

# Control Systems

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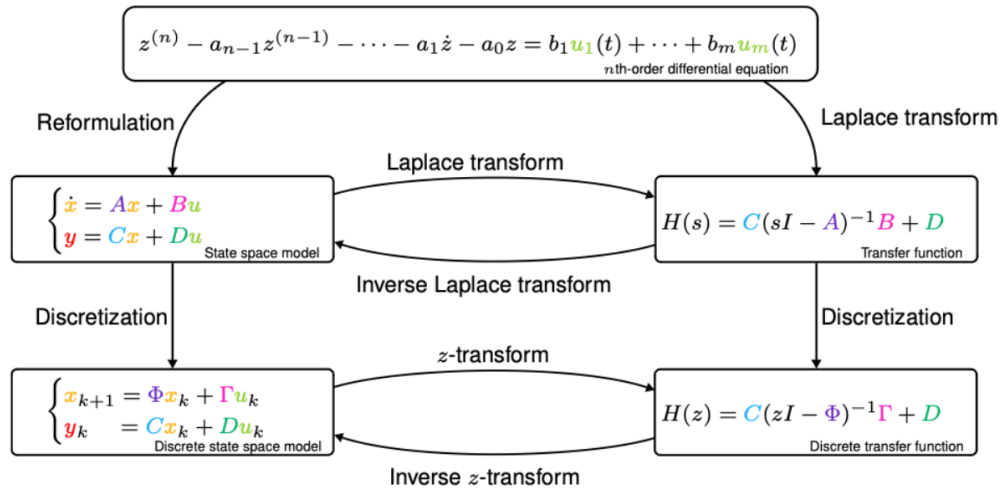
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## Contents

<b>1</b>	<b>Linear Time Invariant Systems</b>	<b>2</b>
1.1	Time-Domain models . . . . .	2
1.2	Frequency-Domain models . . . . .	3
1.2.1	State space to transfer function . . . . .	3
1.2.2	Transfer function to state space . . . . .	3
1.2.3	Discrete-time transfer function . . . . .	3
1.3	Examples . . . . .	4

# 1 Linear Time Invariant Systems

## Overview



## 1.1 Time-Domain models

### Linear Map

The map  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is said to be linear if for any  $x, y \in \mathbb{R}^n$  and  $\alpha \in \mathbb{R}$ , the following conditions hold

$$f(x + y) = f(x) + f(y) \quad \text{Super position}$$

$$f(ax) = \alpha f(x) \quad \text{Homogeneity}$$

The function has to go through (0,0) in 2D for it to be linear due to homogeneity.

### Time-Invariant System

Let  $\sigma : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^p$  define the input-output behavior of a system model  $\Sigma$ . The system  $\Sigma$  is time-invariant if for any input signal  $u : \mathbb{R} \rightarrow \mathbb{R}^m$  and any delay  $\tau \in \mathbb{R}$  the following relation holds:

$$y(t - \tau) = \sigma(t, u(t - \tau))$$

for all times  $t \in \mathbb{R}$ , where  $y$  denotes the output signal of the system.

The importance is that the system does not change its behavior due to time. This can be seen as a canon firing at 8am it will not fire different compared to if you do the same at 5pm.

Two types of linear time-domain models.

Continuous-time state space models (based on differential equations)

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

- $x \in \mathbb{R}^n$  is state. e.g. position or velocity
- $u \in \mathbb{R}^m$  is input
- $y \in \mathbb{R}^p$  is output
- $A \in \mathbb{R}^{n \times n}$  is system matrix

- $B \in \mathbb{R}^{n \times m}$  is input matrix
- $C \in \mathbb{R}^{p \times n}$  is output matrix
- $D \in \mathbb{R}^{p \times m}$  is the direct feedthrough matrix

Discrete-time state space models (based on difference equations)

$$x_{k+1} = \Phi x_k + \Gamma u_k$$

$$y_k = Cx_k + Du_k$$

## 1.2 Frequency-Domain models

Transfer function:

$$G(s) = \frac{Q(s)}{P(s)}$$

where  $Q(s)$  and  $P(s)$  are polynomials in  $s$ .

- The roots of  $P(s)$  are called the **poles** of  $G(s)$
- The roots of  $Q(s)$  are called the **zeros** of  $G(s)$

### 1.2.1 State space to transfer function

Taking Laplace transforms of the system and assuming  $x_0 = 0$ :

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

yields:

$$sX(s) = AX(s) + BU(s)$$

$$Y(s) = CX(s) + DU(s)$$

Which can be rewritten as ( $I$  is the identity matrix):

$$X(s) = (sI - A)^{-1} BU(s)$$

$$Y(s) = \left( C (sI - A)^{-1} B + D \right) U(s)$$

where

$$Y(s) = G(s)U(s)$$

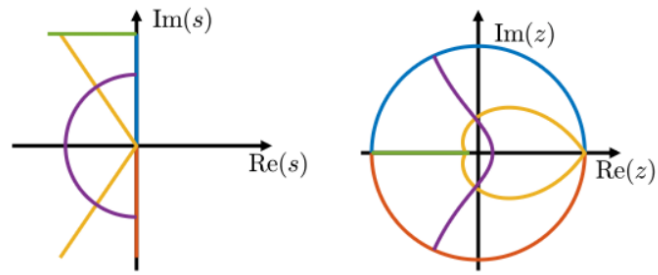
$$G(s) = C (sI - A)^{-1} B + D$$

### 1.2.2 Transfer function to state space

### 1.2.3 Discrete-time transfer function

Discretization from  $s$ -domain to  $z$ -domain can be done using:

- Matched  $z$ -transform
- Bilinear  $z$ -transform
- Impulse invariance  $z$ -transform



### 1.3 Examples