Lectures on Relativity

Non-Inertial Special Relativity and General Relativity

Licence 3 - Master - Engineering schools

2025 - Mathieu ROUAUD

Foreword

This document compiles the courses taught over one semester at *Universitad Mayor San Simon* in Cochabamba, Bolivia. There are many excellent books and PDFs on general relativity, but here is what characterizes and can sets this one apart:

1. Numerous exercises with detailed answers accompany the lectures.

2. Special relativity is often studied only in the inertial context, but in the absence of gravity, it also applies to non-inertial reference frames. We study the cases of uniformly accelerated rectilinear motion and rotating frames. This is a very rich subject that allows for a smooth transition to general relativity, by familiarizing oneself with the concept of coordinate time, tensor calculus, and Lagrangian formalism.

3. In cartography, we use various metrics to represent the sphere on a plane. The shortest path is no longer necessarily a straight line. We use this analogy to understand geodesics that maximize proper time in general relativity.

4. Some chapters are based on a research paper and help to prepare to adopt the researcher's approach.

Good reading!

Always happy to receive mail from my readers: ecrire@incertitudes.fr

For special relativity, reference will regularly be made to the book by the same author:

[SR] *Special Relativity, A Geometric Approach, followed by the conference Interstellar travel and antimatter., 2020, 536 pages.*

Files on github.com/Mathieu-ROUAUD

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Special Relativity

Mathieu : Hello. Let's begin.
Since birth, we have lived in the social fiction of absolute time. However, as we will explain today, time is relative. This new concept needs to be clearly explained, otherwise it can lead to a lot of confusion. For example, the relativity of time in no way calls causality into question. It is impossible to travel into the past, or for someone from the future to visit us. However, as we will see later, we can travel into the future of others.

Arkaitz : Sir, but why not start directly with general relativity? We have already had special relativity classes since the bachelor's degree.

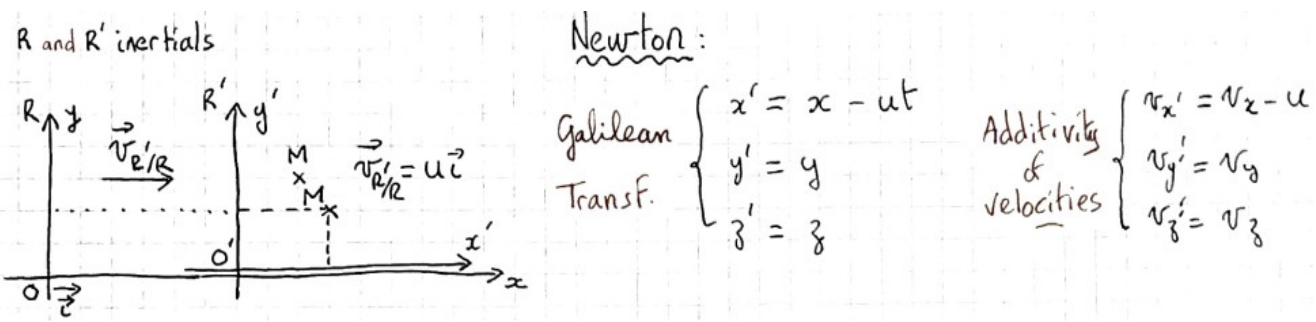
Mathieu : Indeed, it may seem unnecessary to repeat what we already know. But every teacher has a different approach to special relativity. Today, for example, we will look at a visual tool for determining time dilation that you have never seen before: the time triangle. In addition, this will be an opportunity for us to introduce notation, and in the third lecture we will introduce the tensor approach necessary for general relativity. With a new definition of special relativity that breaks away from the historical bias of light, and which, thanks to symmetries, has made it possible to construct the Standard Model.

Cristian : All right, and when will we begin general relativity itself?

Mathieu : We have a total of 18 lectures, only the last 6 of which will deal exclusively with general relativity. Before that, we need to complete your lectures on special relativity. You are not yet familiar with tensor calculus, and, for example, we are going to put Maxwell's equations into tensor form, which will then guide us by analogy for gravitational waves. We will discuss special relativity in accelerated reference frames, which will also be new to you. This will be an opportunity to introduce metric vector spaces and the Lagrangian.

Any other questions?

Okay, then... Historically, the concept of relative time is the result of a contradiction between two theories. On the one hand, we have Newton's theory from 1687, which describes the motion of objects and matter in general, and on the other, Maxwell's theory from 1864, which describes the behavior of light in terms of electromagnetic waves. These two theories, which have been thoroughly verified experimentally, state their laws in inertial reference frames. However, the first predicts the additivity of velocities, which contradicts the second theory's assertion that the speed of light in a vacuum is constant.



at t=0, O=O'

<u>Maxwell</u> (vacuum):

- Non invariance of eq. under Galileo
  R: $\vec{\nabla}\cdot\vec{E} = 0 \xrightarrow[\text{SR p433}]{\text{dem.}}$ R': $\vec{\nabla}'\cdot\vec{E}' \neq 0$
- $\vec{\nabla}\wedge\vec{E} = -\partial\vec{B}/\partial t$ and $\vec{\nabla}\wedge\vec{B} = \mu_0\varepsilon_0\,\partial\vec{E}/\partial t \Rightarrow \Box\vec{E} = \vec{0}$
  $\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \Delta\varphi = 0$ with $c = 1/\sqrt{\mu_0\varepsilon_0}$

An idea, to put everyone in agreement and save the two theories, would be that light propagates in a particular referential named *ether*. However, the experiments of Michelson-Morley, carried out as early as 1887, prove that ether, supposed medium of propagation of light, does not exist.

Which leads Einstein to propose a new theory in 1905. This theory is much more satisfactory because it integrates matter and light in the same framework. Here are the two main **postulates** of Einstein's theory of special relativity:

1. *Principle of relativity*: The laws of physics are the same in all inertial frames.

   Note the word *physics* and not *mechanics* as with Newton:
   physics = mechanics + electromagnetism = matter + light

2. *Universality of the speed of light*: The speed of light in a vacuum is the same for all inertial observers.

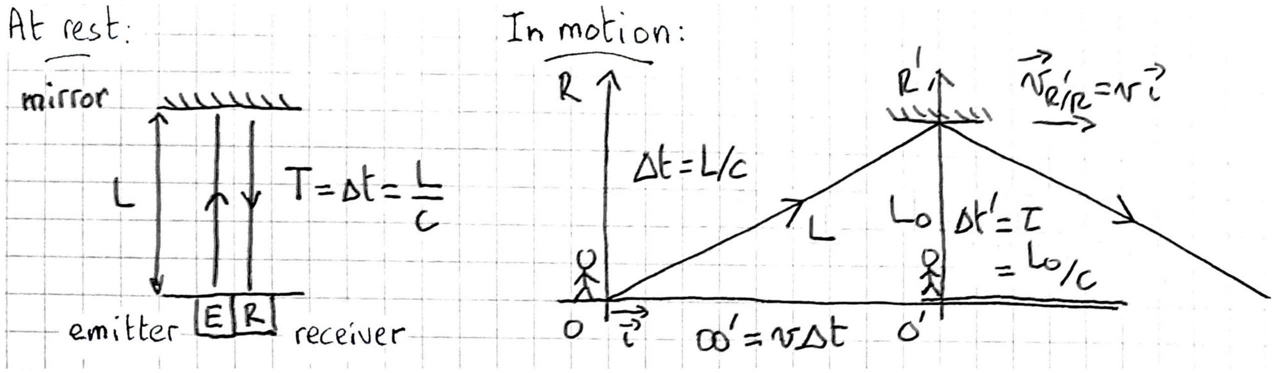
   c = 299 792 458 m/s (universal constant fixed by special relativity)

Arkaitz : It's surprising that light plays such a central role. Gravitational waves, as verified in 2017, also go to velocity c, and any particle of zero mass. I find it strange that the content and the container are linked.

Mathieu : Yes, the second postulate is actually useless. It is a consequence of simple assumptions about symmetries: 3 dimensions of space, one of time, homogeneity of time and space, isotropy of space, principles of relativity and causality. These assumptions seem natural and allow to find the Lorentz transform in a few pages of calculation[1]. The constant c then appears as a structural constant specific to space-time, unrelated to any physical object.
It is interesting to note that Newton had discovered the mathematical tools of differential calculus that could have allowed him to find this most general change of coordinates between two inertial frames of reference. Without realizing it, he restricted himself to the limit case where c is infinite, and space and time, absolute and disconnected.

Let's continue nevertheless with the 2 historical postulates of Einstein and describe the ***light clock***. Let's imagine a clock that uses light rays. A light ray is emitted, reflects on a mirror and returns to the same point after a *2T* duration:

1 Well explained and demonstrated by Jean-Marc Lévy-Leblond, « One more derivation of the Lorentz transformation », American Journal of Physics 44 (3) 271 - 277 (1976).



The clock is at rest in R' and moves at the velocity $\vec{v}$ with respect to R. The distance traveled by the ray in R is therefore larger, while keeping the speed constant:

$$\text{so} \quad L^2=L_0^2+OO'^2 \quad \Rightarrow \quad c^2\Delta t^2=c^2\Delta t'^2+v^2\Delta t^2 \quad \Rightarrow \quad \Delta t=\frac{1}{\sqrt{1-v^2/c^2}}\Delta t'$$

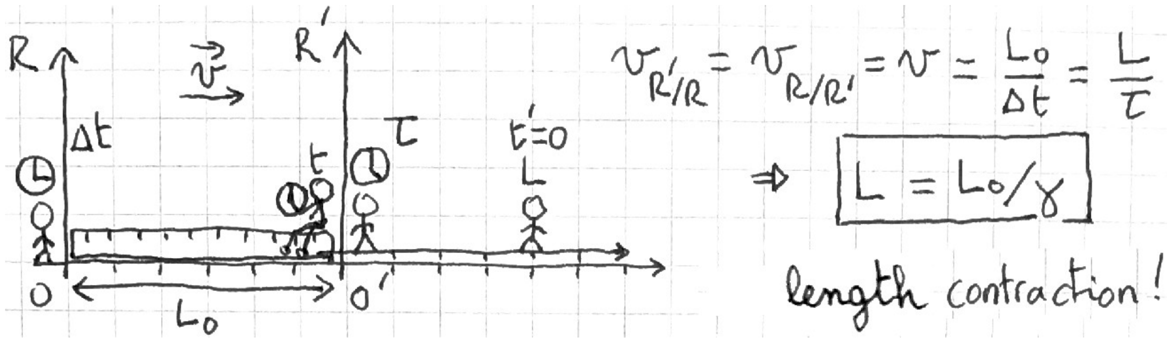
$$\gamma=\frac{1}{\sqrt{1-\beta^2}} \quad , \quad \beta=\frac{v}{c} \quad , \quad 0\leq\beta<1 \quad , \quad \gamma\geq 1 \quad \text{then} \quad \boxed{\Delta t=\gamma\tau} \Rightarrow \textbf{time dilation}!$$

Adriana : ... so an observer of R sees the clock of R' going more slowly, and it's relative, that of R' also sees a clock at rest in R going more slowly.

Mathieu : Yes, the situation is symmetrical. But, no, it's not what is seen but what is measured. We will see later the methods for measuring times and distances. Here the time dilation is the same for frames that move closer or further away. On the other hand, for what we see, it is different, because it is necessary to take into account, in addition, the propagation of light. We will see that in a next class, it's the Doppler effect.

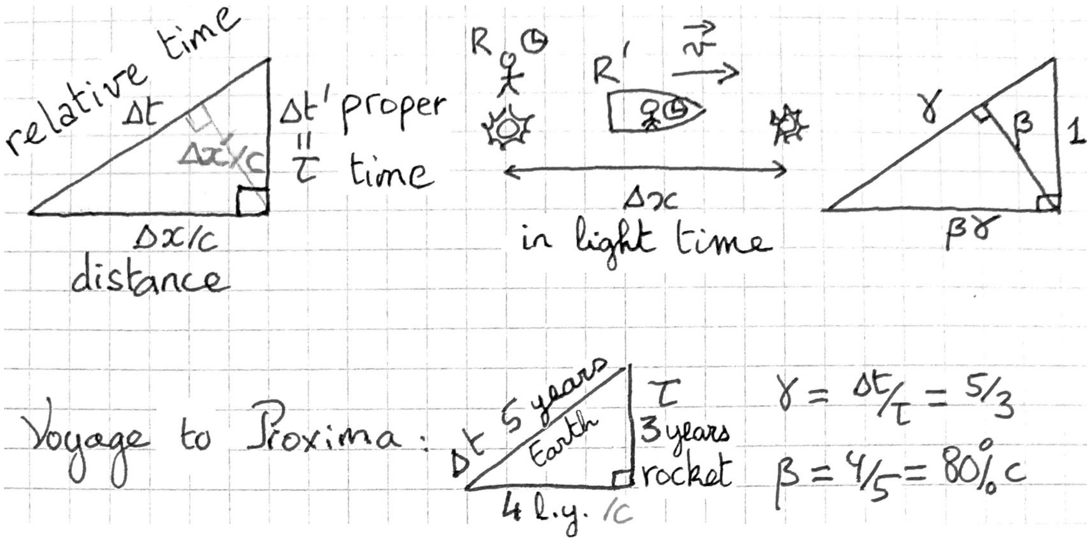
Time dilation can make one think of a spatio-temporal perspective effect. An analogy, from where I am, I can "hold" a student at the back of the room between my thumb and forefinger, he is very small, like a Smurf! But, he also sees me as tiny, it's mutual. In this case, we have a spatial perspective effect. And, of course, we actually both have the same size, it's just an illusion.
In the case of relativity it is more complex, because, unlike space, time is not isotropic.

Where were we? Ah yes! Moreover, time dilation is associated with a **length contraction**. Let's consider a ruler at rest in R, and, as before, a clock at rest in R':



To measure a length in a given reference frame, it is necessary to identify the position of each end at the same moments.

Relying on the Einstein's light clock, let us now present the **time triangle** that summarizes these effects:



We are ready to address the twin paradox: imagine twins who are 20 years old, one stays on Earth, and the other goes in his spaceship at a speed of 80% of *c* for Proxima Centauri located 4 ly. The traveling twin arrives, on the exoplanet *Proxima b*, 3 years later, while it has been on Earth for 5 years. When he returns to Earth, it flows again 3 years in the ship and 5 years on Earth. So, when they meet again, the traveling twin will be 26, while the sedentary twin will be 30! If we imagine the situation symmetrically, it seems paradoxical. But one is undergoing acceleration to return, unlike the other. And ultimately, this is not just an illusion, a perspective effect, but a physical and measurable difference.

Luna : Here, the rocket always travels at the same speed, which is not realistic! You can't go from 0 to 80% of *c* instantly, and even less turn around suddenly by reversing speed. However, it is because of these accelerated phases that the traveler ages less than someone on Earth. Because the rest of the time, the situation is symmetrical. Do you think it is conceivable that such a rocket could be built one day?

Mathieu : Yes, excellent points. For greater realism, in the next lesson we will consider a uniformly accelerated rocket. The rocket's frame of reference is then non-inertial, and Einstein's postulates do not apply. The metric is no longer Minkowskian but of Rindler. Such a rocket is conceivable, but with many “ifs”. If we were able to collect, store, and use antimatter efficiently, a rocket the size of a *Saturn V* would enable a trip to *Proxima b* with an acceleration of 1 *g*, taking 3 years for the traveler and 5 Earth years.

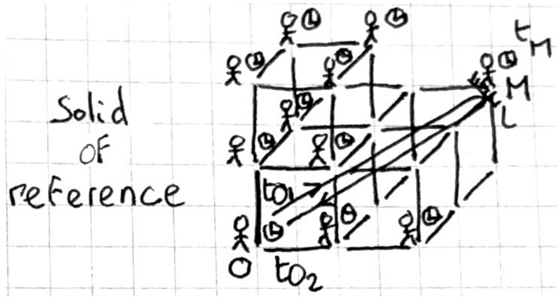
To conclude today, let's look at a fundamental point: the definition of an inertial reference frame in special relativity, the only reference frames in which the postulates are verified.

To locate ourselves in space, we proceed as in Newtonian mechanics. We place one-meter rulers perpendicular to each other to form a grid that covers the entire space. We also want to have clocks at each node of the grid, with all clocks constantly synchronized. In addition, we imagine an observer at each node that records events occurring at its level using labels (ct, x, y, z).
The difference with classical mechanics concerns the synchronization of clocks. If you place all the clocks at node *O*, set them to zero, and then move them to their respective nodes, they will become desynchronized during the motion, because they will then have a non-zero velocity and their time will dilate.

Hence ***Einstein's synchronization method***: the clocks are first placed at their nodes. Arbitrarily, the clock at O, H_O, is chosen as a reference. At time t_{O1} on H_O, the observer at O emits a light beam toward a clock at M to be synchronized. The ray is reflected by a mirror at M and returns to O. When the ray is reflected, the observer at M notes the time t_M indicated by his clock. When the ray returns, O notes t_{O2}. O communicates to M the values t_{O1} and t_{O2}, and M deduces the synchronized t_M.

By symmetry of the trajectories between the outward and return paths (isotropy of space):

$(t_M)_{sync}=\frac{t_{O1}+t_{O2}}{2}$, H_M to be advanced by $(t_M)_{sync}-t_M$.



Cristian : Okay, so the clocks are initially synchronized. But what guarantees that they will remain so?

Mathieu : The homogeneity of space and time required in an inertial reference frame. No point should be favored; due to symmetry, there is no reason why one clock should start behaving differently from the others.
A major advantage of special relativity is that we have an experimental method for determining whether a reference frame is inertial: once synchronized, clocks remain synchronized. If the reference frame is non-inertial, they become desynchronized over time.
In Newtonian mechanics, time is absolute, and a reference frame is Galilean (inertial) if Newton's laws are verified, but the laws are only verified in a Galilean reference frame, so the definition is circular. In classical mechanics, if an object has a curved trajectory, we do not know whether this is due to the action of a force or the non-inertial nature of the reference frame.
In relativity, we use the clock crystal, and this clarifies the issue.

When we talk about measuring time and length, we are referring to this reference solid and its associated clock crystal. Each reference frame has a set of observers, each with their own rulers and clocks. Following an experiment, the observers gather all the data on the various events they have collected to reconstruct what happened[2].

Consider a ruler of length L_0 and a clock with period τ, both at rest in R. When a frame R' moves with the velocity $\vec{v}$, an observer in R' will sometimes coincide with a given observer in R. For example, there are two observers A_1 and A_2 of R at rest at each end of the ruler, and in R' at the same time t' two observers A'_1 and A'_2 will coincide with them:

$$\text{for all } t,\ L_0=A_1A_2 \quad ; \quad \text{same } t',\ L=A'_1A'_2=\sqrt{(x'_{A'_2}-x'_{A'_1})^2+(y'_{A'_2}-y'_{A'_1})^2+(z'_{A'_2}-z'_{A'_1})^2}.$$

For the clock, an observer of R is constantly at rest, and two observers of R' will coincide, one at the beginning and the other at the end of the period:

$$\text{same } x,\ \tau=t_2-t_1 \text{ (one clock)} \quad ; \quad \text{different } x',\ \Delta t'=t'_{A'_2}-t'_{A'_1} \text{ (two clocks).}$$

And it is according to these protocols that we have $L=L_0/\gamma$ and $\Delta t'=\gamma\tau$.

Thank you for your attention.

2 It should also be noted that clocks at rest are thus synchronized in a given inertial reference frame, but *a priori* they do not appear to be synchronized with respect to another reference frame in which they are in motion.

Summary of the notions covered today:

Lecture 1 - SPECIAL RELATIVITY

- Einstein's Postulates
- Light clock: Time Dilation
- Length Contraction
- Time Triangle
- Twin Paradox
- Definition of an inertial frame: Einstein's synchronization method

To review and practice:

- This lecture covers Chapter 1 of the SR book, which includes exercises 1 to 8 with their answers.