Multi Variable Calculus

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Contents

1 Fourier

1.1 Fourier Series

A periodic function with period $2L$ and let $f(x)$ and $f'(x)$ be piecewise continuous on the interval $-L < x < L$

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right) = \sum_{n=-\infty}^{\infty} c_n e^{jn\pi x/L}$$

The coefficients are given by:

$$\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx \qquad && n \geq 0 \\
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx \qquad && n > 0 \\
c_n &= \frac{1}{2}(a_n - jb_n) && n > 0
\end{aligned}$$

1.2 Fourier Transform

If $h(t)$ is a periodic function then the Fourier transform is given by:

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\, dt$$

Inverse Fourier tranformation of $H(\omega)$:

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{j\omega t}\, d\omega$$

1.3 Examples

1.3.1 Example 1: Fourier Series

Find the Fourier coefficients and Fourier Series for the square wave shown below:

$$f(t) = \begin{cases} 1 & \text{for } t - a > 0 \\ 0 & \text{for } t - a < 0 \end{cases}$$

and

$$f(x+2) = f(x)$$

Ans:

$$f(x) = \frac{1}{2} + \sum_{n=1,3,5,\ldots} \frac{2}{n\pi}\sin(n\pi x)$$

1.3.2 Example 2: Fourier Transform

The unit step function is defined as:

$$u(t-a) = \begin{cases} 1 & \text{for } t - a > 0 \\ 0 & \text{for } t - a < 0 \end{cases}$$

is used to define the rectangular pulse function:

$$x(t) = u(t-a) - u(t-b) \qquad \text{where } a < b$$

Ans:

$$X(\omega) = \frac{e^{-j\omega a} - e^{-j\omega b}}{j\omega}$$

1.3.3 Example 3: Inverse Fourier Transform

Consider the signal: $x(t) = \sin(@omega_0 t)$ where ω_0 is a constant. Find the Fourier Transform of this signal to find $X(\omega)$.

Ans:

$$X(\omega) = -j\pi(\delta(\omega - \omega_0) - \delta(\omega + \omega_0))$$

2 Laplace

Is a generalisation of the Fourier transform and defined as:

$$H(s) = \mathcal{L}\{h(t)\} = \int_{0^-}^{\infty} h(t)e^{-st}\,dt \qquad s \in \mathbb{C}$$

s is a complex number $s = \sigma + j\omega$ and is identical with Fourier transform, if s is set to $j\omega$.
Inverse Laplace tranformation:

$$f(t) = \frac{1}{2\pi j}\int_{\sigma_c - j\infty}^{\sigma_c + j\infty} F(s)e^{st}\,ds$$

2.1 Examples

2.1.1 Example 1: Laplace Transform

Using the Laplace transform, find the solution for the following equation:

$$\frac{\partial^2}{\partial t^2}y(t) + 2\frac{\partial}{\partial t}y(t) + 2y(t) = 0$$

with initial conditions $y(0) = 1$ and $Dy(0) = -1$

2.1.2 Example 2: Trabsfer Function

Consider a mass-spring-damper system with the following differential equation:

$$m\ddot{x} = -kx - b\dot{x} + f$$

Find the transfer function for the system with input f and output x.

Ans:

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

3 Several-Variables

3.1 Examples

3.1.1 Example 1: Gradient

Find the rate of change of $f(x, y) = y^4 + 2xy^3 + x^2y^2$ at $(0, 1)$ in each of the following directions:

1. $\mathbf{i} + 2\mathbf{j}$
2. $\mathbf{j} - 2\mathbf{i}$
3. $3\mathbf{i}$
4. $\mathbf{i} + \mathbf{j}$

3.1.2 Example 2: Jacobian

Find the Jacobian $Df(1, 0)$ for the transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ given by:

$$\mathbf{f}(x, y) = (xe^y + \cos(\pi y)\,, x^2, x - e^y)$$

3.1.3 Example 3: Chain Rule

If $z = \sin(x^2y)$ where $x = st^2$ and $y = s^2 + \frac{1}{t}$, $y = e^{-t}$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ using chain rule.

3.1.4 Example 4: Substitution

If $z = \sin(x^2y)$ where $x = st^2$ and $y = s^2 + \frac{1}{t}$, $y = e^{-t}$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ using substitution.

3.1.5 Example 5: Partial Differentiation

Calculate $f_{223}(x, y, z)$, $f_{232}(x, y, z)$, and $f_{322}(x, y, z)$ of the following function:

$$f(x, y, z) = e^{x-2y+3z}$$

4 Double-Integrals

4.1 Riemann Sum

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta A_i$$

$$I = \iint_D f(x,y)dA \qquad \text{where } D \text{ is a region in } \mathbb{R}^2 \text{ and } dA \text{ is } dxdy$$

4.2 Double Integrals over General domains

If $f(x,y)$ is defined and bounded on domain D, then $\hat{f}(x,y)$ is zero outside D.

$$\iint_D f(x,y)\,dA = \iint_R \hat{f}(x,y)\,dA$$

4.3 Iteration of Double Integrals

If $f(x,y)$ is continuous on the bounded y-simple domain D given by $a \le x \le b$ and $c(x) \le y \le d(x)$, then:

$$\iint f(x,y)\,dA = \int_a^b dx \int_{c(x)}^{d(x)} f(x,y)dy$$

If $f(x,y)$ is continuous on the bounded x-simple domain D given by $c \le x \le d$ and $a(x) \le y \le b(x)$, then:

$$\iint f(x,y)\,dA = \int_c^d dx \int_{a(x)}^{b(x)} f(x,y)dy$$

4.4 Double Integrals in Polar Coordinates

$$dA = dxdy = r\ drd\theta$$

$$x = r\cos(\theta) \qquad r^2 = x^2 + y^2$$
$$y = r\sin(\theta) \qquad \tan(\theta) = \tfrac{y}{x}$$

4.4.1 Limits for Polar Coordinates

r is the radius from origin to the point

$$r \ge 0$$

θ is the angle in the positive direction of the xy-plane

$$0 \le \theta \le 2\pi$$

4.5 Change of Variables in Double Integrals

If x and y are given as a function of u and v:

$$x = x(u,v)$$
$$y = y(u,v)$$

These can be transformed or mapped from points (u, v) in the uv-plane to points (x, y) in the xy-plane.
The inverse transformation is given by:

$$u = u(x, y)$$
$$v = v(x, y)$$

Scaled area element:

$$dA = dxdy = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dudv$$

where the Jacobian is:

$$\left|\frac{\partial(x, y)}{\partial(u, v)}\right| = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Let $x(u, v)$ and $y(u, v)$ be a one-to-one transformation from a domain S in the uv-plane onto a domain D xy-plane.
Suppose, that function x and y, and first partial derivatives with respect to u and v are continuous in S. If $f(x, y)$ is integrable on D, then $g(u, v) = f(x(u, v), y(u, v))$ is integrable on S and:

$$\iint_D f(x, y)dA = \iint_S g(u, v)\left|\frac{\partial(x, y)}{\partial(u, v)}\right| dudv$$

4.6 Examples

4.6.1 Example 1: Change of Variables

Evaluate the double integral:

$$\iint (x - 3y)dA$$

where R is triangular region with vertices $(0, 0)$, $(2, 1)$, and $(1, 2)$ using the transformation:

$$x = 2u + v$$
$$y = u + 2v$$

As x and y are dependent on u and v the transformation of dA is given by

$$dA = dxdy = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dudv$$

Using:

$$\left|\frac{\partial(x, y)}{\partial(u, v)}\right| = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

the Jacobian can be calculated:

$$\frac{\partial x}{\partial u} = 2 \quad \frac{\partial y}{\partial v} = 2$$
$$\frac{\partial x}{\partial v} = 1 \quad \frac{\partial y}{\partial u} = 1$$
$$2 * 2 - 1 * 1 = 3$$

Find the boundaries for the double integral
$y_1 : (0, 0) \to (2, 1)$ is $y = \frac{1}{2}x$
$y_2 : (0, 0) \to (1, 2)$ is $y = 2x$
$y_3 : (1, 2) \to (2, 1)$ is $y = 3 - x$

Replace x and y with their transformation:
$u + 2v = \frac{2u+v}{2} \qquad v = 0$
$u + 2v = 4u + 2v \qquad u = 0$
$u + 2v = 3 - 2u - v \qquad u = 1 - v$

Therefore $0 \leq u \leq 1 - v$ and $0 \leq v \leq 1$
Now transform the original function $x - 3y$:

$$\begin{aligned} x - 3y &= 2u + v - 3(u + 2v) \\ &= 2u - 3u + v - 6v \\ &= -u - 5v \end{aligned}$$

$$\begin{aligned} \int_0^1 \int_0^{1-v} (-u - 5v) \cdot 3 \; dudv &= -3 \int_0^1 \left[\frac{u^2}{2} + 5uv \right]_0^{1-v} dv \\ &= -\frac{3}{2} \int_0^1 \left(\frac{27v^2}{2} - 12v \right) dv \\ &= \left[\frac{9v^3}{2} - 6v^2 - \frac{3v}{2} \right]_0^1 \\ &= \boxed{-3} \end{aligned}$$

4.6.2 Example 2: Double integral

Evaluate the double integral by iteration

$$\iint_R (x^2 + y^2) \, dA$$

where R is the rectangle $0 \leq x \leq a$, $0 \leq y \leq b$

Insert the limits and solve the integral:

$$\begin{aligned} \int_0^b \int_0^a (x^2 + y^2) dxdy &= \int_0^b \left[\frac{x^3}{3} + xy^2 \right]_0^a dy \\ &= \int_0^b \left(\frac{a^3}{3} + ay^2 \right) dy \\ &= \boxed{\frac{a^3 b}{3} + \frac{ab^3}{3}} \end{aligned}$$

4.6.3 Example 3: By iteration

Evaluate the double integral by iteration:

$$\iint_D x \cos y \, dA$$

where D is the finite region in the first quadrant bounded by the coordinate axes and the curve $y = 1 - x^2$.

Given the region the minimum for x and y must be 0 and for x the maximum is 1:

$$
\begin{aligned}
\int_0^1 \int_0^{1-x^2} (x \cos(y)) dy dx &= \int_0^1 x \sin\left(1 - x^2\right) dx \\
&= \left[\frac{1}{2} \sin(1) \sin\left(x^2\right) + \frac{1}{2} \cos(1) \cos\left(x^2\right)\right]_0^1 \\
&= \boxed{\sin^2\left(\frac{1}{2}\right)}
\end{aligned}
$$

4.6.4 Example 4: Polar coordinates

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5 Tripple-Integrals

5.1 Riemann Sum

5.2 Tripple Integrals over General domains

5.3 Change of Variables

$$dV = dx\ dy\ dz = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\ dv\ dw$$

$$\iiint f(x,y,z)\, dx\ dy\ dz = \iiint g(u,v,w) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\ dv\ dw$$

5.4 Cylyndrical Coordinates

$$\begin{aligned} x &= r\cos(\theta) \quad r^2 = x^2 + y^2 \\ y &= r\sin(\theta) \quad \tan\theta = (\tfrac{y}{x}) \\ z &= z \end{aligned}$$

From the Jacobian using change of variables:

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = r$$

$$dV = dx\ dy\ dz = r\ dr\ d\theta\ dz$$

5.4.1 Limits for Cylyndrical Coordinates

r is the radius from origin to the point

$$r \geq 0$$

θ is the angle in the positive direction of the x-axis

$$0 \leq \theta \leq 2\pi$$

5.5 Spherical Coordinates

$$\begin{aligned} x &= \rho\sin(\phi)\cos(\theta) \quad \rho^2 = x^2 + y^2 + z^2 \\ y &= \rho\sin(\phi)\sin(\theta) \quad \cos(\phi) = \tfrac{z}{\rho} \\ z &= \rho\cos(\phi) \qquad\qquad \tan\theta = \tfrac{y}{x} \end{aligned}$$

$$dxdydz = dV = \rho^2\sin(\phi)d\rho d\phi d\theta$$

5.5.1 Limits for Spherical Coordinates

ρ is the distance from the origin O to the point P

$$\rho \geq 0$$

ϕ is the angle by the radial line OP to the positive direction of the z-axis

$$0 \leq \phi \leq \pi$$

θ is the angle in the positive direction of the x-axis to the point P in the xy-plane

$$0 \leq \theta \leq 2\pi$$

5.6 Examples

5.6.1 Example 1: Tripple Integral

Find $\iiint(x^2+y^2+z^2)dV$, where the region is bounded by $z=c\sqrt{(x^2+y^2)}$ and $x^2+y^2+z^2=a^2$

5.6.2 Example 2: Sphereical Coordinates

Find the volume of:

$$\iiint \sqrt{x^2+y^2+z^2}dxdydz$$

where the region is bounded by $x^2+y^2+z^2 \leq 1$ in spherical domain.

Transform the region:

$$\rho^2 = x^2+y^2+z^2 \leq 1 \qquad \text{Therefore: } 0 \leq \rho \leq 1$$

For θ and ϕ we have:

$$0 \leq \theta \leq 2\pi \qquad 0 \leq \phi \leq \pi$$

and for dV:

$$dV = dxdydz = \rho^2 \sin\phi d\rho d\phi d\theta$$

Transform the function:

$$\sqrt{x^2+y^2+z^2} = \sqrt{\rho^2} = \rho$$

Solve the integral:

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho\cdot\rho^2\sin(\phi)d\rho d\phi d\theta &= \int_0^{2\pi}\int_0^{\pi}\left[\frac{1}{4}p^4\sin(\phi)\right]_0^1 d\phi d\theta \\
&= \int_0^{2\pi}\int_0^{\pi}\frac{\sin(\phi)}{4}d\phi d\theta \\
&= \int_0^{2\pi}\left[-\frac{\cos(\phi)}{4}\right]_0^{\pi} d\theta \\
&= \frac{1}{2}\int_0^{2\pi} d\theta \\
&= \frac{1}{2}\cdot 2\pi \\
&= \boxed{\pi}
\end{aligned}$$

5.6.3 Example 3: Transformations

6 Fields-Curve

6.1 Curve & Parameterization

Representation of a curve in 3 space by using its position vector is given as:

$$r = r(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \qquad \text{where } a \leq t \leq b$$

6.2 Vector Fields

$$\mathbf{F}(x,y,z) = \underbrace{f_x(x,y,z)}_{\text{Scaler function}}\ \mathbf{i} + \underbrace{f_y(x,y,z)}_{\text{Scaler function}}\ \mathbf{j} + \underbrace{f_z(x,y,z)}_{\text{Scaler function}}\ \mathbf{k}$$

$$\frac{\partial f}{\partial x} = f_1 = f_x \qquad \tfrac{\partial f}{\partial y} = f_2 = f_y \qquad \frac{\partial f}{\partial z} = f_3 = f_z$$

Position vector:

$$r = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Unit vector with magnitude 1:

$$r = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

6.2.1 Scalar field

$$f(x,y,z) = f_1(x,y,z) = f_2(x,y,z) = f_3(x,y,z)$$

The gradient of a scalar field is a vector field:

$$\nabla f = \text{grad } f(x,y,z) = f_x(x,y,z)\mathbf{i} + f_y(x,y,z)\mathbf{j} + f_z(x,y,z)\mathbf{k}$$

6.2.2 Field lines

6.2.3 Convervation field

If $\mathbf{F}(x,y,z) = \nabla\phi(x,y,z)$ in a domain D, then $\mathbf{F}$ is a conservative vector field in D and function ϕ is the potential function.

$$\mathbf{F}(x,y,z) = \nabla\phi(x,y,z) = \phi_x(x,y,z)\mathbf{i} + \phi_y(x,y,z)\mathbf{j} + \phi_z(x,y,z)\mathbf{k}$$

If the vetor field is conservative, then all the following equations are true:

$$\begin{aligned}
\frac{\partial f_x}{\partial y} &= \frac{\partial f_y}{\partial x} \\
\frac{\partial f_x}{\partial z} &= \frac{\partial f_z}{\partial x} \\
\frac{\partial f_y}{\partial z} &= \frac{\partial f_z}{\partial y}
\end{aligned}$$

6.2.4 Vector field in Polar Coordinates

$$\mathbf{F} = f(r,\theta) = f_r(r,\theta)\hat{\mathbf{r}} + f_\theta(r,\theta)\hat{\theta}$$

where:

$$\hat{\mathbf{r}} = \cos(\theta)i + \sin(\theta)j$$

$$\hat{\theta} = -\sin(\theta)i + \cos(\theta)j$$

6.3 Line Integral

$$f(x,y)ds = \text{Area (tiny point)}$$

$$\text{Length of } \mathcal{C} = \int_{\mathcal{C}} f(x,y,z)\,ds = \int_a^b f(r(t)) \left|\frac{dr}{dt}\right| dt$$

6.3.1 Line integral of a vector field

$$W = \int_{\mathcal{C}} F.\hat{T}\,ds = \int F\,dr = \int_{\mathcal{C}} f_1(x,y,z)dx + f_2(x,y,z)dy + f_3(x,y,z)dz$$

6.4 Examples

6.4.1 Example 1: Conservative vector field and potential

Determine whether the given vector field is conservative, and find a potential function if it is:

$$\mathbf{F}(x,y,z) = (2xy - z^2)\mathbf{i} + (2yz + x^2)\mathbf{j} - (2zx - y^2)\mathbf{k}$$

6.4.2 Example 2: Line integral

Evaluate $\oint x^2y^2\ dx + x^3y\ dy$ counterclockwise around the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$

6.4.3 Example 3: Line integral

Evalute the line integral for $f(x,y) = x^2y^2$ along a straight line from origin to the point $(2,1)$

Ans: $(5\sqrt{5})/3$

6.4.4 Example 4: Line integral vector field

Evaluate the line integral for $\mathbf{F}(x,y) = y^2\mathbf{i} + 2xy\mathbf{j}$ from $(0,0)$ to $(1,1)$
Along the line $y = x$
Allong the line $y = x^2$

6.4.5 Example 5: Gradient

6.4.6 Example 6: Parametrize a curve

Use $t = y$ to parametrize the part of the line of intersection of the two planes:
Plane 1: $y = 2x - 4$
Plane 2: $z = 3x + 1$ from $(2,0,7)$ to $(3,2,10)$

Ans: $r(t) = \left(\frac{1}{2}(t+4)\right)\mathbf{i} + (t)\mathbf{j} + \left(\frac{3}{2}t + 7\right)\mathbf{k}$

7 Surface-Integrals

7.1 Parametric Surface

For curve parametrization:

$$r = r(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \qquad \text{where } a \leq t \leq b$$

For surface parametrization:

$$r = r(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k} \qquad \text{where } a \leq u \leq b, \quad c \leq v \leq d$$

7.2 Surface Area

For a surface the area is given by:

$$\iint_S f(x,y,z)dS$$

$$dS = \left|\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right| dudv = \sqrt{\left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2} dudv$$

For a parametrized surface S given by $r = r(u,v)$, where (u,v) is in the domain D in the uv-plane, the surface area is given by:

$$\iint_S f\ dS = \iint_D f(r(u,v)) \left|\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right| dudv$$

$$= \iint_b f(x(u,v), y(u,v), z(u,v)) \sqrt{\left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2}\, du\, dv$$

For a surface S given by $z = g(x,y)$, where (x,y) is in the domain D in the xy-plane, the surface area is given by:

$$\iint_S f(x,y,z)dS = \iint_D f(x,y,z(x,y)) \sqrt{1 + \left(\frac{\partial g(x,y)}{\partial x}\right)^2 + \left(\frac{\partial g(x,y)}{\partial y}\right)^2} dxdy$$

The projection of normal vector onto the xy-plane is given by:

$$\cos(\gamma) = \frac{1}{\sqrt{1 + \left(\frac{\partial g(x,y)}{\partial x}\right)^2 + \left(\frac{\partial g(x,y)}{\partial y}\right)^2}} \qquad \text{hence } dS = \frac{1}{\cos(\gamma)} dxdy$$

7.3 Oriented Surface

- A smooth surface S in 3-space is said to be orientable if there exists a unit vector field $\widehat{N}(P)$.
- $\widehat{N}(P)$ defined on S that varies continuously as P ranges over S and that is everywhere normal to S.
- Any such vector field $\widehat{N}(P)$ determines an orientation of S.
- The oriented surface must have two sides.
- $\widehat{N}(P)$ can have only one value at each point P with two sides.

7.4 Flux

$$\mathbf{F} = f_1(x,y,z)\mathbf{i} + f_2(x,y,z)\mathbf{j} + f_3(x,y,z)\mathbf{k}$$

Given any continuous vector field $\mathbf{F}$, flux of $\mathbf{F}$ across the orientable surface S is integral of the normal component of $\mathbf{F}$ over S

$$\iint_S \mathbf{F} \cdot dS = \iint_S (\mathbf{F} \cdot \widehat{\mathbf{N}}) dS$$

If the surface is closed, then the flux is given by:

$$\oiint_S \mathbf{F} \cdot dS = \oiint_S (\mathbf{F} \cdot \widehat{\mathbf{N}}) dS$$

If S is a parametrized surface given by $r = r(u,v)$, where (u,v) is in the domain D in the uv-plane, then the flux is given by:

$$\iint_S \mathbf{F} \cdot dS = \iint_B \mathbf{F} \cdot \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right) du\, dv$$

$$= \iint_D \left(f_1 \frac{\partial(y,z)}{\partial(u,v)} + f_2 \frac{\partial(z,x)}{\partial(u,v)} + f_3 \frac{\partial(x,y)}{\partial(u,v)} \right) du\ dv$$

For a surface S given by $z = g(x,y)$, where (x,y) is in the domain D in the xy-plane, the flux is given by:

$$\iint_S \mathbf{F} \cdot dS = \iint_D \left(-f_1 \frac{\partial z}{\partial x}\mathbf{i} - f_2 \frac{\partial z}{\partial y}\mathbf{j} + f_3 \mathbf{k} \right) dx\, dy$$

7.5 Examples

8 Theorems

8.1 Differential Operators

8.1.1 Gradient

The gradient of a scalar field is a vector field that points in the direction of the steepest increase of the scalar field.

$$\text{grad } f(x,y,z) = \nabla f(x,y,z) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

$$\mathbf{F}(x,y,z) = f_x(x,y,z)\mathbf{i} + f_y(x,y,z)\mathbf{j} + f_z(x,y,z)\mathbf{k}$$

8.1.2 Divergence

The divergence of a velocity field represents the net flow of fluid out of a small volume in a scalar field.

$$\text{div } \mathbf{F}(x,y,z) = \nabla \cdot \mathbf{F}(x,y,z) = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$$

8.1.3 Curl

The curl or field circulation of the electric field gives the rate of change of the magnetic field.

$$\text{curl } \mathbf{F}(x,y,z) = \nabla \times \mathbf{F}(x,y,z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

$$= \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)\mathbf{k}$$

8.2 Green's Theorem

Let R be a regular, closed region in the xy-plane whose boundary, C, consists of one or more piecewise smooth, simple closed curves that are positively oriented with respect to R.

$$\oint_C f_1(x,y)dx + f_2(x,y)dy = \iint_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dA$$

8.3 Stokes' Theorem

Let S be a piecewise smooth, oriented surface in 3-space, having unit normal field $\widehat{N}$ and boundary C consisting of one or more piecewise smooth, closed curves with orientation inherited from S.

$$\oint_C F \cdot dr = \iint_S \text{curl } F \cdot \widehat{N} dS$$

8.4 Divergence Theorem

Let S, be a closed piecewise smooth surface, which is the boundary of V with normal $\widehat{N}$ pointing outwards.

$$\oiint_s (F \cdot \widehat{N}) dS = \iiint_V \text{div } F dV$$

More variants:

$$\iiint_D \text{curl } F dV = -\oiint_s (F \times \widehat{N}) dS$$

$$\iiint_D \text{grad } \phi \, dV = \oiint_s \phi \, dS$$

8.5 Examples

8.5.1 Example 1: Div and Curl

Calculate the divergence and curl of the following vector field:

$$\mathbf{F} = \cos x\,\mathbf{i} - \sin y\,\mathbf{j} + z\,\mathbf{k}.$$

Divergence: Curl:

8.5.2 Example 2: Green's Theorem

Using Green's Theorem evaluate $\oint_e (x^2y)\,dx + (xy^2)dy$, clockwise boundary of the region:

$$0 \leq y \leq \sqrt{9 - x^2}$$

8.5.3 Example 3: Stokes' Theorem

8.5.4 Example 4: Divergence Theorem

Use the Divergence Theorem to calculate the flux of the given vector field out of the sphere s with equation $x^2 + y^2 + z^2 = a^2$, where $a > 0$ and

$$\mathbf{F} = (x^2 + y^2)\mathbf{i} + (y^2 - z^2)\mathbf{j} + z\mathbf{k}$$

9 PDE

Partial Differential Equations are equations with multiple variables and derivatives. They are used to model many physical phenomena, such as heat, sound, and light. The totality of solutions to a PDE is called its general solution, and there can be a lot.

9.1 Classification of PDEs

General representation of a PDE:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

Conditions:

Linear: A, B, C, D, E, F are only function of x,y variables, not u.

Quasi-linear: A, B, C, D, E, F may be function of (x, y, u, u_x, u_y)

Fully non-linear: A, B, C, D, E, F may be function of $(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy})$

9.2 Characteristics of PDEs

$B^2 - 4AC > 0$	2 real roots	2 characteristics	**Hyperbolic PDE**
$B^2 - 4AC = 0$	1 real roots	1 characteristics	**Parabolic PDE**
$B^2 - 4AC < 0$	0 real roots	0 characteristics	**Elliptic PDE**

Tyoes of varius PDEs:

Wave Equation: Hyperbolic PDE

Heat Equation: Parabolic PDE

Laplace Equation: Elliptic PDE

$B^2 - 4AC > 0$	2 real roots	2 characteristics	**Hyperbolic PDE**
$B^2 - 4AC = 0$	1 real roots	1 characteristics	**Parabolic PDE**
$B^2 - 4AC < 0$	0 real roots	0 characteristics	**Elliptic PDE**

9.3 Important Second-Order PDEs

$\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$	One-dimensional wave equation
$\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$	One-dimensional heat equation
$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$	Two-dimensional Laplace equation
$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f(x, y)$	Two-dimensional Poisson equation
$\dfrac{\partial^2 u}{\partial t^2} = c^2 \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} \right)$	Two-dimensional wave equation
$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$	Three-dimensional Laplace equation

9.4 Initial and Boundary Conditions

9.5 Wave Equation (1D)

One dimensional wave equation is given by:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{where } c^2 = \left[\frac{T(x,t)}{\mu_x} \right] \tag{1}$$

With two boundary conditions $x = 0$ and $x = L$:

$$u(0, t) = 0 \qquad u(L, t) = 0 \qquad \text{For all } t \geq 0$$

And two initial conditions, initial displacement and initial velocity at time $t = 0$:

$$u(x, 0) = f(x) \qquad \frac{\partial u}{\partial t}(x, 0) = g(x) \qquad \text{For all } 0 \leq x \leq L$$

Steps to solve:

1. Method of Separation of Variables $\quad u(x, t) = X(x)T(t)$
2. Satisfy the Boundary Conditions test
3. Fourier Series Validation

9.5.1 D'Alembert's Solution of the Wave Equation

His solution is given by eq. (1) but extended to two variables:

$$v = (x - ct) \qquad w = (x + ct) \tag{2}$$

I.e. $u(v, w)$. Partial derivatives from chain rule:

$$u_x = u_v \cdot v_x + u_w \cdot w_x = u_v + u_w$$

For double derivatives:

$$u_{xx} = (u_v + u_w)_x = (u_v + u_w)_v v_x + (u_v + u_w)_w w_x = u_{vv} + 2u_{vw} + u_{ww}$$

With respect to t:

$$u_{tt} = c^2 u_{xx} = c^2(u_{vv} + 2u_{vw} + u_{ww})$$

From eq. (1) and eq. (2):

$$u_{vw} = \frac{\partial^2 u}{\partial w \partial v} = 0$$

This can be solved by integrating with respect to v and w:

$$\frac{\partial u}{\partial v} = h(v) \text{ and } u = \int h(v)\ dv + \psi(w)$$

Here, $h(v)$ and $\psi(w)$ are arbitrary functions of v and w, respectively. The solution in term for x:

$$u = \phi(v) + \psi(w)$$

This is d'Alembert's solution, which is the general solution to the wave equation.

$$u(x,t) = \phi(x + ct) + \psi(x - ct)$$

This solution satisfies the wave equation and the initial conditions:

9.6 Heat Equation (1D)

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{3}$$

Conditions:

- PDE is linear and homogeneous.
- Boundary conditions are linear and homogeneous.The two for $u(x,t)$ is $u(0,t) = 0$ and $u(L,t) = 0$ for all $t > 0$.
- One initial condition at time ($t = 0$): $u(x,0) = f(x)$.

Solve the

Solution:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda_n^2 t}$$

where

$$\lambda_n = \frac{cn\pi}{L}$$

and

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \qquad \text{for } n = 1, 2, 3 \ldots$$

9.7 Examples

9.7.1