Exercice 1

$$\det(A) = a_{31}\cdot\det(A_{31}) - a_{32}\cdot\det(A_{32}) + a_{33}\cdot\det(A_{33}) - a_{34}\cdot\det(A_{34})$$
$$= \cancel{a}\det(A_{31}) + 4\cdot\det(A_{33})$$

Définissons $B = A_{31} = \begin{pmatrix} 5 & -2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ et $C = A_{33} = \begin{pmatrix} 3 & 5 & 1 \\ 0 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$. Alors,

$$\det(A) = [b_{13}\cdot\det(B_{13}) - b_{23}\cdot\det(B_{23}) + b_{33}\cdot\det(B_{33})] + 4\cdot[c_{13}\cdot\det(C_{13}) - c_{23}\cdot\det(C_{23}) + c_{33}\cdot\det(C_{33})]$$
$$= \det(B_{13}) - \det(B_{23}) + \det(B_{33}) + 4\cdot\det(C_{13}) - 4\cdot\det(C_{23}) + 4\cdot\det(C_{33})$$
$$= (2-3) - (5+2) + (15+4) + 4\cdot(0+4) - 4\cdot(3+10) + 4\cdot(6-0)$$
$$= -1 - 7 + 19 + 16 - 52 + 24$$
$$= 16 + 19 + 24 - (1 + 7 + 52)$$
$$= 59 - 60$$
$$= -1$$

Exercice 2

$$A: \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ -2 & 0 & -8 & 0 & 0 & 1 \end{array}\right) \underset{L_3+2L_1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 4 & -2 & 2 & 0 & 1 \end{array}\right) \underset{L_2\leftrightarrow L_3}{\overset{\det\cdot(-1)}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 4 & -2 & 2 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{array}\right) \underset{L_3\cdot(-1)}{\overset{\det\cdot(-1)}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 4 & -2 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{array}\right) \underset{\substack{L_2+2L_3 \\ L_1-3L_3}}{\sim}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 3 & 0 \\ 0 & 4 & 0 & 2 & -2 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{array}\right) \underset{L_2\cdot\frac{1}{4}}{\overset{\det\cdot 4}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 3 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 1/4 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{array}\right) \underset{L_1-2L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 4 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1/4 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{array}\right)$$

Donc, $A^{-1} = \begin{pmatrix} 0 & 4 & -1/2 \\ 1/2 & 1/2 & 1/4 \\ 0 & -1 & 0 \end{pmatrix}$ et $\det(A) = 4\cdot(-1)\cdot(-1) = 4$

$$B: \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \underset{L_2-L_1}{\sim} \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \underset{L_3\leftrightarrow L_2}{\overset{\det\cdot(-1)}{\sim}} \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \underset{L_4-L_3}{\sim} \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right) \underset{L_2-L_4}{\sim}$$

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right) \underset{L_2-L_3}{\sim} \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right) \underset{L_1-L_2}{\sim} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right)$$

$B^{-1} = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}$ et $\det(B) = -1$

Exercice 3

A: $m_{11} = \det(A_{11}) = 0$ $\quad m_{12} = -\det(A_{21}) = -(-16) = 16$ $\quad m_{13} = \det(A_{31}) = \cancel{0}-2$ $\quad$ Par Ex. 2, $\det(A) = 4$.

$m_{21} = -\det(A_{12}) = -(-2) = 2$ $\quad m_{22} = \det(A_{22}) = -2$ $\quad m_{23} = -\det(A_{32}) = \cancel{4}-(-1) = 1$

$m_{31} = \det(A_{13}) = 0$ $\quad m_{32} = -\det(A_{23}) = -(\overset{-4}{-4}) = 1$ $\quad m_{33} = \det(A_{33}) = 0$

$$A^{-1} = \frac{1}{\det(A)}\cdot M = \frac{1}{4}\cdot\begin{pmatrix} 0 & 16 & -2 \\ 2 & -2 & 1 \\ 0 & -4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 4 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} \\ 0 & -1 & 0 \end{pmatrix}$$

B: $m_{11} = 1$ $\quad m_{12} = -x$ $\quad m_{13} = xy - z$ $\quad \det(B) = 1\cdot 1\cdot 1 = 1$

$m_{21} = 0$ $\quad m_{22} = 1$ $\quad m_{23} = -y$

$m_{31} = 0$ $\quad m_{32} = 0$ $\quad m_{33} = 1$

$$B^{-1} = \frac{1}{\det(B)}\cdot M = 1\cdot\begin{pmatrix} 1 & -x & xy-z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -x & xy-z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix}$$

Exercice 3 (cont.)

C:

$m_{11} = 0 \quad m_{12} = 0 \quad m_{13} = 0 \quad m_{14} = -1$

$m_{21} = -1 \quad m_{22} = 0 \quad m_{23} = 0 \quad m_{24} = 0$

$m_{31} = 0 \quad m_{32} = -1 \quad m_{33} = 0 \quad m_{34} = 0$

$m_{41} = 0 \quad m_{42} = 0 \quad m_{43} = -1 \quad m_{44} = 0$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \overset{\det\cdot(-1)}{\sim} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overset{\det\cdot(-1)}{\sim} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overset{\det\cdot(-1)}{\sim} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

donc $\det(C) = (-1)\cdot(-1)\cdot(-1) = -1.$

$$C^{-1} = \frac{1}{\det(C)} \cdot M = -1 \cdot \begin{pmatrix} 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Exercice 5

(a)
$$\begin{pmatrix} 1 & 1 & m \\ 1 & m & 1 \\ m & 1 & 1 \end{pmatrix} \underset{\substack{L_2 - L_1 \\ L_3 - mL_1}}{\sim} \begin{pmatrix} 1 & 1 & m \\ 0 & m-1 & 1-m \\ 0 & 1-m & 1-m^2 \end{pmatrix} \underset{L_3 + L_2}{\sim} \begin{pmatrix} 1 & 1 & m \\ 0 & m-1 & 1-m \\ 0 & 0 & 2-m^2-m \end{pmatrix}$$

Si $m = 1$, $C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ et $\text{rang}(C) = 1$

Si $m = -2$, $C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ et $\text{rang}(C) = 2$

Sinon, $\text{rang}(C) = 3$.

(b) $\det(C) = 1 \cdot (m-1)(-m^2 - m + 2) = 0$ ssi $x = 1$ ou $x = -2$.

La valeur de $\det(C)$ permet donc de dériver une partie de la réponse de (a), à savoir les cas où $\text{rang}(C) < 3$, mais pas ~~le~~ ~~d~~ rang exact de C dans ces cas.

Exercice 6

$$\det(A) = \det\left(\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & 0 \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{pmatrix}\right) = \det\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix} \cdot \det\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & 0 \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{pmatrix} = (1 \cdot 4 \cdot 6) \cdot \left(1 \cdot \frac{1}{3} \cdot \frac{1}{6}\right) = \frac{4}{3}.$$