Control Systems

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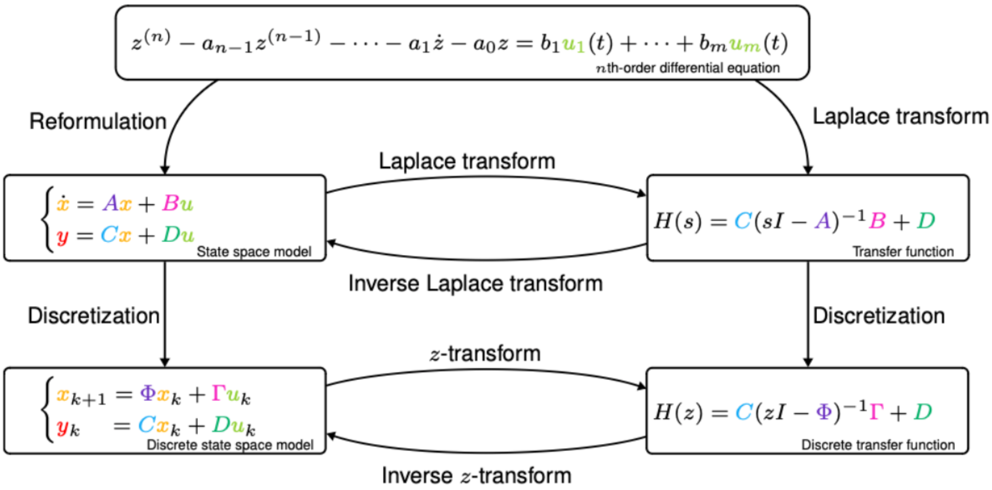
1 Linear Time Invariant Systems

Overview



1.1 Time-Domain models

Linear Map

The map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be linear if for any $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, the following conditions hold

$$f(x + y) = f(x) + f(y) \qquad \text{Super position}$$
$$f(ax) = \alpha f(x) \qquad \text{Homogeneity}$$

The function has to go through (0,0) in 2D for it to be linear due to homogeneity.

Time-Invariant System

Let $\sigma : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^p$ define the input-output behavior of a system model Σ. The system Σ is time-invariant if for any input signal $u : \mathbb{R} \rightarrow \mathbb{R}^m$
and any delay $\tau \in \mathbb{R}$ the following relation holds:

$$y(t - \tau) = \sigma(t, u(t - \tau))$$

for all times $t \in \mathbb{R}$, where y denotes the output signal of the system.

The importance is that the system does not change its behavior due to time. This can be seen as a canon firing at 8am it will not fire different compared to if you do the same at 5pm.

Two types of linear time-domain models.
Continous-time state space models (based on differential equations)

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

- $x \in \mathbb{R}^n$ is state. e.g. position or velocity
- $u \in \mathbb{R}^m$ is input
- $y \in \mathbb{R}^p$ is output
- $A \in \mathbb{R}^{n \times n}$ is system matrix

- $B \in \mathbb{R}^{n \times m}$ is input matrix
- $C \in \mathbb{R}^{p \times n}$ is output matrix
- $D \in \mathbb{R}^{p \times m}$ is the direct feedthrough matrix

Discrete-time state space models (based on difference equations)

$$x_{k+1} = \Phi x_k + \Gamma u_k$$

$$y_k = C x_k + D u_k$$

1.2 Frequency-Domain models

Transfer function:

$$G(s) = \frac{Q(s)}{P(s)}$$

where $Q(s)$ and $P(s)$ are polynomials in s.

- The roots of $P(s)$ are called the **poles** of $G(s)$
- The roots of $Q(s)$ are called the **zeros** of $G(s)$

1.2.1 State space to transfer function

Taking Laplace tranforms of the system and assuming $x_0 = 0$:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

yields:

$$sX(s) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

Which can be rewritten as (I is the identity matrix):

$$X(s) = (sI - A)^{-1} BU(s)$$

$$Y(s) = \left(C (sI - A)^{-1} B + D\right) U(s)$$

where

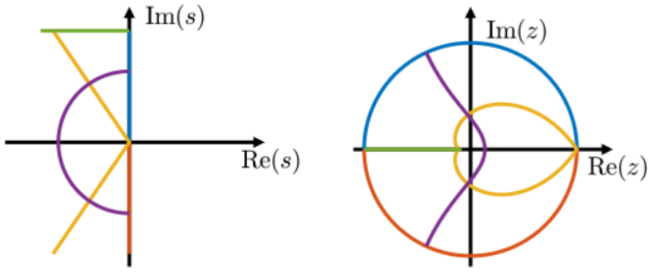
$$Y(s) = G(s)U(s)$$
$$G(s) = C (sI - A)^{-1} B + D$$

1.2.2 Transfer function to state space

1.2.3 Discrete-time transfer function

Discretization from s-domain to z-domain can be done using:

- Matched z-transform
- Bilinear z-transform
- Impulse invariance z-transform



1.3 Examples