Numerical Methods: 24 Hours Take-home examination, June 2025

This examination consists of 5 exercises. Exercises 1–4 are for all students. Exercise 5a is for students taking the 5 ECTS version ONLY, and Exercise 5b is ONLY for students taking the 7 ECTS version. If you answer the wrong exercise of 5a and 5b, you will get zero points for that. Each question has been assigned a weight (points). The total number of points is 100.

For the exam, you will need to be able to read matrices and vectors from files having the following format for an $M \times N$ matrix:

$$\begin{array}{r} M \\ N \\ Row1 \\ \dots \\ RowM \end{array}$$

where vectors always have $N = 1$. An example with a 3×2 matrix could be

$$\begin{array}{ll} 3 & \\ 2 & \\ 3.04464 & 5.06464 \\ -0.6454 & 0.435435 \\ 4.05454 & -1222.933435 \end{array}$$

You must hand in a zip file containing a report and all your used code. Concerning the report, it needs to be CLEARLY readable (unreadable parts will be assessed as wrong). Whenever the word "state" is used in the questions, the answer MUST be present in the report. Wherever it says "Submit the used code", the used code MUST be handed in. Otherwise, you will get zero points for your answer. Clearly name your code files so that it is easy to see what exercise the code is used for.

Notice the following rules:

- It is allowed to use general purpose methods, text or code that are part of NR or elsewhere publicly available, including methods, text or code that has been uploaded by Jens or me during the course. However, there must for each use be a CLEAR marking **stated in the report** including where the methods/text/used code was taken from. In all cases, you are solely responsible for the correctness of used methods, text and code. All methods, text and code that **explicitly handles the problems from the exam exercise** MUST be written by yourself.
- It is NOT allowed to share your answers (or parts of these), or to communicate with other people about the exercise. This includes other NM students taking this 24 hours take-home examination.

I wish you all the best for the forthcoming 24 hours!

Best regards, Henrik

Exercise 1 (20 points)

Consider a Linear Least Squares problem with a 40×8 design matrix $\mathbf{A}$ and the right hand side $\mathbf{b}$ that are given in *Ex1A.dat* and *Ex1b.dat* respectively.

i) (5 points) Find the Singular Value Decomposition $\mathbf{A} = \mathbf{U}\mathbf{W}\mathbf{V}^T$. State the diagonal elements in $\mathbf{W}$. Submit the used code.

ii) (5 points) There is a single element in $\mathbf{W}$ that is basically zero. Use the information from the SVD matrices to state a unit vector in the null space of $\mathbf{A}$.

The SVD method automatically removes the singularity when computing the solution.

iii) (5 points) Use the Singular Value Decomposition to compute the solution $\mathbf{x}$ to $\mathbf{A}\mathbf{x} = \mathbf{b}$. State the solution $\mathbf{x}$. Submit the used code.

iv) (5 points) State an estimate of the accuracy on the solution $\mathbf{x}$. State an explanation of how you computed the accuracy. Submit the used code.

Exercise 2 (15 points)

Consider the equations

$$\begin{aligned}3x_0 + x_1 \sin(x_2) - \cos(x_0) + \cos\left(x_1^2\right) + 4.2 &= 0\\ 3x_1 + x_0 x_2 x_3 + \sin(x_1) - 5.1 &= 0\\ -x_1^2 + x_2 x_3^2 + 3x_2 + 5.2 &= 0\\ x_0 + 3x_3 + \sin\left(x_2^2 x_3^2\right) + \cos(x_1) - 2.3 &= 0\end{aligned}$$

i) (3 points) With $x_0 = -0.7$, $x_1 = 1.2$, $x_2 = 2.3$, $x_3 = -4.1$ state (with at least 7 digits) the values of the left hand side of the four equations. (HINT: you should get something around $(2.36, 6.03, 49.32, -14.12)$.

ii) (6 points) Search for a solution to the equations by performing 7 iterations with the globally convergent Newton method with initial guess $(x_0, x_1, x_2, x_3) = (0, 0, 0, 0)$. State for each iteration the values of x_0, x_1, x_2, x_3 and for each iteration whether backtracking was applied, and if so, what the value of λ was. Submit the used code.

iii) (6 points) State an estimate the accuracy after 7 iterations. The estimate must be based on the data obtained from the 7 iterations and must be stated with a clear argument of how you computed it. Without such an argument, there is no points for the answer.

Exercise 3 (25 points)

Consider lane driving without the possibility of overtaking. A behavior of a car driver depending on the car ahead can be formulated as the differential equation

$$x''(t) = a_{max}\left[1 - \left[\frac{x'(t)}{v_{des}}\right]^4 - \left(\frac{D_{des}(x'(t), x'(t) - X'_F(t))}{X_F(t) - x(t)}\right)^2\right]$$

where $x(t)$ is the position of the car, $X_F(t)$ is the position of the car ahead, v_{des} is the desired velocity that the car driver would choose if there was no other car ahead, a_{max} is the maximal acceleration of the car and $D_{des}(v, \Delta v) \geq 0$ is the car drivers preferred distance to the car ahead as function of the velocity of the drivers own car and the difference in velocity to the car ahead. This function can be modeled as

$$D_{des}(v, \Delta v) = D_0 + \text{Max}\{0, vT_{react} + \frac{v\Delta v}{2a_{com}}\}$$

where D_0 is the closest distance the car driver would like, T_{react} is the driver's reaction time and $a_{com} > 0$ is a deceleration where the driver feels comfortable. Insertion yields

$$x''(t) = a_{max}\left[1 - \left[\frac{x'(t)}{v_{des}}\right]^4 - \left(\frac{D_0 + \text{Max}\left\{0, x'(t)T_{react} + \frac{x'(t)(x'(t) - X'_F(t))}{2a_{com}}\right\}}{X_F(t) - x(t)}\right)^2\right] \tag{1}$$

i) (5 points) Rewrite Eq.(1) into a set of two first order ODE's. State the two first order ODE's.

We from now on choose $a_{max} = 4$, $v_{des} = 25$, $D_0 = 50$, $T_{react} = 1.5$ and $a_{com} = 2$. The initial conditions are $t_0 = -10$, $x(t_0) = 0$, $x'(t_0) = 15$. The car ahead has the trajectory

$$\begin{aligned} X_F(t) &= 250 + 15t - 5\sqrt{1+t^2} \\ X'_F(t) &= 15 - \frac{5t}{\sqrt{1+t^2}} \end{aligned}$$

ii) (5 points) State with at least 7 digits the value of $x''(t_0)$. Submit the used code. HINT: You should get $x''(t_0) \simeq -1.20$.

Around $t = 0$, the car ahead has to perform a reduction in speed as there is queue further ahead (see the figure). We wish to study the trajectory of the drivers car both absolutely and with respect to the car ahead. We now study the behaviour of the drivers car between $t = -10$ and $t = 10$.

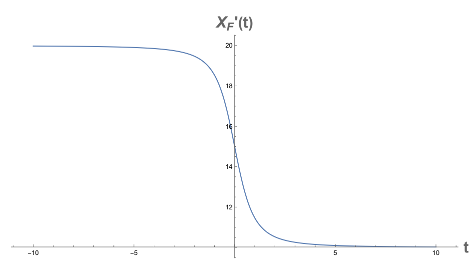

Figure 1: The velocity $X_F(t)$ of the car ahead

iii) (5 points) Use the Midpoint method with $N = 80$; $h = 0.25$ to generate a solution for $x(t)$; $-10 \leq t \leq 10$. State the value of $x(t)$ at $t = 10$. State plots of $x(t)$, $x'(t)$ and $X_F(t) - x(t)$. Submit the used code.

iv) (3 points) State an assessment of whether your results in iii) seem correct.

You may in the last question assume that the global order is 2 as expected.

v) (7 points) Use now the Midpoint Method with the usual approach with subdivisions $N = 20, 40, 80, 160, \ldots$; $h = 20.0/N$. Use as few number of subdivisions as possible to reach an estimated accuracy better than 2×10^{-5} on $x(10)$.

Exercise 4 (25 points)

Consider the integral

$$\int_0^2 f(x)dx$$

$$\text{where} \quad f(x) = \exp(x^3)\sqrt{x(2-x)}$$

We wish to approximate the integral using the Simpson method. For this, we as usual split the integration interval into N equidistant subintervals.

i) (8 points) With $N-1 = 2^k$; $\quad k = 1, \dots, 20$ use the Simpson Method method to approximate the integral. State the results in a table similar to those used during the course. Submit the used code.

ii) (4 points) Use Richardson extrapolation to estimate the order at $N-1 = 2^{20}$. State the result. Submit the used code.

iii) (4 points) The estimated order is different than the expected order. State an explanation for the difference and your guess on the exact order that you have estimated.

iv) (4 points) State the estimated accuracy on the result at $N-1 = 2^{20}$ using the estimated order. State clearly how you compute the accuracy estimate.

v) (5 points) State what other method could have been used to achieve high accuracy with likely much fewer $f(x)$ computations?

Exercise 5a (15 points). (Only for students taking the 5 ECTS version)

Consider the problem

$$\begin{cases} y''(x) = \sin(y(x)) + y'(x)^2 - 0.3x^2\cos(y'(x)); & -4 < x < 4 \\ y(-4) = -0.6; \quad y(4) = 0.3 \end{cases}$$

i) (8 points) Use the Finite Difference method to find an approximation to the solution curve $y(x)$. Use $N = 8, 16, 32, \ldots, 32768$). State the numerical estimate of $y(0)$ for each N with at least 10 digits. Submit the used code.

ii) (7 points) Determine the smallest N at which you can obtain an estimated accuracy on $y(0)$ of less than 10^{-5}. State N, state your estimate of the accuracy, and state an explanation on how you found this estimate.

Exercise 5b (15 points). (Only for students taking the 7 ECTS version)

Consider the problem

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} = 4\frac{\partial^2 u(x,t)}{\partial x^2} + \sin(\pi x)\exp(-t); && 0 < t; \quad 0 < x < 1 \\ &u(x,0) = x^2; && 0 \le x \le 1 \\ &u(0,t) = 0; \quad u(1,t) = 1 + \sin(t); && 0 < t \end{aligned}$$

i) (5 points) Consider $N = 2$. State an analytical expression of the semidiscrete form for this problem. State also the value of $\frac{du_1(t)}{dt}$ for $t = 0$.

ii) (10 points) Estimate $u(\frac{1}{2}, 10)$ with a proven accuracy better than 10^{-7} using Crank-Nicolson. Start with $\Delta x = \Delta t = \frac{1}{4}$ and perform subdivisions as suggested in the course. State at which subdivision you were able to reach the desired accuracy, your approximation for $u(\frac{1}{2}, 10)$ at that subdivision and your accuracy estimate. State a clear argument for how you arrived at the accuracy estimate. Submit the used code.