Exponential sum estimates using Vinogradov Mean Value Theorem

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Abstract

Assuming the reader has no background in estimating exponential sums, we introduce the main techniques. All the statements are proved and we provide some numerical analysis to assess the effectiveness of the estimations. The first objective is to introduce van der Corput processes culminating in the k-th derivative estimate for exponential sums. The first non trivial results we then prove are Theorem 1 from [Hea16] and Theorem 11.1 from the Heath-Brown's appendix in [DGW20], writing up every details which were left to the reader. The use Heath-Brown made of Vinogradov's mean value integral allows us to connect analytic number theory to harmonic analysis, more precisely to decoupling theory. Using breakthroughs from Wooley ([Woo12], [Woo18]), Bourgain, Demeter, Guth, Wang ([BDG16], [DGW20]), Guth-Maldague, Maldague-Oh ([GM24], [MO24]) and Oh-Yeon ([OY25]), we slightly improve known bounds of exponential sums. Moreover, up to Conjecture 2.5 from [DGW20] and Brandes (unpublished), we give an estimate for the general case. We conclude with a study of the growth rate of the Riemann zeta function on vertical lines.

Information. Ce rapport a été et sera complété de diverses manières au cours de l'été:

- une version compactifiée contenant des résultats non présentés ici (notamment une amélioration d'un résultat de chimie quantique [CF24, Lemmes 1.1 et 4.1]) avec Julia Brandes pour publication;
- un compendium de résultats d'analyse harmonique, à la fois nécessaire pour établir les principaux énoncés du ℓ^2-*decoupling* et leur lien avec les sommes exponentielles ainsi que pour servir d'assise à une suggestion de travail de Xavier Lamy ([Ala+23], *decay estimates and elliptic systems*);
- une brève introduction à l'une des méthodes du cercle avec Anna Evenson (M2, Chalmers);
- une rapide introduction à la théorie analytique des nombres (15 pages) contenant une preuve complète du théorème de Dirichlet sur les nombres premiers en progression arithmétique;
- sur suggestion de Norbert Verdier, un court article sur une brève histoire des sommes exponentielles (pour Quadrature) et une introduction aux principales techniques (pour la RMS).

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Follow the bushes.
Here is the bridge!
$0 - 0 - 0 - 0$. *Serve.*

Contents

1 Introduction

The aim of Section 1.1 is to unfold some landmarks of the story of exponential sums, starting from Gauss sums and going up to the complete proof of the Vinogradov Mean Value Theorem in 2015 and 2016 by Wooley, Bourgain, Demeter and Guth. We will thus start with very general considerations, for more details see [Vin75, Chandrasekharan's article]. Section 1.2 motivates the study of exponential sum estimates and we conclude in Section 1.3 presenting typical situations where one encounters exponential sums. Especially, this last section includes specific examples showing a relatively rare phenomenon: cases where one can compute exactly exponential sums. Sections 2 and 3 are devoted to give nontrivial estimates for cases by no means exactly computable.

1.1 Historical background

Exponential sums arise in many different contexts in number theory, algebra and analysis. This ubiquity is now well known and very fruitful to solve a wide range of problems. A first type of exponential sum was studied by Gauss in [Gau+86, Article 356]. Amongst many substantial contributions, Gauss determined the absolute value of the so-called (quadratic) Gauss sum $G(2)$ in the case of p an odd prime [Pat07], i.e. a sum of the form $\sum_{0\leq j\leq p-1}\exp(2i\pi j^2/p)$. Thereafter, he conjectured its sign. It took Gauss 4 years to prove his conjecture and is now very standard, see for instance [Bou21] for a proof with elementary methods of the following theorem.

Theorem 1.1. *Let p be an odd prime. The Gauss sum $G(2)$ is equal to $\sqrt{p}$ if $p\equiv 1 \bmod 4$ and $i\sqrt{p}$ if $p\equiv 3 \bmod 4$.*

One of the most remarkable applications of Theorem 1.1 is the proof of the law of quadratic reciprocity.

Theorem 1.2 (Quadratic reciprocity)**.** *Let p and q be distinct odd prime numbers. Define the Legendre symbol as $\left(\frac{q}{p}\right)$ to be 1 if $n^2\equiv q \bmod p$ for some integer n and -1 otherwise, then*

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right)=(-1)^{(p-1)/2\cdot(q-1)/2}.$$

See for instance [Gol04] for a quick proof. The apparition of Gauss sums is in no way restricted to algebra and arithmetic. For instance, it is profoundly astonishing to trace the impact of Gauss sums in harmonic analysis, especially in the twisted Poisson summation formula

$$\sum_{n\in\mathbf{Z}}f(n)\chi(n)=\frac{\tau(\chi)}{N}\sum_{\ell\in\mathbf{Z}}\widehat{f}(\ell/N)\overline{\chi(\ell)}$$

which holds for any $f:\mathbf{R}\longrightarrow\mathbf{R}$ satisfying $|f(t)|+|\widehat{f}(t)|\ll(1+|t|)^{-1-\delta}$ for some $\delta>0$ with χ a primitive Dirichlet character of modulus N and τ a natural generalisation of the quadratic Gauss sum. To a broader extent, most of character theory, harmonic analysis on finite (or locally compact) abelian groups, analytic number theory, algebraic number theory and algebraic geometry were deeply influenced by the study of Gauss sums and various similar exponential sums. Each of these fields developed their own techniques and methods. We will focus only on analytic methods and pay no attention to techniques coming from algebraic number theory or algebraic geometry.

The first breakthrough is due to Weyl [Wey16, p. 324]. Under suitable rational approximation conditions, exponential sums with one-variable polynomial argument must be relatively small. To prove such a statement, he needed the now known Weyl-van der Corput inequality and Weyl differencing (A Process).

Lemma 1.3 (Weyl-van der Corput inequality). *Suppose that $\xi(n)$ is a complex valued function such that $\xi(n) = 0$ if $n \notin I$. Let H be a positive integer. Then, we have*

$$\left|\sum_{n\in I} \xi(n)\right|^2 \leq \frac{|I| + H}{H} \sum_{|h|<H} \left(1 - \frac{|h|}{H}\right) \sum_{n\in I} \overline{\xi(n+h)}\xi(n).$$

See the proof of Lemma 2.15. □

Proposition 1.4 (Weyl differencing, A Process). *Let $f : I \to \mathbf{R}$ and H be an integer such that $H \leq |I|$. Then*

$$\left|\sum_{n\in I} \exp(2i\pi f(n))\right|^2 \leq \frac{2|I|^2}{H} + \frac{2|I|}{H} \sum_{1\leq|h|<H} \left|\sum_{n\in I} \exp(2i\pi\Delta_h f(n))\right|$$

with $\Delta_h f(n) = f(n+h) - f(n)$.

See the proof of Proposition 2.16. □

Weyl then cleverly iterated the process and computed $|\sum_{n\in I} e(f(n))|^4$, $|\sum_{n\in I} e(f(n))|^8$ and so on [Wey16, p. 328, 329, 330] to get an estimate of $|\sum_{n\in I} e(f(n))|^Q$ with $Q = 2^q$ for f a polynomial of degree $q+1$ with one irrational coefficient [Wey16, Satz 9]. This iteration procedure became very useful and can be generalised to a broader class of functions to give the classical van der Corput k-th derivative estimate: under suitable conditions on f and supposing the existence of appropriate constants λ_k and A, one has

$$\sum_{n\leq N} \exp(2i\pi f(n)) \ll A^{2^{2-k}} N\lambda_k^{1/(2^k-2)} + N^{1-2^{2-k}}\lambda_k^{-1/(2^k-2)}.$$

This statement will be made clear in Section 2.4.2. The main aim of Weyl differencing is to allow relatively good estimates given a sufficiently smooth function whose derivative can be controlled. Unfortunately, this process has no impact on the size of the interval under consideration. This is why van der Corput [Cor21] introduced B process to shorten the size of the interval. Combining A and B processes allows one to deal with exponential sum estimates in a fine way. The most impressive results are obtained via the theory of exponent pairs, some are presented in Section 1.2 and a brief introduction to this theory is given in Section 2.6. Morally speaking, given an exponent pair (k,l) we can generate new pairs using successively A and B processes (e.g. AABAA(k,l), AAAAABAAABAA(k,l)...). This is a tremendous way to obtain new estimates. One of the main concerns is obviously the characterisation of exponent pairs.

Less than twenty years after Weyl's and van der Corput's outstanding results, Vinogradov [VRD04] made significant and very fruitful contributions. For a complete description of Vinogradov's method see [IK21], especially Step 7. His method historically allows to obtain an estimate for sums of the form $\sum_{P<n\leq P+Q} \exp(2i\pi f(n))$ where P, Q are integers and f is sufficiently smooth with an additional regularity condition. Vinogradov's first success was the following inequality: under suitable conditions, one has

$$\left|\sum_{P<n\leq P+Q} \exp(2i\pi f(n))\right| \ll Q^{1-\rho} \exp(k\log^2 k), \quad \rho = \frac{1}{70k^2\log k}. \tag{1}$$

His method is completely different from the one of van der Corput as it is linked with counting problems in number theory. As it will be explained in Section 3.1, it is worth of interest to study the Vinogradov mean value integral

$$J_{s,k}(X) = \int_{[0,1]^k} \left|\sum_{n\leq X} \exp\Big(2i\pi\Big(\alpha_1 n + \cdots + \alpha_k n^k\Big)\Big)\right|^{2s} \mathrm{d}\boldsymbol{\alpha}$$

for $s, k, X \geq 1$. This quantity counts the number of integer solutions to a certain system of equations (this can be reduced to finding the number of solutions of a system of congruences modulo prime powers, see [Cha70]). Such setting led to the p-adic methods of Linnik, Karatsuba, Stetchkin and Wooley [Woo18] and conduced to one of the proof of the Vinogradov Mean Value Theorem.

Theorem 1.5. *For all integers $s, k \geq 1$,*

$$J_{s,k}(X) \ll_{s,k,\varepsilon} X^{\varepsilon}\left(X^s + X^{2s-k(k+1)/2}\right)$$

for all $X \geq 1$ and every $\varepsilon > 0$.

See for instance [Hea15] explaining a proof of Wooley in action in the cubic case: the proof is reduced to seven fundamental lemmas, Hölder's inequality and Hensel's lemma [Hea15, Lemma 5] are determining.

Independently, Bourgain, Demeter and Guth [BDG16] proved this theorem using decoupling theory. This spectacular achievement is rooted in harmonic analysis, see [Pie20] for more details. Even if the two methods seem disjoints, one can hope for a dictionary between efficient congruencing and decoupling, for some perspectives see [Tao17] and [Li19, Chapter 3].

1.2 Some motivations

Exponential sums appear naturally in analytic number theory as one bounds series. For instance, the classical Riemann zeta function admits the following estimate: for $s = \sigma + it$ with $0 < \varepsilon \leq \sigma \ll 1$ and $1 \ll |t|$, we have

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \ll_{\varepsilon} \log(2+|t|) \sup_{1 \leq M \leq N \ll |t|} N^{1-\sigma} \left| \frac{1}{N} \sum_{N \leq n < N+M} \exp(-it \log n) \right|$$

see for instance [Tao15]. This is a first step towards the study of the growth rate of the Riemann zeta function on vertical lines, i.e. finding $\mu(\sigma) = \inf\{\xi \in \mathbf{R}_+ \ : \ \zeta(\sigma + it) = \mathcal{O}(|t|^{\xi})\}$, see [TH86, Chapter 5]. As a consequence of Theorem 1.5 and a new fourth derivative estimate of exponential sums (Theorem 3.9), Heath-Brown was able to obtain the following bound.

Theorem 1.6 ([DGW20, Theorem 11.2])**.** *For any fixed $\varepsilon > 0$, we have $\zeta(11/15 + it) \ll_{\varepsilon} (|t|+1)^{1/15+\varepsilon}$. As a consequence, $\mu(11/15) \leq 1/15$.*

For other values of σ, Bourgain [Bou16, Theorem 5] used the approximate functional equation of the Riemann zeta function together with exponential sum bounds to obtain $\mu(1/2) \leq 13/84$. It is still far away from the today unreachable bound conjectured by Lindelöf hypothesis, i.e. $\mu(1/2) = 0$. See the Analytic Number Theory Exponent Database for the newest results. Note that, due to a result of Backlund [Bac16], the Lindelöf hypothesis is deeply rooted and linked with fundamental conjectures in number theory such as the Riemann hypothesis and the exponent pair conjecture stating that, for all $\varepsilon > 0$, the pair $(\varepsilon, 1/2 + \varepsilon)$ is an exponent pair. These two conjectures imply the Lindelöf hypothesis. Therefore, any substantial contribution or improvement on bounds of exponential sums might contribute towards future breakthroughs.

Furthermore, exponential sums are obviously interesting to themselves. From an algebraic perspective, Weil [Wei48] showed that all exponential sums in one variable over finite fields can be realized as the trace of Frobenius maps [Mil15]. From deep algebraic geometry machinery, Deligne and later Laumon and Katz [Kat80] were able to obtain new estimates of exponential sums over finite fields. See [Ser77] for an account on Deligne method. Today, some of the most striking results in this field are obtained by Kowalski, Michel, Fresán, Forey, Fouvry and especially Sawin, see for instance [Fou+19], [KMS20] and [Saw+22].

To conclude this section we name just some problems, concepts, methods and theories where exponential sums are involved at a critical level or of great interest. The finite field world is resourceful. Many cohomology theories were developed since Grothendieck and Deligne foundational works, see [Ser77] for an account. Some very important results in equidistribution theory by Deligne and Katz [FRE19] and other works of Katz, for instance on hypergeometric sheaves [KT25], considerably developed the theory of exponential sums on finite fields. This setting is by far the most developed. The association of algebra and analysis also contributed to bring out new tools, theory and problems, for instance see [IK21] for a general survey. Some striking results related to character sums, summation formulas, (relative) trace formula and L functions generalise well-known theorems from classical analytic number theory, see for instance [Gol+21]. Variations over non-Archimedean fields exist, see for instance the Igusa stationary phase method. A tremendous interplay between the p-adic setting and harmonic analysis is found in the efficient congruencing method. Many more interesting results emerges as part of harmonic analysis and its discrete analogues [Kra22]. And, last but not least, the Waring problem and the circle methods have got an important role in the development of analytic number theory.

1.3 Some examples

Let $e(x) = \exp(2i\pi x)$. One of the first example of exponential sums one encounters is the sum of roots of unity. Let $\zeta_k := e(k/n)$ be a solution to $X^n - 1 = 0$ for $n \in \mathbf{N}^*, k \in \{0, \dots, n-1\}$. Then, summing the terms of a finite geometric series, we have

$$\sum_{k=0}^{n-1} \zeta_k = \sum_{k=0}^{n-1} e^{2i\pi k/n} = 0.$$

Without any extra effort, we prove that the sum of all the solutions to $X^n - \alpha = 0$ is also zero for $\alpha \in \mathbf{C}$. More generally, consider a polynomial $P \in \mathbf{C}_n[X]$ whose roots are all of the form $e(\theta)$ for θ a real number. For simplicity, assume that P is a monic polynomial, then we have the following factorisation

$$P(X) = \prod_{i=1}^{n} (X - e(\theta_i)) = X^n - (e(\theta_1) + \cdots + e(\theta_n))X^{n-1} + \cdots + (-1)^n e(\theta_1) \dots e(\theta_n).$$

Each coefficient of the polynomial corresponds to a combination of terms of the form $e(\theta_{i_1} + \cdots + \theta_{i_k})$ for $1 \leq k \leq n$ and up to the sign. The second highest order term is exactly an exponential sum. In that case, the Viète coefficients-roots formula encodes an exponential sum and mixed exponential terms.

We conclude this introduction trying to give a first print of some of the classical situations where one encounters exponential sums. Exponential sums arising from an algebraic and geometric context won't be considered in depth. We will only consider a fairly reachable example of Gauss sums. One must remember that there are essentially three situations where exponential sums arise: over finite fields (and algebraic generalisations), over subintervals of $\mathbf{Z}$ and coming from Fourier analysis. From a general perspective, first note that every sum of the form $\sum \lambda_n$ for $\lambda_n > 0$ can be transformed into an exponential sum using the representation $\lambda_n = \exp(2i\pi(\log(\lambda_n)/(2i\pi)))$. The factor $2i\pi$ will be made clear in next sections.

Character sums and orthogonality In classical Fourier analysis, the following orthogonality relation is fundamental

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(inx)\overline{\exp(imx)}\mathrm{d}x = \begin{cases} 1 & \text{if n = m,} \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Since the exponential map is a group morphism and especially a character, it is natural to ask if there exists Fourier-like theory in a broader context. A first answer is harmonic analysis on finite (abelian) groups and more generally on locally compact groups. We restrict here to a finite abelian group G. Using the structure theorem for finite abelian groups, it is (nearly) sufficient to limit oneself to $G = \mathbf{Z}/q\mathbf{Z}$ where q is a positive integer. Let χ be a group character of G extended to a Dirichlet character $\chi : \mathbf{Z} \to \mathbf{C}$.

As [IK21, Chapter 3] points it out, the *raison d'être* of these characters is to give rise to discrete equivalents of (2) given by

$$\frac{1}{|G|}\sum_{g\in G}\chi(g) = \begin{cases} 1 & \text{if } \chi = \chi_0, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \frac{1}{|\widehat{G}|}\sum_{\chi\in\widehat{G}}\chi(g) = \begin{cases} 1 & \text{if g = 1,} \\ 0 & \text{otherwise} \end{cases}$$

with χ_0 the trivial character and $\widehat{G}$ the dual of G. More precisely, for any characters χ_1 and χ_2 modulo q, we have

$$\frac{1}{\varphi(q)}\sum_{n=1}^{q}\chi_1(n)\overline{\chi_2(n)} = \begin{cases} 1 & \text{if } \chi_1 = \chi_2, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \frac{1}{\varphi(q)}\sum_{\chi \bmod q}\chi(n)\overline{\chi(a)} = \begin{cases} 1 & \text{if } n \equiv a \bmod q,\, (a,q)=1, \\ 0 & \text{otherwise} \end{cases}$$

with φ Euler's totient function and a, n being integers. It is thus possible to detect congruences using these orthogonality relations. Unfortunately $\exp(2i\pi n)$ or $\exp(2i\pi n/q)$ are not Dirichlet characters. However character sums, exponential sums and Dirichlet characters are linked by the (multiplicative) Fourier transform and (general) Gauss sums, see for instance [IK21, Section 3.4]. We focus on an example from [DB05, Proof of Lemma 6.1] to show this interplay. We compute the Gauss sum $T(\psi) = \sum_{t\in\mathbf{Z}/q\mathbf{Z}}\psi(t)e(at/q)$ for χ a primitive character modulo q a prime number not dividing a. We restrict to a prime number and not consider q to be p^m for a certain m. First, recall from [Con07, Equation 4.10] the finite Fourier transform on $\mathbf{Z}/q\mathbf{Z}$ of a function $f \in \mathrm{L}^1(\mathbf{Z}/q\mathbf{Z})$

$$\widehat{f}(a) := \sum_{t\in\mathbf{Z}/q\mathbf{Z}} f(t)e(-at/q).$$

Hence $T(\psi) = \widehat{\psi}(-a)$. One could compute this sum using convolution properties of the Fourier transform or using Parseval's identity for the (finite) Fourier transform. However, very basic computations are sufficient. We have

$$T(\psi) = \overline{\psi(a)}\sum_{t\in\mathbf{Z}/q\mathbf{Z}}\psi(t)e(t/q).$$

Then, when considering the modulus of T, one can take a arbitrary since it does not contribute. Let $a = 1$. First, note that

$$|T(\psi)|^2 = \left(\sum_{t\in\mathbf{Z}/q\mathbf{Z}}\psi(t)e(t/q)\right)\overline{\left(\sum_{u\in\mathbf{Z}/q\mathbf{Z}}\psi(u)e(u/q)\right)} = \sum_{t\in\mathbf{Z}/q\mathbf{Z}}\sum_{u\in\mathbf{Z}/q\mathbf{Z}}\psi(t)\overline{\psi(u)}e((t-u)/q).$$

We want to show that $|T(\psi)|^2 = q$. Make the change of variable $t = uv$. Then

$$|T(\psi)|^2 = \sum_{v\in(\mathbf{Z}/q\mathbf{Z})^\times}\psi(uv)\sum_{u\in\mathbf{Z}/q\mathbf{Z}}\overline{\psi(u)}e((v-1)u/q) = \sum_{v\in(\mathbf{Z}/q\mathbf{Z})^\times}\psi(v)\sum_{u\in\mathbf{Z}/q\mathbf{Z}}e((v-1)u/q).$$

Since χ is a primitive character modulo q, it follows that

$$|T(\psi)|^2 = \sum_{v \in \mathbf{Z}/q\mathbf{Z}} \psi(v) \sum_{u \in \mathbf{Z}/q\mathbf{Z}} e((v-1)u/q).$$

Due to the orthogonality relations, the inner sum is zero except if $v = 1$. Then

$$|T(\psi)|^2 = q\psi(1) - \sum_{v \in (\mathbf{Z}/q\mathbf{Z} - \{1\})} \psi(v) \cdot 0 = q.$$

This concludes the proof. More generally, one can consider the sum $T(\psi, P) = \sum_{t \in \mathbf{Z}/q\mathbf{Z}} \psi(t) e(P(t)/q)$ with P a polynomial and q a prime number not dividing $P^{(n)}(0)/n!$. This is highly non-trivial, even when ψ is the trivial character. Nevertheless, estimates of such sums called Weyl sums are possible.

Weyl sums Let P be a polynomial of degree d with real coefficients and N an integer. The exponential sum $\Sigma = \sum_{n=1}^{N} e(P(n))$ is called a Weyl sum. It seems to be exactly computable only when $d = 1$ and in certain cases for higher degrees. Nevertheless, much more is possible over finite fields. For instance, if $P(n) = a/Nx^2 + b/Nx$ (i.e. a Gauss sum) for a, b integers such that $(2a, N) = 1$, we have $|\Sigma| = \sqrt{N}$. Otherwise, apparently, one can not do more than estimations. The $d = 1$ case is easy to treat. The computations are really similar to an other exponential sum: the Dirichlet kernel $D_n(x) = \sum_{-n \le k \le n} e^{ikx}$. Let $P(n) = an + b$. By summing the terms of a finite geometric series and factoring by half angle, we have

$$\begin{aligned}
\Sigma &= \sum_{n=1}^{N} e(an+b) = e(b) \sum_{n=1}^{N} \left(e^{2i\pi a}\right)^n = e(b) \left[\frac{1 - e^{2i\pi a(N+1)}}{1 - e^{2i\pi a}} - 1\right] = e(b) \left[\frac{e^{2i\pi a} - e^{2i\pi a(N+1)}}{1 - e^{2i\pi a}}\right] \\
&= e(b) \left[\frac{1 - e^{2i\pi aN}}{e^{-2i\pi a} - 1}\right] = e(b) \left[\frac{e^{-i\pi aN} - e^{i\pi aN}}{e^{-i\pi a} - e^{i\pi a}}\right] \frac{2i}{2i} = \frac{\sin \pi aN}{\sin \pi a} e\left(\frac{a}{2}(N+1) + b\right).
\end{aligned}$$

If one plots the magnitude of Σ for different values of N, the shape of the resulting plots seem to stay relatively regular when N grows.

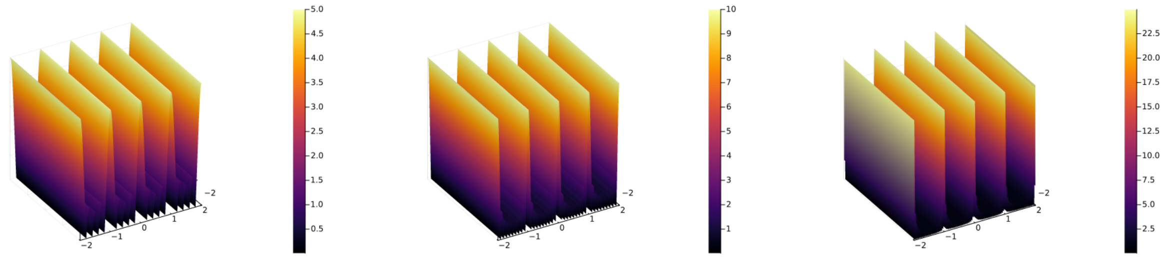


Figure: Plot of magnitude of Σ with $-2 \le a, b \le 2$, $N = 5, 10, 25$.

Nothing happens in the b-direction but, in the a-direction, the sum is bounded by $\min\{N, ||a||^{-1}\}$ which is 1-periodic. For higher degrees, this seems not to be the case. For instance, the graph of the Weyl sum associated to $P(n) = ax^2 + bx$ for real numbers a and b is as follows.

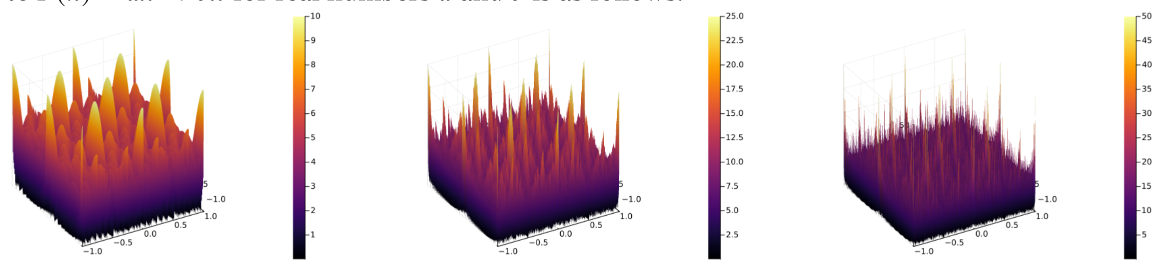


Figure: Plot of magnitude of Weyl sum for $P(n) = ax^2 + bx$ with $-0.001 \le a, b \le 0.001$, $N = 10, 20, 50$.

It clearly gets worse for various other polynomials. The higher the degree of the polynomial P is, the highest seem the oscillations. It would then be interesting to study local, mean, global oscillation of exponential sums. The most interesting and accessible case would be the local one, i.e. $\omega_N(x) = \inf\{\mathrm{diam}(\sum_{1\leq n\leq N} e(P(\mathscr{U}))) : \mathscr{U} \text{ is a neighbourhood of } x\}$ for x in a periodic domain centered at the origin.

For instance, for a sixth degree polynomial, we have the following plot.

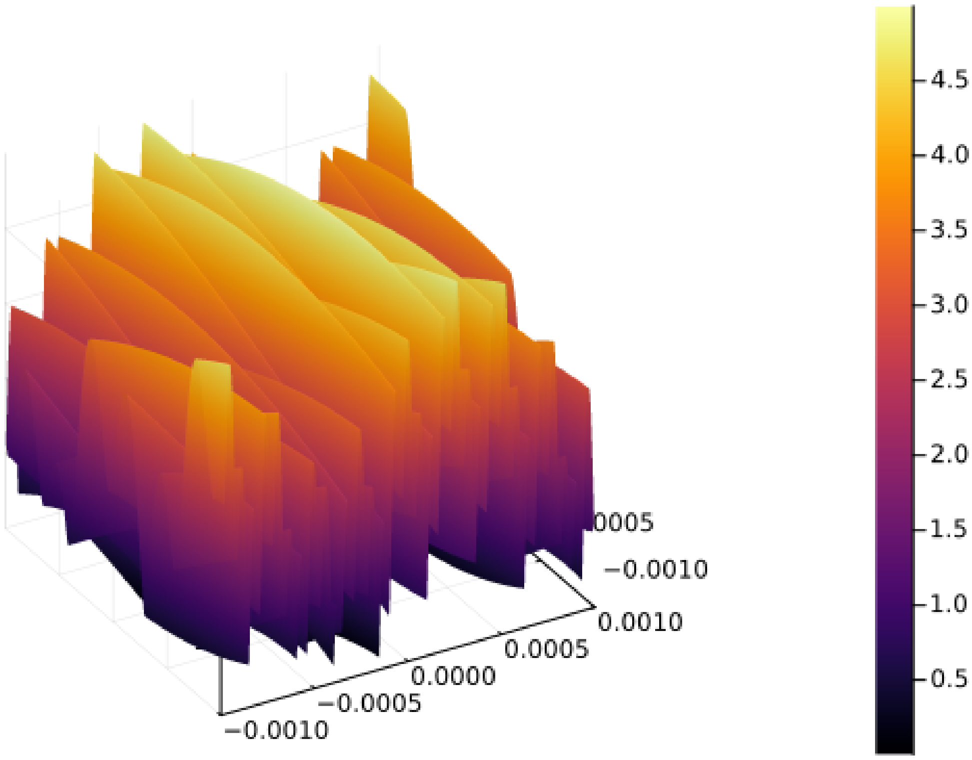


Figure: Plot of magnitude of Weyl sum for $P(n) = an^6 + bn^5$ with $-0.001 \leq a, b \leq 0.001$, $N = 5$.

Observe that the degree 1 case is *sufficient* to estimate any exponential sum of degree d polynomial P with coefficients α_k (but not necessarily to get the finest results, for higher degrees, we don't have enough appropriate tools). Indeed, using a variant of Theorem 2.13, proceeding by induction and based on the following identity

$$P(n+h) - P(n) = \sum_{k=0}^{d} \alpha_k\Big((n+h)^k - n^k\Big) = \sum_{k=0}^{d-1} \alpha_k\Big((n+h)^k - n^k\Big) + \alpha_d\Big((n+h)^d - n^d\Big)$$

$$\begin{aligned} P(n+h) - P(n) &= \sum_{k=0}^{d-1} \alpha_k\Big((n+h)^k - n^k\Big) + \alpha_d\left(\sum_{j=0}^{d} \binom{d}{j} n^j h^{d-j} - n^d\right) \\ &= \sum_{k=0}^{d-1} \alpha_k\Big((n+h)^k - n^k\Big) + \alpha_d\left(\sum_{k=0}^{d-1} \binom{d}{j} n^j h^{d-j} + \Big(n^d - n^d\Big)\right) \end{aligned}$$

the use of the A process (Proposition 1.4) allows to decrease the degree of the needed polynomial to estimate the initial exponential sum. One then gets the following result.

Proposition 1.7 ([IK21, Proposition 8.2]). *If P is a d degree polynomial with dominant term a, then*

$$\left|\sum_{n\leq N} e(P(n))\right| \ll N\left\{N^{-d} \sum_{-N<\ell_1<N} \cdots \sum_{-N<\ell_{d-1}<N} \min\left(N, ||ad!\ell_1 \ldots \ell_{d-1}||^{-1}\right)\right\}^{2^{1-d}}$$

with $||\cdot||$ the distance to the nearest integer.

It is worth noting that only the dominant term of P is taken into account to bound the exponential sum. For an arbitrary function, what could replace the dominant term ? Since the dominant term encodes most of the regularity of the polynomial, it would then be natural to consider a condition on the *highest* derivative to replace the dominant term. That is what will be done in Theorem 2.1 and Theorem 2.2.

Rational exponential sums For rational exponential sums $\sum_{n\leq N} e(a/qf(n))$ with a a rational and f a polynomial, one can reduce to q a prime power via the Chinese remainder theorem and to primes via Hensel's lemma. See for instance works of Cochrane. Then, using very deep tools from algebraic geometry concerning curves over finite fields, one can get astonishing results. Also, there are Burgess-style bounds for sums whose length is much shorter than the modulus. However, in this report, we are interested in another situation where a is not necessarily rational.

2 Introduction to exponential sum estimates

In the last section, we studied very specific cases. Since most of the time exponential sums can not be computed exactly we will be forced to obtain some estimates based on the regularity and properties of exponential sums under consideration. The primary objective of this section is to state and prove the classical van der Corput k-th derivative estimate for exponential sums.

Theorem 2.1. *Let $N \in \mathbf{N}$, $k \geq 2$ an integer and suppose that $f(x) : [0, N] \to \mathbf{R}$ has continuous derivatives of order up to k on $(0, N)$. Suppose further that there exists λ_k and $A \geq 1$ such that $0 < \lambda_k \leq |f^{(k)}(x)| \leq A\lambda_k$ on $(0, N)$. Then*

$$\sum_{n\leq N} e(f(n)) \ll A^{2^{2-k}} N\lambda_k^{1/(2^k-2)} + N^{1-2^{2-k}} \lambda_k^{-1/(2^k-2)}$$

with an implied constant independent of k.

In many problems in number theory, the summation interval is more generally $[N, N_1]$ with N_1 a positive integer such that $N < N_1 \leq 2N$. The obtained bound is of the same order of magnitude, see [BB20, Theorem 6.24]. Using Heath-Brown counting function method [Hea16, Lemma 2], Theorem 2.1 will then be refined in Section 3 to the following form.

Theorem 2.2 ([Hea16, Theorem 1]). *Suppose the same assumptions as in Theorem 2.1 hold. Then*

$$\sum_{n\leq N} e(f(n)) \ll_{A,k,\varepsilon} N^{1+\varepsilon}\left(\lambda_k^{1/(k(k-1))} + N^{-1/(k(k-1))} + N^{-2/(k(k-1))}\lambda_k^{-2/(k^2(k-1))}\right)$$

where the $\varepsilon > 0$ comes from Vinogradov's mean value integral estimate [Hea16, Lemma 1].

2.1 Notations and basics

Notations. The classical symbol e expresses the exponential function as found in analytic number theory $e : \mathbf{R} \to \mathbf{C}$ such that $x \mapsto e^{2i\pi x}$. Nothing would prevent us from considering $e : \mathbf{C} \to \mathbf{C}$ instead, as made in [Bai24] for instance, but we should pay attention to the modulus of $e(\cdot)$ which would no more necessarily be 1 (for instance, $|e(i)| = e^{-2\pi} \approx 10^{-3}$). If not stated otherwise, $||\cdot||$ is not a norm but the distance to the nearest integer. Strictly speaking, $||\cdot||$ is not even a distance on $\mathbf{R}$ but it is on the quotient space $\mathbf{R}/\mathbf{Z}$. We use the notation $[\cdot]$ for the floor function. We will constantly use Vinogradov and Big Oh notation, see Chapter 1 from James Maynard lecture notes [May22] for instance. Recall briefly that $f(x) = \mathcal{O}(g(x))$ is equivalent to $|f(x)| \leq Cg(x)$ for some suitable constant $C > 0$ and for x under consideration. In addition, $f(x) \ll g(x)$ if and only if $f(x) = \mathcal{O}(g(x))$. The former notation is generally easier to handle in actual calculations. For instance, the ε considerations are easier to manage: we will constantly use the following fact $\alpha\varepsilon \ll \varepsilon$ ($\varepsilon, \alpha > 0$). The size of a finite interval I is interchangeably denoted $|I|$ or $\#I$. As it will not be always restated, we fix I a bounded interval (with integer-valued bounds a and b) of $\mathbf{R}$ and if the context is sufficiently clear the constant A is supposed to be greater than or equal to 1. At last, for the purpose of notation, a sum $\sum_{n\in I}$ is just a convenient way to consider a sum indexed by the integer values lying in I.

Basics. Let I be any bounded interval of $\mathbf{R}$. For simplicity, by bounded interval we implicitly mean bounded interval with integer-valued bounds. We want to study and estimate $\sum_{n\in I} e(f(n))$ for a given function f. More specially, we will be interested in the following setting: let $N \in \mathbf{N}$ and $f : [0, N] \to \mathbf{R}$ a function, define Σ to be $\sum_{1\leq n\leq N} e(f(n))$. Based on the regularity of f, we expect to get non-trivial bounds on $|\Sigma|$. A bound is said to be **trivial** if it follows from the triangle inequality up to a multiplicative constant, i.e. $\Sigma \ll |I|$. In the most trivial situation, we have

$$\left|\sum_{n\in I} e(f(n))\right| \leq \sum_{n\in I} |e(f(n))| = \sum_{n\in I} 1 = \#I.$$

Note that the inequality becomes an equality if $e(f(n)) = 1$, i.e. $f(n) \in \ker e = \mathbf{Z}$. We thus expect non-trivial behaviour as soon as f takes at least one non-integer value. Especially, when f is non-constant and at least continuous, the situation starts to get interesting. More generally, the smoother f is, the better are the estimates. Note that one cannot expect any improvement coming from the Cauchy-Schwarz inequality or more generally Hölder's inequality

$$\sum_{n\in I} e(f(n)) \ll \sum_{n\in I} |e(f(n))| \leq \left(\sum_{n\in I} 1^p\right)^{1/p} \left(\sum_{n\in I} |e(f(n))|^q\right)^{1/q} = (\#I)^{1/p}(\#I)^{1/q} = \#I$$

where p and q are the conjugated exponents.

Observations. We now collect straightforward results and principles that will be used in the incoming sections. A first observation is the periodicity of e, i.e. $e(x + k) = e(x)$ for $x \in \mathbf{R}$ and $k \in \mathbf{N}$. The period is obviously 1. Even if the summand and then the sum is periodic, we don't have to consider its Fourier series since Σ is a finite sum of trigonometric functions. Nevertheless, the periodicity is important to restrict the analysis of Σ to the *nearest integer neighbourhood*, that is the real points lying between $f(\cdot)$ and the nearest integer. This explains why we consider $||f||$ and derived quantities in the following subsections.

Here is now a collection of results one can see as warm up exercises. They will be freely used in the incoming sections.

Lemma 2.3. *Let I be a bounded interval of $\mathbf{R}$ and $f : I \to \mathbf{R}$ a function. Then*

$$\left|\sum_{n\in I} e(f(n))\right| = \left|\sum_{n\in I} e(-f(n))\right|.$$

Lemma 2.4. *Let $\varepsilon > 0$ and $X \in \mathbf{R}_{\geq 1}$. Then $\log X \ll X^\varepsilon$.*

Lemma 2.5. *Let A and B be positive real numbers, then $(A+B)^2 \ll A^2 + B^2$.*

Lemma 2.6 (Concavity inequality)**.** *Let (a_i) be a sequence of positive real numbers with finite support, then*

$$\left|\sum a_i\right|^r \ll \sum |a_i|^r$$

for $0 \leq r \leq 1$.

As a consequence of the Hölder's (or Jensen's) inequality, one can generalise Lemmas 2.5 and 2.6.

Lemma 2.7. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers and p_1, p_2 be conjugated exponents such that $p_1 \geq p_2$, then*

$$\sum_{i=1}^{n} a_i^{p_2} \leq n^{1-p_2/p_1} \left(\sum_{i=1}^{n} a_i^{p_1} \right)^{p_2/p_1}.$$

As a direct consequence, we have

$$\left(\frac{1}{n} \sum_{i=1}^{n} a_i^{p_2} \right)^{1/p_2} \leq \left(\frac{1}{n} \sum_{i=1}^{n} a_i^{p_1} \right)^{1/p_1}.$$

Lemma 2.8. *Let x and y be two positive numbers and N a positive integer. Then*

$$N^{-\min(x,y)} \leq N^{-x} + N^{-y}.$$

Lemma 2.9 (Lemma 2.4 from [GK91])**.** *Let H_1 and H_2 be two real numbers such that $H_1 \leq H_2$. Let $(A_i)_i$, $(a_i)_i$, $(B_j)_j$, $(b_j)_j$ be sequences of positive elements. Suppose that*

$$L(H) = \sum_{i=1}^{m} A_i H^{a_i} + \sum_{j=1}^{n} B_j H^{-b_j}.$$

Then, there is some real number H such that $H_1 \leq H \leq H_2$ and

$$L(H) \ll_{m,n} \sum_{i=1}^{m} \sum_{j=1}^{n} \left(A_i^{b_j} B_j^{a_i} \right)^{1/(a_i+b_j)} + \sum_{i=1}^{m} A_i H_1^{a_i} + \sum_{j=1}^{n} B_j H_2^{-b_j}.$$

Proposition 2.10. *Let $(a_n)_n \in \mathbf{C}^{\mathbf{N}}$ be a complex sequence, and $f : \mathbf{R} \to \mathbf{C}$ continuously differentiable on the interval $[a, b]$. Let $A(t) = \sum_{n \leq t} a_n$. Then*

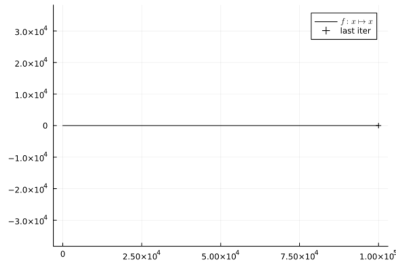
$$\sum_{a<n\leq b} a_n f(n) = A(b)f(b) - A(a)f(a) - \int_a^b A(t) f'(t) \mathrm{d}t.$$

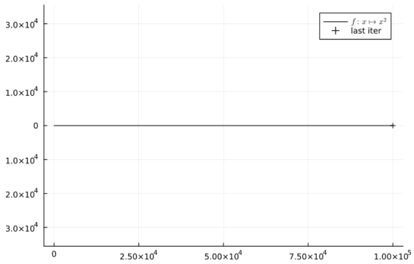
Remark 2.11. The condition $0 < \lambda \leq ||f'||$ implies that λ must be lower than $1/2$. One finds this hypothesis in Theorem 2.13 for instance.

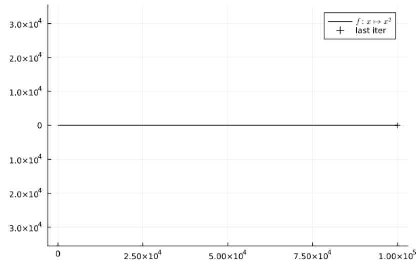
See for instance [Pat21, Section 2.1] for various other lemmata.

2.2 Numerical experiments

One has to be careful. For instance, one shall not draw any hasty conclusions from the following plots.







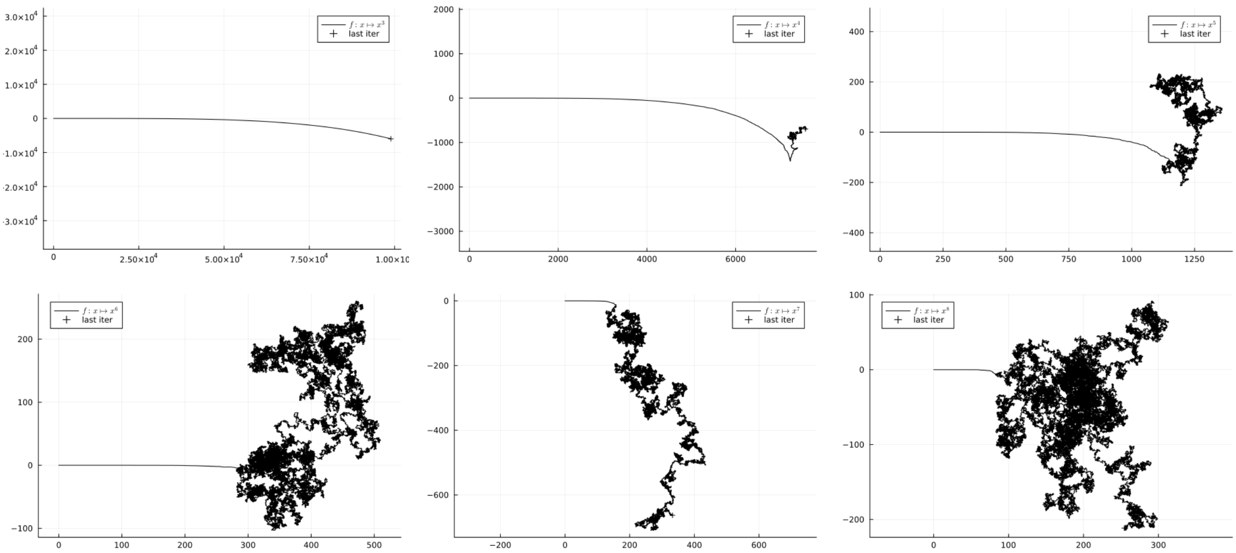


Figure: Plots (real and imaginary parts) of exponential sums with $f(x) = x^i$ for $i = 1, \ldots, 9$.

This is simply the result of accumulation of numerical approximations. It means that these plots do not represent the graph of exponential sums. Only the three first graphs are accurate. The last six plots should also contain only a straight line. (To solve the problem in Julia, use 'BigFloat'.) Due to a remark from last section, we knew that the imaginary parts must be 0 and the real part must increase as N the number of elements summed.

Exponential sums can be quite erratic and also present some phase transitions (see for example the family of functions $f_i : x \mapsto \log(x)^i$).

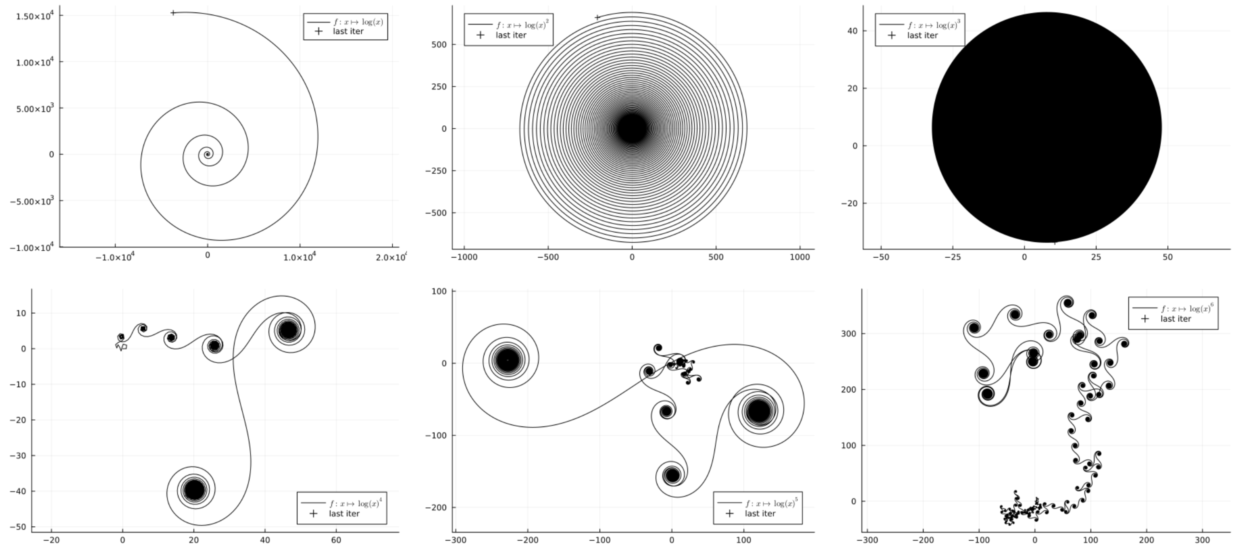
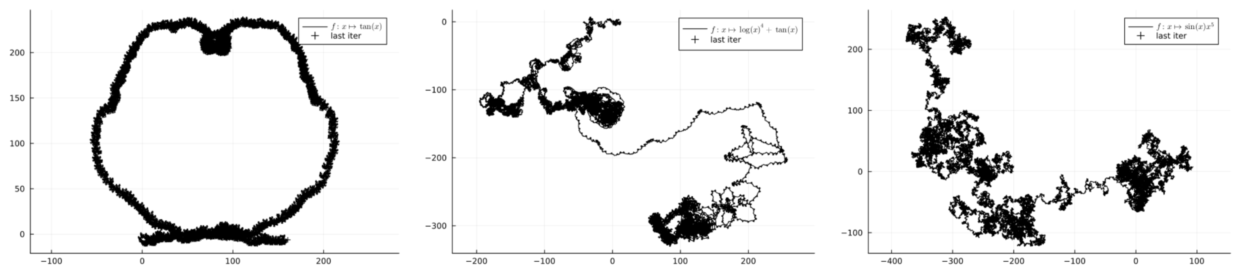


Figure: Plots of exponential sums with $f(x) = \log(x)^i$ for $i = 1, \ldots, 6$.



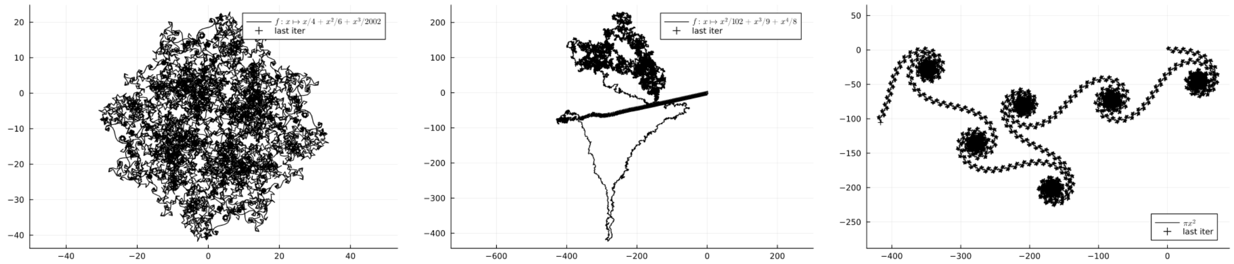


Figure: Plots of exponential sums for $f(x)$ equal $\tan(x)$, $\log(x)^4 + \tan(x)$, $\sin(x)x^5$, $x/2 + x^2/4 + x^3/2002$, $x^2/102 + x^3/9 + x^4/8$ and πx^2.

The main sources for the next sections are [GK91, chapter 2], [BB20, chapter 6] and [Bai24]. Several changes and some adaptations of the proofs from these sources where necessaries. In the incoming sections, to be able to do computations, the constants lying and lost in the Vinogradov notation must be recovered. To get explicit constants, one might use [GR96], [Rey24] and the TME-EMT Project. It would be interesting to do the computations with the explicit constants integrated.

2.3 The Kusmin-Landau bound

The Kusmin-Landau bound (Theorem 2.13) is crucial to establish a first van der Corput type bound.

Lemma 2.12. *Let $x \in \mathbf{R} \setminus \mathbf{Z}$, we have $|\cot \pi x| \ll ||x||^{-1}$.*

Proof. By definition and since the cosine is bounded by 1, one finds $|\cot \pi x| = \left|\frac{\cos \pi x}{\sin \pi x}\right| \leq \frac{1}{|\sin \pi x|}$. To conclude, it then suffices to show that $||x|| \ll |\sin \pi x|$. For $x \in [0, 1/2]$, we have $2x \leq \sin \pi x$. By the periodicity of $x \mapsto |\sin \pi x|$ and because $\sin \pi(1-x) = \sin \pi x$, we have $2||x|| \leq 2 \sin \pi x$. □

Theorem 2.13 (Kusmin-Landau)**.** *Let f be a continuously differentiable function and let I be a real bounded interval. If f' is monotonic and $0 < \lambda \leq ||f'||$ on I, then*

$$\sum_{n \in I} e(f(n)) \ll \lambda^{-1}.$$

Proof. Using Lemma 2.3, we can assume without loss of generality that f' is increasing. The hypothesis $0 < \lambda \leq ||f'||$ implies that there exists an integer k such that $\lambda \leq f'(x) - k \leq 1 - \lambda$ for all $x \in I$. Define the difference operator $\Delta_n f = f(n+1) - f(n)$, then by the mean value theorem there exists $x_n \in (n, n+1)$ such that $\Delta_n f = f'(x_n)$. One now wants to express $e(f(n))$ as a combination of exponentials and difference operators. One finds

$$e(f(n)) = (e(f(n+1)) - e(f(n)))\frac{e(f(n))}{e(f(n+1)) - e(f(n))} = \frac{e(f(n+1)) - e(f(n))}{e(\Delta_n f) - 1} = (e(f(n+1)) - e(f(n)))c_n.$$

with $c_n := 1/(e(\Delta_n f) - 1)$. By trigonometric considerations, we express c_n as a cotangent function. Recall the definition of the cotangent function

$$\cot \pi \Delta_n f = \frac{1}{\tan \pi \Delta_n f} = \frac{\cos \pi \Delta_n f}{\sin \pi \Delta_n f} = \frac{e^{i\pi\Delta_n f} + e^{-i\pi\Delta_n f}}{2} \frac{2i}{e^{i\pi\Delta_n f} - e^{-i\pi\Delta_n f}} = i\frac{1 + e(\Delta_n f)}{e(\Delta_n f) - 1}.$$

Thus $c_n = -\frac{1}{2}(1 + i\cot(\pi\Delta_n f))$. Notice that we are not using the same expression as in [GK91, Theorem 2.1] (in a certain sense, their choice is optimal). Let $I = [a, b]$. This allows to rewrite the exponential sum as

$$\begin{aligned}\sum_{n\in I} e(f(n)) &= \sum_{n=a}^{b-1}(e(f(n+1)) - e(f(n)))c_n + e(f(b)) = \sum_{n=a}^{b-1} e(f(n+1))c_n - \sum_{n=a}^{b-1} e(f(n))c_n + e(f(b)) \\ &= \sum_{n=a+1} e(f(n))c_{n-1} - \sum_{n=a+1} e(f(n))c_n - e(f(a))c_a + e(f(b))c_{b-1} + e(f(b)) \\ &= \sum_{n=a+1}^{b-1} e(f(n))(c_{n-1} - c_n) - e(f(a))c_a + e(f(b))(1 + c_{b-1}).\end{aligned}$$

Notice that Kolesnik and Graham [GK91, Theorem 2.1] exclude the lower bound a from the interval I. Applying the triangle inequality two times, we have

$$\left|\sum_{n\in I} e(f(n))\right| \leq \sum_{n=a+1}^{b-1} |e(f(n))(c_{n-1} - c_n)| + |e(f(a))c_a| + |e(f(b))(1 + c_{b-1})| = \sum_{n=a+1}^{b-1} |c_{n-1} - c_n| + |c_a| + |1 + c_{b-1}|.$$

Note that

$$c_{n-1} - c_n = -1/2(1 + i\cot(\pi\Delta_{n-1}f) - (i\cot(\pi\Delta_n f) + 1)) = -i/2(\cot(\pi\Delta_{n-1}f) - \cot(\pi\Delta_n f)).$$

Consequently

$$\left|\sum_{n\in I} e(f(n))\right| \leq \frac{1}{2}\sum_{n=a+1}^{b-1} |\cot(\pi\Delta_{n-1}f) - \cot(\pi\Delta_n f)| + |c_a| + |1 + c_{b-1}|.$$

Since $x \mapsto \cot(\pi x)$ is a decreasing function on $(k, k+1)$ for each $k \in \mathbf{Z}$ and f' was assumed to be increasing, their composition is a decreasing function and is well defined since $0 < \lambda \leq ||f'||$. Thus $\cot(\pi\Delta_n f) - \cot(\pi\Delta_{n-1}f) \leq 0$, i.e. $\cot(\pi\Delta_{n-1}f) - \cot(\pi\Delta_n f) \geq 0$. Then, the absolute value of the summand can be removed

$$\left|\sum_{n\in I} e(f(n))\right| \leq \frac{1}{2}\sum_{n=a+1}^{b-1} \cot(\pi\Delta_{n-1}f) - \cot(\pi\Delta_n f) + |c_a| + |1 + c_{b-1}|.$$

By telescoping the terms of the sum, we obtain

$$\left|\sum_{n\in I} e(f(n))\right| \leq 1/2(\cot(\pi\Delta_a f) - \cot(\pi\Delta_{b-1}f)) + |c_a| + |1 + c_{b-1}|.$$

To conclude, use Lemma 2.12 to get

$$1/2(\cot(\pi\Delta_a f) - \cot(\pi\Delta_{b-1}f)) + |c_a| + |1 + c_{b-1}| \ll \lambda^{-1}$$

since $||f'||^{-1} \leq \lambda^{-1}$ and because

$$|c_a| = 1/2|1 + i\cot(\pi\Delta_a f)| \leq 1/2(1 + |\cot(\pi\Delta_a f)|) \ll 1/2(1 + \lambda^{-1}) \ll \lambda^{-1}.$$

Moreover, we have

$$|1 + c_{b-1}| = |1/2 - i/2\cot(\pi_{b-1}f)| \leq 1/2(1 + |\cot(\pi_{b-1}f)|) \ll \lambda^{-1}.$$

This concludes the proof. $\square$

One should note that applicability conditions of Kusmin-Landau, i.e. f' is monotonic and $0 < \lambda \leq ||f'||$ on I a bounded interval of $\mathbf{R}$, are rather restrictive and might give some absurdly large results. For instance, consider the family of functions $f : x \mapsto x^\alpha$ for x in the compact interval $I = [1, \kappa]$ with $\kappa \geq 1$ and $\alpha \in (0, 1/2]$. The derivative of f is monotonic and for each κ there exists a $\lambda := \lambda_\kappa$ such that $0 < \lambda \leq ||f'||$ on I. If one takes α to be small enough and κ to be large enough (for instance $(\alpha, \kappa) = (1/5, 100)$), then the Kusmin-Landau bound (Theorem 2.13) gives worst result than the trivial estimate. Nevertheless, Theorem 2.13 gives decent estimates when $\alpha = 1/2$ for instance.

See [Rey20] and [Pat21, Section 2.1.1] for more information about the historical background and a notorious error made.

2.4 The van der Corput bound and A process

As a direct application of the Kusmin-Landau bound (Theorem 2.13), we can prove this first estimate of exponential sums.

Theorem 2.14 (van der Corput, $k = 2$)**.** *Suppose f is a real valued function and has two continuous derivatives on an interval I. Suppose further that there exists λ_2 and $A \geq 1$ such that $0 < \lambda_2 \leq |f^{(2)}(x)| \leq A\lambda_2$ on I. Then*

$$\sum_{n \in I} e(f(n)) \ll A|I|\lambda_2^{1/2} + \lambda_2^{-1/2}.$$

Proof. We can assume without loss of generality that the function $f^{(2)}$ is positive on $I = [a, b]$ (otherwise, consider $-f^{(2)}$). Then f' is monotonically increasing on I. We will partition I in subintervals depending on a parameter δ intended to be optimised. Let $\delta > 0$. Consider $I_\delta = \{x \in I \ : \ ||f'(x)|| \geq \delta\}$, then we have

$$\sum_{n \in I_\delta} e(f(n)) \ll \delta^{-1} \tag{3}$$

by the Kusmin-Landau bound (Theorem 2.13). The intervals not of the form I_δ are estimated trivially, and can thus be bounded above by $2\delta/\lambda_2$. Indeed, denote by J such an interval not of the form I_δ, then by the mean value theorem for the left hand side and by the triangle inequality for the right hand side we have

$$|J|\lambda_2 \leq |J| \inf |f^{(2)}(x)| \leq |f'(\sup J) - f'(\inf J)| < 2\delta$$

since J is of the form $\{x \in I \ : \ ||f'(x)|| < \delta\}$. Therefore

$$\sum_{n \in J} e(f(n)) \leq |J| < \frac{2\delta}{\lambda_2}. \tag{4}$$

One then can do a formal partition of I with intervals of the form I_δ and J (see [Bai24, Proof of Theorem 1.7] for a similar explicit construction). Using (3) and (4), we then have

$$\begin{aligned} \sum_{n \in I} e(f(n)) &\ll (\text{\# of intervals } I_\delta \text{ or } J \text{ needed}) \left(\sum_{n \in I_\delta} e(f(n)) + \sum_{n \in J} e(f(n)) \right) \\ &\ll (|I|A\lambda_2 + 1)\left(\delta^{-1} + \frac{\delta}{\lambda_2}\right) \end{aligned}$$

where the last bound is obtained via the mean value theorem. Indeed, by definition of I_δ and J, the number of intervals I_δ or J needed is controlled by the regularity and variations of f' in I. Since, f' is monotonically increasing on I, we can consider its total variation $f'(b) - f'(a)$. By the mean value theorem (one could also use the fundamental theorem of calculus), we have

$$(\text{\# of intervals } I_\delta \text{ or } J \text{ needed}) \leq [f'(b) - f'(a)] + 1 \leq f'(b) - f'(a) + 1 \leq |I|A\lambda_2 + 1.$$

We add one to ensure we don't have zero intervals. With that bound on the range of f', we consequently obtain a bound on the required minimal number of intervals needed to cover I in that setting. Optimising in δ (i.e. finding roots of the derivative of $\delta \mapsto \delta^{-1} + \delta/\lambda_2$), one finds that the best value is $\delta = \lambda_2^{1/2}$. This concludes the proof. □

2.4.1 A process: the Weyl differencing

To prove the familiar van der Corput k-th derivative estimate for exponential sums (Theorem 2.1), the Kusmin-Landau bound (Theorem 2.13) is not sufficiently efficient. We first state required results and will then proceed by induction making a critical use of the A process (Proposition 2.16).

Lemma 2.15 (Weyl-van der Corput inequality)**.** *Let H be a positive integer, $I = [a, b]$ and suppose that ξ is a complex valued function defined on I such that $\xi(n) = 0$ if $n \notin I$. Then, we have*

$$\left|\sum_{n\in I} \xi(n)\right|^2 \leq \frac{|I| + H}{H} \sum_{|h|<H} \left(1 - \frac{|h|}{H}\right) \sum_{n\in I} \overline{\xi(n+h)}\xi(n).$$

Proof. Note the following identity

$$H\sum_{n=a}^{b} \xi(n) = \left(\sum_{1\leq h\leq H} 1\right)\left(\sum_{n=a}^{b} \xi(n)\right) = \sum_{1\leq h\leq H}\ \sum_{a-h\leq n\leq b-h} \xi(n+h) = \sum_{a-H\leq n\leq b-1}\ \sum_{1\leq h\leq H} \xi(n+h)$$

where we just changed the indices to a new range (subsequently, we adapted the summand) and we used a characteristic function argument and the Fubini Theorem to interchange the sums

$$\sum_{1\leq h\leq H}\ \sum_{a-h\leq n\leq b-h} \xi(n+h) = \sum_{1\leq h\leq H}\ \sum_{n\in Z} \xi(n+h)\mathbb{1}_{a-h\leq n\leq b-h}$$

getting thus a nonzero sum when $a - H \leq n \leq b - 1$. By the Cauchy-Schwarz inequality, we find

$$\begin{aligned} H^2\left|\sum_{n\in I} \xi(n)\right|^2 = \left|H\sum_{n\in I} \xi(n)\right|^2 &= \left|\sum_{a-H\leq n\leq b-1} 1 \cdot \sum_{1\leq h\leq H} \xi(n+h)\right|^2 \\ &\leq \left(\sum_{a-H\leq n\leq b-1} 1\right)\left(\sum_{a-H\leq n\leq b-1}\left|\sum_{1\leq h\leq H} \xi(n+h)\right|^2\right). \end{aligned}$$

Thus

$$\begin{aligned} H^2\left|\sum_{n\in I} \xi(n)\right|^2 &\leq (|I| + H) \sum_{a-H\leq n\leq b-1}\left|\sum_{1\leq h\leq H} \xi(n+h)\right|^2 \\ &= (|I| + H) \sum_{a-H\leq n\leq b-1}\ \sum_{1\leq h\leq H}\ \sum_{1\leq h'\leq H} \overline{\xi(n+h)}\xi(n+h'). \end{aligned}$$

Do the change of variable $m = n + h'$ to get

$$\begin{aligned} \sum_{a-H\leq n\leq b-1}\ \sum_{1\leq h\leq H}\ \sum_{1\leq h'\leq H} \overline{\xi(n+h)}\xi(n+h') &= \sum_{a+1-H\leq m\leq b-1+H}\ \sum_{1\leq h,h'\leq H} \overline{\xi(m+h-h')}\xi(m) \\ &= \sum_{a+1-H\leq m\leq b-1+H}\ \sum_{|h^*|\leq H} r(h^*)\overline{\xi(m+h^*)}\xi(m) \\ &= \sum_{|h^*|\leq H} (H - |h^*|) \sum_{a+1-H\leq m\leq b-1+H} \overline{\xi(m+h^*)}\xi(m) \end{aligned}$$

with $r(h^*) = \#\{(h,h') \in [1,H]^2 : h - h' = h^*\} = H - |h^*|$ by induction. Remark that when $|h^*| = H$, we have

$$(H - |h^*|) \sum_{a+1-H \le m \le b-1+H} \overline{\xi(m+h^*)}\xi(m) = 0.$$

Then, the summation set becomes $\{h^* \ : \ |h^*| < H\}$. Dividing by H^2, we thus conclude that

$$\left|\sum_{n\in I} \xi(n)\right|^2 \le \frac{|I| + H}{H} \sum_{|h^*|<H} \left(1 - \frac{|h^*|}{H}\right) \sum_{a+1-H \le m \le b-1+H} \overline{\xi(m+h^*)}\xi(m)$$

as desired. Remark that, as soon as $H \ge 1$, we have $I \subset [a+1-H, b-1+H]$ and use the hypothesis $\xi(n) = 0$ if $n \notin I$ to get the exact same form as stated. □

More generally, one could use indicator function in order to avoid the condition $\xi(n) = 0$ if $n \notin I$. We now apply Lemma 2.15 in the case of obvious interest to us $\xi(\cdot) := e(\cdot)$. We do not forget this technical detail: the support of e is I. Define $\Delta_h f(n) = f(n) - f(n+h)$.

Proposition 2.16 (Weyl differencing, A Process)**.** *Let $f : I \to \mathbf{R}$ and H be an integer such that $H \le |I|$. Then*

$$\left|\sum_{n\in I} e(f(n))\right|^2 \le \frac{2|I|^2}{H} + \frac{2|I|}{H} \sum_{1\le|h|<H} \left|\sum_{n\in I} e(\Delta_h f(n))\right|. \tag{5}$$

Proof. Each term of the right hand side of (5) corresponds to a certain contribution. The first term corresponds to the $h = 0$ contribution, the other stacks all the remaining contributions. Following this idea, by Lemma 2.15, we have

$$\begin{aligned}
\left|\sum_{n\in I} e(f(n))\right|^2 &\le \frac{|I| + H}{H} \sum_{|h|<H} \left(1 - \frac{|h|}{H}\right) \sum_{n\in I} \overline{e(f(n+h))}e(f(n)) \\
&= \frac{|I| + H}{H}\left(\sum_{n\in I} \overline{e(f(n))}e(f(n)) + \sum_{1\le|h|<H} \left(1 - \frac{|h|}{H}\right) \sum_{n\in I} \overline{e(f(n+h))}e(f(n))\right).
\end{aligned}$$

By basic algebraic manipulations, we can now conclude that

$$\begin{aligned}
\left|\sum_{n\in I} e(f(n))\right|^2 &= \frac{|I| + H}{H}\left(\sum_{n\in I} 1 + \sum_{1\le|h|<H} \left(1 - \frac{|h|}{H}\right) \sum_{n\in I} \overline{e(f(n+h))}e(f(n))\right) \\
&= \frac{|I|^2 + |I|H}{H} + \frac{|I| + H}{H} \sum_{1\le|h|<H} \left(1 - \frac{|h|}{H}\right) \sum_{n\in I} \overline{e(f(n+h))}e(f(n)) \\
&\le \frac{2|I|^2}{H} + \frac{2|I|}{H} \sum_{1\le|h|<H} \left(1 - \frac{|h|}{H}\right) \sum_{n\in I} e(\Delta_h f(n))
\end{aligned}$$

because $H \le |I|$. This gives the desired result since $1 - |h|/H \le 1$. □

Note that Proposition 2.16 can be generalised with not much efforts to an arbitrary real number $H \le |I|$, not necessarily an integer (see for instance [GK91, Page 11]). However, in order to do so, one must generalise Lemma 2.15 and add an absolute value to the inner sum. This will be necessary to prove the third derivative test (Theorem 2.17). Moreover, by conjugation, one can set $\Delta_h f(n)$ to be $f(n+h) - f(n)$. Moreover, we abuse of the notation $n \in I$ in the inner sum. Implicitly, I is intended in that case to depend on h in order to have a well defined sum. Hence, one should replace $n \in I$ by $n \in I_h := \{n \in \mathbf{Z} \ : \ n \in I, n + h \in I\}$. We now prove Theorem 6.9 from [Ten22] following ideas from [GK91].

Theorem 2.17 (van der Corput, $k = 3$). *Suppose that f is a real valued function which has three continuous derivatives on an interval I. Suppose further that there exists λ_3 and $A \geq 1$ such that $0 < \lambda_3 \leq |f^{(3)}(x)| \leq A\lambda_3$ on I. Then*

$$\sum_{n\in I} e(f(n)) \ll A^{1/2}|I|\lambda_3^{1/6} + |I|^{1/2}\lambda_3^{-1/6}.$$

Proof. As in the proof of Theorem 2.14, assume $f^{(3)}$ to be positive on I. Let $H \leq |I|$ be a real number. By the A process (Proposition 2.16), we have

$$\left|\sum_{n\in I} e(f(n))\right|^2 \ll \frac{|I|^2}{H} + \frac{|I|}{H}\sum_{1\leq|h|<H}\left|\sum_{n\in I} e(\Delta_h f(n))\right|.$$

In order to apply Theorem 2.14, we must show that there exists λ_2 and A such that $0 < \lambda_2 \leq |(\Delta_h f(x))^{(2)}| \leq A\lambda_2$ on I. To do so, by the fundamental theorem of calculus, we have

$$(\Delta_h f(x))^{(2)} = f''(x+h) - f''(x) = \int_0^h f'''(x+y)\mathrm{d}y.$$

Thus

$$0 < h\lambda_3 = \int_0^h \inf|f^3(x)|\mathrm{d}y \leq \left|(\Delta_h f(x))^{(2)}\right| \leq \int_0^h \sup|f^3(x)|\mathrm{d}y = Ah\lambda_3.$$

Set $\lambda_2 = h\lambda_3$ to get from Theorem 2.14

$$\begin{aligned}\left|\sum_{n\in I} e(f(n))\right|^2 &\ll \frac{|I|^2}{H} + \frac{|I|}{H}\sum_{1\leq|h|<H}\left(A|I|(h\lambda_3)^{1/2} + (h\lambda_3)^{-1/2}\right) \leq \frac{|I|^2}{H} + 2\frac{|I|}{H}\sum_{1\leq h<H}\left(A|I|(h\lambda_3)^{1/2} + (h\lambda_3)^{-1/2}\right)\\ &\ll \frac{|I|^2}{H} + \frac{|I|}{H}\left(A|I|\lambda_3^{1/2}H^{3/2} + \lambda_3^{-1/2}H^{1/2}\right) = \frac{|I|^2}{H} + A|I|^2\lambda_3^{1/2}H^{1/2} + |I|\lambda_3^{-1/2}H^{-1/2}.\end{aligned}$$

Then

$$\sum_{n\in I} e(f(n)) \ll |I|H^{-1/2} + A^{1/2}|I|\lambda_3^{1/4}H^{1/4} + |I|^{1/2}\lambda_3^{-1/4}H^{-1/4}$$

by Lemma 2.6. To conclude, we optimise in H. Instead of computing a derivative, we unite *similar* terms such that they will be of the same order of magnitude. Since the first and second term of the right hand side have $|I|$ to the same power, we look for an H verifying $H^{-1/2} = \lambda_3^{1/4}H^{1/4}$. We then choose $H = \lambda^{-1/3}$. Since $A \geq 1$, we have

$$\sum_{n\in I} e(f(n)) \ll A^{1/2}|I|\lambda_3^{1/6} + |I|^{1/2}\lambda_3^{-1/6}.$$

This concludes the proof. □

Note that [GK91, Theorem 2.6] finds under the same conditions the following bound

$$\sum_{n\in I} e(f(n)) \ll A^{1/3}|I|\lambda_3^{1/6} + A^{1/4}|I|^{3/4} + |I|^{1/4}\lambda_3^{-1/4}$$

whereas, [Bai24, Theorem 13] with the extra condition $|I|^3 \leq A^2\lambda_3$ finds

$$\sum_{n\in I} e(f(n)) \ll A^{1/3}|I|\lambda_3^{1/6} + A^{1/6}|I|^{1/2}\lambda_3^{-1/6}.$$

It seems there are no further improvements appearing in the Analytic Number Theory Exponent Database (see `https://teorth.github.io/expdb/`).

2.4.2 Classical van der Corput k-th derivative estimate

We now prove by induction a generalisation of Theorems 2.14 and 2.17. The proof massively relies on an idea of Weyl explained in Section 1.1 : compute $|\sum_{n\in I} e(f(n))|^4$, $|\sum_{n\in I} e(f(n))|^8$ and so on to get an estimate of $|\sum_{n\in I} e(f(n))|^K$ with $K = 2^k$. We treat by hand the case $k = 2$ to clearly see what will happen during the induction. From the A process, we have

$$\left|\sum_{n\in I} e(f(n))\right|^2 \ll \frac{|I|^2}{H_1} + \frac{|I|}{H_1} \sum_{1\le|h_1|<H_1} \left|\sum_{n\in I} e(\Delta_{h_1} f(n))\right| \tag{6}$$

with $\Delta_h f(n) = f(n+h) - f(n)$. We square it and use Lemma 2.5 on (6) to get

$$\left|\sum_{n\in I} e(f(n))\right|^4 \ll \frac{|I|^4}{H_1^2} + \frac{|I|^2}{H_1^2}\left(\sum_{1\le|h_1|<H_1} 1 \cdot \left|\sum_{n\in I} e(\Delta_{h_1} f(n))\right|\right)^2.$$

By the Cauchy-Schwarz inequality, we can conclude that

$$\left|\sum_{n\in I} e(f(n))\right|^4 \ll \frac{|I|^4}{H_1^2} + \frac{|I|^2}{H_1} \sum_{1\le|h_1|<H_1} \left|\sum_{n\in I} e(\Delta_{h_1} f(n))\right|^2. \tag{7}$$

Apply Proposition 2.16 to get

$$\left|\sum_{n\in I} e(\Delta_{h_1} f(n))\right|^2 \ll \frac{|I|^2}{H_2} + \frac{|I|}{H_2} \sum_{1\le|h_2|<H_2} \left|\sum_{n\in I} e(\Delta_{h_2}(\Delta_{h_1} f(n)))\right|.$$

It follows that

$$\begin{aligned}\left|\sum_{n\in I} e(f(n))\right|^4 &\ll \frac{|I|^4}{H_1^2} + \frac{|I|^2}{H_1} \sum_{1\le|h_1|<H_1} \frac{|I|^2}{H_2} + \frac{|I|^2}{H_1} \sum_{1\le|h_1|<H_1} \frac{|I|}{H_2} \sum_{1\le|h_2|<H_2} \left|\sum_{n\in I} e(\Delta_{h_2}(\Delta_{h_1} f(n)))\right| \\ &\ll \frac{|I|^4}{H_1^2} + \frac{|I|^4}{H_2} + \frac{|I|^3}{H_1 H_2} \sum_{1\le|h_1|<H_1} \sum_{1\le|h_2|<H_2} \left|\sum_{n\in I} e(\Delta_{h_2}(\Delta_{h_1} f(n)))\right|.\end{aligned}$$

If we assume $H_2 \ge H_1^2$, we can omit the second term of the right hand side. This concludes the $k = 2$ case. Before proving a fundamental lemma, we record the following useful integral representation

$$\begin{aligned}\Delta_{h_2}(\Delta_{h_1} f(n)) = \Delta_{h_2}(f(n+h_1) - f(n)) &= f(n+h_1+h_2) - f(n+h_1) - f(n+h_2) + f(n) \\ &= \int_0^1 \int_0^1 \frac{\partial^2}{\partial t_1 \partial t_2} f(n + t_1 h_1 + t_2 h_2) \mathrm{d}t_1 \mathrm{d}t_2.\end{aligned}$$

Such representation was already necessary to prove Theorem 2.17. For convenience, we use the notation

$$\Delta_{h_i, h_{i-1}, \dots, h_2, h_1} f(n) := \Delta_{h_i}(\Delta_{h_{i-1}}(\dots(\Delta_{h_2}(\Delta_{h_1} f(n))))).$$

Lemma 2.18. *Let k be a positive integer and let $K = 2^k$. If $H \le |I|$, $H_1 = H$, $H_2 = H^{1/2}, \dots, H_k = H^{2/K}$, then*

$$\left|\sum_{n\in I} e(f(n))\right|^K \ll \frac{|I|^K}{H} + \frac{|I|^{K-1}}{H_1 \dots H_k} \sum_{1\le h_1<H_1} \cdots \sum_{1\le h_k<H_k} \left|\sum_{n\in I} e(f_k(n; \boldsymbol{h}))\right| \tag{8}$$

where $\boldsymbol{h} = (h_1, \dots h_k)$ and

$$f_k(n; \boldsymbol{h}) = \int_0^1 \cdots \int_0^1 \frac{\partial^k}{\partial t_1 \partial t_2 \dots \partial t_k} f(n + \boldsymbol{h} \cdot \boldsymbol{t}) \mathrm{d}t_1 \dots \mathrm{d}t_k.$$

Proof. Again, we abuse of notations: the inner indices of the sum of the right hand side $n \in I$ are intended to depend on $h_1, \dots, h_k$ such that $n \in I$ means $\{n \in \mathbf{Z} \,:\, n \in I, n+h_1+\dots+h_k \in I \text{ for all } 1 \le h_i \le H_i, 1 \le i \le k\}$. We proceed by induction. Initialisation is proved (twice, $k = 1, K = 2$ and $k = 2, K = 4$). Suppose there exists a k such that the statement is true, we prove the result also holds for $k+1$. We square (8), use Lemma 2.5 and apply the Cauchy-Schwarz inequality to get

$$\begin{aligned}\left|\sum_{n\in I} e(f(n))\right|^{2K} &\ll \frac{|I|^{2K}}{H_1^2} + \frac{|I|^{2K-2}}{H_1^2 \dots H_k^2}\left(\sum_{1\le h_1<H_1} \cdots \sum_{1\le h_k<H_k} 1 \cdot \left|\sum_{n\in I} e(f_k(n;\boldsymbol{h}))\right|\right)^2 \\ &\ll \frac{|I|^{2K}}{H_1^2} + \frac{|I|^{2K-2}}{H_1^2 \dots H_k^2} H_1 \dots H_k \cdot \sum_{1\le h_1<H_1} \cdots \sum_{1\le h_k<H_k} \left|\sum_{n\in I} e(f_k(n;\boldsymbol{h}))\right|^2 \\ &\ll \frac{|I|^{2K}}{H_1^2} + \frac{|I|^{2K-2}}{H_1 \dots H_k} \sum_{1\le h_1<H_1} \cdots \sum_{1\le h_k<H_k} \left|\sum_{n\in I} e(f_k(n;\boldsymbol{h}))\right|^2.\end{aligned}$$

Apply Proposition 2.16 to get

$$\left|\sum_{n\in I} e(f(n))\right|^{2K} \ll \frac{|I|^{2K}}{H^2} + \frac{|I|^{2K}}{H_1 \dots H_k H_{k+1}} + \frac{|I|^{2K-1}}{H_1 \dots H_k H_{k+1}} \sum_{1\le h_1<H_1} \cdots \sum_{1\le h_{k+1}<H_{k+1}} \left|\sum_{n\in I} e(f_{k+1}(n;\boldsymbol{h}))\right|$$

which proves the induction hypothesis and thus concludes since by taking $H_{k+1} = H^{1/2^k}$ we have $1/2 + 1/4 + \dots + 1/2^k \le 1$ and then one can remove the second term of the right hand side. □

Following [GK91], one finds this first version of the van der Corput k-th derivative estimate.

Theorem 2.19. *Let $N \in \mathbf{N}$, $k \ge 2$ an integer and suppose that $f(x) : [0, N] \to \mathbf{R}$ has continuous derivatives of order up to k on $(0, N)$. Suppose further that there exists λ_k and $A \ge 1$ such that $0 < \lambda_k \le |f^{(k)}(x)| \le A\lambda_k$ on $(0, N)$. Then*

$$\sum_{n\le N} e(f(n)) \ll N(A^2\lambda_k)^{1/(2^k-2)} + N^{1-1/2^{k-1}} A^{1/2^{k-1}} + N^{1-2/2^{k-2}+1/(2^{k-2})^2} \lambda_k^{-1/2^{k-1}}$$

with an implied constant independent of k.

Proof. The integral representation of Lemma 2.18 is crucial since it links the regularity condition to the estimate. For commodity, let $q = k - 2$. One has

$$f_q(n;\boldsymbol{h}) = \int_0^1 \cdots \int_0^1 \frac{\partial^q}{\partial t_1 \partial t_2 \dots \partial t_q} f(n + \boldsymbol{h}\cdot\boldsymbol{t}) \mathrm{d}t_1 \dots \mathrm{d}t_q = \int_0^1 \cdots \int_0^1 h_1 \dots h_q f^{(q)}(n + \boldsymbol{h}\cdot\boldsymbol{t}) \mathrm{d}t_1 \dots \mathrm{d}t_q$$

therefore

$$h_1 \dots h_q \lambda_k \le |f_q^{(2)}(n;\boldsymbol{h})| \le h_1 \dots h_q A \lambda_k.$$

Let $I = [1, N]$ such that $|I| = N$. By Theorem 2.14, we have

$$\sum_{n\in I} e(f_q(n;\boldsymbol{h})) \ll A|I|\left(h_1 \dots h_q \lambda_k\right)^{1/2} + \left(h_1 \dots h_q \lambda_k\right)^{-1/2}. \tag{9}$$

Before summing over $[1, H_1) \times [1, H_2) \times \dots \times [1, H_q)$, notice that

$$\sum_{1\le h<H} h^{1/2} \le H^{1/2} \sum_{1\le h\le H} 1 = H^{3/2} \tag{10}$$

and

$$\sum_{1\le h<H} h^{-1/2} \le \sum_{1\le h\le H} \int_{h-1}^{h} x^{-1/2} \mathrm{d}x = \int_0^H x^{-1/2} \mathrm{d}x = 2H^{1/2}. \tag{11}$$

Then, as a direct application of the Fubini theorem, from (10) and (11), it holds that

$$\sum_{1\le h_1<H_1}\cdots\sum_{1\le h_q<H_q}(h_1\dots h_q)^{1/2}\le H_1^{3/2}\dots H_q^{3/2}$$

and

$$\sum_{1\le h_1<H_1}\cdots\sum_{1\le h_q<H_q}(h_1\dots h_q)^{-1/2}\le 2^qH_1^{1/2}\dots H_q^{1/2}\ll H_1^{1/2}\dots H_q^{1/2}.$$

Let $\boldsymbol{h}=(h_1,\dots h_k)$ and $Q=2^q$. Putting this altogether and using Lemma 2.18 and (9), we deduce

$$\left|\sum_{n\in I}e(f(n))\right|^Q \ll \frac{|I|^Q}{H}+\frac{|I|^{Q-1}}{H_1\dots H_q}\sum_{1\le h_1<H_1}\cdots\sum_{1\le h_q<H_q}\left|\sum_{n\in I}e(f_k(n;\boldsymbol{h}))\right| \tag{12}$$

$$\ll \frac{|I|^Q}{H}+\frac{|I|^{Q-1}}{H_1\dots H_q}\sum_{1\le h_1<H_1}\cdots\sum_{1\le h_q<H_q}\left(A|I|(h_1\dots h_q\lambda_k)^{1/2}+(h_1\dots h_q\lambda_k)^{-1/2}\right) \tag{13}$$

$$\ll \frac{|I|^Q}{H}+\frac{|I|^{Q-1}}{H_1\dots H_q}\left(A|I|\lambda_k^{1/2}H_1^{3/2}\dots H_q^{3/2}+\lambda_k^{-1/2}H_1^{1/2}\dots H_q^{1/2}\right) \tag{14}$$

$$=\frac{|I|^Q}{H}+A|I|^Q\lambda_k^{1/2}H_1^{1/2}\dots H_q^{1/2}+|I|^{Q-1}\lambda_k^{-1/2}H_1^{-1/2}\dots H_q^{-1/2} \tag{15}$$

$$\ll |I|^Q\left(H^{-1}+A\lambda_k^{1/2}H^{1-1/Q}+|I|^{-1}\lambda_k^{-1/2}H^{-1+1/Q}\right) \tag{16}$$

by definition of $H_1,\dots H_q$ from Lemma 2.18 and by routine considerations about geometric series. To conclude, apply Lemma 2.9 with $A_1=A\lambda_k^{1/2}$, $a_1=1-1/Q$, $B_1=|I|^{-1}\lambda_k^{-1/2}$, $B_2=1$, $b_1=1-1/Q$, $b_2=1$ and $H_1=0$, $H_2=|I|$ to get an H such that

$$\left|\sum_{n\in I}e(f(n))\right|^Q\ll |I|^Q\left((A^2\lambda_k)^{(Q-1)/(4Q-2)}+|I|^{-2+1/Q}\lambda_k^{-1/2}+A^{1/2}|I|^{-1/2}\right)$$

and then take the Q-th root. It gives the desired result since $(Q-1)/Q\le 1$. □

In [TH86], an alternative version of Theorem 2.19 is given.

Theorem 2.1. *Let $N\in\mathbf{N}$, $k\ge 2$ an integer and suppose that $f(x):[0,N]\to\mathbf{R}$ has continuous derivatives of order up to k on $(0,N)$. Suppose further that there exists λ_k and $A\ge 1$ such that $0<\lambda_k\le|f^{(k)}(x)|\le A\lambda_k$ on $(0,N)$. Then*

$$\sum_{n\le N}e(f(n))\ll A^{2^{2-k}}N\lambda_k^{1/(2^k-2)}+N^{1-2^{2-k}}\lambda_k^{-1/(2^k-2)}$$

with an implied constant independent of k.

Proof. If $\lambda_k\ge 1$, the theorem is trivial. Suppose then λ_k to be strictly less than 1. We proceed by induction: suppose the theorem holds true for every integers up to $k-1$. Let $g(x)=f(x+h)-f(x)$ for x and $x+h$ lying in $[0,N]$ with h an integer. Differentiating with respect to x, we get $g^{(k-1)}(x)=f^{(k-1)}(x+h)-f^{(k-1)}(x)$. By virtue of the mean value theorem, there exists $\xi\in(x,x+h)$ such that $g^{(k-1)}(x)=hf^{(k)}(\xi)$. Then, by hypothesis

$$0<h\lambda_k\le\left|g^{(k-1)}(x)\right|\le Ah\lambda_k.$$

We plan to apply the A process on $g(x)=\Delta_hf(x)$ (using the notation of Proposition 2.16) to get consequently an estimate of f. Let $I=[1,N]$ such that $|I|=N$. By the induction hypothesis applied at the rank $k-1$, we deduce

$$\sum_{n\in I}e(g(n))=\sum_{n\in I}e(\Delta_hf(n))\ll A^{2^{3-k}}N(h\lambda_k)^{1/(2^{k-1}-2)}+N^{1-2^{3-k}}(h\lambda_k)^{-1/(2^{k-1}-2)}$$

for clarity, we recall that we are using an abuse of notations when writing $n \in I$, to be perfectly formal we should write $n \in I_h$ as defined in remarks following the statement of Proposition 2.16. Let H be an integer, we then have

$$\begin{aligned}\sum_{1\le h<H}\left|\sum_{n\in I} e(\Delta_h f(n))\right| &< A^{2^{3-k}} N\lambda_k^{1/(2^{k-1}-2)} \sum_{1\le h<H} h^{1/(2^{k-1}-2)} + N^{1-2^{3-k}}\lambda_k^{-1/(2^{k-1}-2)}\sum_{1\le h<H} h^{-1/(2^{k-1}-2)}\\ &\ll A^{2^{3-k}} N\lambda_k^{1/(2^{k-1}-2)} H^{1+1/(2^{k-1}-2)} + N^{1-2^{3-k}}\lambda_k^{-1/(2^{k-1}-2)} H^{1-1/(2^{k-1}-2)}\end{aligned}$$

since, by a similar computation as in (11), we have

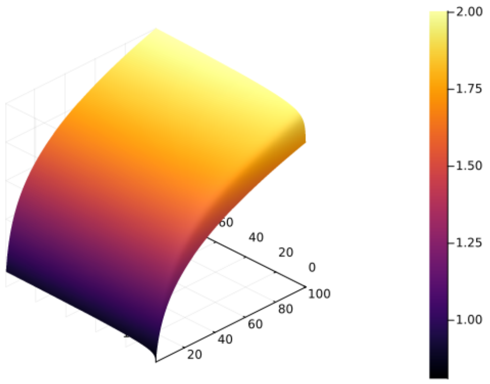
$$0 < \sum_{1\le h<H} h^{-1/(2^{k-1}-2)} < \int_0^H x^{-1/(2^{k-1}-2)}\mathrm{d}x = \frac{H^{1-1/(2^{k-1}-2)}}{1-1/(2^{k-1}-2)} \le 2H^{1-1/(2^{k-1}-2)}$$

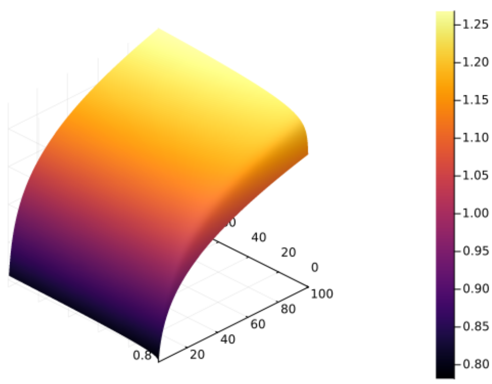
for $k \ge 3$. If $k = 2$, the theorem is also true (see Theorem 2.14). Same computations give the result for the other sum $\sum_{1\le h<H} h^{1/(2^{k-1}-2)}$, it generalises what we have done in (10). We now apply Proposition 2.16, take the square root and we thus get

$$\begin{aligned}\left|\sum_{n\in I} e(f(n))\right| &\ll \frac{N}{H^{1/2}} + \left(\frac{N}{H}\sum_{1\le|h|<H}\left|\sum_{n\in I} e(\Delta_h f(n))\right|\right)^{1/2}\\ &\ll \frac{N}{H^{1/2}} + \frac{N^{1/2}}{H^{1/2}}\left\{A^{2^{3-k}} N\lambda_k^{1/(2^{k-1}-2)} H^{1+1/(2^{k-1}-2)} + N^{1-2^{3-k}}\lambda_k^{-1/(2^{k-1}-2)} H^{1-1/(2^{k-1}-2)}\right\}^{1/2}\\ &\ll \frac{N}{H^{1/2}} + \frac{N^{1/2}}{H^{1/2}} A^{2^{2-k}} N^{1/2}\lambda_k^{1/(2^k-4)} H^{1/2+1/(2^k-4)} + \frac{N^{1/2}}{H^{1/2}} N^{1/2-2^{2-k}}\lambda_k^{-1/(2^k-4)} H^{1/2-1/(2^k-4)}\\ &= NH^{-1/2} + A^{2^{2-k}} N\lambda_k^{1/(2^k-4)} H^{1/(2^k-4)} + N^{1-2^{2-k}}\lambda_k^{-1/(2^k-4)} H^{-1/(2^k-4)}\end{aligned}$$

by Lemma 2.6. We now choose H such that the two first terms are of the same order, i.e. $H^{-1/2} = \lambda_k^{1/(2^k-4)} H^{1/(2^k-4)}$. In order to have an integer valued H, we find $\left\lfloor \lambda_k^{-1/(2^{k-1}-1)} \right\rfloor + 1$. Record that value for future developments, especially in the proof of Lemma 3.5 in Section 3. Recall we assumed $\lambda_k < 1$. Since $\mathbf{R}$ is an Archimedean field, we have $H \ll \lambda_k^{-1/(2^{k-1}-1)}$. By direct computations, this concludes the induction and gives the desired result. □

In certain situations Theorem 2.19 might be asymptotically sharper than Theorem 2.1. This is not the case in general as seen in the following plots (the magnitude can be greater than 1, see the bars). One finds that when $k = 2, 3, 4$, Proposition 2.19 tends to give sharper results than Theorem 2.1.





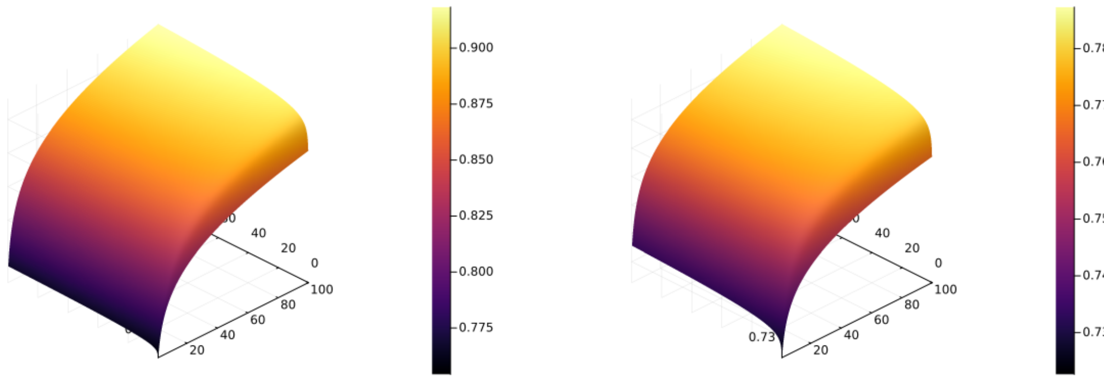


Figure: Plots of bound from Theorem 2.1 over bound from Proposition 2.19 for $N = 300$, (in order) $k = 3, 4, 5, 6$, $A \in [1, 100]$ and $\lambda_k \in [0, 100]$.

2.5 The B process through the stationary phase method

The main references for this section could have been [GK91, Section 3.2] or [Bai24, Lectures 4, 5]. Since their proofs require only skills to calculate swiftly, consist in lengthy computations and because the results of this sections are not particularly relevant for the next section (but they are in the theory of exponential sums!), the choice was made to present the B process and exponent pairs theory from a broader point of view letting most of the proofs to the reader (but giving references). We introduce basic ideas from the theory of oscillatory integrals [SM93, Chapters VIII and IX] and insist on analogies from this theory with results from latter sections. See [BB20, Section 6.5] to get explicit constants.

Whereas the A process allows to estimate an exponential sum in terms of an other phase-changed exponential sum, it has no substantial impact on the interval of summation if H is of the same order as $|I|$ (see Proposition 2.16). The aim of the B process is to shorten the interval of summation and to replace sums by integrals in order to perform asymptotic analysis.

We briefly highlighted in Section 1.3 a link between Fourier analysis and a class of exponential sums. One was able to write a Gauss sum as the Fourier transform of a character. With more fragrance, Fourier analysis will play a central role in the theory of oscillatory integral. As Stein [SM93, Chapter VIII] points it out, the Fourier transform is itself an oscillatory integral *par excellence*. We now introduce the two kinds of oscillatory integrals. We restrict to the one dimensional case, the reader might find analogous discussion for several variables in [SM93, Section VIII.2].

Definition 2.20. Let the phase ϕ be a real-valued smooth function and ψ be a complex-valued smooth function with compact support $[a, b]$. Let λ be a positive real number. An oscillatory integral of the first kind is of the form

$$I(\lambda) = \int_a^b e^{i\lambda\phi(x)}\psi(x)\mathrm{d}x.$$

One is interested in the contribution of the critical points of ϕ to $I(\lambda)$, scaling properties and the asymptotic behaviour of $I(\lambda)$. A typical example of oscillatory integral of the first kind are the Bessel functions

$$J_m(\lambda) := \frac{1}{2\pi}\int_0^{2\pi} e^{i\lambda\sin x}e^{-imx}\mathrm{d}x$$

for m an integer, see [SM93, Section VIII.1.4]. They are a sort of cylindrical counterpart of the spherical harmonics. The name of these integrals is very well chosen since they oscillate. For instance, see the following graph.

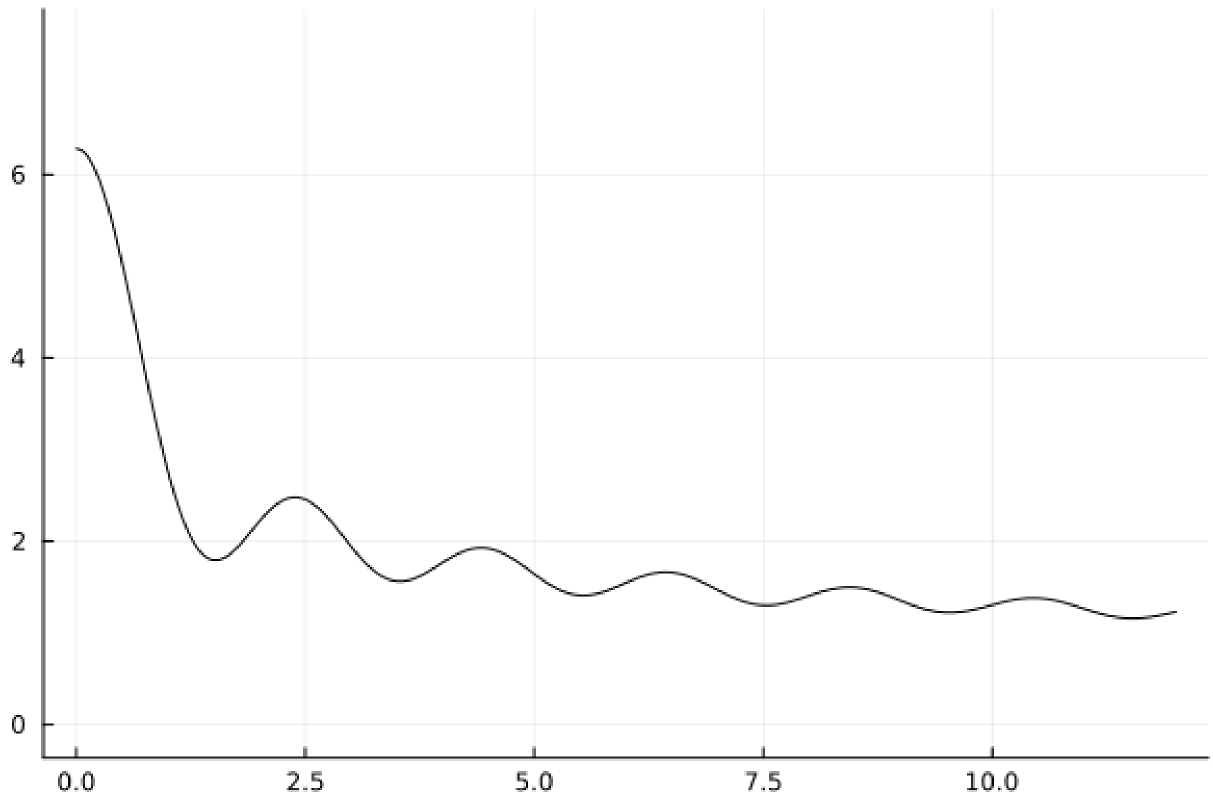


Figure: Plot of $I(\lambda)$ for $\phi(\lambda) = \sin(\lambda) - \lambda$, $\psi(\lambda) = 1$ and $[a, b] = [-\pi, \pi]$.

Localization We first see that outside of critical points, the behaviour of $I(\lambda)$ is well understood.

Proposition 2.21. *Let $I(\lambda)$ be an oscillatory integral of the first kind such that ψ has compact support in (a, b) and $\phi'(x) \neq 0$ for all $x \in [a, b]$. Then, for all $N \geq 0$, as $\lambda \to \infty$, we have*

$$I(\lambda) = \mathcal{O}(\lambda^{-N}).$$

This condition of a nonzero derivative was already present in van der Corput estimates (Theorems 2.14, 2.17, 2.1). In fact, it is a consequence of the hypothesis made in the Kusmin-Landau bound (Theorem 2.13). We will now give estimates based on this critical condition. There are two cases to consider: if $\phi'(x)$ avoid 0, for instance if we have $|\phi'(x)| \geq 1$, or if $\phi'(x) = 0$ has a solution in $[a, b]$.

Scaling We now suppose that $\phi'(x)$ avoid 0 and generalise the situation to more regular functions.

Proposition 2.22 ([SM93, p. 322])**.** *Suppose the phase ϕ is real-valued and smooth in (a, b) and $|\phi^{(k)}(x)| \geq 1$ for some integer k and for all $x \in (a, b)$. Then*

$$\int_a^b e^{i\lambda\phi(x)} \mathrm{d}x \ll_k \lambda^{-1/k}$$

if $k = 1$ and ϕ' is monotonic, or if $k \geq 2$. This holds for a constant independent of ϕ and λ.

Proposition 2.22 generalizes Lemmas 3.1 and 3.2 from [GK91]. As a direct consequence, using integration by parts, one has the following corollary.

Corollary 2.23. *Under the same assumptions as in Proposition 2.22, we have*

$$I(\lambda) = \int_a^b e^{i\lambda\phi(x)}\psi(x)\mathrm{d}x \ll_k \lambda^{-1/k}\left\{|\psi(b)| + \int_a^b |\psi'(x)|\mathrm{d}x\right\}.$$

As we have used the scaling property of van der Corput lemma in (13) in the proof of Theorem 2.19, the condition $|\phi^{(k)}(x)| \geq 1$ is somewhat arbitrary and can be replaced by $|\phi^{(k)}(x)| \geq c$ for c a positive real number.

Asymptotics We now suppose for commodity that the support of ψ is small enough to contain only one critical point of ϕ. We then can determine the asymptotic behaviour of $I(\lambda)$.

Proposition 2.24 ([SM93, p. 334]). *Suppose $k \geq 2$ and suppose there exists an x_0 such that*

$$\phi(x_0) = \phi'(x_0) = \cdots = \phi^{(k-1)}(x_0) = 0, \quad \phi^{(k)}(x_0) \neq 0.$$

If ψ is supported in a sufficiently small neighbourhood of x_0, then

$$I(\lambda) = \int_a^b e^{i\lambda\phi(x)}\psi(x)\mathrm{d}x \sim \lambda^{-1/k}\sum_{j=0}^{\infty} a_j\lambda^{-j/k}$$

in the sense that, for all nonnegative integers N and r, as $\lambda \longrightarrow \infty$, we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^r\left[I(\lambda) - \lambda^{-1/k}\sum_{j=0}^{\infty} a_j\lambda^{-j/k}\right] = \mathcal{O}(\lambda^{-r-(N+1)/k}).$$

For instance, in particular, when $k = 2$, we have

$$a_0 = \left(\frac{2\pi}{-i\phi^{(2)}(x_0)}\right)^{1/2}\psi(x_0). \tag{17}$$

Note that, according to a more physical approach, we could have used a time dependence $\int_a^b e^{i\lambda\phi(t)}\psi(t)\mathrm{d}t$ instead of the spatial variable x. It would then have been more natural to talk of *stationary* point (i.e. not time-dependent, of zero time derivative) in place of *critical* points. This fact motivates the name *stationary* phase method, giving the leading asymptotic behaviour of $I(\lambda)$ for a phase having stationary points.

Application to exponential sums When studying exponential sums, it is natural to have more hypothesis on the phase function (see for instance Theorem 2.1). As a consequence, one can get slightly better results than Proposition 2.24 and (17). For instance, see the following proposition from [GK91, Theorem 3.6] and [Bai24, Theorem 20].

Proposition 2.25 (Stationary phase). *Suppose ϕ is a real valued function with four continuous derivatives on $[a, b]$. Suppose also that $\phi''(x) \geq \lambda_2 > 0$ on $[a, b]$ and $\phi'(x_0) = 0$ for some $x_0 \in (a, b)$. Finally, assume that*

$$|\phi^{(3)}(x)| \leq \lambda_3, \quad |\phi^{(4)}(x)| \leq \lambda_4$$

on $[a, b]$. Then

$$\int_a^b e(\phi(x))\mathrm{d}x = \frac{e(1/8 + \phi(x_0))}{\phi''(x_0)^{1/2}} + \mathcal{O}(R_1 + R_2)$$

where

$$R_1 := \min\left(\frac{1}{\lambda_2(x_0-a)}, \frac{1}{\lambda_2^{1/2}}\right) + \min\left(\frac{1}{\lambda_2(b-x_0)}, \frac{1}{\lambda_2^{1/2}}\right)$$

and

$$R_2 = (b-a)\lambda_4\lambda_2^{-2} + (b-a)\lambda_3^2\lambda_2^{-3}.$$

The B Process is a corollary of the stationary phase method. See [Bai24, Theorems 21, 22] for a proof.

Theorem 2.26 (B Process)**.** *Suppose that f has four continuous derivatives on $I = [a,b]$. Suppose further that $[a,b] \subset [N,2N]$. Assume that there is some $F > 0$ such that*

$$FN^{-2} \ll f''(x) \ll FN^{-2}, \quad f^{(3)}(x) \ll FN^{-3}, \quad f^{(4)}(x) \ll FN^{-4}$$

on $[a,b]$. Let x_h be defined by the relation $f'(x_h) = h$, and let $\theta(h) = f(x_h) - hx_h$. If $f''(x) > 0$ on I, then

$$\sum_{n\in I} e(f(n)) = \sum_{\alpha\le h\le\beta} \frac{e(1/8+\theta(h))}{f''(x_h)^{1/2}} + \mathcal{O}(\log(2+FN^{-1}) + F^{-1/2}N).$$

with $\alpha = f'(a)$ and $\beta = f'(b)$. If $f''(x) < 0$ on I, then

$$\sum_{n\in I} e(f(n)) = \sum_{\alpha\le h\le\beta} \frac{e(-1/8+\theta(h))}{|f''(x_h)|^{1/2}} + \mathcal{O}(\log(2+FN^{-1}) + F^{-1/2}N)$$

with $\alpha = f'(b)$ and $\beta = f'(a)$.

For commodity, consider only the first case. By summation by parts, one finds

$$\sum_{\alpha\le h\le\beta} \frac{e(1/8+\theta(h))}{f''(x_h)^{1/2}} \ll NF^{-1/2} \sup_{\alpha\le x\le\beta}\left|\sum_{\alpha\le h\le x} e(\theta(h))\right|.$$

It thus allows to bound an exponential sum using a shorter one. One could then wonder if it is possible to apply inductively the B process as we did for the Weyl A process. No. For technical analytical reasons (essentially because applying the Poisson summation formula twice has no effect, see [Bai24, Lecture 6]), the B process is an involution. Expressed in the theory of exponent pairs, it is obvious that the B process is an involution, see Proposition 2.29. For a generalisation of van der Corput lemma, see the ATS theorem [KK07].

2.6 Exponent pairs

We do not give the formal definition of an exponent pair as given for instance in [GK91, Section 3.3], [Hea16, Section 1] or [TTY25, Blueprint, Section 5] since the function class $\boldsymbol{F}$ is a little bit too demanding to construct in view of our purpose. For simplicity, we will call an element of $\boldsymbol{F}$ an admissible phase function.

Definition 2.27. An exponent pair is an element (k,ℓ) of the triangle

$$\{(k,\ell) \in \mathbf{R}^2 \ : \ 0 \le k \le 1/2 \le \ell \le 1, \ k+\ell \le 1\}$$

with the following property: for all admissible phase functions F, all $T \ge N$ with $N \ge 1$ and all intervals $I \subset [N,2N]$, one has

$$\sum_{n\in I} e(TF(n/N)) \ll (T/N)^{k+o(1)} N^{\ell+o(1)}.$$

Obviously, since exponent pairs give estimate of exponential sums they are very important. It is then a famous problem to characterise them or, at least, find ways to generate new exponent pairs using old. Since the A and B processes allow to bound an exponential sum in term of other exponential sums, it seems natural to ask if these processes have counterparts in the theory of exponent pairs. One can show (Lemmas 3.7, 3.9 and Theorems 3.8, 3.10 from [GK91]) this is true. See the following propositions.

Proposition 2.28 (van der Corput, A process)**.** *If (k,ℓ) is an exponent pair, then so is*

$$A(k,\ell) := \left(\frac{k}{2k+2}, \frac{\ell}{2k+2} + \frac{1}{2}\right).$$

Proposition 2.29 (van der Corput, B process)**.** *If (k,ℓ) is an exponent pair, then so is*

$$B(k,\ell) := \left(\ell - \frac{1}{2}, k + \frac{1}{2}\right).$$

3 Improvements *à la Heath-Brown*

We now use results from the theory of exponential sums to improve the k-th derivative estimate. All the improvements presented here rely on the Vinogradov Mean Value Theorem.

3.1 The Vinogradov Mean Value Theorem after [Pie20]

As it was stated in Section 1.1, Vinogradov's method contributed to give another perspective to estimate exponential sums. Let $X, s, k \geq 1$ be integers and consider the following system of k equations in $2s$ variables.

$$\begin{cases} x_1 + x_2 + \cdots + x_s = x_{s+1} + x_{s+2} + \cdots + x_{2s}, \\ x_1^2 + x_2^2 + \cdots + x_s^2 = x_{s+1}^2 + x_{s+2}^2 + \cdots + x_{2s}^2, \\ \vdots \\ x_1^k + x_2^k + \cdots + x_s^k = x_{s+1}^k + x_{s+2}^k + \cdots + x_{2s}^k \end{cases}$$

with $1 \leq x_i \leq X$ for $1 \leq i \leq 2s$. Let $J_{s,k}(X)$ count the number of integral solutions to the so-called Vinogradov system. One can express $J_{s,k}(X)$ as an integral, justifying the name of Vinogradov mean value integral.

Lemma 3.1. *One can express $J_{s,k}(X)$ in an integral form as*

$$J_{s,k}(X) = \int_{[0,1]^k} \left| \sum_{x \leq X} e(\alpha_1 x + \cdots + \alpha_k x^k) \right|^{2s} \mathrm{d}\boldsymbol{\alpha}.$$

Proof. By definition, we have

$$\begin{aligned} J_{s,k}(X) &= \int_{[0,1]^k} \left(\sum_{x \leq X} e(\alpha_1 x + \cdots + \alpha_k x^k) \right)^s \overline{\left(\sum_{y \leq X} e(\alpha_1 y + \cdots + \alpha_k y^k) \right)^s} \mathrm{d}\boldsymbol{\alpha} \\ &= \int_{[0,1]^k} \left(\sum_{x_1,\ldots,x_s \leq X} \prod_{1 \leq j \leq s} e(\alpha_1 x_j + \cdots + \alpha_k x_j^k) \right) \overline{\left(\sum_{y_1,\ldots,y_s \leq X} \prod_{1 \leq j \leq s} e(\alpha_1 y_j + \cdots + \alpha_k y_j^k) \right)} \mathrm{d}\boldsymbol{\alpha}. \end{aligned}$$

Note that

$$\begin{aligned}\prod_{1\le j\le s} e(\alpha_1 x_j + \cdots + \alpha_k x_j^k) &= (e(\alpha_1 x_1)\dots e(\alpha_1 x_s))(e(\alpha_2 x_1^2)\dots e(\alpha_2 x_s^2))\dots(e(\alpha_k x_1^k)\dots e(\alpha_k x_s^k))\\ &= e(\alpha_1(x_1+\cdots+x_s))e(\alpha_2(x_1^1+\cdots+x_s^1))\dots e(\alpha_k(x_1^k+\cdots+x_s^k))\\ &= \prod_{1\le j\le k} e(\alpha_j(x_1^j+\cdots+x_s^j)).\end{aligned}$$

Hence

$$\begin{aligned}J_{s,k}(X) &= \int_{[0,1]^k} \sum_{\substack{x_1,\dots,x_s\le X\\ y_1,\dots,y_s\le X}} \prod_{1\le j\le k} \Big(e(\alpha_j(x_1^j+\cdots+x_s^j))\Big)\Big(e(-\alpha_j(y_1^j+\cdots+y_s^j))\Big)\mathrm{d}\boldsymbol{\alpha}\\ &= \sum_{\substack{x_1,\dots,x_s\le X\\ y_1,\dots,y_s\le X}} \prod_{1\le j\le k} \int_{[0,1]^k} e(\alpha_j(x_1^j+\cdots+x_s^j-y_1^j-\cdots-y_s^j))\mathrm{d}\boldsymbol{\alpha}.\end{aligned}$$

By the orthogonality relations, the inner integral equals 1 if and only if $x_1^j+\cdots+x_s^j-y_1^j-\cdots-y_s^j=0$ and 0 otherwise. Thus, it counts solutions to the Vinogradov system. Let $x_{s+j}=y_j$, this concludes the proof. ◻

More generally, if one considers this slight variation of Vinogradov system [IK21, Step 3]

$$\begin{cases} x_1+x_2+\cdots+x_s-x_{s+1}-x_{s+2}-\cdots-x_{2s}=a_1,\\ x_1^2+x_2^2+\cdots+x_s^2-x_{s+1}^2-x_{s+2}^2-\cdots-x_{2s}^2=a_2,\\ \qquad\vdots\\ x_1^k+x_2^k+\cdots+x_s^k-x_{s+1}^k-x_{s+2}^k-\cdots-x_{2s}^k=a_k\end{cases}$$

for given a_i, the number of solutions is encoded in the integral

$$\int_{[0,1]^k}\left|\sum_{x\le X} e(\alpha_1 x+\cdots+\alpha_k x^k)\right|^{2s} e(\alpha_1 a_1+\cdots+\alpha_k a_k)\mathrm{d}\boldsymbol{\alpha}.$$

A highly non trivial bound of $J_{s,k}(X)$ is given by the Vinogradov Mean Value Theorem (Theorem 1.5).

Theorem 1.5. *For all integers $s,k\ge 1$,*

$$J_{s,k}(X) \ll_{s,k,\varepsilon} X^{\varepsilon}\Big(X^s + X^{2s-k(k+1)/2}\Big)$$

for all $X\ge 1$ and every $\varepsilon>0$.

When $s>k(k+1)/2$, the factor X^{ε} can be omitted [Pie20, Section 3.4]. This is a significant improvement of the trivial bound X^{2s} and is the sharpest bound one can hope for in such a general case. The study of Theorem 1.5 can be reduced to critical cases, i.e. when $s=k(k+1)/2$ meaning that the terms X^s and $X^{2s-k(k+1)/2}$ are of the same order of magnitude.

Lemma 3.2. *Suppose there exists an integer k such that $J_{k(k+1)/2,k}\ll_{k,\varepsilon} X^{k(k+1)/2+\varepsilon}$, then Theorem 1.5 holds for all $s\ge 1$ and such k.*

Proof. Let $s_k=k(k+1)/2$ denote the critical exponent. Suppose $s>s_k$, then

$$\begin{aligned}J_{s,k}(X) &= \int_{[0,1]^k}\left|\sum_{x\le X} e(\alpha_1 x+\cdots+\alpha_k x^k)\right|^{2s-k(k+1)}\left|\sum_{x\le X} e(\alpha_1 x+\cdots+\alpha_k x^k)\right|^{k(k+1)}\mathrm{d}\boldsymbol{\alpha}\\ &\le \sup_{\boldsymbol{\alpha}\in[0,1]^k}\left|\sum_{x\le X} e(\alpha_1 x+\cdots+\alpha_k x^k)\right|^{2s-k(k+1)}\int_{[0,1]^k}\left|\sum_{x\le X} e(\alpha_1 x+\cdots+\alpha_k x^k)\right|^{k(k+1)}\mathrm{d}\boldsymbol{\alpha}\\ &\ll X^{2s-k(k+1)}X^{k(k+1)/2+\varepsilon} = X^{2s-k(k+1)/2+\varepsilon}\end{aligned}$$

and this proves Theorem 1.5 hold in that case. Suppose $s < s_k$, applying Hölder inequality, we get

$$J_{s,k}(X) = \int_{[0,1]^k} \left| \sum_{x \le X} e(\alpha_1 x + \cdots + \alpha_k x^k) \right|^{2s} \cdot 1 \mathrm{d}\boldsymbol{\alpha} \le \left(\int_{[0,1]^k} \left| \sum_{x \le X} e(\alpha_1 x + \cdots + \alpha_k x^k) \right|^{2sp} \mathrm{d}\boldsymbol{\alpha} \right)^{1/p} \left(\int_{[0,1]^k} 1^q \mathrm{d}\boldsymbol{\alpha} \right)^{1/q}$$

with the conjugated exponents $p = s_k/s$ and $q = p/(p-1)$. This reduces to

$$J_{s,k}(X) \le \left(\int_{[0,1]^k} \left| \sum_{x \le X} e(\alpha_1 x + \cdots + \alpha_k x^k) \right|^{2s_k} \mathrm{d}\boldsymbol{\alpha} \right)^{1/p} \ll \left(X^{k(k+1)/2+\varepsilon} \right)^{s/s_k} \ll X^{s+\varepsilon}.$$

This concludes the proof. □

Even if the exponential sum dependence of $J_{s,k}$ is absolutely obvious, it might still be a little bit unclear why counting solutions of Diophantine equations might help to deduce nontrivial estimates of exponential sums. In the introduction, the equation (1) gave a first reason. We now make a clearer statement, see [BB20, Section 6.8.2] for further information or [Pie20, Section 2.1] for a general overview.

Theorem 3.3. *Let $k \ge 5$ be an integer and $f \in C^{k+1}([N, 2N])$ such that there exists a real number $\lambda_{k+1} \in [N^{-2}, N^{-1}]$ such that, for all $x \in [N, 2N]$, we have $\lambda_{k+1} \ll f^{(k+1)}(x)/(k+1)! \ll \lambda_{k+1}$. Then*

$$\sum_{N < n \le 2N} e(f(n)) \ll N^{1 - \frac{1}{6k^2 \log k}} \log N.$$

The proof relies crucially on the following lemma, which is a consequence of Lemma 3.1.

Lemma 3.4 ([BB20, Lemma 6.14])**.** *Set $\varepsilon = e^{-s/k^2}$. Then*

$$J_{s,k}(X) \ll X^{2s-(1-\varepsilon)k(k+1)/2}.$$

See [BB20, Section 6.8.2] for a proof of Theorem 3.3. Thus, it is natural to use the Vinogradov Mean Value Theorem (Theorem 1.5) in order to find new bounds on exponential sums. This will be done in the next section to prove Lemma 3.5, Proposition 3.12 and Proposition 3.14.

3.2 Heath-Brown seminal papers

Following [Hea16] and [DGW20], we explain how to improve the classical van der Corput estimate to get the following bound.

Theorem 2.2 ([Hea16, Theorem 1])**.** *Suppose the same assumptions as in Theorem 2.1 hold. Then*

$$\sum_{n \le N} e(f(n)) \ll_{A,k,\varepsilon} N^{1+\varepsilon} \left(\lambda_k^{1/(k(k-1))} + N^{-1/(k(k-1))} + N^{-2/(k(k-1))} \lambda_k^{-2/(k^2(k-1))} \right)$$

where the $\varepsilon > 0$ comes from Vinogradov's mean value integral estimate [Hea16, Lemma 1].

We fill all the gaps and plainly clarify the so-called «trivial» details in the proof of theorem 2.2 from [Hea16]. We then repeat the same process on theorem 11.1 (first, assuming theorem 3.3) from [DGW20]. The initial stages are the same and can be stated with no restrictions on the parameter k.

3.2.1 Proof of Theorem 1 from [Hea16].

Here are the main ingredients of the proof : estimate the sum in terms of Vinogradov mean value integral, a counting function involving Taylor-like j-th terms and all what we have done previously. In order to prove Theorem 2.2, we need three lemmas. For brevity, we use Σ to denote the exponential sum under consideration. Let P be an integer, introduce the Vinogradov mean value integral

$$J_{s,l}(P) = \int_{[0,1]^l} \left| \sum_{n \le P} e(\alpha_1 n + \cdots + \alpha_l n^l) \right|^{2s} \mathrm{d}\boldsymbol{\alpha} \tag{18}$$

with $\mathrm{d}\boldsymbol{\alpha} := \mathrm{d}\alpha_l \dots \mathrm{d}\alpha_2 \mathrm{d}\alpha_1$, $s \ge 1$ and $l \ge 1$.

Lemma 3.5. *Let $k \ge 2$ be an integer, and suppose that $f(x) : [0,N] \to \mathbf{R}$ has continuous derivatives of order up to k on $(0,N)$. Suppose further that $0 < \lambda_k \le f^{(k)}(x) \le A\lambda_k \le 1/4$ for $x \in (0,N)$. Then, for all $s \ge 1$, we have*

$$\Sigma := \sum_{n \le N} e(f(n)) \ll H + k^2 N^{1-1/s} \mathcal{N}^{1/(2s)} \left\{ H^{-2s+k(k-1)/2} J_{s,k-1}(H) \right\}^{1/(2s)}$$

where $H = \left\lceil (A\lambda_k)^{-1/k} \right\rceil$ and $\mathcal{N} = \#\left\{ m, n \le N \ : \ \left\| \frac{f^{(j)}(m)}{j!} - \frac{f^{(j)}(n)}{j!} \right\| \le 2H^{-j} \text{ for } 1 \le j \le k-1 \right\}$.

Lemma 3.6. *Let N be a positive integer, and suppose $g(x) : [0,N] \to \mathbf{R}$ has a continuous derivative on $(0,N)$. Suppose further that $0 < \mu \le g'(x) \le A_0 \mu$ for $x \in (0,N)$. Then, we have*

$$\#\{n \le N \ : \ ||g(n)|| \le \theta\} \ll (1 + A_0 \mu N)(1 + \mu^{-1}\theta).$$

Lemma 3.7. *When $k \ge 3$, we have*

$$\mathcal{N} \ll ((k-1)!A)^4 (N + \lambda_k N^2 + \lambda_k^{-2/k}) \log N.$$

First, assume Lemmas 3.5, 3.6 and 3.7 are true. We show how to prove Theorem 2.2.

Proof of Theorem 2.2. Let $k \ge 3$ be an integer and suppose that $f(x) : [0,N] \to \mathbf{R}$ has continuous derivatives of order up to k on $(0,N)$. Suppose further that $0 \le \lambda_k \le f^{(k)}(x) \le A\lambda_k$ for $x \in (0,N)$. From Lemma 3.5, using notations previously introduced, we get

$$\Sigma \ll H + k^2 N^{1-1/s} \mathcal{N}^{1/(2s)} \left\{ H^{-2s+k(k-1)/2} J_{s,k-1}(H) \right\}^{1/(2s)}. \tag{19}$$

Using Lemma 3.7, we bound $\mathcal{N}$ by $((k-1)!A)^4 (N + \lambda_k N^2 + \lambda_k^{-2/k}) \log N$ up to a multiplicative constant. Moreover, works from Wooley, Bourgain, Demeter and Guth ([Woo16], [BDG16], see also Theorem 1.5) allow one to bound $J_{s,l}(P)$ the following way $J_{s,l}(P) \ll_{\varepsilon,l} P^{2s-l(l+1)/2+\varepsilon}$ for $s \ge l(l+1)/2, l \ge 1, \varepsilon > 0$. Take $l = k-1$ and $P = H$, we thus get $J_{s,k-1}(H) \ll_{\varepsilon,l} H^{2s-(k-1)k/2+\varepsilon}$. Inserting it in (19) and bounding $\mathcal{N}$ using Lemma 3.7, we get

$$\Sigma \ll H + k^2 N^{1-1/s} \Big(((k-1)!A)^4 (N + \lambda_k N^2 + \lambda_k^{-2/k}) \log N \Big)^{1/(2s)} \left\{ H^{-2s+k(k-1)/2} H^{2s-(k-1)k/2+\varepsilon} \right\}^{1/(2s)}$$

for any $s \ge k(k-1)/2$. Cleaning up the right hand side, we obtain

$$\Sigma \ll H + k^2 N^{1-1/s} \Big(((k-1)!A)^4 (N + \lambda_k N^2 + \lambda_k^{-2/k}) \log N \Big)^{1/(2s)} H^{\varepsilon/(2s)}.$$

Recall we have $H \le (A\lambda_k)^{-1/k}$ and choose $s = (k-1)k/2$ (see later the proof of Lemma 3.5), thus

$$\begin{aligned} \Sigma &\ll_{A,k} \lambda_k^{-1/k} + k^2 N^{1-1/s} \Big(((k-1)!A)^4 (N + \lambda_k N^2 + \lambda_k^{-2/k}) \log N \Big)^{1/(2s)} N^{\varepsilon/(2s)} \\ &\ll_{A,k} \lambda_k^{-1/k} + N^{1-2/((k-1)k)} \Big(\Big(N + \lambda_k N^2 + \lambda_k^{-2/k}\Big) \log N \Big)^{1/((k-1)k)} N^{\varepsilon/((k-1)k)}. \end{aligned}$$

Moreover, in Lemma 3.7, we have the assumption that $k \geq 3$. Hence, $(k-1)k \geq 6$ and $1/((k-1)k) \leq 1/6 \leq 1$. Thus $N^{\varepsilon/((k-1)k)} \leq N^{\varepsilon}$. Or, more simply, this is trivial by ε considerations for Vinogradov notations (see Section 2.1). Then

$$\Sigma \ll_{A,k,\varepsilon} N^{\varepsilon}\left(\lambda_k^{-1/k} + N^{1-1/(k(k-1))} + N\lambda_k^{1/(k(k-1))} + N^{1-2/(k(k-1))}\lambda_k^{-2/(k^2(k-1))}\right)$$

by a concavity inequality from Lemma 2.6 and $\log N$ being absorbed by N^{ε} as a consequence of Lemma 2.4.

To get the desired result, split the analysis in two cases and compare N with the last term. Firstly, if $N \leq N^{1-2/(k(k-1))}\lambda_k^{-2/(k^2(k-1))}$, then one can drop the term $\lambda_k^{-1/k}$ because, by assumption, we have $N^{2/(k(k-1))} \leq \lambda_k^{-2/(k^2(k-1))}$ and then $N \leq \lambda_k^{-1/k}$. In this regime, the first and last terms are bigger than what would give the trivial estimate. Since all terms are positive, the first term can be dominated by the third one to obtain

$$\Sigma \ll_{A,k,\varepsilon} N^{\varepsilon}\left(N^{1-1/(k(k-1))} + N\lambda_k^{1/(k(k-1))} + N^{1-2/(k(k-1))}\lambda_k^{-2/(k^2(k-1))}\right). \tag{20}$$

Otherwise, suppose $N \geq N^{1-2/(k(k-1))}\lambda_k^{-2/(k^2(k-1))}$. By the same computations as before, we find $\lambda_k^{-1/k} \leq N$ and then

$$\begin{aligned}\Sigma &\ll_{A,k,\varepsilon} N^{\varepsilon}\left(N + N^{1-1/(k(k-1))} + N\lambda_k^{1/(k(k-1))} + N^{1-2/(k(k-1))}\lambda_k^{-2/(k^2(k-1))}\right) \\ &\ll_{A,k,\varepsilon} N^{1+\varepsilon}\left(N^{-1/(k(k-1))} + \lambda_k^{1/(k(k-1))} + N^{-2/(k(k-1))}\lambda_k^{-2/(k^2(k-1))}\right).\end{aligned} \tag{21}$$

Once again one can drop the term $\lambda_k^{-1/k}$. This concludes the proof of Theorem 2.2. □

Finally, notice the hypothesis $k \geq 3$ in Lemma 3.7 and notice that Theorem 2.2 holds for every $k \geq 2$. This is indeed not a problem since Theorem 2.2 is also true for $k = 2$ because it has the same assumptions and is weaker than Theorem 2.14 in that case. This is checked by direct calculations.

We now prove the three fundamental lemmas.

Proof (Lemma 3.5). We split the proof in two cases : if $N \leq H$, as the exponential sum is bounded by N using the triangle inequality, the statement would be trivial. Suppose then for the proof that $H \leq N$. We adapt ideas from the Weyl differencing method (A process, see Section 2.4.1). Consider $H\Sigma$ and note the following identity

$$H\Sigma = H\sum_{n\leq N} e(f(n)) = \left(\sum_{1\leq h\leq H} 1\right)\left(\sum_{n\leq N} e(f(n))\right) = \sum_{1\leq h\leq H}\ \sum_{-h<n\leq N-h} e(f(n+h))$$

where we just changed the indices to a new range (and subsequently we adapted the summand). We will make an error term coming out of this sum. To do so, use the following sum cutting

$$\begin{aligned}\sum_{1\leq h\leq H}\ \sum_{-h<n\leq N-h} e(f(n+h)) &= \sum_{1\leq h\leq H}\left(\sum_{-h<n\leq 0} + \sum_{0<n\leq N-H} + \sum_{N-H<n\leq N-h}\right) e(f(n+h)) \\ &= \sum_{1\leq h\leq H}\ \sum_{0<n\leq N-H} e(f(n+h)) + \sum_{1\leq h\leq H}\left(\sum_{-h<n\leq 0} + \sum_{N-H<n\leq N-h}\right) e(f(n+h)).\end{aligned}$$

We thus have

$$\sum_{1\leq h\leq H}\ \sum_{-h<n\leq N-h} e(f(n+h)) = \sum_{1\leq h\leq H}\ \sum_{0<n\leq N-H} e(f(n+h)) + \mathcal{O}(H^2)$$

because the two following inequalities hold

$$\left|\sum_{1\le h\le H}\sum_{-h<n\le 0} e(f(n+h))\right| \le \sum_{1\le h\le H}\sum_{-h<n\le 0} 1 = \sum_{h\le H} h = \frac{H(H+1)}{2} = \mathcal{O}(H^2)$$

$$\left|\sum_{1\le h\le H}\sum_{N-H<n\le N-h} e(f(n+h))\right| \le \sum_{1\le h\le H}\sum_{N-H<n\le N-h} 1 = \sum_{h\le H} 2H = 2H^2 = \mathcal{O}(H^2)$$

where we used that $|e(\cdot)|\le 1$. Dividing by H, we obtain

$$\Sigma = H^{-1}\sum_{1\le h\le H}\sum_{0<n\le N-H} e(f(n+h)) + \mathcal{O}(H). \tag{22}$$

To study the exponential sum Σ, we thus need to understand the behaviour and have some control on the exponential term $e(f(n+h))$. Due to the smoothness hypothesis we can estimate, or more precisely approximate, $f(n+h)$ up to a certain margin of error using the Taylor formula. It gives us the following expansion

$$f_n(h) := f(n) + f'(n)h + \cdots + \frac{f^{(k-1)}(n)}{(k-1)!}h^{k-1}.$$

Since we have a hypothesis on the k-th derivative, we stopped at the $(k-1)$-th order. Notice that $f(n+h) = f_n(h) + \mathcal{O}(h^n)$, the objective is now to bound $\sum_{h\le H} e(f(n+h))$. To do so, record the summation by parts formula as stated in 2.10. Let $g_n(x) = f(n+x) - f_n(x)$. We split the analysis in a main part and an error part

$$\sum_{h\le H} e(f(n+h)) = \sum_{h\le H} e(f_n(h) + g_n(h)) = \sum_{h\le H} e(f_n(h))e(g_n(h))$$

apply Proposition 2.10 to get

$$\begin{aligned}
\sum_{h\le H} e(f(n+h)) &= \left(\sum_{h\le H} e(f_n(h))\right)e(g_n(H)) - \int_0^H \sum_{h\le x} e(f_n(h))(e(g_n(x)))'\mathrm{d}x \\
&= \left(\sum_{h\le H} e(f_n(h))\right)e(g_n(H)) - \int_0^H \sum_{h\le x} e(f_n(h))2i\pi g_n'(x)e(g_n(x))\mathrm{d}x \\
&\ll \sum_{h\le H} e(f_n(h)) + \int_0^H \left|\sum_{h\le x} e(f_n(h))g_n'(x)\right|\mathrm{d}x
\end{aligned}$$

therefore, we have

$$\sum_{h\le H} e(f(n+h)) \ll |S_n(H)| + \int_0^H \left|S_n(x)g_n'(x)\right|\mathrm{d}x \tag{23}$$

with $S_n(x) = \sum_{h\le x} e(f_n(h))$. The term $|g_n'(x)|$ can be estimated. In order to do so, by the Taylor-Lagrange theorem, there exists some $\xi \in (n, n+x) \subset (0, N)$ for $x \in [0, H]$ such that

$$f(n+x) = f(n) + f'(n)x + \frac{f^{(2)}(n)}{2!}x^2 + \cdots + \frac{f^{(k-1)}(n)}{(k-1)!}x^{k-1} + \frac{f^{(k)}(\xi)}{k!}x^k. \tag{24}$$

Deriving the relation with respect to x, we obtain

$$f'(n+x) = f'(n) + f^{(2)}(n)x + \cdots + \frac{f^{(k-1)}(n)}{(k-2)!}x^{k-2} + \frac{f^{(k)}(\xi)}{(k-1)!}x^{k-1} = f_n'(x) + \frac{f^{(k)}(\xi)}{(k-1)!}x^{k-1}.$$

It then follows that

$$g_n'(x) = f'(n+x) - f_n'(x) = \frac{f^{(k)}(\xi)}{(k-1)!}x^{k-1} \le \frac{A\lambda_k}{(k-1)!}H^{k-1} \ll A\lambda_k H^{k-1}$$

because $1/(k-1)! \ll 1$, $x \leq H$ and by hypothesis on $f^{(k)}(\cdot)$. It is at that moment one chooses the value of H such that (23) becomes

$$\sum_{h \leq H} e(f(n+h)) \ll |S_n(H)| + H^{-1} \int_0^H |S_n(x)| \mathrm{d}x. \tag{25}$$

To do so, take $H = \left[(A\lambda_k)^{-1/k}\right]$. This will be useful to neglect the integral. Using (22), we get

$$\begin{aligned}\Sigma &\ll H + H^{-1} \sum_{0<n\leq N-H} \left(|S_n(H)| + H^{-1} \int_0^H |S_n(x)| \mathrm{d}x\right) \\ &\ll H + H^{-1} \sum_{0<n\leq N-H} |S_n(H_0)| \end{aligned} \tag{26}$$

because there exists a $H_0 \leq H$ such that $|S_n(x)| \leq |S_n(H_0)|$ for every $x \in I$ and fixed n.

Recall $f_n(h) := f(n) + f'(n)h + \cdots + \frac{f^{(k-1)}(n)}{(k-1)!}h^{k-1}$. Now, let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{k-1}) \in [0,1]^{k-1}$ be such that $||f^{(j)}(n)/j! - \alpha_j|| \leq H^{-j}$ for $1 \leq j \leq k-1$ and introduce $f(h;\boldsymbol{\alpha}) = \alpha_1 h + \cdots + \alpha_{k-1}h^{k-1}$. The multiset $\boldsymbol{\alpha}$ will be used to control and locally approximate $f_n(h)$. We condition $f_n(h)$ such that it fits in the arithmetic setting implemented. The constant term $f(n)$ has no effect when taking the absolute value of $S_n(H_0)$, then it can be removed. The distance to the nearest integer operator can also be removed by introducing a collection of elements c_j such that $f^{(j)}(n)/j! - c_j \in \mathbf{Z}$. We thus have $|c_j - \alpha_j| \leq H^{-j}$ for $1 \leq j \leq k-1$ and we denote the resulting polynomial by $f_n^*(h)$. As a consequence, one can replace $|S_n(H_0)|$ by $|S_n^*(H_0)|$ defined as

$$\left|S_n^*(H_0)\right| := \left|\sum_{n\leq H_0} e(f_n^*(h))\right| = \left|\sum_{n\leq H_0} e(f_n(h))\right| = |S_n(H_0)|. \tag{27}$$

We bound $|S_n^*(H_0)|$ in the exact same way as we did in (25). We recall the main steps and show the needed changes. Using summation by parts (Proposition 2.10), doing the exact same steps as in (23), we get

$$S_n^*(H_0) = \sum_{n\leq H_0} e(f_n^*(h)) \ll |S(H_0,\boldsymbol{\alpha})| + \int_0^{H_0} \left|S(x;\boldsymbol{\alpha})g_n'(x)\right| \mathrm{d}x \tag{28}$$

with $S(x;\boldsymbol{\alpha}) = \sum_{h\leq x} e(f(h;\boldsymbol{\alpha}))$ where $g_n(x) = f(x;\boldsymbol{\alpha}) - f_n^*(x)$. By direct computations, we have

$$\begin{aligned} g_n'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(f(x;\boldsymbol{\alpha}) - f_n^*(x)\right) = \frac{\mathrm{d}}{\mathrm{d}x}\sum_{j=1}^{k-1}(\alpha_j - c_j)x^j = \sum_{j=1}^{k-1}(\alpha_j - c_j)jx^{j-1} \\ &\ll \max_{1\leq j\leq k-1}|\alpha_j - c_j| \sum_{j=1}^{k-1} jx^{j-1} \ll \max_{1\leq j\leq k-1}|\alpha_j - c_j| H^{j-1} \sum_{j=1}^{k-1} j \\ &\ll \max_{1\leq j\leq k-1}|\alpha_j - c_j| k^2 H^{j-1} \ll k^2 H^{-1} \end{aligned}$$

since x was supposed to be lower than H in (24) and by construction of the c_j. Using this, (28) becomes

$$S_n^*(H_0) \ll |S(H_0,\boldsymbol{\alpha})| + k^2H^{-1}\int_0^{H_0} |S(x;\boldsymbol{\alpha})| \mathrm{d}x.$$

We now take into account all the contributions coming from $\boldsymbol{\alpha}$. Thus

$$\int_{\boldsymbol{\alpha}} S_n^*(H_0)\mathrm{d}\boldsymbol{\alpha} \ll \int_{\boldsymbol{\alpha}} \left\{|S(H_0,\boldsymbol{\alpha})| + k^2H^{-1}\int_0^{H_0} |S(x;\boldsymbol{\alpha})|\mathrm{d}x\right\}\mathrm{d}\boldsymbol{\alpha}.$$

We abused of notations by confusing $\boldsymbol{\alpha}$ as a multiset of indices and $\boldsymbol{\alpha}$ as a domain of integration, the context makes the distinction clear. Note that by construction of the domain $\boldsymbol{\alpha}$, we have

$$\begin{aligned}\int_{\boldsymbol{\alpha}} 1 \mathrm{d}\boldsymbol{\alpha} &= \int_0^1 \cdots \int_0^1 \mathbb{1}_{\{\boldsymbol{\alpha}\,:\,||f^{(j)}(n)/j!-\alpha_j||\leq H^{-j},\ 1\leq j\leq k-1\}} \mathrm{d}\boldsymbol{\alpha} \\ &= \int_0^1 \cdots \int_0^1 \mathbb{1}_{\bigcap_{1\leq j\leq k-1}\{\boldsymbol{\alpha}\,:\,||f^{(j)}(n)/j!-\alpha_j||\leq H^{-j}\}} \mathrm{d}\boldsymbol{\alpha} \\ &= \prod_{1\leq j\leq k-1} \int_0^1 \mathbb{1}_{\{\boldsymbol{\alpha}\,:\,||f^{(j)}(n)/j!-\alpha_j||\leq H^{-j}\}} \mathrm{d}\alpha_j \\ &= \prod_{1\leq j\leq k-1} 2H^{-j} = 2^{k-1}H^{-k(k-1)/2} \end{aligned} \tag{29}$$

by direct geometric considerations since $||\cdot||$ acts as a distance as it was recalled in Section 2.1. Thus, by linearity and dividing by that nonzero integral, we find

$$S_n^*(H_0) \ll 2^{1-k}H^{k(k-1)/2}\left\{\int_{\boldsymbol{\alpha}} |S(H_0,\boldsymbol{\alpha})|\mathrm{d}\boldsymbol{\alpha} + k^2H^{-1}\int_0^{H_0}\int_{\boldsymbol{\alpha}} |S(x;\boldsymbol{\alpha})|\mathrm{d}x\mathrm{d}\boldsymbol{\alpha}\right\}. \tag{30}$$

We now reexpress (30) in order to bound $\sum_{n\leq N-H}|S_n(H_0)|$ using (27). Let

$$I(x) = \int_{[0,1]^{k-1}} |S(x;\boldsymbol{\alpha})|\nu(\boldsymbol{\alpha})\mathrm{d}\boldsymbol{\alpha}$$

where

$$\nu(\boldsymbol{\alpha}) = \#\left\{n \leq N-H \ :\ ||f^{(j)}(n)/j! - \alpha_j|| \leq H^{-j} \text{ for } 1\leq j\leq k-1\right\}.$$

It follows that

$$\sum_{n\leq N-H} |S_n(H_0)| \ll 2^{1-k}H^{k(k-1)/2}\left\{I(H_0) + k^2H^{-1}\int_0^{H_0} I(x)\mathrm{d}x\right\} \tag{31}$$

as $\nu(\boldsymbol{\alpha})$ was supposed non-empty after (26) and because, as we did in (29), we have

$$\int_0^1\cdots\int_0^1 \nu(\alpha)\mathrm{d}\boldsymbol{\alpha} = \sum_{n\leq N-H}\int_{\boldsymbol{\alpha}} 1\mathrm{d}\boldsymbol{\alpha} = 2^{k-1}H^{-k(k-1)/2}(N-H). \tag{32}$$

By a similar argument (one could use indicator function to be fully rigorous), we obtain

$$\nu(\boldsymbol{\alpha})^2 \leq \#\left\{m,n\leq N \ :\ ||f^{(j)}(m)/j! - \alpha_j|| \leq H^{-j},\ ||f^{(j)}(n)/j! - \alpha_j|| \leq H^{-j} \text{ for } 1\leq j\leq k-1\right\}$$

therefore

$$\int_0^1\cdots\int_0^1 \nu(\boldsymbol{\alpha})^2\mathrm{d}\boldsymbol{\alpha} \leq \int_0^1\cdots\int_0^1 \#\left\{m,n\leq N \ :\ ||f^{(j)}(m)/j! - f^{(j)}(n)/j!|| \leq 2H^{-j},\ 1\leq j\leq k-1\right\}\mathrm{d}\boldsymbol{\alpha}$$

since, by the triangle inequality, we have

$$||f^{(j)}(m)/j!-f^{(j)}(n)/j!|| = ||f^{(j)}(m)/j!-\alpha_j+\alpha_j-f^{(j)}(n)/j!|| \leq ||f^{(j)}(m)/j!-\alpha_j||+||\alpha_j-f^{(j)}(n)/j!|| \leq 2H^{-j}.$$

Thus, using the definition of $\mathcal{N}$ from Lemma 3.5, we get

$$\int_0^1\cdots\int_0^1 \nu(\boldsymbol{\alpha})^2\mathrm{d}\boldsymbol{\alpha} \leq 2^{k-1}H^{-k(k-1)/2}\mathcal{N}. \tag{33}$$

Let $s\geq 1$. We then use (33) and the Hölder's inequality to obtain

$$I(x) \ll 2^{k-1}H^{-k(k-1)/2}N^{1-1/s}\mathcal{N}^{1/(2s)}\left\{H^{k(k-1)/2}J_{s,k-1}(H)\right\}^{1/(2s)}. \tag{34}$$

Indeed, we have

$$I(x) = \int_{[0,1]^{k-1}} |S(x;\boldsymbol{\alpha})| \nu(\boldsymbol{\alpha}) \mathrm{d}\boldsymbol{\alpha} \leq \left(\int_{[0,1]^{k-1}} |S(x;\boldsymbol{\alpha})|^{2s} \mathrm{d}\boldsymbol{\alpha} \right)^{1/(2s)} \left(\int_{[0,1]^{k-1}} \nu(\boldsymbol{\alpha})^{2s/(2s-1)} \mathrm{d}\boldsymbol{\alpha} \right)^{1-1/(2s)} . \tag{35}$$

Recalling the definition of the Vinogradov mean value integral (18) and, since it is a non-decreasing function, the first integral of the right hand side is bounded by $J_{s,k-1}(H)$. In brief,

$$\left(\int_{[0,1]^{k-1}} |S(x;\boldsymbol{\alpha})|^{2s} \mathrm{d}\boldsymbol{\alpha} \right)^{1/(2s)} = J_{s,k-1}(x)^{1/(2s)} \leq J_{s,k-1}(H)^{1/(2s)}. \tag{36}$$

To bound the remaining integral, note that $2s/(2s-1) \leq 2$ when $s \geq 1$. Thus, by (33), we have

$$\left(\int_{[0,1]^{k-1}} \nu(\boldsymbol{\alpha})^{2s/(2s-1)} \mathrm{d}\boldsymbol{\alpha} \right)^{1-1/(2s)} \leq \left(\int_{[0,1]^{k-1}} \nu(\boldsymbol{\alpha})^{2} \mathrm{d}\boldsymbol{\alpha} \right)^{1-1/(2s)} \leq \left(2^{k-1} H^{-k(k-1)/2} \mathcal{N} \right)^{1-1/(2s)} \tag{37}$$

since $\nu(\alpha)$ is 0 or strictly larger than 1. Moreover, note that $2^{(1-k)/(2s)} H^{k/2(k-1)/(2s)} \mathcal{N} \ll N^{1-1/s}$ since $\mathcal{N} \leq N^2$ by construction and $N \ll_k H^{-k(k-1)/(8s)} 2^{(k-1)/(4s)}$ because these quantities are fixed and finite. Putting this altogether yields exactly (34). It remains to bound Σ using (31). Note that the estimate from (34) has no x dependence, therefore we can bound (26) by linearity of the integral. Inserting (34) in (31), we find

$$\begin{aligned} \sum_{n \leq N-H} |S_n(H_0)| &\ll 2^{1-k} H^{k(k-1)/2} \left\{ k^2 2^{k-1} H^{-k(k-1)/2} N^{1-1/s} \mathcal{N}^{1/(2s)} \left\{ H^{k(k-1)/2} J_{s,k-1}(H) \right\}^{1/(2s)} \right\} \\ &\ll k^2 N^{1-1/s} \mathcal{N}^{1/(2s)} \left\{ H^{k(k-1)/2} J_{s,k-1}(H) \right\}^{1/(2s)} \end{aligned} \tag{38}$$

since $H_0 \leq H$. It now suffices to combine (38) and (26), it then comes

$$\begin{aligned} \sigma &\ll H + H^{-1} \sum_{0<n \leq N-H} |S_n(H_0)| \\ &\ll H + k^2 N^{1-1/s} \mathcal{N}^{1/(2s)} \left\{ H^{-2s+k(k-1)/2} J_{s,k-1}(H) \right\}^{1/(2s)} . \end{aligned}$$

This concludes the proof. □

Proof (Lemma 3.6). Use the following lemma.

Lemma 3.8 ([TH86, Lemma 6.11]). *Let M and N be integers, $N > 1$ and let $\phi(n)$ be a real function of n defined for $M \leq n \leq M+N-1$ such that $\delta \leq \phi(n+1) - \phi(n) \leq c\delta$ where $\delta > 0$, $c \geq 1$, $c\delta \leq 1/2$. Let $W > 0$ and let $||x||$ denote the distance to the nearest integer. Then, the number of values of n for which $||\phi(n)|| \leq W\delta$ is less than $(1 + Nc\delta)(1 + 2W)$.*

To conclude, the proof of Lemma 3.8 must be adapted in a continuous setting: replacing $\phi(n+1) - \phi(n)$ by $\phi'(x)$. This is just a matter of considering the natural generalisation of the number of elements for non discrete sets, namely the Lebesgue measure. With the identification $\delta = \mu$, $c\delta = A_0\mu$, $W\delta = \theta$ one thus has $Nc\delta = A_0\mu N$ and $2W = \mu^{-1}\theta$ and then Lemma 3.6 is proved. □

Proof (Lemma 3.7). Assume $k \geq 3$. Recall the definition of $\mathcal{N}$

$$\mathcal{N} = \#\left\{ m, n \leq N \ : \ \left\| \frac{f^{(j)}(m)}{j!} - \frac{f^{(j)}(n)}{j!} \right\| \leq 2H^{-j} \text{ for } 1 \leq j \leq k-1 \right\}. \tag{39}$$

We want to consider a easier-to-handle counting function to study $\mathcal{N}$. Denote by $\mathscr{N}$ the set counted by $\mathcal{N}$. Rewrite $\mathscr{N}$ to be

$$\mathscr{N} = \bigcap_{1 \leq j \leq k-1} \left\{ m, n \leq N \ : \ \left\| \frac{f^{(j)}(m)}{j!} - \frac{f^{(j)}(n)}{j!} \right\| \leq 2H^{-j} \right\}. \tag{40}$$

We have the obvious inclusion

$$\mathscr{N} \subset \bigcap_{j=\{k-2,k-1\}} \left\{ m,n \le N \ : \ \left\| \frac{f^{(j)}(m)}{j!} - \frac{f^{(j)}(n)}{j!} \right\| \le 2H^{-j} \right\}. \tag{41}$$

By cardinality consideration (the sets are finite), it results

$$\mathcal{N} = \#\mathscr{N} \le \mathcal{N}_1 := \#\left\{ m,n \le N \ : \ \left\| \frac{f^{(j)}(m)}{j!} - \frac{f^{(j)}(n)}{j!} \right\| \le 2H^{-j} \text{ for } j = k-2, k-1 \right\}. \tag{42}$$

Then, $\mathcal{N}$ is at most $\mathcal{N}_1$.

We now introduce the following functions

$$g_1(x) = \frac{f^{(k-2)}(x)}{(k-2)!}, \quad g_2(x) = \frac{f^{(k-1)}(x)}{(k-1)!}, \quad \phi(x,y) = \max\big(1 - B^{-1}||x||, 0\big)\max\big(1 - C^{-1}||y||, 0\big)$$

where $B = 4H^{2-k}$ and $C = 4H^{1-k}$. The function ϕ is doubly-periodic. We can get its Fourier series. Nevertheless, it is sufficient to know its existence (the exact value of the coefficients is irrelevant). This function is interesting because it bounds $\mathcal{N}_1$. To see that, notice that

$$||g_1(m) - g_1(n)|| \le 2H^{2-k} = B/2, \quad ||g_2(m) - g_2(n)|| \le 2H^{1-k} = C/2.$$

Then, if $\phi(\cdot,\cdot)$ is nonzero, we have

$$\phi(g_1(m) - g_1(n), g_2(m) - g_2(n)) = \left(1 - \frac{||g_1(m) - g_1(n)||}{B}\right)\left(1 - \frac{||g_2(m) - g_2(n)||}{C}\right) \ge \frac{1}{2}\frac{1}{2} = \frac{1}{4}.$$

It follows that

$$\mathcal{N}_1 \le 4 \sum_{m,n \le N} \phi(g_1(m) - g_1(n), g_2(m) - g_2(n))$$

by a direct counting argument. Thus, using the Fourier expansion

$$\phi(x,y) = \sum_{r,s \in \mathbf{Z}} c_{r,s} e(rx + sy)$$

we find that

$$\begin{aligned} \mathcal{N}_1 &\ll \sum_{m,n \le N} \sum_{r,s \in \mathbf{Z}} c_{r,s} e(r(g_1(m) - g_1(n)) + s(g_2(m) - g_2(n))) \\ &\ll \sum_{r,s \in \mathbf{Z}} c_{r,s} \sum_{m,n \le N} e(r(g_1(m) - g_1(n)) + s(g_2(m) - g_2(n))) \end{aligned}$$

where we used linearity since the sum on m and n is finite. Now, carefully note that

$$\begin{aligned} \left| \sum_{n \le N} e(rg_1(n) + sg_2(n)) \right|^2 &= \left(\sum_{m \le N} e(rg_1(m) + sg_2(m)) \right) \left(\overline{\sum_{n \le N} e(rg_1(n) + sg_2(n))} \right) \\ &= \sum_{m,n \le N} e(rg_1(m) + sg_2(m)) \overline{e(rg_1(n) + sg_2(n))} \\ &= \sum_{m,n \le N} \exp(2i\pi(rg_1(m) + sg_2(m))) \overline{\exp(2i\pi(rg_1(n) + sg_2(n)))} \\ &= \sum_{m,n \le N} e(r(g_1(m) - g_1(n)) + s(g_2(m) - g_2(n))). \end{aligned}$$

We will make use of that to bound $\mathcal{N}_1$. Let K be a positive integer parameter, to be chosen later. We partition the range $(0, N]$ into K intervals $I_i = (a_i, b_i]$ for $i \le K$ having integer endpoints and length

$b_i - a_i \leq 1 + N/K$. Such a partition will be useful to localise the magnitude of $g_2(m) - g_2(n)$ and concentrate the study in the interval $[0,1]$, thus to get rid of the $||\cdot||$ operator. We now bound the sum according to the cutting. First, note that $\sum_{n\leq N} = \sum_{i\leq K}\sum_{n\in I_i}$. It then follows, by Cauchy-Schwarz

$$\mathcal{N}_1 \ll \sum_{r,s\in\mathbb{Z}} c_{r,s}\left|\sum_{i\leq K}\left(1\sum_{n\in I_i} e(rg_1(n)+sg_2(n))\right)\right|^2 \leq \sum_{r,s\in\mathbb{Z}} c_{r,s}\left(\sum_{i\leq K} 1\right)\left(\sum_{i\leq K}\left|\sum_{n\in I_i} e(rg_1(n)+sg_2(n))\right|^2\right).$$

We rearrange the sum to get

$$\mathcal{N}_1 \ll K\sum_{i\leq K}\sum_{r,s\in\mathbb{Z}} c_{r,s}\left|\sum_{n\in I_i} e(rg_1(n)+sg_2(n))\right|^2 = K\sum_{i\leq K}\sum_{r,s\in\mathbb{Z}} c_{r,s}\sum_{m,n\in I_i} e(r(g_1(m)-g_1(n))+s(g_2(m)-g_2(n))).$$

By the definition of ϕ, we conclude

$$\mathcal{N}_1 \ll K\sum_{i\leq K}\sum_{m,n\in I_i}\phi(g_1(m)-g_1(n), g_2(m)-g_2(n)) \ll K\sum_{\substack{m,n\leq N\\ |m-n|\leq 1+N/K}}\phi(g_1(m)-g_1(n), g_2(m)-g_2(n)) \ll K\mathcal{N}_2$$

since, by construction, we clearly have $\sum_{i\leq K}\sum_{m,n\in I_i} \leq \sum_{m,n\leq N, |m-n|\leq 1+N/K}$ and with $\mathcal{N}_2$ defined to be

$$\#\left\{m,n\leq N \;:\; |m-n|\leq 1+N/K, \left\|\frac{f^{(j)}(m)}{j!}-\frac{f^{(j)}(n)}{j!}\right\| \leq 4H^{-j} \text{ for } j=k-2, j-1\right\}.$$

Note that no substantial improvement comes from considering finer sets, i.e. letting j lying in $\{k-\ell,\dots,k-2,k-1\}$ for $\ell\geq 2$ instead of $\{k-2,k-1\}$. Indeed, direct computations give the same result up to a multiplicative constant depending on the number of terms involved (in Heath-Brown's setting, only two terms are involved). The aim is now to estimate $\mathcal{N}_2$. Applying the mean value theorem for m,n such that $|m-n|\leq 1+N/K$, we have

$$\left|\frac{f^{(k-1)}(m)}{(k-1)!}-\frac{f^{(k-1)}(n)}{(k-1)!}\right| \leq \frac{|m-n|}{(k-1)!}\sup|f^{(k)}(x)| \leq A\lambda_k(1+N/K) \tag{43}$$

since $1/((k-1)!)\leq 1$. We proceed to choose K such that we get rid of the *nearest integer* operator, i.e. we need to find a real number α such that

$$\left|\frac{f^{(k-1)}(m)}{(k-1)!}-\frac{f^{(k-1)}(n)}{(k-1)!}\right| \leq \frac{1}{\alpha} < 1.$$

Say $\alpha = 2$, we then seek K such that $A\lambda_k(1+N/K) = 1/2$. Direct computations give

$$\frac{2AN\lambda_k}{1-2A\lambda_k} = K.$$

We make use of the hypothesis $A\lambda_k \leq 1/4$ to obtain $K\leq 4AN\lambda_k$. In order to have an integer, we choose K to be $[4AN\lambda_k]$. For such K, by definition of $\mathcal{N}_2$, we then have

$$\left|\frac{f^{(k-1)}(m)}{(k-1)!}-\frac{f^{(k-1)}(n)}{(k-1)!}\right| = \left\|\frac{f^{(k-1)}(m)}{(k-1)!}-\frac{f^{(k-1)}(n)}{(k-1)!}\right\| \leq 4H^{1-k}. \tag{44}$$

Using the mean value theorem we can bound the distance between pairs (m,n) counted by $\mathcal{N}_2$

$$\left|\frac{f^{(k-1)}(m)}{(k-1)!}-\frac{f^{(k-1)}(n)}{(k-1)!}\right| \geq \frac{|m-n|}{(k-1)!}\inf|f^{(k)}(x)| \geq \lambda_k\frac{|m-n|}{(k-1)!}. \tag{45}$$

Combining (44) and (45), we therefore have

$$|m-n| \leq \frac{4(k-1)!}{\lambda_k H^{k-1}}. \tag{46}$$

We now split the counting procedure to estimate $\mathcal{N}_2$ in three cases: $m = n$, $m > n$, $m < n$. The last two cases are dual and produce the same estimate by elementary arithmetic considerations. It thus suffices to study the two first cases. The first case is rather simple because (46) is always satisfied. We thus get N different pairs from that case. To study the remaining case, we will make use of Lemma 3.6.

Introduce d such that $m = n + d$ with $1 \le d \le D$ where, according to the definition of $\mathcal{N}_2$ and (46), D is defined to be

$$D = \min\left(N, \left[\frac{4(k-1)!}{\lambda_k H^{k-1}}\right]\right).$$

We estimate $\mathcal{N}_2$ separately for each d and then will sum for all possible values of d. First, fix d an integer in $[1, D]$ and consider

$$g(x) = \frac{f^{(k-2)}(x+d) - f^{(k-2)}(x)}{(k-2)!}.$$

Then, by linearity of the differentiation, the derivative with respect to x is

$$g'(x) = \frac{f^{(k-1)}(x+d) - f^{(k-1)}(x)}{(k-2)!}$$

so that, by proceeding as for (43) and (45), we get

$$d\frac{\lambda_k}{(k-2)!} \le d\frac{\inf|f^{(k)}(x)|}{(k-2)!} \le g'(x) \le d\frac{\sup|f^{(k)}(x)|}{(k-2)!} \le d\frac{A\lambda_k}{(k-2)!}.$$

Applicability conditions for Lemma 3.6 are then verified. With $\mu = \lambda_k d/(k-2)!$ and $A_0 = A$, for each integer $d \in [1, D]$, we have a contribution for each d bounded by

$$\begin{aligned}(k-2)!(1 + A\lambda_k dN)(1 + \lambda_k^{-1}d^{-1}H^{2-k}) &\ll (k-2)!A(1 + \lambda_k DN)(D\lambda_k H^{k-2} + 1)\lambda_k^{-1}H^{2-k}d^{-1} \\ &\ll ((k-1)!)^3 A(1 + \lambda_k\lambda_k^{-1}H^{1-k}N)(\lambda_k^{-1}H^{1-k}\lambda_k H^{k-2} + 1)\lambda_k^{-1}H^{2-k}d^{-1} \\ &\ll ((k-1)!)^3 A(1 + NH^{1-k})\lambda_k^{-1}H^{2-k}d^{-1} \\ &\ll ((k-1)!A)^3(1 + N\lambda_k^{1-1/k})\lambda_k^{-2/k}d^{-1}\end{aligned}$$

since $1 \le D/d$, using (46), the definition of H from Lemma 3.5 and noticing some quantities are trivially bounded by 1. Summing for d, we therefore find that

$$\begin{aligned}\mathcal{N}_2 &\ll N + ((k-1)!A)^3(1 + N\lambda_k^{1-1/k})\lambda_k^{-2/k}\sum_{1\le d\le D} d^{-1} \\ &\ll N + ((k-1)!A)^3(1 + N\lambda_k^{1-1/k})\lambda_k^{-2/k}\log D \\ &\ll ((k-1)!A)^3(\lambda_k^{-2/k} + N)\log N\end{aligned}$$

by a classic series to integral comparison and since $k \ge 3$, $\lambda_k \le 1$, $D \le N$, $A \ge 1$. To conclude the proof of Lemma 3.7, it suffices to see that

$$\begin{aligned}\mathcal{N} \le \mathcal{N}_1 \ll K\mathcal{N}_2 &\ll (1 + A\lambda_k N)((k-1)!A)^3(\lambda_k^{-2/k} + N)\log N \\ &= ((k-1)!A)^3(\lambda_k^{-2/k} + N + A\lambda_k^{1-2/k}N + A\lambda_k N^2)\log N \\ &\ll ((k-1)!A)^4(\lambda_k^{-2/k} + N + \lambda_k^{1-2/k}N + \lambda_k N^2)\log N\end{aligned}$$

and that $N\lambda_k^{1-2/k} \le N$ again because $k \ge 3$ and $\lambda_k \le 1$. This proves the lemma. $\square$

3.2.2 Proof of Theorem 11.1 from [DGW20].

Following [DGW20, Section 11.1], we show how to improve the fourth derivative estimate (i.e. when $k = 4$ in Theorem 2.1).

Theorem 3.9. *Let $f(x) : [0, N] \to \mathbf{R}$ have a continuous fourth derivative on $(0, N)$ with $0 < \lambda_4 \leq f^{(4)}(x) \leq A\lambda_4$ for some constant $A \geq 1$. Write $\lambda_4 = N^{-\varpi}$ and suppose that $N^{-2} \ll \lambda_4 \ll N^{-1}$. Then, for any fixed $\varepsilon > 0$*

$$\sum_{n\leq N} e(f(n)) \ll_{A,\varepsilon} N^{1-\varpi/(4\varpi+8)+\varepsilon} + N^{8/9+\varepsilon}.$$

The proof massively relies on previous results and the improvement by Demeter, Guth and Wang [DGW20, Theorem 3.3] of Vinogradov Mean Value Theorem.

Theorem 3.10 (Theorem 3.3 from [DGW20])**.** *Let $N \in \mathbf{N}$. For each $0 \leq \beta \leq 3/2$, each interval I of length $1/N^\beta$ and each $a_j \in \mathbf{C}$ with $|a_j| = 1$, we have*

$$\int_{[0,1]\times[0,1]\times I} \left| \sum_{k=1}^{N} a_j e(kx_1 + k^2x_2 + k^3x_3) \right|^{12-2\beta} \mathrm{d}\boldsymbol{x} \ll_\varepsilon N^{6-2\beta+\varepsilon}. \tag{47}$$

We will mimic the proof of Theorem 2.2 and will only adapt the proof of Lemma 3.5 to fit this new estimate of a quantity closely related to $J_{2s,3}$. Nevertheless, the initial stages of the proof of Theorem 3.9 need no restriction to $k = 4$. As that will be of a significant importance to consider the general case in Proposition 3.16, we start by considering $k \geq 3$. Obviously, the hypothesis becomes $0 < \lambda_k \leq f^{(k)}(x) \leq A\lambda_k$ for $A > 1$.

Proof (Theorem 3.9). Let $k \geq 3$ and recall $H = \left\lceil (A\lambda_k)^{-1/k} \right\rceil$ as usual. For $\boldsymbol{\alpha} \in [0, 1]^{k-1}$, we define

$$\nu(\boldsymbol{\alpha}) = \#\left\{ n \leq N - H,\ \left\| f^{(j)}(n)/j! - \alpha_j \right\| \leq H^{-j} \text{ for } 1 \leq j \leq k-1 \right\}.$$

We proceed as in the last part of the proof of Lemma 3.5. Set $\alpha^* = f^{(k-1)}(0)/(k-1)!$, assume $\nu(\boldsymbol{\alpha}) \neq 0$ and let n be a number counted by $\nu(\boldsymbol{\alpha})$, thus

$$\begin{aligned}
|\alpha_{k-1} - \alpha^*| &= \left| \alpha_{k-1} - \frac{f^{(k-1)}(n)}{(k-1)!} + \frac{f^{(k-1)}(n)}{(k-1)!} - \alpha^* \right| \leq \left| \alpha_{k-1} - \frac{f^{(k-1)}(n)}{(k-1)!} \right| + \left| \frac{f^{(k-1)}(n)}{(k-1)!} - \alpha^* \right| \\
&\leq H^{-(k-1)} + \left| \frac{f^{(k-1)}(n)}{(k-1)!} - \frac{f^{(k-1)}(0)}{(k-1)!} \right| = H^{1-k} + \left| \frac{f^{(k-1)}(n) - f^{(k-1)}(0)}{(k-1)!} \right|
\end{aligned}$$

where we used the definition of $\nu(\boldsymbol{\alpha})$. By the mean value theorem to the $(k-1)$-th order, there exists a ξ such that $f^{(k-1)}(n) - f^{(k-1)}(0) = nf^{(k)}(\xi)$. By the hypotheses, $n \leq N - H \leq N$ and $f^{(k)}(\xi) \leq A\lambda_k$. Then

$$|\alpha_{k-1} - \alpha^*| \ll_{A,k} N\lambda_k + H^{1-k} \ll_{A,k} N\lambda_k + \lambda_k^{(k-1)/k}.$$

Because there exists a positive real number ζ such that $|\alpha_{k-1} - \alpha^*| \leq \zeta$ with the constraint $\zeta \ll_{A,k} N\lambda_k + \lambda_k^{(k-1)/k}$, one can replace the domain of integration and adapt the proof of Lemma 3.5. This enables us to localize the contribution from α_{k-1} by restricting the box $[0, 1]$ for the $(k-1)$-th variable from Lemma 3.5 to the finer one $[\alpha^* - \zeta, \alpha^* + \zeta]$. By changing the domain of integration of (35), (36) and using the positivity of $\nu(\boldsymbol{\alpha})$ in (37), we finally get this new bound which is very similar to Lemma 3.5 but the Vinogradov mean value integral J differs

$$\sum_{n\leq N} e(f(n)) \ll_{A,\varepsilon} H + k^2 N^{1-1/s} \mathfrak{N}^{1/(2s)} \left\{ H^{-2s+k(k-1)/2} J \right\}^{1/(2s)} \ll_{A,\varepsilon,k} H + N^{1-1/s} \mathfrak{N}^{1/(2s)} \left\{ H^{-2s+k(k-1)/2} J \right\}^{1/(2s)} \tag{48}$$

with $\mathcal{N}$ defined as in Lemma 3.7 and

$$J := \int_{[0,1]^{k-2}\times[\alpha^*-\zeta,\alpha^*+\zeta]} \left|\sum_{n\leq H} e(\alpha_1 n + \cdots + \alpha_{k-1}n^{k-1})\right|^{2s} \mathrm{d}\alpha_{k-1}\mathrm{d}\alpha_{k-2}\ldots\mathrm{d}\alpha_1.$$

Since the bound (47) is uniform in β (see the proof of Theorem 11.1 from [DGW20]), we can now define β in the range $[0,3/2]$ to be such that $H^\beta = \min\{c(N\lambda_4)^{-1}, H^{3/2}\}$ where c comes from the hypothesis $\lambda_4 \leq cN^{-1}$ with $c > 0$ being the implied constant. As required, we have $0 \leq \beta \leq 3/2$ because $1 = \lambda_4\lambda_4^{-1} \leq c(N\lambda_4)^{-1}$ and $1 = H^0$. Recall the value of $H = \left\lceil (A\lambda_4)^{-1/4} \right\rceil$, it follows that $H \leq (A\lambda_4)^{-1/4}$. Moreover, we have

$$(A\lambda_4)^{-1/4} = (AN^{-\varpi})^{-1/4} = N^{\varpi/4 - \frac{\log A}{4\log N}}$$

because $A = N^{\log A/\log N}$. Thus $H = N^{\varpi/4 + \mathcal{O}(1/\log N)}$ and therefore $N = H^{4/\varpi + \mathcal{O}(1/\log N)}$. Thus

$$c(N\lambda_4)^{-1} = cH^{-4/\varpi+\mathcal{O}(1/\log N)}N^{\varpi} = cH^{-4/\varpi+\mathcal{O}(1/\log N)}H^{4+\mathcal{O}(1/\log N)} = cH^{4(\varpi-1)/\varpi+\mathcal{O}(1/\log N)}.$$

We hence choose the value of β to be

$$\beta = \min\left\{\frac{4(\varpi-1)}{\varpi}, \frac{3}{2}\right\}. \tag{49}$$

Heath-Brown defines β to be $\min\left\{\frac{4(\varpi-1)}{\varpi}, \frac{3}{2}\right\} + \mathcal{O}(1/\log N)$. However, using that definition, as Julia Brandes noted it, β could be bigger than $3/2$. Putting this altogether, we are now ready to bound Σ. By Theorem 3.10, for $s = 6-\beta$, we have

$$J \ll_\varepsilon H^{6-2\beta+\varepsilon} = H^{2s-6+\varepsilon}.$$

Moreover for $k = 4$, by Lemma 3.7 and using Lemma 2.4, the following holds

$$\mathcal{N} \ll_A (N + \lambda_4 N^2 + \lambda_4^{-1/2})\log N \ll_{A,\varepsilon} (N + \lambda_4 N^2 + \lambda_4^{-1/2})N^\varepsilon.$$

We now make use of the hypothesis $N^{-2} \ll \lambda_4 \ll N^{-1}$ to get $\mathcal{N} \ll_{A,\varepsilon} N^{1+\varepsilon}$ in this regime. We therefore conclude that

$$\sum_{n\leq N} e(f(n)) \ll_{A,\varepsilon} \lambda_4^{-1/4} + N^{1-1/s}N^{(1+\varepsilon)/(2s)}\left\{H^{-2s+6}H^{2s-6+\varepsilon}\right\}^{1/(2s)} \ll_{A,\varepsilon} \lambda_4^{-1/4} + N^{1-1/(2s)+\varepsilon} \tag{50}$$

using the definition of H. However

$$\frac{1}{2s} = \frac{1}{12-2\beta} = \min\left\{\frac{\varpi}{4\varpi+8}, \frac{1}{9}\right\}.$$

To conclude, apply Lemma 2.8 and use the identification $\lambda_4 = N^{-\varpi}$. It gives

$$\sum_{n\leq N} e(f(n)) \ll_{A,\varepsilon} N^{\varpi/4} + N^{1-\varpi/(4\varpi+8)+\varepsilon} + N^{8/9+\varepsilon}.$$

Notice one can drop the term $N^{\varpi/4}$ using the same procedure as in (20) and (21) in the proof of Theorem 2.2 using the additional hypothesis $N^{-2} \ll N^{-\varpi} \ll N^{-1}$. □

3.3 Some new improvements

We follow the trend set by Heath-Brown in the appendix of [DGW20]: each improvement is based on the derivation of a new bound for $J_{s,k}$ in various settings. The proof is thus exactly the same as for Theorem 3.9, except that the estimate of Theorem 3.10 is replaced by Theorem 3.11 for $\beta \in [0,2]$ and by Theorem

3.13 for $\beta \geq 0$. After such slight improvements, we consider general conjectures by Demeter-Guth-Wang [DGW20] and Brandes (private communication) to obtain *conjecturally* new bounds.

We first introduce a general procedure in order to get new bounds in a rather general setting. This procedure being implicit and mostly non-constructive, we then restrict and study the case of an explicit linear fractional improvement (Proposition 3.12). There are no conceptual difficulties to extend Proposition 3.12 to broader families of functions. We restrict to $k = 4$, see Proposition 3.16 for arbitrary values of k. Let $N \in \mathbf{N}$, $\beta > 0$, $s \geq 1$ and I be an interval of length $1/N^{\beta}$. Suppose given a function $B(N, s, \beta, \varepsilon, I)$ of the form $N^{\varepsilon} \sum_j N^{\sigma_j(s,\beta)}$ for an adequate finite family of functions σ_j verifying

$$\int_{[0,1]^2\times I} \left| \sum_{k=1}^{N} e(kx_1 + k^2x_2 + k^3x_3) \right|^{2s} \mathrm{d}\boldsymbol{x} \ll_{\varepsilon} B(N, s, \beta, \varepsilon, I).$$

and such that there exists $(s^*, \beta^*) \in \mathbf{R}_{\geq 1} \times \mathbf{R}_+^*$ a solution of $\sigma_{j_1}(s^*, \beta^*) = \sigma_{j_2}(s^*, \beta^*)$ for every j_1 and j_2. If the hypotheses of the implicit function theorem hold, there exists a function φ such that $s^* = \varphi(\beta^*)$. Denote by $\sigma^*(\beta^*)$ the quantity $\sigma_j(\varphi(\beta^*), \beta^*)$ for any j. Then

$$\sum_{n\leq N} e(f(n)) \ll_{A,\varepsilon} \lambda_4^{-1/4} + N^{\sigma^*(\beta^*)/(2s)-2/(3s)+\varepsilon}\left(N^{1/(2s)} + \lambda_4^{1/(2s)}N^{1/s} + \lambda_4^{-1/(4s)}\right). \tag{51}$$

In order to prove (51), one must adapt Lemma 3.2 to show it is sufficient to consider only the critical case $\sigma^*(\beta^*)$. From (48), we find that

$$\sum_{n\leq N} e(f(n)) \ll_{A,\varepsilon} H + N^{1-1/s}\mathcal{N}^{1/(2s)}\left\{N^{-2s+6}B(N, s, \beta, \varepsilon, I)\right\}^{1/(2s)}.$$

Notice that $B(N, s, \beta, \varepsilon, I) \ll N^{\varepsilon}N^{\sigma^*(\beta^*)}$ by criticality. Then, by Lemmas 2.4, 2.6 and 3.7, we have

$$\begin{aligned}\sum_{n\leq N} e(f(n)) &\ll_{A,\varepsilon} H + N^{1-1/s}\left\{\left(N + \lambda_4N^2 + \lambda_4^{-1/2}\right)N^{\varepsilon}\right\}^{1/(2s)} N^{-1+1/(3s)}N^{\sigma^*(\beta^*)/(2s)+\varepsilon} \\ &\ll_{A,\varepsilon} \lambda_4^{-1/4} + N^{\sigma^*(\beta^*)/(2s)-2/(3s)+\varepsilon}\left\{N + \lambda_4N^2 + \lambda_4^{-1/2}\right\}^{1/(2s)} \\ &\ll_{A,\varepsilon} \lambda_4^{-1/4} + N^{\sigma^*(\beta^*)/(2s)-2/(3s)+\varepsilon}\left(N^{1/(2s)} + \lambda_4^{1/(2s)}N^{1/s} + \lambda_4^{-1/(4s)}\right)\end{aligned}$$

With no explicit relations between $\sigma^*(\beta^*)$ and s, one cannot go any further. The next step would be to use relation between these two quantities to optimise the result as we did in (49).

Note that, most of the time, Lemma 3.7 is not taken in its full form: only the regime $N^{-2} \ll \lambda_k \ll N^{-1}$ is considered in order to have a neat bound on $\mathcal{N}$. Using the notation of Heath-Brown as in Theorem 3.9, (51) becomes

$$\sum_{n\leq N} e(f(n)) \ll_{A,\varepsilon} N^{\varpi/4} + N^{\sigma^*(\beta^*)/(2s)-2/(3s)+\varepsilon}\left(N^{1/(2s)} + N^{1/s-\varpi/(2s)} + N^{\varpi/(4s)}\right).$$

We now consider the procedure for σ_j being essentially linear or linear fractional functions.

3.3.1 The $k = 4$, $\beta \geq 0$ case.

To improve Theorem 3.9, we consider this new estimate of $J_{s,3}$.

Theorem 3.11 (2 from [GM24]). *Let $N \in \mathbf{N}$, $0 \le \beta \le 2$, $s \ge 1$ and I be an interval of length $1/N^{\beta}$, we have*

$$J = \int_{[0,1]^2 \times I} \left| \sum_{k=1}^{N} a_k e(kx_1 + k^2 x_2 + k^3 x_3) \right|^{2s} \mathrm{d}\boldsymbol{x} \ll_\varepsilon N^\varepsilon \left(N^{s-\beta} + N^{2s-6} \right) \tag{52}$$

for any $a_k \in \mathbf{C}$ satisfying $|a_k| \ll 1$.

Note that this is not exactly $J_{s,3}$ since the domain of integration is a bit more general. To conclude, we proceed as before, we redefine β and then use Theorem 3.11. As a direct consequence of the procedure (51) and more precisely its Proposition 3.12, this allows to prove this slight improvement.

Proposition 3.12. *Let $f(x) : [0, N] \to \mathbf{R}$ have a continuous fourth derivative on $(0, N)$ with $0 < \lambda_4 \le f^{(4)}(x) \le A\lambda_4$ for some constant $A \ge 1$. Write $\lambda_4 = N^{-\varpi}$ and suppose that $N^{-2} \ll \lambda_4 \ll N^{-1}$. Then, for any fixed $\varepsilon > 0$, we have*

$$\sum_{n \le N} e(f(n)) \ll_{A,\varepsilon} N^{1-\varpi/(4\varpi+8)+\varepsilon} + N^{7/8+\varepsilon}.$$

Proof. Due to Theorem 3.11, we may now redefine β of (49) as

$$\beta = \min\left\{ \frac{4(\varpi - 1)}{\varpi}, 2 \right\}.$$

Fix $k = 4$, from (48) in Theorem 3.9, we have

$$\sum_{n \le N} e(f(n)) \ll_{A,\varepsilon} H + N^{1-1/s} \mathcal{N}^{1/(2s)} \left\{ H^{-2s+6} J \right\}^{1/(2s)} \ll_{A,\varepsilon} N^{\varpi/4} + N^{1-1/(2s)+\varepsilon} \left\{ H^{-2s+6} J \right\}^{1/(2s)}.$$

By virtue of Theorem 3.11, one gets

$$\left\{ H^{-2s+6} J \right\}^{1/(2s)} \ll_\varepsilon \left\{ N^{-2s+6} N^\varepsilon \left(N^{s-\beta} + N^{2s-6} \right) \right\}^{1/(2s)} = \left\{ N^\varepsilon (N^{6-\beta-s} + 1) \right\}^{1/(2s)}$$

remembering that $H \le N$, otherwise the bound would be trivial (actually, one might not even use the inequality $H \le N$ and stay in the H world, but it is a fact one should not forget about). The critical case occurs when $s = 6 - \beta$, thus

$$\sum_{n \le N} e(f(n)) \ll_{A,\varepsilon} N^{\varpi/4} + N^{1-1/(2s)+\varepsilon} \tag{53}$$

giving the same result as (50). One then concludes the same way, we find

$$\frac{1}{2s} = \frac{1}{12 - 2\beta} = \min\left\{ \frac{\varpi}{4\varpi + 8}, \frac{1}{8} \right\}.$$

Hence, by Lemma 2.8

$$\sum_{n \le N} e(f(n)) \ll_{A,\varepsilon} N^{\varpi/4} + N^{1-\varpi/(4\varpi+8)+\varepsilon} + N^{7/8+\varepsilon}.$$

As we have done in the proof of Theorem 3.9, the term $N^{\varpi/4}$ can be dropped. This concludes the proof. □

This proposition can be slightly improved using Theorem 1.1 from [MO24].

Theorem 3.13. *Let $N \in \mathbf{N}$, $\beta \ge 0$, $s \ge 1$, we have*

$$\left(\int_{[0,1]^2 \times [0, 1/N^\beta]} \left| \sum_{k=1}^{N} a_k e(kx_1 + k^2 x_2 + k^3 x_3) \right|^{2s} \mathrm{d}\boldsymbol{x} \right)^{1/(2s)} \ll_\varepsilon N^\varepsilon D_{2s}(\beta, N) \left(\sum_{k=1}^{N} |a_k|^{2s} \right)^{1/(2s)}$$

for any $a_k \in \mathbf{C}$, in which

$$D_{2s}(\beta, N) = \begin{cases} N^{1/2-(\beta+1)/(2s)} + N^{1-7/(2s)} & \text{if } 0 \le \beta < 5/2, \\ N^{1/2-(\beta+1)/(2s)} + N^{\beta/3-7\beta/(6s)} + N^{1-7/(2s)} & \text{if } 5/2 \le \beta < 3, \\ N^{1/2-\beta/(2s)} + N^{1-(\beta+3)/(2s)} & \text{if } 3 \le \beta. \end{cases} \tag{54}$$

To get a more familiar estimate, we consider the case $a_k = 1$, for all $k = 1, \dots, N$ and raise the expression to the power $2s$. We find

$$J = \int_{[0,1]^2\times[0,1/N^\beta]} \left|\sum_{k=1}^{N} e(kx_1 + k^2x_2 + k^3x_3)\right|^{2s} \mathrm{d}\boldsymbol{x} \ll_\varepsilon N^{1+\varepsilon} D_{2s}^{2s}(\beta, N). \tag{55}$$

The most interesting case appears to be $0 \le \beta \le 5/2$. Proceeding exactly as in the proof of Proposition 3.12, one defines

$$\beta = \min\left\{\frac{4(\varpi-1)}{\varpi}, 5/2\right\}.$$

and by the exact same considerations: note that the critical case appears again to be $s = 6 - \beta$, thus (53) is unchanged since

$$\begin{aligned}\left\{H^{-2s+6}J\right\}^{1/(2s)} &\ll_\varepsilon \left\{H^{-2s+6}H^{1+\varepsilon}D_{2s}^{2s}(\beta, H)\right\}^{1/(2s)} = \left\{H^{-2s+6}H^{1+\varepsilon}\left(H^{1/2-(\beta+1)/(2s)} + H^{1-7/(2s)}\right)^{2s}\right\}^{1/(2s)} \\ &\ll_\varepsilon \left\{H^{-2s+6}H^{1+\varepsilon}H^{2s-7}\right\}^{1/(2s)} \ll H^\varepsilon\end{aligned}$$

where we used the bound (55) from Theorem 3.13 and criticality. Using these facts, one then shows the following proposition.

Proposition 3.14. *Suppose the same assumptions as in Proposition 3.12 hold. Then*

$$\sum_{n\le N} e(f(n)) \ll_{A,\varepsilon} N^{1-\varpi/(4\varpi+8)+\varepsilon} + N^{6/7+\varepsilon}.$$

Remark that as soon as $\varpi > 2$ i.e. $\lambda_4 < N^{-2}$, one finds that $N^{1-\varpi/(4\varpi+8)+\varepsilon}$ is negligible in front of $N^{\varpi/4}$. We have thus replaced the exponent $8/9$ of Theorem 3.9 by $7/8$ and then by $6/7$. A global improvement of nearly 0.0317. Note that, for $\beta \in [5/2, 3)$ in (54), one can not bound $D_{2s}(\beta, N)$ using only the method of Heath Brown since there is no critical case. The remaining case ($\beta \ge 3$) is not considered since the bound is worst than (52) from Theorem 3.11.

To conclude this section, it shall be noted that not all Vinogradov mean value type bounds are able to sharpen Theorem 3.9. For instance, Oh and Yeon [OY25] obtain

$$\int_{[0,1]\times[0,N^{-\beta}]\times[0,1]} \left|\sum_{k=1}^{N} e(kx_1 + k^2x_2 + k^3x_3)\right|^{2s} \mathrm{d}\boldsymbol{x} \ll N^\varepsilon\left(N^{2s-6} + N^{s-\beta}\right) \tag{56}$$

for $0 < \beta \le 1$ and $s > 0$. The critical exponent is again $s = 6 - \beta$ and one thus finds again

$$\begin{aligned}\sum_{n\le N} e(f(n)) &\ll_{A,\varepsilon} H + N^{1-1/s}\mathcal{N}^{1/(2s)}\left\{H^{-2s+6}H^\varepsilon\left(H^{2s-6} + H^{s-\beta}\right)\right\}^{1/(2s)} \ll_{A,\varepsilon} H + N^{1-1/s+\varepsilon}\mathcal{N}^{1/(2s)}\left\{H^{2\beta-6}H^{6-2\beta}\right\}^{1/(2s)} \\ &\ll_{A,\varepsilon} H + N^{1-1/s}\mathcal{N}^{1/(2s)}.\end{aligned}$$

The quantity β is then defined as usual in order to have

$$\frac{1}{2s} = \min\left\{\frac{\varpi}{4\varpi+8}, \frac{1}{10}\right\}$$

therefore, in the same regime as usual, one finds

$$\sum_{n\le N} e(f(n)) \ll_{A,\varepsilon} N^{1-\varpi/(4\varpi+8)+\varepsilon} + N^{9/10+\varepsilon} \tag{57}$$

which is worse than what was previously found. It would be interesting to have an improved version of (56) for larger β in order to fully compare it with Propositions 3.12 and 3.14. Nevertheless, even if it does not sharpen Theorem 3.9, this result is interesting since it might set on the way to think that the substantial contribution comes from the dominant term (as it may be the case for Gauss sums or for Weyl sums, see Section 1.3 for examples and, more precisely, see Proposition 1.7).

3.3.2 Conjecturally, the general case.

When k is arbitrary (not necessarily equal to 4) and the regime is not necessarily chosen in order to get a nice estimate of $\mathcal{N}$ (see page 43), we can obtain a conjectural bound (see Proposition 3.16) using Conjecture 2.5 from [DGW20]. See [Dem20, Conjecture 13.6] for a natural generalisation of Conjecture 3.15 over a slightly different domain.

Conjecture 3.15. *Let $N \in \mathbf{N}$, $\varepsilon > 0$. For each $k \geq 2$, $0 \leq \beta \leq k-1$ and $s \geq 1$, we have*

$$\int_{[0,1]^{k-1}\times[0,1/N^\beta]} \left| \sum_{\ell=1}^{N} e(\ell x_1 + \ell^2 x_2 + \cdots + \ell^k x_k) \right|^{2s} \mathrm{d}\boldsymbol{x} \ll_\varepsilon N^\varepsilon \Big(N^{s-\beta} + N^{2s-k(k+1)/2} \Big).$$

From which we get the following estimate.

Proposition 3.16. *Let $f(x) : [0,N] \to \mathbf{R}$ have a continuous k-th derivative on $(0,N)$ with $0 < \lambda_k \leq f^{(k)}(x) \leq A\lambda_k$ for some constant $A \geq 1$. Write $\lambda_k = N^{-\varpi_k}$. Then, using notations from Conjecture 3.15, we have*

$$\sum_{n \leq N} e(f(n)) \ll_{A,\varepsilon} N^{\varpi_k/k} + k^2((k-1)!)^{2/(k(k-1)/2-\beta)} N^{1+\varepsilon} \Big\{ N^{(1-\varpi_k)/(k(k-1)-2\beta)} + N^{(2\varpi_k/k-1)/(k(k-1)-2\beta)} \Big\}.$$

Proof. We follow the same procedure as before. Since Lemmas 3.5 and 3.7 are true for every $k \geq 2$, we have

$$\begin{aligned}\sum_{n \leq N} e(f(n)) &\ll H + k^2 N^{1-1/s} \mathcal{N}^{1/(2s)} \Big\{ H^{-2s+k(k-1)/2} J_{s,k-1}(H) \Big\}^{1/(2s)} \\ &\ll_{A,\varepsilon} H + k^2 N^{1-1/s+\varepsilon} \Big\{ ((k-1)!)^4 (N + \lambda_k N^2 + \lambda_k^{-2/k}) \log N \cdot H^{-2s+k(k-1)/2} \Big(H^{s-\beta} + H^{2s-k(k-1)/2} \Big) \Big\}^{1/(2s)}\end{aligned}$$

using Conjecture 3.15. The critical case is $s := s_{k-1} = k(k-1)/2 - \beta$. By criticality and since $\log N \leq N$, it comes

$$\begin{aligned}\sum_{n \leq N} e(f(n)) &\ll_{A,\varepsilon} H + k^2 N^{1-1/s+\varepsilon} \Big\{ ((k-1)!)^4 (N + \lambda_k N^2 + \lambda_k^{-2/k}) \log N \cdot H^{2\beta-k(k-1)/2} H^{k(k-1)/2-2\beta} \Big\}^{1/(2s)} \\ &\ll_{A,\varepsilon} H + k^2 N^{1-1/(2s)+\varepsilon} \Big\{ ((k-1)!)^4 (N + \lambda_k N^2 + \lambda_k^{-2/k}) \Big\}^{1/(2s)}.\end{aligned}$$

With no condition on $\lambda_k = N^{-\varpi_k}$, one cannot use the β strategy. However, using Lemma 2.6, one finds

$$\sum_{n \leq N} e(f(n)) \ll_{A,\varepsilon} N^{\varpi_k/k} + k^2((k-1)!)^{2/s} N^{1+\varepsilon} \Big\{ N^{(1-\varpi_k)/(2s)} + N^{(2\varpi_k/k-1)/(2s)} \Big\}. \tag{58}$$

Using the Lambert W special function, one could study the minimum value of the right hand side of (58) as a function of s and thus optimizing for β. □

For the sake of completion, Conjecture 3.15 can be generalised in at least two directions. We present both directions and then explain how to connect them.

Conjecture 3.17 (Julia Brandes, private communication)**.** *For $1 \leq j \leq k$, let $\delta_j > 0$. We have*

$$\begin{aligned}\int_{[0,\delta_1]\times\cdots\times[0,\delta_k]} \left| \sum_{\ell=1}^{N} e(\ell x_1 + \ell^2 x_2 + \cdots + \ell^k x_k) \right|^{2s} \mathrm{d}\boldsymbol{x} &\ll \delta_1 \ldots \delta_k N^\varepsilon \Big(N^s + N^{2s-k(k+1)/2} (\delta_1 \ldots \delta_k)^{-1} \Big) \\ &\quad + \delta_1 \ldots \delta_k N^\varepsilon \Bigg(\sum_j \delta_{i(j)}^{-\frac{2s-i(1)-\cdots-i(j)}{i(j)}} \big(\delta_{i(1)} \ldots \delta_{i(j)} \big)^{-1} \Bigg)\end{aligned} \tag{59}$$

where $i(1), \ldots, i(k)$ are such that $\delta_{i(1)}^{-1/i(1)} \leq \delta_{i(2)}^{-1/i(2)} \leq \cdots \leq \delta_{i(k)}^{-1/i(k)}$.

Consider the usual case, i.e. $\delta_1 = \cdots = \delta_{k-1} = 1$ and $\delta_k = N^{-\beta}$. We then have

$$1 = \delta_1^{-1} \leq \delta_2^{-1/2} \leq \cdots \leq \delta_k^{-1/k}.$$

It follows that the right hand side of (59) is

$$N^{\varepsilon}\Big(N^{s-\beta} + N^{2s-k(k+1)/2}\Big) + \delta_k N^{\varepsilon}\Big(1 + \cdots + 1 + \delta_k^{-\frac{2s-1-\cdots-k}{k}}\delta_k^{-1}\Big) \ll N^{\varepsilon}\Big(N^{s-\beta} + N^{2s-k(k+1)/2} + N^{2s\beta/k-\beta(k+1)/2}\Big)$$

Assuming $\beta < k$, one would recover the exact same result as in Conjecture 3.15 using the hypothesis $\beta < k-1 < k$. Indeed, the following inequality $2s\beta/k - \beta(k+1)/2 \leq \max\{s-\beta, 2s-k(k+1)/2\}$ holds for every $s \geq k(k+1)/4$. For comparison, Conjecture 3.17 is more general than the following conjecture because it takes into consideration directions and arithmetic-interest information.

Conjecture 3.18 ([Woo22, Conjecture 8.2]). *Suppose that $k \in \mathbf{N}$ and $\mathfrak{B} \subseteq [0,1)^k$ is measurable. Then whenever s is a positive number and $N^{1-k(k+1)/4} \ll \mathrm{mes}(\mathfrak{B})$, one has*

$$\int_{\mathfrak{B}} \left| \sum_{\ell=1}^{N} e(\ell x_1 + \ell^2 x_2 + \cdots + \ell^k x_k) \right|^{2s} \mathrm{d}\boldsymbol{x} \ll N^{\varepsilon}\Big(N^s \mathrm{mes}(\mathfrak{B}) + X^{2s-k(k+1)/2}\Big).$$

It is immediate that Conjecture 3.18 implies Conjecture 3.15. Technically, Conjecture 3.17 does not imply Conjecture 3.18 since $\mathfrak{B}$ is amorphous and the domain considered in Conjecture 3.17 is constituted by rectangular boxes. Nevertheless, for $\mathfrak{B}$ sufficiently regular and a shifted version of Conjecture 3.17 (i.e. intervals not necessarily starting at 0), one could decompose and reconstruct $\mathfrak{B}$ using enough rectangular boxes. It would then *suffices* to use the linearity of the integral to get a concoction of those two conjectures. Finally, note that Conjecture 3.18 can be stated with more generality: for s a real number, see [Woo22, Conjecture 8.1]. The case $s \geq k(k+1)/4 + 1$ is sufficient to recover the whole real range. To do so, employ the same strategy as in the proof of Lemma 3.2. Moreover, Conjecture 3.18 is not sharp enough since it does not permit to make the difference between a contribution coming from x_{k-1} or x_k. We previously saw that the difference can be notable, compare for instance Theorem 3.9 and how (56) gives (57).

3.3.3 Growth rate of the Riemann zeta function on vertical lines

In this section, we provide a detailed analysis of the arguments in [DGW20, Appendix, proof of Theorem 11.2] and we show that some direct generalizations are possible, see Proposition 3.20. This section will be rewritten and generalized in the incoming months. Recall from Section 1.2 the Lindelöf hypothesis, i.e. to determine

$$\mu(\sigma) = \inf\{\xi \in \mathbf{R}_+ \;:\; \zeta(\sigma + it) = \mathcal{O}(|t|^{\xi})\}.$$

Using the theory of exponential sums, especially Theorem 3.9, Heath Brown was able to prove that $\mu(11/15) \leq 1/15$ [DGW20, Theorem 11.2]. Any substantial improvement of that result might require refinements of k-th derivative estimate for $k \geq 4$. Using the improvements in the $k = 4$ case (see Proposition 3.14), we try to optimise the Heath Brown method. We do not survey the theory of subconvexity bounds, see for instance [Kum11], especially Section 4.1.1.

Let σ_0 be a real number in $[1/2, 1)$ and let t be a positive real number. By the following lemma, we reduce the problem to a bound on exponential sums.

Lemma 3.19. *Let $\alpha, \varepsilon > 0$, $s = \sigma_0 + it$ and N an integer. Suppose the following bound holds*

$$\sum_{N < n \leq 2N} n^{it} \ll_{\varepsilon} N^{\sigma_0} t^{\alpha+\varepsilon} \tag{60}$$

then $\zeta(s) \ll_{\varepsilon} t^{\alpha+\varepsilon}$ for t sufficiently large.

Proof. Let $N_j = 2^j$ and $\sigma \in [1/2, 1)$. Truncate the Riemann zeta function up to order $T \in \mathbf{N}$ and use a dyadic decomposition as in [Tao15] and [Tao16] to get

$$\sum_{n \le T} \frac{1}{n^\sigma} e^{-it\log n} = \sum_{0 \le j \le \log_2 T} \sum_{N_j < n \le 2N_j} \frac{1}{n^\sigma} e^{-it\log n}.$$

Using Lemma 2.3, trivial bounds and the hypothesis, one finds

$$\sum_{N_j < n \le 2N_j} \frac{1}{n^\sigma} e^{-it\log n} \ll_\varepsilon N_j^{-\sigma} \left| \sum_{N_j < n \le 2N_j} n^{it} \right| \ll_\varepsilon N_j^{\sigma_0 - \sigma} t^{\alpha+\varepsilon}.$$

Let $s = \sigma + it$. We now combine this altogether in the approximate functional equation of the Riemann zeta function [TH86, Section 4.17]

$$\zeta(s) = \sum_{n \le t^{1/2}} \frac{1}{n^s} + \chi(s) \sum_{n \le t^{1/2}} \frac{1}{n^{1-s}} + R(s)$$

with $R(s)$ the error term and $\chi(s)$ is defined in [TH86, Equation 4.12.3]. The dual sum can be controlled. For $2N \le t^{1/2}$, i.e. t sufficiently large, we have $(N, 2N] \subset [1, t^{1/2}]$. It then follows that, for $\sigma = \sigma_0$, we have

$$\zeta(s) \ll_\varepsilon \sum_{0 \le j \le \log_2 T} N_j^{\sigma_0 - \sigma} t^{\alpha+\varepsilon} = t^{\alpha+\varepsilon}(1 + \log_2 T) \ll_\varepsilon t^{\alpha+\varepsilon} \log t \ll_\varepsilon t^{\alpha+\varepsilon}$$

by Lemma 2.4. This concludes the proof. □

The Heath Brown setting [DGW20, Appendix] is $\sigma_0 = 11/15$ and $\alpha = 1/15$. The author is wondering if one can break or bypass the $t^{1/2}$ barrier. Nevertheless, using Lemma 3.19, one is able to prove subconvexity bounds. The method is the following: study the bounds given by the van der Corput estimates and find the optimal ranges to apply them. For instance, Heath Brown proves Theorem 1.6 by subdividing $[0, 1/2]$ in five different intervals ($[0, 1/4]$, $[1/4, 1/3]$, $[1/3, 5/12]$, $[5/12, 3/7]$ and $[3/7, 1/2]$). This subdivision is a consequence of the approximate functional equation. Indeed, one needs to prove the bound holds for $1 = t^0 \le N \le t^{1/2}$. He then applies in each case the appropriate estimate in order to get $\sum_{N<n\le 2N} n^{it} \ll_\varepsilon N^{11/15} t^{1/15+\varepsilon}$. The theorem then follows by Lemma 3.19. See the Analytic Number Theory Exponent Database ([TTY25, Table 6.1]) for the newest results. In June 2025, Heath Brown's improvements are the latest ones.

We generalize Heath Brown's method in order to determine a bound on $\mu(\sigma_0)$ for various σ_0. In Corollary 3.23, we slightly improve Heath Brown's previous result (i.e. the case $\sigma_0 = 11/15$).

Proposition 3.20. *Let $\alpha, \varepsilon > 0$, $s = \sigma_0 + it$ and N an integer. Let γ be a real number such that $\gamma > \max(6/7, \sigma_0)$ and k be an integer. The bound from (60) holds in the following ranges:*

- *it holds trivially for $1 \le N \le t^{\alpha/(1-\sigma_0)}$ with no restrictions,*
- *by various fourth derivative estimates, it holds for $t^{\alpha/(1-\sigma_0)} \le N \le t^{1/3}$ with the constraint on α and σ being such that $11/36 \le \sigma_0/3 + \alpha$,*
- *by a generalization of Heath Brown's improvements, it holds for $t^{1/3} \le N \le t^{(4\gamma-3)/(24\gamma-20)}$,*
- *by improvements from Section 3.3, it holds for $t^{(4\gamma-3)/(24\gamma-20)} \ll N \le t^{\alpha/(\gamma-\sigma_0)}$ and one can choose γ such that*

  $$t^{\alpha/(\gamma-\sigma_0)} = t^{(1/(2^k-2)-\alpha)/(\sigma_0+k/(2^k-2)-1)},$$

- *by the classical k-th derivative estimate, it holds for $t^{(1/(2^k-2)-\alpha)/(\sigma_0+k/(2^k-2)-1)} \le N \le t^{1/2}$.*

Note that, sometimes, the ranges may overlap. Moreover, the condition $\gamma > \sigma_0$ can be changed and one could consider $\gamma < \sigma_0$. The only crucial condition is to have $\gamma \neq \sigma_0$. It wasn't made explicit but γ has some extra constraints in order to respect the order of the boundaries (for instance, to have $1/3 \leq (4\gamma - 3)/(24\gamma - 20)$ and not the opposite).

Proof. One needs to prove the proposition holds for the five different intervals. All along the proof, in the end of an argument, we might specialize to specific values of α and σ_0 in order to confirm the generalization gives the same result as Heath Brown's one.

- In $[0, \alpha/(1-\sigma_0)]$. The bound (60) holds trivially for $N \leq t^{\alpha/(1-\sigma_0)}$. The lemma follows by direct computations. In order to do so, one must find a a positive real number such that $N = t^a$ implies $N \leq N^{\sigma_0} t^{\alpha+\varepsilon}$. The Heath Brown setting corresponds to $\alpha/(1-\sigma_0) = 1/4$. One now concentrate on the remaining range, i.e. $t^{\alpha/(1-\sigma_0)} \leq N \leq t^{1/2}$.

- In $[\alpha/(1-\sigma_0), 1/3]$. Let α and σ be such that $11/36 \leq \sigma_0/3 + \alpha$. Following Heath Brown [DGW20, page 62], using the usual notations, we have

$$\sum_{N<n\leq 2N} n^{it} \ll N^{1+\varepsilon-\varpi/12} = N^{1+\varepsilon}\lambda_4^{1/12} \ll N^{1+\varepsilon}N^{-4/12}t^{1/12} = N^{2/3+\varepsilon}t^{1/12} \leq N^{\varepsilon}t^{11/36}$$

  since $N \leq t^{1/3}$. Furthermore, note that $N^{\sigma_0}t^{\alpha} \leq t^{\sigma_0/3+\alpha}$. Then, up to a multiplicative constant non-uniform in α and σ_0, we have $N^{2/3+\varepsilon}t^{1/12} \ll N^{\sigma_0+\varepsilon}t^{\alpha}$. Hopefully, in Corollary 3.23 we won't have to compute this multiplicative constant since the inequality will in fact be an equality.

- In $[1/3, (4\gamma-3)/(24\gamma-20)]$. We prove the bound (60) holds in the range $t^{1/3} \leq N \leq t^{(4\gamma-3)/(24\gamma-20)}$. For such N, we find that $N^{(8\gamma-8)/(4\gamma-3)} \ll \lambda_4 \ll N^{-1}$ which is precisely the range from (62). In this case, it follows that

$$\sum_{N<n\leq 2N} n^{it} \ll_{\varepsilon} N^{1-\varpi/(4\varpi+8)+\varepsilon}.$$

  Write $t = N^{\tau}$, we thus have $(24\gamma-20)/(4\gamma-3) \leq \tau \leq 3$. Moreover, since $\lambda_4 = N^{-\varpi} \ll tN^{-4} = N^{\tau-4}$, we find that $\varpi = 4 - \tau + \mathcal{O}(1/\log N)$. We now need to find out in which circumstances the bound (60) holds. It suffices to determine for which α and σ_0 the following inequality holds

$$1 - \frac{\varpi}{4\varpi+8} = 1 - \frac{4-\tau}{24-4\tau} \leq \sigma_0 + \alpha\tau$$

  whenever $(24\gamma-20)/(4\gamma-3) \leq \tau \leq 3$, up to a multiplicative constant (which is absorbed by Vinogradov's notation). Many cases are possible:

  - if $6 < \tau$, then

$$P(\tau) := 4\alpha\tau^2 + \tau(4\sigma_0 - 24\alpha - 8) - 24(\sigma_0 - 1) \geq 0. \tag{61}$$

    We compute the discriminant of the polynomial P and get

$$\Delta = (4\sigma_0 - 24\alpha - 8)^2 - 384\alpha(\sigma_0 - 1) = 64 + 576\alpha^2 - 64\sigma_0 + 192\alpha\sigma_0 + 16\sigma_0^2.$$

    Three cases are now possible.

    * $\Delta = 0$. The set of solutions $\{(\alpha, \sigma_0)\}$ is a parabola. In our setting, we only consider $\alpha > 0$ and thus get solutions of the form $(\alpha, 2(1 \pm i\sqrt{6\alpha} - 3\alpha))$. Impossible, solutions can not be complex.

* $\Delta \geq 0$. This is equivalent to $(6\alpha+\sigma_0)^2+4 \geq 4\sigma_0$. Using that a^2+b^2 is always greater than $2ab$, the inequality is always true for $\alpha > 0$. Since $4\alpha > 0$, we have that (61) is verified for $\tau \in (-\infty, \tau_0] \cup [\tau_1, +\infty)$ with τ_0 and τ_1 the roots of P given by

$$\tau_{0,1} = \frac{-(\sigma_0-6\alpha-2) \pm \sqrt{(\sigma_0-6\alpha-2)^2-24\alpha(\sigma_0-1)}}{2\alpha}.$$

* $\Delta \leq 0$. Since $\alpha > 0$, using a combination of arguments from the two previous cases, there are no real positive solutions $\{(\alpha, \sigma_0)\}$.

– otherwise, if $6 > \tau$, we have $4\alpha\tau^2+\tau(4\sigma_0-24\alpha-8)-24(\sigma_0-1) \leq 0$ and we proceed the same way.

Note that if $6 < \tau$, then $N < t^{1/6}$ by definition. Unfortunately, as in the Heath Brown setting, this case is already covered by the trivial estimate (see the first case). Indeed, $1/6 \leq \alpha/(1-\sigma_0)$ implies $1-\sigma_0 \leq 6\alpha$. When it is not the case, i.e. $6\alpha \leq 1-\sigma_0$, one can thus expect some new improvements. The other case ($6 > \tau$) is even more interesting because in that setting we consider ranges such that $t^{1/6} < N$.

- In $[(4\gamma-3)/(24\gamma-20), \alpha/(\gamma-\sigma_0)]$. We now see what we can get from the improvement of Proposition 3.14. For a technical reason explained hereafter, consider this weaker bound

$$\sum_{n \leq N} e(f(n)) \ll_{A,\varepsilon} N^{1-\varpi/(4\varpi+8)+\varepsilon} + N^{\gamma+\varepsilon}$$

for $\gamma \geq 6/7$. Recall we are in the regime $N^{-2} \ll \lambda_4 = N^{-\varpi} \ll N^{-1}$ and λ_4 is of the same order as tN^{-4}. We want to find which term dominates depending on a subrange (to be found). We have

$$\sum_{n \leq N} e(f(n)) \ll_{A,\varepsilon} \begin{cases} N^{1-\varpi/(4\varpi+8)+\varepsilon}, & N^{(8\gamma-8)/(4\gamma-3)} \ll \lambda_4 \ll N^{-1}, \\ N^{\gamma+\varepsilon}, & N^{-2} \ll \lambda_4 \ll N^{(8\gamma-8)/(4\gamma-3)}. \end{cases} \tag{62}$$

When $\gamma = 8/9$ (see Theorem 3.9) we precisely find back [DGW20, Appendix, Equation 71]. By the hypotheses, in the range where $N^{\gamma+\varepsilon}$ is dominant, we have $tN^{-4} \ll \lambda_4 \ll N^{(8\gamma-8)/(4\gamma-3)}$. Then $t^{(4\gamma-3)/(24\gamma-20)} \ll N$. Notice that $(4\gamma-3)/(24\gamma-20) \leq 1/2$ for $\gamma \geq 7/8$ or $\gamma < 5/6$. In the latter case, we don't have such good estimates for the moment. Being less than $1/2$ is a necessary condition due to the technical hypothesis in the approximate functional equation of the Riemann zeta function, see the proof of Lemma 3.19.

For clarity, suppose $\gamma > \sigma_0$ as in Heath Brown setting (there would be no problem to consider that $\gamma < \sigma_0$). If $N \leq t^{\alpha/(\gamma-\sigma_0)}$, then $N^{\gamma+\varepsilon} \leq N^{\sigma_0}t^{\alpha+\varepsilon}$, meaning that for $t^{(4\gamma-3)/(24\gamma-20)} \ll N \leq t^{\alpha/(\gamma-\sigma_0)}$, (60) is verified and then in that range $\zeta(s) \ll_\varepsilon t^{\alpha+\varepsilon}$ for t sufficiently large. With the usual values, one finds back the exact same result as in Heath Brown setting. One wants to choose γ such that

$$t^{\alpha/(\gamma-\sigma_0)} = t^{(1/(2^k-2)-\alpha)/(\sigma_0+k/(2^k-2)-1)}$$

under the constraint $\gamma \geq 6/7$ (and $\gamma > \sigma_0$). This is achieved for

$$\gamma = \frac{\alpha(k-2^k+2)+\sigma_0}{1-\alpha(2^k-2)}$$

when it is well-defined.

- In $[(1/(2^k-2)-\alpha)/(\sigma_0+k/(2^k-2)-1), 1/2]$. From the classical third derivative estimate (Theorem 2.17) applied at the indefinitely differentiable function $f : x \mapsto t \log x/(2\pi)$, one finds that

$$\sum_{N<n\leq 2N} n^{it} \ll_A N\lambda_3^{1/6} + N^{1/2}\lambda_3^{-1/6} \ll Nt^{1/6}N^{-1/2} + N^{1/2}t^{-1/6}N^{1/2} = N^{1/2}t^{1/6} + Nt^{-1/6}$$

since $\left|\partial_x^3 f(x)\right|$ is of order tx^{-3}, thus λ_3 is of order tN^{-3}. By direct computations (find b a positive real number such that $N = t^b$ implies $N^{1/2}t^{1/6}$ is of the same order as $N^{\sigma_0}t^{\alpha}$ noting that the term $Nt^{-1/6}$ is negligible since we are in a regime where t is a sufficiently large number and N is fixed), the third derivative estimate allows to prove (60) in the range $t^{(1/6-\alpha)/(\sigma_0-1/2)} \leq N \leq t^{1/2}$. Putting $\sigma_0 = 11/15$ and $\alpha = 1/15$, one finds the range $t^{3/7} \leq N \leq t^{1/2}$ as in Heath Brown setting. Using the classical van der Corput k-th derivative estimate (for $k \neq 3$) may make the range larger. From Theorem 2.1, one finds that

$$\sum_{N<n\leq 2N} n^{it} \ll_A N\lambda_k^{1/(2^k-2)} + N^{1-2^{2-k}}\lambda_k^{-1/(2^k-2)}.$$

By a direct induction, one gets that λ_k is of the same order as tN^{-k}. Thus

$$\begin{aligned}\sum_{N<n\leq 2N} n^{it} &\ll_A N\left(tN^{-k}\right)^{1/(2^k-2)} + N^{1-2^{2-k}}\left(tN^{-k}\right)^{-1/(2^k-2)} \\ &= N^{1-k/(2^k-2)}t^{1/(2^k-2)} + N^{1-2^{2-k}+k/(2^k-2)}t^{-1/(2^k-2)}.\end{aligned}$$

Again, for t sufficiently large, the last term of the right hand side is negligible. By the same reasoning as before, the k-th derivative estimate allows to prove (60) in the range

$$t^{(1/(2^k-2)-\alpha)/(\sigma_0+k/(2^k-2)-1)} \leq N \leq t^{1/2}.$$

This concludes the proof. □

Remark 3.21. Notice the following limit appearing in the last step of the proof of Proposition 3.20

$$\lim_{k\to\infty} \frac{1/(2^k-2)-\alpha}{\sigma_0+k/(2^k-2)-1} = \frac{\alpha}{1-\sigma_0}.$$

The limit is exactly the value of the exponent up to which the bound (60) is trivial (see the first case). This let think that using (possibly infinitely many) van der Corput estimates might increase a bit more known subconvexity bounds.

Remark 3.22. One should use Proposition 3.16 to improve step 2 from the previous proof and replace $\lambda_4^{1/12}$ by something of the form $\lambda_k^{1/(k(k-1))}$. Unfortunately, such improvement relies on Conjecture 3.15.

As a consequence, we obtain a slight improvement of Theorem 1.6.

Corollary 3.23. *For any fixed $\varepsilon > 0$, we have $\zeta(11/15+it) \ll_\varepsilon (|t|+1)^{11/180+\varepsilon}$. As a consequence, $\mu(11/15) \leq 11/180$.*

Proof. We apply Proposition 3.20 with $\alpha = 11/180$, $\sigma_0 = 11/15$ and $\gamma = 33/38 > 6/7 > 5/6$. We need to prove the statement holds for $1 \leq N \leq t^{1/2}$. By the trivial estimate, it holds for $1 \leq N \leq t^{11/48}$ (with $11/48 \approx 0.2291$). Using the generalized form of Heath Brown's improvement, it holds for $t^{1/3} \leq N \leq t^{1/2}$. It remains to prove it holds in the range $t^{11/48} \leq N \leq t^{1/3}$. This is done using the various fourth derivative estimates, the same as Heath Brown [DGW20, page 62] or step 2 in the previous proof. To apply it, one only needs to verify that $11/36 \leq \sigma_0/3 + \alpha$. In this case, this is even an equality. This concludes the proof. □

We remark that the improvements from Section 3.3 simplify the proof (in comparison with the one from Heath-Brown). Indeed, less intervals are needed to prove the statement.

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