Control Systems

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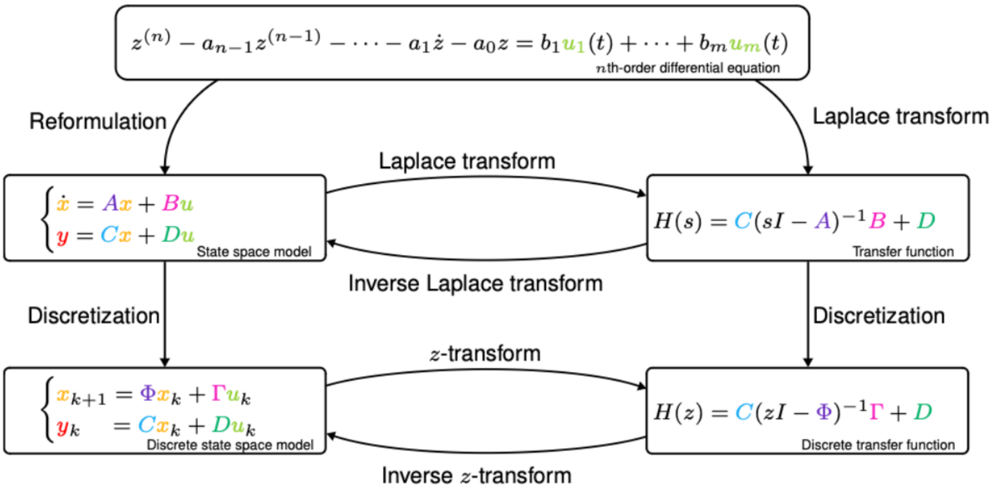
1 Linear Time Invariant Systems

Overview



1.1 Time-Domain models

Linear Map
The map $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be linear if for any $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, the following conditions hold

$$f(x + y) = f(x) + f(y) \qquad \text{Super position}$$
$$f(ax) = \alpha f(x) \qquad \text{Homogeneity}$$

The function has to go through (0,0) in 2D for it to be linear due to homogeneity.

Time-Invariant System
Let $\sigma : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^p$ define the input-output behavior of a system model Σ. The system Σ is time-invariant if for any input signal $u : \mathbb{R} \to \mathbb{R}^m$
and any delay $\tau \in \mathbb{R}$ the following relation holds:

$$y(t - \tau) = \sigma(t, u(t - \tau))$$

for all times $t \in \mathbb{R}$, where y denotes the output signal of the system.

The importance is that the system does not change its behavior due to time. This can be seen as a canon firing at 8am it will not fire different compared to if you do the same at 5pm.

Two types of linear time-domain models.
Continous-time state space models (based on differential equations)

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

- $x \in \mathbb{R}^n$ is state. e.g. position or velocity
- $u \in \mathbb{R}^m$ is input
- $y \in \mathbb{R}^p$ is output
- $A \in \mathbb{R}^{n \times n}$ is system matrix

- $B \in \mathbb{R}^{n \times m}$ is input matrix
- $C \in \mathbb{R}^{p \times n}$ is output matrix
- $D \in \mathbb{R}^{p \times m}$ is the direct feedthrough matrix

Discrete-time state space models (based on difference equations)

$$x_{k+1} = \Phi x_k + \Gamma u_k$$

$$y_k = C x_k + D u_k$$

1.2 Frequency-Domain models

Transfer function:

$$G(s) = \frac{Q(s)}{P(s)}$$

where $Q(s)$ and $P(s)$ are polynomials in s.

- The roots of $P(s)$ are called the **poles** of $G(s)$
- The roots of $Q(s)$ are called the **zeros** of $G(s)$

1.2.1 State space to transfer function

Taking Laplace tranforms of the system and assuming $x_0 = 0$:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

yields:

$$sX(s) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

Which can be rewritten as (I is the identity matrix):

$$X(s) = (sI - A)^{-1} BU(s)$$

$$Y(s) = \left(C (sI - A)^{-1} B + D\right) U(s)$$

where

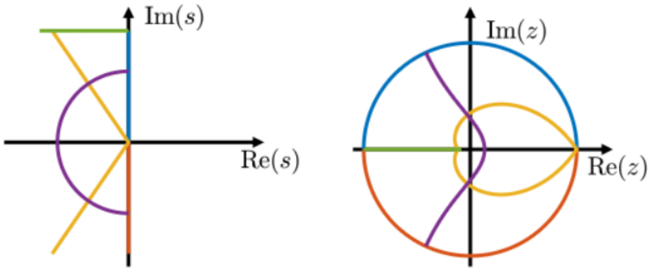
$$Y(s) = G(s)U(s)$$
$$G(s) = C (sI - A)^{-1} B + D$$

1.2.2 Transfer function to state space

1.2.3 Discrete-time transfer function

Discretization from s-domain to z-domain can be done using:

- Matched z-transform
- Bilinear z-transform
- Impulse invariance z-transform



1.3 Examples

2 Stability and Performance Analysis

2.1 Basic System Classes

2.1.1 First Order Systems

State-space representation of first order system:

$$\dot{x} = -\frac{1}{\tau}x + \frac{k}{\tau}u$$
$$y = x$$

A first-order system has one pole and is described by:

$$H(s) = \frac{k}{\tau s + 1}$$

Where k is the DC-gain and τ is the time-constant. The system has a pole in $s = -\frac{1}{\tau}$ i.e., the smaller time-constant, the faster system response.

2.1.2 Second Order Systems

The transfer funciton of a second-order system is given by:

$$H(s) = \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Where $\omega_n > 0$ is the natural frequency and $\zeta > 0$ is the damping ratio and k is the gain.

The system has two poles, which are $s \in \mathbb{C}$ where:

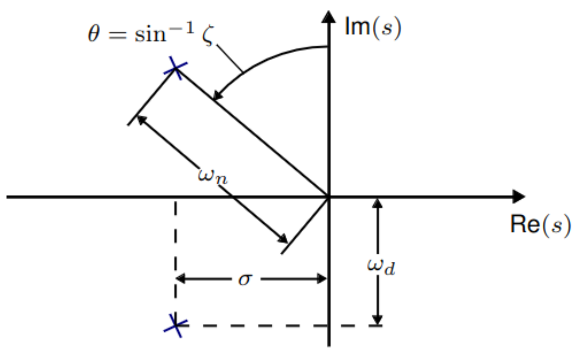
$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

The values of s is given by:

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

When $\zeta = 1$ the system is critically damped and $H(s)$ has a double pole in $s = -\zeta\omega_n$, when $0 < \zeta < 1$ the system is underdamped and has complex poles. When $\zeta > 1$ the system is overdamped and has real and distinct poles.

$$-\underbrace{\zeta\omega_n}_{\sigma} \pm j\,\underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d}.$$



Impulse response of a underdamped second-order system:

$$h(t) = k\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\sigma t}sin(\omega_d t)1(t)$$

The step resonse of a underdamped second-order system:

$$y(t) = k(1 - e^{-\sigma t}(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)))$$

Impulse response of a critically damped second order system:

$$h(t) = k\omega_n^2 te^{-\omega_n t}$$

Having a overdamped system with a damping ration that is greater than one leads to a slower impulse response.

The step response of a critically damped second order system:

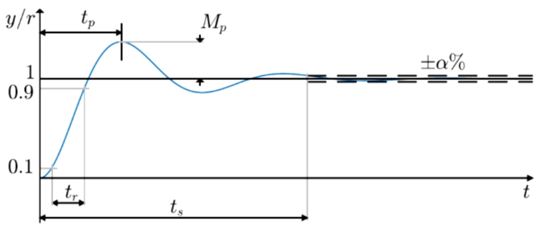
$$y(t) = k(1 - e^{-\omega_n t} - \omega_n te^{-\omega_n t})$$

2.2 Performance specifications

Three different performance measurements of dynamical systems are:

1. The rise time t_r (stigetid)
2. The settling time t_s (indsvingingstid)
3. The overshoot M_p (oversving)

The rise time is the time it takes for the system to go from 10% to 90% of the final value. The settling time is the time it takes for the system to reach and stay within a certain % of the final value. The overshoot is the maximum percentage that the system goes over the final value.



The peak time is denoted t_p which is the time where the signal reaches its maximum value.

For a second-order system the rise time can be approximated as:

$$t_r = \frac{1.8}{\omega_n}$$

For a second-order system the settling time can be approximated as:

$$t_s = \frac{-log(\alpha/100)}{\zeta\omega_n}$$

Log is ln in the equation.

2.3 Poles and Zeros of Space-State Models

Using thee state-space matrices:

$$G(s) = C(sI - A)^{-1}B + D$$

Poles can be found using the following equation:

When $G(s) \to \infty$ the system has a pole at $s \to p$.

$$\det(pI - A) = 0$$

Here p is the eigenvalue of A.

Zeros can be found using the following equation:

When $G(s) = 0$ the system has a zero at $s = z$.

$$\begin{vmatrix} A - zI & B \\ C & D \end{vmatrix} = 0$$

2.4 Stability

The stability of the dynamical system can be determined from the eigenvalues of A in the time domain. This is the equivalent of poles in the frequency domain.

$$\dot{x} = Ax + Bu$$

When the eigenvalues of A have a negative real part, the system is stable.

Definition A linear discrete-time system

$$x_{k+1} = \Phi x_k$$

The state of the next time step is given by ϕ mutliplied by x_k.

Where $x_k \in \mathbb{R}^{n \times n}$ is asymptotically (asymptotisk) stable if

$$\lim_{k \to \infty} x_k = 0$$

for any $x_0 \in \mathbb{R}^n$.

A linear discrete-time system is stable if all the eigenvalues are inside the unit circle.

2.5 Examples