MCMC Exercises (Systems of Systems)

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November 17, 2022

1 Exercise 1

A robot is being controlled by a velocity command v over a wireless network. At time $t = 0$, the robot position is $x(0) = 0$ and the operator sends a command $v = 1$ over the network. The command reaches the robot after a delay of d unit of time. Afterwards, we take N measurements $Y = [y_1, \ldots, y_N]$ of the robot position $x(T) = v \times (T - d)$ at time $T = 1$ unit of time. We know that the measurements can be modelled as

$$y_i = x(T) + \varepsilon_i,\ \varepsilon_i \sim \mathcal{N}(0, \sigma^2), \tag{1}$$

with the measurement noise being independent and identically distributed with $\sigma = 0.1$. Additionally, our prior knowledge on the delay is such that it is distributed according to an exponential distribution with mean $1/\lambda$ unit of time

$$\mathrm{d} \sim \mathrm{Exp}(\lambda)$$

Given that the probability density function of a Gaussian random variable z with mean μ and variance σ^2 is $p_{\mathrm{z}}(z) = \frac{1}{\sigma\sqrt{2\pi}}\left(e^{\frac{-(z-\mu)^2}{2\sigma^2}}\right)$ and that the probability density function of an exponential random variable z with mean $1/\lambda$ is $p_{\mathrm{z}}(z) = \lambda e^{-\lambda z}$,

- 1) Write the likelihood function $p_{\mathrm{Y}}(Y|d)$.
- 2) Write a function $f_Y(d)$ which is proportional to the posterior $p_{\mathrm{d}}(d|Y)$.
- 3) Write a Metropolis Hasting code to sample from the posterior $p_{\mathrm{d}}(d|Y)$ in the Matlab file MHEx.m (*Hint*: for efficient implementation, use the log of $f_Y(d)$).

2 Exercise 2

Consider a sequence of observations $X = [x_1, \ldots, x_N]$ obtained according to:

$$\begin{aligned} \mathrm{x}_i &\sim Ber(q_f), \quad 1 \le i \le n \\ \mathrm{x}_i &\sim Ber(q_s), \quad n+1 \le i \le N \end{aligned}$$

with positive integers N and n. Assume a beta prior for $\mathrm{q}_f \sim Beta(\alpha_f, \beta_f)$ and for $\mathrm{q}_s \sim Beta(\alpha_s, \beta_s)$. Moreover, assume a discrete uniform prior for n over $\{1, \ldots, N\}$. It is desired to estimate $\theta = [\mathrm{q}_f, \mathrm{q}_s, \mathrm{n}]$.

- 1) Show that the likelihood is in the form:

$$p_{\mathrm{X}|\theta}(X|\theta) = q_f^{\sum_{i=1}^n x_i}(1-q_f)^{n-\sum_{i=1}^n x_i} q_s^{\sum_{i=n+1}^N x_i}(1-q_s)^{N-n-\sum_{i=n+1}^N x_i} \tag{2}$$

  (*Hint*: the random variables x_i are independent and identically distributed. If u and v are independent then $p_{\mathrm{u,v}}(u, v) = p_{\mathrm{u}}(u)p_{\mathrm{v}}(v)$).

- 2) Show that the posterior is proportional to:

$$\begin{aligned} p_{\theta|\mathrm{X}}(\theta|X) \propto\ & q_f^{\alpha_f - 1 + \sum_{i=1}^n x_i}(1-q_f)^{\beta_f - 1 + n - \sum_{i=1}^n x_i} \\ & q_s^{\alpha_s - 1 + \sum_{i=n+1}^N x_i}(1-q_s)^{\beta_s - 1 + N - n - \sum_{i=n+1}^N x_i} \mathbb{1}(q_f, q_s, n) \end{aligned}$$

  with $\mathbb{1}(q_f, q_s, n) = \mathbb{1}_{[0,1]}(q_f)\mathbb{1}_{[0,1]}(q_s)\mathbb{1}_{[1,N]}(n)$.

- 3) Show that the conditional densities for q_f and q_s are:

$$
\begin{aligned}
p(q_f|X, q_s, n) \propto & Beta(\alpha_f + \sum_{i=1}^{n} X_i, \beta_f + n - \sum_{i=1}^{n} X_i) \\
p(q_s|X, q_f, n) \propto & Beta(\alpha_s + \sum_{i=n+1}^{N} X_i, \beta_s + N - n - \sum_{i=n+1}^{N} X_i)
\end{aligned}
$$

- 4) Show that the conditional probability density for n can be found by:

$$
p(n|X, q_f, q_s) = \frac{L(n, q_f, q_s, X)}{\sum_{n=1}^{N} L(n, q_f, q_s, X)}, \forall n \in 1, \ldots, N
$$

  with

$$
\begin{aligned}
L(n, q_f, q_s, X) = & q_f^{\alpha_f - 1 + \sum_{i=1}^{n} x_i} (1 - q_f)^{\beta_f - 1 + n - \sum_{i=1}^{n} x_i} \\
& q_s^{\alpha_s - 1 + \sum_{i=n+1}^{N} x_i} (1 - q_s)^{\beta_s - 1 + N - n - \sum_{i=n+1}^{N} x_i} \mathbb{1}_{[1,N]}(n)
\end{aligned}
$$

- 5) Write a Gibbs sampling algorithm in the Matlab file GibbsEx.m to sample from the posterior.