Multi Variable Calculus

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Last updated: December 22, 2023

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1 Fourier

2 Laplace

2.1 Examples

2.1.1 Example 1: Laplace Transform

Using the Laplace transform, find the solution for the following equation:

$$\frac{\partial^2}{\partial t^2}y(t) + 2\frac{\partial}{\partial t}y(t) + 2y(t) = 0$$

with initial conditions $y(0) = 1$ and $Dy(0) = -1$

3 Several-Variables

4 Double-Integrals

4.1 Riemann Sum

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta A_i$$

$$I = \iint_D f(x,y)dA \qquad \text{where } D \text{ is a region in } \mathbb{R}^2 \text{ and } dA \text{ is } dxdy$$

4.2 Double Integrals over General domains

If $f(x,y)$ is defined and bounded on domain D, then $\hat{f}(x,y)$ is zero outside D.

$$\iint_D f(x,y)\, dA = \iint_R \hat{f}(x,y)\, dA$$

4.3 Iteration of Double Integrals

4.4 Double Integrals in Polar Coordinates

$$dA = dxdy = r\ drd\theta$$

$$\begin{array}{ll} x = r\cos(\theta) & r^2 = x^2 + y^2 \\ y = r\sin(\theta) & \tan(\theta) = \frac{y}{x} \end{array}$$

4.5 Change of Variables in Double Integrals

If x and y are given as a function of u and v:

$$\begin{array}{l} x = x(u,v) \\ y = y(u,v) \end{array}$$

These can be transformed or mapped from points (u,v) in the uv-plane to points (x,y) in the xy-plane.
The inverse transformation is given by:

$$\begin{array}{l} u = u(x,y) \\ v = v(x,y) \end{array}$$

Scaled area element:

$$dA = dxdy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

where the Jacobian is:

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Let $x(u,v)$ and $y(u,v)$ be a one-to-one transformation from a domain S in the uv-plane onto a domain D xy-plane.
Suppose, that function x and y, and first partial derivatives with respect to u and v are continuous in S. If $f(x,y)$ is integrable on D, then $g(u,v) = f(x(u,v), y(u,v))$ is integrable on S and:

$$\iint_D f(x,y)dA = \iint_S g(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

4.6 Examples

4.6.1 Example 1: Change of Variables

Evaluate the double integral:

$$\iint (x - 3y)dA$$

where R is triangular region with vertices $(0, 0)$, $(2, 1)$, and $(1, 2)$ using the transformation:

$$\begin{aligned} x &= 2u + v \\ y &= u + 2v \end{aligned}$$

4.6.2 Example 2: Riemann Sum

4.6.3 Example 3: By iteration

4.6.4 Example 4: Double integral

Find the area within the region:

$$\iint (\sin x + \cos y)dA \qquad R : \left\{ x, y \,\middle|\, \begin{array}{l} 0 \leq x \leq \pi/2 \\ 0 \leq y \leq \pi/2 \end{array} \right.$$

5 Tripple-Integrals

5.1 Examples

5.1.1 Example 1: Tripple Integral

Find $\iiint(x^2+y^2+z^2)dV$, where the region is bounded by $z = c\sqrt{(x^2+y^2)}$ and $x^2+y^2+z^2=a^2$

6 Fields-Curve

6.1 Examples

6.1.1 Example 1: Conservative vector field and potential

Determine whether the given vector field is conservative, and find a potential function if it is:

$$\mathbf{F}(x, y, z) = (2xy - z^2)\mathbf{i} + (2yz + x^2)\mathbf{j} - (2zx - y^2)\mathbf{k}$$

6.1.2 Example 2: Line integral

Evaluate $\oint x^2y^2\ dx + x^3y\ dy$ counterclockwise around the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$

7 Theorems

7.1 Examples

7.1.1 Example 1: Div and Curl

Calculate the divergence and curl of the following vector field:

$$\mathbf{F} = \cos x\,\mathbf{i} - \sin y\,\mathbf{j} + z\,\mathbf{k}.$$

Divergence: Curl:

7.1.2 Example 2: Green's Theorem

Using Green's Theorem evaluate $\oint_e (x^2 y)\,dx + (xy^2)dy$, clockwise boundary of the region:

$$0 \leq y \leq \sqrt{9 - x^2}$$

7.1.3 Example 3: Stokes' Theorem

7.1.4 Example 4: Divergence Theorem

Use the Divergence Theorem to calculate the flux of the given vector field out of the sphere s with equation $x^2 + y^2 + z^2 = a^2$, where $a > 0$ and

$$\mathbf{F} = (x^2 + y^2)\mathbf{i} + (y^2 - z^2)\mathbf{j} + z\mathbf{k}$$

8 PDE

Partial Differential Equations are equations with multiple variables and derivatives. They are used to model many physical phenomena, such as heat, sound, and light. The totality of solutions to a PDE is called its general solution, and there can be a lot.

8.1 Classification of PDEs

General representation of a PDE:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

Conditions:

Linear: A, B, C, D, E, F are constants
Quasi-linear: A, B, C are constants
Fully non-linear: A, B, C, D, E, F are functions of u and its partial derivatives

8.2 Characteristics of PDEs

$B^2 - 4AC > 0$	2 real roots	2 characteristics	**Hyperbolic PDE**
$B^2 - 4AC = 0$	1 real roots	1 characteristics	**Parabolic PDE**
$B^2 - 4AC < 0$	0 real roots	0 characteristics	**Elliptic PDE**

Tyoes of varius PDEs:
Wave Equation: Hyperbolic PDE
Heat Equation: Parabolic PDE
Laplace Equation: Elliptic PDE

8.3 Initial and Boundary Conditions

8.4 Wave Equation (1D)

One dimensional wave equation is given by:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{where } c^2 = \left[\frac{T(x,t)}{\mu_x}\right] \tag{1}$$

Steps to solve:

1. Method of Separation of Variables $\quad u(x,t) = X(x)T(t)$
2. Satify the Boundary Conditions test
3. Fourier Series Validation

8.4.1 D'Alembert's Solution of the Wave Equation

His solution is given by eq. (1) but extended to two variables:

$$v = (x - ct) \qquad\qquad w = (x + ct) \tag{2}$$

I.e. $u(v, w)$. Partial derivatives from chain rule:

$$u_x = u_v \cdot v_x + u_w \cdot w_x = u_v + u_w$$

For double derivatives:

$$u_{xx} = (u_v + u_w)_x = (u_v + u_w)_v v_x + (u_v + u_w)_w w_x = u_{vv} + 2u_{vw} + u_{ww}$$

With respect to t:

$$u_{tt} = c^2 u_{xx} = c^2(u_{vv} + 2u_{vw} + u_{ww})$$

From eq. (1) and eq. (2):

$$u_{vw} = \frac{\partial^2 u}{\partial w \partial v} = 0$$

This can be solved by integrating with respect to v and w:

$$\frac{\partial u}{\partial v} = h(v) \text{ and } u = \int h(v)\ dv + \psi(w)$$

Here, $h(v)$ and $\psi(w)$ are arbitrary functions of v and w, respectively. The solution in term for x:

$$u = \phi(v) + \psi(w)$$

This is d'Alembert's solution, which is the general solution to the wave equation.

$$u(x,t) = \phi(x + ct) + \psi(x - ct)$$

This solution satisfies the wave equation and the initial conditions:

8.5 Heat Equation (1D)

$$\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2} \tag{3}$$

Conditions:

- PDE is linear and homogeneous.
- Boundary conditions are linear and homogeneous.The two for $u(x,t)$ is $u(0,t) = 0$ and $u(L,t) = 0$ for all $t > 0$.
- One initial condition at time ($t = 0$): $u(x,0) = f(x)$.

Solve the
Solution:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda_n^2 t}$$

where

$$\lambda_n = \frac{cn\pi}{L}$$

and

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx \qquad \text{for } n = 1, 2, 3 \ldots$$

8.6 Examples