Convex Optimization M2

Lecture 1

Today

- Convex optimization: introduction
- Course organization and other gory details...
- Convex sets, basic definitions.

Convex Optimization

Convex Optimization

- How do we identify **easy** and **hard** problems?
- **Convexity**: why is it so important?
- Modeling: how do we recognize easy problems in real **applications**?
- Algorithms: how do we solve these problems **in practice**?

Least squares (LS)

$$\text{minimize} \quad \|Ax - b\|_2^2$$

$A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ are parameters; $x \in \mathbb{R}^n$ is variable

- Complete theory (existence & uniqueness, sensitivity analysis . . .)
- Several algorithms compute (global) solution reliably
- We can solve dense problems with $n = 1000$ vbles, $m = 10000$ terms
- By exploiting structure (e.g., sparsity) can solve **far larger** problems

. . . LS is a (widely used) **technology**

Linear program (LP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \end{array}$$

$c,\ a_i \in \mathbb{R}^n$ are parameters; $x \in \mathbb{R}^n$ is variable

- Nearly complete theory
  (existence & uniqueness, sensitivity analysis . . .)
- Several algorithms compute (global) solution reliably
- Can solve dense problems with $n = 1000$ vbles, $m = 10000$ constraints
- By exploiting structure (e.g., sparsity) can solve **far larger** problems

. . . LP is a (widely used) **technology**

Quadratic program (QP)

$$
\begin{array}{ll}
\text{minimize} & \|Fx - g\|_2^2 \\
\text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m
\end{array}
$$

- Combination of LS & LP
- Same story . . . QP is a technology
- Reliability: Programmed on chips to solve **real-time** problems
- Classic application: **portfolio optimization**

The bad news

- LS, LP, and QP are **exceptions**
- Most optimization problems, even some very simple looking ones, are **intractable**
- The objective of this class is to show you how to recognize the nice ones. . .
- Many, many applications across all fields. . .

Polynomial minimization

$$\text{minimize} \quad p(x)$$

p is polynomial of degree d; $x \in \mathbb{R}^n$ is variable

- Except for special cases (e.g., $d = 2$) this is a **very difficult problem**
- Even sparse problems with size $n = 20$, $d = 10$ are essentially intractable
- All algorithms known to solve this problem require effort exponential in n

What makes a problem easy or hard?

Classical view:

- **linear** is easy
- **nonlinear** is hard(er)

What makes a problem easy or hard?

Emerging (and correct) view:

. . . the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity.

— R. Rockafellar, SIAM Review 1993

Convex optimization

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_1(x) \leq 0, \ldots, f_m(x) \leq 0 \end{array}$$

$x \in \mathbb{R}^n$ is optimization variable; $f_i : \mathbb{R}^n \to \mathbb{R}$ are **convex**:

$$f_i(\lambda x + (1-\lambda)y) \leq \lambda f_i(x) + (1-\lambda) f_i(y)$$

for all x, y, $0 \leq \lambda \leq 1$

- includes LS, LP, QP, and **many others**
- like LS, LP, and QP, convex problems are **fundamentally tractable**

Example: Stochastic LP

Consider the following stochastic LP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{Prob}(a_i^T x \leq b_i) \geq \eta, \quad i = 1, \ldots, m \end{array}$$

coefficient vectors a_i IID, $\mathcal{N}(\overline{a}_i, \Sigma_i)$; η is required reliability

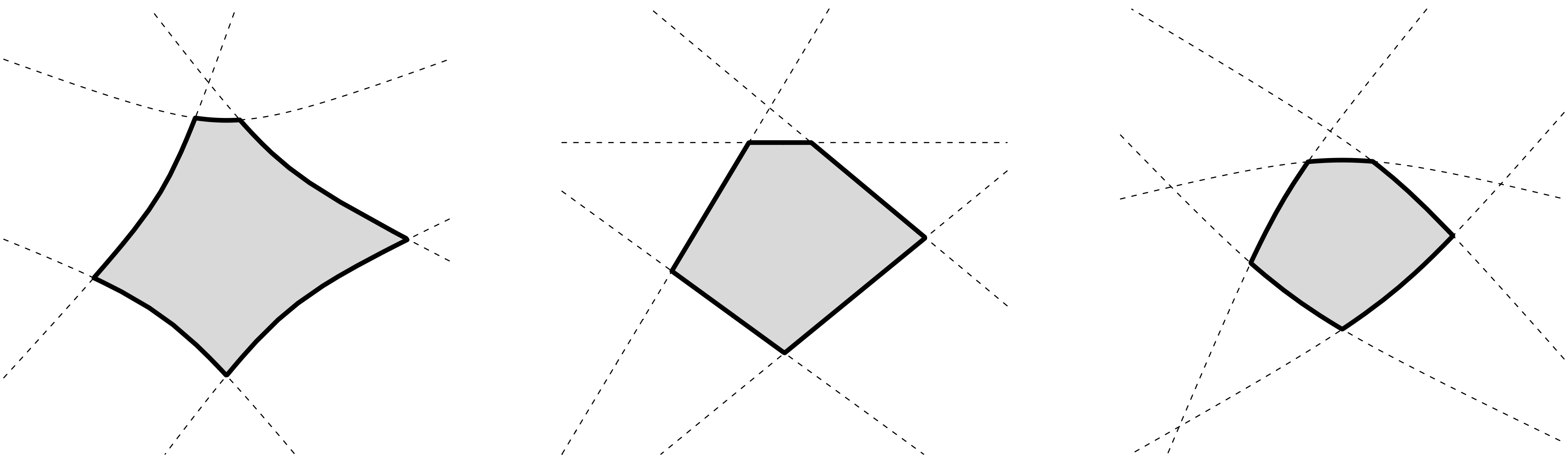
- for fixed x, $a_i^T x$ is $\mathcal{N}(\overline{a}_i^T x, x^T \Sigma_i x)$
- so for $\eta = 50\%$, stochastic LP reduces to LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \overline{a}_i^T x \leq b_i, \quad i = 1, \ldots, m \end{array}$$

  and so is easily solved
- what about other values of η, *e.g.*, $\eta = 10\%$? $\eta = 90\%$?

Hint

$$\{x \mid \mathbf{Prob}(a_i^T x \leq b_i) \geq \eta,\ i = 1, \ldots, m\}$$



$\eta = 10\%$ $\eta = 50\%$ $\eta = 90\%$

Convexity again

stochastic LP with reliability $\eta = 90\%$ is convex, and **very easily solved**

stochastic LP with reliability $\eta = 10\%$ is not convex, and **extremely difficult**

moral: **very difficult** and **very easy** problems can look **quite similar**

(to the untrained eye)

Convex Optimization

A brief history. . .

- The field is about 50 years old.
- Starts with the work of Von Neumann, Kuhn and Tucker, etc
- Explodes in the 60's with the advent of "relatively" cheap and efficient computers. . .
- Key to all this: fast linear algebra
- Some of the theory developed before computers even existed. . .

Convex optimization: history

- Convexity $\Longrightarrow$ low complexity:

  "... *In fact the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity.*" **T. Rockafellar**.

- True: Nemirovskii and Yudin [1979].

- Very true: Karmarkar [1984].

- Seriously true: convex programming, Nesterov and Nemirovskii [1994].

Standard convex complexity analysis

- All convex minimization problems with: a first order oracle (returning $f(x)$ and a subgradient) can be solved in polynomial time in size and number of precision digits.

- Proved using the **ellipsoid method** by Nemirovskii and Yudin [1979].

- Very slow convergence in practice.

Linear Programming

- Simplex algorithm by Dantzig (1949): exponential worst-case complexity, very efficient in most cases.

- Khachiyan [1979] then used the ellipsoid method to show the polynomial complexity of LP.

- Karmarkar [1984] describes the first efficient polynomial time algorithm for LP, using interior point methods.

From LP to structured convex programs

- Nesterov and Nemirovskii [1994] show that the interior point methods used for LPs can be applied to a larger class of structured convex problems.

- The **self-concordance** analysis that they introduce extends the polynomial time complexity proof for LPs.

- Most operations that preserve convexity also preserve self-concordance.

- The complexity of a certain number of elementary problems can be directly extended to a much wider class.

Symmetric cone programs

- An important particular case: linear programming on symmetric cones

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax - b \in \mathcal{K} \end{array}$$

- These include the LP, second-order (Lorentz) and semidefinite cone:

$$\begin{array}{ll} \text{LP:} & \{x \in \mathbb{R}^n : x \geq 0\} \\ \text{Second order:} & \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : \|x\| \leq y\} \\ \text{Semidefinite:} & \{X \in \mathbf{S}^n : X \succeq 0\} \end{array}$$

- Again, the class of problems that can be represented using these cones is extremely vast.

Course Organization

Course Plan

- Convex analysis & modeling
- Duality
- Algorithms: interior point methods, first order methods.
- Applications

Grading

Course website with lecture notes, homework, etc.

`http://www.cmap.polytechnique.fr/~aspremon/OptConvexeM2.html`

- A few homeworks, will be posted online.
- A final project (a few exercises, some code).

Short blurb

- Contact info on `http://www.cmap.polytechnique.fr/~aspremon`
- Email: `alexandre.daspremont@m4x.org`
- Dual PhDs: Ecole Polytechnique & Stanford University
- Interests: Optimization, machine learning, statistics & finance.

Recruiting PhD students starting next year. Full ERC funding for 3 years.

References

- All lecture notes will be posted online
- Textbook: **Convex Optimization** by Lieven Vandenberghe and Stephen Boyd, available online at:

  `http://www.stanford.edu/~boyd/cvxbook/`

- See also Ben-Tal and Nemirovski [2001], "Lectures On Modern Convex Optimization: Analysis, Algorithms, And Engineering Applications", SIAM.

  `http://www2.isye.gatech.edu/~nemirovs/`

- Nesterov [2003], "Introductory Lectures on Convex Optimization", Springer.
- Nesterov and Nemirovskii [1994], "Interior Point Polynomial Algorithms in Convex Programming", SIAM.

Convex Sets

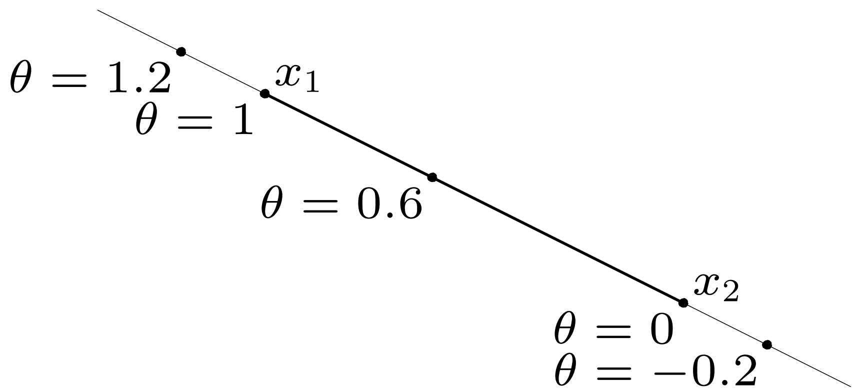
Convex Sets

- affine and convex sets
- some important examples
- operations that preserve convexity
- generalized inequalities
- separating and supporting hyperplanes
- dual cones and generalized inequalities

Affine set

line through x_1, x_2: all points

$$x = \theta x_1 + (1 - \theta) x_2 \qquad (\theta \in \mathbb{R})$$



affine set: contains the line through any two distinct points in the set

example: solution set of linear equations $\{x \mid Ax = b\}$

Convex set

line segment between x_1 and x_2: all points

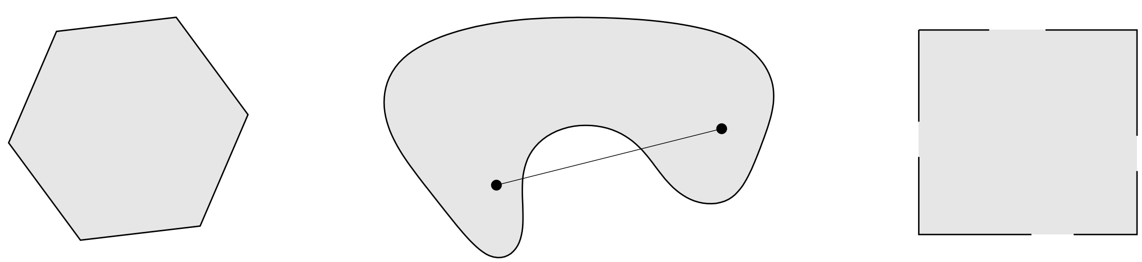
$$x = \theta x_1 + (1-\theta)x_2$$

with $0 \leq \theta \leq 1$

convex set: contains line segment between any two points in the set

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \Longrightarrow \quad \theta x_1 + (1-\theta)x_2 \in C$$

examples (one convex, two nonconvex sets)



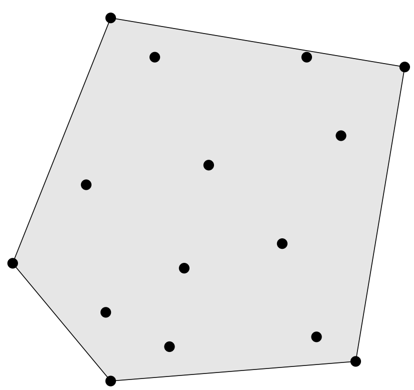
Convex combination and convex hull

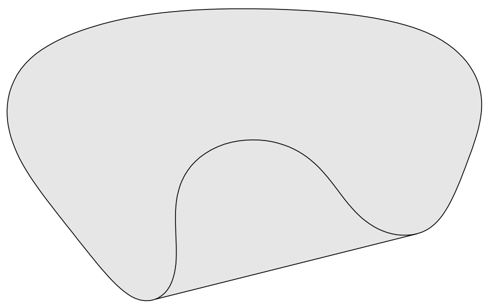
convex combination of x_1,. . . , x_k: any point x of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$$

with $\theta_1 + \cdots + \theta_k = 1$, $\theta_i \geq 0$

convex hull $\mathbf{Co}S$: set of all convex combinations of points in S



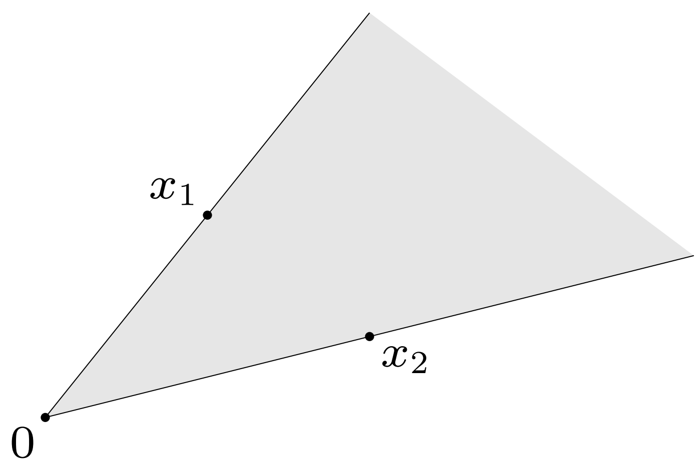


Convex cone

conic (nonnegative) combination of x_1 and x_2: any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2$$

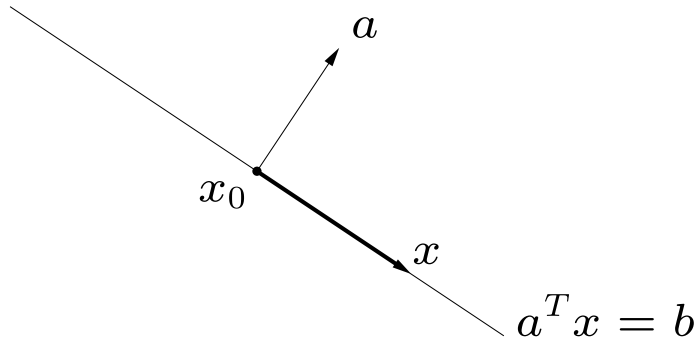
with $\theta_1 \geq 0$, $\theta_2 \geq 0$



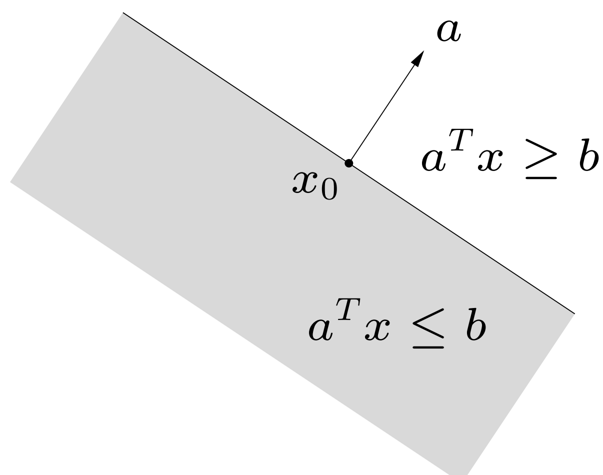
convex cone: set that contains all conic combinations of points in the set

Hyperplanes and halfspaces

hyperplane: set of the form $\{x \mid a^T x = b\}$ $(a \neq 0)$



halfspace: set of the form $\{x \mid a^T x \leq b\}$ $(a \neq 0)$



- a is the normal vector
- hyperplanes are affine and convex; halfspaces are convex

Euclidean balls and ellipsoids

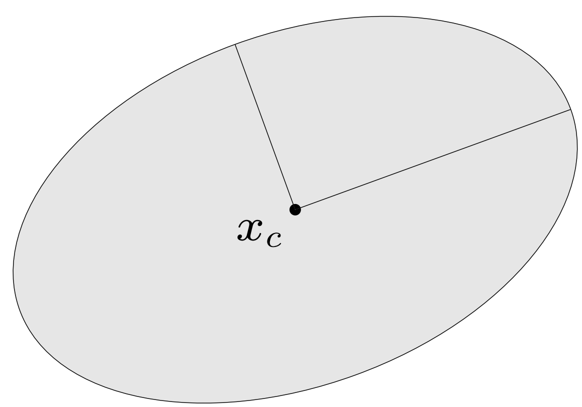
(Euclidean) ball with center x_c and radius r:

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \le r\} = \{x_c + ru \mid \|u\|_2 \le 1\}$$

ellipsoid: set of the form

$$\{x \mid (x - x_c)^T P^{-1} (x - x_c) \le 1\}$$

with $P \in \mathbf{S}^n_{++}$ (*i.e.*, P symmetric positive definite)



other representation: $\{x_c + Au \mid \|u\|_2 \le 1\}$ with A square and nonsingular

Norm balls and norm cones

norm: a function $\|\cdot\|$ that satisfies

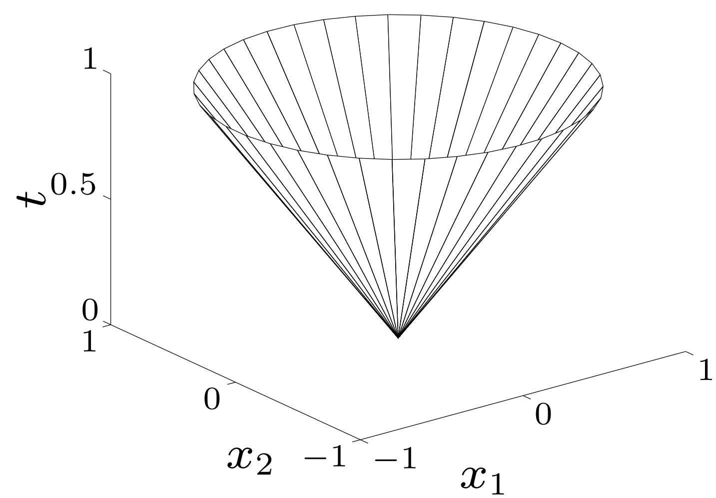
- $\|x\| \geq 0$; $\|x\| = 0$ if and only if $x = 0$
- $\|tx\| = |t| \, \|x\|$ for $t \in \mathbb{R}$
- $\|x + y\| \leq \|x\| + \|y\|$

notation: $\|\cdot\|$ is general (unspecified) norm; $\|\cdot\|_{\text{symb}}$ is particular norm

norm ball with center x_c and radius r: $\{x \mid \|x - x_c\| \leq r\}$

norm cone: $\{(x,t) \mid \|x\| \leq t\}$

Euclidean norm cone is called second-order cone



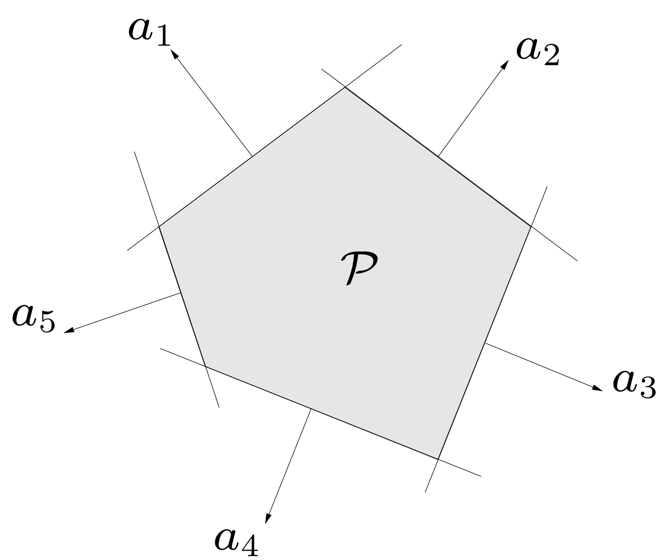
norm balls and cones are convex

Polyhedra

solution set of finitely many linear inequalities and equalities

$$Ax \preceq b, \qquad Cx = d$$

($A \in \mathbb{R}^{m \times n}$, $C \in \mathbb{R}^{p \times n}$, $\preceq$ is componentwise inequality)



polyhedron is intersection of finite number of halfspaces and hyperplanes

Positive semidefinite cone

notation:

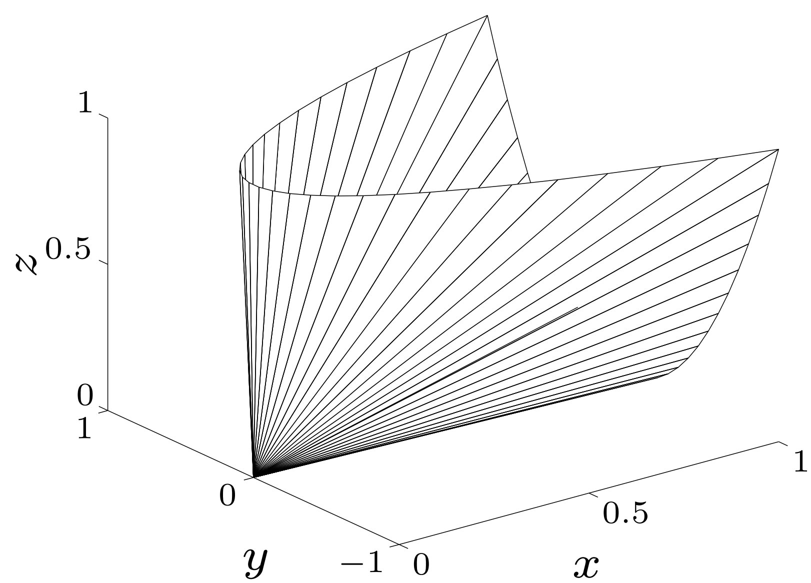
- $\mathbf{S}^n$ is set of symmetric $n \times n$ matrices
- $\mathbf{S}^n_+ = \{X \in \mathbf{S}^n \mid X \succeq 0\}$: positive semidefinite $n \times n$ matrices

$$X \in \mathbf{S}^n_+ \quad \Longleftrightarrow \quad z^T X z \geq 0 \text{ for all } z$$

  $\mathbf{S}^n_+$ is a convex cone

- $\mathbf{S}^n_{++} = \{X \in \mathbf{S}^n \mid X \succ 0\}$: positive definite $n \times n$ matrices

example: $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}^2_+$



Operations that preserve convexity

practical methods for establishing convexity of a set C

1. apply definition

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \Longrightarrow \quad \theta x_1 + (1-\theta)x_2 \in C$$

2. show that C is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, . . .) by operations that preserve convexity
   - intersection
   - affine functions
   - perspective function
   - linear-fractional functions

Intersection

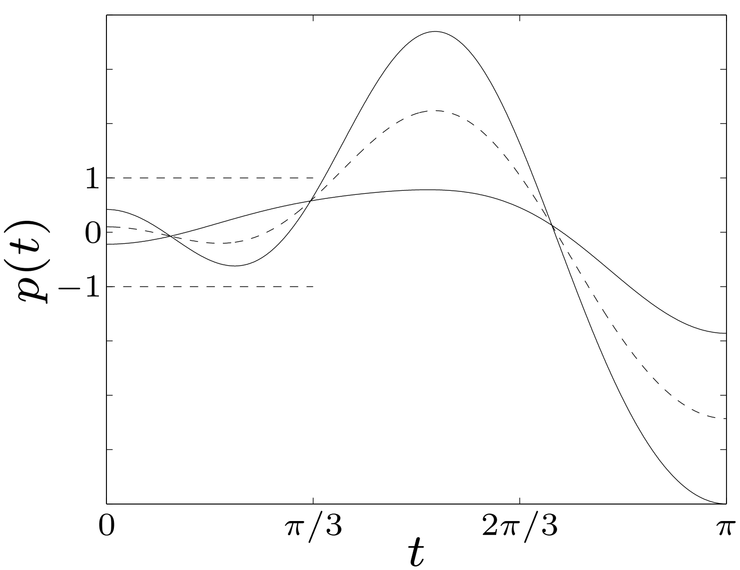
the intersection of (any number of) convex sets is convex

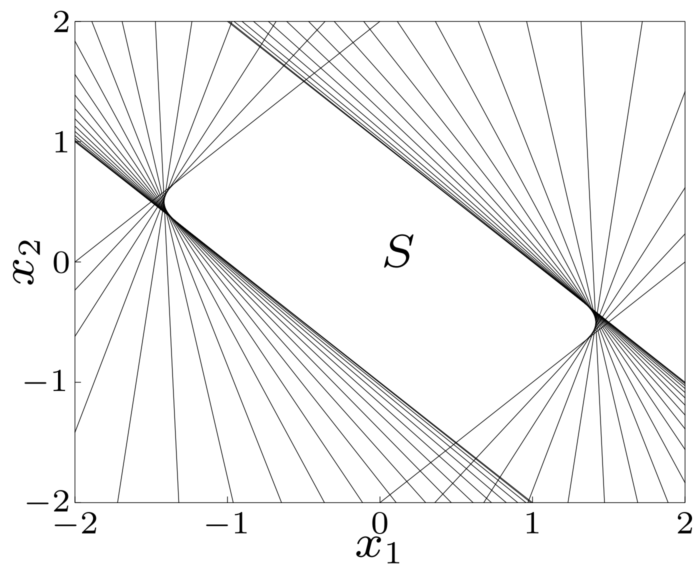
example:

$$S = \{x \in \mathbb{R}^m \mid |p(t)| \le 1 \text{ for } |t| \le \pi/3\}$$

where $p(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_m \cos mt$

for $m = 2$:





Affine function

suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is affine ($f(x) = Ax + b$ with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$)

- the image of a convex set under f is convex

$$S \subseteq \mathbb{R}^n \text{ convex} \quad \Longrightarrow \quad f(S) = \{f(x) \mid x \in S\} \text{ convex}$$

- the inverse image $f^{-1}(C)$ of a convex set under f is convex

$$C \subseteq \mathbb{R}^m \text{ convex} \quad \Longrightarrow \quad f^{-1}(C) = \{x \in \mathbb{R}^n \mid f(x) \in C\} \text{ convex}$$

examples

- scaling, translation, projection
- solution set of linear matrix inequality $\{x \mid x_1 A_1 + \cdots + x_m A_m \preceq B\}$ (with $A_i, B \in \mathbf{S}^p$)
- hyperbolic cone $\{x \mid x^T P x \leq (c^T x)^2,\ c^T x \geq 0\}$ (with $P \in \mathbf{S}^n_+$)

Perspective and linear-fractional function

perspective function $P : \mathbb{R}^{n+1} \to \mathbb{R}^n$:

$$P(x,t) = x/t, \qquad \mathbf{dom}\, P = \{(x,t) \mid t > 0\}$$

images and inverse images of convex sets under perspective are convex

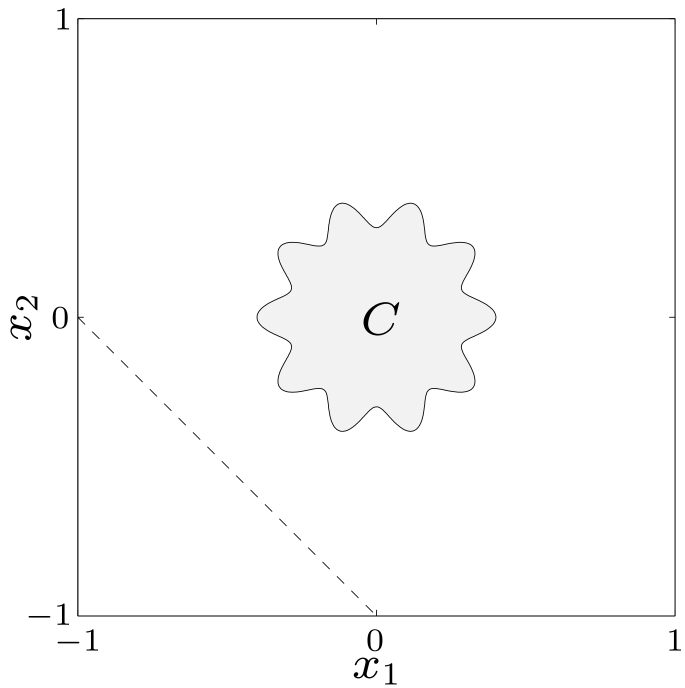
linear-fractional function $f : \mathbb{R}^n \to \mathbb{R}^m$:

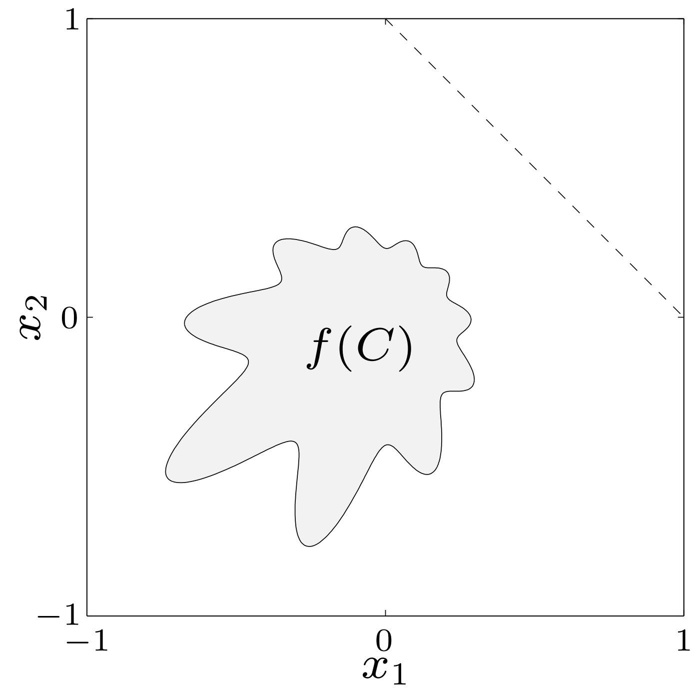
$$f(x) = \frac{Ax+b}{c^T x + d}, \qquad \mathbf{dom}\, f = \{x \mid c^T x + d > 0\}$$

images and inverse images of convex sets under linear-fractional functions are convex

example of a linear-fractional function

$$f(x) = \frac{1}{x_1 + x_2 + 1} x$$





Generalized inequalities

a convex cone $K \subseteq \mathbb{R}^n$ is a **proper cone** if

- K is closed (contains its boundary)
- K is solid (has nonempty interior)
- K is pointed (contains no line)

examples

- nonnegative orthant $K = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_i \geq 0, i = 1, \ldots, n\}$
- positive semidefinite cone $K = \mathbf{S}^n_+$
- nonnegative polynomials on $[0, 1]$:

$$K = \{x \in \mathbb{R}^n \mid x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1} \geq 0 \text{ for } t \in [0, 1]\}$$

generalized inequality defined by a proper cone K:

$$x \preceq_K y \quad \Longleftrightarrow \quad y - x \in K, \qquad\qquad x \prec_K y \quad \Longleftrightarrow \quad y - x \in \mathbf{int}\, K$$

examples

- componentwise inequality ($K = \mathbb{R}^n_+$)

$$x \preceq_{\mathbf{R}^n_+} y \quad \Longleftrightarrow \quad x_i \leq y_i, \quad i = 1, \ldots, n$$

- matrix inequality ($K = \mathbf{S}^n_+$)

$$X \preceq_{\mathbf{S}^n_+} Y \quad \Longleftrightarrow \quad Y - X \text{ positive semidefinite}$$

these two types are so common that we drop the subscript in $\preceq_K$

properties: many properties of $\preceq_K$ are similar to $\leq$ on $\mathbb{R}$, *e.g.*,

$$x \preceq_K y, \quad u \preceq_K v \quad \Longrightarrow \quad x + u \preceq_K y + v$$

Minimum and minimal elements

$\preceq_K$ is not in general a *linear ordering*: we can have $x \not\preceq_K y$ and $y \not\preceq_K x$

$x \in S$ is **the minimum element** of S with respect to $\preceq_K$ if

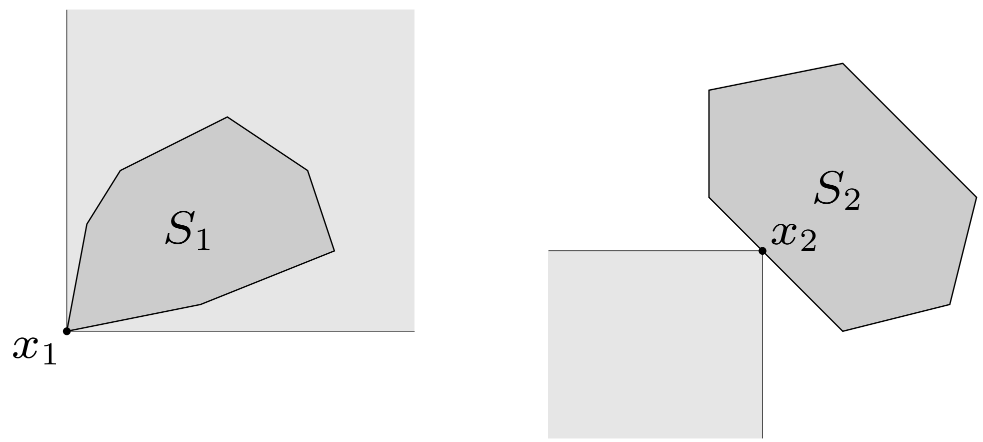
$$y \in S \quad \Longrightarrow \quad x \preceq_K y$$

$x \in S$ is **a minimal element** of S with respect to $\preceq_K$ if

$$y \in S, \quad y \preceq_K x \quad \Longrightarrow \quad y = x$$

example $(K = \mathbb{R}^2_+)$

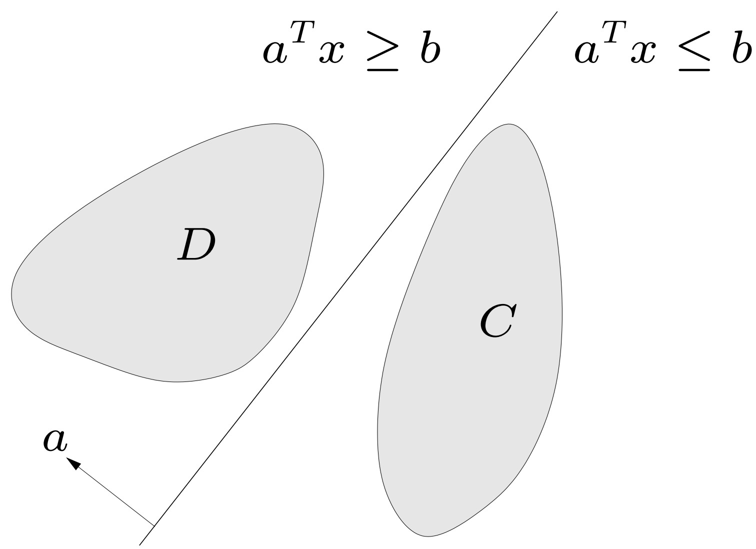
x_1 is the minimum element of S_1
x_2 is a minimal element of S_2



Separating hyperplane theorem

if C and D are disjoint convex sets, then there exists $a \neq 0$, b such that

$$a^T x \leq b \text{ for } x \in C, \qquad a^T x \geq b \text{ for } x \in D$$



the hyperplane $\{x \mid a^T x = b\}$ separates C and D

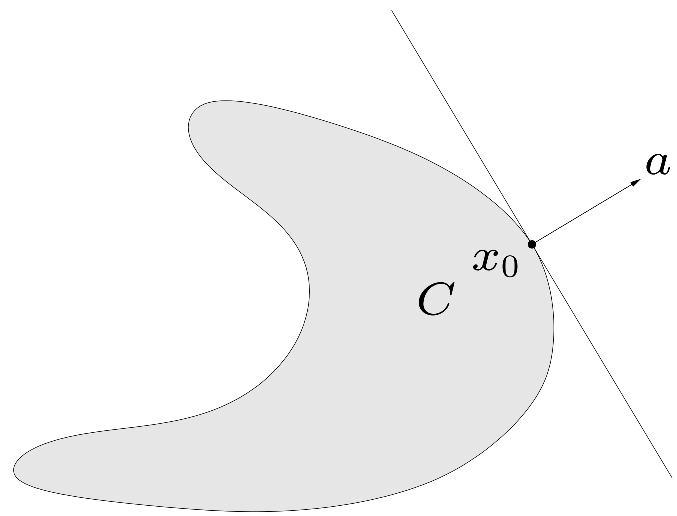
strict separation requires additional assumptions ($e.g.$, C is closed, D is a singleton)

Supporting hyperplane theorem

supporting hyperplane to set C at boundary point x_0:

$$\{x \mid a^T x = a^T x_0\}$$

where $a \neq 0$ and $a^T x \leq a^T x_0$ for all $x \in C$



supporting hyperplane theorem: if C is convex, then there exists a supporting hyperplane at every boundary point of C

Dual cones and generalized inequalities

dual cone of a cone K:

$$K^* = \{y \mid y^T x \geq 0 \text{ for all } x \in K\}$$

examples

- $K = \mathbb{R}^n_+$: $K^* = \mathbb{R}^n_+$
- $K = \mathbf{S}^n_+$: $K^* = \mathbf{S}^n_+$
- $K = \{(x,t) \mid \|x\|_2 \leq t\}$: $K^* = \{(x,t) \mid \|x\|_2 \leq t\}$
- $K = \{(x,t) \mid \|x\|_1 \leq t\}$: $K^* = \{(x,t) \mid \|x\|_\infty \leq t\}$

first three examples are **self-dual** cones

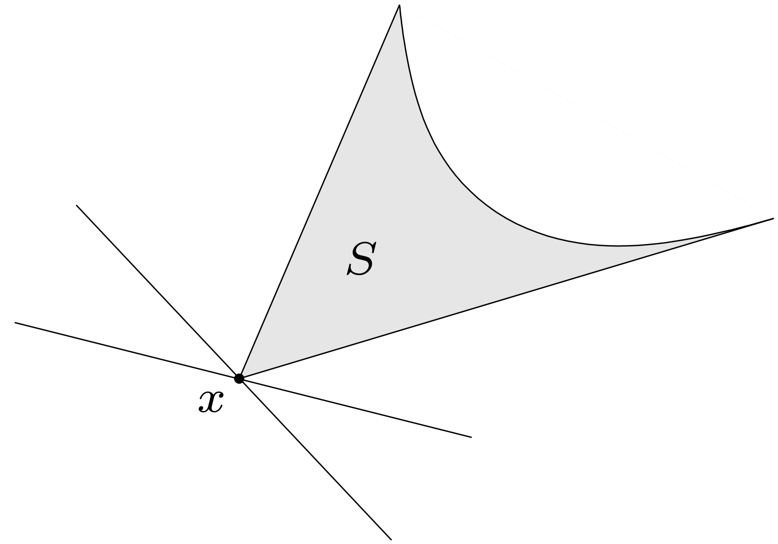
dual cones of proper cones are proper, hence define generalized inequalities:

$$y \succeq_{K^*} 0 \quad \Longleftrightarrow \quad y^T x \geq 0 \text{ for all } x \succeq_K 0$$

Minimum and minimal elements via dual inequalities

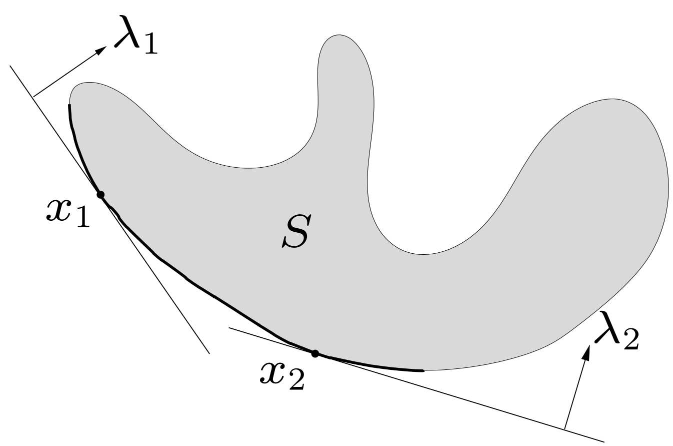
minimum element w.r.t. $\preceq_K$

x is minimum element of S iff for all $\lambda \succ_{K^*} 0$, x is the unique minimizer of $\lambda^T z$ over S



minimal element w.r.t. $\preceq_K$

- if x minimizes $\lambda^T z$ over S for some $\lambda \succ_{K^*} 0$, then x is minimal



- if x is a minimal element of a *convex* set S, then there exists a nonzero $\lambda \succeq_{K^*} 0$ such that x minimizes $\lambda^T z$ over S

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