Mandatory 3

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Exercise 5

NR3's implementation of `DErule` from "derule.h" is used as a base for solving the integral in question 5.3

The equations for Extended Midpoint and DErule are set up as follows:

```
// For extended midpoint
double eqn(double x) { return (cos(pow(x, 3)) * exp(-x)) / sqrt(x); }

// For DErule
double eqn_derule(double x, double delta) {
  // If x is small use delta instead to avoid division by zero
  if (abs(x) < 1e-6) {
    return (cos(pow(x, 3)) * exp(-x)) / sqrt(delta);
  } else {
    return (cos(pow(x, 3)) * exp(-x)) / sqrt(x);
  }
}
```

5.1

The analytical solution using Extended Midpoint is:

$$\int_a^b \frac{\cos(x^3)\exp(-x)}{\sqrt{x}}dx$$

$$\approx h \cdot \sum_{i=0}^{N-2} \left(\frac{\cos\left(\left(a+h\cdot i+\frac{h}{2}\right)^3\right)\exp\left(-(a+h\cdot i+\frac{h}{2})\right)}{\sqrt{(a+h\cdot i+\frac{h}{2})}} \right) + O\left(\frac{1}{N^2}\right) \quad \text{where} \quad h=\tfrac{(b-a)}{N-1}$$

$$= (b-a)\cdot \left(\frac{\cos\left(\left(a+\frac{(b-a)}{2-1}\cdot 0+\frac{\frac{(b-a)}{2-1}}{2}\right)^3\right)\exp\left(-(a+\frac{(b-a)}{2-1}\cdot 0+\frac{\frac{(b-a)}{2-1}}{2})\right)}{\sqrt{a+\frac{(b-a)}{2-1}\cdot 0+\frac{\frac{(b-a)}{2-1}}{2}}} \right) + O\left(\frac{1}{2^2}\right) \quad \text{for } N=2$$

$$= (b-a)\cdot \left(\frac{\cos\left(\left(a+\frac{b-a}{2}\right)^3\right)\exp\left(-a-\frac{b-a}{2}\right)}{\sqrt{a+\frac{b-a}{2}}} \right) + O\left(\frac{1}{4}\right)$$

5.2

The accuracy is computed as follows using Richardson Extrapolation Error:

```
std::vector<double>
richardson_extrapolation_error(const std::vector<double> &A_k,
                               const double alpha_k_order_expected) {
  std::vector<double> A_R = {NAN}; // No error on the first
  for (size_t i = 1; i < A_k.size(); i++) {
    const double A_1 = A_k[i - 1];
    const double A_2 = A_k[i];
    // pow(2, alpha_k_order) as we use N-1=1,2,4,8
    double error = (A_2 - A_1) / (pow(2, alpha_k_order_expected) - 1);
    A_R.push_back(error);
  }
  return A_R;
}
```

The generated table is (using 'utils/quadrature_table.h'):

```
////////////////////////////////////// Extended Midpoint: ////////////////////////////////
|  i   |      A(i)     | A(i-1)-A(i)  |  alpha^k    | Rich error | Order est. | f comps  |
|------|---------------|--------------|-------------|------------|------------|----------|
|  1   |  -0.055695    |              |             |            |     1      |
|  2   |   0.380737    |   -0.436432  |             |  0.145477  |            |     2    |
|  3   |   0.629306    |   -0.248568  |  1.755781   |  0.082856  |   0.81211  |     4    |
|  4   |   1.010516    |   -0.381209  |  0.652052   |  0.127069  |  -0.61693  |     8    |
|  5   |   1.135519    |   -0.125003  |  3.049604   |  0.041667  |   1.60862  |    16    |
|  6   |   1.214732    |  -0.0792131  |  1.578058   |  0.026404  |   0.65815  |    32    |
|  7   |   1.277964    |  -0.0632325  |  1.252728   |  0.021077  |   0.32507  |    64    |
|  8   |   1.322794    |  -0.0448294  |  1.410512   |  0.014943  |   0.49621  |   128    |
|  9   |   1.354316    |  -0.0315223  |  1.422148   |  0.010507  |   0.50807  |   256    |
|  10  |   1.376536    |  -0.0222198  |  1.418658   |  0.007406  |   0.50452  |   512    |
|  11  |   1.392222    |  -0.0156861  |  1.416521   |  0.005228  |   0.50235  |   1024   |
|  12  |   1.403305    |  -0.0110825  |  1.415391   |  0.003694  |   0.50120  |   2048   |
|  13  |   1.411138    |  -0.0078332  |  1.414810   |  0.002611  |   0.50060  |   4096   |
|  14  |   1.416676    |  -0.0055377  |  1.414514   |  0.001845  |   0.50030  |   8192   |
|  15  |   1.420591    |  -0.0039153  |  1.414364   |  0.001305  |   0.50015  |  16384   |
|  16  |   1.423360    |  -0.0027684  |  1.414289   |  0.000922  |   0.50007  |  32768   |
```

The function computations for each N-1=pow(2,i) is shown in the table.

5.3

DErule found the integral to be:

$$1.4300433455$$

using 127 f-calculation.