

1 Point estimation

Context

Our engineering team just landed a consulting contract with a company interested in the electricity consumption of its machines. In a first part, we would like to determine how electricity consumption is evenly distributed across the different machines of the same type. To this end, we use the Gini coefficient. In a nutshell, it is an index ranging from 0 to 1 measuring the inequality featured in a distribution. A value of 0 denotes that all our machines use the same amount of electricity while a value of 1 means that all the electricity is used by a single machine. We assume that all of the n machines operate independently and their daily electricity consumption (in MWh) can be modelled as a random variable X with the following probability density function (PDF),

$$f_{\theta_1, \theta_2}(x) = \begin{cases} \frac{\theta_1 \theta_2^{\theta_1}}{x^{\theta_1+1}}, & x \geq \theta_2 \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

with $\theta_1 > 2$ and $\theta_2 > 0$. This is the PDF of the **Pareto distribution**.

(a) Derive the quantile function of X

We're looking to solve $P(X \leq x_t) = t$ for x_t .

First let's compute the cumulative distribution function (CDFa) $P(X \leq x_t)$,

$$\begin{aligned} P(X \leq x_t) &= \int_{-\infty}^{x_t} f_{\theta_1, \theta_2}(x) dx \\ &= \int_{\theta_2}^{x_t} \theta_1 \theta_2^{\theta_1} x^{-(\theta_1+1)} dx \\ &= -\frac{\theta_1 \theta_2^{\theta_1}}{\theta_1} [x^{-\theta_1}]_{x=\theta_2}^{x=x_t} \\ &= -\theta_2^{\theta_1} (x_t^{-\theta_1} - \theta_2^{-\theta_1}) \\ &= 1 - \left(\frac{\theta_2}{x_t}\right)^{\theta_1} \end{aligned}$$

Let's solve $P(X \leq x_t) = t$ for x_t ,

$$\begin{aligned} 1 - \left(\frac{\theta_2}{x_t}\right)^{\theta_1} &= t \iff (1-t)^{1/\theta_1} = \frac{\theta_2}{x_t} \\ \iff x_t &= \frac{\theta_2}{(1-t)^{1/\theta_1}} \end{aligned}$$

Therefore we have,

$$Q_{\theta_1, \theta_2}(t) = \frac{\theta_2}{(1-t)^{1/\theta_1}} \quad (2)$$

(b) Derive the Gini coefficient of X .

The Gini coefficient is defined as,

$$G_{\theta_1, \theta_2} = 2 \int_0^1 \left(p - \frac{\int_0^p Q(t) dt}{E(X)} \right) dp \quad (3)$$

Let's first compute the expectation value of X ,

$$\begin{aligned} E(X) &= \int_{-\infty}^{+\infty} x \cdot f_{\theta_1, \theta_2}(x) dx \\ &= \int_{\theta_2}^{+\infty} x \frac{\theta_1 \theta_2^{\theta_1}}{x^{\theta_1+1}} dx \\ &= \theta_1 \theta_2^{\theta_1} \int_{\theta_2}^{+\infty} x^{-\theta_1} dx \\ &= -\frac{\theta_1 \theta_2^{\theta_1}}{(\theta_1 - 1)} \left[x^{-(\theta_1-1)} \right]_{\theta_2}^{+\infty} \\ &= \begin{cases} -\frac{\theta_1 \theta_2^{\theta_1}}{(\theta_1-1)} \left(-\frac{1}{\theta_2^{-(\theta_1-1)}} \right), & \theta_1 > 1 \\ +\infty, & \theta_1 \leq 1 \end{cases} \\ &= \begin{cases} \frac{\theta_1 \theta_2}{(\theta_1-1)}, & \theta_1 > 1 \\ +\infty, & \theta_1 \leq 1 \end{cases} \end{aligned}$$

So the Gini coefficient is defined for $\theta_1 > 1$,

$$G_{\theta_1, \theta_2} = 2 \left(\int_0^1 p dp - \int_0^1 \frac{\int_0^p Q_{\theta_1, \theta_2}(t) dt}{E(X)} dp \right)$$

We compute each integral separately,

$$\int_0^1 p dp = \frac{1}{2}$$

Then,

$$\int_0^p Q_{\theta_1, \theta_2}(t) dt = \theta_2 \int_0^p \frac{1}{(1-t)^{1/\theta_1}}$$

We use the change of variable $u = 1 - t \implies du = -dt$

The boundaries becomes,

$$\begin{cases} t = 0 & \implies u_1 \equiv 1 \\ t = p & \implies u_2 \equiv 1 - p \end{cases}$$

Then,

$$\begin{aligned}
 \int_0^p Q_{\theta_1, \theta_2}(t) dt &= -\theta_2 \int_{u_1}^{u_2} \frac{1}{(u)^{1/\theta_1}} du \\
 &= -\theta_2 \left[\frac{(u)^{-(1/\theta_1-1)}}{-((1/\theta_1)-1)} \right]_{u_1}^{u_2} \\
 &= \frac{\theta_2}{(1/\theta_1)-1} \left(\frac{1}{(1-p)^{1/\theta_1-1}} - \frac{1}{1^{1/\theta_1-1}} \right) \\
 &= \frac{\theta_2}{(1/\theta_1)-1} \left(\frac{1}{(1-p)^{1/\theta_1-1}} - 1 \right)
 \end{aligned}$$

Therefore for $\theta_1 > 1$,

$$\begin{aligned}
 \frac{\int_0^p Q_{\theta_1, \theta_2}(t) dt}{E(X)} &= \frac{\frac{\theta_2}{(1/\theta_1)-1} \left(\frac{1}{(1-p)^{1/\theta_1-1}} - 1 \right)}{\frac{\theta_1 \theta_2}{(\theta_1-1)}} \\
 &= \frac{\theta_2}{(1/\theta_1)-1} \left(\frac{1}{(1-p)^{1/\theta_1-1}} - 1 \right) \frac{(\theta_1-1)}{\theta_1 \theta_2} \\
 &= \frac{\theta_1(1-(1/\theta_1))}{((1/\theta_1)-1)\theta_1} \left(\frac{1}{(1-p)^{1/\theta_1-1}} - 1 \right) \\
 &= - \left(\frac{1}{(1-p)^{(1/\theta_1)-1}} - 1 \right) \\
 &= 1 - \frac{1}{(1-p)^{(1/\theta_1)-1}}
 \end{aligned}$$

Then,

$$\int_0^1 \frac{\int_0^p Q_{\theta_1, \theta_2}(t) dt}{E(X)} dp = \underbrace{\int_0^1 1 dp}_A - \underbrace{\int_0^1 \frac{1}{(1-p)^{(1/\theta_1)-1}} dp}_B$$

Computing integral A and B.

$$A = \int_0^1 1 dp = 1$$

$$B = \int_0^1 \frac{1}{(1-p)^{(1/\theta_1)-1}} dp$$

We use the change of variable $u = 1 - p \implies du = -dp$.

The boundaries become,

$$\begin{cases} p = 0 & \implies u_1 \equiv 1 \\ p = 1 & \implies u_2 \equiv 0 \end{cases}$$

Then,

$$\begin{aligned}
 \int_0^1 \frac{1}{(1-p)^{(1/\theta_1)-1}} dp &= - \int_{u_1}^{u_2} \frac{1}{(u)^{(1/\theta_1)-1}} du \\
 &= - \int_{u_1}^{u_2} u^{-((1/\theta_1)-1)} du \\
 &= \frac{1}{((1/\theta_1) - 1) - 1} \left[(u)^{((1/\theta_1)-1-1)} \right]_1^0 \\
 &= - \frac{1}{(1/\theta_1) - 2} \\
 &= \frac{1}{2 - (1/\theta_1)}
 \end{aligned}$$

Eventually the Gini coefficient is (for $\theta_1 > 0$),

$$\begin{aligned}
 G_{\theta_1, \theta_2} &= 2 \left(\frac{1}{2} - \frac{1}{2 - (1/\theta_1)} \right) \\
 &= 2 \left(\frac{1}{2} \left[1 - \frac{1}{1 - (1/2\theta_1)} \right] \right) \\
 &= 1 - \frac{1}{1 - (1/2\theta_1)} \\
 &= \frac{1/2\theta_1}{1 - (1/2\theta_1)} \\
 &= \frac{1}{2\theta_1 \left(1 - \frac{1}{2\theta_1} \right)} \\
 &= \frac{1}{2\theta_1 - 1}
 \end{aligned}$$

(c) Derive the maximum likelihood estimator (MLE) of G_{θ_1, θ_2} . Call this estimator \hat{G}_{MLE}

Let's first compute the likelihood function $L(\theta_1, \theta_2)$,

$$\begin{aligned}
 L(\theta_1, \theta_2) &:= \prod_{i=1}^n f_{\theta_1, \theta_2}(x) \\
 &= \prod_{i=1}^n \frac{\theta_1 \theta_2^{\theta_1}}{x^{\theta_1+1}} \cdot I(X_i \geq \theta_2 > 0) \\
 &= \theta_1^n \theta_2^{n\theta_1} \frac{1}{\prod_{i=1}^n X_i^{\theta_1+1}} \cdot I(X_{(1)} \geq \theta_2 > 0)
 \end{aligned}$$

where $X_{(1)} \equiv \min(X_1, \dots, X_n)$.

We notice that $L(\theta_1, \theta_2)$ is not continuous along θ_2 and then not differentiable in θ_2 . However, we observe that $L(\theta_1, \theta_2)$ increase with θ_2 . Therefore, we have to take θ_2 the largest possible in order to maximize $L(\theta_1, \theta_2)$ respecting the condition $X_{(1)} \leq \theta_2 > 0$ otherwise we would have $L(\theta_1, \theta_2) = 0$,

$$\hat{\theta}_2 = X_{(1)}$$

For $\hat{\theta}_1$ we can compute the log-likelihood function $l(\theta_1, \theta_2)$,

$$\begin{aligned}
 l(\theta_1, \theta_2) &:= \ln(L(\theta_1, \theta_2)) \\
 &= \ln(\theta_1^n) + \ln(\theta_2^{n\theta_1}) + \ln(1) - \ln(\pi_{i=1}^n X_i^{(\theta_1+1)}) \\
 &= n \ln(\theta_1) + n\theta_1 \ln(\theta_2) - \left(\sum_{i=1}^n \ln(X_i^{(\theta_1+1)}) \right) \\
 &= n \ln(\theta_1) + n\theta_1 \ln(\theta_2) - \sum_{i=1}^n (\theta_1 + 1) \ln(X_i)
 \end{aligned}$$

We differentiate with respect to θ_1 in order to find the maximum,

$$\frac{\partial l(\theta_1, \theta_2)}{\partial \theta_1} = \frac{n}{\theta_1} + n \ln(\theta_2) - \sum_{i=1}^n \ln(X_i)$$

Then,

$$\begin{aligned}
 \frac{\partial l(\theta_1, \theta_2)}{\partial \theta_1} = 0 &\iff \hat{\theta}_1 = \frac{n}{\sum_{i=1}^n (\ln(X_i)) - n \ln(\hat{\theta}_2)} \\
 &= \frac{n}{\sum_{i=1}^n (\ln(X_i) - \ln(X_{(1)}))} \\
 &= \frac{n}{\sum_{i=1}^n \ln\left(\frac{X_i}{X_{(1)}}\right)}
 \end{aligned}$$

Now we can compute \hat{G}_{MLE} ,

$$\begin{aligned}
 \hat{G}_{\text{MLE}} &:= G_{\hat{\theta}_1, \hat{\theta}_2} \\
 &= \frac{1}{2\hat{\theta}_1 - 1} \\
 &= \frac{1}{\left(\frac{2n}{\sum_{i=1}^n \ln\left(\frac{X_i}{X_{(1)}}\right)} \right) - 1}
 \end{aligned}$$

(d) Propose a method of moment estimator of G_{θ_1, θ_2} . Call this estimator \hat{G}_{MME}

We already have computed the expectation value of X ,

$$E(X) = \begin{cases} \frac{\theta_1 \theta_2}{(\theta_1 - 1)}, & \theta_1 > 1 \\ +\infty, & \theta_1 \leq 1 \end{cases}$$

We know that,

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i \equiv E(X)$$

Let's solve for θ_1 ,

$$\begin{aligned}\bar{X} = \frac{\hat{\theta}_1 \hat{\theta}_2}{(\hat{\theta}_1 - 1)} &\iff \bar{X} \hat{\theta}_1 - \bar{X} = \hat{\theta}_1 \hat{\theta}_2 \\ &\iff \hat{\theta}_1 (\bar{X} - \hat{\theta}_2) = \bar{X} \\ &\iff \hat{\theta}_1 = \frac{\bar{X}}{(\bar{X} - \hat{\theta}_2)}\end{aligned}$$

In order to estimate $\hat{\theta}_2$ we know that the CDF is given by,

$$F_{\theta_1 \theta_2}(x) = P(X \leq x) = 1 - \left(\frac{\theta_2}{x}\right)^{\theta_1}$$

Therefore,

$$\begin{aligned}P(X > x) &= 1 - P(X \leq x) \\ &= \left(\frac{\theta_2}{x}\right)^{\theta_1}\end{aligned}$$

The probability that all random variables (X_1, \dots, X_n) are greater than x is,

$$\begin{aligned}P((X_1, \dots, X_n) > x) &= \Pi_{i=1}^n P(X > x) \\ &= \left(\frac{\theta_2}{x}\right)^{n\theta_1}\end{aligned}$$

Then, the probability that the minimum random variable $X_{(1)} \equiv \min(X_1, \dots, X_n)$ is greater than x is also,

$$P(X_{(1)} > x) = \left(\frac{\theta_2}{x}\right)^{n\theta_1}$$

Therefore,

$$P(X_{(1)} \leq x) = 1 - \left(\frac{\theta_2}{x}\right)^{n\theta_1}$$

The corresponding probability density function is,

$$\begin{aligned}f_{\theta_1, \theta_2}(x) &= F'_{\theta_1, \theta_2}(x) \\ &= \frac{d}{dx} \left(1 - \left(\frac{\theta_2}{x}\right)^{n\theta_1} \right) \\ &= -\theta_2^{n\theta_1} \frac{d}{dx} (x^{-n\theta_1}) \\ &= n\theta_1 \theta_2^{n\theta_1} x^{-(n\theta_1+1)} \\ &= \frac{n\theta_1 \theta_2^{n\theta_1}}{x^{(n\theta_1+1)}}, \quad x \geq \theta_2\end{aligned}$$

The corresponding expectation value is,

$$\begin{aligned}
 E(X) &= \int_{\theta_2}^{+\infty} x \cdot f_{\theta_1, \theta_2}(x) dx \\
 &= \int_{\theta_2}^{+\infty} x \cdot \frac{n\theta_1\theta_2^{n\theta_1}}{x^{(n\theta_1+1)}} dx \\
 &= n\theta_1\theta_2^{n\theta_1} \int_{\theta_2}^{+\infty} x^{(-n\theta_1)} dx \\
 &= \frac{n\theta_1\theta_2^{n\theta_1}}{-(n\theta_1-1)} \left(-\frac{1}{\theta_2^{-(n\theta_1-1)}} \right) \\
 &= \frac{n\theta_1\theta_2}{(n\theta_1-1)}
 \end{aligned}$$

Setting expectation value $E(X)$ to be equal the minimum random variable $X_{(1)}$,

$$X_{(1)} = \frac{n\theta_1\theta_2}{(n\theta_1-1)} \iff \hat{\theta}_2 = X_{(1)} \frac{(n\hat{\theta}_1-1)}{n\hat{\theta}_1}$$

Therefore,

$$\begin{aligned}
 \hat{\theta}_1 &= \frac{\bar{X}}{(\bar{X} - \hat{\theta}_2)} \\
 &= \frac{\bar{X}}{\bar{X} - X_{(1)} \frac{(n\hat{\theta}_1-1)}{n\hat{\theta}_1}} \\
 \iff \bar{X} &= \hat{\theta}_1 \left(\bar{X} - X_{(1)} \frac{(n\hat{\theta}_1-1)}{n\hat{\theta}_1} \right) \\
 &= \hat{\theta}_1 \bar{X} - \hat{\theta}_1 X_{(1)} \frac{(n\hat{\theta}_1-1)}{n\hat{\theta}_1} + \hat{\theta}_1 X_{(1)} \frac{1}{n\hat{\theta}_1} \\
 &= \hat{\theta}_1 (\bar{X} - X_{(1)}) + \frac{X_{(1)}}{n} \\
 \iff \hat{\theta}_1 &= \frac{\bar{X} - (X_{(1)}/n)}{(\bar{X} - X_{(1)})} \\
 &= \frac{n\bar{X} - X_{(1)}}{n(\bar{X} - X_{(1)})}
 \end{aligned}$$

Now we can compute \hat{G}_{MME} ,

$$\begin{aligned}
 \hat{G}_{\text{MME}} &:= G_{\hat{\theta}_1, \hat{\theta}_2} \\
 &= \frac{1}{2\hat{\theta}_1 - 1} \\
 &= \frac{1}{\left(\frac{2(n\bar{X} - X_{(1)})}{n(\bar{X} - X_{(1)})} \right) - 1}
 \end{aligned}$$

(e) Set $\theta_1^0 = 3$ and $\theta_2^0 = 1$. Generate an i.i.d sample of size $n = 20$ from the density $f_{\theta_1^0, \theta_2^0}$. In order to achieve this, you can make use of the inverse transform sampling. Using this sample, compute \hat{G}_{MLE} and \hat{G}_{MME} .

We have,

$$f_{\theta_1^0, \theta_2^0} = \begin{cases} \frac{3 \cdot 1^3}{x^{3+1}} = \frac{3}{x^4}, & x \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

We compute the CDF of X ,

$$\begin{aligned} F_{\theta_1, \theta_2}(x) &= \int_1^x \frac{3}{t^4} dt = 3 \left[\frac{t^{-3}}{-3} \right]_1^x \\ &= - \left(\frac{1}{x^3} - \frac{1}{1^3} \right) \\ &= 1 - \frac{1}{x^3} \end{aligned}$$

The inverse is,

$$F_{\theta_1, \theta_2}^{-1}(y) = \frac{1}{(1-y)^{1/3}}$$

Using the following R code,

```
number_of_samples <- 20

# generate randoms numbers from the uniform distribution U(0,1)
U <- runif(number_of_samples)

# compute X = F^{-1}(u)
X <- 1 / (1-U)^(1/3)

# compute Gini coefficients
G_MLE <- 1 / ((2 * number_of_samples) / (sum(log(X / min(X)))) - 1)
G_MME <- 1 / ((2 * (number_of_samples) * mean(X) - min(X)) / (number_of_samples * (mean(X) - min(X))))

G_MLE
G_MME
```

We get the following estimations,

$$\hat{G}_{MLE} = 0.1728057 \quad ; \quad \hat{G}_{MME} = 0.1770613 \quad (4)$$

(f) Repeat this data generating process $N = 1000$ times (with the same sample size $n = 20$ and the same (θ_1^0, θ_2^0)). Hence, you obtain a sample of size N of each estimator of G_{θ_1, θ_2} . Make a **histogram** and a **boxplot** of these two samples. What can you conclude?

```
# Generate N = 1000 times the sample

N <- 1000
```



```

G_MLE_array <- numeric(N)
G_MME_array <- numeric(N)

for (i in 1:1000) {
  U <- runif(number_of_samples)
  X <- 1 / (1-U)^(1/3)
  G_MLE <- 1 / ((2 * number_of_samples) / (sum(log(X / min(X)))) - 1)
  G_MME <- 1 / ((2 * (number_of_samples) * mean(X) - min(X)) / (number_of_samples * (mean(X) - min(X))))

  G_MLE_array[i] <- G_MLE
  G_MME_array[i] <- G_MME
}

hist(G_MLE_array)
hist(G_MME_array)

boxplot(G_MLE_array)
boxplot(G_MME_array)

```

(g) Use the samples obtained in (f) to estimate the **bias**, the **variance** and the **mean squared error (MSE)** of both estimators. What can you conclude?

(h) Repeat the calculations in (f) for $n = 20, 40, 60, 80, 100, 150, 200, 300, 400, 500$. Compare the **biases**, the **variances** and the **mean squared errors** of both estimators graphically (make a separate plot for each quantity as a function of n). What can you conclude? Which estimator is the best? Justify your answer.

(i) Create an histogram for $\sqrt{n}(\hat{G}_{MLE} - G_{\theta_1^0, \theta_2^0})$, for $n = 20, n = 100$ and $n = 500$. What can you conclude?
