Control Systems

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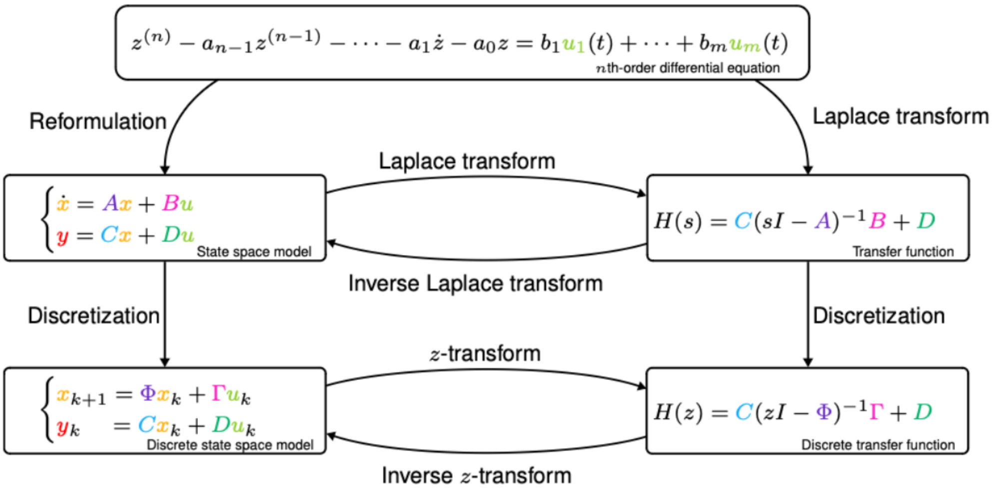
1 Linear Time Invariant Systems

Overview



1.1 Time-Domain models

Linear Map
The map $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be linear if for any $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, the following conditions hold

$$f(x + y) = f(x) + f(y) \qquad \text{Super position}$$
$$f(ax) = \alpha f(x) \qquad \text{Homogeneity}$$

The function has to go through (0,0) in 2D for it to be linear due to homogeneity.

Time-Invariant System
Let $\sigma : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^p$ define the input-output behavior of a system model Σ. The system Σ is time-invariant if for any input signal $u : \mathbb{R} \to \mathbb{R}^m$
and any delay $\tau \in \mathbb{R}$ the following relation holds:

$$y(t - \tau) = \sigma(t, u(t - \tau))$$

for all times $t \in \mathbb{R}$, where y denotes the output signal of the system.

The importance is that the system does not change its behavior due to time. This can be seen as a canon firing at 8am it will not fire different compared to if you do the same at 5pm.

Two types of linear time-domain models.
Continous-time state space models (based on differential equations)

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

- $x \in \mathbb{R}^n$ is state. e.g. position or velocity
- $u \in \mathbb{R}^m$ is input
- $y \in \mathbb{R}^p$ is output
- $A \in \mathbb{R}^{n \times n}$ is system matrix

- $B \in \mathbb{R}^{n \times m}$ is input matrix
- $C \in \mathbb{R}^{p \times n}$ is output matrix
- $D \in \mathbb{R}^{p \times m}$ is the direct feedthrough matrix

Discrete-time state space models (based on difference equations)

$$x_{k+1} = \Phi x_k + \Gamma u_k$$

$$y_k = C x_k + D u_k$$

1.2 Frequency-Domain models

Transfer function:

$$G(s) = \frac{Q(s)}{P(s)}$$

where $Q(s)$ and $P(s)$ are polynomials in s.

- The roots of $P(s)$ are called the **poles** of $G(s)$
- The roots of $Q(s)$ are called the **zeros** of $G(s)$

1.2.1 State space to transfer function

Taking Laplace tranforms of the system and assuming $x_0 = 0$:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

yields:

$$sX(s) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

Which can be rewritten as (I is the identity matrix):

$$X(s) = (sI - A)^{-1} BU(s)$$

$$Y(s) = \Big(C (sI - A)^{-1} B + D\Big) U(s)$$

where

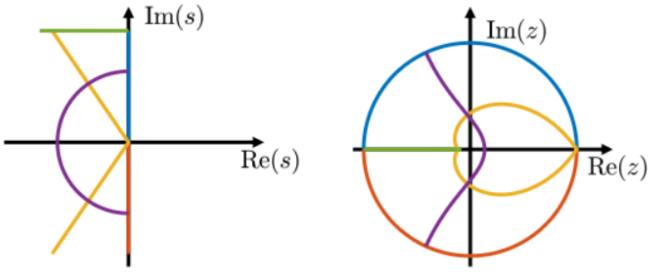
$$Y(s) = G(s)U(s)$$
$$G(s) = C (sI - A)^{-1} B + D$$

1.2.2 Transfer function to state space

1.2.3 Discrete-time transfer function

Discretization from s-domain to z-domain can be done using:

- Matched z-transform
- Bilinear z-transform
- Impulse invariance z-transform



1.3 Examples

2 Stability and Performance Analysis

2.1 Basic System Classes

2.1.1 First Order Systems

State-space representation of first order system:

$$\dot{x} = -\frac{1}{\tau}x + \frac{k}{\tau}u$$
$$y = x$$

A first-order system has one pole and is described by:

$$H(s) = \frac{k}{\tau s + 1}$$

Where k is the DC-gain and τ is the time-constant. The system has a pole in $s = -\frac{1}{\tau}$ i.e., the smaller time-constant, the faster system response.

2.1.2 Second Order Systems

The transfer funciton of a second-order system is given by:

$$H(s) = \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Where $\omega_n > 0$ is the natural frequency and $\zeta > 0$ is the damping ratio and k is the gain.

The system has two poles, which are $s \in \mathbb{C}$ where:

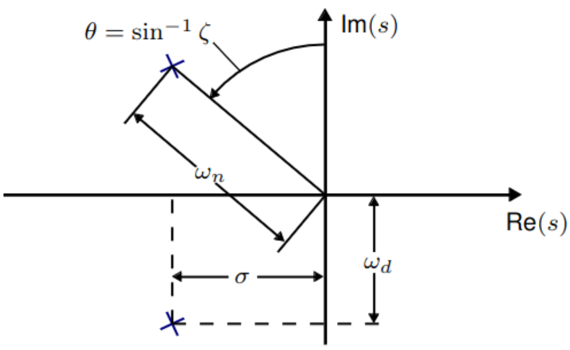
$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

The values of s is given by:

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

When $\zeta = 1$ the system is critically damped and $H(s)$ has a double pole in $s = -\zeta\omega_n$, when $0 < \zeta < 1$ the system is underdamped and has complex poles. When $\zeta > 1$ the system is overdamped and has real and distinct poles.

$$-\underbrace{\zeta\omega_n}_{\sigma} \pm j\,\underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d}.$$



Impulse response of a underdamped second-order system:

$$h(t) = k\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\sigma t}sin(\omega_d t)1(t)$$

The step resonse of a underdamped second-order system:

$$y(t) = k(1 - e^{-\sigma t}(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)))$$

Impulse response of a critically damped second order system:

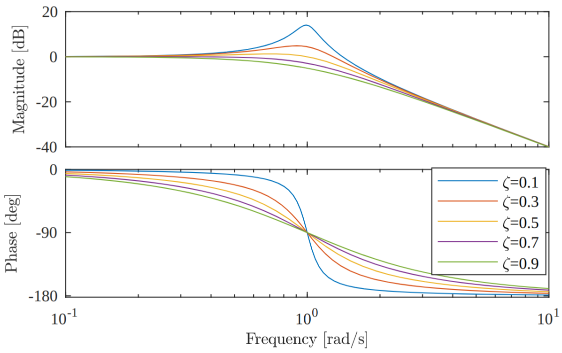
$$h(t) = k\omega_n^2 te^{-\omega_n t}$$

Having a overdamped system with a damping ration that is greater than one leads to a slower impulse response.

The step response of a critically damped second order system:

$$y(t) = k(1 - e^{-\omega_n t} - \omega_n te^{-\omega_n t})$$

Looking at the bode plot of a second-order system depends on the damping ratio. For every pole the magnitude of the bode plot decreases by 20dB/decade. And the phase of the bode plot decreases by 90 degrees for every pole.

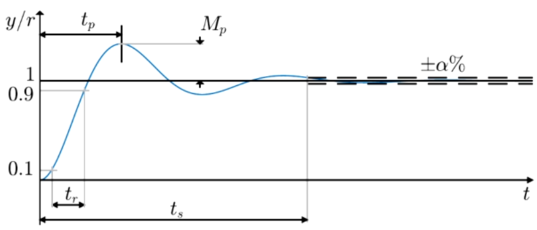


2.2 Performance specifications

Three different performance measurements of dynamical systems are:

1. The rise time t_r (stigetid)
2. The settling time t_s (indsvingingstid)
3. The overshoot M_p (oversving)

The rise time is the time it takes for the system to go from 10% to 90% of the final value. The settling time is the time it takes for the system to reach and stay within a certain % of the final value. The overshoot is the maximum percentage that the system goes over the final value.



The peak time is denoted t_p which is the time where the signal reaches its maximum value.

For a second-order system the rise time can be approximated as:

$$t_r = \frac{1.8}{\omega_n}$$

For a second-order system the settling time can be approximated as:

$$t_s = \frac{-log(\alpha/100)}{\zeta\omega_n}$$

Log is ln in the equation.

The peak time can be found using:

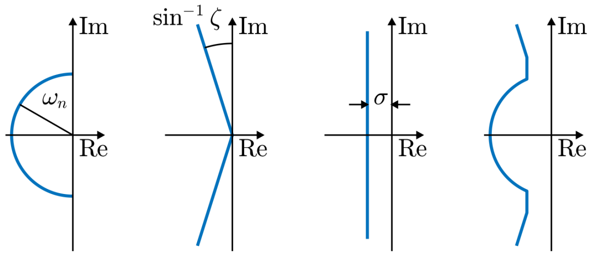
$$t_p = \frac{\pi}{\omega_d}$$

The overshoot is computed from the step response at the peak time. This gives an expression for the overshoot:

$$M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

for $0 < \zeta < 1$. The overshoot is given in percentage.

Overview for determining performance variables



Rise time | Overshoot | Settling time | Combined

To obtain a rise rise shorter than t_r

$$\omega_n \geq \frac{1.8}{t_r}$$

To obtain an overshoot that is smaller than M_p

$$\zeta \geq \sqrt{\frac{(\frac{\log(M_p)}{-\pi})^2}{1 + (\frac{\log(M_p)}{\pi})^2}}$$

To obtain an α%-settling time shorter than t_s

$$\sigma \geq \frac{-\log(\alpha/100)}{t_s}$$

2.3 Poles and Zeros of Space-State Models

Using thee state-space matrices:

$$G(s) = C(sI - A)^{-1}B + D$$

Poles can be found using the following equation:

When $G(s) \to \infty$ the system has a pole at $s \to p$.

$$\det(pI - A) = 0$$

Here p is the eigenvalue of A.

Zeros can be found using the following equation:

When $G(s) = 0$ the system has a zero at $s = z$.

$$\begin{vmatrix} A - zI & B \\ C & D \end{vmatrix} = 0$$

2.4 Stability

The stability of the dynamical system can be determined from the eigenvalues of A in the time domain. This is the equivalent of poles in the frequency domain.

$$\dot{x} = Ax + Bu$$

When the eigenvalues of A have a negative real part, the system is stable.

Definition A linear discrete-time system

$$x_{k+1} = \Phi x_k$$

The state of the next time step is given by ϕ mutliplied by x_k.

Where $x_k \in \mathbb{R}^{n \times n}$ is asymptotically (asymptotisk) stable if

$$\lim_{k \to \infty} x_k = 0$$

for any $x_0 \in \mathbb{R}^n$.

A linear discrete-time system is stable if all the eigenvalues are inside the unit circle.

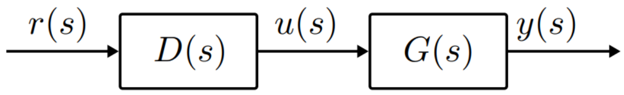
The closer the poles are to origin the more dominating the pole is in the system.

2.5 Examples

3 Introduction to Control

3.1 Open Loop Control

(åbensløjfe regulering)



Steady-State Value of Time Function:

Suppose that Y(s) is the Laplace transform of y(t). Then the final value of y(t) is either:

- Ubounded. If Y(s) has any poles in the open right half-plane (unstable)
- Undefined. If Y(s) has a pole pair on the imaginary axis.
- Constant. If all poles of Y(s) are in the open left half-plane, except for one at s=0

The Final Value Theorem (slutværdi-sætningen)

If all poles of sY(s) are in the open left half-plane, then:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$$

The Final Value Theorem determines the constant value that the impulse response of a stable system converges to. The theorem can also be used to determine the DC gain of a system, i.e., the output when a step input is applied to the system.

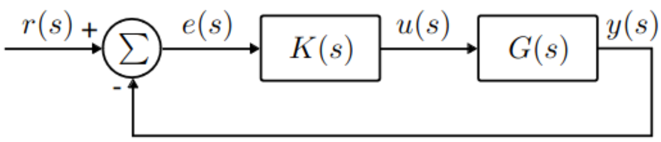
By integrating an impulse response, the step response is obtained. The step response is the integral of the impulse response, and the impulse response is the derivative of the step response.

If the system has a disturbance an open loop control system will not be able to compensate for this.

3.2 Feedback Control

(tilbagekobling). Tilbagekobling anvendes for at elminere forstyrrelsesundertrykkelse.

The connection of a controller K(s) and a system (also called plant) G(s) is called a closed-loop system.



e(s) is the error signal, and is the difference between the reference signal r(t) and the output signal y(t). The error signal is used to adjust the input signal u(t) to the system. The controller K(s) is used to adjust the input signal.

The closed-loop transfer function is given by:

$$\frac{y(s)}{r(s)} = \frac{G(s)K(s)}{1 + G(s)K(s)}$$

K(s) can be designed to meet certain requirements, such as stability, disturbance rejection, and sensitivity analysis.

Stability

The closed-loop system is stable if all the closed-loop poles are in the open left half-plane.

The **loop gain (sløjfe-forstærkning)** is defined as: $L(s) = G(s)K(s)$. The closed-loop poles are given by:

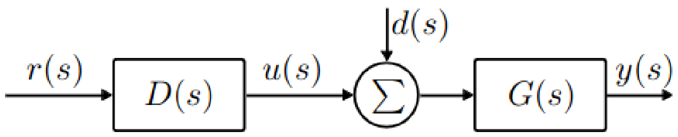
$$1 + L(s) = 0$$

The closed loop poles satisfy:

$$1 + L(s) = 0 \Rightarrow L(s) = -1$$

Disturbance Rejection

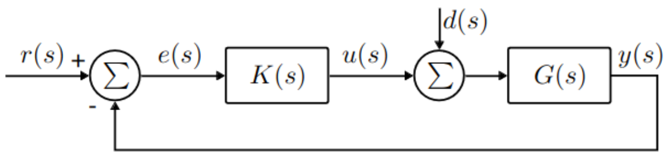
Open loop control system:



The output is given by:

$$y_m(s) = G(s)d(s) + G(s)D(s)r(s)$$

Closed loop control system:



The output is given by (superposition can be applied for deriving the expression):

$$y_m(s) = \frac{G(s)}{1 + G(s)K(s)}d(s) + \frac{G(s)K(s)}{1 + G(s)K(s)}r(s)$$

The larger the amplification of the feedback loop, the smaller the effect of the disturbance on the output.

Sensitivity Analysis

The sensitivity S of the open-loop system is given by the ratio of $\delta T_{ol}/T_o l$ to $\delta G_0/G_0$. For the open loop system S=1. δ is the relative change in the transfer function.

For the closed loop the sensitivity is given by:

$$S = \frac{1}{1 + G_0 K_0}$$

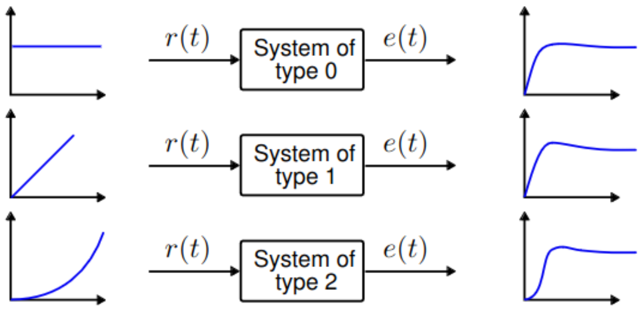
This means that a large amplification leads to a low sensitivity.

Linearization Taylor approximation, YES.

3.3 Steady State Tracking

Definition of system type

Stable systems can be classified according to its system type, defined to be the degree of the polynomial for which the steady-state error in a nonzero finite constant.



An expression for the tracing error is computed to determine the steady-state error. The error is given by:

$$e(s) = \frac{1}{1 + L(s)} r(s)$$

Where r(s) is the reference signal and L(s) is the loop gain.

The amount of poles at s=0 in the denominator of L(s) determines the system type.

Final value theorem is used to determine the steady-state error when the reference signal is a polynomial $r(t) = t^k 1(t)$.

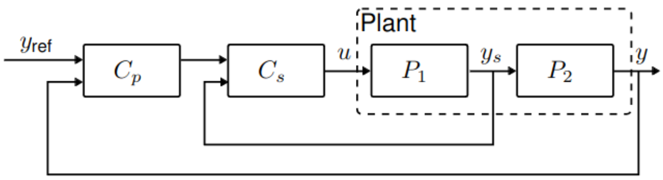
$$e_s s = \lim_{t \to \infty} e(t) = \lim_{s \to 0} e(s)s$$

$$\lim_{s \to 0} e(s)s = \lim_{s \to 0} s \frac{1}{1 + L(s)} r(s)$$

$$\lim_{s \to 0} s \frac{1}{1 + L(s)} \frac{1}{s^{k+1}}$$

3.4 Cascade Control

A cascade control uses the output from one controller as the input to another controller.



The nested control loops are called the inner loop (secondary) and the outer loop (primary). A fundamental reason for applying cascade control is to obtain better disturbance rejection and lower sensitivity to parameter variations.

3.5 DC Motor Dynamics

3.6 Examples

4 Design of PID Controllers

4.1 P-controller

The control law for the proportional controller is given by

$$u(t) = K_p e(t)$$

where K_p is the proportional gain and $e(t)$ is the error signal.

The controller applies a control signal to the systen, which depends linearly on the error signal.

To eliminate the steady state error, feedforward can be added to the P-controller.

$$u(t) = K_p e(t) + u_{ff}$$

Where the term u_{ff} is also called reset. We choose the feedforward according to the DC gain of the system, i.e.

$$u(t) = K_p e(t) + \frac{1}{G(0)} r(t)$$

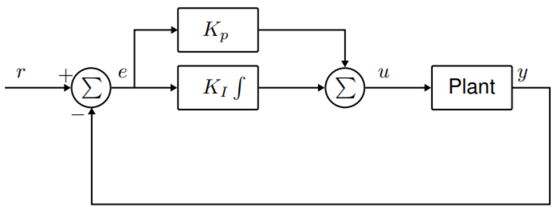
The feedforward does not affect the poles of the closed loop system.

4.2 PI-controller

The control law of a proportional-integral feedback controller is given by:

$$u = K_p e + K_I \int_{t_0}^{t} e(\tau) d\tau$$

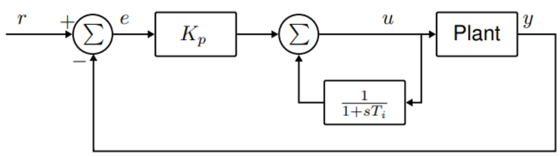
The controller adds a term, which is proportional to the integral of the error e to the P-control term.



The purpose of adding the integral term is to eliminate the steady state error of the system without the need for feed forward, i.e., integral control is less sensitive to modelling errors. Feed forward cannot eliminate modelling errors, but integral control can.

Comparing the PI-controller to the P-controller, the PI-controller has a better response to disturbances, that are unknown or cannot be measured. This is because the feedforward control can only eliminate the steady state error if the model and disturbances are known.

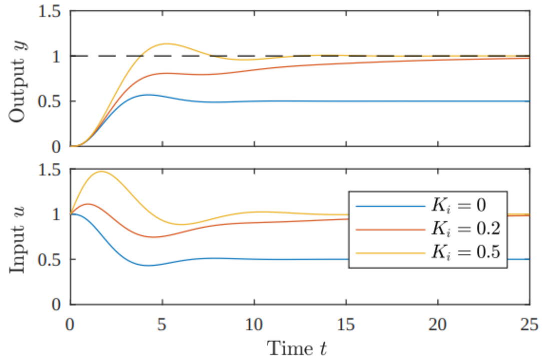
Implementation of a PI-controller is called automatic reset, and is shown in the following figure.



On the figure, T_i is the integration time (integraltid). The transfer function from e to u is given by:

$$T_{ue} = K_p \frac{1 + sT_i}{sT_i} = K_p + \frac{K_p}{sT_i}$$

On the figure below the step respones are shown for $K_i = 0, 0.2, 0.5$ and $K_p = 1$



From the unit step response it can be seen that the integral action removes the steady state error. A large K_i will give a fast response, but also a large overshoot. A small K_i will give a slow response, but also a small overshoot.

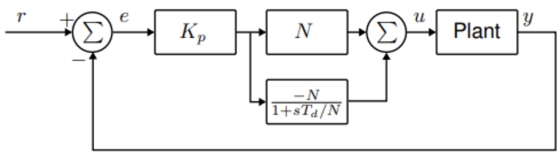
4.3 PD-controller

The derivative term provides an anticipatory action to the control, by doing feedback based on the trend of the error, i.e.

$$u(t) = K_p e(t) + K_d \frac{de(t)}{dt} = K_p \left(e + T_d \frac{de(t)}{dt} \right)$$

Where k_d is the derivative gain and T_d is the derivative time constant and e_p is a prediction of the error (T_d forwards in time). The parenthesis in the equation on the right is a prediction of the error called e_p.

PD-implementation



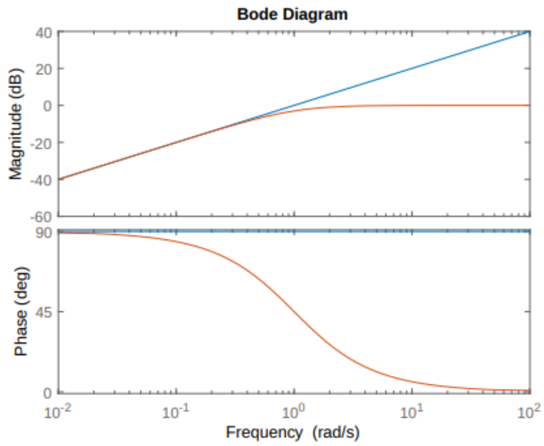
Noise is an issue for controllers that include a derivative term. Therefore, they can be implemented in a low-pass filtered version.

The transfer function of the controller is

$$T_{ue}(s) = K_p \left(N - \frac{N}{1 + sT_d/N} \right) = K_p \frac{sT_d}{1 + sT_d/N}$$

Where N is a filter constant (typical values of N are 2 to 20), and T_d is the derivative time constant.

The bode plot of an ideal PD-controller and a filtered PD controller are similar for low frequencies. Using a low-pass filter, we can at a certain frequency, reduce the gain of the derivative term, and thereby reduce the noise amplification.

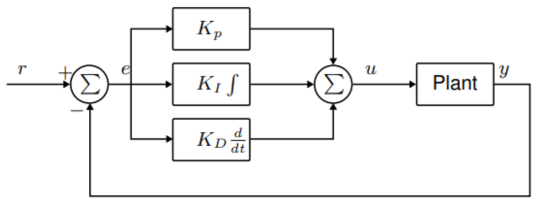


4.4 PID-controller

The control law of a PID feedback controller is

$$u(t) = K_p e(t) + K_i \int_{t_0}^{t} e(\tau) d\tau + K_d \frac{de(t)}{dt}$$

A block diagram of the controller is given below:



Alternatively, the PID-controlelr with filter on the D-term is

$$K(s) = K_p \left(1 + \frac{1}{T_i s} + \frac{sT_d}{1 + sT_d/N} \right)$$

4.5 Tuning a PID Controller

4.5.1 Pole placement

If a model of the system is available, then it is possible to compute an expression for the characteristic polynomial of the closed-loop system. Based on this polynomial, it may be possible to place the poles at desired locations.

4.5.2 Ziegler-Nichols method

Sometimes a model of the plant is not availabe, then the controller should be tuned by only studying the input-output behaviour of the system. Ziegler and Nichols has proposed two methods for tuning PID controllers without explicit use of a plant model.

- Ziegler-Nichols tuning based on step response

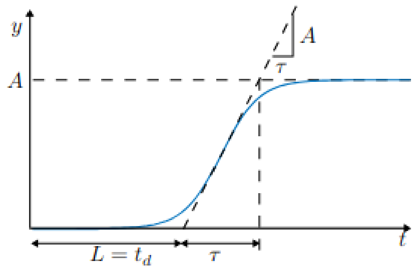
- Ziegler-Nichols tuning - The ultimate sensitivity method

Tuning based on step response

The considered system is assumed to have the transfer function

$$\frac{y(s)}{u(s)} = \frac{A}{\tau s + 1} e^{-s t_d}$$

This is a first-order system with a time delay of t_d.



The parameters for the PID controller can be found using the following table:

Controller type	Gains
P	$K_p = \frac{1}{RL}$
PI	$K_p = \frac{0.9}{RL}$ $T_i = \frac{L}{0.3}$
PID	$K_p = \frac{1.2}{RL}$ $T_i = 2L$ $T_d = 0.5L$

where $R = A/\tau$ is the amplitude of the oscillation, and $L = t_d$.

The ultimate sensitivity method

Start using a P-controller, and increase the gain until the system starts to oscillate. The value of K_p when the output oscillates with a constant amplitude is called the ultimate gain K_u. The period of the oscillation is called the ultimate period P_u.

The parameters for the PID controller can be found using the following table:

Controller type	Gains
P	$K_p = 0.5K_u$
PI	$K_p = 0.45K_u$ $T_i = \frac{P_u}{1.2}$
PID	$K_p = 0.6K_u$ $T_i = 0.5K_u$ $T_d = \frac{1}{8}P_u$

4.6 Examples

5 Root Locus

Rudkurve metoden. Grafisk metode til desgin af regulatorer.

Rule 1

Consider the characteristic equation:

$$1 + KG(s) = 1 + K\frac{Q(s)}{P(s)} = 0$$

This can be written as:

$$P(s) + KQ(s) = 0$$

This is a polynoial of degree $N = max(m, n)$ where m is the number of poles and n is the number of zeros. Lemma. A univariate polynomial of degree d has d roots in $\mathbb{C}$.

There are N lines (loci) where $N = max(m, n)$. Where m is the number of poles and n is the number of zeros.

Rule 2

Similar to previously, the characteristic equation is rewritten as:

$$P(s) + KQ(s) = 0$$

Let $K = 0$, then we observe that the roots of the characteristic equation are the poles of the open-loop system.

Let $K \to \infty$, then we observe that the roots of the characteristic equation are the zeros of the open-loop system.

$$\frac{P(s)}{K} + Q(s) = 0$$

Defintion: As K increases from 0 to ∞, the root move from the poles of $G(s)$ to the zeros of $G(s)$.

Rule 3

When roots are complex they occur in conjugate pairs.

Rule 4

We study the rewritten characteristic equation:

$$\frac{Q(s)}{P(s)} = -\frac{1}{K}$$

And see that the phase of $\frac{Q(s)}{P(s)}$ is 180° to satisfy the equation.

The transfer function can be written as:

$$T(s) = \frac{Q(s)}{P(s)} = \frac{(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)}$$

where z_i are the zeros and p_i are the poles. Let $z \in \mathbb{C}$ then $zz* = |z|^2$; hence, complex pole pairs and pairs of complex conjugated zeros do not affect the phase of $T(s)$ for $s \in \mathbb{R}$.

The phase of $(s - z_m)$ when $s, z_m \in \mathbb{R}$ is

$$\angle(s - z_m) = \begin{cases} 180° & \text{if } s > z_m \\ 0° & \text{otherwise} \end{cases} \tag{1}$$

The portion of the real axis to the left of an odd number of open loop poles and zeros are part of the loci.

Rule 5

Lines leave and enter the real axis at $90°$ angles.

Rule 6

For very large values of s the equation:

$$1 + K\frac{s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n} = 0$$

can be approximated to:

$$1 + K\frac{1}{(s-\alpha)^{n-m}}$$

The phase of this expression should be $180°$ for this expression to hold. This implies that:

$$(n-m)\phi_l = 180° + 360°(l-1)$$

where ϕ_l is the phase of the expression.

let $m < n-1$ then:

$$-\sum r_i = -\sum p_i$$

where r_i are the closed-loop poles and p_i are the open-loop poles.

For s going to ∞, it is known that m closed-loop poles go towards the open-loop zeros, and $n-m$ closed-loop poles go towards α, i.e.

$$-\sum r_i = -(n-m)\alpha - \sum z_i = -\sum p_i$$

where z_i are the open-loop zeros.

Thus,

$$\alpha = \frac{\sum z_i - \sum p_i}{n-m}$$

5.1 Examples

6 The Nyquist Stability Criterion

6.1 Frequency response

The frequency response of a system is the steady-state response of a system to a sinusoidal input. The output of a time-invariant system will have the same frequency as the sinusoidal input, but possibly with a different amplitude and phase.

The output of a time-invariant system is given by:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau$$

Where y is the output, u is the input, and h is the impulse response of the system.

To obtain the frequency response, only sinusoidal inputs are considered.

The frequency response of H(s) is given by the magnitude and phase of $H(j\omega)$

$$M = |H(j\omega)|$$
$$\phi = \angle H(j\omega)$$

The M is the amplitude ratio and ϕ is the phase shift.

The bandwidth of a closed-loop system $T(s)$ is defined to be the maximum frequency at which the output y of a system will track a sinusoidal input r in a satisfactory manner. Output attaenuated to $1/\sqrt{2}$ of the input amplitude. Formally the bandwidth ω_{BW} of $T(s)$ is the maximal frequency such that:

$$T(j\omega) >= 1/\sqrt{2}$$

The maximal value of the frequency response is called the resonant peak,

6.2 Bode Plots

Pole decreases the magnitude of the frequency response by 20dB/decade. Zero increases the magnitude of the frequency response by 20dB/decade.

Bode Form of transfer function

In the follwing, we consider the open-loop transfer function

$$KG(s) = K\frac{(s-z_1)(s-z_2)\cdots}{(s-p_1)(s-p_2)\cdots}$$

For convenience, the transfer function is evaluated at $s = j\omega$ and rewritten into Bode form as follows:

$$KG(j\omega) = K_0\frac{(j\omega\tau_1+1)(j\omega\tau_2+1)\cdots}{(j\omega\tau_a+1)(j\omega\tau_b+1)\cdots}$$

Classes of terms in transfer function

Class 1: $K_0(j\omega)^n$ for $n \in \mathbb{Z}$

Example: For the term $K_0(j\omega)^n$ we have:

$$logK_0|(j\omega)| = logK_0 + nlog\omega$$

This means that the magniutde plot is a straight line with slope $n \cdot (20dB/dec)$ The phase of $(j\omega)^n$ is constant and given by: $\phi = n \cdot 90°$ $n = -1$ means that there is a pole at the origin. This therefore leads to a slope of $-20dB/dec$ and a phase shift of $-90°$

Class 2: $(j\omega\tau + 1)^{\pm 1}$

For $\omega\tau << 1, j\omega\tau \approx 1$

For $\omega\tau >> 1, j\omega\tau + 1 \approx j\omega\tau$

In addition, for $\omega = 1/\tau$, the gain is $sqrt2$ - an increase of 3dB compared to the DC gain. The point $\omega = 1/\tau$ is called the break point. At the break point the angle is $45°$.

Class 3: $((j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1)^{\pm 1}$

For $\omega << \omega_n$, the amplitde is approximately 1. In addition, at the break point $\omega = \omega_n$, the magnitude is $|G(j\omega)| = 1/(2\zeta)$ and the phase is $\pm 90°$

Summary of Bode Plot Rules

1. Rewrite the considered transfer function to bode form
2. Determine the value of the $K_0(j\omega)^n$ term. Plot the low frequency magnitude asymptote through the point K_0 at $\omega = 1$ and with slope of $n \cdot 20dB/dec$
3. Complete the composite magnitude asymptotes by extending the low-frequency asymptote until the first frequency break point. Then change the slope according to the behavior at the break point, and continue the procedure for the remaining break points.
4. Sketch the approximate magnitude curve by increasing the asymptote value by a factor $\sqrt{2}$ at first-order numerator break and decreasing it by a factor $1/\sqrt{2}$ at denominator break.
5. Plot the low-frequency asymptote of the phase $phi = n \cdot 90°$
6. Change the phase at the phase points, and correct the phase according to the slope at the phase point.

6.3 Nyquist Stability Criterion

A contour map of a complex function will encircle the origin Z-P times where Z is the number of zeros and P is the number of poles of the function inside the contour.

To verify the stability of a system, one needs to determine the number of closed-loop poles in the right halp plane. Thus, the Nyquist plot is a map of a contour that encircles the entire right-half plane.

The number of closed-loop poles in the right half-plane equals the number of right half-plane zeros of the characteristic equation. The characteristic equation is given by:

$$1 + KG(s) = 0$$

Let N denote the number of clockwise encirclements of -1. Then the number of zeros in the right half plane Z (closed-loop poles) minus the number of open-loop poles in the right half plane P.

$$N = Z - P$$

6.4 Examples

7 Dynamic Compensators and Stability Margins

7.1 Stability Margins

Gain margin: factor by which the gain can be rained before a system becomes unstable.

Phase margin: The amount by which the phase can be increased before the system becomes unstable. (exceeds $-180°$)

7.2 Dynamic compensation

Lead compensator is equivalent to a d-part for a PID controller. A lead compensator has a positive phase.

Lag compensator is equivalent to an i-part for a PID controller.

Both compensators are given by the transfer function:

$$D(s) = K\frac{s+z}{s+p}$$

Where z and p are the zero and pole of the transfer function respectively.

- If $z < p$, then $D(s)$ is called a lead compensation
- If $z > p$, then $D(s)$ is called a lag compensation

7.2.1 Lead compensation

Used for dynamic properties such as rise time, overshoot and settling time. It has relation to the phase margin, where we can lift the phase margin by adding a lead compensator.

A lead compensation is given by:

$$D(s) = \frac{Ts+1}{\alpha Ts+1}$$

and $1/\alpha$ is called the lead ratio.

We have the following properties:

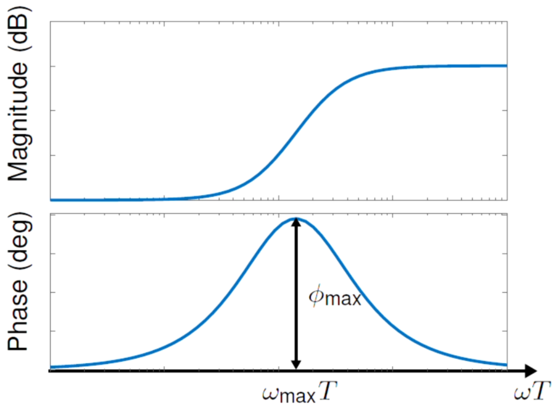
$$\omega_{max} = \frac{1}{T\sqrt{\alpha}} = \sqrt{|z||p|}$$

,

$$T = \frac{1}{\omega_{max}\sqrt{\alpha}}$$

$$sin(\phi_{max}) = \frac{1-\alpha}{1+\alpha}$$

ϕ_{max} is the maximum phase lead. Which means that it is the maximum phase that the lead compensator can provide. And ω_{max} is the frequency at which the phase lead is maximum.



7.2.2 Lag compensation

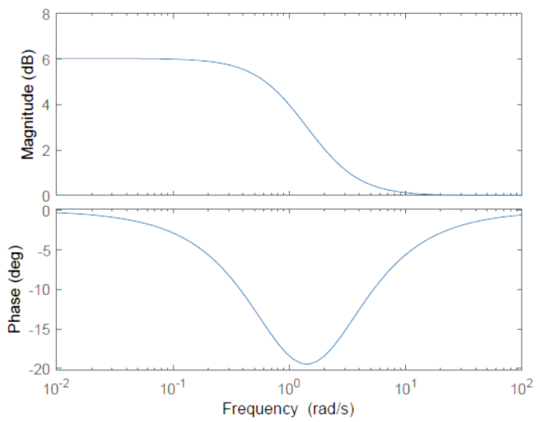
A lag compensator can be used for minimizing the steady state error without affecting the other dynamics. Only looking at stationary error. It doesn't eliminate the error but reduces it.

A lag compensator can be written as:

$$D(s) = K_0 \frac{\alpha T s + 1}{\alpha T s + 1}$$

where K_0 is the gain of the compensator, $z > p \in \mathbb{R}$ and $\alpha > 1$.

The gain decreases and the phase gets negative.



7.3 Examples

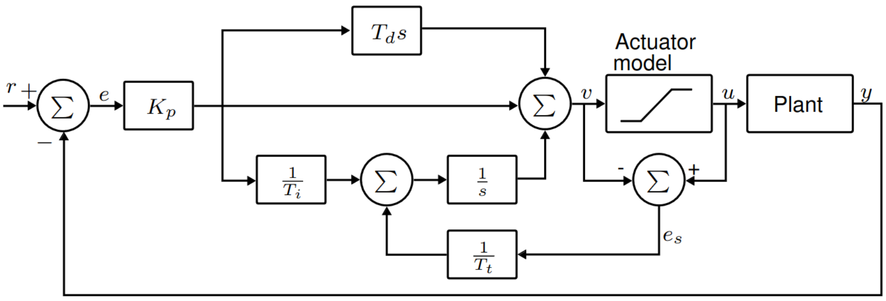
8 Implementation

8.1 Anti-windup

Real electromechanical systems has saturations on their physical variables. In particular, the output of any actuator is limited from above and below. (Clipping). If you stay between the limits, the system will behave as a linear system.

Back-calculation

An anti-windup scheme that recomputes the integral term when the system input is saturated. The value of the integrator output is not changed instantaneously, but it is changed based on the tracking time constant.



The input to the integrator is given by

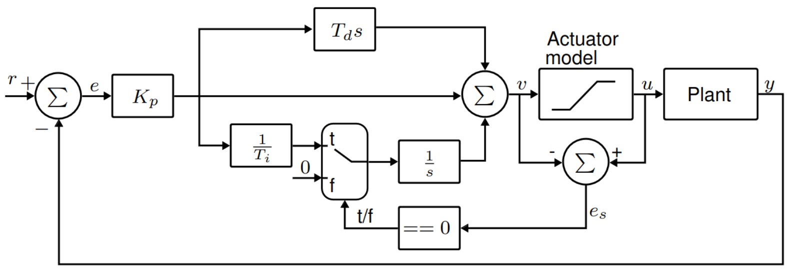
$$\frac{1}{T_t}e_s + \frac{K_p}{T_i}e$$

Where e_s is zero when the system is not saturated. In steady state, the output of the integrator is constant: hence, its input must be zero, i.e.

$$e_s = -\frac{K_p T_t}{T_i}e$$

Conditional integration

Also known as clamping is a bit simpler, and just stops integreating when the system is in saturation.



Setpoint weighting

To avoid large control signals when changing the reference rapidly, setpoint weiths can be introduced. These modify the PID controller to be:

$$u(t) = K_p(\beta r(t) - y(t)) + K_i \int_0^t (r(\tau) - y(\tau))d\tau + K_d(\gamma\frac{dr}{dt} - \frac{dy}{dt})$$

where γandβ are setpoint weights. Typically $\gamma = 0$ and β takes values between 0 and 1.

Filtering

To avoid large control signal noise on the measured output y, one adds a filter on the derivative term. Often a first order filter is sufficient, but this is application dependent.

$$u_d = k_d s \frac{1}{1 + sT_f}$$

Sometimes it is favorable to filter the control signal directly, thus the controller gets the form

$$C(s) = K_p(1 + \frac{1}{sT_i} + sT_d)\frac{1}{1 + sT_f + (sT_f)^2/2}$$

8.2 Digitalization

Most control systems are sample data systems, i.e., they consist of both discrete and continous signals. The sampling frequency used for the discrete controlelr should be above 20 times the closed-loop bandwidth.

8.2.1 Emulation

Design continous controller $K(s)$ and approximate it with $K(z)$ obtained via e.g. Tustin's method. Tustin's method also called the trapezoidal rule means to replace the variable s with

$$\frac{2}{T}\frac{z-1}{z+1}$$

The control output of the discrete PID controller can be written using 3 terms.

$$u_p(kT + T) = k_p e(kT + T)$$
$$u_I(kT + T) = u_I(KT) + K_i \frac{T}{2}(e(kT + T) + e(kT))$$
$$u_D(kT + T) = k_D \frac{2}{T}(e(kT + T) - e(kT)) - u_D(kT)$$

$$u(kT + T) = u_p(kT + T) + u_I(kT + T) + u_D(kT + T)$$

To analyze the system, the I-term and D-term are z-transformed:

$$zu_I(z) = u_I(z) + k_i \frac{T}{2}(ze(z) + e(z))$$

$$zu_D(z) = k_D \frac{2}{T}(ze(z) - e(z)) - u_D(z)$$

This gives the following expression for the controller

$$u(z) = (k_P + k_I \frac{T}{2}\frac{z+1}{z-1} + k_D \frac{2}{T}\frac{z-1}{z+1})e(z)$$

Compensation for sampling effects

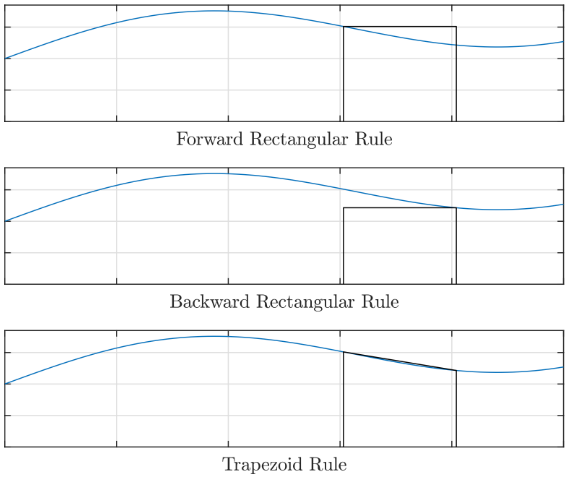
Design procedure

A discrete controller can be designed by emulation for the system $G(s)$ according to the next procedure:

1. Design continuous compensation for the system $G_d(s)G(s)$, where $G_d(s)$ approximates a delay of $T/2$
2. Derive the discrete controller by applying Tustin's rule or the matched pole-zero method
3. Analyze the design by simulation or experimentally

8.2.2 Numerical Integration Methods

There is three methods to convert a continous controller to a discrete controller:



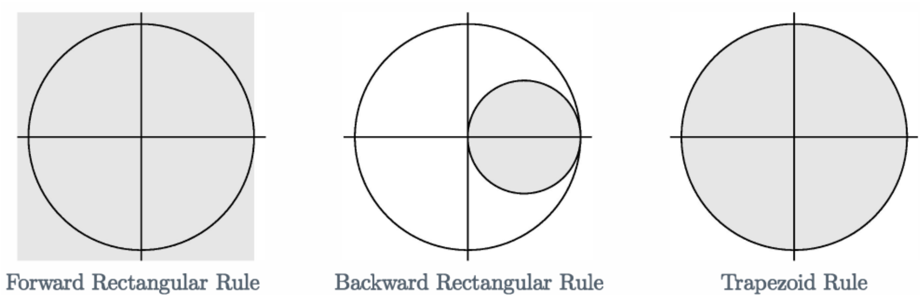
Get an approximation of the discrete transfer function by replacing s with:

Method:	**Approximation:**
Forward rule:	$s \leftarrow \frac{z-1}{T}$
Backward rule:	$s \leftarrow \frac{z-1}{Tz}$
Trapezoidal rule:	$s \leftarrow \frac{2}{T}\frac{z-1}{z+1}$

Get an approximation of the discrete transfer function by replacing z with:

Method:	**Approximation:**
Forward rectangular rule:	$z \leftarrow 1 + Ts$
Backward rectangular rule:	$z \leftarrow \frac{1}{1-Ts}$
Bilinear rule:	$z \leftarrow \frac{1+Ts/2}{1-Ts/2}$

The left half-plane is mapped to different regions in the z-plane for the different methods.



This means that the discretization of a stable system using the forward rule can result in an unstable system.

8.2.3 Discrete Design

Design the discrete controller directly, without computing $K(s)$ first. Therefore it relies on a discretized plant model. The discrete transfer function of a system $G(s)$ and preceding zero-order hold is

$$G(z) = (1 - z^{-1})Z\left\{\frac{G(s)}{s}\right\}$$

where Z denotes the Z-transform.

The system can subsequently be analyzed based on the clode-loop transfer function:

$$T_{cl}(z) = \frac{G(z)K(z)}{1 + G(z)K(z)}$$

Since the structure of the continuous and discrete closed-loop transfer functions are the same, the stability of the system can be studied via the characteristic equation:

$$1 + G(z)K(z) = 0$$

Consequently, the controller K(z) can be designed via root locus or any of the other methods.

Design procedure

1. Transform the continuous-time plant into discrete time as follows:

$$G_d(z) = (1 - z^{-1})Z\left\{\frac{G(s)}{s}\right\}$$

2. Design the feedback controller $K(z)$ using the same approaches as for a continuous-time.
3. Verify the design on the sampled data system.

8.3 Examples

Emulation example of PID controller

$$K(s) = 1.4\frac{s+6}{s}$$

Use Tustin's method to emulate the controller in discrete time.

$$K(z) = K(\frac{2}{T}\frac{z-1}{z+1}) = 1.4\frac{\frac{2}{T}\frac{z-1}{z+1} + 6}{\frac{2}{T}\frac{z-1}{z+1}} = 1.4\frac{(1+3T)z - (3T-1)}{z-1}$$

In practice you implement the difference equation - not a discrete transfer function.

9 State Feedback Control

9.1 Controllability (styrbarhed)

A continuous time system

$$\dot{x} = Ax(t) + Bu(t), x(0) = 0$$

is said to be controllable if and only if for any $\zeta \in \mathbb{R}^n$, there exists $u(t)$ such that for some $T > 0, x(T) = \zeta$

A discrete time system

$$x_{k+1} = \Phi x_k + \Gamma u_k, x_0 = 0$$

is said to be controllable if and only if for any $\zeta \in \mathbb{R}^n$, there exists $(u_0, u_1, \cdots)$ such that for some $N > 0, x_N = \zeta$

From iterating through the discrete time system, we get:

$$x_n = \Phi^{n-1}\Gamma u_0 + \cdots + \Phi\Gamma u_{n-2} + \Gamma u_{n-1}$$

The controllability matrix (styrbarheds matricen):

$$x_n = \begin{bmatrix} \Gamma & \Phi\Gamma & \cdots & \phi^{n-1}\Gamma \end{bmatrix}$$

A system where $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ is controlable if and only if:

Having only 1 input signal, the system is controllable if:

$$\det \begin{Bmatrix} B & AB & \cdots & A^{n-1}B \end{Bmatrix} \neq 0$$

9.2 Controllable Cannonical Form

9.3 State Feedback and Pole Assignment

9.4 Examples

10 Observer Design

10.1 Observability

Estimation of state.

Observability matrix

10.2 Full Order Observer

Estimation of all states. (tilstandsestimator)

$$\dot{\hat{x}} = A\hat{x} + Bu + L(C\hat{x} - y)$$
$$\hat{y} = C\hat{x}$$

Error, $e = \hat{x} - x$:

$$\begin{aligned}\dot{e} = \dot{\hat{x}} - \dot{x} &= A\hat{x} + Bu + L(C\hat{x} - y) - (Ax + Bu)\\ &= A(\hat{x} - x) + L(C\hat{x} - Cx)\\ &= (A + LC)e\end{aligned}$$

Eigenvalues for A+LC should be placed in the left half plane for the error to converge to zero. This has to be done by only changing L.

Theorem

A full order observer for the system with observer gain L is stable, if and only if the eigenvalues of the matrix $A + LC$ all have a negative real part. Moreover, such an L always exists, if (A,C) is observable.

Observable Canonical Form

10.3 Observer Design

10.4 Observer Based Control

Combination of observer and state feedback. This gives 2n poles because of the extra eigenvalues. The state feedback poles should be placed 2 to 6 times further to the left than the observer poles.

10.5 Examples

11 Integral Control and Optimal Control

11.1 Integral Control

11.2 Anti-windup

Must always be applied when having an integrator in the controller.

Designing Saturation Gain

Dynamics of controller during saturation

$$\dot{\hat{x}} = (A + LC + MF)\hat{x}$$

Determining M can be recognized as an observer gain design problem:

$$\dot{\hat{x}} = (\tilde{A} + \tilde{L}\tilde{C})\hat{x}$$

With $\tilde{A} = A + LC, \tilde{L} = M, \tilde{C} = F$. From which the unknown $\tilde{L} = M$ can be chosen to assign any desired poles to the saturated controller.

11.3 Optimal Control

Consider a linear control system of the form:

$$\dot{x} = Ax + Bu, x(0) = x_0$$

$$y = Cx$$

A control law for such a system is said to be optimal, if it minimizes the cost functional:

$$J = \int_0^\infty (x^T Q x + u^T R u) dt$$

Where $Q = Q^T$ is a positive semi-definite matrix (eigenvalues larger than or equal to 0) and $R = R^T$ is a positive definite matrices.

Algebraic Riccati Equation is a second order matrix equation in an indeterminate $P = P^T \in \mathbb{R}^{n \times n}$:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

P is called a stabilizing solution to the ARE, if it satisfies the equation, and further satisfies that the eigenvalues of $A - BR^{-1}B^T P$ have negative real parts.

The optimal state feedback law is given by:

$$u = Fx$$

Where F is given by: $F = -R^{-1}B^T P$

11.4 Examples