Example 1

Self-interacting Klein-Gordon field on Minkowski metric

Tartalomjegyzék

[1. Example problem 2](#_Toc392767210)

[1.1. Analytic formulation 2](#_Toc392767211)

[1.1.1. Lagrangian and action 2](#_Toc392767212)

[1.1.2. Chosing of coordinates 2](#_Toc392767213)

[1.1.3. Euler-Lagrange 2](#_Toc392767214)

[1.1.4. Equation of motion 2](#_Toc392767215)

[1.1.5. Sommerfeld outgoing radiation boundary condition 3](#_Toc392767216)

[1.1.6. First order reduction 3](#_Toc392767217)

[1.1.7. Decomposition into spherical harmonics 3](#_Toc392767218)

[1.2. Numerical formulation 4](#_Toc392767219)

[1.2.1. Treating the boundaries 4](#_Toc392767220)

[1.2.1.1. The origin 4](#_Toc392767221)

[1.2.1.2. Next to the origin 5](#_Toc392767222)

[1.2.1.3. The outer sphere 5](#_Toc392767223)

[1.2.2. Numerical viscosity 6](#_Toc392767224)

[1.3. Energy conservation 6](#_Toc392767225)

[2. Results 7](#_Toc392767226)

# Example problem

A relevant use-case of the GridRipper template library is the Klein-Gordon equation with the mass and self-interacting terms taken into account, taken on flat Minkowski metric.

## Analytic formulation

### Lagrangian and action

Our Lagrangian reads as the following:

Note that the Lagrangian holds the inverse metric. Our field variable if complex, but the potential is real.

### Chosing of coordinates

Now let us use the Boyer-Lindquist coordinates, which in turn will render our metric and the inverse of the metric respectively into the form

### Euler-Lagrange

The minimal of the action is obtained by the variation of the the Euler-Lagrange, which will take the form of the well known Klein-Gordon equation. This reads as:

In our calculations we made use of the fact that in the case of scalar fields, when the metric has no off-diagonal elements, the covariant derivate is identical to the regular partial derivate. We also took a shortcut in our calculations by making use of the knowledge on the Laplace operator in various coordinate systems, in our case in spherical coordinates. The operator is the Laplace-Beltrami operator on the unit 2-sphere and collects the derivates in the angular directions. This will be the most convenient form as later we will see.

### Equation of motion

Rearranging the equation will yield our equation of motion to be the following:

Chosing the potential to be the self-interacting term of a scalar Higgs particle, and will read as

which will give us our final equation of motion:

### Sommerfeld outgoing radiation boundary condition

In order to increase stability on the borders, but more importantly eliminate the singularity of the origin, we use the widely adopted Sommerfeld reformulation, which substitutes the field variables with a non-physical quantity. Making the

substitution, one arrives at the equation

### First order reduction

To be able to handle the equation of motion numerically, we have to use the usual first-order reduction of the partial derivate. Introducing

the equation decomposes into a partial differential equation system governing the evolution of

The constraint of the initial condition is conserved throughout the evolution.

### Decomposition into spherical harmonics

The choice of Boyer-Lindquist coordinates to express our metric is not by chance. The field can be decomposed on the base of spherical harmonics in the following way:

where are the coefficients of the various spherical harmonic functions, henceforth called multipole coefficients. With this, we have separated the radial and angular part of our equations.

Substituting back this formula into our equation system, we get the following:

where correspond to the multipole coefficients of respectively. Plus we have made use of the Eigen values of the spherical Laplace-Beltrami operator in spherical coordinates.

The time evolution equation can be obtained by scalar multiplying the equations by which will result in

So it would seem that the equations fully decouple along the distinct multipole directions, however coupling of the various multipole coefficients arise in the multiplication operator of the field coefficients. The multiplication of fields are defined by

The double integral on the right side can be pre-computed. Due to the parity of , it’s sole integral is zero, not counting the mode . The cross-integral however is not trivially zero. It’s values are the Gaunt-coefficients and they can be written in matrix form:

The matrix of Gaunt-coefficients is a sparse matrix, thus each multipole direction is only connected to a handful of other multipole directions. However, taking high-order coefficients into account, this multiplication will tend to dominate our numerical calculations.

## Numerical formulation

### Treating the boundaries

As with all lattice calculations, the treating of the origin requires extra care.

#### The origin

The continuity requirement of the field quantities ensure that spatial derivates in the origin are finite. The spatial part of the Laplacian

and applying l’Hôspital’s rule we obtain the equalities

which in turn mean

for our substituted field variables. Given that all multipole directions vanish in both value and first derivate, the time derivate of these will vanish too. Only will differ in this regard, where only the second derivate vanishes, but not the first. Summarizing, our field equations in the origin will look as following:

#### Next to the origin

While the field equations hold true as normal next to the origin, calculating the radial derivates requires obtaining field values from the far side of the origin. These values can be obtained by

The first parity is that of the multipole direction’s, while the second is the parity of the variable, and the resulting parity is the combination of the two.

These however although are analytically correct, the multipole coefficients next to the origin are ill conditioned. Sufficiently large radial resolution will yield high noise close to the origin due to the inverse powers of in the field equations, as well as the Eigen values of the spherical Laplace operator further magnify numeric inaccuracies.

To eliminate this noise, we make use of the constraints on the field variables at the origin. Taking the symmetric fourth order finite difference stencil

and substituting in plus our knowledge that the field variables are exactly 0 in the origin, we obtain the relations

The above formulation trivially holds true for the multipole coefficients as well.

#### The outer sphere

There is no compactification involved, therefore the outer end of the lattice is a free boundary. As such, calculating radial derivates becomes problematic. Investigations (CITATION) showed that somewhat counter-intuitively using less precise derivates provides greater stability than using the more and more asymmetric finite difference stencils. Former papers referred to this method as the method, referring to the order of correctness decreasing towards the boundary of the lattice.

### Numerical viscosity

In order to suppress high-order oscillations to wreck the stability of the evolution, one needs to introduce a higher order dissipative factor into the evolution, than that of the driving finite difference method.

Since the application uses finite differencing precise to the 4th order in the separation of coordinates, we introduce a 6th order derivate

## Energy conservation

As the first and most important test of the simulation one needs to check whether energy is conserved throughout the evolution. The energy-impulse tensor generally is defined as

where we can make use of the analogies

Substituting the differential Lagrangian into the equation and using the formulas above we get

The energy inside the evolution is conserved if

between any two time points holds true. Having one free boundary on the lattice, the total energy of the system will be the sum of the energy that is still inside the lattice and of the energy that has been radiated out through the free boundary up until the given time point. This can be interpreted as a relativistic continuity equation. The energy thus at any given can be written as

where

given that are the time-like and radial Killing-vectors of the metric. Note that our continuity equation did not depend on any tangential coordinates, as the surface boundary of the volume is a sphere. In order to keep our evolution simple, we expressed the field variables on the basis of spherical harmonics, which induced that the multiplication of field variables were expressed using the sparse matrix of Gaunt-coefficients. This pre-computed matrix will also come in handy when we wish to integrate our field variables over a complete sphere.

Recalling our knowledge on the values of Gaunt-coefficients and how the expansion over spherical harmonics went, we can rewrite the above equation.

The vector of angular integral of the Spherical Harmonics is all zero but the element corresponding to which is equal to 0.282095.

# Results