Spheric analogs of fullerenes

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I. 8 families of standard $(\{a, b\}, k)$-spheres

(R, k)-spheres: curvature $C_i = \frac{1}{k} + \frac{1}{i} - \frac{1}{2}$ of i-gons

- Fix $R \subset \mathbb{N}$, an (R, k)-sphere is a k-regular, $k \geq 3$, map on $\mathbb{S}^2$ whose faces are i-gons, $i \in R$. Let $m =$ min and $M = \max_{i \in R}$.
- Let v, e and $f = \sum_i p_i$ be the numbers of vertices, edges and faces of S, where p_i is the number of i-gonal faces. Clearly, $kv = 2e = \sum_i i p_i$ and Euler formula $v - e + f = 2$ become $2 = \sum_i i p_i C_i$, where $C_i = \frac{1}{k} + \frac{1}{i} - \frac{1}{2}$ is the curvature of i-gons.
- So, $m < \frac{2k}{k-2}$. For $m \geq 3$, it implies $3 \leq m, k \leq 5$, i.e. 5 Platonic pairs of parameters $(m, k) = (3, 3), (4, 3), (3, 4), (5, 3), (3, 5)$.
- If $M < \frac{2k}{k-2}$ ($\min_{i \in R} C_i > 0$), then $M \leq 5$, $k = 3$ or $M \leq 3$, $k \in \{4, 5\}$ So, for $m \geq 3$, they are only Octahedron, Icosahedron and 11 $(\{3, 4, 5\}, 3)$-spheres: 8 dual deltahedra, Cube and its truncations on 1 or 2 opposite vertices (Dürer octahedron).

Standard (R, k)-spheres

- An (R, k)-sphere is standard if $M = \frac{2k}{k-2}$, i.e. $\min_{i \in R} C_i = 0$. So, $(M, k) = (6, 3), (4, 4), (3, 6)$ (Euclidean parameter pairs). Exclusion of i-faces $C_i < 0$ simplifies enumeration, while the number p_M of flat ($C_M = 0$) M-faces not being restricted, there is an infinity of such (R, k)-spheres.

- The number of such v-vertex (R, k)-spheres with $|R| = 2$ increases polynomially with v; their set is countable. Such spheres admit parametrization and description in terms of rings of (Gaussian if $k = 4$ and Eisenstein if $k = 3, 6$) integers. All eight series of such spheres will be considered in detail.

- Remaining (R, k)-spheres (with $M > \frac{2k}{k-2}$) not admit above, in general. The number of such v-vertex $(\{3, 4\}, 5)$-spheres grows at least exponentially with v.

8 families of standard $(\{a, b\}, k)$-spheres

- An $(\{a, b\}, k)$-sphere is an (R, k)-sphere with $R = \{a, b\}$, $1 \le a < b$. It has $v = \frac{1}{k}(ap_a + bp_b)$ vertices.
- Such standard sphere has $b = \frac{2k}{k-2}$; so, $(b, k) = (6, 3), (4, 4), (3, 6)$ and Euler formula become $2 = aC_a p_a = a(\frac{1}{a} + \frac{1}{k} - \frac{1}{2})p_a = (1 - \frac{a}{b})p_a$.
- So, $p_a = \frac{2b}{b-a}$ and all possible (a, p_a) are:
  $(5, 12), (4, 6), (3, 4), (2, 3)$ for $(b, k) = (6, 3)$;
  $(3, 8), (2, 4)$ for $(b, k) = (4, 4)$;
  $(2, 6), (1, 3)$ for $(b, k) = (3, 6)$.
- Those 8 families can be seen as spheric analogs of the regular plane partitions $\{6^3\}, \{4^4\}, \{3^6\}$ with p_a a-gonal "defects", disclinations added to get the curvature of the sphere $\mathbb{S}^2$.

8 families: existence criterions

Grűnbaum-Motzkin, 1963: criterion for $k=3 \leq a$; Grűnbaum, 1967: for $(\{3,4\},4)$-spheres; Grűnbaum-Zaks, 1974: for other cases.

k	(a,b)	smallest one	it exists if and only if	p_a	v
3	$(5,6)$	Dodecahedron	$p_6 \neq 1$	12	$20 + 2p_6$
3	$(4,6)$	Cube	$p_6 \neq 1$	6	$8 + 2p_6$
4	$(3,4)$	Octahedron	$p_4 \neq 1$	8	$6 + p_4$
6	$(2,3)$	$6 \times K_2$	p_3 is even	6	$2 + \frac{p_3}{2}$
3	$(3,6)$	Tetrahedron	p_6 is even	4	$4 + 2p_6$
4	$(2,4)$	$4 \times K_2$	p_4 is even	4	$2 + p_4$
3	$(2,6)$	$3 \times K_2$	$p_6 = (k^2 + kl + l^2) - 1$	3	$2 + 2p_6$
6	$(1,3)$	Trifolium	$p_3 = 2(k^2 + kl + l^2) - 1$	3	$\frac{1+p_3}{2}$
5	$(3,4)$	Icosahedron	$p_4 \neq 1$	$2p_4 + 20$	$2p_4 + 12$

$(\{3,6\},3)$- (Grűnbaum-Motzkin, 1963) and $(\{2,4\},4)$-spheres (Deza-Shtogrin, 2003) admit a simple 2-parametric description.

8 families of standard $(\{a, b\}, k)$-spheres

- Let us denote $(\{a, b\}, k)$-sphere with v vertices by $\{a, b\}_v$.
- $(\{5, 6\}, 3)$- and $(\{4, 6\}, 3)$-spheres are (geometric) fullerenes and *boron nitrides*. $\{5, 6\}_{60}(I_h)$: a new *carbon allotrope* C_{60}. $\{5, 6\}_{620}(I) = GC_{5,1}(\{5, 6\}_{20}) \approx$ *Callaway golf ball* $\{5, 6\}_{660}$.
- $(\{a, b\}, 4)$-spheres are minimal projections of alternating links, whose components are their *central circuits* (those going only ahead) and crossings are the verices.
- By smallest member Dodecahedron $\{5, 6\}_{20}$, Cube $\{4, 6\}_8$, Tetrahedron $\{3, 6\}_4$, Octahedron $\{3, 4\}_6$ and $3 \times K_2$ $\{2, 6\}_2$, $4 \times K_2$ $\{2, 4\}_2$, $6 \times K_2$ $\{2, 3\}_2$, Trifolium $\{1, 3\}_1$, we call eight families: dodecahedrites, cubites, tetrahedrites, octahedrites and 3-bundelites, 4-bundelites, 6-bundelites, trifoliumites.
- b-icosahedrites $((\{3, b\}, 5)$-spheres) are not standard if $b \geq 3$, $p_b \geq 0$, since $p_3 = p_b(3b\text{-}10) + 20$ and $C_b = \frac{10 - 3b}{10b} < 0$.

Digression on Rose of Three Petals

- The polar equation of the rose (or *rhodonea*) is $r=\cos(n\theta)$. $\{1,3\}_1$ models its case $n=3$: *quartic* (algebraic of degree 4) plane curve Trifolium $(x^2+y^2)^2=x(x^2-3y^2)$ shown below.
- It models also sextic $(x^2+y^2)^3=2x(x^2-3y^2)$ or $r^3=2\cos(3\theta)$: Kiepert curve $d(x,A)d(x,B)d(x,C)=1$ for reg. triangle ABC
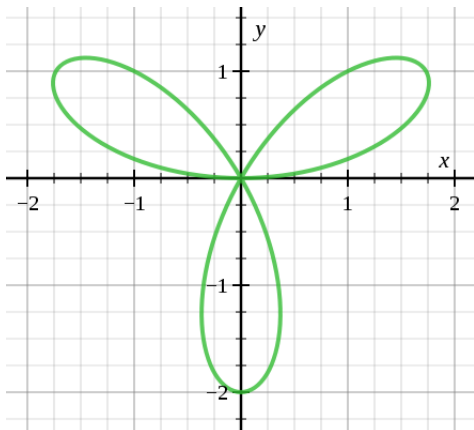
Generation of standard $(\{a, b\}, k)$-spheres

- $(\{2, 3\}, 6)$-spheres, except $2 \times K_2$ and $2 \times K_3$, are the duals of $(\{3, 4, 5, 6\}, 3)$-spheres with six new vertices put on edge(s). Exp: $(\{5, 6\}, 3)$-spheres with 5-gons organized in six pairs.

- $(\{1, 3\}, 6)$-spheres, except $\{1, 3\}_1$ and $\{1, 3\}_3$, are as above but with 3 edges changed into 2-gons enclosing one 1-gon.

- $(\{2, 6\}, 3)$-spheres are given by the *Goldberg-Coxeter construction* from Bundle$_3 = 3 \times K_2$ $\{2, 6\}_2$.

- $(\{1, 3\}, 6)$-spheres come by the *Goldberg-Coxeter construction* (extended below on 6-regular spheres) from Trifolium $\{1, 3\}_1$.

Computer generation of the families

Main technique: exhaustive search. Sometimes, speedup by proving that a group of faces cannot be completed to the desired graph.

- The program CPF by Brinkmann-Delgado-Dress-Harmuth, 1997 generates 3-regular plane graphs with specified p-vector.
- ENU by Brinkmann-Harmuth-Heidemeier, 2003 and Heidemeier, 1998 does the same for 4-regular plane graphs. Dutour adapted ENU to deal with 2-gonal faces also.
- CGF by Harmuth generates 3-regular orientable maps with specified genus and p-vector.
- Plantri by Brinkmann-McKay deals with general graphs.
- The package CaGe by Brinkmann-Delgado-Dress-Harmuth, 1997 is used for plane graph drawings.
- The package PlanGraph by Dutour, 2002 is used for handling planar graphs in general.

II. Connectedness of $(\{a, b\}, k)$-spheres

- A graph is called *k-connected* if after removing any set of $k-1$ vertices it remains connected.
- The skeleton of a polytope P is the graph $G(P)$ formed by its vertices, with two vertices adjacent if they generate a face.
- Steinitz Theorem: a graph is the skeleton of a polyhedron (3-polytope) if and only if it is planar and 3-connected.
- A polyhedron is usually represented by the *Schlegel diagram* of its skeleton, the program used for this is CaGe.
- The dual graph G^* of a plane graph G is the plane graph formed by the faces of G, with two faces adjacent if they share an edge. The skeletons of dual polyhedra are dual.

- Any ($\{a, b\}, k$)-sphere is 2-connected. But some infinite series of ($\{1, 2, 3\}, 6$)-spheres with (p_1, p_2)=(2, 2) are *not*.
- Any ($\{a, 6\}, 3$)-sphere is 3-connected if $a = 4, 5$ and not if $a = 2$ (one can delete two vertices adjacent to a 2-gon).
- Except the following series, ($\{3, 6\}, 3$)-spheres (moreover, all ($\{3, 4, 5, 6\}, 3$)-spheres) are 3-connected.
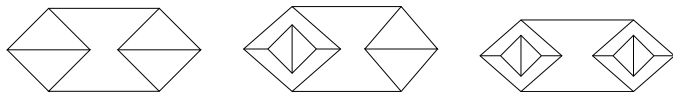
3-connectedness of ($\{a, b\}, 6$)- and ($\{a, b\}, 4$)-spheres

- Any ($\{a, b\}, 6$)-sphere is 3-connected, except ($\{2, 3\}, 6$)- ones which are duals of only 2-connected ($\{3, 6\}, 3$)-spheres, with six vertices of degree 2 added on edges.
- Any ($\{a, b\}, 4$)-sphere is 3-connected, except the following series of ($\{2, 4\}, 4$)-spheres.
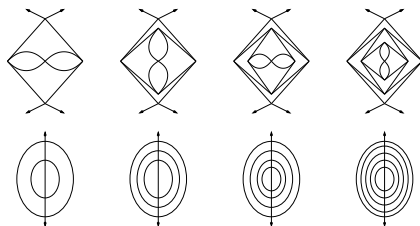
REMARK. $\{2, 4\}_v (D_{2d}, D_{2h})$ are *k-inflations* of above. D_4, D_{4h} are $GC_{k,l}(4 \times K_2)$. Remaining D_2: 2 complex or 3 natural parameters.

- Grűnbaum-Zaks, 1974: all $(\{1, 3\}, 6)$- and $(\{2, 4\}, 4)$-spheres are Hamiltonian, but $(\{2, 6\}, 3)$- with $v \equiv 0 \pmod{4}$ are not
- Goodey, 1977: $(\{3, 6\}, 3)$- and $(\{4, 6\}, 3)$- are Hamiltonian.
- Conjecture: an Hamiltonian circuit exists in all other cases.

To check hamiltonicity of a $(\{a, b\}, k)$-map on the projective plane $\mathbb{P}^2$, the following theorem (Thomas-Yu, 1994) could help: every 4-connected graph on $\mathbb{P}^2$ has a *contractible* (i.e. being a boundary of 2-cell) Hamiltonian circuit.

II'. $(\{a, b\}, k)$-spheres with small p_b: listings

$(\{a, b\}, k)$-spheres with $p_b \leq 2 < a < b$

- Remind: $(a, k) = (3, 3), (4, 3), (3, 4), (5, 3), (3, 5)$ if $k, a \geq 3$.
- The only $(\{a, b\}, k)$-spheres with $p_b \leq 1$ are 5 Platonic (a^k): Tetrahedron, Cube $(Prism_4)$, Octahedron $(APrism_3)$, Dodecahedron (snub $Prism_5$), Icosahedron (snub $APrism_3$).
- There exists unique trivial 3-connected $(\{a, b\}, k)$-sphere with $p_b = 2$ for $(\{4, b\}, 3)$-, $(\{3, b\}, 4)$-, $(\{5, b\}, 3)$-, $(\{3, b\}, 5)$-: D_{bh} $Prism_b$ and D_{bd} $APrism_b$, snub $Prism_b$, snub $APrism_b$: two b-gons separated by b-ring of 4-gons, $2b$-ring of 3-gons, two b-rings of 5-gons, two $3b$-rings of 3-gons.
- Also, for $t \geq 2$, 10 non-trivial $(\{a, at\}, k)$-spheres with $p_{at} = 2$: 5 $(\{a, ta\}, k)$-spheres are (D_{th}) necklaces of polycycles $\{a^k\}$-e, 3 are (D_{th}) necklaces of t v-split $\{3^4\}$ and e-split $\{5^3\}$, $\{3^5\}$, $(\{3, 3t\}, 5)$-spheres C_{th}, D_t are necklaces of t v-, f-split $\{3^5\}$.
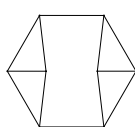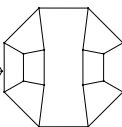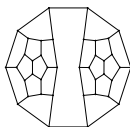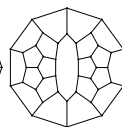
D_{2h}: $a=3$ $a=4$ $a=5$ $a=5$
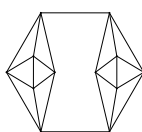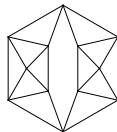
$a=3$ D_{2h} $a=3$ D_{2h}
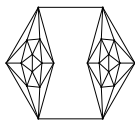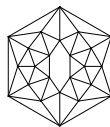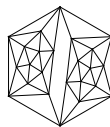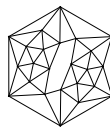
$a=3$ D_{2h} $a=3$ D_{2h} $a=3$ C_{2h} $a=3$ D_2

Proof method: elementary (a, k)-polycycles

- A (a, k)-polycycle is a 2-connected plane graph with faces partitioned in a-gonal proper faces and holes, exterior face among them, so that vertex degrees are in $\{2, \ldots, k\}$ and can be $< k$ only for a vertex lying on the boundary of a hole.

- Any (a, k)-polycycle decomposes uniquely along its bridges (non-boundary going hole-to-hole, possibly, same, edges) into elementary ones. Cf. integer factorisation into primes.

- We listed them for $\frac{1}{a} + \frac{1}{k} - \frac{1}{2} \geq 0$. Othervise, continuum...
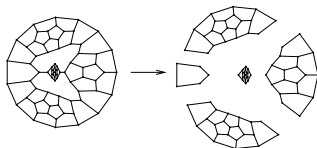
This $(\{5, 15\}, 3)$-sphere with $p_{15}=3$ is a 3-holes $(\{5\}, 3)$-polycycle It decomposes into five 1-hole elementary $(\{5\}, k)$-polycycles.

- $(\{a, b\}, k)$-sphere with $p_b = 3$ exists if and only if $b \equiv 2, a, 2a - 2 \, (mod \, 2a)$ and $b \equiv 4, 6 \, (mod \, 10)$ if $a=5$.
- Such sphere are unique if b is not $\equiv a \, (mod \, 2a)$ and then their symmetry is D_{3h}, except when $(a, k) = (3, 5)$ when the symmetry is D_3.
- There are 7 such spheres with $t = \lfloor \frac{b}{6} \rfloor = 0$ and $3+4+5+17$ of them for any $t \geq 1$.

- Such ($\{3, b\}, 3$)-sphere exists iff $b \equiv 2, 3, 4 \, (mod \, 6)$:
  unique (D_{3h}, C_{3v}, D_{3h}), respectively, and comes by putting
  $t = \lfloor \frac{b}{6} \rfloor \{3^3\}$-$e$'s on 3 edges of $3K_2$, $Prism_3$, Tetrahedron $\{3^3\}$.
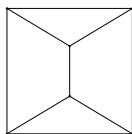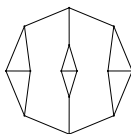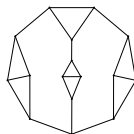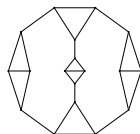
b=4, D_{3h} 2+6, D_{3h} 3+6, C_{3v} 4+6, D_{3h}
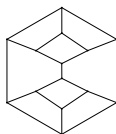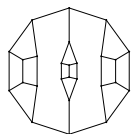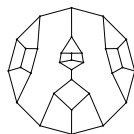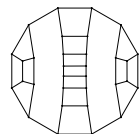
b=6, D_{3h} 2+8, D_{3h} 4+8, C_{3v} 4+8, C_{2v}

- Such ($\{4, b\}, 3$)-sphere exists iff $b \equiv 2, 4, 6 \, (mod \, 8)$: 2 ($C_{3v}$,
  C_{2v}) if $b \equiv 4 \, (mod \, 8)$ and unique (D_{3h}), otherwise.

$(\{(3, b), 4\})$-spheres with $p_b = 3$

- Such $(\{3, b\}, 4)$-sphere exists iff $b \equiv 2, 3, 4 \,(mod\, 6)$:
  3 (C_{3v}, C_s, C_s) if $b \equiv 3 \,(mod\, 6)$ and unique (D_{3h}), otherwise.
- For $b = 2+6t$, $4+6t$ and C_{3v}-case of $3+6t$, it is $3K_2$, 9-vertex $(\{3, b\}, 4)$-sphere, Octahedron $\{3^4\}$ with 3 edges replaced by t v-split $\{3^4\}$'s. It is 3-connected iff $b = 2, 4$.
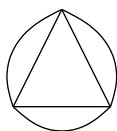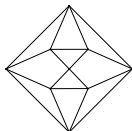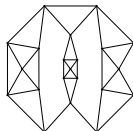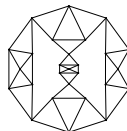
D_{3h}, $b=2$ $b=4$ $b=2+6$ $b=4+6$
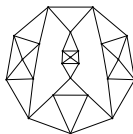
3+6, C_{3v} 3+6, C_s 3+6, C_s
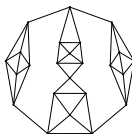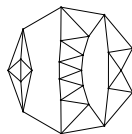
Such ($\{5, b\}, 3$)-sphere exists iff $b \equiv 2, 4, 5, 6, 8 \, (mod \, 10)$:
5 (2 C_{3v} and 2 C_s) if $b \equiv 5 \, (mod \, 10)$ and unique (D_{3h}), otherwise.
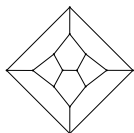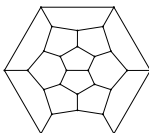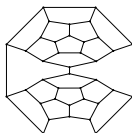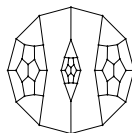
D_{3h}: b=4   b=6   b=8   b=2+10
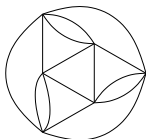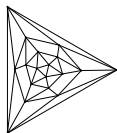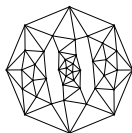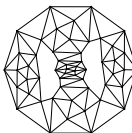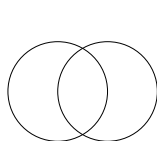
D_3: b=2   b=4   b=2+6   b=4+6

Such ($\{3, b\}, 5$)-sphere exists if $b \equiv 2, 3, 4 \, (mod \, 6)$:
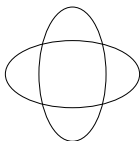15 if $b \equiv 3 \, (mod \, 6)$ and one (D_3), otherwise.
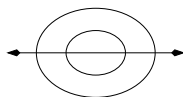
III. 8 standard families:
4 smallest members
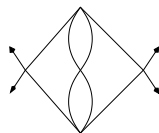
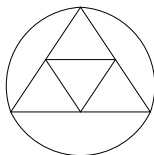# First four ($\{2,4\},4$)- and ($\{3,4\},4$)-spheres

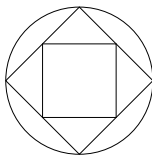D_{4h} 2_1^2 (2^2) D_{4h} 4_1^2 (4^2) D_{2h} $2 \times 2_1^2$ ($2^2,4$) D_{2d} 6_2^2 (6^2)
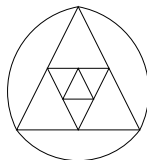
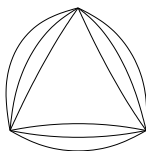O_h 6_2^3 (4^3) D_{4d} 8_{18} (16) D_{3h} 9_{40} (18) D_2 10_{56}^2
Borr. rings   (Herschel)*   (6;14)
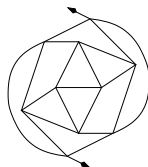
Above links/knots are given in Rolfsen, 1976 and 1990 notation.
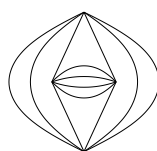Herschel graph: the smallest non-Hamiltonian polyhedral graph.
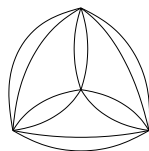
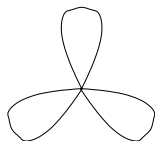# First four $(\{2,3\},6)$- and $(\{1,3\},6)$-spheres
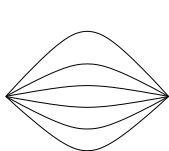
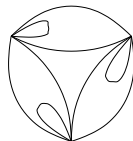D_{6h} (2^3) D_{3h} $(3;6)$ D_{2d} $(2^2;8)$ T_d (3^4)
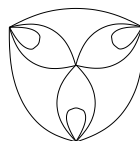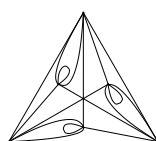
C_{3v} (3) C_{3h} $(3;6)$ C_{3v} (6^2) C_3 (21)

Grűnbaum-Zaks, 1974: $\{1,3\}_v$ exists iff $v = k^2 + kl + l^2$ for integers $0 \leq l \leq k$. We show that the number of $\{1,3\}_v$'s is the number of such representations of v, i.e. found $GC_{k,l}(\{1,3\}_1)$.

First four $(\{2,6\},3)$- and $(\{3,6\},3)$-spheres

Number of $(\{2,6\}_v$'s is nr. of representations $v=2(k^2 + kl + l^2)$,
$0 \leq l \leq k$ ($GC_{k,l}(\{2,6\}_2)$). It become 2 for $v=7^2=5^2+15+3^2$.
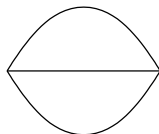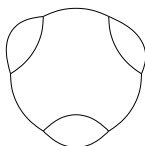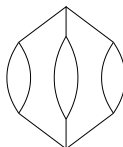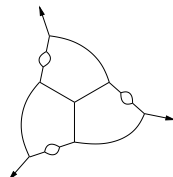
D_{3h} (6) D_{3h} (6^3) D_{3h} (12^2) D_3 (42)
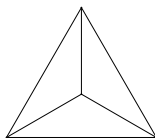
T_d (4^3) D_{2h} ($8^2, 4^2$) T_d (12^3) T_d (8^6)
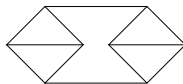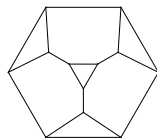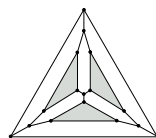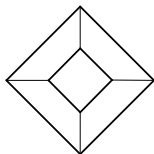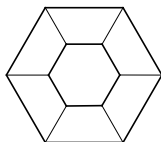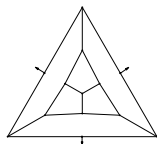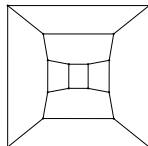
O_h (6^4) D_{6h} (18^2) D_{3h} $(6^2; 30)$ D_{2d} (24^2)
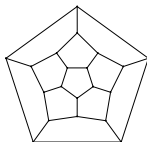
I_h (10^6) D_{6d} $(12; 60)$ D_{3h} $(12^3; 42)$ T_d (12^7)

IV. Symmetry groups of $(\{a, b\}, k)$-spheres

All finite groups of isometries of 3-space $\mathbb{E}^3$ are classified.
In Schoenflies notations, they are:

- C_1 is the trivial group

- C_s is the group generated by a plane reflexion

- $C_i = \{I_3, -I_3\}$ is the inversion group

- C_m is the group generated by a rotation of order m of axis Δ

- C_{mv} ($\simeq$ dihedral group) is the group generated by C_m and m reflexion containing Δ

- $C_{mh} = C_m \times C_s$ is the group generated by C_m and the symmetry by the plane orthogonal to Δ

- S_{2m} is the group of order $2m$ generated by an antirotation, i.e. commuting composition of a rotation and a plane symmetry

- D_m ($\simeq$ dihedral group) is the group generated of C_m and m rotations of order 2 with axis orthogonal to Δ
- D_{mh} is the group generated by D_m and a plane symmetry orthogonal to Δ
- D_{md} is the group generated by D_m and m symmetry planes containing Δ and which does not contain axis of order 2
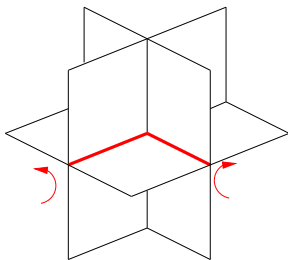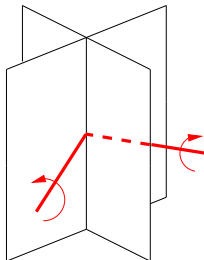
D_{2h} D_{2d}

Remaining 7 finite isometry groups

- $I_h = H_3$ is the group of isometries of Dodecahedron;
  $I_h \simeq Alt_5 \times C_2$
- $I \simeq Alt_5$ is the group of rotations of Dodecahedron
- $O_h = B_3$ is the group of isometries of Cube
- $O \simeq Sym(4)$ is the group of rotations of Cube
- $T_d = A_3 \simeq Sym(4)$ is the group of isometries of Tetrahedron
- $T \simeq Alt(4)$ is the group of rotations of Tetrahedron
- $T_h = T \cup -T$

While (point group) $Isom(P) \subset Aut(G(P))$ (combinatorial group),
Mani, 1971: for any 3-polytope P, there is a map-isomorphic
3-polytope P' (so, with the same skeleton $G(P') = G(P)$), such
that the group $Isom(P')$ of its isometries is isomorphic to $Aut(G)$.

8 families: symmetry groups

- 28 for $\{5,6\}_v$: C_1, C_s, C_i; C_2, C_{2v}, C_{2h}, S_4; C_3, C_{3v}, C_{3h}, S_6; D_2, D_{2h}, D_{2d}; D_3, D_{3h}, D_{3d}; D_5, D_{5h}, D_{5d}; D_6, D_{6h}, D_{6d}; T, T_d, T_h; I, I_h (Fowler-Manolopoulos, 1995)

- 16 for $\{4,6\}_v$: C_1, C_s, C_i; C_2, C_{2v}, C_{2h}; D_2, D_{2h}, D_{2d}; D_3, D_{3h}, D_{3d}; D_6, D_{6h}; O, O_h (Deza-Dutour, 2005)

- 5 for $\{3,6\}_v$: D_2, D_{2h}, D_{2d}; T, T_d (Fowler-Cremona,1997)

- 2 for $\{2,6\}_v$: D_3, D_{3h} (Grünbaum-Zaks, 1974)

- 18 for $\{3,4\}_v$: C_1, C_s, C_i; C_2, C_{2v}, C_{2h}, S_4; D_2, D_{2h}, D_{2d}; D_3, D_{3h}, D_{3d}; D_4, D_{4h}, D_{4d}; O, O_h (Deza-Dutour-Shtogrin, 2003)

- 5 for $\{2,4\}_v$: D_2, D_{2h}, D_{2d}; D_4, D_{4h}, all in $[D_2, D_{4h}]$ (same)

- 3 for $\{1,3\}_v$: C_3, C_{3v}, C_{3h} (Deza-Dutour, 2010)

- 22 for $\{2,3\}_v$: C_1, C_s, C_i; C_2, C_{2v}, C_{2h}, S_4; C_3, C_{3v}, C_{3h}, S_6; D_2, D_{2h}, D_{2d}; D_3, D_{3h}, D_{3d}; D_6, D_{6h}; T, T_d, T_h (same)

1. 38 for icosahedrites ($\{3,4\}, 5$)- (same, 2011).

8 families: Goldberg-Coxeter construction $GC_{k,l}(.)$

With $\mathbf{T}=\{T, T_d, T_h\}$, $\mathbf{O}=\{O, O_h\}$, $\mathbf{I}=\{I, I_h\}$, $\mathbf{C_1}=\{C_1, C_s, C_i\}$, $\mathbf{C_m}=\{C_m, C_{mv}, C_{mh}, S_{2m}\}$, $\mathbf{D_m}=\{D_m, D_{mh}, D_{md}\}$, we get

- for $(\{5,6\}, 3)$-: $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{C_3}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\mathbf{D_5}$, $\mathbf{D_6}$, $\mathbf{T}$, $\mathbf{I}$
- for $(\{2,3\}, 6)$-: $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{C_3}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\{D_6, D_{6h}\}$, $\mathbf{T}$
- for $(\{4,6\}, 3)$-: $\mathbf{C_1}$, $\mathbf{C_2}\backslash S_4$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\{D_6, D_{6h}\}$, $\mathbf{O}$
- for $(\{3,4\}, 4)$-: $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\mathbf{D_4}$, $\mathbf{O}$
- for $(\{3,6\}, 3$-: $\mathbf{D_2}$, $\{T, T_d\}$ $\{D_3, D_{3h}\}$
- for $(\{2,4\}, 4)$-: $\mathbf{D_2}$, $\{D_4, D_{4h}\}$
- for $(\{2,6\}, 3)$-: $\{D_3, D_{3h}\}$
- for $(\{1,3\}, 6)$-: $\mathbf{C_3}\backslash S_6=\{C_3, C_{3v}, C_{3h}\}$
1. if $(\{3,4\}, 5)$-: $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{C_3}$, $\mathbf{C_4}$, $\mathbf{C_5}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\mathbf{D_4}$, $\mathbf{D_5}$, $\mathbf{T}$, $\mathbf{O}$, $\mathbf{I}$.

With $\mathbf{T}=\{T, T_d, T_h\}$, $\mathbf{O}=\{O, O_h\}$, $\mathbf{I}=\{I, I_h\}$, $\mathbf{C_1}=\{C_1, C_s, C_i\}$, $\mathbf{C_m}=\{C_m, C_{mv}, C_{mh}, S_{2m}\}$, $\mathbf{D_m}=\{D_m, D_{mh}, D_{md}\}$, we get

- for $(\{5,6\}, 3)$-: $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{C_3}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\mathbf{D_5}$, $\mathbf{D_6}$, $\mathbf{T}$, $\mathbf{I}$
- for $(\{2,3\}, 6)$-: $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{C_3}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\{D_6, D_{6h}\}$, $\mathbf{T}$
- for $(\{4,6\}, 3)$-: $\mathbf{C_1}$, $\mathbf{C_2}\setminus S_4$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\{D_6, D_{6h}\}$, $\mathbf{O}$
- for $(\{3,4\}, 4)$-: $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\mathbf{D_4}$, $\mathbf{O}$
- for $(\{3,6\}, 3$-: $\mathbf{D_2}$, $\{T, T_d\}$ $\{D_3, D_{3h}\}$
- for $(\{2,4\}, 4)$-: $\mathbf{D_2}$, $\{D_4, D_{4h}\}$
- for $(\{2,6\}, 3)$-: $\{D_3, D_{3h}\}$
- for $(\{1,3\}, 6)$-: $\mathbf{C_3}\setminus S_6 = \{C_3, C_{3v}, C_{3h}\}$
1. if $(\{3,4\}, 5)$-: $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{C_3}$, $\mathbf{C_4}$, $\mathbf{C_5}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\mathbf{D_4}$, $\mathbf{D_5}$, $\mathbf{T}$, $\mathbf{O}$, $\mathbf{I}$.

Spheres of blue symmetry are $GC_{k,l}$ from 1st such; so, given by one complex (Gaussian for $k=4$, Eisenstein for $k=3,6$) parameter. Goldberg, 1937 and Coxeter, 1971: $\{5,6\}_v(I, I_h)$, $\{4,6\}_v(O, O_h)$, $\{3,6\}_v(T, T_d)$. Dutour-Deza, 2004 and 2010: for other cases.
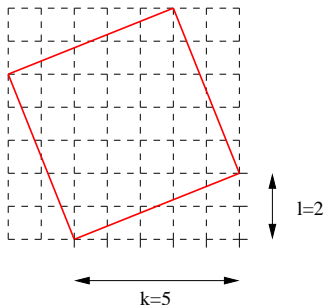
V. Goldberg–Coxeter construction

Goldberg-Coxeter construction $GC_{k,l}(.)$

- Take a 3- or 4-regular plane graph G. The faces of dual graph G^* are triangles or squares, respectively.
- Break each face into pieces according to parameter (k, l). *Master polygons* below have area $\mathcal{A}(k^2+kl+l^2)$ or $\mathcal{A}(k^2+l^2)$, where $\mathcal{A}$ is the area of a small polygon.
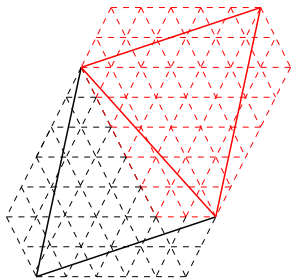

l=2

k=5

3–valent case

l=2

k=5
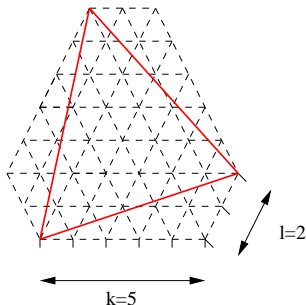
4–valent case

- Gluing the pieces so that, say, 2 non-triangles, coming from subdivision of neighboring triangles, form a small triangle, we obtain another triangulation or quadrangulation of the plane.

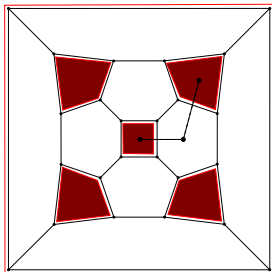- The dual is a 3- or 4-regular plane graph, denoted $GC_{k,l}(G)$; we call it Goldberg-Coxeter construction.
- It works for any 3- or 4-regular map on oriented surface.

$GC_{k,l}(Cube)$ for $(k, l) = (1, 0), (1, 1), (2, 0), (2, 1)$

1,0
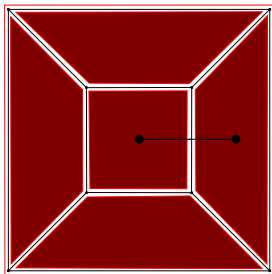
1,1
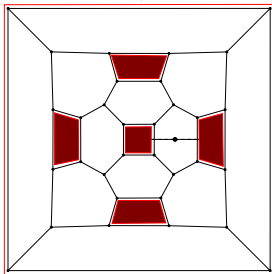
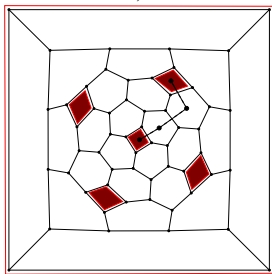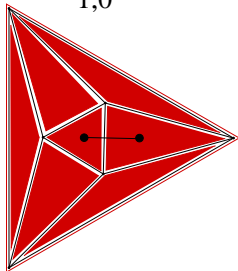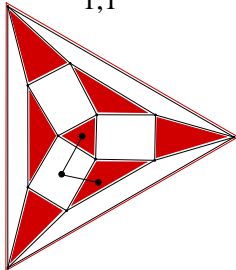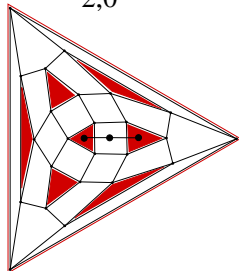2,0

2,1

Goldberg-Coxeter construction from Octahedron

1,0
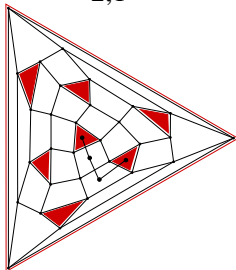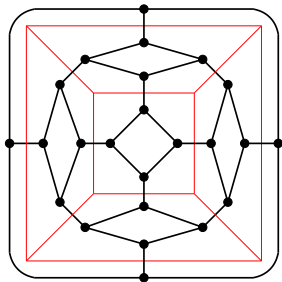
1,1

2,0

2,1

3-regular case
$GC_{1,1}$ is called leapfrog
($\frac{1}{3}$-truncation of the dual)
truncated Octahedron
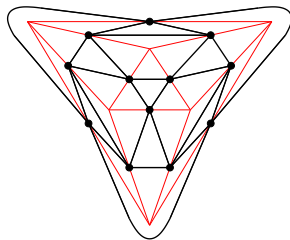
4-regular case
$GC_{1,1}$ is called medial
($\frac{1}{2}$-truncation)
Cuboctahedron

Chamfering (*quadrupling*) $GC_{2,0}(G)$ of 8 1st $(\{a, b\}, k)$-spheres, $(a, b) = (2, 6), (3, 6), (4, 6), (5, 6)$ and $(2, 4), (3, 4), (1, 3), (2, 3)$, are:
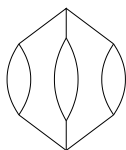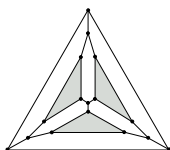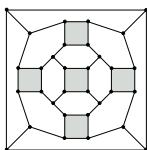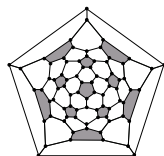
D_{3h} (12^2) T_d (8^6) O_h (12^8) I_h (20^{12})
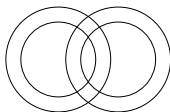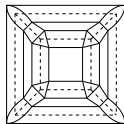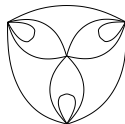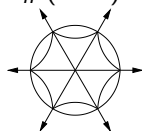
D_{4h} (4^4) O_h (8^6) C_{3v} (6^2) D_{6h} $(4^3, 6^2)$

For 4-regular G, $GC_{2k^2,0}(G) = GC_{k,k}(GC_{k,k}(G))$ by $(k + ki)^2 = 2k^2 i$.

All $(\{2,6\}, 3)$-spheres are $G_{k,l}(3 \times K_2)$: D_{3h}, D_{3h}, D_3 if $l=0$, k, else.
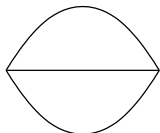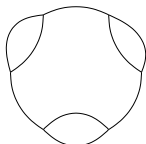
| D_{3h} $3 \times K_2$ | D_{3h} leapfrog | D_{3h} $G_{2,0}$ | D_3 $G_{2,1}$ |

| D_{4h} $4 \times K_2$ | D_{4h} medial | D_{4h} $G_{2,0}$ | D_4 $G_{2,1}$ |
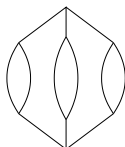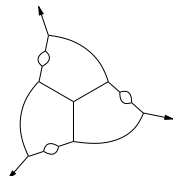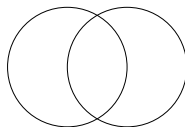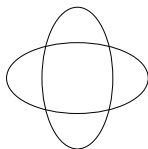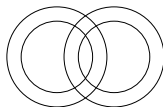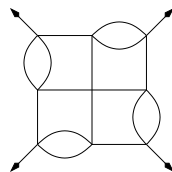
First four $GC_{k,l}(6 \times K_2)$ and $GC_{k,l}(Trifolium)$
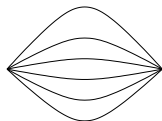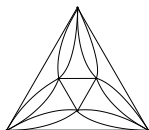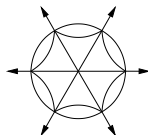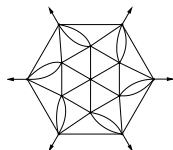
D_{6h}	$D_{3d}\ G_{1,1}$	$D_{6h}\ G_{2,0}$	$D_6\ G_{2,1}$
C_{3v}	$C_{3h}\ G_{1,1}$	$C_{3v}\ G_{2,0}$	$C_3\ G_{2,1}$
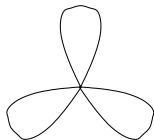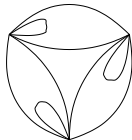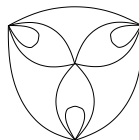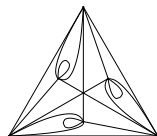
All $(\{2,3\},6)$-spheres are $G_{k,l}(6 \times K_2)$: C_{3v}, C_{3h}, C_3 if $l=0, k$, else.

Plane tilings $\{4^4\}$, $\{3^6\}$ and complex rings $\mathbb{Z}[i]$, $\mathbb{Z}[w]$

- The vertices of regular plane tilings $\{4^4\}$ and $\{3^6\}$ form each, convenient algebraic structures: lattice and ring. Path-metrics of those graphs are l_1- 4-*metric* and *hexagonal* 6-*metric*.

- $\{4^4\}$: square lattice $\mathbb{Z}^2$ and ring $\mathbb{Z}[i]=\{z=k+li : k, l \in \mathbb{Z}\}$ of Gaussian integers with norm $N(z)=z\bar{z}=k^2+l^2=||(k, l)||^2$.

- $\{3^6\}$: hexagonal lattice $A^2=\{x \in \mathbb{Z}^3 : x_0+x_1+x_2=0\}$ and ring $\mathbb{Z}[w]=\{z=k+lw : k, l \in \mathbb{Z}\}$, where $w=e^{i\frac{\pi}{3}}=\frac{1}{2}(1+i\sqrt{3})$, of Eisenstein integers with norm $N(z)=z\bar{z}=k^2+kl+l^2=\frac{1}{2}||x||^2$ We identify points $x=(x_0, x_1, x_2) \in A^2$ with $x_0+x_1 w \in \mathbb{Z}[w]$.

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  We identify points $x=(x_0,x_1,x_2) \in A^2$ with $x_0+x_1w \in \mathbb{Z}[w]$.

- A natural number $n = \prod_i p_i^{\alpha_i}$ is of form $n=k^2+l^2$ if and only if any α_i is even, whenever $p_i \equiv 3 \pmod 4$ (*Fermat Theorem*). It is of form $n = k^2 + kl + l^2$ if and only if $p_i \equiv 2 \pmod 3$.

- The first cases of non-unicity with $gcd(k,l)=gcd(k_1,l_1)=1$ are $91=9^2+9+1^2=6^2+30+5^2$ and $65=8^2+1^2=7^2+4^2$.
  The first cases with $l=0$ are $7^2=5^2+15+3^2$ and $5^2=4^2+3^2$.

The bilattice of vertices of hexagonal plane tiling $\{6^3\}$

- We identify the *hexagonal lattice* A^2 (or *equilateral triangular lattice* of the vertices of the *regular plane tiling* $\{3^6\}$) with *Eisenstein ring* (of Eisenstein integers) $\mathbb{Z}[w]$.
- The hexagon centers of $\{6^3\}$ form $\{3^6\}$. Also, with vertices of $\{6^3\}$, they form $\{3^6\}$, rotated by $90°$ and scaled by $\frac{1}{3}\sqrt{3}$.
- The complex coordinates of vertices of $\{6^3\}$ are given by vectors $v_1=1$ and $v_2=w$. The lattice $L=\mathbb{Z}v_1+\mathbb{Z}v_2$ is $\mathbb{Z}[w]$.
- The vertices of $\{6^3\}$ form bilattice $L_1 \cup L_2$, where the bipartite complements, $L_1=(1+w)L$ and $L_2=1+(1+w)L$, are stable under multiplication. Using this,

$GC_{k,l}(G)$ for 6-regular graph G can be defined similarly to 3- and 4-regular case, but only for $k + lw \in L_2$, i.e. $k \equiv l \pm 1 \, (mod \, 3)$.

Ring formalism

$\mathbb{Z}[i]$ (Gaussian integers) and $\mathbb{Z}[\omega]$ (Eisenstein integers) are *unique factorization* rings

Dictionary

	3-regular G	4-regular G	6-regular G
the ring	Eisenstein $\mathbb{Z}[\omega]$	Gaussian $\mathbb{Z}[i]$	Eisenstein $\mathbb{Z}[\omega]$
Euler formula	$\sum_i (6-i)p_i = 12$	$\sum_i (4-i)p_i = 8$	$\sum_i (3-i)p_i = 6$
curvature 0	hexagons	squares	triangles
ZC-circuits	zigzags	central circuits	both
$GC_{11}(G)$	leapfrog graph	medial graph	or. tripling

Goldberg-Coxeter operation in ring terms

- Associate $z=k+lw$ (Eisenstein) or $z=k+li$ (Gaussian integer) to the pair (k,l) in 3-,6- or 4-regular case. Operation $GC_z(G)$ correspond to scalar multiplication by $z=k+lw$ or $k+li$.

- Writing $GC_z(G)$, instead of $GC_{k,l}(G)$, one has:

$$GC_z(GC_{z'}(G)) = GC_{zz'}(G)$$

- If G has v vertices, then $GC_{k,l}(G)$ has $vN(z)$ vertices, i.e., $v(k^2+l^2)$ in 4-regular and $v(k^2+kl+l^2)$ in 3- or 6-reg. case.

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- $GC_z(G)$ has all rotational symmetries of G in 3- and 4-regular case, and all symmetries if $l = 0, k$ in general case.

- $GC_z(G) = GC_{\overline{z}}(\overline{G})$ where $\overline{G}$ differs by a plane symmetry only from G. So, if G has a symmetry plane, we reduce to $0 \leq l \leq k$; otherwise, graphs $GC_{k,l}(G)$ and $GC_{l,k}(G)$ are not isomorphic.

$GC_{k,l}(G)$ for 6-regular plane graph G and *any* k, l

- Bipartition of G^* gives vertex 2-coloring, say, red/blue of G.
- Truncation $Tr(G)$ of $\{1, 2, 3\}_v$ is a 3-regular $\{2, 4, 6\}_{6v}$.
- Coloring white vertices of G gives face 3-coloring of $Tr(G)$. White faces in $Tr(G)$ correspond to such in $GC_{k,l}(Tr(G))$.
- For $k \equiv l \pm 1 \,(mod\, 3)$, i.e. $k + lw \in L_2$, define $GC_{k,l}(G)$ as $GC_{k,l}(Tr(G))$ with all white faces shrinked.
- If $k \equiv l \,((mod\, 3)$, faces of $Tr(G)$ are white in $GC_{k,l}(Tr(G))$. Among 3 faces around each vertex, one is white. Coloring other red gives unique 3-coloring of $GC_{k,l}(Tr(G))$. Define $GC_{k,l}(G)$ as pair G_1, G_2 with $Tr(G_1) = Tr(G_2) = GC_{k,l}(Tr(G))$ obtained from it by shrinking all red or blue faces.
- $GC_{1,0}(G) = G$ and $GC_{1,1}(G)$ is oriented tripling.

Oriented tripling $GC_{1,1}(G)$ of 6-regular plane graph G

- Let C_1, C_2 be bipartite classes of G^*. For each C_i, oriented tripling $GC_{1,1}(G)$ is 6-regular plane graph $Or_{C_i}(G)$ coming by each vertex of $G \to 3$ vertices and 4 3-gonal faces of $Or_{C_i}(G)$. Symmetries of $Or_{C_i}(G)$ are symmetries of G preserving C_i.

- Orient edges of C_i clockwise. Select 3 of 6 neighbors of each vertex v: $\{2, 4, 6\}$ are those with directed edge going to v; for $\{1, 5, 5\}$, edges go to them.
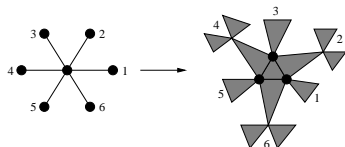
- Any $z = k + lw \neq 0$ with $k \equiv l \pmod 3$ can be written as $(1+w)^s (k' + l'w)w$, where $s \geq 0$ and $k' \equiv l' \pm 1 \pmod 3$. So, it holds reduction $GC_{k,l}(G) = G_{k',l'}(Or^s(G))$.

Examples of oriented tripling $GC_{1,1}(G)$

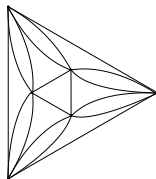Below: $\{2,3\}_2$ and $\{2,3\}_4$ have *unique* oriented tripling.
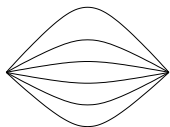
2 D_{6h} **6** D_{3d} **4** T_d **12** T_h

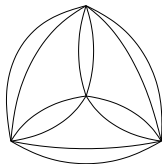## Examples of oriented tripling $GC_{1,1}(G)$

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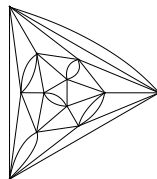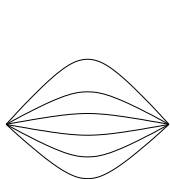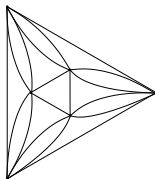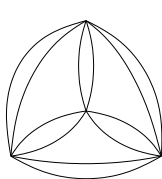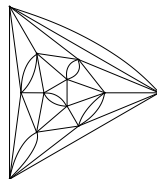**2** D_{6h} **6** D_{3d} **4** T_d **12** T_h
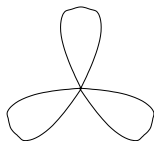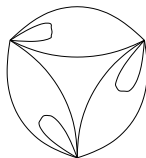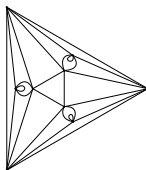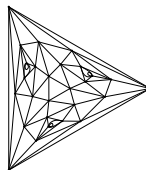
1 C_{3v} **3** C_{3h} **9** C_{3v} **27** C_{3h} **81** C_{3v}
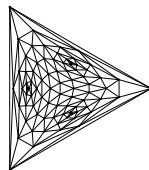
Above: first 4 *consecutive* oriented triplings of the Trifolium.

VI. Parameterizing $(\{a, b\}, k)$-spheres

Example: construction of the ($\{3, 6\}, 3$)-spheres in $Z[\omega]$
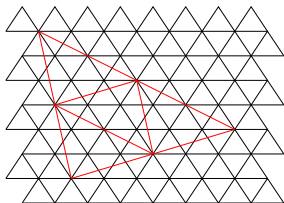
In the central triangle ABC, let A be the origin of the complex plane
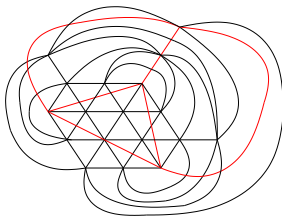
The corresponding triangulation
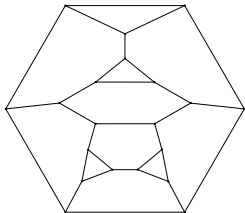
All ($\{3, 6\}, 3$)-spheres come this way; two complex parameters in $Z[\omega]$ defined by the points B and C

Thurston, 1998 implies: ($\{a, b\}, k$)-spheres have p_a-2 parameters and the number of v-vertex ones is $O(v^{m-1})$ if $m = p_a$-2 > 2.

Idea: since b-gons are of zero curvature, it suffices to give relative positions of a-gons having curvature $2k - a(k - 2) > 0$.

At most $p_a - 1$ vectors will do, since one position can be taken 0.

But once $p_a - 1$ a-gons are specified, the last one is constrained.

The number of m-parametrized spheres with at most v vertices is $O(v^m)$ by direct integration. The number of such v-vertex spheres is $O(v^{m-1})$ if $m > 1$, by a *Tauberian* theorem.

Parameterizing standard ($C_b = 0$) ($\{a, b\}, k$)-spheres

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- Goldberg, 1937: $\{a, 6\}_v$ (highest 2 symmetries): 1 parameter
- Fowler and al., 1988: $\{5, 6\}_v$ (D_5, D_6 or T): 2 parameters.
- Grűnbaum-Motzkin, 1963: $\{3, 6\}_v$: 2 parameters.
- Deza-Shtogrin, 2003: $\{2, 4\}_v$; 2 parameters.
- Thurston, 1998: $\{5, 6\}_v$: 10 (again complex) parameters.
  Graver, 1999: $\{5, 6\}_v$: 20 integer parameters.
- Rivin, 1994: parameter desciption by dihedral angles.

Parameterizing (R, k)-spheres with $\min_{i \in R} C_i \geq 0$

Thurston, 1998 parametrized (dually, as triangulations) such $(R, 3)$-spheres, i.e. 19 series of $(\{3, 4, 5, 6\}, 3)$-spheres.
In general, such (R, k)-spheres are given by $m = \sum_{3 \leq i < \frac{2k}{k-2}} p_i - 2$ complex parameters $z_1, \ldots, z_m$.
The number of vertices is expressed as a non-degenerate Hermitian form $q = q(z_1, \ldots, z_m)$ of signature $(1, m-1)$.
Let H^m be the cone of $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$ with $q(z) > 0$.
Given (R, k)-sphere is described by different parameter sets; let $M = M(\{p_3, \ldots, p_m\}, k)$ be the discrete linear group preserving q.
For $k = 3$, the quotient $H^m / (\mathbb{R}_{>0} \times M)$ is of finite covolume (Thurston, 1998, actually, 1993). Sah, 1994 deduced from it that the number of corresponding spheres grows as $O(v^{m-1})$.
Dutour partially generalized above for other k and surface maps.

- $\{5,6\}_v$ $\mathbf{C_1}(10)$, $\mathbf{C_2}(6)$, $\mathbf{C_3}(4)$, $\mathbf{D_2}(4)$, $\mathbf{D_3}(3)$, $\mathbf{D_5}(2)$, $\mathbf{D_6}(2)$, $\mathbf{T}(2)$, $\{I, I_h\}(1)$
- $\{4,6\}_v$ $\mathbf{C_1}(4)$, $\mathbf{C_2}\backslash S_4(3)$, $\mathbf{D_2}(2)$, $\mathbf{D_3}(2)$, $\{D_6, D_{6h}\}(1)$, $\{O, O_h\}(1)$
- $\{3,4\}_v$ $\mathbf{C_1}(6)$, $\mathbf{C_2}(4)$, $\mathbf{D_2}(3)$, $\mathbf{D_3}(2)$, $\mathbf{D_4}(2)$, $\{O, O_h\}(1)$
- $\{2,3\}_v$ $\mathbf{C_1}(4)$, $\mathbf{C_2}(3?)$, $\mathbf{C_3}(3?)$, $\mathbf{D_2}(2?)$, $\mathbf{D_3}(2?)$, $\mathbf{T}(1)$, $\{D_6, D_{6h}\}(1)$
- $\{3,6\}_v$ $\mathbf{D_2}(2)$, $\{T, T_d\}(1)$
- $\{2,4\}_v$ $\mathbf{D_2}(2)$, $\{D_4, D_{4h}\}(1)$
- $\{2,6\}_v$ $\{D_3, D_{3h}\}(1)$
- $\{1,3\}_v$ $\{C_3, C_{3v}, C_{3h}\}(1)$

Thurston, 1998 implies: $(\{a, b\}, k)$-spheres have p_a-2 parameters and the number of v-vertex ones is $O(v^{m-1})$ if $m = p_a$-2 > 1.

Number of complex parameters

$\{5,6\}_v$

Group	#param.
C_1	10
C_2	6
C_3, D_2	4
D_3	3
D_5, D_6, T	2
I	1

$\{3,4\}_v$

Group	#param.
C_1	6
C_2	4
D_2	3
D_3, D_4	2
O	1

$\{4,6\}_v$

Group	#param.
C_1	4
C_2	3
D_2, D_3	2
D_6, O	1

$\{2,3\}_v$

Group	#param.
C_1	4
C_2, C_3	3?
D_2, D_3	2?
D_6, T	1

$\{3,6\}_v$- and $\{2,4\}_v$: 2 complex parameters but 3 natural ones will do: *pseudoroad* length, number of circumscribing *railroads*, *shift*.

VII. Railroads and tight $(\{a, b\}, k)$-spheres

ZC-circuits

- The edges of any plane graph are doubly covered by zigzags (Petri or left-right paths), i.e., circuits such that any two but not three consecutive edges bound the same face.
- The edges of any *Eulerian* (i.e., even-valent) plane graph are partitioned by its central circuits (those going straight ahead).
- A ZC-circuit means *zigzag* or *central circuit* as needed. CC- or Z-vector enumerate lengths of above circuits.

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- A ZC-circuit means *zigzag* or *central circuit* as needed. CC- or Z-vector enumerate lengths of above circuits.
- A railroad in a 3-, 4- or 6-regular plane graph is a circuit of 6-, 4- or 3-gons, each adjacent to neighbors on opposite edges. Any railroad is bound by two "parallel" ZC-circuits. It (any if 4-, simple if 3- or 6-regular) can be collapsed into 1 ZC-circuit.
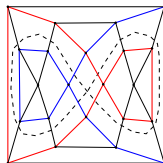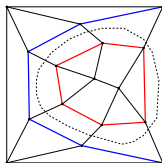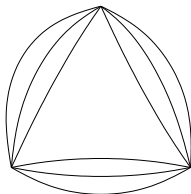
Railroad in a 6-regular sphere: examples

$APrism_3$ with 2 base 3-gons doubled is the $\{2,3\}_6$ (D_{3d}) with
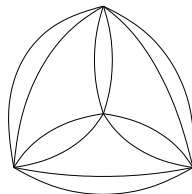CC-vector $(3^2, 4^3)$, all five central circuits are simple.
Base 3-gons are separated by a simple railroad R of six 3-gons,
bounded by two parallel central 3-circuits around them. Collapsing
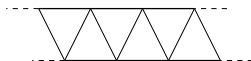R into one 3-circuit gives the $\{2,3\}_3$ (D_{3h}) with CC-vector $(3; 6)$.
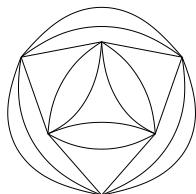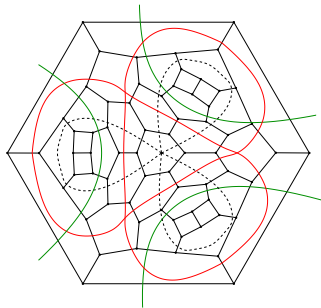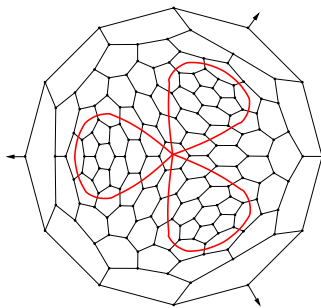

D_{3d} $(3^2, 4^3)$ D_{3h} $(3; 6)$ T_d (3^4)

Above $\{2,3\}_4$ (T_d) has no railroads but it is not strictly tight, i.e.
no any central circut is adjacent to a non-3-gon on each side.

Railroads can be simple or self-intersect, including triply if $k = 3$. First such Dutour ($\{a, b\}, k$)-spheres for $(a, b) = (4, 6), (5, 6)$ are:

$\{4, 6\}_{66}(D_{3h})$ twice $\{5, 6\}_{172}(C_{3v})$

Which plane curves with at most triple self-intersectionss come so?

Number of ZC-circuits in tight $(\{a, b\}, k)$-sphere

- Call an $(\{a, b\}, k)$-sphere tight if it has no railroads.

- ≤ 15 for $\{5, 6\}_v$ Dutour, 2004
- ≤ 9 for $\{4, 6\}_v$ and $\{2, 3\}_v$ Deza-Dutour, 2005 and 2010
- ≤ 3 for $\{2, 6\}_v$ and $\{1, 3\}_v$ same
- ≤ 6 for $\{3, 4\}_v$ Deza-Shtogrin, 2003
- Any $\{3, 6\}_v$ has ≥ 3 zigzags with equality iff it is tight. All $\{3, 6\}_v$ are tight iff $\frac{v}{4}$ is prime and none iff it is even.
- Any $\{2, 4\}_v$ has ≥ 2 central circuits with equality iff it is tight. There is a tight one for any even v.

Number of ZC-circuits in tight $(\{a, b\}, k)$-sphere

- Call an $(\{a, b\}, k)$-sphere tight if it has no railroads.

- ≤ 15 for $\{5, 6\}_v$ Dutour, 2004
- ≤ 9 for $\{4, 6\}_v$ and $\{2, 3\}_v$ Deza-Dutour, 2005 and 2010
- ≤ 3 for $\{2, 6\}_v$ and $\{1, 3\}_v$ same
- ≤ 6 for $\{3, 4\}_v$ Deza-Shtogrin, 2003
- Any $\{3, 6\}_v$ has ≥ 3 zigzags with equality iff it is tight. All $\{3, 6\}_v$ are tight iff $\frac{v}{4}$ is prime and none iff it is even.
- Any $\{2, 4\}_v$ has ≥ 2 central circuits with equality iff it is tight. There is a tight one for any even v.

First tight ones with max. of ZC-circuits are $GC_{21}(\{a, b\}_{min})$: $\{5, 6\}_{140}(I)$, $\{4, 6\}_{56}(O)$, $\{2, 6\}_{14}(D_3)$, $\{3, 4\}_{30}(0)$; $\{2, 3\}_{44}(D_{3h})$ and $\{a, b\}_{min}$: $\{3, 6\}_4(T_d)$, $\{2, 4\}_2(D_{4h})$. Besides $\{2, 3\}_{44}(D_{3h})$, ZC-circuits are: (28^{15}), (21^8), (14^3), (10^6), (4^3), (2^2), all simple.

- $M_v = \frac{v}{2} + 1$, $\frac{v}{2} + 2$ for $v \equiv 0, 2 \pmod 4$. It is realized by the series of symmetry D_{2d} with CC-vector $2^{\frac{v}{2}}, 2v_{0,v}$ and of symmetry D_{2h} with CC-vector $2^{\frac{v}{2}}, v_{0, \frac{v-2}{4}}^2$ if $v \equiv 0, 2 \pmod 4$.

- For odd v, M_v is $\lfloor \frac{v}{3} \rfloor + 3$ if $v \equiv 2, 4, 6 \pmod 9$ and $\lfloor \frac{v}{3} \rfloor + 1$, otherwise. Define t_v by $\frac{v - t_v}{3} = \lfloor \frac{v}{3} \rfloor$. M_v is realized by the series of symmetry C_{3v} if $v \equiv 1 \pmod 3$ and D_{3h}, otherwise. CC-vector is $3^{\lfloor \frac{v}{3} \rfloor}, (2\lfloor \frac{v}{3} \rfloor + t_v)_{0, \lfloor \frac{v-2t_v}{9} \rfloor}^3$ if $v \equiv 2, 4, 6 \pmod 9$ and $3^{\lfloor \frac{v}{3} \rfloor}, (2v + t_v)_{0, v+2t_v}$, otherwise.

Smallest CC-knotted or Z-knotted $\{2, 3\}_v$

- The minimal number of central circuits or zigzags, 1, have CC-knotted and Z-knotted $\{2, 3\}_v$. They correspond to plane curves with only triple self-intersection points. For $v \leq 16$, there are $1, 2, 4, 7, 9, 12$ Z-knotted if $v = 3, 7, 9, 11, 13, 15$ and $1, 2, 2, 4, 11, 9, 1, 19$ CC-knotted if $v = 4, 6, 8, 10, 12, 14, 15, 16$.

- Conjecture (holds if $v \leq 54$): any Z-knotted $\{2, 3\}_v$ has odd v and a CC-knotted $\{2, 3\}_v$ is Z-knotted if and only if v is odd.
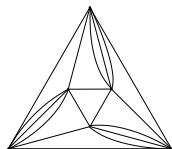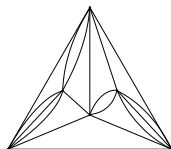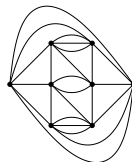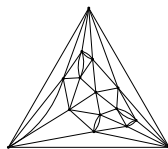

4 D_2 **6** D_3 **6** C_2 **8** D_2 **15** C_1

VIII. Tight pure $(\{a, b\}, k)$-spheres

Tight $(\{a, b\}, k)$-spheres with only simple ZC-circuits

- Call $(\{a, b\}, k)$-sphere pure if any of its ZC-circuits is *simple*, i.e. has no self-intersections. Such ZC-circuit can be seen as a Jordan curve, i.e. a plane curve which is topologically equivalent to (a homeomorphic image of) the unit circle.

- Any $(\{3, 6\}, 3)$- or $(\{2, 4\}, 4)$-sphere is pure. They are tight if and only if have three or, respectively, two ZC-circuits.

- Any ZC-circuit of $\{2, 6\}_v$ or $\{1, 3\}_v$ self-intersects.

Tight $(\{a, b\}, k)$-spheres with only simple ZC-circuits

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- Any $(\{3, 6\}, 3)$- or $(\{2, 4\}, 4)$-sphere is pure. They are tight if and only if have three or, respectively, two ZC-circuits.
- Any ZC-circuit of $\{2, 6\}_v$ or $\{1, 3\}_v$ self-intersects.

The number of tight pure $(\{a, b\}, k)$-spheres is:

- 9? for $\{5, 6\}_v$ computer-checked for $v \leq 300$ by Brinkmann
- 2 for $\{4, 6\}_v$ Deza-Dutour, 2005
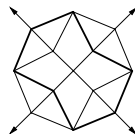- 8 for $\{3, 4\}_v$ same
- 5 for $\{2, 3\}_v$ same, 2010
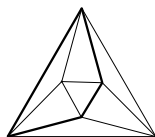
6 O_h (4^3)
Octahedron
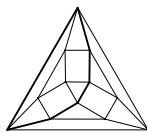
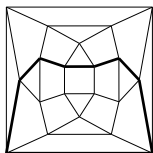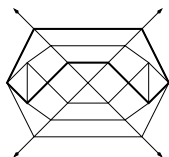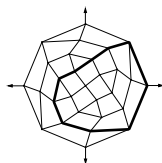**12** O_h (6^4)
$GC_{1,1}(Oct.)$
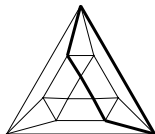
12 D_{3h} (6^4)

14 D_{4h}
($6^2, 8^2$)
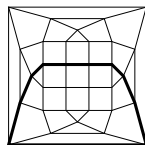
20 D_{2d} (8^5)

22 D_{2h}
($8^3, 10^2$)

30 O (10^6)
$GC_{2,1}(Oct.)$

32 D_{4h}
($10^4, 12^2$)

There are exactly two such spheres: Cube and its leapfrog $GC_{11}(Cube)$, truncated Octahedron.

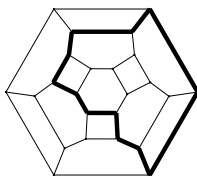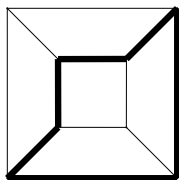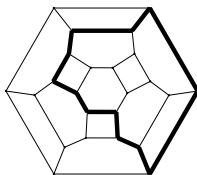**6** O_h (6^4)   **24** O_h (10^6)

All tight ($\{4,6\}, 3$)-spheres with only simple zigzags

There are exactly two such spheres: Cube and its leapfrog $GC_{1,1}(Cube)$, truncated Octahedron.
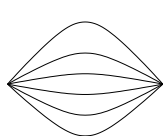

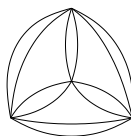**6** O_h (6^4)   **24** O_h (10^6)

Proof is based on a) The size of intersection of two simple zigzags in any ($\{4,6\}, 3$)-sphere is $0, 2, 4$ or 6 and
b) Tight ($\{4,6\}, 3$)-sphere has at most 9 zigzags.

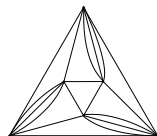For ($\{2,3\}, 6$)-spheres, a) holds also, implying a similar result.
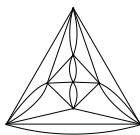
Tight ({2, 3}, 6)-spheres with only simple ZC-circuits

2 D_{6h} (2^3)
(6^2)
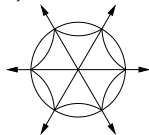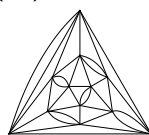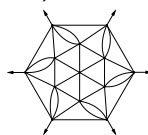
4 T_d (3^4)
(6^4)

6 D_3 no
($12, 8^3$)

8 D_{2d} ($5^4, 4$)
no

D_{6h} ($4^3, 6^2$)
(8^6) no

12 T_h (6^6)
(12^6)

14 D_6 no
(14^6)

All CC-pure, tight: Nrs. 1,2,4,5,6 (Nrs. 3,7 are not CC-pure).
All Z-pure, tight: Nrs. 1,2,3,6,7 (4 is not Z-pure, 5 is not Z-tight).
1st, 3rd are strictly CC-, Z-tight: all ZC-circuits sides touch 2-gons

7 tight ({5, 6}, 3)-spheres with only simple zigzags

20 I_h (10^6)   **28** T_d (12^7)   **48** D_3 (16^9)   **76** D_{2d} ($22^4, 20^7$)

88 T (22^{12})   **92** T_h ($24^6, 22^6$)   **140** I, (28^{15})
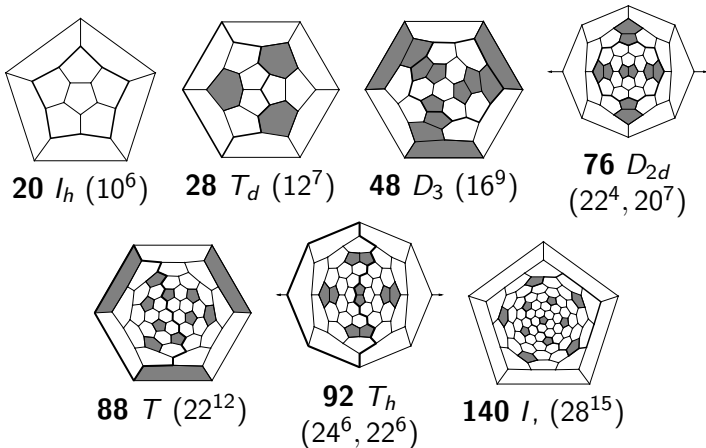
The zigzags of $1, 2, 3, 5, 7$th above and next two form 7 Grűnbaum arrangements of Jordan curves, i.e. any two intersect in 2 points. The groups of $1, 5, 7$th and $\{5, 6\}_{60}(I_h)$ are zigzag-transitive.

Two other such ($\{5, 6\}, 3$)-spheres
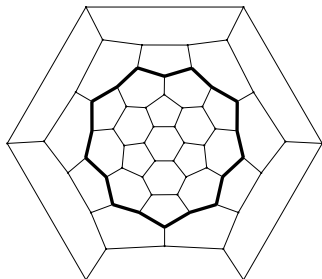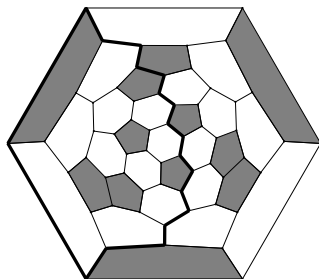
60 I_h (18^{10})   **60** D_3 (18^{10})

This pair was first answer on a question in Grünbaum, 1967, 2003 *Convex Polytopes* about existence of *simple* polyhedra with the same p-vector but different zigzags. The groups of above $\{5, 6\}_{60}$ have, acting on zigzags, 1 and 3 orbits, respectively.

IX. Other fullerene analogs: $(\{a, b, c\}, k)$-disks ($p_c=1$)

Related non-standard (R, k)-spheres with $\frac{1}{k} + \frac{1}{\max_{i \in R} i} < \frac{1}{2}$ are:

- G-fulleroids (Deza-Delgado, 2000; Jendrol-Trenkler, 2001 and Kardos, 2007): $(\{5, b\}, 3)$-spheres with $b \geq 7$ and symmetry G.

- b-Icosahedrites: $(\{3, b\}, 5)$-spheres. So, they have $p_3 = (3b - 10)p_b + 20$ 3-gons and $v = 2(b - 3)p_b + 12$ vertices. Snub Cube and Snub Dodecahedron are the cases $(b, v; group) = (4, 24; O)$ and $(5, 60; I)$.

- Haeckel, 1887: $(\{5, 6, c\}, 3)$-spheres with $c = 7, 8$ representing skeletons of radiolarian zooplankton Aulonia hexagona.

- $(\{a, b, c\}, k)$-disk is an $(\{a, b, c\}, k)$-sphere with $p_c = 1$; so, its $v = \frac{2}{k-2}(p_a - 1 + p_b) = \frac{2}{2k - a(k-2)}(a + c + p_b(b-a))$ and (setting $b' = \frac{2k}{k-2}$) $p_a = \frac{b' + c}{b' - a} + p_b \frac{b - b'}{b' - a}$. So, $p_a = \frac{b+c}{b-a}$ if $b = b'$ (8 families).

- Fullerene c-disk is the case $(a, b, c; k) = (5, 6, c; 3)$ of above. So, they have $p_5 = c + 6$ and $v = 2(p_6 + c + 5)$ vertices.
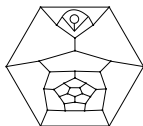
Minimal fullerene $(\{5,6\},3)$) c-disks for $1 \le c \le 8$

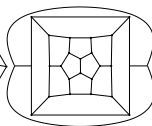It is 1-vertex-, 1-edge-truncated, usual F_{20}, F_{24} for $c=3,4,5,6$.
It comes from minimal 4-disk for $c=2$: add edge with 2-gon on it,
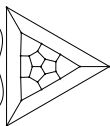from C_s-minimal 3-disk for $c=1$: add edge with 1-gon on its end.
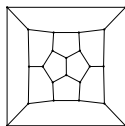For $1 \le c \le 8$, minimal c-disks are unique and $p_6 = 6, 6, 3, 2, 0, 1, 3, 4$.
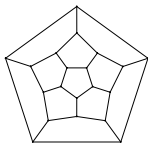

1 28 C_s 2 26 C_{2v} 3 22 C_{3v} 4 22 C_{2v}
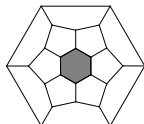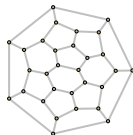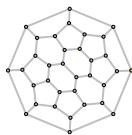
5 20 I_h 6 24 D_{6d} 7 30 C_s 8 34 C_{2v}

Minimal fullerene c-disks for $c \geq 9$

$2, 3, 10, 1$ minimal c-disks and $p_6 = 6, 7, 8, 5$ for $c = 9, 10, 11, 12$.

Conjecture: for $c \geq 13$, the only minimal c-disk is c-pentatube $B + Hex_3 + Pen_{c-12} + Hex_3 + B$ (symmetry C_s / C_2 for odd/even c).
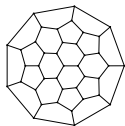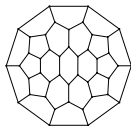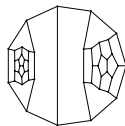
9-2 40 C_{3v} 10-3 44 C_{2v} 11-10 48 C_s 12 44 C_{2v}
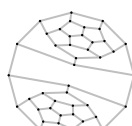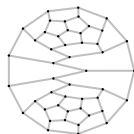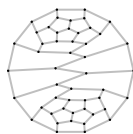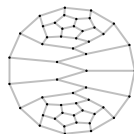
13 48 C_s 14 50 C_2 15 52 C_s 16 54 C_2

Symmetries of fullerene c-disks: $(\{5, 6, c\}, 3)$ with $p_c = 1$

- Their groups: C_m, C_{mv} with $m \equiv 0 (mod\ c)$ (since any symmetry should stabilize unique c-gonal face) and $m \in \{1, 2, 3, 5, 6\}$ since the axis pass by a vertex, edge or face.
- The minimal such 8- and 9-disks are given below.
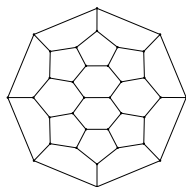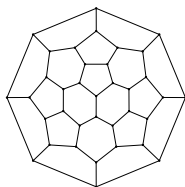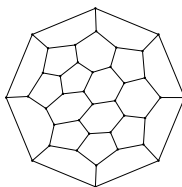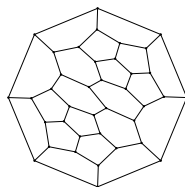
8 34 C_{2v} 8 36 C_s 8 38 C_1 8 38 C_2
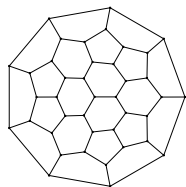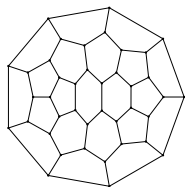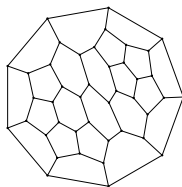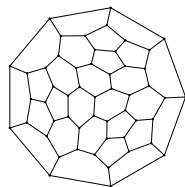
9 40 C_{3v} 9 40 C_s 9 42 C_1 9 52 C_3

X. Icosahedrites: $(\{3, 4\}, 5)$-spheres

Icosahedrites, i.e., $(\{3,4\},5)$-spheres

- They have $p_3 = 2p_b + 20$ and $v = 2p_b + 12$ vertices.
- Their number is $1, 0, 1, 1, 5, 12, 63, 246, 1395, 7668, 45460$ for $v = 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32$. It grows at least exponentially with v.
- p_a is fixed in for standard $(\{a,b\},k)$-spheres permitting Goldberg-Coxeter construction and parametrization of graphs which imply the polynomial growth of their number. It does not happen for icosahedrites; no parametrization for them.
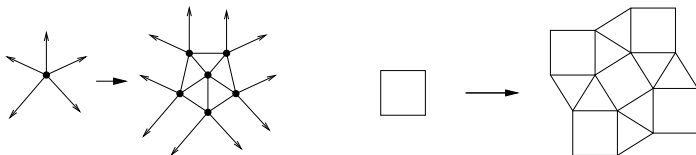
A-operation keeps symmetries; B-operation: only rotational ones.

Proof for the number of icosahedrites

A weak zigzag ia a left/right, but never extreme, edge-circuit.
If a v-vertex icosahedrite has a simple weak zigzag of length 6, a
$(v+6)$-vertex one come by inserting a corona (6-ring of three
4-gons alternated by three pairs of adjacent 3-gons) instead of it.
But such spheres exist for $v=18$, 20, 22; so, for $v \equiv 0, 2, 4 \pmod 6$.
There are two options of inserting corona; so, the number of
v-vertex icosahedrites grows at least exponentially.
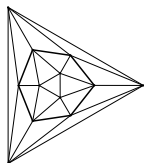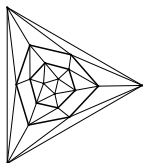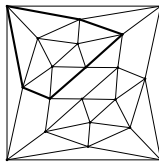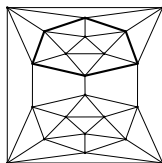
12 I_h	18 D_3	20 D_{2d}	22 D_{5h}
$wZ=6^{10}$	$6^2, 8^3; 54_{0,9}$	$6^4, 20^2; 18^2_{0,3}$	$6^{10}; 50_{15,0}$
$Z=10^6$	$90_{27,18}$	$10^4; 30^2_{3,0}$	$10^2; 90_{15,20}$

An usual (strong) zigzag is a left/right, both extreme, edge-circuit.

38 symmetry groups of icosahedrites

- Agregating $\mathbf{C_1} = \{C_1, C_s, C_i\}$, $\mathbf{C_m} = \{C_m, C_{mv}, C_{mh}, S_{2m}\}$, $\mathbf{D_m} = \{D_m, D_{mh}, D_{md}\}$, $\mathbf{T} = \{T, T_d, T_h\}$, $\mathbf{O} = \{O, O_h\}$, $\mathbf{I} = \{I, I_h\}$, all 38 symmetries of $(\{3,4\}, 5)$-spheres are:
  $\mathbf{C_1}$, $\mathbf{C_m}$, $\mathbf{D_m}$ for $2 \leq m \leq 5$ and $\mathbf{T}$, $\mathbf{O}$, $\mathbf{I}$.
- Any group appear an infinite number of times since one gets an infinity by applying A-operation iteratively.
- Group limitations came from k-fold axis only. Is it occurs for all $(\{a, b\}, k)$-spheres with b-faces of negative curvature?
- Examples (minimal whenever $v \leq 32$) are given below:
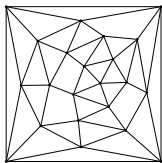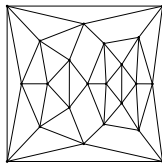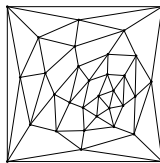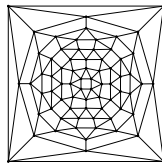
22 C_1 22 C_s 32 C_i 72 O_h

Minimal ($\{3,4\},5$)-spheres of 5-fold symmetry

It exists iff $p_4 \equiv 0\,(mod\,5)$, i.e., $v = 2p_4 + 12 \equiv 2\,(mod\,10)$.
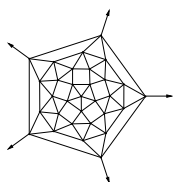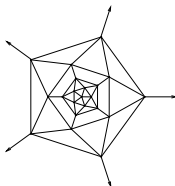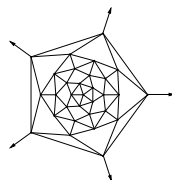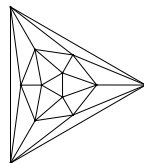
32 D_5 22 D_{5h} 32 D_{5d} 12 I_h Icosahed.
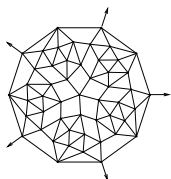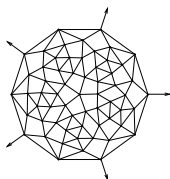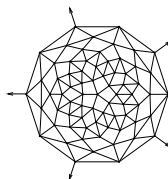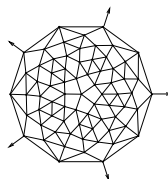
52 C_5 62 C_{5h} 72 C_{5v} 72 S_{10}

Minimal $(\{3, 4\}, 5)$-spheres of 4-fold symmetry

It exists iff $p_4 \equiv 2 \, (mod \, 4)$, i.e., $v = 2p_4 + 12 \equiv 0 \, (mod \, 8)$.
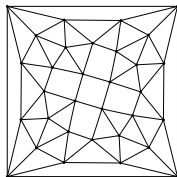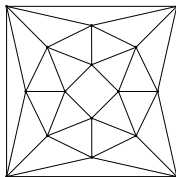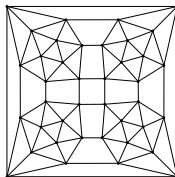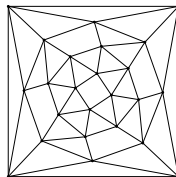
32 D_4 16 D_{4d} 40 D_{4h} 24 O Snub cube
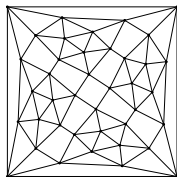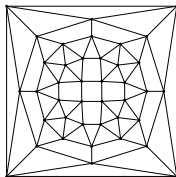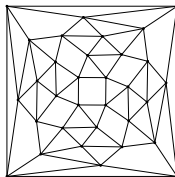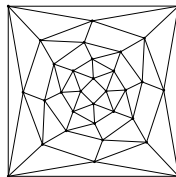
40 C_4 40 C_{4v} 32 C_{4h} 32 S_8

Icosahedron, Snub Cube and, with $(b, v; G) = (5, 60; I)$, Snub Dodecahedron are the only vertex-transitive $(\{3, b\}, 5)$-spheres.

Minimal $(\{3,4\},5)$-spheres of 3-fold symmetry

It exists iff $p_4 \equiv 0 \,(mod\, 3)$, i.e., $v = 2p_4 + 12 \equiv 0 \,(mod\, 6)$.
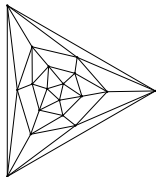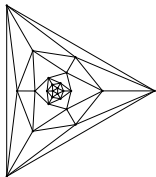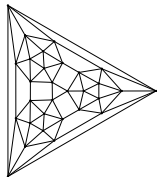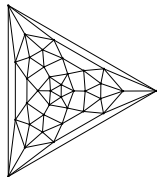
18 D_3 24 D_{3d} 30 D_{3h} 36 T_d
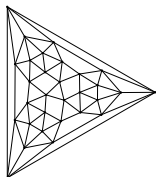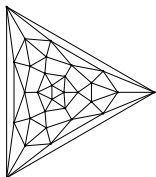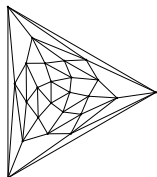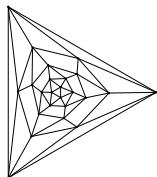
30 C_3 30 C_{3v} 24 C_{3h} 24 S_6

Minimal $(\{3,4\},5)$-spheres of 2-fold symmetry
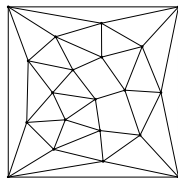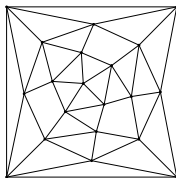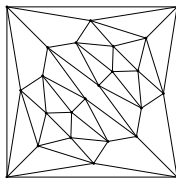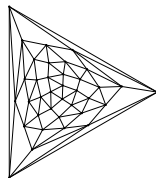
20 D_2 20 D_{2d} 24 D_{2h} 36 T_h
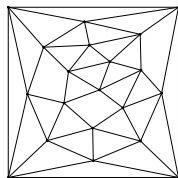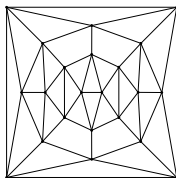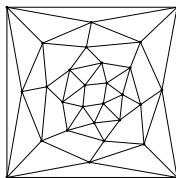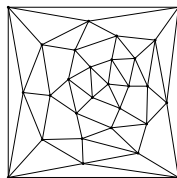
20 C_2 22 C_{2v} 28 C_{2h} 28 S_4

Face-regular $(\{3, b\}, 5)$-spheres

- A 3-connected map (on sphere or torus) is *pR_i face-regular* if any *p*-gonal face is adjacent to exactly *i* *p*-gons.
- No $(\{3, b\}, 5)$-sphere, besides Icosahedron $3R_3$, is $3R_i$.
- Clearly, bR_j $(\{3, b\}, 5)$-sphere has $j\frac{p_b}{2}$ $(b - b)$-edges. So, bR_j with odd *j* implies that 4 divides $v = 2p_b(b - 3) + 12$.
- There is infinity of bR_j $(\{3, b\}, 5)$-spheres for $j = 0, 1, 2$.
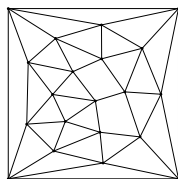
| 20 D_2 $4R_1$ | 36 T_d $4R_1$ | 22 D_{5h} $4R_2$ | 24 D_{3d} $4R_2$ |
| $G_4 = 2K_2$ | $G_4 = 6K_2$ | $G_4 = C_5$ | $G_4 = C_6$ |
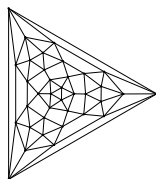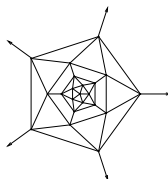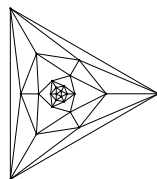
- Icosahedron (snub $APrism_3$) is regular. So, let $p_b > 0$.
- Snub $APrism_b$ has $v = 4b$ vertices (2 orbits of size $2b$), 2 b-gons (1 orbit) and $6b$ 3-gons (2 orbits of size $3b$). Its group G is D_{bd} for $b \geq 4$.
- With $(b, v; G) = (4, 24; O), (5, 60; I)$, Snub Cube and Snub Dodecahedron are only vertex-transitive $(\{3, b\}, 5)$-spheres. They are also b-gon-transitive and have 2 orbits of triangles.
- Do other b-gon-transitive $(\{3, b\}, 5)$-spheres or $(\{3, b\}, 5)$-spheres with at most 3 orbits of faces exist?

XI. Standard $(\{a, b\}, k)$-maps on surfaces

Standard (R, k)-maps

- Given $R \subset \mathbb{N}$ and a surface $\mathbb{F}^2$, an (R, k)-$\mathbb{F}^2$ is a k-regular map M on surface $\mathbb{F}^2$ whose faces have gonalities $i \in R$.
- Euler characteristic $\chi(M)$ is $v - e + f$, where v, e and $f = \sum_i p_i$ are the numbers of vertices, edges and faces of M.
- Since $kv = 2e = \sum_i ip_i$, Euler formula $\chi = v - e + f$ becomes Gauss-Bonnet-like one $\chi(M) = \sum_i ip_i C_i$.
- Again, let our maps be standard, i.e., $\min_{i \in R}(\frac{1}{k} + \frac{1}{i} - \frac{1}{2} = 0)$. So, $M = max\{i \in R\} = \frac{2k}{k-2}$ and $(M, k) = (6, 3), (4, 4), (3, 6)$.
- There are infnity of standard maps (R, k)-$\mathbb{F}^2$, since the number p_M of flat ($C_M = 0$) faces is not restricted.
- Also, $\chi \geq 0$ with $\chi = 0$ if and only if $R = \{m\}$. So, $\mathbb{F}^2$ is $\mathbb{S}^2$, $\mathbb{T}^2$, $\mathbb{P}^2$, $\mathbb{K}^2$ with $\chi = 2, 0, 1, 0$, respectively.
- Such $(\{a, b\}, k)$-$\mathbb{F}^2$ map has $b = \frac{2k}{k-2}$, $p_a = \frac{\chi b}{b-a}$, $v = \frac{1}{k}(ap_a + bp_b)$ So, $(a = b, k) = (6, 3), (3, 6), (4, 4)$ if $\mathbb{F}^2$ is $\mathbb{T}^2$ or $\mathbb{K}^2$.
- But $\chi = \frac{p_3 - 2p_4}{10}$ for icosahedrite maps $(\{3, 4\}, 5)$ (non-standard) So, $\chi < 0$ is possible and $\chi = 0$ (i.e., $\mathbb{F}^2 = \mathbb{T}^2, \mathbb{K}^2$) iff $p_3 = 2p_4$.

Digression on interesting non-standard $(\{5,6,c\},3)$-maps

Such maps, generalizing fullerenes, have $c \geq 7$. Examples are:

- *G*-fulleroids (Deza-Delgado, 2000; Jendrol-Trenkler, 2001 and Kardos, 2007): $(\{5,b\},3)$-spheres with $b \geq 7$ and symmetry G
- Haeckel, 1887: $(\{5,6,c\},3)$-spheres with $c = 7,8$ representing skeletons of radiolarian zooplankton Aulonia hexagona.
- Azulenoids: $(\{5,6,7\},3)$-tori; so $g = 1, p_5 = p = 7$.
- Schwarzits: $(\{5,6,c\},3)$-maps on minimal surfaces of constant negative curvature ($g \geq 2$) with $c = 7,8$. Knor-Potocnik-Siran-Skrekovski, 2010: such $(\{6,c\},3)$-maps exist for any $g \geq 2, p_6 \geq 0$ and $c = 7,8,9,10,12$. For $c = 7,8$ such polyhedral maps exist.

The $(\{a, b\}, k)$-maps on torus and Klein bottle

The connected *closed* (compact and without boundary) irreducible surfaces are: sphere $\mathbb{S}^2$, torus $\mathbb{T}^2$ (two orientable), real projective plane $\mathbb{P}^2$ and Klein bottle $\mathbb{K}^2$ with $\chi = 2, 0, 1, 0$, respectively.

The maps $(\{a, b\}, k)$-$\mathbb{T}^2$ and $(\{a, b\}, k)$-$\mathbb{K}^2$ have $a = b = \frac{2k}{k-2}$; so, $(a = b, k)$ should be $(6, 3), (3, 6)$ or $(4, 4)$.

We consider only polyhedral maps, i.e. no loops or multiple edges (1- or 2-gons), and any two faces intersect in edge, point or $\emptyset$ only.

The $(\{a, b\}, k)$-maps on torus and Klein bottle

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The maps $(\{a, b\}, k)$-$\mathbb{T}^2$ and $(\{a, b\}, k)$-$\mathbb{K}^2$ have $a = b = \frac{2k}{k-2}$; so, $(a = b, k)$ should be $(6, 3), (3, 6)$ or $(4, 4)$.

We consider only polyhedral maps, i.e. no loops or multiple edges (1- or 2-gons), and any two faces intersect in edge, point or $\emptyset$ only.

Smallest $\mathbb{T}^2$ and $\mathbb{K}^2$-embeddings for $(a=b, k)=(6, 3), (3, 6), (4, 4)$:

as 6-regular triangulations: K_7 and $K_{3,3,3}$ ($p_3 = 14, 18$);

as 3-regular polyhexes: Heawood graph (dual K_7) and dual $K_{3,3,3}$;

as 4-regular quadrangulations: K_5 and $K_{2,2,2}$ ($p_4 = 5, 6$).

K_5 and $K_{2,2,2}$ are also smallest $(\{3, 4\}, 4)$-$\mathbb{P}^2$ and $(\{3, 4\}, 4)$-$\mathbb{S}^2$, while K_4 is the smallest $(\{4, 6\}, 3)$-$\mathbb{P}^2$ and $(\{3, 6\}, 3)$-$\mathbb{S}^2$.

3-regular polyhexes on $\mathbb{T}^2$, cylinder, Möbius surface, $\mathbb{K}^2$ are $\{6^3\}$'s quotients by fixed-point-free group of isometries, generated by: two translations, a transl., a glide reflection, transl. *and* glide reflection.

The point symmetry groups with inversion operation are: T_h, O_h, I_h, C_{mh}, D_{mh} with even m and D_{md}, S_{2m} with odd m. So, they are

- 9 for $\{5,6\}_v$: C_i, C_{2h}, D_{2h}, D_{3d}, D_{6h}, S_6, T_h, D_{5d}, I_h
- 7 for $\{2,3\}_v$: C_i, C_{2h}, D_{2h}, D_{3d}, D_{6h}, S_6, T_h
- 6 for $\{4,6\}_v$: C_i, C_{2h}, D_{2h}, D_{3d}, D_{6h}, O_h
- 6 for $\{3,4\}_v$: C_i, C_{2h}, D_{2h}, D_{3d}, D_{4h}, O_h
- 2 for $\{2,4\}_v$: D_{2h}, D_{4h}
- 1 for $\{3,6\}_v$: D_{2h}
- 0 for $\{2,6\}_v$ and $\{1,3\}_v$
- Cf. 12 for icosahedrites (($\{3,4\},5$)-spheres):
  C_i, C_{2h}, C_{4h}, D_{2h}, D_{4h}, D_{3d}, D_{5d}, S_6, S_{10}, T_h, O_h, I_h

(R,k)-maps on the projective plane are the antipodal quotients of centrosymmetric (R,k)-spheres; so, halving their p-vector and v.

Smallest $(\{a, b\}, k)$-maps on the projective plane

- The smallest ones for $(a, b) = (4, 6), (3, 4), (3, 6), (5, 6)$ are: K_4 (smallest $\mathbb{P}^2$-quadrangulation), K_5, 2-truncated K_4, dual K_6 (Petersen graph), i.e., the antipodal quotients of Cube $\{4, 6\}_8$, $\{3, 4\}_{10}(D_{4h})$, $\{3, 6\}_{16}(D_{2h})$, Dodecahedron $\{5, 6\}_{20}$.
- The smallest ones for $(a, b) = (2, 4), (2, 3)$ are points with 2, 3 loops; smallest without loops are $4 \times K_2$, $6 \times K_2$ but on $\mathbb{P}^2$.
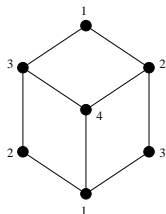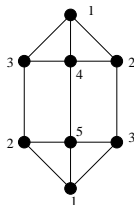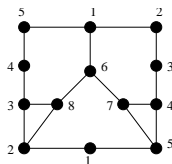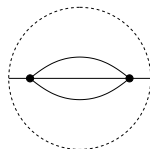
$\{4, 6\}_4 \qquad \{3, 4\}_5 \qquad \{3, 6\}_8 \qquad \{2, 4\}_2$

The Petersen graph (in positive role) is the smallest $\mathbb{P}^2$-fullerene.
Its $\mathbb{P}^2$-dual, K_6, is the smallest $\mathbb{P}^2$-icosahedrite (half-Icosahedron).
K_6 is also the smallest (with 10 triangles) triangulation of $\mathbb{P}^2$.

- $\{5,6\}_v$: C_i, C_{2h}, D_{2h}, S_6, D_{3d}, D_{6h}, T_h, D_{5d}, I_h
- $\{2,3\}_v$: C_i, C_{2h}, D_{2h}, S_6, D_{3d}, D_{6h}, T_h
- $\{4,6\}_v$: C_i, C_{2h}, D_{2h}, D_{3d}, D_{6h}, O_h
- $\{3,4\}_v$: C_i, C_{2h}, D_{2h}, D_{3d}, D_{4h}, O_h
- $\{2,4\}_v$: D_{2h}, D_{4h}
- $\{3,6\}_v$: D_{2h}

- $\{5, 6\}_v$: C_i, C_{2h}, D_{2h}, S_6, D_{3d}, D_{6h}, T_h, D_{5d}, I_h
- $\{2, 3\}_v$: C_i, C_{2h}, D_{2h}, S_6, D_{3d}, D_{6h}, T_h
- $\{4, 6\}_v$: C_i, C_{2h}, D_{2h}, D_{3d}, D_{6h}, O_h
- $\{3, 4\}_v$: C_i, C_{2h}, D_{2h}, D_{3d}, D_{4h}, O_h
- $\{2, 4\}_v$: D_{2h}, D_{4h}
- $\{3, 6\}_v$: D_{2h}

$(\{2, 3\}, 6)$-spheres T_h and D_{6h} are $GC_{k,k}(2 \times \text{Tetrahedron})$ and, for $k \equiv 1, 2 \pmod 3$, $GC_{k,0}(6 \times K_2)$, respectively. Other spheres of blue symmetry are $GC_{k,l}$ with $l = 0$, k from the first such sphere.

So, each of 7 blue-symmetric families is described by one natural parameter k and contains $O(\sqrt{v})$ spheres with at most v vertices.

$(\{a,b\},k)$-maps on Euclidean plane and 3-space

- An $(\{a,b\},k)$-$\mathbb{E}^2$ is a k-regular tiling of $\mathbb{E}^2$ by a- and b-gons.
- $(\{a,b\},k)$-$\mathbb{E}^2$ have $p_a \leq \frac{b}{b-a}$ and $p_b = \infty$. It follows from Alexandrov, 1958: any metric on $\mathbb{E}^2$ of non-negative curvature can be realized as a metric of convex surface on $\mathbb{E}^3$. In fact, consider plane metric such that all faces became regular in it. Its curvature is 0 on all interior points (faces, edges) and ≥ 0 on vertices. A convex surface is at most half-$\mathbb{S}^2$.
- There are ∞ of $(\{a,b\},k)$-$\mathbb{E}^2$ if $2 \leq p_a \leq \frac{b}{b-a}$ and 1 if $p_a = 0, 1$.
- The plane fullerenes (or nanocones) $(\{5,6\},k)$-$\mathbb{E}^2$ are classified by Klein and Balaban, 2007: the number of *equivalence* (isomorphism up to a finite induced subgraph) classes is 2,2,2,1 for $p_5 = 2, 3, 4, 5$, respectively.

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- An $(\{a, b\}, k)$-$\mathbb{E}^3$ is a 3-periodic k'-regular face-to-face tiling of the Euclidean 3-space $\mathbb{E}^3$ by $(\{a, b\}, k)$-spheres.
- Next, we will mention such tilings by 4 special fullerenes, which are important in Chemistry and Crystallography. Then we consider extension of $(\{a, b\}, k)$-maps on manifolds.

XII. Beyond surfaces

Frank-Kasper ($\{a, b\}, k$)-spheres and tilings

- A ($\{a, b\}, k$)-sphere is Frank-Kasper if no b-gons are adjacent.
- All cases are: smallest ones in 8 families, 3 ($\{5, 6\}, 3$)-spheres (24-, 26-, 28-vertex fullerenes), ($\{4, 6\}, 3$)-sphere $Prism_6$, 3 ($\{3, 4\}, 4$)-spheres ($APrism_4$, $APrism_3^2$, Cuboctahedron), ($\{2, 4\}, 4$)-sphere doubled square and two ($\{2, 3\}, 6$)-spheres (tripled triangle and doubled Tetrahedron).
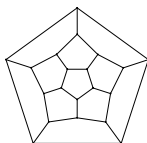
20, I_h 24 D_{6d} 26, D_{3h} 28, T_d

FK space fullerenes

A FK space fullerene is a 3-periodic 4-regular face-to-face tiling of 3-space $\mathbb{E}^3$ by four Frank-Kasper fullerenes $\{5, 6\}_v$.

They appear in crystallography of alloys, clathrate hydrates, zeolites and bubble structures. The most important, A_{15}, is below.
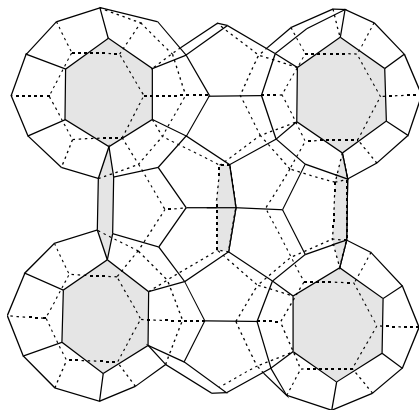
Weaire-Phelan, 1994: best known solution of weak Kelvin problem

Other $\mathbb{E}^3$-tilings by $(\{a, b\}, k)$-spheres

- An $(\{a, b\}, k)$-$\mathbb{E}^3$ is a 3-periodic k'-regular face-to-face $\mathbb{E}^3$-tiling by $(\{a, b\}, k)$-spheres. Some examples follow.
- Deza-Shtogrin, 1999: first known non-FK space fullerene $(\{5, 6\}, 3)$-$\mathbb{E}^3$: 4-regular $\mathbb{E}^3$-tiling by $\{5, 6\}_{20}$, $\{5, 6\}_{24}$ and its elongation $\simeq \{5, 6\}_{36}$ (D_{6h}) in proportion 7:2:1.

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- space cubite $(\{4, 6\}, 3)$-$\mathbb{E}^3$: tiling by $Prism_4$, $Prism_6$ with bipyramidal star. Examples: 5- and 6-regular $\mathbb{E}^3$-tilings by $Prism_6$ and by Cube (Voronoi tilings of lattices $A_2 \times \mathbb{Z}$ and $\mathbb{Z}^3$ with stars $Prism_3^*$ and $\beta_3 = Prism_4^*$), respectively.

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- space octahedrite $(\{3, 4\}, 4)$-$\mathbb{E}^3$: 6-regular (star-Octahedron) tiling by Octahedron, Cuboctahedron in proportion 1:1. It is uniform (vertex-transitive and with Archimedean tiles) and Delaunay tiling of J-complex (mineral **perovskite** structure).
- Cf. $\mathbb{H}^3$-tilings: 6-regular $\{5, 3, 4\}$ by $\{5, 6\}_{20}$, (Löbell, 1931) by $\{5, 6\}_{24}$ and 12-reg. $\{5, 3, 5\}$ by $\{5, 6\}_{20}$, $\{4, 3, 5\}$ by Cube.

- Given $3 \leq a < b \leq 6$, $\{a, b\}$-manifold is a $(d-1)$-dimensional d-valent compact connected *manifold* (locally homeomorphic to $\mathbb{R}^{d-1}$) whose 2-faces are only a- or b-gonal.

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Fullerene manifolds

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- The smallest polyhex is 6-gon on $\mathbb{T}^2$. The "greatest": $\{633\}$, the convex hull of vertices of $\{63\}$, realized on a horosphere.
- Prominent 4-fullerene (600-vertex on $\mathbb{S}^3$) is 120-cell ($\{533\}$). The "greatest" polypent: $\{5333\}$, tiling of $\mathbb{H}^4$ by 120-cells.
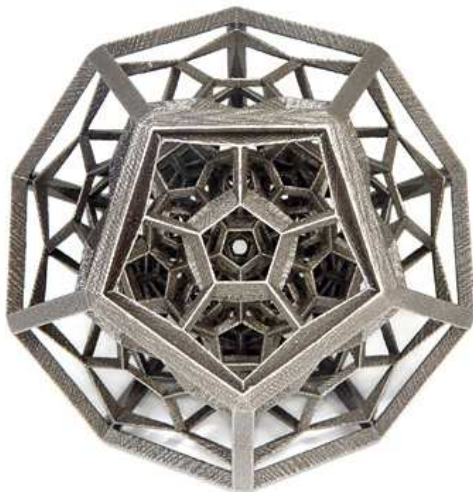
{533}: 600 vertices, 120 dodecahedral facets, $|Aut| = 14400$

- All known finite 4-fullerenes are "mutations" of 120-cell by interfering in one of ways to construct it: tubes of 120-cells, coronas, inflation-decoration method, etc.
  Some putative facets: $\simeq \{5,6\}_v(G)$ with $(v, G){=}(20, I_h)$, $(24, D_{6h})$, $(26, D_3)$, $(28, T_d)$, $(30, D_{5h})$, $(32, D_{3h})$, $(36, D_{6h})$.
- $(\{5,6\}, 3)\text{-}\mathbb{E}^3$: example of interesting infinite 4-fullerenes.

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- $(\{5, 6\}, 3)$-$\mathbb{E}^3$: example of interesting infinite 4-fullerenes.
- All known 5-fullerenes come from $\{5333\}$'s by following ways.
  With 6-gons also: glue two $\{5333\}$'s on some 120-cells and delete their interiors. If it is done on only one 120-cell, it is $\mathbb{R} \times \mathbb{S}^3$ (so, simply-connected).
  Finite compact ones: the quotients of $\{5333\}$ by its symmetry group (partitioned into 120-cells) and gluings of them.

Quotient d-fullerenes

- Selberg, 1960, Borel, 1963: if a discrete group of motions of a symmetric space has a compact fundamental domain, then it has a torsion-free normal subgroup of finite index.

- So, the *quotient* of a d-fullerene by such symmetry group (its points are group orbits) is a finite d-fullerene.

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- Exp 2: Poincaré dodecahedral space: the quotient of 120-cell by I_h ; so, its f-vector is $(5, 10, 6, 1) = \frac{1}{120}$f(120-cell).

- Cf. 6-, 12-regular $\mathbb{H}^3$-tilings $\{5, 3, 4\}$, $\{5, 3, 5\}$ by $\{5, 6\}_{20}$ and 6-regular $\mathbb{H}^3$-tiling by (right-angled) $\{5, 6\}_{24}$. Seifert-Weber, 1933 and Löbell, 1931 spaces are quotients of last 2 with f-vectors $(1, 6, p_5=6, 1)$, $(24, 72, 48+8=p_5+p_6, 8)$.