

1 Point estimation

Context

Our engineering team just landed a consulting contract with a company interested in the electricity consumption of its machines. In a first part, we would like to determine how electricity consumption is evenly distributed across the different machines of the same type. To this end, we use the Gini coefficient. In a nutshell, it is an index ranging from 0 to 1 measuring the inequality featured in a distribution. A value of 0 denotes that all our machines use the same amount of electricity while a value of 1 means that all the electricity is used by a single machine. We assume that all of the n machines operate independently and their daily electricity consumption (in MWh) can be modelled as a random variable X with the following density function,

$$f_{\theta_1, \theta_2}(x) = \begin{cases} \frac{\theta_1 \theta_2^{\theta_1}}{x^{\theta_1+1}}, & x \geq \theta_2 \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

with $\theta_1 > 2$ and $\theta_2 > 0$.

(a) Derive the quantile function of X

We're looking to solve $P(X \leq x_t) = t$ for x_t .

First let's compute $P(X \leq x_t)$,

$$\begin{aligned} P(X \leq x_t) &= \int_{-\infty}^{x_t} f_{\theta_1, \theta_2}(x) dx \\ &= \int_{\theta_2}^{x_t} \theta_1 \theta_2^{\theta_1} x^{-(\theta_1+1)} dx \\ &= -\frac{\theta_1 \theta_2^{\theta_1}}{\theta_1} [x^{-\theta_1}]_{x=\theta_2}^{x=x_t} \\ &= -\frac{\theta_1 \theta_2^{\theta_1}}{\theta_1} (x_t^{-\theta_1} - \theta_2^{-\theta_1}) \end{aligned}$$

Let's solve $P(X \leq x_t) = t$ for x_t ,

$$\begin{aligned} -\frac{\theta_1 \theta_2^{\theta_1}}{\theta_1} (x_t^{-\theta_1} - \theta_2^{-\theta_1}) &= t \iff x_t^{\theta_1} = \frac{t \theta_1}{\theta_1 \theta_2^{\theta_1}} - \theta_2^{-\theta_1} \\ \iff x_t &= \left(\frac{t \theta_1}{\theta_1 \theta_2^{\theta_1}} - \theta_2^{-\theta_1} \right)^{1/\theta_1} \equiv Q_{\theta_1, \theta_2}(t) \end{aligned}$$

(b) Derive the Gini coefficient of X .

The Gini coefficient is defined as,

$$G_{\theta_1, \theta_2}(t) = 2 \int_0^1 \left(p - \frac{\int_0^p Q(t) dt}{E(X)} \right) dp \quad (2)$$

Let's first compute the mean of X ,

$$\begin{aligned} E(X) &= \int_{-\infty}^{+\infty} x f(x) dx \\ &= \int_{\theta_2}^{+\infty} x \frac{\theta_1 \theta_2^{\theta_1}}{x^{\theta_1+1}} dx \\ &= \theta_1 \theta_2^{\theta_1} \int_{\theta_2}^{+\infty} x^{-\theta_1} dx \\ &= -\frac{\theta_1 \theta_2^{\theta_1}}{(\theta_1 - 1)} \left[x^{-(\theta_1-1)} \right]_{\theta_2}^{+\infty} \end{aligned}$$