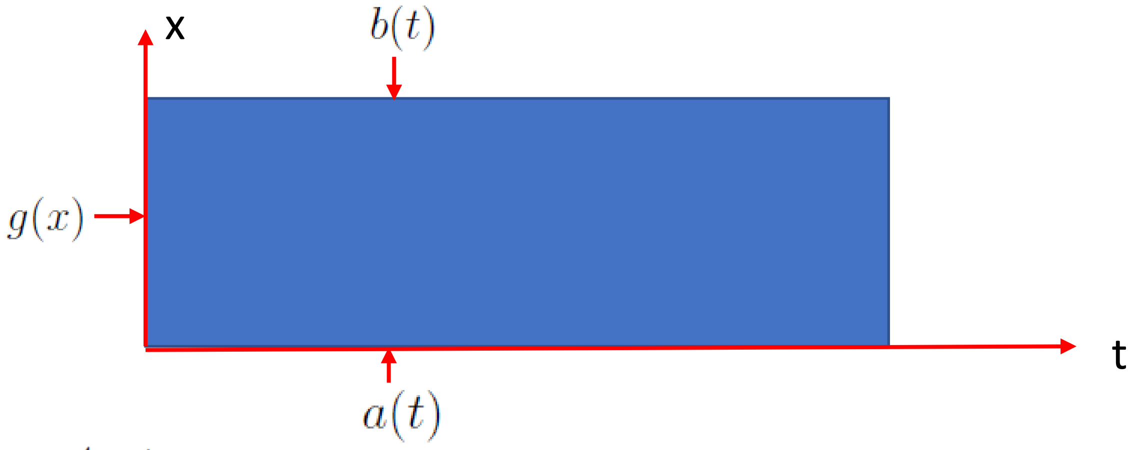
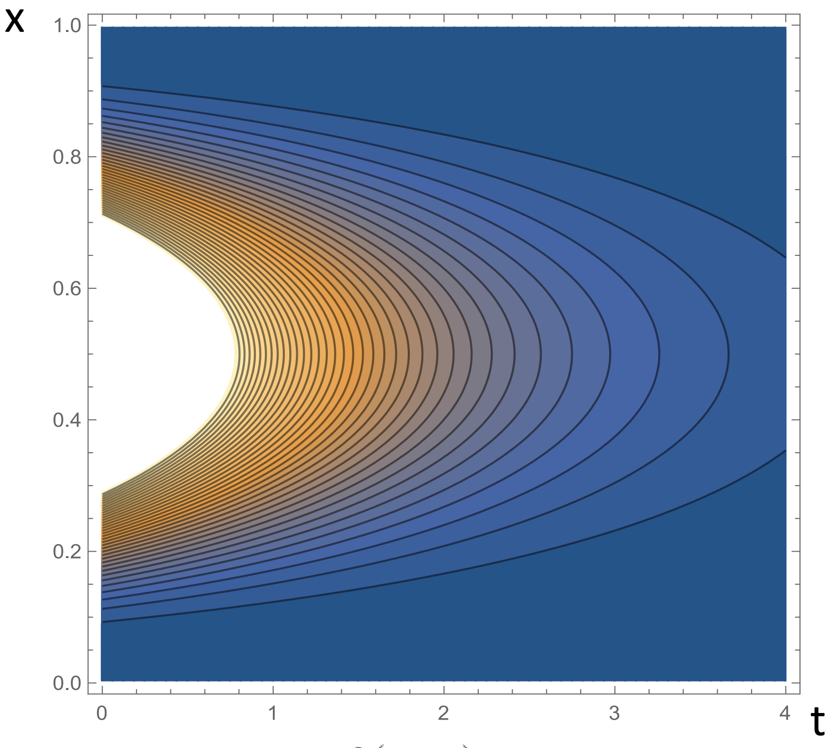
Parabolic PDE's. Model problem:

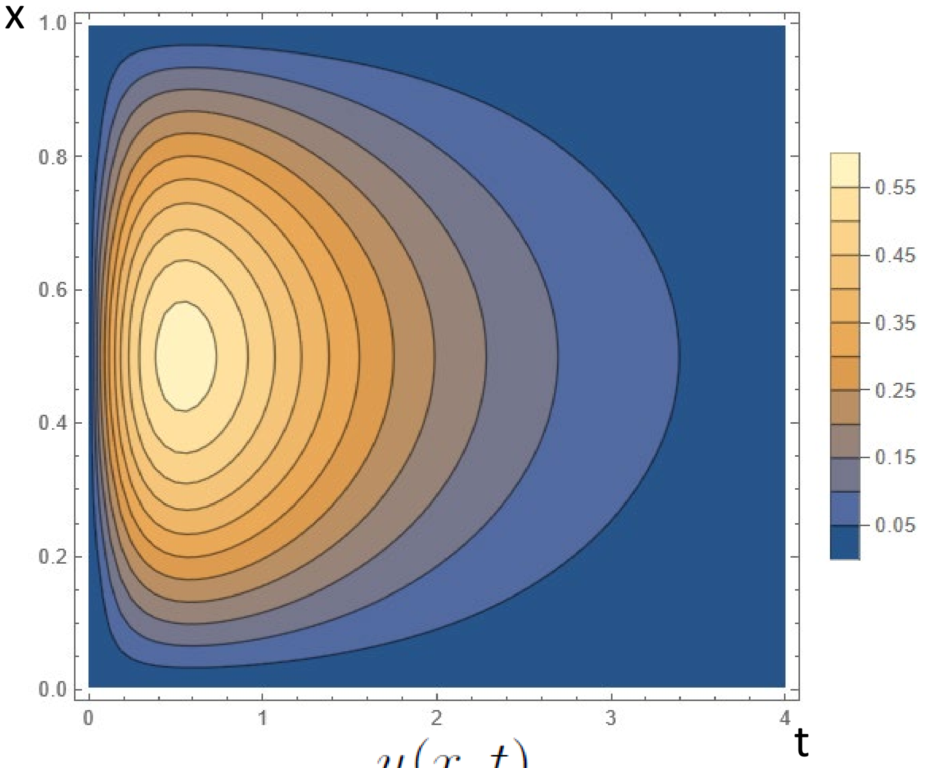
$$\begin{cases} \dfrac{\partial u}{\partial t} = \overset{\text{Diffusion constant}}{\boxed{\alpha}} \dfrac{\partial^2 u}{\partial x^2} + \overset{\text{Source}}{\boxed{f(x,t)}} & 0 < t,\ 0 < x < 1 \qquad \alpha > 0 \\ u(x,0) = g(x), & 0 \le x \le 1 \\ u(0,t) = a(t),\ u(1,t) = b(t), & 0 < t \end{cases}$$



Example: $g(x) = 0, a(t) = 0, b(t) = 0, \alpha = 0.3$ and $f(x,t) = 1000x^4(1-x)^4e^{-t}$



$f(x,t)$



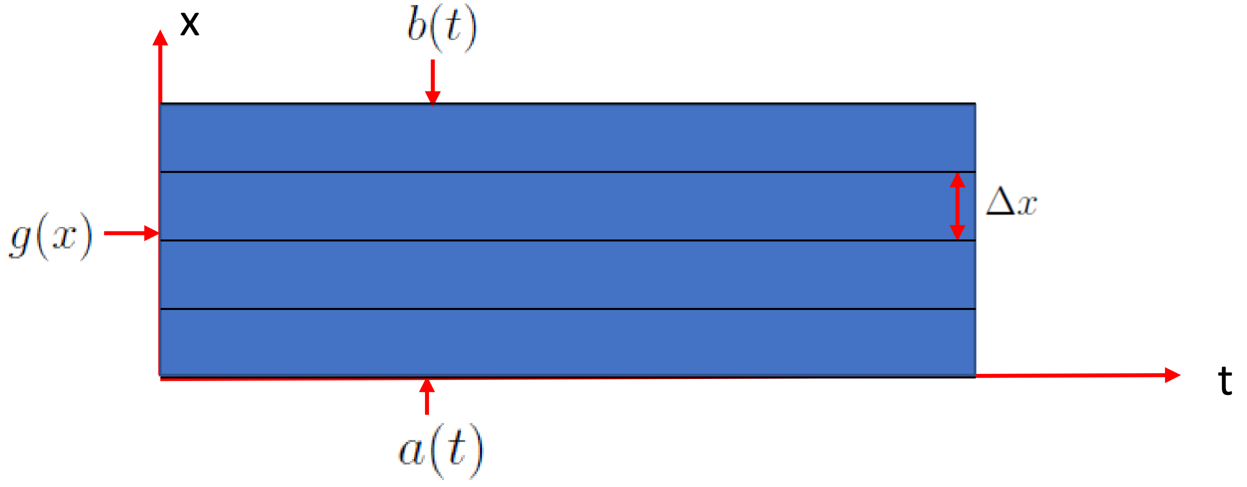
$u(x,t)$

Parabolic PDE's. Model problem:

Diffusion constant

Source

$$\begin{cases} \dfrac{\partial u}{\partial t} = \boxed{\alpha}\,\dfrac{\partial^2 u}{\partial x^2} + \boxed{f(x,t)} & 0 < t,\ 0 < x < 1 \qquad \alpha > 0 \\ u(x,0) = g(x), & 0 \le x \le 1 \\ u(0,t) = a(t),\ u(1,t) = b(t), & 0 < t \end{cases}$$



$$\Delta x = 1/N, \quad x_j = j\,\Delta x$$

$$u_j(t) \simeq u(x_j,t) \qquad f_j(t) \equiv f(x_j,t)$$

Semidiscrete form (discretization in x but not in t):

$$\begin{cases} \dfrac{du_j}{dt}(t) = \dfrac{\alpha}{(\Delta x)^2}\big(u_{j-1}(t) - 2u_j(t) + u_{j+1}(t)\big) + f_j(t), & j = 1, \cdots, N-1 \\ u_j(0) = g(x_j) & j = 0, \cdots, N \\ u_0(t) = a(t),\ u_N(t) = b(t), & t > 0 \end{cases}$$

$$
\begin{cases}
\dfrac{du_j}{dt}(t) = \dfrac{\alpha}{(\Delta x)^2}\big(u_{j-1}(t) - 2u_j(t) + u_{j+1}(t)\big) + f_j(t), & j = 1, \cdots, N-1 \\
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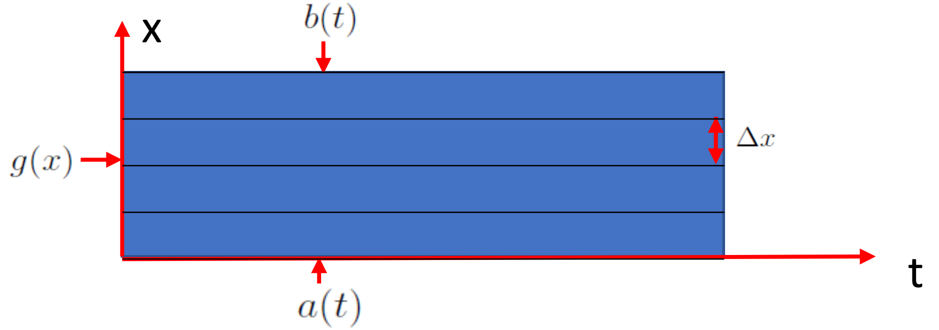
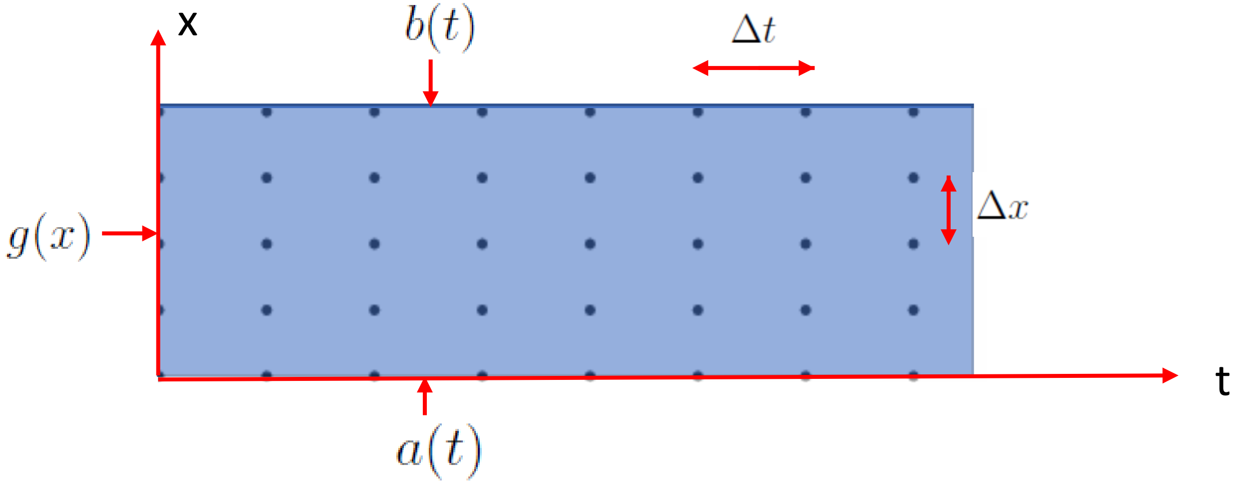
u_0(t) = a(t), \; u_N(t) = b(t), & t > 0
\end{cases}
$$



Semidiscrete form is a set of coupled ODE's (initial value problem). Any method is valid. Here we consider Euler (for pedagogical reasons) and the Trapezoidal method (here called Crank-Nicolson),



$$t_n = n\,\Delta t$$

$$u_j^n \approx u_j(t_n) \qquad\qquad f_j^n \equiv f(x_j, t_n)$$

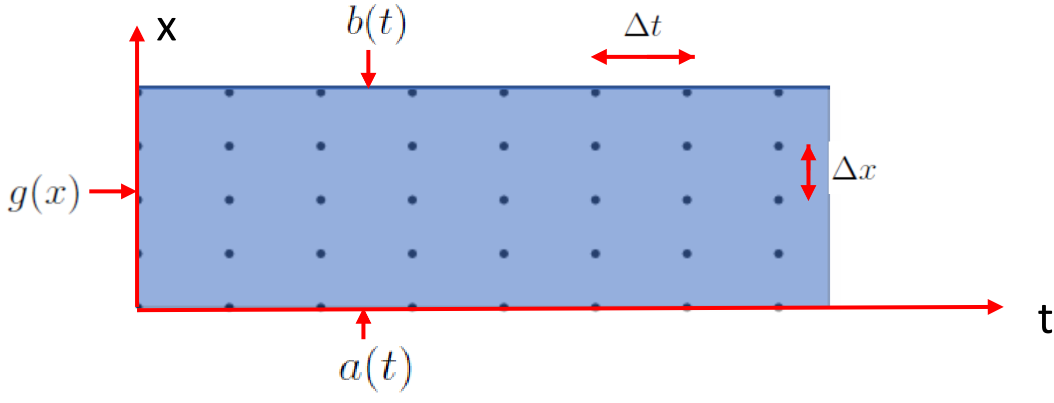
Euler:

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$$
\begin{cases}
u_j^{n+1} = u_j^n + \alpha\dfrac{\Delta t}{(\Delta x)^2}\left(u_{j-1}^n - 2u_j^n + u_{j+1}^n\right) + \Delta t\, f_j^n & 1 \le j \le N-1 \\
u_j^0 = g(x_j) \\
u_0^n = a(t_n)\;, \quad u_N^n = b(t_n)
\end{cases}
$$

Analysis of Euler's method:

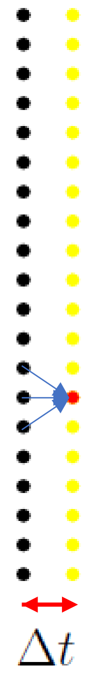
$$\begin{cases} u_j^{n+1} &= u_j^n + \alpha \dfrac{\Delta t}{(\Delta x)^2}\left(u_{j-1}^n - 2u_j^n + u_{j+1}^n\right) + \Delta t\, f_j^n \\ u_j^0 &= g(x_j) \\ u_0^n &= a(t_n)\ , \quad u_N^n = b(t_n) \end{cases}$$



Limit in information diffusion in a single time step with Eulers method

Discretization error: $O(\Delta t) + O((\Delta x)^2)$

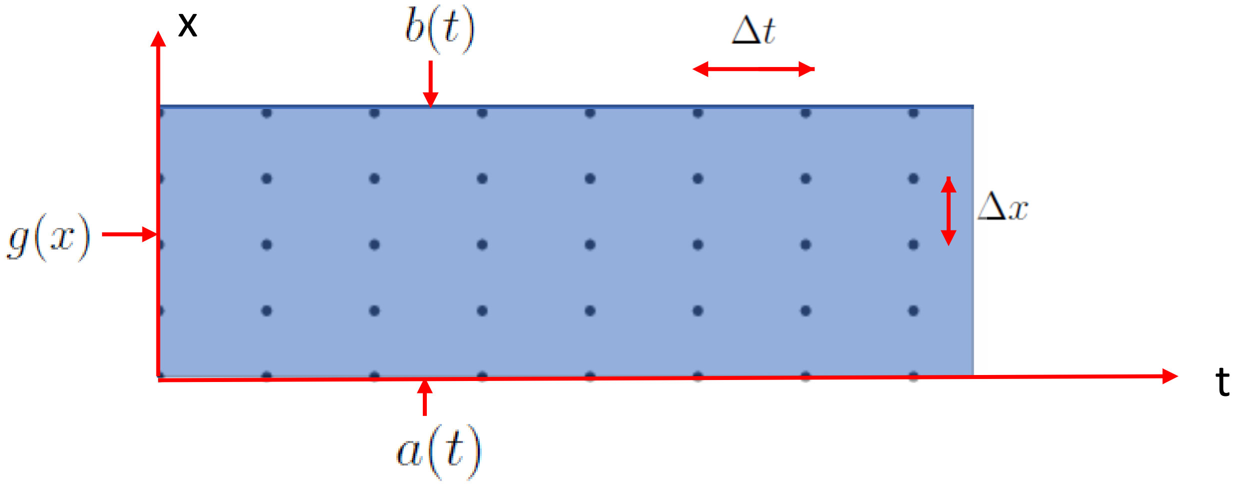
Stability criterion: $\alpha \frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{2}$ (can be proved)



Advantage: Explicit formulas for the u_j^{n+1}'s

Disadvantage: Very small Δt's needed to achieve good accuracy and in particular stability !!!

$$
\begin{cases}
\dfrac{du_j}{dt}(t) = \dfrac{\alpha}{(\Delta x)^2}\left(u_{j-1}(t) - 2u_j(t) + u_{j+1}(t)\right) + f_j(t), & j = 1, \cdots, N-1 \\
u_j(0) = g(x_j) & j = 0, \cdots, N \\
u_0(t) = a(t),\ u_N(t) = b(t), & t > 0
\end{cases}
$$



Semidiscrete form is a set of coupled ODE's (initial value problem). Any method is valid. Here we consider Euler (for pedagogical reasons) and the Trapezoidal method (for usage),

$$t_n = n\,\Delta t$$

$$u_j^n \approx u_j(t_n) \qquad f_j^n = f(u_j^n, x_j, t_n)$$

Trapezoidal (Crank-Nicolson):

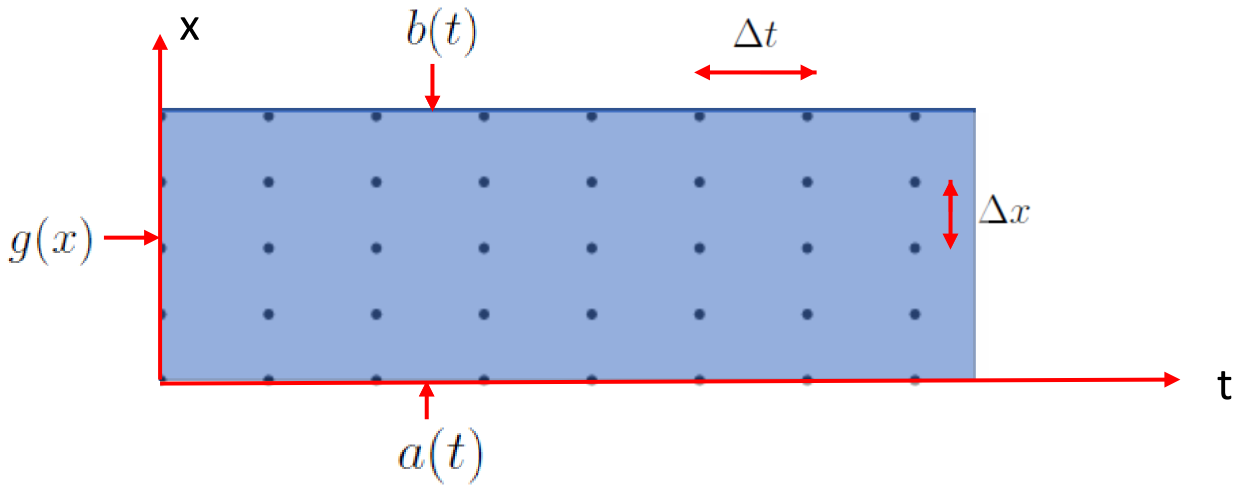
$$y_{n+1} = y_n + \frac{h}{2}\left(f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right) + \mathcal{O}(h^3)$$

Abbreviation:

$$D_x^2 u_j^m \equiv \frac{1}{(\Delta x)^2}(u_{j-1}^m - 2u_j^m + u_{j+1}^m)$$

$$
\begin{cases}
u_j^{n+1} = u_j^n + \dfrac{\alpha \Delta t}{2}\left(D_x^2 u_j^{n+1} + D_x^2 u_j^n\right) + \dfrac{\Delta t}{2}\left(f_j^{n+1} + f_j^n\right) \\
u_j^0 \quad = \quad g(x_j) \\
u_0^n \quad = \quad a(t_n)\ , \quad u_N^n = b(t_n)
\end{cases}
$$

$$\begin{cases} u_j^{n+1} = u_j^n + \dfrac{\alpha\Delta t}{2}\left(D_x^2 u_j^{n+1} + D_x^2 u_j^n\right) + \dfrac{\Delta t}{2}\left(f_j^{n+1} + f_j^n\right) \\ u_j^0 \quad = \quad g(x_j) \\ u_0^n \quad = \quad a(t_n)\ , \quad u_N^n = b(t_n) \end{cases}$$



Terms with unknowns u_j^{n+1} collected on the left hand side:

$$u_j^{n+1} - \frac{\alpha\Delta t}{2} D_x^2 u_j^{n+1} = u_j^n + \frac{\alpha\Delta t}{2} D_x^2 u_j^n + \frac{\Delta t}{2}\left(f_j^{n+1} + f_j^n\right)$$

$$r = \alpha\Delta t/(\Delta x)^2 \qquad D_x^2 u_j^m \equiv \frac{1}{(\Delta x)^2}\left(u_{j-1}^m - 2u_j^m + u_{j+1}^m\right)$$

$$\begin{cases} -\dfrac{1}{2} r\, u_{j-1}^{n+1} + (1+r) u_j^{n+1} - \dfrac{1}{2} r\, u_{j+1}^{n+1} = \dfrac{1}{2} r\, u_{j-1}^n + (1-r) u_j^n + \dfrac{1}{2} r\, u_{j+1}^n + \dfrac{\Delta t}{2}\left(f_j^{n+1} + f_j^n\right) \qquad j = 1, \ldots, N-1 \\ u_j^0 \quad = \quad g(x_j) \\ u_0^n \quad = \quad a(t_n)\ , \quad u_N^n = b(t_n) \end{cases}$$

$$
\begin{cases}
-\frac{1}{2}r\,u_{j-1}^{n+1} + (1+r)u_j^{n+1} - \frac{1}{2}r\,u_{j+1}^{n+1} = \frac{1}{2}r\,u_{j-1}^{n} + (1-r)u_j^{n} + \frac{1}{2}r\,u_{j+1}^{n} + \frac{\Delta t}{2}\left(f_j^{n+1} + f_j^{n}\right) & j = 1, \ldots, N-1 \\
u_j^0 \quad = \quad g(x_j) & \\
u_0^n \quad = \quad a(t_n)\,, \quad u_N^n = b(t_n) & r = \alpha \Delta t/(\Delta x)^2
\end{cases}
$$

Discretization error: $O((\Delta t)^2) + O((\Delta x)^2)$

Run with N=2,4,8,16, etc. and keep $\Delta t/\Delta x$ =c where c is a constant.

With $h = \Delta x$, we get a discretization error:

$$O((\Delta t)^2) + O((\Delta x)^2) \equiv O(h^2(c^2+1)) \equiv O(h^2)$$

Advantages: No stability issues because of usage of the Trapezoidal method and we can run with larger Δt 's because of the second order accuracy.

Need to solve a system of linear equations. But matrix is tridiagonal (-r/2, (1+r), -r/2), so still O(N) complexity.

ONLY LACK OF KNOWLEDGE IS THE REASON WHY EULER'S METHOD IS SO WIDELY APPLIED !!!

Exercise:

Consider

$$\begin{cases} \dfrac{\partial u}{\partial t} = \alpha \dfrac{\partial^2 u}{\partial x^2} + f(x,t) & 0 < t,\ 0 < x < 1 \qquad \alpha > 0 \\ u(x,0) = g(x), & 0 \le x \le 1 \\ u(0,t) = a(t),\ u(1,t) = b(t), & 0 < t \end{cases}$$

with $g(x) = x^4$, $a(t) = 0$, $b(t) = 1$, $\alpha = 1$ and $f(x,t) = x(1-x)\cos(t)\exp[-t/10]$.

i) Write down the semidiscrete form for this problem

ii) Write down the coefficient matrix and the right hand side for the system of linear equations for this problem corresponding to the Crank-Nicolson method.

iii) Apply Crank-Nicolson to obtain an accuracy on $u(\frac{1}{2}, 20)$ of 10^{-4}. Use $N = 2, 4, 8, 16, \ldots$ and $\Delta t = \Delta x$ (corresponding to $c = 1$).