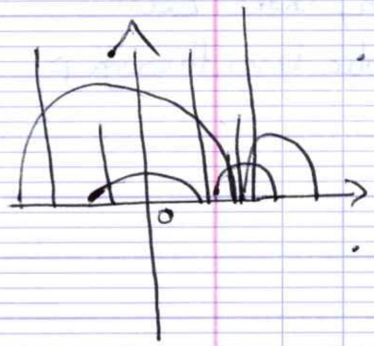
SL - GéoHyperbo

• upperhalf-plane model

$$\mathbb{C} \supset \mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \}$$

Let $\mathbb{S}^1 = \{ z \in \mathbb{C} , |z| = 1 \}$ be the unit circle of $\mathbb{C}$.

Definition 1.1 There are two seemingly different types of hyperbolic lines:

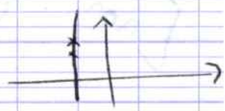


• one is the intersection of $\mathbb{H}$ with a euclidean line in $\mathbb{C}$ perpendicular to the real axis $\mathbb{R}$ in $\mathbb{C}$

• the other is the intersection of $\mathbb{H}$ with a euclidean circle centred on the real axis $\mathbb{R}$.

→ exists a way of unifying these two types of hyperbolic lines.

Proposition 1.2 For each pair p and q of distincts points in $\mathbb{H}$, there exists a unique hyperbolic line ℓ in $\mathbb{H}$ passing through p and q.

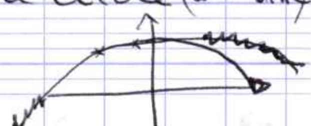


Proof Two cases: • same real parts

↳ euclidean line

• $\neq$ real parts

↳ construct a circle (centred on $\mathbb{R}$)



uniqueness comes from the euclidean setting.

Is hyperbolic geo different from euclidean geo since it is based on it?

YES ! A LOT. Will see how much.

$\swarrow$

First example

Definition 1.3 Two hyperbolic lines in $\mathbb{H}$ are parallel if they are disjoint. [par analogie avec la géométrie euclidienne]

Theorem 1.4. Let ℓ be a hyperbolic line in $\mathbb{H}$ and let p be a point in $\mathbb{H}$ not on ℓ. Then, there exist infinitely many distinct hyperbolic lines through p that are parallel to ℓ.

if two are constructed take points in between.

Proof Two cases ✓.

Hyperbolic geometry do not respect Euclidean parallel postulate, i.e. given Euclidean line L and a point not on L, ~~there~~ there exists a unique Euclidean line through p and parallel to L.

[axiomatique]

[non euclidean geometry]

$\Downarrow$

$\exists$ at least two in hyperbo (infinitely many)

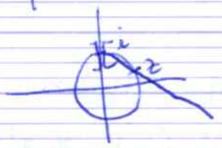
Need to define hyperbolic length, hyperbolic distance, hyperbolic area

$\Rightarrow$ group of transformations of $\mathbb{H}$ taking hyperbolic lines to hyperbolic lines

Beltrami, Klein, Poincaré

1.2. The Riemann Sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

$\longrightarrow$ unify hyperbolic lines as one single object



[one-point compactification]

Consider $\xi : \mathbb{S}^1 - \{i\} \to \mathbb{R}$ assigning $z \mapsto \mathbb{R} \cap K_z$ where K_z is the euclidean line passing through i and $z \in \mathbb{S}^1 - \{i\}$
[stereographic projection]
$\hookrightarrow$ Rend raisonnable d'appeler le compactifié la sphère de Riemann

One has : $\xi(z) = \dfrac{\mathrm{Re}(z)}{1 - \mathrm{Im}(z)}$ $(, z \neq i)$
[bc of line equation of K_z]

$$\xi^{-1}(x) = \frac{2x}{x^2+1} + i\,\frac{x^2-1}{x^2+1}$$

$\Rightarrow$ ξ is a bijection between $\mathbb{S}^1 - \{i\}$ and $\mathbb{R}$
$\Rightarrow$ because $\mathbb{R}$ is $\mathbb{S}^1$ minus a point, then (alike) $\mathbb{R}$ might be $\mathbb{S}^1$ plus a point

Topoly is $\mathbb{H}$ is essentially the same, except for the ∞ point.

For regular points: $U_\varepsilon(z) = \{w \in \mathbb{C} \mid |w - z| < \varepsilon\}$
For ∞ point : $U_\varepsilon(\infty) = \{w \in \mathbb{C} \mid |w| > \varepsilon\} \cup \{\infty\}$

Def A set X is open if for each point x of X, there exists some $\varepsilon > 0$ (possibly depending on x, X) so that $U_\varepsilon(x) \subset X$

D open of $\mathbb{C} \Rightarrow D$ open of $\overline{\mathbb{C}}$

eg. $\mathbb{H}$ is an open of $\overline{\mathbb{C}}$
$U_1(\varepsilon)$ is open in $\overline{\mathbb{C}}$
$\mathbb{S}^1$ is not open

Def A set X is closed in $\overline{\mathbb{C}}$ if its complement $\overline{\mathbb{C}} - X$ is open in $\overline{\mathbb{C}}$.

eg $\mathbb{S}^1$ is closed in $\overline{\mathbb{C}}$ because
$$\overline{\mathbb{C}} - \mathbb{S}^1 = \mathcal{U}_1(0) \cup \mathcal{U}_1(\infty)$$
(union of opens is open)

Def A sequence $\{z_n\}_n$ of $\overline{\mathbb{C}}$ converges to $z \in \overline{\mathbb{C}}$ if for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ so that $z_n \in \mathcal{U}_\epsilon(z)$, $\forall n \geq N$

[adhérence]

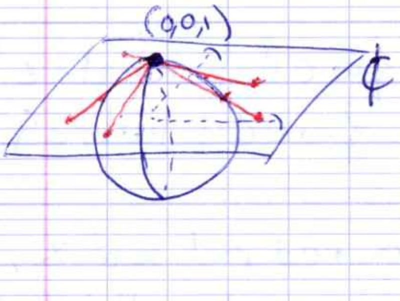
Def Let $X \subset \overline{\mathbb{C}}$. The closure of X in $\overline{\mathbb{C}}$, denoted $\overline{X}$, is
$$\overline{X} = \{z \in \overline{\mathbb{C}} \mid \mathcal{U}_\epsilon(z) \cap X \neq \emptyset, \text{ for all } \epsilon > 0\}.$$

Note $X \subset \overline{X}$ because $\{x\} \subset \mathcal{U}_\epsilon(x) \cap X$
$\lim_n x_n \in \overline{X}$ for $(x_n) \in X^{\mathbb{N}}$.

Def A circle in $\overline{\mathbb{C}}$ is either a Euclidean circle in $\mathbb{C}$ or the union of a Euclidean line in $\mathbb{C}$ with $\{\infty\}$.

$\overline{L} = L \cup \{\infty\}$
notation

eg $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is a circle in $\overline{\mathbb{C}}$



le pôle nord

$$\xi : \mathbb{S}^2 - \{(0,0,1)\} \longrightarrow \mathbb{C}$$

[generalization of stereographic projection]

$\forall P \in \mathbb{S}^2 - \{(0,0,1)\}$, let L_P line passing through $\{(0,0,1)\}$ and P and let $\xi(P)$ be the intersection between L_P and $\mathbb{C}$

cercles, droites, sphères ⇒ noms raisonnables même si ça ne le paraît pas totalement au début

ξ is bijective.

Equation circles in $\overline{\mathbb{C}}$: $\alpha z\overline{z} + \beta z + \overline{\beta}\,\overline{z} + \gamma = 0$, $\alpha, \gamma \in \mathbb{R}$, $\beta \in \mathbb{C}$

Rq : un cercle sur $\mathbb{S}^2$ est l'intersection d'un plan non tangent avec $\mathbb{S}^2$
les cercles de $\overline{\mathbb{C}}$ sont exactement les images de ces cercles par la projection (clair pour cercles passant par N, moins pour les autres)
($\hookrightarrow$ soit preuve bourrin calcul, soit dans quelques séances argument plus abstrait

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Def A function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is continuous at $z \in \overline{\mathbb{C}}$ if for each $\epsilon > 0$, there exist $\delta > 0$ (depending on ϵ, z) so that $w \in U_\delta(z)$ implies $f(w) \in U_\epsilon(f(z))$.

[usual operations ~~Rea~~ continuous function]
[slight difference between $\mathbb{R}$-continuity and $\overline{\mathbb{C}}$-continuity]

Proposition The function $J : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ defined by
$$J(z) = \frac{1}{z} \text{ for } z \in \mathbb{C}^*, \ J(0) = \infty, \ J(\infty) = 0$$
is continuous on $\overline{\mathbb{C}}$.
Moreover, it is a homeomorphism of $\overline{\mathbb{C}}$

on va le revoir cette fonction

$J \circ J(z) = z, \ z \in \overline{\mathbb{C}}$

$\Downarrow$ a bijection and both f and f^{-1} are continuous

We note $\mathrm{Homeo}(\overline{\mathbb{C}}) = \{ f : \overline{\mathbb{C}} \to \overline{\mathbb{C}} \mid f \text{ is a homeomorphism} \}$
$\hookrightarrow$ forms a group.

$\to$ terminologie deviendra claire + tard

1.3 The Boundary at infinity of $\mathbb{H}$.

NOTE : complement of circles in $\overline{\mathbb{C}}$ have two components
eg. $(\mathbb{S}^1)^c$: $U_1(0)$ and $U_1(\infty)$
$(\overline{\mathbb{R}})^c$: $\mathbb{H}$ and $\mathbb{H}^c_-$

Def A disc D in $\overline{\mathbb{C}}$ is one of the components of the complement of a circle A in $\overline{\mathbb{C}}$. For such D and A, we refer to A as the circle determining the disc D.

eg $\overline{\mathbb{R}}$ is the circle determining $\mathbb{H}$ $\to$ $\overline{\mathbb{R}}$ = "boundary at infinity of $\mathbb{H}$"
$\to$ pts of $\overline{\mathbb{R}}$ = "points at infinity of $\mathbb{H}$"

cercle est le bord du disque !
pq dist "à l'infini" ? $\to$ distance.
~~ça manque du bord~~.

every circle on $\overline{\mathbb{C}}$ determines two disjoint discs in $\overline{\mathbb{C}}$.
a disc determines an unique circle.