INTRODUCTION TO ANALYTIC NUMBER THEORY (II)
PROOF OF DIRICHLET THEOREM

MATHIAS GARNIER

ABSTRACT. We prove Dirichlet theorem on primes in arithmetic progression (Analytic number theory course, lecture given by Anders Södergren).

Written during the first week in Göteborg, Sweden. **State: draft. Need to pay more attention later (when it will be properly written).**

To prove the classical Dirichlet theorem (theorem 1) we will study the behaviour of an arithmetic-interest function (theorem 2).

Theorem 1. *If $a \in \mathbf{Z}$, $q \in \mathbf{N}$ are such that $(a, q) = 1$, then :*

$$\sum_{p \equiv a \bmod q} \frac{1}{p} = +\infty.$$

Theorem 2. *Under the same assumptions as in theorem 1, then :*

$$\sum_{\substack{p \leq x \\ p \equiv a \bmod q}} \frac{1}{p} = \frac{1}{\phi(q)} \log \log x + O_q(1) \quad \text{for } x \geq 3.$$

For the moment, we will use the following theorem as a black box. It will be proved afterwards.

Theorem 3. *Let $q \in \mathbf{N}$, if $\chi \neq \chi_0$ is a Dirichlet character modulo q, then $\mathrm{L}(1, \chi) \neq 0$.*

Let's prove theorem 2, the classical Dirichlet theorem 1 on primes in arithmetic progressions is an immediate corollary.

Let $a \in \mathbf{Z}$ and $q \in \mathbf{N}$ be such that $(a, q) = 1$. Define the following quantities :

$$S_{a,q}(x) = \sum_{\substack{p \leq x \\ p \equiv a \bmod q}} \frac{1}{p} \quad \text{for } x \geq 2,$$

$$F_{a,q}(s) = \sum_{p \equiv a \bmod q} \frac{1}{p^s} \quad \text{for } \mathrm{Re}(s) > 1.$$

Recall that in the upper half-plane $\mathrm{Re}(s) > 1$, there is absolute and uniform convergence on compact subsets. We will have to go through six steps.

Step 1. It is sufficient to show that :

$$F_{a,q}(\sigma) = \frac{1}{\phi(q)} \log \frac{1}{\sigma - 1} + O_q(1), \quad 1 < \sigma \leq 2. \tag{1}$$

Later on, we will see that the bound 2 is essentially decorative. The major problems arise near $s = 1$.

Let $x \geq 3$ and choose $\sigma := \sigma(x) = 1 + \dfrac{1}{\log x}$. With such σ in equation 1, we get the same main terms. It remains to show that :

$$|S_{a,q}(x) - F_{a,q}(\sigma(x))| = O_q(1).$$

UNIVERSITÉ PAUL SABATIER, TOULOUSE, FRANCE and CHALMERS TEKNISKA HÖGSKOLA, UNIVERSITY OF GÖTEBORG, SWEDEN.

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Note that :

$$S_{a,q}(x) - F_{a,q}(\sigma(x)) = \sum_{\substack{p\leq x \\ p\equiv a \bmod q}} \frac{1-p^{1-\sigma(x)}}{p} - \sum_{\substack{p>x \\ p\equiv a \bmod q}} \frac{1}{p^{\sigma(x)}}.$$

Moreover :

$$1-p^{1-\sigma(x)} = 1-e^{(1-1-1/\log x)\log p} = 1-e^{-\log p/\log x} \ll \log p/\log x$$

by a Taylor expansion for $p \leq x$.

Hence :

$$\sum_{\substack{p\leq x \\ p\equiv a \bmod q}} \frac{1-p^{1-\sigma(x)}}{p} \ll \frac{1}{\log x}\sum_{p\leq x}\frac{\log p}{p} \ll 1$$

by Mertens theorem. We now need our best friend summation by parts (always in the bag pocket) to control the last part (the sum for $p > x$). Thus :

$$\begin{aligned}
\sum_{\substack{p>x \\ p\equiv a \bmod q}} \frac{1}{p^{\sigma(x)}} \leq \sum_{p>x}\frac{1}{p^{\sigma(x)}} &= -\frac{\pi(x)}{x^{\sigma(x)}} + \sigma(x)\int_x^\infty \frac{\pi(u)}{u^{\sigma(x)+1}}\mathrm{d}u \\
&\ll \frac{1}{\log x} + \int_x^\infty \frac{1}{u^{\sigma(x)+1}\log u}\mathrm{d}u \\
&\ll \frac{1}{\log x} + \frac{1}{\log x} + \int_x^\infty \frac{1}{u^{\sigma(x)+1}}\mathrm{d}u \\
&\ll \frac{1}{\log x} + \frac{1}{\log x}\frac{x^{1-\sigma(x)}}{\sigma(x)-1} \\
&\ll 1
\end{aligned}$$

for x sufficiently large. Notice that, in the second line, we used Chebyshev theorem or the Prime Number Theorem; and, in the third line, we used that $1/\log x$ is non-increasing ($x \geq 3$).

Thus :

$$|S_{a,q}(x) - F_{a,q}(\sigma(x))| = O_q(1)$$

and this shows this is sufficient to study the behaviour of $F_{a,q}(\sigma)$.

Step 2. Let χ be a Dirichlet character modulo q, define:

$$F_\chi(s) = \sum_p \frac{\chi(p)}{p^s}, \quad \mathrm{Re}(s) > 1.$$

Never forget that, in this domain, we have absolute and uniform convergence of the series. Now, let $\sigma > 1$, one has :

$$\begin{aligned}
F_{a,q}(\sigma) &= \sum_{p\equiv a \bmod q} \frac{1}{p^\sigma} \\
&= \sum_p \frac{1}{p^\sigma}\frac{1}{\phi(q)}\sum_{\chi \bmod q}\chi(p)\overline{\chi(a)} \\
&= \frac{1}{\phi(q)}\sum_{\chi \bmod q}\overline{\chi(a)}\sum_p\frac{\chi(p)}{p^\sigma} \\
&= \frac{1}{\phi(q)}\sum_{\chi \bmod q}\overline{\chi(a)}F_\chi(\sigma).
\end{aligned}$$

We applied a very prolific strategy to make congruences disappear : use orthogonality relations of Dirichlet series! And, again, we use the fact that there is only a finite number of Dirichlet characters modulo q (there are exactly $\phi(q)$) and that we can rearrange a finite sum of absolutely convergent series. Thus, one do not care about summation order.

Just recall a small thing :

$$\chi(x)^{-1} = \frac{1}{\chi(x)} = \frac{\overline{\chi(x)}}{|\chi(x)|^2} = \overline{\chi(x)}.$$

Step 3. We know want to express $F_\chi(s)$ in terms of $\log \mathrm{L}(s,\chi)$. Recall the logarithmic formula :

$$\log \mathrm{L}(s,\chi) = -\sum_p \log\left(1 - \frac{\chi(p)}{p^s}\right).$$

It is a direct consequence of the Euler product formula for such L function. We now use Taylor expansion :

$$\begin{aligned}\log \mathrm{L}(s,\chi) &= \sum_p \sum_{m=1}^{\infty} \frac{\chi(p)^m}{mp^{ms}} \\ &= \sum_p \frac{\chi(p)}{p^s} + \sum_p \sum_{m=2}^{\infty} \frac{\chi(p)^m}{mp^{ms}} \\ &= F_\chi(s) + \text{remainder}.\end{aligned}$$

We were able to make such an expansion because $|\chi(p)/p^s| \leq |1/p^s| \leq 1/2 < 1$. Estimate the remainder trivially :

$$\begin{aligned}\left|\sum_p \sum_{m=2}^{\infty} \frac{\chi(p)^m}{mp^{ms}}\right| &\leq \sum_p \sum_{m=2}^{\infty} \frac{1}{mp^m} \leq \frac{1}{2}\sum_p \frac{1}{p^2}\sum_{m=0}^{\infty}\frac{1}{p^m} \\ &\leq \frac{1}{2}\sum_{n\geq 2}\frac{1}{n^2}\sum_{m=0}^{\infty}\frac{1}{n^m}\frac{1}{2}\sum_{n\geq 2}\frac{1}{n^2}\frac{n}{n-1} \\ &\leq \frac{1}{2}\sum_{n\geq 2}\frac{1}{n(n-1)} = O(1).\end{aligned}$$

We conclude that :

$$F_\chi(s) = \log \mathrm{L}(s,\chi) + O(1), \quad \sigma > 1.$$

Step 4. We now prove the following statement :

$$\exists \sigma_0 > 1,\ \log \mathrm{L}(s,\chi_0) = \log \frac{1}{\sigma - 1} + O_q(1), \quad 1 < \sigma \leq \sigma_0.$$

Note that σ_0 depends on q. If one wonders why proving such statement, think about step 1!

Recall that $L(s,\chi_0)$ is analytic in $\sigma > 0$ apart from the simple pole $s = 1$, with residue $\phi(q)/q$. Thus :

$$\mathrm{L}(s,\chi_0) = \frac{\phi(q)}{q}\frac{1}{s-1} + A(s)$$

where A is analytic in $\sigma > 0$. Since, $A(s)$ is bounded on compacts in $\sigma > 0$, we get :

$$|A(\sigma)| \leq \frac{\phi(q)}{2q(\sigma - 1)}, \quad 1 < \sigma \leq \sigma_0 \tag{2}$$

for some $\sigma_0 := \sigma_0(q) > 1$. We decorated the bound such that the incoming computations look nice. Pay attention that $]1,\sigma_0]$ is not compact! It has no repercussion there since, if σ is close to 1, the quantity $1/(\sigma - 1)$ will be large (so no problem). Thus, using log properties and tricking quantities :

$$\begin{aligned}\log \mathrm{L}(s,\chi) &= \log\left(\frac{\phi(q)}{q}\frac{1}{\sigma-1}\left(1 + A(\sigma)(\sigma-1)\frac{q}{\phi(q)}\right)\right) \\ &= \log\left(\frac{\phi(q)}{q}\frac{1}{\sigma-1}\right) + \log\left(1 + A(\sigma)(\sigma-1)\frac{q}{\phi(q)}\right) \\ &= \log\left(\frac{\phi(q)}{q}\right) + \log\left(\frac{1}{\sigma-1}\right) + \log\left(1 + A(\sigma)(\sigma-1)\frac{q}{\phi(q)}\right).\end{aligned}$$

We will restrict to real numbers, otherwise we should care about branches of the logarithm.

Note that, due to equation 2, we have :

$$\left| A(\sigma)(\sigma-1)\frac{q}{\phi(q)} \right| \leq \frac{1}{2}.$$

Hence :

$$\log \mathrm{L}(s,\chi_0) = \log \frac{1}{\sigma-1} + O_q(1), \quad 1 < \sigma \leq \sigma_0.$$

From step 3, we deduce :

$$F_{\chi_0}(\sigma) = \log \mathrm{L}(s,\chi_0) + O(1) = \log \frac{1}{\sigma-1} + O(1), \quad 1 < \sigma \leq \sigma_0. \tag{3}$$

Also, equation 3 holds more generally for any $\sigma > 1$, if we remove the log term for large σ :

$$|F_{\chi_0}(\sigma)| = \left| \sum_p \frac{\chi_0(p)}{p^\sigma} \right| \leq \sum_p \frac{1}{p^\sigma} = O_q(1), \quad \sigma > \sigma_0. \tag{4}$$

Step 5. We now prove the following statement on the remaining characters : if $\chi \neq \chi_0$ is a Dirichlet character modulo q, then $F_\chi(\sigma) = O_q(1)$ in $\sigma > 1$.

Recall that, if $\chi \neq \chi_0$, then $\mathrm{L}(s,\chi)$ is analytic in $\sigma > 0$ (the situation is even better than the χ_0 case). Moreover, from theorem 3, we have $\mathrm{L}(s,\chi) \neq 0$ for $\chi \neq \chi_0$. Thus, $\log \mathrm{L}(s,\chi)$ is well-defined, analytic and then continuous in a neighbourhood of $s = 1$. Hence, there exists a $\sigma_0 > 1$ such that $\log \mathrm{L}(s,\chi)$ is bounded in $1 < \sigma \leq \sigma_0$. From step 3, we deduce :

$$F_\chi(s) = O_q(1), \quad 1 < \sigma \leq \sigma_0$$

The domain can be extended to $\sigma > 1$ by a same argument as in equation 4.

Step 6. We now put the pieces together. From step 2, we have :

$$\begin{aligned} F_{a,q}(\sigma) = \sum_{p \equiv a \bmod q} \frac{1}{p^\sigma} &= \frac{1}{\phi(q)} \sum_{\chi \bmod q} \overline{\chi(a)} F_\chi(\sigma) \\ &= \frac{1}{\phi(q)} \overline{\chi_0(a)} F_{\chi_0}(\sigma) + O_q(1) \\ &= \frac{1}{\phi(q)} \log \frac{1}{\sigma-1} + O_q(1) \end{aligned}$$

where the $O_q(1)$ is the contribution from non trivial characters (step 5). From step 1, this proves theorem 2.

We now need to prove theorem 3 :

let $q \in \mathbf{N}$, if $\chi \neq \chi_0$ is a Dirichlet character modulo q, then $\mathrm{L}(1,\chi) \neq 0$.

We will consider two cases : complex and real characters.

Complex characters. Recall that a character is complex if there exists an $n \in \mathbf{Z}$ such that $\chi(n) \notin \mathbf{R}$. Note for $\sigma > 1$, we have :

$$\begin{aligned} \sum_{\chi \bmod q} \log \mathrm{L}(s,\chi) &= - \sum_{\chi \bmod q} \sum_p \log\left(1 - \frac{\chi(p)}{p^s}\right) \\ &= \sum_{\chi \bmod q} \sum_p \sum_{m=1}^{\infty} \frac{\chi(p)^m}{mp^{ms}} \\ &= \sum_p \sum_{m=1}^{\infty} \frac{1}{p^{ms}} \sum_{\chi \bmod q} \chi(p)^m. \end{aligned}$$

However, by the orthogonality relations,

$$\sum_{\chi \bmod q} \chi(p)^m = \sum_{\chi \bmod q} \chi(p^m) = \begin{cases} \phi(q) & \text{if } p^m \equiv 1 \bmod q \\ 0 & \text{else.} \end{cases}$$

Thus, since we sum non-negative terms, we conclude :

$$\sum_{\chi \bmod q} \log \mathrm{L}(\sigma, \chi) \geq 0, \quad \sigma > 1.$$

We have already done these calculations in step 3 (see there for any justification). Take the exponential :

$$\prod_{\chi \bmod q} \mathrm{L}(\sigma, \chi) \geq 1, \quad \sigma > 1. \tag{5}$$

Now be really careful (*ne pas aller trop vite en besogne!*), suppose $\mathrm{L}(1, \chi_1) = 0$ for a complex character χ_1. Then :

$$L(1, \overline{\chi_1}) = \sum_{n=1}^{\infty} \frac{\overline{\chi_1}(n)}{n} = \overline{\sum_{n=1}^{\infty} \frac{\chi_1(n)}{n}} = \overline{\mathrm{L}(1, \chi_1)}$$

by absolute convergence. Note that χ_1 and $\overline{\chi_1}$ are distinct since χ_1 is a complex character. We thus have at least two factors in equation 5 that are zero at $s = 1$. Next, write :

$$\prod_{\chi \bmod q} \mathrm{L}(s, \chi) = \mathrm{L}(s, \chi_0)\mathrm{L}(s, \chi_1)\mathrm{L}(s, \overline{\chi_1}) \prod_{\chi \bmod q \chi \neq \chi_0, \chi_1, \overline{\chi_1}} \mathrm{L}(s, \chi). \tag{6}$$

We have four different terms on the right hand side : a **simple** pole at $s = 1$, a **double** zero at $s = 1$ and an analytic part at $s = 1$. The whole product is meromorphic in $\sigma > 0$ and equation 6 implies that $s = 1$ is a removable singularity. Then, the product will extend to an analytic function in $\sigma > 0$ with a zero at $s = 1$. Taking the limit, we get :

$$\lim_{s \to 1^+} \prod_{\chi \bmod q} \mathrm{L}(s, \chi) = 0.$$

This contradicts equation 5. Then, the assertion is proved for complex characters.

Real characters. It is a little bit harder. Again, suppose $\mathrm{L}(s, \chi) = 0$. Remark that $\mathrm{L}(s, \chi)\mathrm{L}(s, \chi_0)$ is analytic at $s = 1$ so in $\sigma > 0$ (its only pole has been cancelled). Also, $\mathrm{L}(2s, \chi_0)$ is analytic in $\sigma > 1/2$ and non-zero. Introduce now the analytic function Ψ :

$$\Psi(s) = \frac{\mathrm{L}(s, \chi)\mathrm{L}(s, \chi_0)}{\mathrm{L}(2s, \chi_0)}, \quad \sigma > 1/2.$$

Note that $\lim_{s \to 1/2^+} \mathrm{L}(2s, \chi_0) = +\infty$, thus $\lim_{s \to 1/2^+} \Psi(s) = 0$ because the denominator is finite when $s \to 1/2^+$.

To find a contradiction, we need to understand Ψ better. For $\sigma > 1$, we use the Euler products :

$$\Psi(s) = \prod_p \frac{\left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \left(1 - \frac{\chi_0(p)}{p^s}\right)^{-1}}{\left(1 - \frac{\chi_0(p)}{p^{2s}}\right)^{-1}} = \prod_p \lambda(\chi, p, s).$$

Note that $\chi_0(p) = 1$ if $p \not| q$, $\chi_0(p) = 0$ if $p|q$ and $\chi(p) = \pm 1$ if $p \not| q$, $\chi(p) = 0$ if $p|q$ because χ is a real character. Thus :

$$\begin{cases} \lambda(\chi, p, s) = 1 & \text{if } p|q \\ \lambda(\chi, p, s) = \dfrac{\left(1 + \frac{\chi(p)}{p^s}\right)^{-1} \left(1 - \frac{\chi_0(p)}{p^s}\right)^{-1}}{\left(1 - \frac{\chi_0(p)}{p^{2s}}\right)^{-1}} = 1 & \text{if } p \not| q \text{ and } \chi(p) = -1 \\ \lambda(\chi, p, s) = \dfrac{\left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \left(1 - \frac{\chi_0(p)}{p^s}\right)^{-1}}{\left(1 - \frac{\chi_0(p)}{p^{2s}}\right)^{-1}} & \text{else.} \end{cases}$$

As a direct consequence :

$$\Psi(s) = \prod_{p,\ \chi(p)=1} \frac{\left(1-\frac{\chi(p)}{p^s}\right)^{-1}\left(1-\frac{\chi_0(p)}{p^s}\right)^{-1}}{\left(1-\frac{\chi_0(p)}{p^{2s}}\right)^{-1}}$$
$$= \prod_{p,\ \chi(p)=1} \left(\frac{1+\frac{1}{p^s}}{1-\frac{1}{p^s}}\right)$$
$$= \prod_{p,\ \chi(p)=1} \left(1+\frac{1}{p^s}\right)\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\dots\right), \quad \sigma > 1$$

using the geometric series expansion. Note that such prime ($\chi(p) = 1$) exists, otherwise the product would be empty and then equal to 1. Taking the limit, it would still be 1. Moreover, we would be able to extend the product to a large region of the complex problem (no problem since it would be a constant function). This is a contradiction since the limit of $\Psi(s)$ is 0 as long as $s \to 1/2^+$ and not 1.

Now, we claim and give only a rough but nearly complete idea of the proof that $\Psi(s)$ can be expanded as a Dirichlet series :

$$\Psi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad \sigma > 1.$$

where $a_1 = 1$ and $a_n \geq 0$ for all $n \in \mathbf{N}$. This series is also convergent in $\sigma > 1$ and uniformly convergent on compact subsets of $\sigma > 1$.

It is not trivial since we start from a product and go to a sum (usually, we do the contrary). When $\sigma > 1$, we have $|1/p^s| < 1$ for all p and we compute :

$$\Psi(s) = \prod_{p,\ \chi(p)=1} \left(1 + 2p^{-s} + 2p^{-2s} + 2p^{-3s} + \dots\right).$$

Define, for fixed $s \in \mathbf{C}$ with $\sigma > 1$, $f : \mathbf{N} \to \mathbf{C}$ to be multiplicative and satisfying :

$$f(p^k) = \begin{cases} 0 & \text{if } \chi(p) = 0 \\ 2p^{-ks} & \text{if } \chi(p) = 1. \end{cases}$$

Then :

$$f(n) = n^{-s} \prod_{p|n} \left\{ \begin{matrix} 0 & \text{if } \chi(p) = 0 \\ 2p^{-ks} & \text{if } \chi(p) = 1. \end{matrix} \right\}$$

One still has to prove that the Dirichlet series defined is absolutely convergent. We conclude that :

$$\Psi(s) = \sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} n^{-s} \prod_{p|n} \left\{ \begin{matrix} 0 & \text{if } \chi(p) = 0 \\ 2p^{-ks} & \text{if } \chi(p) = 1. \end{matrix} \right\}$$

and verify that $f(1) = 1$ since the product is empty and $f(n) \geq 0$.

Now, recall that Ψ is analytic in the half-plane $\sigma > 1/2$. Hence, because of analyticity, we can expand Ψ in power series around $s = 2$ (say, somewhere in $\sigma > 1/2$) with radius of convergence at most[1] $3/2$:

$$\Psi(s) = \sum_{m=0}^{\infty} \frac{1}{m!} \Psi^{(m)}(2)(s-2)^m.$$

We need to understand the derivatives of Ψ. Note that we can differentiate termwise since it is a Dirichlet series convergent in $\sigma > 1$. We have :

$$\Psi^{(m)}(2) = (-1)^m \sum_{n=1}^{\infty} a_n (\log n)^m n^{-2} = (-1)^m b_m$$

[1]Verify! I guess this is at most but I wrote at least.

where b_m is non-negative. Thus :

$$\Psi(s) = \sum_{m=0}^{\infty} \frac{1}{m!} b_m (2-s)^m.$$

Notice that the $(-1)^m$ was put inside $(s-2)^m$. Hence, if we restrict to real s such that $1/2 < s < 2$, then all terms in the Taylor series are non-negative and thus $\Psi(s) \geq \Psi(2) \geq 1$ by the product formulas. In such case, Ψ could never tend to 0. This is a contradiction (of the $1/2^+$ limit).

As a conclusion, if χ is a real or complex character different from χ_0, then $\mathrm{L}(1,\chi) \neq 0$.

It is now time to end this section with some information about L functions, what comes next (in Montgomery's textbook) and an incursion in modern number theory.

A. Analytic continuation and functional equation. Let χ be a Dirichlet character modulo q. We need to express $n \mapsto \chi(n)$ as a linear combination of $n \mapsto e^{2i\pi mn/q}$ for $m = 0, \ldots, q-1$.

For any χ modulo q, we define the Gauss sum $\tau(\chi)$ by :

$$\tau(\chi) = \sum_{m \in \mathbf{Z}/q\mathbf{Z}} \chi(m) e^{2i\pi m/q}.$$

If $(n,q) = 1$, then :

$$\chi(n)\tau(\overline{\chi}) = \chi(n) \sum_{m \in \mathbf{Z}/q\mathbf{Z}} \overline{\chi}(m) e^{2i\pi m/q} = \sum_{m \in \mathbf{Z}/q\mathbf{Z}} \overline{\chi}(n^{-1}m) e^{2i\pi m/q} = \sum_{h \in \mathbf{Z}/q\mathbf{Z}} \overline{\chi}(h) e^{2i\pi hn/q}.$$

This is the desired decomposition of $\chi(n)$.

Note that, if χ is a primitive character, then $|\tau(\chi)| = \sqrt{q}$.

Theorem 4. *Let χ be a primitive character modulo $q \geq 3$. Then $\mathrm{L}(s,\chi)$ has an analytic continuation to an entire function. Furthermore, $\mathrm{L}(s,\chi)$ satisfies*

$$\xi(1-s,\overline{\chi}) = \frac{i^a\sqrt{q}}{\tau(x)} \xi(s,\chi)$$

where $\xi(s,\chi) = \left(\frac{\pi}{a}\right)^{-1/2(s+a)} \Gamma(1/2(s+a))\mathrm{L}(s,\chi)$, $a = 0$ if $\chi(-1) = 1$ and $a = 1$ if $\chi(-1) = -1$. Note that the absolute value of the factor in the functional equation is 1.

The proof is very similar to the one for $\zeta(s)$. In particular, we start looking at the function $(s,\chi) \mapsto \pi^{-s/2} q^{s/2} \Gamma(s/2) L(s,\chi)$ and representing it with an integral (the Γ integral representation). Then, it is exactly the same idea but not the same details (the theta identity is a little bit trickier).

Remark that we get information about zeros of $\xi(s,\chi)$ for $0 \leq \sigma \leq 1$ and $\mathrm{L}(s,\chi)$ (trivial and non trivial).

B. Zero-free regions for $\mathrm{L}(s,\chi)$. It is relatively easy to generalize what is done for $\zeta(s)$ to $\mathrm{L}(s,\chi)$ with a **fixed** χ modulo q. However, we would like estimates with explicit q-dependence. This is way more difficult.

Theorem 5.

i) *There exists an absolute constant $c > 0$ such that, for all q in $\mathbf{N}$ and every **complex** character modulo q, $\mathrm{L}(s,\chi)$ has no zeros in the following region :*

$$\sigma \geq \begin{cases} 1 - \dfrac{c}{\log(q|t|)} & |t| \geq 1, \\ 1 - \dfrac{c}{\log q} & |t| \leq 1. \end{cases}$$

ii) *There exists an absolute constant $c > 0$ such that, for all q in $\mathbf{N}$ and every **real non trivial** character modulo q, $\mathrm{L}(s,\chi)$ has at most one zero in the same σ region. If such a zero exist, then this is a simple real zero (and it is a potential threat to any generalization of the Riemann hypothesis). These zeros are called Siegel zeros.*

C. The PNT for arithmetic progressions. This is an asymptotic formula for $\pi(x;q,a) = \#\{p \leq x \ : \ p \equiv a \bmod q\}$ (p is obviously a prime). As for the PNT, it is more convenient to work with a Ψ function. We introduce :

$$\Psi(x;q,a) = \sum_{\substack{n \leq x \\ n \equiv a \bmod q}} \Lambda(n)$$

where Λ is the von Mongoldt function. How to relate Ψ with characters ? Obviously, use orthogonality ! If $(a,q)=1$, then :

$$\Psi(x;q,a) = \frac{1}{\phi(q)} \sum_{\chi \bmod q} \overline{\chi}(a) \Psi(x,\chi)$$

with $\Psi(x,\chi) = \sum_{n \leq x} \chi(n)\Lambda(n)$ is the Ψ function related to $\mathrm{L}(s,\chi)$.

Theorem 6. *There exists an absolute constant $c_1 > 0$ such that, for all $x \geq 2$ and all pairs $\{q,a\}$ with $(a,q)=1$, we have :*

$$\Psi(x;q,a) = \frac{x}{\phi(q)} - \frac{\overline{\chi_1}(a)x^{\beta_1}}{\phi(q)\beta_1} + O(xe^{-c_1\sqrt{\log x}})$$

where χ_1 is a real character modulo q (if it exists) for which $\mathrm{L}(s,\chi_1)$ has a Siegel zero β_1. If there is no Siegel zero, the second terms of the right hand side ***should*** *be removed.*

We know very few things about Siegel zeros.

Theorem 7. *Let $c_1 > 0$ be as in theorem 6. For all $x \geq 2$ and all admissible a and q, we have :*

$$\pi(x;q,a) = \frac{1}{\phi(q)}\mathrm{L}(x) - \frac{\overline{\chi}(a)}{\phi(q)}\mathrm{Li}(x^{\beta_1}) + O(xe^{-c_1\sqrt{\log x}})$$

Cf. Montgomery's book for further results.