Exam Numerical Methods 2025

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Code used for the exercises can be found in "exam/", "utils/", and "Numerical-Recipes/".

Exercise 1

i

Using method of SVD from NR3 headers.

```
SVD svd_solver(A);
util::print(svd_solver.w, "w");
```

`w` is as follows:
11.951932409770585
5.397266515466196
4.719596639028403
3.953906595603368
3.6230909930002446
3.368300112475192
3.0916664682435697
$4.811863543913732 \cdot 10^{-15}$

ii

Using method of SVD from NR3 headers.

```
double threshold = 1e-14;
SVD svd_solver(A);
auto nullspace = svd_solver.nullspace(threshold);
util::print(nullspace, "nullspace");
```

This could also be done by finding the index of the 0 element in w and then taking the corresponding column of V.

The unit vector in the null space of A is:
0.30414953233623704
-0.6995439243733448
$6.934350372534523 \cdot 10^{-17}$
$-7.275658594236239 \cdot 10^{-17}$
0.42580934527073205
$-3.3958678618647428 \cdot 10^{-18}$
$3.11158887096989 \cdot 10^{-17}$
-0.4866392517379791

iii

Using the SVD solver from NR3 headers.

```
double threshold = 1e-14;
SVD svd_solver(A);
VecDoub x_sol(A_width);
svd_solver.solve(b, x_sol, threshold);
util::print(x_sol, "x_sol");
```

x is as follows:
−47.248363616439306
0.3688425226349671
79.65376424734224
−4.71197096346749
40.40427152514301
69.74266521165164
−42.73505350336738
5.29329919793784

iv

Code for linear regression error calculations can be found in "utils/linreg_error.h"

The residual error of $5.42797258495411 \cdot 10^{-6}$ is considarably smaller compared to the random fitting of 0.9128709291752769, which a good starting point for accuracy validation.

The std. dev. of the estimated parameters:
0.08468563169399769
0.16099885651146362
0.27703518823386664
0.2750301024010811
0.22661912057496514
0.2597660651878672
0.2503745120251377
0.22830996340629964

The std. dev. of the estimated parameters are relativly small indicating the results have approced the correct values. The std. dev. of the estimated parameters are calculated as follows using `V` and `w` from the SVD solver:

```
VecDoub calculate_standard_deviation_svd(SVD &svd_solver, double threshold) {
  auto w = svd_solver.w;
  auto V = svd_solver.v;
  auto sigma = VecDoub(w.size());
  double sum;
  for (int j = 0; j < w.size(); j++) {
    sum = 0;
    for (int i = 0; i < w.size(); i++) {
      if (w[i] > threshold) {
        sum += pow(V[j][i] / w[i], 2);
      }
    }
    sigma[j] = sqrt(sum);
  }
  return sigma;
}
```

Exercise 2

```
Vector function setup:
VecDoub vecfunc(VecDoub_I x) {
  assert(x.size() == 4);

  VecDoub f(4);

  f[0] = 3 * x[0] + x[1] * sin(x[2]) - cos(x[0]) + cos(pow(x[1], 2)) + 4.2;
  f[1] = 3 * x[1] + x[0] * x[2] * x[3] + sin(x[1]) - 5.1;
  f[2] = -pow(x[1], 2) + x[2] * pow(x[3], 2) + 3 * x[2] + 5.2;
  f[3] = x[0] + 3 * x[3] + sin(pow(x[2], 2) * pow(x[3], 2)) + cos(x[1]) - 2.3;
  return f;
}
```

i

When evaluating the function with $x_0 = -0.7 \quad x_1 = 1.2 \quad x_2 = 2.3 \quad x_3 = -4.1$ the result is:
2.3604277760657215
6.033039085967225
49.32299999999999
-14.118275390737853

ii

k	x_k	d_k	C	e	lambda
1					No BT
1(0)	-1.40000000851				
1(1)	1.27500000775				
1(2)	-1.73333334387				
1(3)	0.899999998808				
2					No BT
2(0)	-2.52981246183	-1.12981245332			
2(1)	0.206707041083	-1.06829296667			
2(2)	-1.6594756569	0.0738576869645			
2(3)	0.892282003669	-0.0077179951390			
3			0.10966935	0.007745490374	0.111687
3(0)	-2.37494239018	0.15487007165			
3(1)	0.382163475623	0.17545643454			
3(2)	-1.69981398652	-0.0403383296173			
3(3)	0.772996796054	-0.119285207615			
4			20.347090	42.0178871606	No BT
4(0)	-1.2746127961	1.10032959407			
4(1)	1.22281361303	0.840650137408			
4(2)	-1.36896314904	0.330850837483			
4(3)	0.577609614918	-0.195387181136			
5			0.20976732	0.039362012827	No BT
5(0)	-0.97107267384	0.303540122259			
5(1)	1.11854284125	-0.104270771779			
5(2)	-1.11800236852	0.250960780519			
5(3)	0.724768551128	0.147158936209			
6			0.14970545	0.000118138606	No BT
6(0)	-0.96667857465	0.00439409918669			
6(1)	1.13440513728	0.0158622960234			

6(2)	-1.10244628986	0.0155560786568			
6(3)	0.741388663637	0.0166201125093			
7			0.59612044	1.31919941e-07	No BT
7(0)	-0.96634644910	0.00033212555338			
7(1)	1.13461605039	0.00021091311314			
7(2)	-1.10221138134	0.00023490852128			
7(3)	0.741495076632	0.00010641299509			

iii

The accuracy is estimated as $e = 1.31919941514e{-}07$ with the convergence constant $C = 0.596120441159$

The error is calculated as follows (d_k is a vector of the difference in x from last iteration to current iteration):

$$C = \frac{||d_k||}{||d_{k-1}||^2}$$

The convergence constant is then used to calculate the error:

$$C \cdot ||d_k||^2$$

The code for error calculation is in "utils/multi_roots.h"

Exercise 3

i

The 2nd order system is:

$$x''(t) = a_{max}\left(1 - \left(\frac{x'(t)}{v_{des}}\right)^4 - \left(\frac{D_0 + \text{Max}\left(0, x'(t)T_{react} + \frac{x'(t)(x'(t)-X_F'(t))}{2a_{com}}\right)}{X_F(t) - x(t)}\right)^2\right)$$

From the 2nd order system we can make 2 1st order systems with new variables $x_1(t) = x(t)$, $x_2(t) = x'(t)$:

$$x_1'(t) = x_2(t)$$
$$x_2'(t) = a_{max}\left(1 - \left(\frac{x_2(t)}{v_{des}}\right)^4 - \left(\frac{D_0 + \text{Max}\left(0, x_2(t)T_{react} + \frac{x_2(t)(x_2(t)-X_F'(t))}{2a_{com}}\right)}{X_F(t) - x_1(t)}\right)^2\right)$$

ii

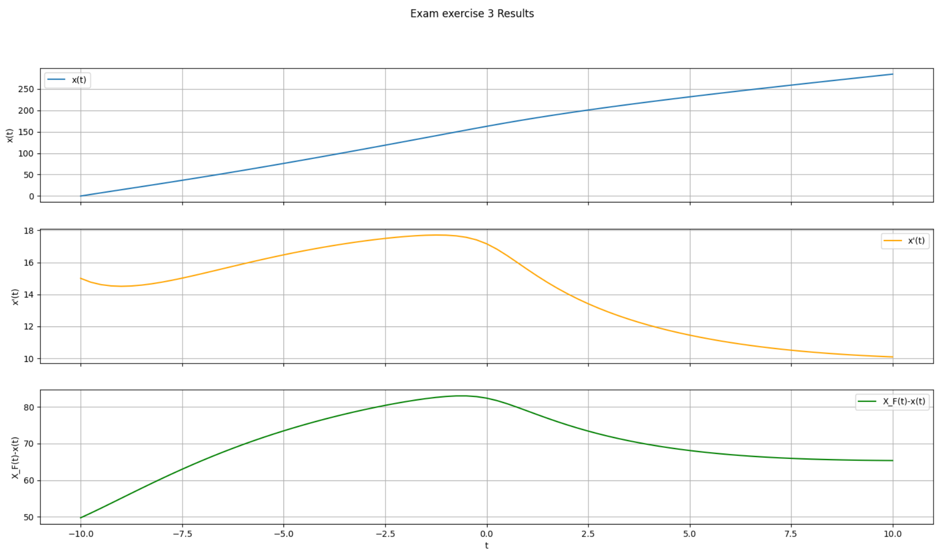
Setup of the ODE system:

```
double a_max = 4.0;
double v_des = 25.0;
double D_0 = 50.0;
double T_react = 1.5;
double a_com = 2.0;

double X_F(double t) { return 250 + 15 * t - 5 * sqrt(1 + pow(t, 2)); }
double X_F_prime(double t) { return 15 - (5 * t) / sqrt(1 + pow(t, 2)); }

VecDoub derivs(const Doub t, VecDoub_I &x) {
  double x1 = x[0];
  double x2 = x[1];
  VecDoub_O dxdt(2);
  dxdt[0] = x2;
  dxdt[1] = a_max * (1 - pow((x2 / v_des), 4) - pow((D_0 + std::max(0.0, x2 * T_react +
             (x2 * (x2 - X_F_prime(t))) / (2 * a_com))) / (X_F(t) - x1), 2));
  return dxdt;
}
```

The resulting value of the initial conditions with $t_0 = -10$ is:
$x'(-10) = 15.0$ and $x''(-10) = -1.2035371947583346$.

iii

The plot can be seen here and is made using "exam/exam3_plot.py":



$x(10) = 284.3842346596431$ and $x'(10) = 10.092177928387363$

Code:

```
// Midpoint method
VecDoub second_order_runge_kuttea_method_plot(double low, double high,
                                              int steps, VecDoub_I &x,
                                              VecDoub derivs(const Doub t,
                                                             VecDoub_I &x)) {
  std::vector<std::vector<double>> plotting;
  const double h = (high - low) / (double)steps;
  VecDoub x_n = x;
  plotting.push_back({low, x_n[0], x_n[1], (X_F(low) - x_n[0])});
  for (double t_n = low; t_n < high; t_n += h) {
    f_comps_current += 2;
    auto k1 = h * derivs(t_n, x_n);
    auto k2 = h * derivs(t_n + 0.5 * h, x_n + 0.5 * k1);
    auto x_n_next = x_n + k2;
    x_n = x_n_next;
    plotting.push_back({t_n + h, x_n[0], x_n[1], (X_F(t_n + h) - x_n[0])});
  }

  std::ofstream file;
  file.open("exam3_results.txt");
  for (const auto &vec : plotting) {
    std::println(file, "{}\t{}\t{}\t{}", vec[0], vec[1], vec[2], vec[3]);
  }
  file.close();

  return x_n;
}
```

iv

Based on the provided graph we see the car in front slows down from around 20 to around 10, which matches the plot from exercise iii vel. We both see the car slowing down and the distance between the car and the car in front is decreasing at around $t = 0$ until the speeds match again.

v

For x(10)

N	A(N)	A(N/2)-A(N)	alpha^k	Rich error	Order est.	f comps
20	284.30333445					40
40	284.37167114	-0.06833668906		0.0227788963566		80
80	284.38423466	-0.01256351530	5.43929	0.00418783843344	2.443420160	160
160	284.38702585	-0.00279119785	4.50111	0.00093039928557	2.170283890	320
320	284.38768707	-0.00066122196	4.22127	0.000220407322293	2.077677835	640
640	284.38784818	-0.00016110802	4.10421	5.37026762875e-05	2.037106261	1280
1280	284.38788796	-3.9773799e-05	4.05060	1.3257933252e-05	2.018138092	2560

For x'(10)

N	A(N)	A(N/2)-A(N)	alpha^k	Rich error	Order est.	f comps
20	10.116751370					40
40	10.096355282	0.02039608843		-0.00679869614505		80
80	10.092177928	0.004177354050	4.88253	-0.00139245135016	2.2876311694	160
160	10.091213181	0.000964746857	4.33000	-0.00032158228567	2.1143670800	320
320	10.090980755	0.000232425688	4.15077	-7.7475229565e-05	2.0533809082	640
640	10.090923676	5.70791552e-05	4.07198	-1.9026385067e-05	2.0257336429	1280
1280	10.090909531	1.41453933e-05	4.03517	-4.7151311086e-06	2.0126316896	2560

From the table we can see that with $N = 1280$ or $h = 0,015625$ the estimated error for $x(10)$ is better than $2 \cdot 10^{-5}$. Error estimation is done using Richardson.

Exercise 4

Equation is set up as follows:

```
double eqn(double x) { return exp(pow(x, 3)) * sqrt(x * (2 - x)); }
```

i

its	A(i)	A(i-1)-A(i)	alpha^k	Rich error	Order est.	f comps
2	3.6243757712					3
4	18.432965615	-14.8085898439		0.98723932292		5
8	54.698586304	-36.2656206892	0.408336	2.41770804595	-1.292168275	9
16	86.718876557	-32.0202902529	1.132582	2.13468601686	0.1796161555	17
32	100.11196736	-13.3930908053	2.390806	0.89287272035	1.2574974468	33
64	104.35400456	-4.24203720557	3.157230	0.28280248037	1.6586597591	65
128	105.70663979	-1.35263522555	3.136128	0.09017568170	1.6489844440	129
256	106.15990895	-0.453269157456	2.984176	0.03021794383	1.5773329282	257
512	106.31622467	-0.156315726242	2.899702	0.01042104841	1.535905068	513
1024	106.37085226	-0.0546275897805	2.861479	0.00364183931	1.5167612523	1025
2048	106.39005897	-0.0192067023301	2.844194	0.00128044682	1.5080199328	2049
4096	106.39683117	-0.00677220593843	2.836107	0.00045148039	1.5039120881	4097
8192	106.39922231	-0.00239113439314	2.832214	0.00015940895	1.5019306266	8193
16384	106.40006714	-0.000844831994243	2.830307	5.6322132e-05	1.5009588421	16385
32768	106.40036573	-0.000298594312255	2.829363	1.9906287e-05	1.5004777866	32769
65536	106.40047128	-0.000105551581555	2.828894	7.0367721e-06	1.5002384889	65537
131072	106.40050860	-3.73150385116e-05	2.828660	2.4876692e-06	1.5001191155	131073
262144	106.40052179	-1.31923135172e-05	2.828543	8.7948756e-07	1.5000595849	262145
524288	106.40052645	-4.66409086641e-06	2.828485	3.1093939e-07	1.5000297894	524289
1048576	106.40052810	-1.6489892829e-06	2.828454	1.0993261e-07	1.5000138731	1048577

ii

First we find $alpha_k$:

$$\frac{106.40052179 - 106.40052645}{106.40052645 - 106.40052810} = 2.8245$$

Then since we double each time we find order=$\log_2(alpha_k)$. The order estimate is $\log_2(2.8245) = 1.49786$. The same is shown in the table but with higher precision.

iii

This is far from the expected order of 4. It could be a result of the singularities at the endpoints $x = 0$ and $x = 2$. Here we get $\sqrt{0}$ at both ends of the function which is not good.

iv

The table shows Richardson extrapolation error using the expected order of 4. With the estimated order of around 1.5 the error will be:

$$\frac{106.40052810 - 106.40052645}{(2.8245 - 1)} = 9.04357 \cdot 10^{-7}$$

This error is higher because of the lower computed alpha_k estimate.

This is from the Richardson extrapolation error:

$$\frac{A(h_2) - A(h_1)}{alpha_k - 1}$$

v

Generally Simpson is better because of the higher expected order, where Midpoint and Trapezoid are both second order methods.

The midpoint methods is though expected to outperform the other two methods here since there is no function evaluating at the endpoints (singularities). This makes it more stable in this case.

Exercise 5b

i

$$\frac{\partial u(x,t)}{\partial t} = 4\frac{\partial^2 u(x,t)}{\partial x^2} + \sin(\pi x)\exp(-t)$$
$$u(x,t) = x^2$$
$$u(0,t) = 0$$
$$u(1,t) = 1 + \sin(t)$$

For $N = 2$, $\alpha = 4$, and $\Delta x = 1/N = 0.5$, the semidiscrete form using Euler is:

$$\begin{cases} \frac{du_j(t)}{dt} = \frac{4}{0.5^2}(u_{j-1}(t) - 2u_j(t) + u_{j+1}(t)) + \sin(\pi x_j)\exp(-t) & j = 1, \ldots, 2-1 \\ u_j(0) = g(x_j) & j = 0, \ldots, 2 \\ u_0(t) = a(t) & t > 0 \\ u_2(t) = b(t) & t > 0 \end{cases}$$

For the system:

$$\frac{du_1(t)}{dt}$$

at $t = 0$ we have:

$$\frac{du_1(0)}{dt} = \frac{4}{0.5^2}(u_0(0) - 2u_1(0) + u_2(0)) + \sin(\pi 0.5)\exp(-0)$$

and further reduced:

$$\frac{du_1(0)}{dt} = 16(0 - 2 \cdot 0.5^2 + 1) + 1 \cdot 1 = 9$$

ii

The parabolic PDE system is set up as follows:

```
double u(double x, double t) {
  if (t == 0.0) {
    return pow(x, 2);
  }
  if (x == 0.0) {
    return 0.0;
  }
  if (x == 1.0) {
    return 1 + sin(t);
  }

  assert(false && "Not possible");
  return 0;
}

double f(double x, double t) { return sin(std::numbers::pi * x) * exp(-t); }

double alpha = 4.0;
double x_target = 0.5;
double t_target = 10.0;
```

The code was not able to reach the accuracy of 10^{-7} after 10 hours of computation. But it reached the accuracy of 10^{-6} after reasonable time, and that is used as shown in the table.

N	A(N)	A(N/2)-A(N)	alpha^k	Rich error	Order est.	f comps
4	0.096352855562					246
8	0.107138879158	-0.010786023595		0.0035953411		1134
16	0.113959604474	-0.0068207253164	1.581360206	0.0022735751	0.661166	4830
32	0.11731649895	-0.0033568944762	2.031855741	0.0011189648	1.022797	19902
64	0.118986866406	-0.0016703674554	2.009674257	0.0005567891	1.006961	80766
128	0.119822117869	-0.0008352514628	1.999837808	0.0002784171	0.999882	325374
256	0.120239761946	-0.0004176440777	1.999912143	0.0001392146	0.999936	1306110
512	0.120448589975	-0.0002088280283	1.999942636	6.960934e-05	0.999958	5233662
1024	0.120553005662	-0.0001044156872	1.999967954	3.480522e-05	0.999976	20953086
2048	0.120605213943	-5.220828085e-05	1.999983250	1.740276e-05	0.999987	83849214
4096	0.120631318182	-2.610423899e-05	1.999992448	8.701412e-06	0.999994	335470590
8192	0.120644370273	-1.305209137e-05	2.000004308	4.350697e-06	1.000003	1342029822
16384	0.120650896099	-6.525826005e-06	2.000067327	2.175275e-06	1.000048	5368414206
32768	0.120654158096	-3.261996981e-06	2.000561632	1.087332e-06	1.000405	21474246654
65536	0.12065578548	-1.627384007e-06	2.004442077	5.424613e-07	1.003200	85898166270
131072	0.12065658454	-7.990601963e-07	2.036622541	2.663533e-07	1.026178	343595024382

The estimate for $u(0.5, 10)$ is 0.12065658454 at $N = 131072$.

The error is calculated using Richardson extrapolation error using the expected order of 2 for the method. The estimated order is though only around 1 which could indicate bad convergence.