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1. Introduction

Waves have significant effect on currents and circulation in general. This has led to the development of wave coupling theories ([1, 2]) in which one couples both waves and circulation models. However, the coupling procedure is a computationally expensive procedure that involves two models and requires great care in its implementation. As a consequence approximate approaches are of great interest for practical purposes.

Moreover, for environmental applications, the key parameter is the wave setup, i.e. how much of the sea surface elevation is due to waves. This was especially significant during the Xynthia storm in which the storm surge induced by tides, waves and currents was higher than expected. Therefore, we consider here the problem of estimating the wave setup based on the simplified equations given by [3].

2. The elliptic system

According to the classical Longuet-Higgins theory [1], if one takes the shallow water equations, in order to model the effect of the waves, one needs to introduce the following force $\mathbf{F}_{wave} = (F_{wave,x}, F_{wave,y})$:

$$\begin{cases} F_{wave;x} = -\dfrac{\partial \overline{S}_{xx}}{\partial x} - \dfrac{\partial \overline{S}_{xy}}{\partial y} \\ F_{wave;y} = -\dfrac{\partial \overline{S}_{xy}}{\partial x} - \dfrac{\partial \overline{S}_{yy}}{\partial y} \end{cases}$$

The potentials $\overline{S}_{\alpha\beta}$ are obtained by integration over the wave spectrum. This wave spectrum can be provided by a model such as WaveWatch III or WWMIII and the spectral integration can be done by those models. In order to estimate the induced sea surface elevation, we assume that the sea is static and standard hydrostatic gives us the relation:

$$\mathbf{F}_{wave} + d\,\mathbf{grad}\,z = 0$$

with d the bathymetry and z the sea surface elevation. This equation is over determined as is natural due to our static assumption. In [3] it is asserted that wave-setup is mainly due to the rotation-free part of the force field. Therefore we take the gradient and obtain the following equation:

$$div\left(\mathbf{F}_{wave}\right) + div\left(d\,\mathbf{grad}\,z\right) = 0.$$

In order to get a well posed system, we have to complement this equation with a boundary condition. Here we choose that at any point x of the boundary with normal $\mathbf{n}$ we have the equation:

$$\mathbf{n}\cdot\mathbf{F}_{wave} + d\,\mathbf{n}\cdot\mathbf{grad}\,z = 0$$

and so we get the following combined system, which is known as an elliptic Neumann equation system:

$$\begin{cases} div\left(\mathbf{F}_{wave}\right) + div\left(d\,\mathbf{grad}\,z\right) & = & 0 \\ \mathbf{n}\cdot\mathbf{F}_{wave} + d\,\mathbf{n}\cdot\mathbf{grad}\,z & = & 0 \end{cases}$$

We will apply standard techniques of numerical analysis for solving this system. Before that, we should point out that z is only defined up to some constant term.

2.1 Weak formulation

we are now looking at possible discretizations of the system. Before doing that, one has to look after a weak formulation of the problem. For simplicity we write:

$$\mathbf{G} = \mathbf{F}_{wave} + d\,\mathbf{grad}\,z$$

and we thus have $\mathbf{G}=0$ and $\mathbf{n}\cdot\mathbf{G}=0$ on the boundary. If u is a function defined in the interior Ω of the domain then one has by multiplying $\mathbf{G}=0$ by u and integrating over Ω the relation:

$$\begin{aligned} 0 &= \int_\Omega u\,div\,\mathbf{G} \\ &= \int_\Omega div(u\mathbf{G}) - \mathbf{grad}\,u\cdot\mathbf{G} \end{aligned}$$

So, by using Ostrogradski theorem, we obtain

$$\begin{aligned} \int_\Omega \mathbf{grad}\,u\cdot\mathbf{G} &= \int_\Omega div(u\mathbf{G}) \\ &= \int_{\partial\Omega} u\,\mathbf{n}\cdot\mathbf{G} \\ &= \int_{\partial\Omega} u\times 0 \\ &= 0 \end{aligned}$$

As a consequence, the weak formulation that we will work with is that for any function u in Ω we have

$$\int_\Omega \mathbf{grad}\,u\cdot\left\{\mathbf{F}_{wave} + d\,\mathbf{grad}\,z\right\} = 0$$

2.2 Discretization on unstructured grids

An unstructured grid is formed by a number N_{tri} of triangles together with a number N_{node} of nodes. In the numerical models that we are considering (WWMII and WW3), the model values are located at the nodes. So, for a node n, we need to define the fundamental function ψ_n. The function is defined so that

$$\psi_n(n') = \begin{cases} 1 & \text{if } n = n' \\ 0 & \text{otherwise} \end{cases}$$

We also require that on any triangle T of nodes n_1, n_2, n_3 the function ψ_n varies affinely on T. The functions ψ_n satisfy the fundamental property

$$\sum_{n=1}^{N_{node}} \psi_n = 1$$

on Ω. As a consequence a function f on Ω is approximated by

$$\sum_{n=1}^{N_{node}} f(n)\psi_n$$

We do this approximation for the bathymetry d and the sought free surface elevation z. So, we can reexpress our weak formulation in the case of a finite element formulation in the following way: For any node k we have the equality:

$$\int_\Omega \mathbf{grad}\psi_k \cdot \left\{ \mathbf{F}_{wave} + d\,\mathbf{grad} \sum_n z(n)\psi_n \right\} = 0.$$

The gradients $\mathbf{grad}\psi_k$ are easy to compute: they are constants on the elements. So, the explicit computation of the above formulation requires no more than taking the right averages. So, the system is expressed as:

$$AZ=B$$

It must be pointed out that the obtained matrix equation retains most features of the original system: The matrix A is sparse and symmetric. It is positive semi definite and its kernel corresponds to constant functions. Furthermore, it is easy to see that the vector B has sum 0 and so the system admits a solution.

2.3 Discretization on structured grids

For structured grids we proceed in a similar way. A critical issue is to choose well the basis functions that one uses for the solution of the system. As in the triangular case, the model numerical formulation puts the values at the nodes. So, we have again some function ψn that we need to choose. But, we have no easy way to choose functions varying on the whole space and satisfying.

$$\sum_{n=1}^{N_{node}} \psi_n = 1.$$

The method that we choose is to consider that finite difference schemes are determined by whether or not two points are adjacent. This explains why a river can me modeled in finite difference by a chain of adjacent points.

So, we define functions ψ_n such that

$$\psi_n(n') = \begin{cases} 1 & \text{if } n = n' \\ 0 & \text{otherwise} \end{cases}$$

and we require that ψ_n varies affinely on any edge between two adjacent nodes.

We denote Ω_e the set of all edges of Ω. The weak formulation is then obtained by setting for any node *k* we have the equality

$$\int_{\Omega_e} \mathbf{grad}\psi_k \cdot \left\{ \mathbf{F}_{wave} + d\,\mathbf{grad} \sum_n z(n)\psi_n \right\} = 0.$$

Just as in the case of unstructured grid, one obtains a system of the form

$$AZ=B$$

with *A* a sparse symmetric matrix. The matrix *A* is again positive semidefinite and its kernel is equal to the set of constant functions.

2.4 Solution method

The method that one uses for solving system *AZ=B* for *A* a positive definite matrix is classically the conjugate gradient method [4]. Such methods are iterative, i.e. returns approximate solution with an error that can be fully controlled. For efficient performance, it is often important to have a preconditioner, i.e. an approximate inverse *M* of the matrix *A*. For every node *i*, we denote by *N*(*i*) the set of neighbors of *i* in the network (this number is around 6 for unstructured mesh and is at most 4 for structured meshes). The matrix *A* is expressed as:

$$(Ax)_i = \alpha_i x_i + \sum_{j \in N(i)} c_{ij} x_j$$

and we impose that the approximate inverse has the same sparsity, i.e.

$$(Mx)_i = \beta_i x_i + \sum_{j \in N(i)} d_{ij} x_j$$

A good approximate inverse is, for example

$$\beta_i = \alpha_i^{-1} \; and \; d_{ij} = -c_{ij}\beta_i\beta_j$$

By using *M* we manage to reduce by 2 the number of needed iterations.

2.5 Results on a test case

In this example we set up a classical breakwater test case. To do so, the same parameters as considered in [5] are used, i.e. a 9 m long basin with a maximum depth of 0.45 m has a beach of slope 1:10. The incoming waves were of frequency 1.5 s and a height of 0.181 m. The wave breaking is modeled by $H_s \leq c_B d$ with c_B=0.78. We restricted ourselves to the [1] theory for which it is easy to compute these parameters with high precision. Without wave breaking the wave action flux is conserved and so $H_s^2 c_g$ is conserved, while $H_s = c_B d$ in case of wave breaking. Plots of obtained results with WWMII are shown in Figure 1.

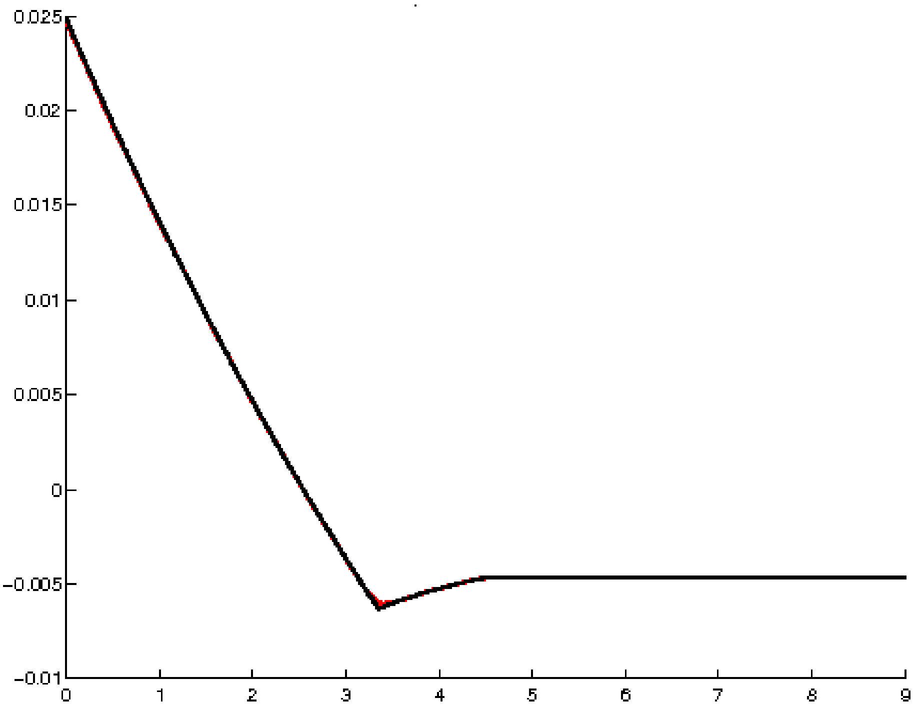

Figure 1: The plots of zeta setup for the WWMII model. Red is model output. Black is analytical solution.

3. Literature

[1] Longuet-Higgins, M.S., Stewart, R.W., 1964. Radiation stresses in water waves; a physical discussion with applications. Deep-Sea Res. 11, 529{562.

[2] Ardhuin, F., Rascle, N., Belibassakis, K.A., 2008. Explicit wave averaged primitive equations using a generalized lagrangian mean. Ocean Model. 20, 35{60.

[3] Dingemans, M.W., 1997. Water wave propagation over uneven bottoms. Part 1 - linear wave propagation. Advanced Series on Ocean Engineering, 13,World Scientic, 471 p.

[4] Shewchuk, J.R., 1994. An introduction to the conjugate gradient method without the agonizing pain. http://www.cs.cmu.edu/quake-papers/ painless-conjugate-gradient.pdf

[5] Xia, H., Xia, Z., Zhu, L., 2004. Vertical variation in radiation stress and wave-induced current. Coastal Eng. 51, 309{321.

4. Summary

We have finished at this time all the technical part of the contract and we are now merging our developments into the latest brunch of NCEP/IFREMER in order to have the most actual version delivered to SHOM. The final report summarizing everything that has been done will be submitted at the beginning of February with the source code and all test cases. The actual version of the source code is attached to the E-Mail of this report.

I remain with my best regards

Dr.-Ing. A. Roland, 10.12.2013, Frankfurt, Germany