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PHYS 512 - A1

#1a) $f(x+\delta) = f(x) + \delta f'(x) + \frac{1}{2}\delta^2 f''(x) + \frac{1}{6}\delta^3 f'''(x) + \frac{1}{24}\delta^4 f''''(x) + \ldots$

$f(x-\delta) = f(x) - \delta f'(x) + \frac{1}{2}\delta^2 f''(x) - \frac{1}{6}\delta^3 f'''(x) + \frac{1}{24}\delta^4 f''''(x) + \ldots$

$f(x+2\delta) = f(x) + 2\delta f'(x) + 2\delta^2 f''(x) + \frac{4}{3}\delta^3 f'''(x) + \frac{2}{3}\delta^4 f''''(x) + \ldots$

$f(x-2\delta) = f(x) - 2\delta f'(x) + 2\delta^2 f''(x) - \frac{4}{3}\delta^3 f'''(x) + \frac{2}{3}\delta^4 f''''(x) + \ldots$

We want all of these except the $f'(x)$ to cancel out, meaning our error is order 5.

$f(x+2\delta) - f(x-2\delta) = 4\delta f'(x) + \frac{8}{3}\delta^3 f'''(x) + O(\delta^5)$

$f(x+\delta) - f(x-\delta) = 2\delta f'(x) + \frac{1}{3}\delta^3 f'''(x) + O(\delta^5)$

We want to cancel the $O(\delta^3$ terms$)$:

$(f(x+2\delta) - f(x-2\delta)) - 8(f(x+\delta) - f(x-\delta)) \approx -12\delta f'(x)$

$\Rightarrow f'(x) \approx \frac{8[f(x+\delta) - f(x-\delta)] - [f(x+2\delta) - f(x-2\delta)]}{12\delta} + \frac{\delta^4}{30} f^{(5)}(x)$

This cancels the next term in the Taylor series

b) $\bar{f} = f(x)(1+g_+\varepsilon)$

To find our error we need to find $|f' - \bar{f}'|$

$$|f' - \bar{f}'| = \Big| f'(x) - \frac{8[f(x+\delta) - f(x-\delta)] - [f(x+2\delta) - f(x-2\delta)]}{12\delta}$$
$$- \frac{8[f(x+\delta)\varepsilon_1 - f(x-\delta)\varepsilon_2] - [f(x+2\delta)\varepsilon_3 - f(x-2\delta)\varepsilon_4]}{12\delta} \Big|$$

The first line here is simply $\frac{\delta^4}{30} f^{(5)}(x)$, so

$$|f' - \bar{f}'| = \left| \frac{\delta^4}{30} f^{(5)}(x) - \frac{8[f(x+\delta)\varepsilon_1 - f(x-\delta)\varepsilon_2] - [f(x+2\delta)\varepsilon_3 - f(x-2\delta)\varepsilon_4]}{12\delta} \right|$$

$$\leq \left| \frac{\delta^4}{30} f^{(5)}(x) \right| + \left| \frac{8(f(x+\delta)\varepsilon_1 - f(x-\delta)\varepsilon_2) - (f(x+2\delta)\varepsilon_3 - f(x-2\delta)\varepsilon_4)}{12\delta} \right|$$

$$\approx \left| \frac{\delta^4}{30} f^{(5)}(x) \right| + \left| \frac{8 f(x)(\varepsilon_1 - \varepsilon_2) - f(x)(\varepsilon_3 - \varepsilon_4)}{12\delta} \right|$$

We choose ε s.t $\varepsilon \geq \varepsilon_1 - \varepsilon_2$, $\varepsilon \geq \varepsilon_3 - \varepsilon_4$ and get:

$$|f' - \bar{f}'| \leq \frac{\delta^4}{30} f^{(5)}(x) + \left| \frac{8 f(x)\varepsilon}{12\delta} \right| + \left| \frac{f(x)\varepsilon}{12\delta} \right| = \frac{\delta^4}{30} f^{(5)}(x) + \frac{3f(x)}{4\delta}\varepsilon$$

We differentiate w.r.t δ and set equal to 0

$$\frac{4\delta^3}{30} f^{(5)}(x) - \frac{3f(x)}{4\delta^2}\varepsilon \Rightarrow \delta = \sqrt[5]{\frac{45 f(x)}{8 f^{(5)}(x)}\varepsilon}$$