Partial Differential Equations (PDE's)

Huge topic in science and engineering. ANYONE with a master degree in engineering should know

- **something about** PDE's
- **something about** how to formulate boundary conditions for PDE's
- **something about** how to solve PDE's numerically including **clever** usage of available software.

Classification into

Hyperbolic PDE's with the 1D wave equation as the model example:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = v^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$

Elliptic PDE's with the 2D Poisson's equation as the model example:

$$-\left(\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2}\right) + \lambda u(x,y) = f(x,y)$$

Parabolic PDE's with the 1D diffusion equation as the model example:

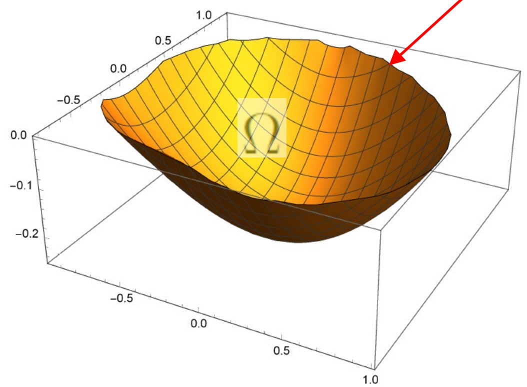
$$\frac{\partial u(x,t)}{\partial t} = \alpha \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t)$$

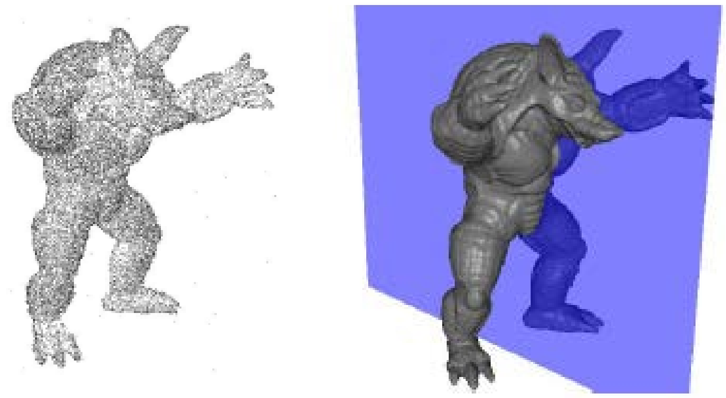
Poissons equation:

$$-\left(\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2}\right) + \lambda u(x,y) = f(x,y); \quad (x,y) \in \Omega \subseteq \mathbb{R}^2; \quad \lambda \geq 0$$

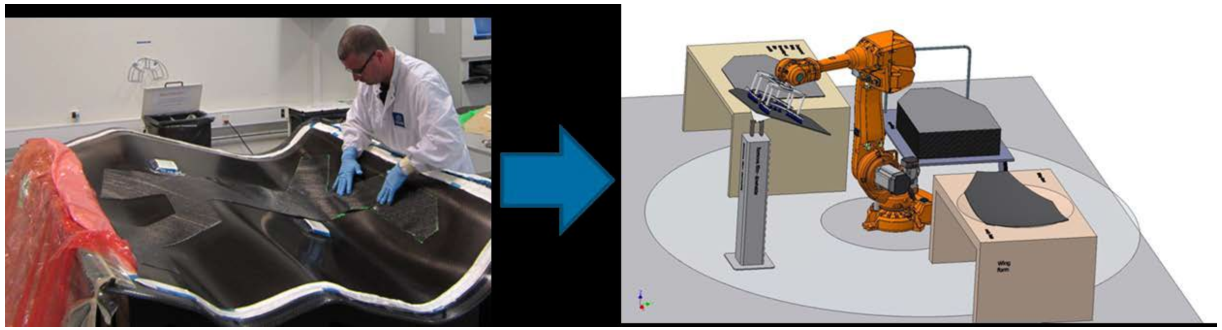
Condition on $u(x,y)$ on the boundary $\delta\Omega$ (discussed on next slide)

Boundary $\delta\Omega$





Poisson Reconstruction of surfaces from point clouds



Model equation for shapes of handled deformable materials

Poissons equation:

$$-\left(\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2}\right) + \lambda u(x,y) = f(x,y); \quad (x,y) \in \Omega \subseteq \mathbb{R}^2; \quad \lambda \geq 0$$

On the boundary of Ω, written as $\delta\Omega$, we have either that $u(x,y)$ is specified

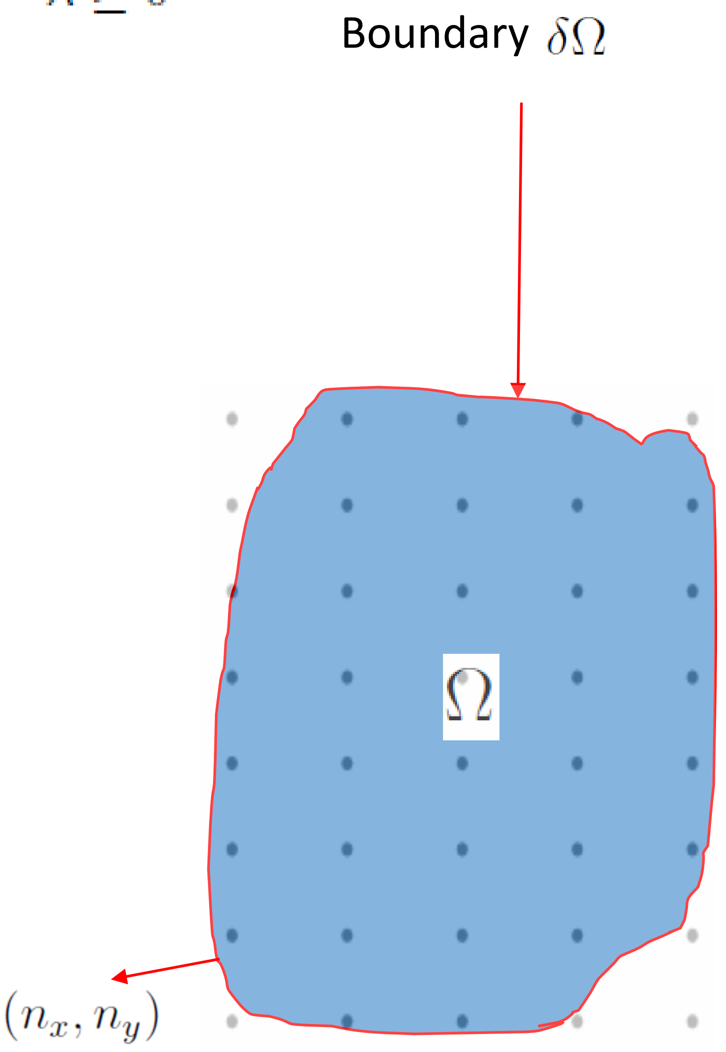
$$u(x,y) = \phi_D(x,y); \quad (x,y) \in \delta\Omega$$

called *Dirichlet boundary conditions.*

Or the derivative is specified

$$\left(\frac{\partial u(x,y)}{\partial x}, \frac{\partial u(x,y)}{\partial y}\right) \cdot (n_x, n_y) = \phi_N(x,y); \quad (x,y) \in \delta\Omega$$

where (n_x, n_y) is a vector perpendicular to the boundary at (x,y). These are called *Neumann boundary conditions.* Often $\phi_N(x,y) = 0$, which we call *free Neumann boundary conditions.* It can also be a mixture of the two conditions along the boundary.



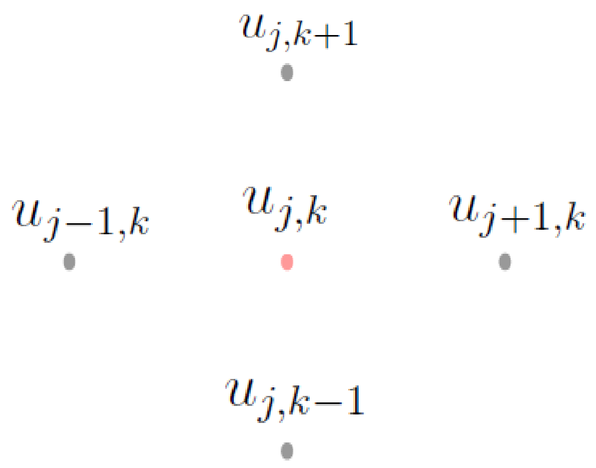
Discretization: Grid size h.

$$x_j = x_0 + jh; \quad y_k = y_0 + kh; \quad u_{j,k} \simeq u(x_j, y_k); \quad f_{j,k} \equiv f(x_j, y_k)$$

Poissons equation:

$$-\left(\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2}\right) + \lambda u(x,y) = f(x,y); \quad (x,y) \in \Omega \subseteq \mathbb{R}^2; \quad \lambda \geq 0$$

$$x_j = x_0 + ih; \quad y_k = y_0 + kh; \quad u_{j,k} \simeq u(x_j, y_k); \quad f_{j,k} \equiv f(x_j, y_k)$$



We then get

$$\frac{\partial^2 u(x_j,y_k)}{\partial x^2} = \frac{u_{j+1,k} - 2u_{j,k} + u_{j-1,k}}{h^2} + \mathcal{O}(h^2)$$
$$\frac{\partial^2 u(x_j,y_k)}{\partial y^2} = \frac{u_{j,k+1} - 2u_{j,k} + u_{j,k-1}}{h^2} + \mathcal{O}(h^2)$$

Inserting into Poisson's equation yields the following equation for every interior point (none of the four neighbours are at or exceeds the boundary of Ω:

$$-u_{j-1,k} - u_{j,k-1} + (4 + h^2\lambda)u_{j,k} - u_{j+1,k} - u_{j,k+1} = h^2 f_{j,k}$$

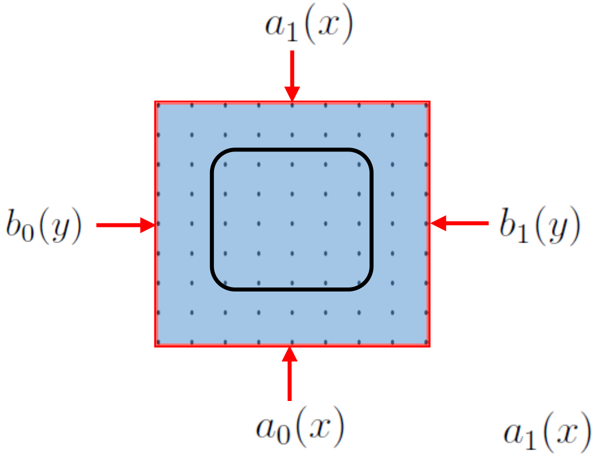
$$-\left(\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}\right)+\lambda u(x,y)=f(x,y); \quad 0\le x,y\le 1; \quad \lambda\ge 0$$

$$u(x,0)=a_0(x); \quad u(x,1)=a_1(x); \quad u(0,y)=b_0(y); \quad u(1,y)=b_1(y)$$

$$\boxed{-u_{j-1,k}-u_{j,k-1}+(4+h^2\lambda)u_{j,k}-u_{j+1,k}-u_{j,k+1}=h^2 f_{j,k}}$$

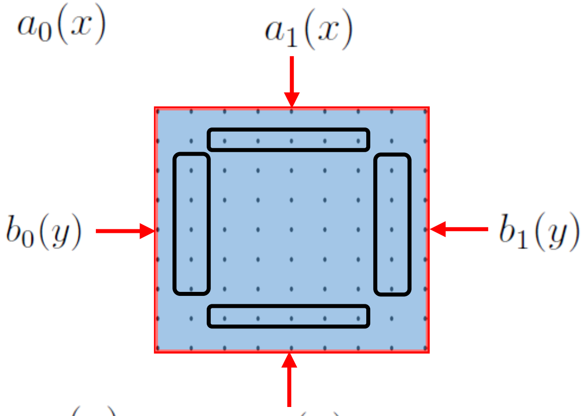
We then get the equation for the interior points

$$-u_{j-1,k}-u_{j,k-1}+(4+h^2\lambda)u_{j,k}-u_{j+1,k}-u_{j,k+1} = h^2 f_{j,k} \quad 2\le j,k\le N-2$$



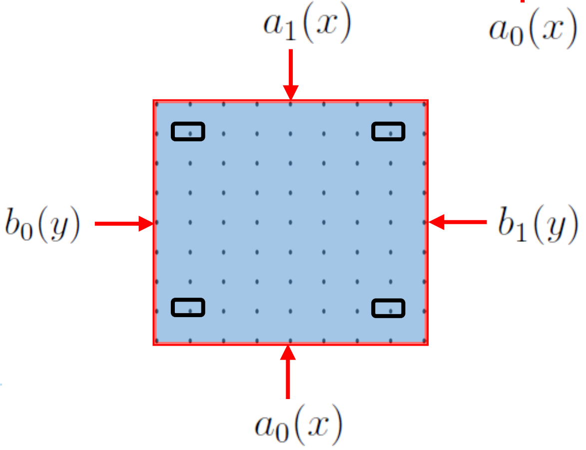
For the edge points, we get

$$\begin{aligned}
-u_{j,k-1}+(4+h^2\lambda)u_{j,k}-u_{j+1,k}-u_{j,k+1} &= h^2 f_{j,k}+b_0(y_k) \quad j=1; \quad 2\le k\le N-2\\
-u_{j-1,k}+(4+h^2\lambda)u_{j,k}-u_{j+1,k}-u_{j,k+1} &= h^2 f_{j,k}+a_0(x_j) \quad k=1; \quad 2\le j\le N-2\\
-u_{j-1,k}-u_{j,k-1}+(4+h^2\lambda)u_{j,k}-u_{j,k+1} &= h^2 f_{j,k}+b_1(y_k) \quad j=N-1; \quad 2\le k\le N-2\\
-u_{j-1,k}-u_{j,k-1}+(4+h^2\lambda)u_{j,k}-u_{j+1,k} &= h^2 f_{j,k}+a_1(x_j) \quad k=N-1; \quad 2\le j\le N-2
\end{aligned}$$

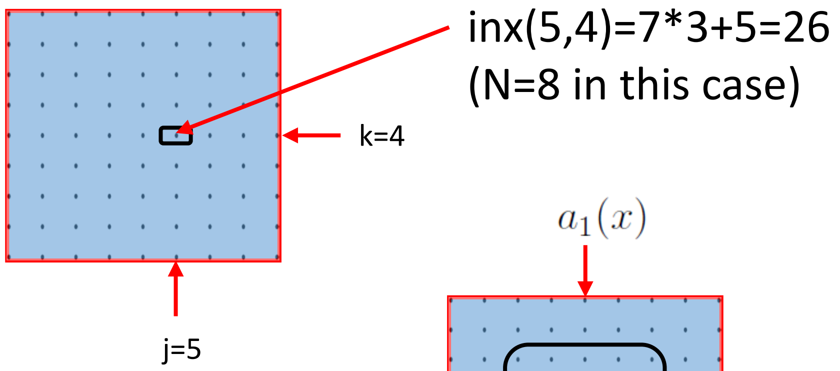


and for the corner points:

$$\begin{aligned}
(4+h^2\lambda)u_{j,k}-u_{j+1,k}-u_{j,k+1} &= h^2 f_{j,k}+a_0(x_j)+b_0(y_k) \quad j=1; \quad k=1\\
-u_{j-1,k}+(4+h^2\lambda)u_{j,k}-u_{j,k+1} &= h^2 f_{j,k}+a_0(x_j)+b_1(y_k) \quad j=N-1; \quad k=1\\
-u_{j,k-1}+(4+h^2\lambda)u_{j,k}-u_{j+1,k} &= h^2 f_{j,k}+a_1(x_j)+b_0(y_k) \quad j=1; \quad k=N-1\\
-u_{j-1,k}-u_{j,k-1}+(4+h^2\lambda)u_{j,k} &= h^2 f_{j,k}+a_1(x_j)+b_1(y_k) \quad j=N-1; \quad k=N-1
\end{aligned}$$



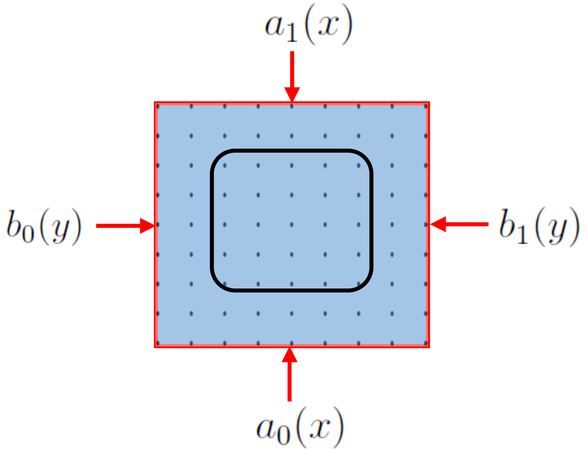
We now define an index function $\boxed{\text{inx}(j,k) = (N-1)(k-1)+j}$. We then get a system of $(N-1)^2$ linear equations in $(N-1)^2$ unknowns $u_{i,j}$ $\quad 1 \le i,j \le N-1$. With $i = inx(j,k)$, we write $w_i \equiv u_{j,k}$. We then write the system of linear equations as $Aw = \phi$. **Notice: Subtract 1 from inx(j,k) if indices run from zero**



For the interior points, we get with $i = \text{inx}(j,k)$

$$A_{i,s} = \begin{cases} -1 & s \in \{i-(N-1), i-1, i+1, i+(N-1)\} \\ 4+h^2\lambda & s = i \\ 0 & \text{otherwise} \end{cases}$$

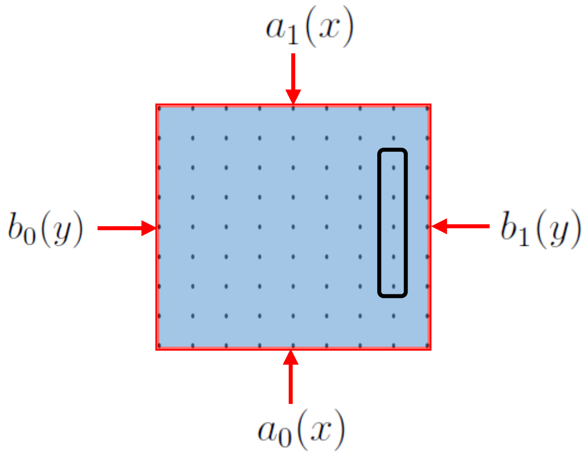
$$\phi_i = h^2 f_{j,k}$$



For the edge points, we get with $i = \text{inx}(N-1,k)$ (the other edges are similar)

$$A_{i,s} = \begin{cases} -1 & s \in \{i-(N-1), i-1, i+(N-1)\} \\ 4+h^2\lambda & s = i \\ 0 & \text{otherwise} \end{cases}$$

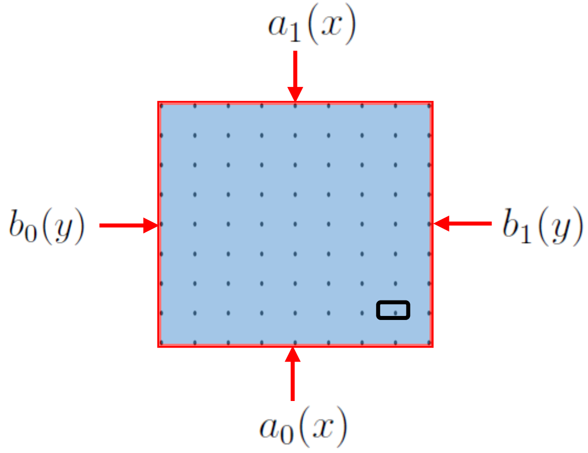
$$\phi_i = h^2 f_{N-1,k} + b_1(y_k)$$

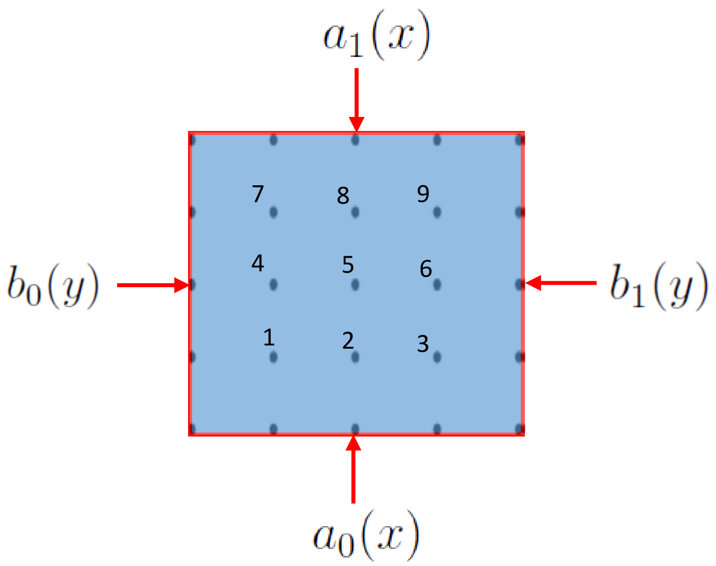


For the corner points, we get with $i = \text{inx}(N-1,1)$ (the other corners are similar)

$$A_{i,s} = \begin{cases} -1 & s \in \{i-1, i+(N-1)\} \\ 4+h^2\lambda & s = i \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_i = h^2 f_{N-1,1} + a_0(x_{N-1}) + b_1(y_1)$$





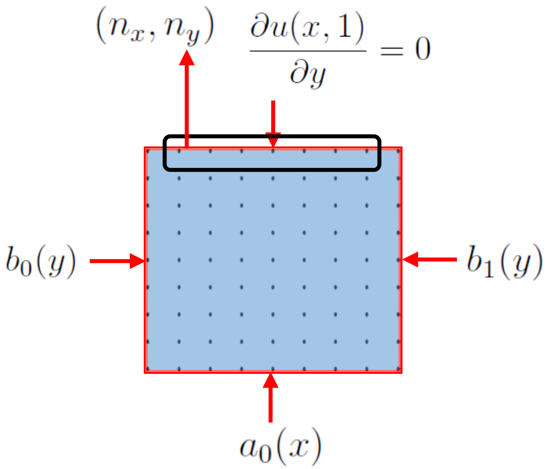
To summarize, we provide the matrix and the right hand side for $N = 4$:

$$\begin{pmatrix} X & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & X & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & X & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & X & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & X & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & X & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & X & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & X & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & X \end{pmatrix}; \quad \phi = \begin{pmatrix} h^2 f_{1,1} + a_0(x_1) + b_0(y_1) \\ h^2 f_{2,1} + a_0(x_2) \\ h^2 f_{3,1} + a_0(x_3) + b_1(y_1) \\ h^2 f_{1,2} + b_0(y_2) \\ h^2 f_{2,2} \\ h^2 f_{3,2} + b_1(y_2) \\ h^2 f_{1,3} + a_1(x_1) + b_0(y_3) \\ h^2 f_{2,3} + a_1(x_2) \\ h^2 f_{3,3} + a_1(x_3) + b_1(y_3) \end{pmatrix} \qquad \text{where } X = 4 + h^2\lambda.$$

Let us now consider a model problem with combined fixed and free boundary conditions where we have free boundary conditions at $y = 1$:

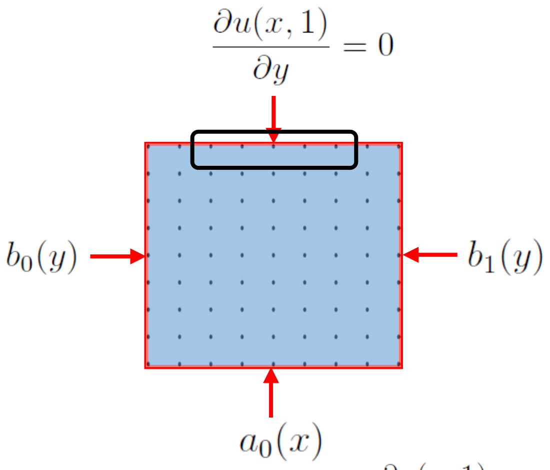
$$-\left(\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2}\right) + \lambda u(x,y) = f(x,y); \quad 0 \leq x, y \leq 1; \quad \lambda \geq 0$$

$$\boxed{\frac{\partial u(x,1)}{\partial y} = 0;} \quad u(x,0) = a_0(x); \quad u(0,y) = b_0(y); \quad u(1,y) = b_1(y)$$



We implement the free boundary conditions by introducing numerical Poissons equation at the free boundary. For the edge at $y = 1$ corresponding to $k = N$, we get

$$-u_{j-1,N} - u_{j,N-1} + (4 + h^2\lambda)u_{j,N} - u_{j+1,N} - \boxed{u_{j,N+1}} = \quad h^2 f_{j,N} \quad 2 \leq j \leq N-2$$

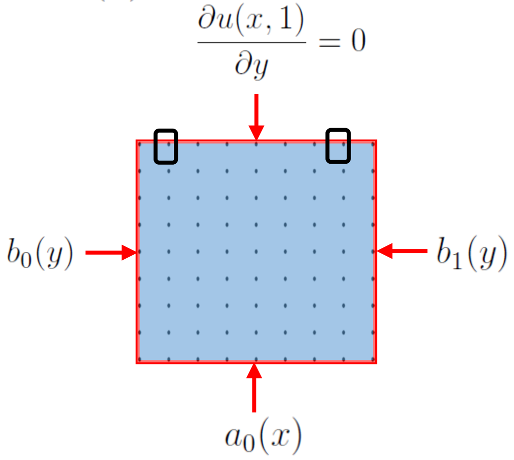


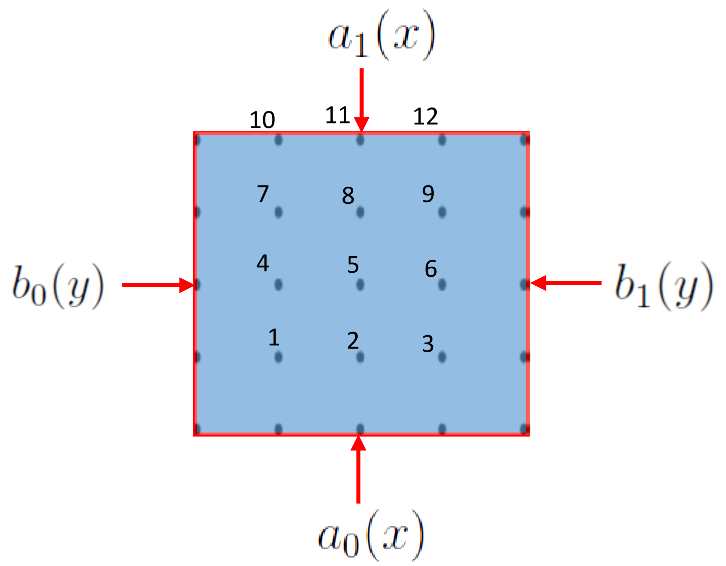
We now implement the boundary condition $\frac{\partial u(x,1)}{\partial y} = 0$ by setting the "phantom point" $\boxed{u_{j,N+1} \equiv u_{j,N-1}}$. We then get

$$-u_{j-1,N} - 2u_{j,N-1} + (4 + h^2\lambda)u_{j,0} - u_{j+1,N} \quad = \quad h^2 f_{j,N} \quad 2 \leq j \leq N-2$$

For the two corner points at the free edge, we get

$$\begin{aligned} -2u_{1,N-1} + (4 + h^2\lambda)u_{1,N} - u_{2,N} &= h^2 f_{1,N} + b_0(y_N) \\ -u_{N-2,N} - 2u_{N-1,N-1} + (4 + h^2\lambda)u_{N-1,N} &= h^2 f_{N-1,N} + b_1(y_N) \end{aligned}$$





Notice that you will typically need to modify the index function inx(j,k) with free boundary conditions as more points are included (this example being the exception). Also non-square shapes lead to different index functions.

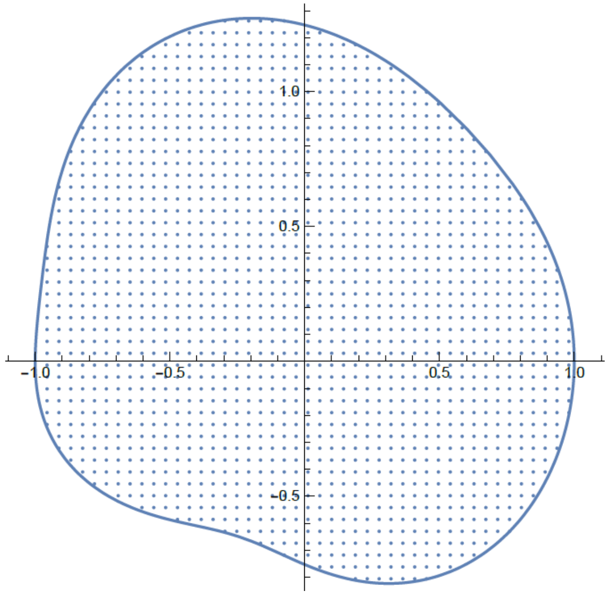
We now get $(N-1)N$ equations in $(N-1)N$ unknowns. The coefficient matrix and right hand side looks as follows for $N = 4$:

$$\begin{pmatrix}
X & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & X & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & X & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & X & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & X & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & -1 & X & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & X & -1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & X & -1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & X & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & X & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & -1 & X & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & -1 & X
\end{pmatrix}
\qquad
\phi = \begin{pmatrix}
h^2 f_{1,1} + a_0(x_1) + b_0(y_1) \\
h^2 f_{2,1} + a_0(x_2) \\
h^2 f_{3,1} + a_0(x_3) + b_1(y_1) \\
h^2 f_{1,2} + b_0(y_2) \\
h^2 f_{2,2} \\
h^2 f_{3,2} + b_1(y_2) \\
h^2 f_{1,3} + b_0(y_3) \\
h^2 f_{2,3} \\
h^2 f_{3,3} + b_1(y_3) \\
h^2 f_{1,4} + b_0(y_4) \\
h^2 f_{2,4} \\
h^2 f_{3,4} + b_1(y_4)
\end{pmatrix}$$

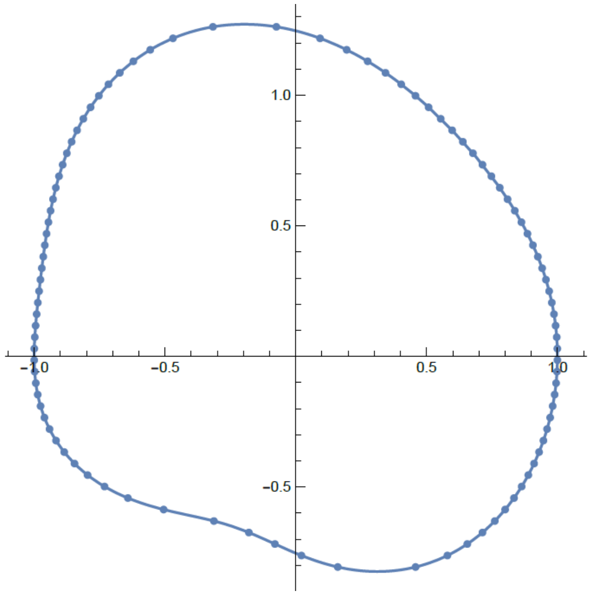
Band matrix

- In all cases, we have a band-width not larger than N from the diagonal. Hence the algorithm in Sec. 2.4.2 p. 58-61 in NR can be applied.
- Complexity will then be O(N^4) instead of O(N^6) for LU decomposition and O(N^3) instead of O(N^4) for forward reduction and back substitution
- Similar band structures occur for arbitrarily shaped regions however with a bandwidth that varies from row to row.
- Hence, it is a numerical crime not to use this computational advantage.

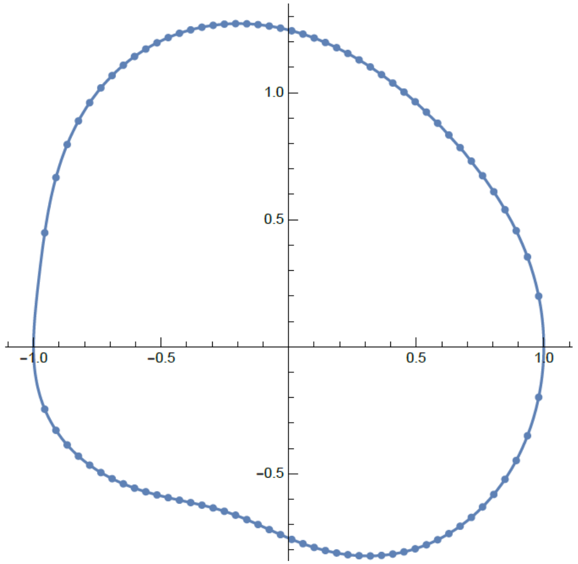
General 2D shapes



Interior grid points

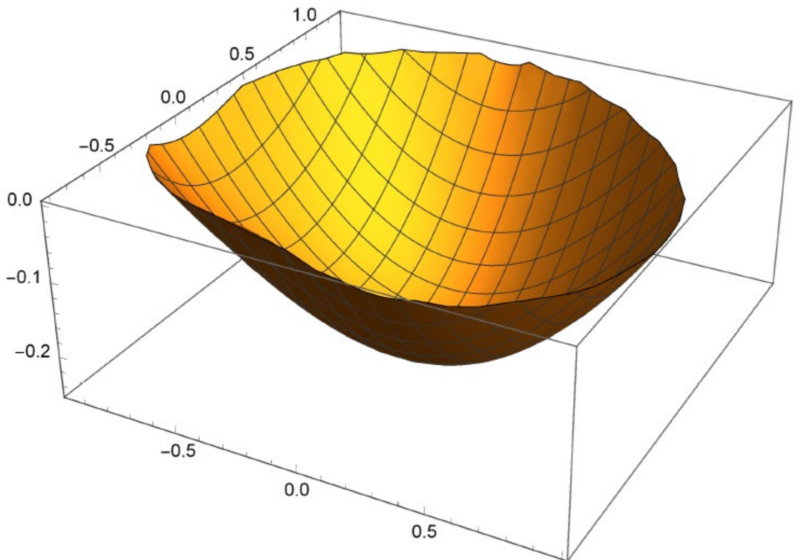


Row boundary points

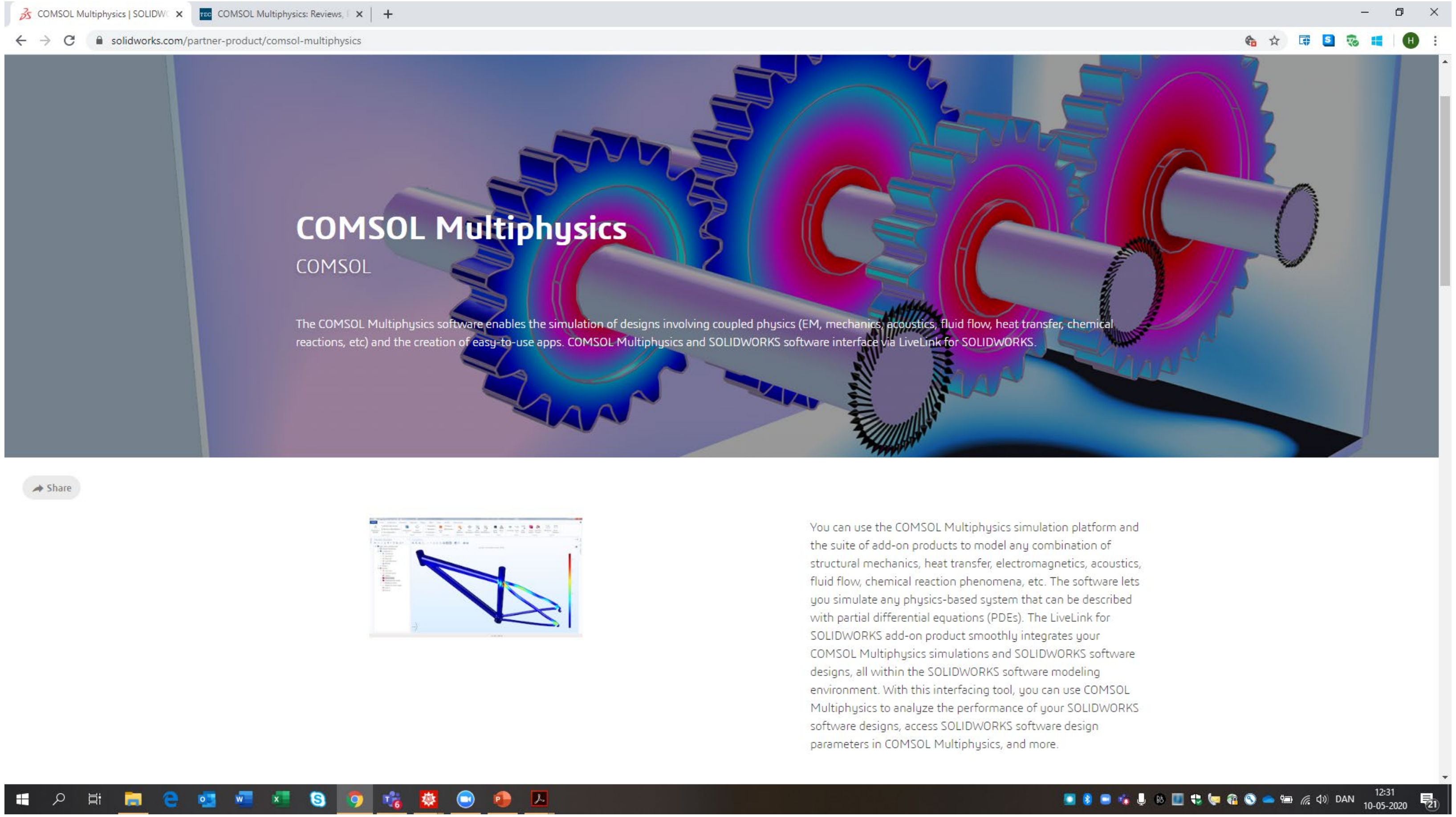


Column boundary points

Implement boundary conditions using phantom points immediately outside the boundary and interpolate (here same idea for Dirichlet and Neumann conditions)



Doing this for complex PDE's and going to PDE's in 3 dimensions is complicated. Software packages are commercially available (expensive) (typically finite element based, but ideas are the same). If you use these packages at some stage, be aware to estimate errors and make sure to use the packages correctly. Buildings have crashed because of wrong use.



Exercise:

Consider the elliptic partial differential equation

$$-\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 1 + x + y \qquad \text{for} \qquad (x, y) \in \Omega$$

where $\Omega = \{(x, y)|0 < x < 1, 0 < y < 1\}$ and $u(x, y) = 0$ for $(x, y) \in \partial\Omega$

- Set up the system of linear equations for N = 4,8,16,... and solve
- Perform Richardson extrapolation and error estimation for $u(0.5, 0.5)$