Multi Variable Calculus

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Contents

1 Fourier **3**
1.1 Fourier Series 3
1.2 Fourier Transform 3
1.3 Examples 3
1.3.1 Example 1: Fourier Series 3
1.3.2 Example 2: Fourier Transform 3
1.3.3 Example 3: Inverse Fourier Transform 4

2 Laplace **5**
2.1 Examples 5
2.1.1 Example 1: Laplace Transform 5
2.1.2 Example 2: Trabsfer Function 5

3 Several-Variables **6**
3.1 Examples 6
3.1.1 Example 1: Gradient 6
3.1.2 Example 2: Jacobian 6
3.1.3 Example 3: Chain Rule 6
3.1.4 Example 4: Substitution 6
3.1.5 Example 5: Partial Differentiation 6

4 Double-Integrals **7**
4.1 Riemann Sum 7
4.2 Double Integrals over General domains 7
4.3 Iteration of Double Integrals 7
4.4 Double Integrals in Polar Coordinates 7
4.4.1 Limits for Polar Coordinates 7
4.5 Change of Variables in Double Integrals 7
4.6 Examples 8
4.6.1 Example 1: Change of Variables 8
4.6.2 Example 2: Double integral 9
4.6.3 Example 3: By iteration 9
4.6.4 Example 4: Polar coordinates 9

5 Tripple-Integrals **10**
5.1 Riemann Sum 10
5.2 Tripple Integrals over General domains 10
5.3 Change of Variables 10
5.4 Cylyndrical Coordinates 10
5.4.1 Limits for Cylyndrical Coordinates 10
5.5 Spherical Coordinates 10

5.5.1 Limits for Spherical Coordinates 10
5.6 Examples 11
5.6.1 Example 1: Tripple Integral 11
5.6.2 Example 2: Sphereical Coordinates 11
5.6.3 Example 3: Transformations 11

6 Fields-Curve **12**
6.1 Curve & Parameterization 12
6.2 Vector Fields 12
6.2.1 Scalar field 12
6.2.2 Field lines 12
6.2.3 Convervation field 12
6.2.4 Vector field in Polar Coordinates 12
6.3 Line Integral 13
6.3.1 Line integral of a vector field 13
6.4 Examples 13
6.4.1 Example 1: Conservative vector field and potential 13
6.4.2 Example 2: Line integral 13
6.4.3 Example 3: Line integral 13
6.4.4 Example 4: Line integral vector field 13
6.4.5 Example 5: Gradient 13
6.4.6 Example 6: Parametrize a curve 13

7 Theorems **14**
7.1 Green's Theorem 14
7.2 Stokes' Theorem 14
7.3 Divergence Theorem 14
7.4 Examples 14
7.4.1 Example 1: Div and Curl 14
7.4.2 Example 2: Green's Theorem 14
7.4.3 Example 3: Stokes' Theorem 14
7.4.4 Example 4: Divergence Theorem 14

8 PDE **15**
8.1 Classification of PDEs 15
8.2 Characteristics of PDEs 15
8.3 Important Second-Order PDEs 15
8.4 Initial and Boundary Conditions 16
8.5 Wave Equation (1D) 16
8.5.1 D'Alembert's Solution of the Wave Equation 16
8.6 Heat Equation (1D) 17
8.7 Examples 17
8.7.1 17

1 Fourier

1.1 Fourier Series

A periodic function with period $2L$ and let $f(x)$ and $f'(x)$ be piecewise continuous on the interval $-L < x < L$

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right) = \sum_{n=-\infty}^{\infty} c_n e^{jn\pi x/L}$$

The coefficients are given by:

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \qquad n \geq 0$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \qquad n > 0$$

$$c_n = \frac{1}{2}(a_n - jb_n) \qquad n > 0$$

1.2 Fourier Transform

If $h(t)$ is a periodic function then the Fourier transform is given by:

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} \, dt$$

Inverse Fourier tranformation of $H(\omega)$:

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{j\omega t} \, d\omega$$

1.3 Examples

1.3.1 Example 1: Fourier Series

Find the Fourier coefficients and Fourier Series for the square wave shown below:

$$f(t) = \begin{cases} 1 & \text{for } t - a > 0 \\ 0 & \text{for } t - a < 0 \end{cases}$$

and

$$f(x + 2) = f(x)$$

Ans:

$$f(x) = \frac{1}{2} + \sum_{n=1,3,5,\ldots} \frac{2}{n\pi} \sin(n\pi x)$$

1.3.2 Example 2: Fourier Transform

The unit step function is defined as:

$$u(t - a) = \begin{cases} 1 & \text{for } t - a > 0 \\ 0 & \text{for } t - a < 0 \end{cases}$$

is used to define the rectangular pulse function:

$$x(t) = u(t - a) - u(t - b) \qquad \text{where } a < b$$

Ans:

$$X(\omega) = \frac{e^{-j\omega a} - e^{-j\omega b}}{j\omega}$$

1.3.3 Example 3: Inverse Fourier Transform

Consider the signal: $x(t) = \sin(@omega_0 t)$ where ω_0 is a constant. Find the Fourier Transform of this signal to find $X(\omega)$.

Ans:

$$X(\omega) = -j\pi(\delta(\omega - \omega_0) - \delta(\omega + \omega_0))$$

2 Laplace

Is a generalisation of the Fourier transform and defined as:

$$H(s) = \mathcal{L}\{h(t)\} = \int_{0^-}^{\infty} h(t)e^{-st}\,dt \qquad s \in \mathbb{C}$$

s is a complex number $s = \sigma + j\omega$ and is identical with Fourier transform, if s is set to $j\omega$.
Inverse Laplace tranformation:

$$f(t) = \frac{1}{2\pi j} \int_{\sigma_c - j\infty}^{\sigma_c + j\infty} F(s)e^{st}\,ds$$

2.1 Examples

2.1.1 Example 1: Laplace Transform

Using the Laplace transform, find the solution for the following equation:

$$\frac{\partial^2}{\partial t^2}y(t) + 2\frac{\partial}{\partial t}y(t) + 2y(t) = 0$$

with initial conditions $y(0) = 1$ and $Dy(0) = -1$

2.1.2 Example 2: Trabsfer Function

Consider a mass-spring-damper system with the following differential equation:

$$m\ddot{x} = -kx - b\dot{x} + f$$

Find the transfer function for the system with input f and output x.

Ans:

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

3 Several-Variables

3.1 Examples

3.1.1 Example 1: Gradient

Find the rate of change of $f(x, y) = y^4 + 2xy^3 + x^2y^2$ at $(0, 1)$ in each of the following directions:

1. $\mathbf{i} + 2\mathbf{j}$
2. $\mathbf{j} - 2\mathbf{i}$
3. $3\mathbf{i}$
4. $\mathbf{i} + \mathbf{j}$

3.1.2 Example 2: Jacobian

Find the Jacobian $Df(1, 0)$ for the transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ given by:

$$\mathbf{f}(x, y) = (xe^y + \cos(\pi y)\,, x^2, x - e^y)$$

3.1.3 Example 3: Chain Rule

If $z = \sin(x^2y)$ where $x = st^2$ and $y = s^2 + \frac{1}{t}$, $y = e^{-t}$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ using chain rule.

3.1.4 Example 4: Substitution

If $z = \sin(x^2y)$ where $x = st^2$ and $y = s^2 + \frac{1}{t}$, $y = e^{-t}$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ using substitution.

3.1.5 Example 5: Partial Differentiation

Calculate $f_{223}(x, y, z)$, $f_{232}(x, y, z)$, and $f_{322}(x, y, z)$ of the following function:

$$f(x, y, z) = e^{x-2y+3z}$$

4 Double-Integrals

4.1 Riemann Sum

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta A_i$$

$$I = \iint_D f(x,y)dA \qquad \text{where } D \text{ is a region in } \mathbb{R}^2 \text{ and } dA \text{ is } dxdy$$

4.2 Double Integrals over General domains

If $f(x,y)$ is defined and bounded on domain D, then $\hat{f}(x,y)$ is zero outside D.

$$\iint_D f(x,y)\,dA = \iint_R \hat{f}(x,y)\,dA$$

4.3 Iteration of Double Integrals

If $f(x,y)$ is continuous on the bounded y-simple domain D given by $a \leq x \leq b$ and $c(x) \leq y \leq d(x)$, then:

$$\iint f(x,y)\,dA = \int_a^b dx \int_{c(x)}^{d(x)} f(x,y)dy$$

If $f(x,y)$ is continuous on the bounded x-simple domain D given by $c \leq x \leq d$ and $a(x) \leq y \leq b(x)$, then:

$$\iint f(x,y)\,dA = \int_c^d dx \int_{a(x)}^{b(x)} f(x,y)dy$$

4.4 Double Integrals in Polar Coordinates

$$dA = dxdy = r\ drd\theta$$

$$\begin{array}{ll} x = r\cos(\theta) & r^2 = x^2 + y^2 \\ y = r\sin(\theta) & \tan(\theta) = \frac{y}{x} \end{array}$$

4.4.1 Limits for Polar Coordinates

r is the radius from origin to the point

$$r \geq 0$$

θ is the angle in the positive direction of the xy-plane

$$0 \leq \theta \leq 2\pi$$

4.5 Change of Variables in Double Integrals

If x and y are given as a function of u and v:

$$x = x(u,v)$$
$$y = y(u,v)$$

These can be transformed or mapped from points (u,v) in the uv-plane to points (x,y) in the xy-plane.
The inverse transformation is given by:

$$u = u(x,y)$$
$$v = v(x,y)$$

Scaled area element:

$$dA = dxdy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

where the Jacobian is:

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Let $x(u,v)$ and $y(u,v)$ be a one-to-one transformation from a domain S in the uv-plane onto a domain D xy-plane.
Suppose, that function x and y, and first partial derivatives with respect to u and v are continuous in S. If $f(x,y)$ is integrable on D, then $g(u,v) = f(x(u,v), y(u,v))$ is integrable on S and:

$$\iint_D f(x,y)dA = \iint_S g(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

4.6 Examples

4.6.1 Example 1: Change of Variables

Evaluate the double integral:

$$\iint (x-3y)dA$$

where R is triangular region with vertices $(0,0)$, $(2,1)$, and $(1,2)$ using the transformation:

$$\begin{aligned} x &= 2u+v \\ y &= u+2v \end{aligned}$$

As x and y are dependent on u and v the transformation of dA is given by

$$dA = dxdy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

Using:

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

the Jacobian can be calculated:

$$\begin{array}{cc} \frac{\partial x}{\partial u} = 2 & \frac{\partial y}{\partial v} = 2 \\ \frac{\partial x}{\partial v} = 1 & \frac{\partial y}{\partial u} = 1 \\ \multicolumn{2}{c}{2*2-1*1=3} \end{array}$$

Find the boundaries for the double integral
$y_1 : (0,0) \rightarrow (2,1)$ is $y = \frac{1}{2}x$
$y_2 : (0,0) \rightarrow (1,2)$ is $y = 2x$
$y_3 : (1,2) \rightarrow (2,1)$ is $y = 3-x$

Replace x and y with their transformation:
$u+2v = \frac{2u+v}{2}$ $\quad v = 0$
$u+2v = 4u+2v$ $\quad u = 0$
$u+2v = 3-2u-v$ $\quad u = 1-v$

Therefore $0 \le u \le 1-v$ and $0 \le v \le 1$
Now transform the original function $x-3y$:

$$\begin{aligned} x-3y &= 2u+v-3(u+2v) \\ &= 2u-3u+v-6v \\ &= -u-5v \end{aligned}$$

$$
\begin{aligned}
\int_0^1 \int_0^{1-v} (-u-5v)\cdot 3 \; dudv &= -3\int_0^1 \left[\frac{u^2}{2}+5uv\right]_0^{1-v} dv \\
&= -\frac{3}{2}\int_0^1 \left(\frac{27v^2}{2}-12v\right) dv \\
&= \left[\frac{9v^3}{2}-6v^2-\frac{3v}{2}\right]_0^1 \\
&= \boxed{-3}
\end{aligned}
$$

4.6.2 Example 2: Double integral

Evaluate the double integral by iteration

$$\iint_R (x^2+y^2)\, dA$$

where R is the rectangle $0 \le x \le a$, $0 \le y \le b$

Insert the limits and solve the integral:

$$
\begin{aligned}
\int_0^b \int_0^a (x^2+y^2) dxdy &= \int_0^b \left[\frac{x^3}{3}+xy^2\right]_0^a dy \\
&= \int_0^b \left(\frac{a^3}{3}+ay^2\right) dy \\
&= \boxed{\frac{a^3 b}{3}+\frac{ab^3}{3}}
\end{aligned}
$$

4.6.3 Example 3: By iteration

Evaluate the double integral by iteration:

$$\iint_D x\cos y \, dA$$

where D is the finite region in the first quadrant bounded by the coordinate axes and the curve $y = 1 - x^2$.

Given the region the minimum for x and y must be 0 and for x the maximum is 1:

$$
\begin{aligned}
\int_0^1 \int_0^{1-x^2} (x\cos(y)) dydx &= \int_0^1 x\sin\left(1-x^2\right) dx \\
&= \left[\frac{1}{2}\sin(1)\sin\left(x^2\right)+\frac{1}{2}\cos(1)\cos\left(x^2\right)\right]_0^1 \\
&= \boxed{\sin^2\left(\frac{1}{2}\right)}
\end{aligned}
$$

4.6.4 Example 4: Polar coordinates

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5 Tripple-Integrals

5.1 Riemann Sum

5.2 Tripple Integrals over General domains

5.3 Change of Variables

$$dV = dx\ dy\ dz = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\ dv\ dw$$

$$\iiint f(x,y,z)\, dx\ dy\ dz = \iiint g(u,v,w) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\ dv\ dw$$

5.4 Cylyndrical Coordinates

$$\begin{array}{ll} x = r\cos(\theta) & r^2 = x^2 + y^2 \\ y = r\sin(\theta) & \tan\theta = (\frac{y}{x}) \\ z = z & \end{array}$$

From the Jacobian using change of variables:

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = r$$

$$dV = dx\ dy\ dz = r\ dr\ d\theta\ dz$$

5.4.1 Limits for Cylyndrical Coordinates

r is the radius from origin to the point

$$r \geq 0$$

θ is the angle in the positive direction of the x-axis

$$0 \leq \theta \leq 2\pi$$

5.5 Spherical Coordinates

$$\begin{array}{ll} x = \rho\sin(\phi)\cos(\theta) & \rho^2 = x^2 + y^2 + z^2 \\ y = \rho\sin(\phi)\sin(\theta) & \cos(\phi) = \frac{z}{\rho} \\ z = \rho\cos(\phi) & \tan\theta = \frac{y}{x} \end{array}$$

$$dxdydz = dV = \rho^2\sin(\phi)d\rho d\phi d\theta$$

5.5.1 Limits for Spherical Coordinates

ρ is the distance from the origin O to the point P

$$\rho \geq 0$$

ϕ is the angle by the radial line OP to the positive direction of the z-axis

$$0 \leq \phi \leq \pi$$

θ is the angle in the positive direction of the x-axis to the point P in the xy-plane

$$0 \leq \theta \leq 2\pi$$

5.6 Examples

5.6.1 Example 1: Tripple Integral

Find $\iiint(x^2+y^2+z^2)dV$, where the region is bounded by $z=c\sqrt{(x^2+y^2)}$ and $x^2+y^2+z^2=a^2$

5.6.2 Example 2: Sphereical Coordinates

Find the volume of:

$$\iiint \sqrt{x^2+y^2+z^2}dxdydz$$

where the region is bounded by $x^2+y^2+z^2 \leq 1$ in spherical domain.

Transform the region:

$$\rho^2 = x^2+y^2+z^2 \leq 1 \qquad \text{Therefore: } 0 \leq \rho \leq 1$$

For θ and ϕ we have:

$$0 \leq \theta \leq 2\pi \qquad 0 \leq \phi \leq \pi$$

and for dV:

$$dV = dxdydz = \rho^2 \sin\phi d\rho d\phi d\theta$$

Transform the function:

$$\sqrt{x^2+y^2+z^2} = \sqrt{\rho^2} = \rho$$

Solve the integral:

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho\cdot\rho^2\sin(\phi)d\rho d\phi d\theta &= \int_0^{2\pi}\int_0^{\pi}\left[\frac{1}{4}p^4\sin(\phi)\right]_0^1 d\phi d\theta \\
&= \int_0^{2\pi}\int_0^{\pi}\frac{\sin(\phi)}{4}d\phi d\theta \\
&= \int_0^{2\pi}\left[-\frac{\cos(\phi)}{4}\right]_0^{\pi} d\theta \\
&= \frac{1}{2}\int_0^{2\pi} d\theta \\
&= \frac{1}{2}\cdot 2\pi \\
&= \boxed{\pi}
\end{aligned}$$

5.6.3 Example 3: Transformations

6 Fields-Curve

6.1 Curve & Parameterization

Representation of a curve in 3 space by using its position vector is given as:

$$r = r(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \qquad \text{where } a \leq t \leq b$$

6.2 Vector Fields

$$\mathbf{F}(x,y,z) = \underbrace{f_x(x,y,z)}_{\text{Scaler function}}\ \mathbf{i} + \underbrace{f_y(x,y,z)}_{\text{Scaler function}}\ \mathbf{j} + \underbrace{f_z(x,y,z)}_{\text{Scaler function}}\ \mathbf{k}$$

$$\frac{\partial f}{\partial x} = f_1 = f_x \qquad \tfrac{\partial f}{\partial y} = f_2 = f_y \qquad \frac{\partial f}{\partial z} = f_3 = f_z$$

Position vector:

$$r = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Unit vector with magnitude 1:

$$r = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

6.2.1 Scalar field

$$f(x,y,z) = f_1(x,y,z) = f_2(x,y,z) = f_3(x,y,z)$$

The gradient of a scalar field is a vector field:

$$\nabla f = \text{grad } f(x,y,z) = f_x(x,y,z)\mathbf{i} + f_y(x,y,z)\mathbf{j} + f_z(x,y,z)\mathbf{k}$$

6.2.2 Field lines

6.2.3 Convervation field

If $\mathbf{F}(x,y,z) = \nabla\phi(x,y,z)$ in a domain D, then $\mathbf{F}$ is a conservative vector field in D and function ϕ is the potential function.

$$\mathbf{F}(x,y,z) = \nabla\phi(x,y,z) = \phi_x(x,y,z)\mathbf{i} + \phi_y(x,y,z)\mathbf{j} + \phi_z(x,y,z)\mathbf{k}$$

If the vetor field is conservative, then all the following equations are true:

$$\begin{aligned} \frac{\partial f_x}{\partial y} &= \frac{\partial f_y}{\partial x} \\ \frac{\partial f_x}{\partial z} &= \frac{\partial f_z}{\partial x} \\ \frac{\partial f_y}{\partial z} &= \frac{\partial f_z}{\partial y} \end{aligned}$$

6.2.4 Vector field in Polar Coordinates

$$\mathbf{F} = f(r,\theta) = f_r(r,\theta)\hat{\mathbf{r}} + f_\theta(r,\theta)\hat{\theta}$$

where:

$$\hat{\mathbf{r}} = \cos(\theta)i + \sin(\theta)j$$

$$\hat{\theta} = -\sin(\theta)i + \cos(\theta)j$$

6.3 Line Integral

$$f(x, y)ds = \text{Area (tiny point)}$$

$$\text{Length of } \mathcal{C} = \int_{\mathcal{C}} f(x, y, z)\, ds = \int_a^b f(r(t)) \left|\frac{dr}{dt}\right| \, dt$$

6.3.1 Line integral of a vector field

$$W = \int_{\mathcal{C}} F.\hat{T}\, ds = \int F\, dr = \int_{\mathcal{C}} f_1(x, y, z)dx + f_2(x, y, z)dy + f_3(x, y, z)dz$$

6.4 Examples

6.4.1 Example 1: Conservative vector field and potential

Determine whether the given vector field is conservative, and find a potential function if it is:

$$\mathbf{F}(x, y, z) = (2xy - z^2)\mathbf{i} + (2yz + x^2)\mathbf{j} - (2zx - y^2)\mathbf{k}$$

6.4.2 Example 2: Line integral

Evaluate $\oint x^2y^2 \; dx + x^3y \; dy$ counterclockwise around the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$

6.4.3 Example 3: Line integral

Evalute the line integral for $f(x, y) = x^2y^2$ along a straight line from origin to the point $(2, 1)$

Ans: $(5\sqrt{5})/3$

6.4.4 Example 4: Line integral vector field

Evaluate the line integral for $\mathbf{F}(x, y) = y^2\mathbf{i} + 2xy\mathbf{j}$ from $(0, 0)$ to $(1, 1)$
Along the line $y = x$
Allong the line $y = x^2$

6.4.5 Example 5: Gradient

6.4.6 Example 6: Parametrize a curve

Use $t = y$ to parametrize the part of the line of intersection of the two planes:
Plane 1: $y = 2x - 4$
Plane 2: $z = 3x + 1$ from $(2, 0, 7)$ to $(3, 2, 10)$

Ans: $r(t) = \left(\frac{1}{2}(t + 4)\right)\mathbf{i} + (t)\mathbf{j} + \left(\frac{3}{2}t + 7\right)\mathbf{k}$

7 Theorems

7.1 Green's Theorem

$$\oint_C f_1(x,y)dx + f_2(x,y)dy = \iint_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dA$$

7.2 Stokes' Theorem

$$\oint_C F \cdot dr = \iint_S \text{curl } F \cdot \widehat{N} dS$$

7.3 Divergence Theorem

$$\nabla \cdot F(x,y,z) = \text{div } F = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$F(x,y,z) = f_1(x,y,z)i + f_2(x,y,z)j + f_3(x,y,z)k$$

$$\oiint_s (F \cdot \widehat{N})dS = \iiint_V \text{div}\, F dV$$

More variants:

$$\iiint_D \text{curl } F dV = -\oiint_s (F \times \widehat{N})dS$$

$$\iiint_D \text{grad } \phi\, dV = \oiint_s \phi\, dS$$

7.4 Examples

7.4.1 Example 1: Div and Curl

Calculate the divergence and curl of the following vector field:

$$\mathbf{F} = \cos x\, \mathbf{i} - \sin y\, \mathbf{j} + z\, \mathbf{k}.$$

Divergence: Curl:

7.4.2 Example 2: Green's Theorem

Using Green's Theorem evaluate $\oint_e (x^2 y)\, dx + (xy^2)dy$, clockwise boundary of the region:

$$0 \leq y \leq \sqrt{9 - x^2}$$

7.4.3 Example 3: Stokes' Theorem

7.4.4 Example 4: Divergence Theorem

Use the Divergence Theorem to calculate the flux of the given vector field out of the sphere s with equation $x^2 + y^2 + z^2 = a^2$, where $a > 0$ and

$$\mathbf{F} = (x^2 + y^2)\mathbf{i} + (y^2 - z^2)\mathbf{j} + z\mathbf{k}$$

8 PDE

Partial Differential Equations are equations with multiple variables and derivatives. They are used to model many physical phenomena, such as heat, sound, and light. The totality of solutions to a PDE is called its general solution, and there can be a lot.

8.1 Classification of PDEs

General representation of a PDE:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

Conditions:

Linear: A, B, C, D, E, F are only function of x,y variables, not u.
Quasi-linear: A, B, C, D, E, F may be function of (x, y, u, u_x, u_y)
Fully non-linear: A, B, C, D, E, F may be function of $(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy})$

8.2 Characteristics of PDEs

$B^2 - 4AC > 0$	2 real roots	2 characteristics	**Hyperbolic PDE**
$B^2 - 4AC = 0$	1 real roots	1 characteristics	**Parabolic PDE**
$B^2 - 4AC < 0$	0 real roots	0 characteristics	**Elliptic PDE**

Tyoes of varius PDEs:

Wave Equation: Hyperbolic PDE

Heat Equation: Parabolic PDE

Laplace Equation: Elliptic PDE

$B^2 - 4AC > 0$	2 real roots	2 characteristics	**Hyperbolic PDE**
$B^2 - 4AC = 0$	1 real roots	1 characteristics	**Parabolic PDE**
$B^2 - 4AC < 0$	0 real roots	0 characteristics	**Elliptic PDE**

8.3 Important Second-Order PDEs

$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$	One-dimensional wave equation
$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$	One-dimensional heat equation
$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$	Two-dimensional Laplace equation
$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$	Two-dimensional Poisson equation
$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$	Two-dimensional wave equation
$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$	Three-dimensional Laplace equation

8.4 Initial and Boundary Conditions

8.5 Wave Equation (1D)

One dimensional wave equation is given by:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{where } c^2 = \left[\frac{T(x,t)}{\mu_x}\right] \tag{1}$$

With two boundary conditions $x = 0$ and $x = L$:

$$u(0,t) = 0 \qquad u(L,t) = 0 \qquad \text{For all } t \geq 0$$

And two initial conditions, initial displacement and initial velocity at time $t = 0$:

$$u(x,0) = f(x) \qquad \frac{\partial u}{\partial t}(x,0) = g(x) \qquad \text{For all } 0 \leq x \leq L$$

Steps to solve:

1. Method of Separation of Variables $\quad u(x,t) = X(x)T(t)$
2. Satify the Boundary Conditions test
3. Fourier Series Validation

8.5.1 D'Alembert's Solution of the Wave Equation

His solution is given by eq. (1) but extended to two variables:

$$v = (x - ct) \qquad w = (x + ct) \tag{2}$$

I.e. $u(v, w)$. Partial derivatives from chain rule:

$$u_x = u_v \cdot v_x + u_w \cdot w_x = u_v + u_w$$

For double derivatives:

$$u_{xx} = (u_v + u_w)_x = (u_v + u_w)_v v_x + (u_v + u_w)_w w_x = u_{vv} + 2u_{vw} + u_{ww}$$

With respect to t:

$$u_{tt} = c^2 u_{xx} = c^2(u_{vv} + 2u_{vw} + u_{ww})$$

From eq. (1) and eq. (2):

$$u_{vw} = \frac{\partial^2 u}{\partial w \partial v} = 0$$

This can be solved by integrating with respect to v and w:

$$\frac{\partial u}{\partial v} = h(v) \text{ and } u = \int h(v)\ dv + \psi(w)$$

Here, $h(v)$ and $\psi(w)$ are arbitrary functions of v and w, respectively. The solution in term for x:

$$u = \phi(v) + \psi(w)$$

This is d'Alembert's solution, which is the general solution to the wave equation.

$$u(x,t) = \phi(x + ct) + \psi(x - ct)$$

This solution satisfies the wave equation and the initial conditions:

8.6 Heat Equation (1D)

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{3}$$

Conditions:

- PDE is linear and homogeneous.
- Boundary conditions are linear and homogeneous.The two for $u(x,t)$ is $u(0,t) = 0$ and $u(L,t) = 0$ for all $t > 0$.
- One initial condition at time $(t = 0)$: $u(x,0) = f(x)$.

Solve the

Solution:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\lambda_n^2 t}$$

where

$$\lambda_n = \frac{cn\pi}{L}$$

and

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \qquad \text{for } n = 1, 2, 3 \ldots$$

8.7 Examples

8.7.1