sigma_1 T} = -e^{-6283.2 \cdot \frac{1}{8000}} = -0.4559 \quad a_{0_1} = 1 + b_{1_1} = 0.5441$$

$$H_1(z) = \frac{0.5441}{1 - 0.4559 z^{-1}}$$

and

$$\begin{aligned}
H_2(z) &= \frac{1}{(z - e^{\sigma_2 T} e^{j\omega_2 T})(z - e^{\sigma_2 T} e^{-j\omega_2 T})} = \frac{a_{0_2}}{1 + b_{1_2} z^{-1} + b_{2_2} z^{-2}} \\
b_{1_2} &= -2e^{\sigma_2 T} \cos(\omega_2 T) = -2e^{-3141.6 \cdot \frac{1}{8000}} \cos(5434.4 \cdot \frac{1}{8000}) = -1.0507 \\
b_{2_2} &= e^{2\sigma_2 T} = e^{2 \cdot -3141.6 \cdot \frac{1}{8000}} = 0.4559 \\
a_{0_2} &= 1 + b_{1_2} + b_{2_2} = 1 - 1.0507 + 0.4559 = 0.4052 \\
H_2(z) &= \frac{0.4559}{1 - 1.0507z^{-1} + 0.4559z^{-2}}
\end{aligned}$$

Combining the two transfer functions:

$$H(z) = H_1(z)H_2(z) = \frac{0.5441}{1 - 0.4559z^{-1}} \frac{0.4052}{1 - 1.0507z^{-1} + 0.4559z^{-2}}$$

### 9.5.3 Example 2: Matched 3. order impulse invariance z-transform

Consider the following Butterworth 3rd order low-pass filter (frequency-normalized filter) and design an equivalent digital low-pass filter with cut-off frequency at 1 kHz and sample frequency at 8 kHz.

$$\tilde{H}(s) = \frac{1}{s+1} \frac{1}{s^2 + s + 1}$$

Using the impulse invariance method

**1. Determine the frequency-normalized and factorized transfer function  $H(s)$  of the analog prototype filter.**

Is in this case it's already given.

**2. Partial fractionally resolve  $H(s)$  into 1st and 2nd order transfer functions (maximum number of 2nd order transfer functions).**

The poles are given by:

$$s_1 = -1 \quad s_2 = s_2^* = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$$

The transfer function is given by:

$$H(s) = \frac{k_1}{s - (s_1)} + \frac{k_2}{s - (s_2)} + \frac{k_2^*}{s - (s_2^*)} = \frac{k_1}{s - (-1)} + \frac{k_2}{s - (-\frac{1}{2} + j\frac{\sqrt{3}}{2})} + \frac{k_2^*}{s - (-\frac{1}{2} - j\frac{\sqrt{3}}{2})}$$

### 9.5.4 Example 3: Bilinear z-transform



## 10 FIR-Filters

**Finite Impulse Response** filters where the impulse response of the system is of finite duration. A FIR filter with  $N$  samples has a impulse response:

$$h(n) = \begin{cases} a_n & \text{for } 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore the filter has  $N$  coefficients  $a_n$  for  $n = 0, 1, \dots, N-1$ .

### 10.1 Linear phase

### 10.2 Frequency response

### 10.3 FIR-Filter design

If there is  $N$  samples in the impulse response, then:

$$M = \frac{N-1}{2}$$

And the transfer function of the filter is:

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{i=0}^{2M} a_i z^{-i}$$

Filter type	$c_0$	$c_m = c_{-m}$	$a_i$
Low-pass	$2Tf_a$	$\frac{1}{m\pi} \sin(2\pi m T f_a)$	$c_{M-i}$
High-pass	$1 - 2Tf_a$	$\frac{1}{m\pi} (\sin(m\pi) - \sin(2\pi m T f_a))$	$c_{M-i}$
Band-pass	$2T(f_{a_2} - f_{a_1})$	$\frac{1}{m\pi} (\sin(2\pi m T f_{a_2}) - \sin(2\pi m T f_{a_1}))$	$c_{M-i}$
Band-stop	$1 - 2T(f_{a_2} - f_{a_1})$	$\frac{1}{m\pi} (\sin(m\pi) + \sin(2\pi m T f_{a_1}) - \sin(2\pi m T f_{a_2}))$	$c_{M-i}$

Using the inverse z-transform, the impulse response is found:

$$y(n) = \sum_{i=0}^{2M} a_i x(n-i)$$

### 10.4 Window functions

Due to the ripples in the passband and stopband, the ideal filter is not realizable. Therefore, the ideal impulse response is multiplied by a window function  $w(n)$  to obtain a realizable impulse response  $h(n)$ :

$$h(n) = h_{\infty}(n)w(n)$$

Following are some common window functions:

Window	$B_n$	$M_{\min}$	Min. stopband attenuation	Min. passband ripple
Rectangular	2	$f_s/\Delta_f$	20 dB	1.5 dB
Barlett	4	$2f_s/\Delta_f$	25 dB	0.1 dB
Hamming	4	$2f_s/\Delta_f$	50 dB	0.05 dB
Hanning	4	$2f_s/\Delta_f$	45 dB	0.1 dB
Kaiser ( $\beta = \pi$ )	2.8	$1.4f_s/\Delta_f$	40 dB	0.2 dB
Kaiser ( $\beta = 2\pi$ )	4.4	$2.2f_s/\Delta_f$	65 dB	0.01 dB

#### 10.4.1 Rectangular window

$$w(n) = \begin{cases} 1 & \text{if } -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

#### 10.4.2 Barlett window

$$w(n) = \begin{cases} 1 - \frac{|n|}{M} & \text{if } -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

#### 10.4.3 Hamming and Hanning window

$$w(n) = \begin{cases} \alpha + (1 - \alpha) \cos\left(\frac{n\pi}{M}\right) & \text{if } -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

where  $\alpha = 0.54$  for Hamming and  $\alpha = 0.5$  for Hanning.

#### 10.4.4 Kaizer window

$$w(n) = \begin{cases} \frac{I_0\left(\beta\sqrt{1-\left(\frac{n}{M}\right)^2}\right)}{I_0(\beta)} & \text{if } -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases}$$

### 10.5 Examples

#### 10.5.1 Low-pass filter

#### 10.5.2 High-pass filter

A FIR high-pass filter with cutoff frequency  $f_a = 1$  kHz, a transition band of  $\Delta f \leq 0.5$  kHz, maximum stopband attenuation of  $H_s \leq -50$  dB and sample frequency  $f_s = 5$  kHz is designed.

#### Which window function should be used?

To satisfy the stopband attenuation, the Hamming or the Kaiser window should be used, as they both have 50 dB or higher stopband attenuation. The Hamming window is chosen, which has a  $B_n = 4$  and minimum stopband attenuation of 50 dB.

#### What is the minimum filter order?

The filter order is given by  $2M$ , where  $M$  is:

$$M = \frac{B_n f_s}{2\Delta f} = \frac{4 \cdot 5000}{2 \cdot 500} = 20$$

This satisfies  $M_{\min}$  of  $20 \geq \frac{2f_s}{\Delta f} = \frac{2 \cdot 5000}{500} = 20$  for the Hamming window. Therefore the minimum filter order is  $2M = 40$ .

**What is the transfer function for the filter?**

Using the following formula:

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{i=0}^{2M} a_i z^{-i}$$

where  $a_i = c_{M-i}$  is the coefficient for the  $z^{-i}$  term, which for a high-pass filter is given by:

$$c_0 = 1 - 2Tf_a \quad c_m = \frac{1}{m\pi}(\sin(m\pi) - \sin(2\pi mTf_a))$$

Calculating the coefficients:

$$\begin{aligned} c_0 &= 1 - 2 \cdot \frac{1}{5000} \cdot 1000 = 0.6 \\ c_1 = c_{-1} &= \frac{1}{1 \cdot \pi} \left( \sin(1 \cdot \pi) - \sin(2\pi \cdot 1 \cdot \frac{1}{5000} \cdot 1000) \right) = -0.3027 \\ c_2 = c_{-2} &= \frac{1}{2 \cdot \pi} \left( \sin(2 \cdot \pi) - \sin(2\pi \cdot 2 \cdot \frac{1}{5000} \cdot 1000) \right) = -0.0935 \\ c_3 = c_{-3} &= \frac{1}{3 \cdot \pi} \left( \sin(3 \cdot \pi) - \sin(2\pi \cdot 3 \cdot \frac{1}{5000} \cdot 1000) \right) = 0.0624 \end{aligned}$$

**10.5.3 Band-pass filter****10.5.4 Band-stop filter**