Control Systems

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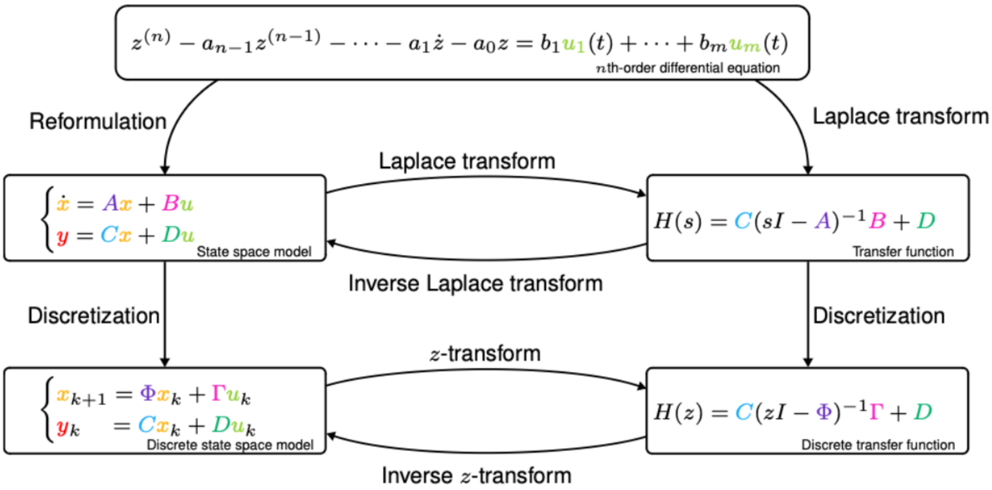
1 Linear Time Invariant Systems

Overview



1.1 Time-Domain models

Linear Map
The map $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be linear if for any $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, the following conditions hold

$$f(x + y) = f(x) + f(y) \qquad \text{Super position}$$
$$f(ax) = \alpha f(x) \qquad \text{Homogeneity}$$

The function has to go through (0,0) in 2D for it to be linear due to homogeneity.

Time-Invariant System
Let $\sigma : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^p$ define the input-output behavior of a system model Σ. The system Σ is time-invariant if for any input signal $u : \mathbb{R} \to \mathbb{R}^m$
and any delay $\tau \in \mathbb{R}$ the following relation holds:

$$y(t - \tau) = \sigma(t, u(t - \tau))$$

for all times $t \in \mathbb{R}$, where y denotes the output signal of the system.

The importance is that the system does not change its behavior due to time. This can be seen as a canon firing at 8am it will not fire different compared to if you do the same at 5pm.

Two types of linear time-domain models.
Continous-time state space models (based on differential equations)

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

- $x \in \mathbb{R}^n$ is state. e.g. position or velocity
- $u \in \mathbb{R}^m$ is input
- $y \in \mathbb{R}^p$ is output
- $A \in \mathbb{R}^{n \times n}$ is system matrix

- $B \in \mathbb{R}^{n \times m}$ is input matrix
- $C \in \mathbb{R}^{p \times n}$ is output matrix
- $D \in \mathbb{R}^{p \times m}$ is the direct feedthrough matrix

Discrete-time state space models (based on difference equations)

$$x_{k+1} = \Phi x_k + \Gamma u_k$$

$$y_k = C x_k + D u_k$$

1.2 Frequency-Domain models

Transfer function:

$$G(s) = \frac{Q(s)}{P(s)}$$

where $Q(s)$ and $P(s)$ are polynomials in s.

- The roots of $P(s)$ are called the **poles** of $G(s)$
- The roots of $Q(s)$ are called the **zeros** of $G(s)$

1.2.1 State space to transfer function

Taking Laplace tranforms of the system and assuming $x_0 = 0$:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

yields:

$$sX(s) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

Which can be rewritten as (I is the identity matrix):

$$X(s) = (sI - A)^{-1} BU(s)$$

$$Y(s) = \left(C (sI - A)^{-1} B + D\right) U(s)$$

where

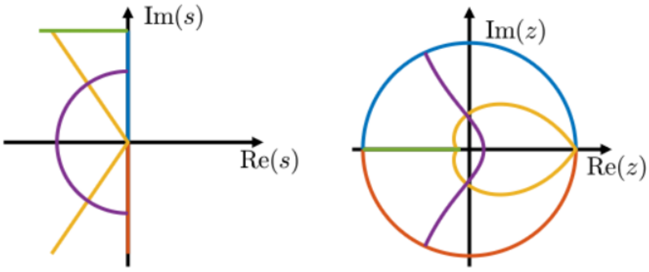
$$Y(s) = G(s)U(s)$$
$$G(s) = C (sI - A)^{-1} B + D$$

1.2.2 Transfer function to state space

1.2.3 Discrete-time transfer function

Discretization from s-domain to z-domain can be done using:

- Matched z-transform
- Bilinear z-transform
- Impulse invariance z-transform



1.3 Examples

2 Stability and Performance Analysis

2.1 Basic System Classes

2.1.1 First Order Systems

State-space representation of first order system:

$$\dot{x} = -\frac{1}{\tau}x + \frac{k}{\tau}u$$
$$y = x$$

A first-order system has one pole and is described by:

$$H(s) = \frac{k}{\tau s + 1}$$

Where k is the DC-gain and τ is the time-constant. The system has a pole in $s = -\frac{1}{\tau}$ i.e., the smaller time-constant, the faster system response.

2.1.2 Second Order Systems

The transfer funciton of a second-order system is given by:

$$H(s) = \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Where $\omega_n > 0$ is the natural frequency and $\zeta > 0$ is the damping ratio and k is the gain.

The system has two poles, which are $s \in \mathbb{C}$ where:

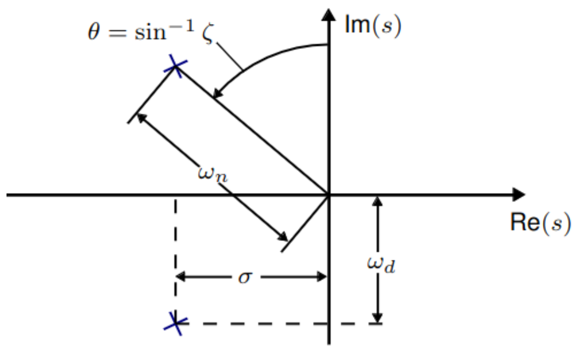
$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

The values of s is given by:

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

When $\zeta = 1$ the system is critically damped and $H(s)$ has a double pole in $s = -\zeta\omega_n$, when $0 < \zeta < 1$ the system is underdamped and has complex poles. When $\zeta > 1$ the system is overdamped and has real and distinct poles.

$$-\underbrace{\zeta\omega_n}_{\sigma} \pm j\,\underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d}.$$



Impulse response of a underdamped second-order system:

$$h(t) = k\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\sigma t}sin(\omega_d t)1(t)$$

The step resonse of a underdamped second-order system:

$$y(t) = k(1 - e^{-\sigma t}(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)))$$

Impulse response of a critically damped second order system:

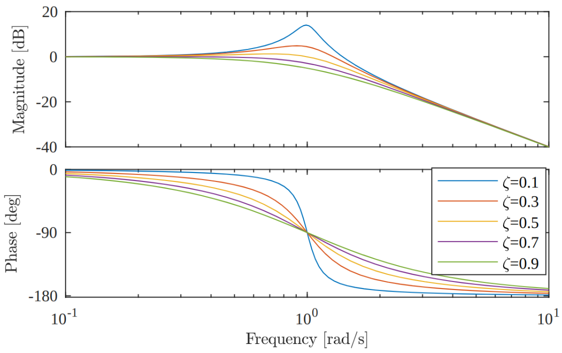
$$h(t) = k\omega_n^2 te^{-\omega_n t}$$

Having a overdamped system with a damping ration that is greater than one leads to a slower impulse response.

The step response of a critically damped second order system:

$$y(t) = k(1 - e^{-\omega_n t} - \omega_n te^{-\omega_n t})$$

Looking at the bode plot of a second-order system depends on the damping ratio. For every pole the magnitude of the bode plot decreases by 20dB/decade. And the phase of the bode plot decreases by 90 degrees for every pole.

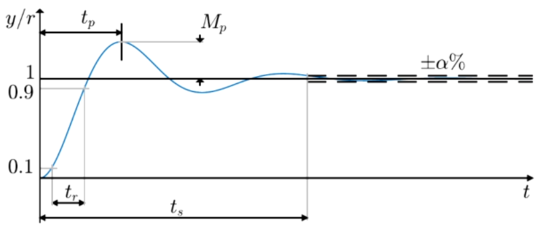


2.2 Performance specifications

Three different performance measurements of dynamical systems are:

1. The rise time t_r (stigetid)
2. The settling time t_s (indsvingingstid)
3. The overshoot M_p (oversving)

The rise time is the time it takes for the system to go from 10% to 90% of the final value. The settling time is the time it takes for the system to reach and stay within a certain % of the final value. The overshoot is the maximum percentage that the system goes over the final value.



The peak time is denoted t_p which is the time where the signal reaches its maximum value.

For a second-order system the rise time can be approximated as:

$$t_r = \frac{1.8}{\omega_n}$$

For a second-order system the settling time can be approximated as:

$$t_s = \frac{-log(\alpha/100)}{\zeta\omega_n}$$

Log is ln in the equation.

The peak time can be found using:

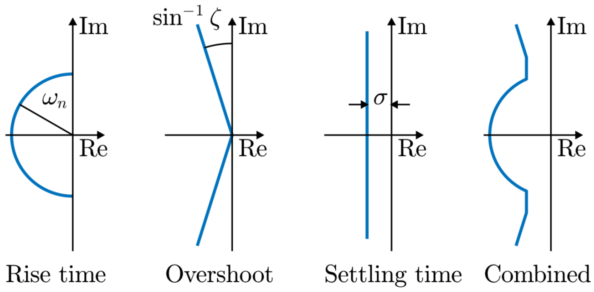
$$t_p = \frac{\pi}{\omega_d}$$

The overshoot is computed from the step response at the peak time. This gives an expression for the overshoot:

$$M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

for $0 < \zeta < 1$. The overshoot is given in percentage.

Overview for determining performance variables



To obtain a rise rise shorter than t_r

$$\omega_n \geq \frac{1.8}{t_r}$$

To obtain an overshoot that is smaller than M_p

$$\zeta \geq \sqrt{\frac{(\frac{\log(M_p)}{-\pi})^2}{1 + (\frac{\log(M_p)}{\pi})^2}}$$

To obtain an α%-settling time shorter than t_s

$$\sigma \geq \frac{-\log(\alpha/100)}{t_s}$$

2.3 Poles and Zeros of Space-State Models

Using thee state-space matrices:

$$G(s) = C(sI - A)^{-1}B + D$$

Poles can be found using the following equation:

When $G(s) \to \infty$ the system has a pole at $s \to p$.

$$\det(pI - A) = 0$$

Here p is the eigenvalue of A.

Zeros can be found using the following equation:

When $G(s) = 0$ the system has a zero at $s = z$.

$$\begin{vmatrix} A - zI & B \\ C & D \end{vmatrix} = 0$$

2.4 Stability

The stability of the dynamical system can be determined from the eigenvalues of A in the time domain. This is the equivalent of poles in the frequency domain.

$$\dot{x} = Ax + Bu$$

When the eigenvalues of A have a negative real part, the system is stable.

Definition A linear discrete-time system

$$x_{k+1} = \Phi x_k$$

The state of the next time step is given by ϕ mutliplied by x_k.

Where $x_k \in \mathbb{R}^{n\times n}$ is asymptotically (asymptotisk) stable if

$$\lim_{k\to\infty} x_k = 0$$

for any $x_0 \in \mathbb{R}^n$.

A linear discrete-time system is stable if all the eigenvalues are inside the unit circle.

The closer the poles are to origin the more dominating the pole is in the system.

2.5 Examples