Exponential sum estimates using Vinogradov Mean Value Theorem

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Contents

1. – **Motivations**

1.1 – Exponential sums

Object of interest:

$$\sum_{n\in I} e(f(n)) := \sum_{n\in I} \exp(2i\pi f(n))$$

for $I \subset \mathbf{R}$ a compact interval and $f : I \to \mathbf{R}$ a *nice* function.

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Is it a regular object ? symmetric ? an erratic one ? smooth ? diffuse or concentrated ? ...

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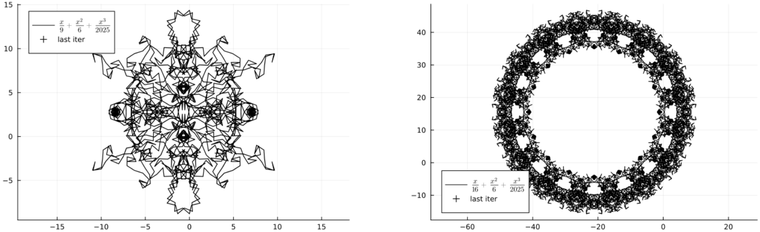

Figure: Scatter plot $\{(\mathfrak{Re}(\Sigma_i), \mathfrak{Im}(\Sigma_i)\}_{i=1,\ldots,100'000}$ for different f.

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Other representations.

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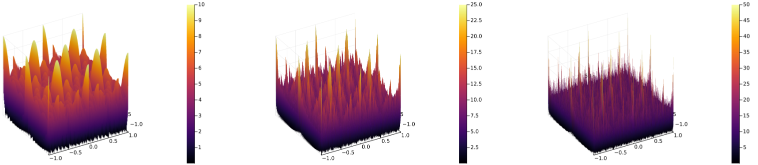

Figure: Magnitude of $\sum_{n=1}^{N} e(an^2 + bn)$ with $-2 \leq a, b \leq 2$, $N = 5, 10, 25$.

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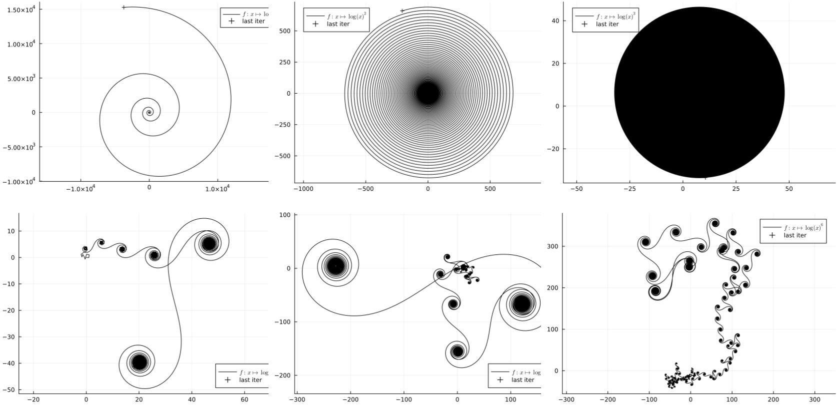

Figure: Plots of exponential sums with $f(x) = \log(x)^i$ for $i = 1, \ldots, 6$.

1.2 – Some applications

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- Bound on the Riemann zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \ll_\varepsilon \log(2+|t|) \sup_{1 \leq M \leq N \ll |t|} N^{1-\sigma} \left| \frac{1}{N} \sum_{N \leq n < N+M} e\left(-\frac{t}{2\pi} \log n\right) \right|$$

for $s = \sigma + it$ with $0 < \varepsilon \leq \sigma \ll 1$ and $1 \ll |t|$.

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- Lindelöf hypothesis: let

$$\mu(\sigma) = \inf\{\xi \in \mathbf{R}_+ \ : \ \zeta(\sigma+it) = \mathcal{O}(|t|^{\xi})\}$$

  it is conjectured that $\mu(1/2) = 0$ (today's best: Bourgain (2017), $\mu(1/2) \leq 13/84$).

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As a consequence, $\mu(11/15) \leq 1/15$. **Improvements?**

2. – **The classical theory in a nutshell**

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Theorem (van der Corput, $k = 2$)

Suppose f is a real valued function and has two continuous derivatives on an interval I. Suppose further that there exists λ_2 and $A \geq 1$ such that $0 < \lambda_2 \leq |f^{(2)}(x)| \leq A\lambda_2$ on I. Then

$$\sum_{n \in I} e(f(n)) \ll A|I|\lambda_2^{1/2} + \lambda_2^{-1/2}.$$

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Theorem (Classical k-th derivative estimate)

Let $N \in \mathbf{N}$, $k \geq 2$ an integer and suppose that $f(x) : [0, N] \to \mathbf{R}$ has continuous derivatives of order up to k on $(0, N)$. Suppose further that there exists λ_k and $A \geq 1$ such that $0 < \lambda_k \leq |f^{(k)}(x)| \leq A\lambda_k$ on $(0, N)$. Then

$$\sum_{n \leq N} e(f(n)) \ll A^{2^{2-k}} N \lambda_k^{1/(2^k-2)} + N^{1-2^{2-k}} \lambda_k^{-1/(2^k-2)}.$$

Can we do better ?

3. – Improvements à la Heath-Brown

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Plan.

- Use Vinogradov Mean Value Theorem (VMVT), count number of solutions of systems of polynomial equations.
- Improve the k-th derivative estimate.
- Improved improvements for $k = 4$.
- Application: study the growth rate of the Riemann zeta function ζ on vertical lines.

3.1 – The Vinogradov Mean Value Theorem

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$$J_{s,k}(X) = \int_{[0,1]^k} \left| \sum_{n \leq X} e(\alpha_1 n + \cdots + \alpha_k n^k) \right|^{2s} \mathrm{d}\boldsymbol{\alpha}.$$

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Theorem (VMVT)

For all integers $s, k \geq 1$*,*

$$J_{s,k}(X) \ll_{s,k,\varepsilon} X^{\varepsilon}\left(X^{s} + X^{2s-k(k+1)/2}\right)$$

for all $X \geq 1$ *and every* $\varepsilon > 0$*.*

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Proof.

Nested efficient congruencing *and* / *or* decoupling theory. $\square$

3.2 – Improved k-th derivative estimate

Recall that, under appropriate hypotheses, the classical k-th derivative estimate gives

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if $f \in \mathcal{C}^k((0,N),\mathbf{R})$ and λ_k, A s.t. $0 < \lambda_k \leq |f^{(k)}(x)| \leq A\lambda_k$ on $(0,N)$.

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Fix $k = 4$, let $\lambda_4 = N^{-\varpi}$. Consider the regime $N^{-2} \leq \lambda_4 \leq N^{-1}$.

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Application?

3.4 – Growth rate of the Riemann zeta function on vertical lines

Recall $\mu(\sigma) = \inf\{\xi \in \mathbf{R}_+ \ : \ \zeta(\sigma + it) = \mathcal{O}(|t|^\xi)\}$.

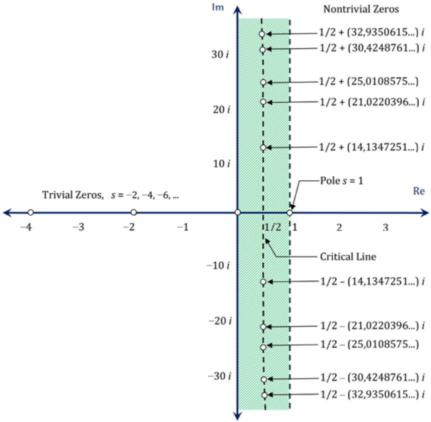

Figure: Critical strip, $\sigma = 1/2$. Source: hdd23.com

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Heath-Brown interest: $\sigma = 11/15$. He proved that

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Proof idea: sufficient to prove that

$$\sum_{N<n\leq 2N} n^{it} \ll_\varepsilon N^{11/15} t^{1/15+\varepsilon}$$

for every $1 = t^0 \leq N \leq t^{1/2}$. Then, use five different estimates on five different subintervals for N in

$$[t^0, t^{1/4}],\ [t^{1/4}, t^{1/3}],\ [t^{1/3}, t^{5/12}],\ [t^{5/12}, t^{3/7}],\ [t^{3/7}, t^{1/2}].$$

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In particular

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Hence $\mu(11/15) \leq 11/180 \approx 0.0611 < 1/15 \approx 0.0666$.

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Proof idea: Optimize Heath-Brown procedure and use previous improvements. Sufficient to prove that

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3.4 – Growth rate of the Riemann zeta function on vertical lines

Proof idea (continued). Sufficient to prove that

$$\sum_{N<n\leq 2N} n^{it} \ll_\varepsilon N^{\sigma_0} t^{\alpha+\varepsilon}.$$

- it holds trivially for $t^0 \leq N \leq t^{11/48}$,
- by fourth derivative estimates, it holds for $t^{11/48} \leq N \leq t^{1/3}$,
- by the 6/7 improvement, it holds for $t^{1/3} \leq N \leq t^{1/2}$.

This "concludes" the "proof".

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- Study ℓ^2-decoupling and more generally harmonic analysis (especially the link with PDEs).
- Continue to discover analytic number theory with Julia Brandes, Régis de la Bretèche and, why not, be introduced to probabilistic number theory...

Thanks!

Any questions?