Single Delaunay in lattices

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January 16, 2010

I. Delaunay polytopes and hypermetrics

Hypermetric inequalities

- We consider functions on $X_n = \{0, \dots, n\}$:

$$d : X_n^2 \to \mathbb{R} \text{ with } d(i,j) = d(j,i) \text{ and } d(i,i) = 0$$

- If $b \in \mathbb{Z}^{n+1}$, $\sum_{i=0}^{n} b_i = 1$ then the hypermetric inequality is

$$H(b)d = \sum_{0 \le i < j \le n} b_i b_j d(i,j) \le 0$$

- If $b = (1, 1, -1, 0, \dots, 0)$ then one gets

$$d(1,2) \le d(1,3) + d(2,3),$$

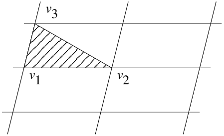
  which is the triangle inequality.
- The hypermetric cone HYP_{n+1} is the set of all d such that $H(b)d \le 0$ for all b
- *dim* $\mathrm{HYP}_{n+1} = \binom{n+1}{2}$
- HYP_{n+1} is defined by an infinite set of inequalities

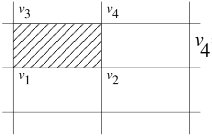
Delaunay polytopes

- If $\mathcal{D}$ is an n dimensional Delaunay polytope with center c, radius r and vertices $\{v_0, \ldots, v_N\}$ then $d(i,j) = \|v_i - v_j\|^2$ satisfies

$$\sum_{i,j} b_i b_j d(i,j) = 2(r^2 - \|\sum_i b_i v_i - c\|^2) \leq 0$$

- So, Delaunay polytope gives hypermetrics and the correspondence goes the other way.
- Moreover $\sum_i b_i v_i$ is a vertex of $\mathcal{D}$ if and only if $H(b)d = 0$





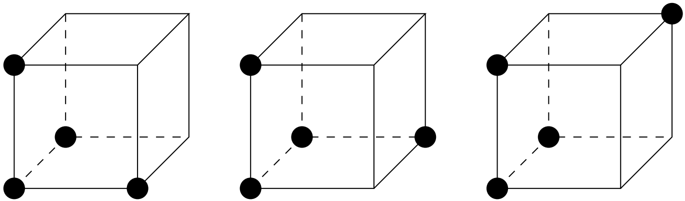
$v_4 = v_2 + v_3 - v_1$

Affine basis

- An affine basis of an n-dimensional polytope P is $\{v_0, \dots, v_n\}$ such that for every vertex v of P, there is

$$\begin{array}{c} b_i \in \mathbb{Z}, \text{such that} \;\; b_0 + \cdots + b_n = 1 \\ \text{and} \;\; b_0 v_0 + b_1 v_1 + \cdots + b_n v_n = v \end{array}$$

- The three types of affine basis of the 3-cube:



- Every Delaunay polytope of dimension ≤ 6 has an affine basis
  - M. Dutour, *The six-dimensional Delaunay polytopes*, European Journal of Combinatorics **25-4** (2004) 535–548.

From an hypermetric to its representations

- The vector $e_i = (0, \dots, 1, \dots, 0)$ gives the inequality $H(e_i) = 0$.
- There is a linear bijection between $n \times n$ symmetric matrices M and metrics on $n+1$ points $(0, e_1, \dots, e_n)$ by writing

$$d_M(v, v') = \|v - v'\|_M^2 = vMv^T - 2v'Mv^T + v'Mv'^T$$

- Let M be a Gram matrix such that $d_M \in \mathrm{HYP}_{n+1}$, then
  - $(0, e_1, \dots, e_n)$ belong to a Delaunay polytope of M.
  - Any other vertices is characterized by $H(b)d_M = 0$.
- So, we can interpret the hypermetric cone HYP_{n+1} as the parameter space of a basic simplex in a n-dimensional lattice.
- Also, all possible n-dimensional Delaunay polytopes and their affine basis are parametrized by the faces of HYP_{n+1}.

Polyhedrality of HYP_n

- Theorem: HYP_n is a polyhedral cone.
  Proof:
  - HYP_n is a convex cone.
  - HYP_n is an union of L-type domains.
  - Given a L-type, we have a finite number of simplices to choose from as basic simplices.
  - There is a finite number of L-types under $\mathrm{GL}_n(\mathbb{Z})$ equivalence and so a finite number of L-types in HYP_n.
  - So, HYP_n is polyhedral.
- if $H(b)d \leq 0$ is a facet then $|b_i| \leq n!$
- (Lovasz) if $H(b)$ defines a facet then $|b_i| \leq \frac{2^n}{\binom{2n}{n}} n!$

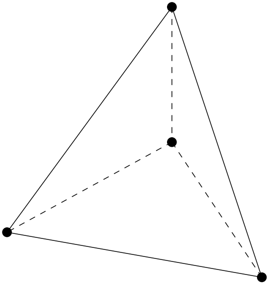
n	# orbit facets	# facets
3	1	3
4	1	12
5	2	40
6	4	210
7	14	3773

Rank of a Delaunay polytope

- Combinatorial types of n-dimensional Delaunay polytope P correspond to faces F of HYP_{n+1}
- One defines $rank(P) = dim\, F$
- $rank(P)$ is the number of degree of freedom.
- $rank\,(P) = \binom{n+1}{2}$, then P is a simplex.
- $rank\,(P) = 1$, then P is an extreme Delaunay polytope.
- The only degrees of freedom of an extreme Delaunay polytope are translation, isometries and homotheties
- If P is extreme then it has at least $\frac{(n+1)(n+2)}{2} - 1$ vertices.
  - ➠ M. Deza, V.P. Grishukhin, and M. Laurent, *Extreme hypermetrics and L-polytopes*, Colloquia Mathematica Societatis János Bolyai, **60** (1992) 157–209.

II. Classifications of lattice Delaunay polytopes

3-dimensional case

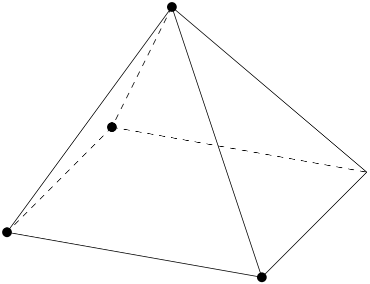


3–simplex

Hypermetric Vectors

Rank: 6

3-dimensional case



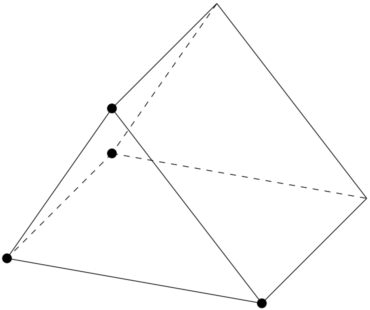
Pyramid

Hypermetric Vectors

$(-1\ ,\ 0\ ,\ 1\ ,\ 1)$

Rank: 5

3-dimensional case



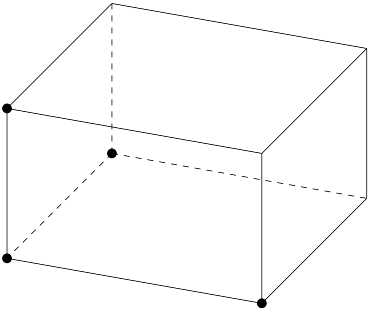
3–Prism

Hypermetric Vectors

(−1 , 0 , 1 , 1)
(−1 , 1 , 0 , 1)

Rank: 4

3-dimensional case



Cube

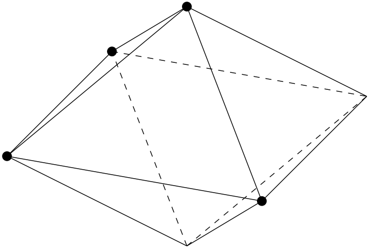
Hypermetric Vectors

(−1 , 0 , 1 , 1)
(−1 , 1 , 0 , 1)
(−1 , 1 , 1 , 0)
(−2 , 1 , 1 , 1)

Rank: 3

$$H(-2, 1, 1, 1)=H(-1, 0, 1, 1)+H(-1, 1, 0, 1)+H(-1, 1, 1, 0)$$

3-dimensional case



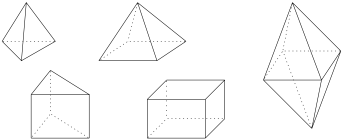
Octahedron

Hypermetric Vectors

(−1 , 0 , 1 , 1)
(0 ,−1 , 1, 1)

Rank: 4

Combinatorial types



dim	Nr of types	Authors
2	2	Dirichlet (1860)
3	5	Fedorov (1885)
4	19	Erdahl-Ryshkov (1987)
5	138	Kononenko (1997)
6	6241	Dutour (2002)

➠ M. Dutour, *The six-dimensional Delaunay polytopes*, European Journal of Combinatorics **25**-**4** (2004) 535–548.

III. The six-dimensional Delaunay polytopes

Cut polytope and cut cone

- If $S \subset X_n = \{1, \dots, n\}$ then we define the cut metric

$$\begin{array}{rcl} \delta_S : X_n^2 & \rightarrow & \mathbb{R} \\ (x, y) & \mapsto & \begin{cases} 1 & \text{if } |S \cap \{x, y\}| = 1 \\ 0 & \text{otherwise} \end{cases} \end{array}$$

  One has $\delta_S = \delta_{X_n - S}$

- The cut polytope CUT_n^P is the convex hull of all δ_S. CUT_n^P has 2^{n-1} vertices, dimension $\frac{n(n-1)}{2}$, $|\operatorname{Aut}(\mathrm{CUT}_n^P)| = 2^{n-1} n!$.
- The cut cone CUT_n is the polyhedral cone with the extreme rays δ_S for $S \neq 0, X_n$. CUT_n has $2^{n-1} - 1$ extreme rays, dimension $\frac{n(n-1)}{2}$, $|\operatorname{Aut}(\mathrm{CUT}_n)| = n!$.
- See for more details:
  - M. Deza and M. Laurent, *Geometry of cuts and metrics*, Springer Verlag.

Relation to the hypermetric cone

- The cut metric δ_S corresponds to the metric on a Delaunay with two vertices $[0,1]$ of $\mathbb{Z}$.
- $\mathrm{CUT}_{n+1} \subset \mathrm{HYP}_{n+1}$ for all n
- $\mathrm{CUT}_{n+1} = \mathrm{HYP}_{n+1}$ if $n \leq 5$, which means that all facets are hypermetric.
- $\Rightarrow$ no other extreme Delaunay polytope in dimension lower than 5
- But $\mathrm{CUT}_7 \neq \mathrm{HYP}_7 \Rightarrow$ there is an extreme six-dimensional Delaunay polytope
  - The Schlafli polytope *Sch* is an extreme six-dimensional Delaunay polytope
  - There are 26 orbits of affine basis of *Sch*.

Facets of HYP_7 and CUT_7

- Baranovski has found 14 orbits of facets of HYP_7
- Method: direct proof that others are redundant

$(1, 1, -1, 0, 0, 0, 0)$	$(1, 1, 1, -1, -1, 0, 0)$
$(1, 1, 1, 1, -1, -2, 0)$	$(2, 1, 1, -1, -1, -1, 0)$
$(1, 1, 1, 1, -1, -1, -1)$	$(2, 2, 1, -1, -1, -1, -1)$
$(1, 1, 1, 1, 1, -2, -2)$	$(2, 1, 1, 1, -1, -1, -2)$
$(3, 1, 1, -1, -1, -1, -1)$	$(1, 1, 1, 1, 1, -1, -3)$
$(2, 2, 1, 1, -1, -1, -3)$	$(3, 1, 1, 1, -1, -2, -2)$
$(3, 2, 1, -1, -1, -1, -2)$	$(2, 1, 1, 1, 1, -2, -3)$

- CUT_7 has 36 orbits of facets
  - 10 orbits are hypermetric.
  - 26 orbits are non-hypermetric.

Extreme rays of HYP_7

The cone CUT_7 has hypermetric and non-hypermetric facets.

- Suppose e is an extreme ray of HYP_7, which is not an extreme ray of CUT_7.
- Then there exist a non-hypermetric facet f of CUT_7, which is violated by e, $f(e) < 0$.
- We define the cone

$$\mathrm{HYP}_{7,f} = \{x \in \mathrm{HYP}_7 \text{ such that } f(x) \leq 0\}$$

  It has $3773 + 1$ inequalities but highly redundant.
- We reduce by linear programming to a set of 21 inequalities.
- $\mathrm{HYP}_{7,f}$ contains 20 cut metric extreme ray and 1 non-cut extreme ray.
- This establish a 1 to 1 correspondence between non-cut extreme rays of HYP_7 and non-hypermetric facets of CUT_7.

6241 six-dimensional Delaunays

rank	Nr. in HYP_7	Nr. in CUT_7			
21	1(simplex)	0	11	686	325
20	9	1	10	417	183
19	30	2	9	218	83
18	95	8	8	108	35
17	233	28	7	52	13
16	500	95	6	21	3
15	814	241	5	8	0
14	1092	434	4	4	0
13	1145	527	3	2	0
12	984	481	2	1	0
			1	1(Schläfli)	0

IV. Examples of extreme Delaunay polytopes

Highly symmetric examples

- In dimension 6, there is only Schlafli polytope
  - M. Deza and M. Dutour, *The hypermetric cone on seven vertices*, Experimental Mathematics **12-4** (2004) 433–440.
- Conjectured list in dimension 7:

$\#vertices$	$\#facets$	$\|Sym\|$
35	228	1440
56	702	2903040(*Gossetpolytope*)

- Conjectured list in dimension 8:

$\|V\|$	$\|G\|$	$\|V\|$	$\|G\|$	$\|V\|$	$\|G\|$	$\|V\|$	$\|G\|$	$\|V\|$	$\|G\|$	$\|V\|$	$\|G\|$
47	24	47	48	52	192	44	288	45	1296	72	80640
46	36	44	72	45	192	46	288	58	1440	79	322560
45	48	44	72	44	240	54	384	44	2880		
44	48	45	144	55	288	54	864	44	10080		
47	48	44	144	49	288	49	960	72	80640		

Infinite sequence of extreme Delaunay

- If n even, $n \geq 6$, there is a n-dimensional extreme Delaunay ED_n formed with 3 layers of D_{n-1} lattice
  - a vertex
  - the $n-1$ half-cube
  - the $n-1$ cross-polytope

  $n=6$: Schläfli polytope
- If n odd, $n \geq 7$, there is a n-dimensional extreme Delaunay ED_n formed with 4 layers of previous lattice
  - a vertex
  - the ED_{n-1} extreme Delaunay
  - the ED_{n-1} extreme Delaunay
  - a vertex

  $n=7$: Gosset polytope
  - M. Dutour, *Infinite series of extreme Delaunay polytope*, European Journal of Combinatorics **26-1** (2005) 129–132.

Other infinite series

- Construction based on lamination over $J(n, 2)$:
  - R.M. Erdahl, K. Rybnikov and A. Ordine, *Perfect Delaunay Polytopes and Perfect Quadratic Functions on Lattices*, Contemporary Mathematics
- Construction of centrally symmetric polytopes:
  - V. Grishukhin, *Infinite series of extreme Delaunay polytopes*, European Journal of Combinatorics, **27-4** (2006) 481-495.
- Another construction:
  - Take $2 \leq p$ and $4p \leq n$.
  - Take as first vertex set $J(n, p)$:

$$J(n, p) = \{x \in \{0, 1\}^n \text{ such that } \sum_i x_i = p\}$$

  - Define the vector $v \in \mathbb{R}^n$ with

$$v_i = \frac{1}{2} \text{ if } i \leq 4p \text{ and } v_i = 0 \text{ otherwise.}$$

  - $J(n, p) \cup v - J(n, p)$ is an extreme $(n-1)$-dimensional Delaunay polytope

V. The Erdahl cone

Limitations of the hypermetric cone

Despite the success in dimension six

- Some Delaunay polytopes have many orbits of affine basis.
- Some Delaunay polytopes have no affine basis.
- There exist variant of the hypermetric cone obtained by taking inequalities $H(b)$ with b a fractional vector for non-basic simplex.
  - S.S. Ryshkov and E.P. Baranovskii, *Repartitioning complexes in n-dimensional lattices (with full description for $n \leq 6$)*, Voronoi impact on modern science, Book 2, Institute of Mathematics, Kyiv (1998) 115–124.

  but anyway this is not simple to work with this.
- It is difficult to obtain the list of facets of the hypermetric cone. The methods used by Baranovski have not been systematized so far.
- For $n \leq 8$, it seems the list of Delaunay polytopes is manageable, but to get a proof of completeness seems difficult.

The Erdahl cone

- Denote by $E_2(n)$ the vector space of degree 2 polynomial functions on $\mathbb{R}^n$:

$$E_2(n) = \{f(x) = b + c^T x + xAx^T\}$$

with $b \in \mathbb{R}$, $c \in \mathbb{R}^n$ and $A \in S^n$.

- The Erdahl cone is defined as

$$Erd(n) = \{f \in E_2(n) \text{ such that } f(x) \geq 0 \text{ for } x \in \mathbb{Z}^n\}$$

One has dim $Erd(n) = \frac{(n+1)(n+2)}{2}$

- If P is a Delaunay polytopes of a lattice $L = \mathbb{Z}v_1 + \dots + \mathbb{Z}v_n$ of empty sphere $S(c, r)$ then the function

$$f_P(x) = \|\sum x_i v_i - c\|^2 - r^2$$

belongs to $Erd(n)$.

- Extreme Delaunay polytopes of dimension $p \leq n$ correspond to vertices of $Erd(n)$.

The program

- We want to enumerate the 7-, 8-dimensional extreme Delaunay polytopes, that is vertices of $Erd(n)$ up to $Aff(\mathbb{Z}^n)$.
- The method will be adjacency decomposition method but with an infinite set of inequalities.
- If P is a n-dimensional extreme Delaunay polytope, then finding vertices adjacent to $f_P \in Erd(n)$ is relatively easy:
  - Find the facets of the cone defined by $f(x) \geq 0$ for $x \in V(P)$.
  - For every facet of the cone, realize the lifting with an iterative scheme.
- More interesting are polytopes of the form $Sch \times \mathbb{Z}$, which are infinite.
  - Their dimension is 34.
  - We apply the adjacency decomposition method again to find the orbits of faces.
  - We are faced with 27 problematic $(Sch - v) \times \mathbb{Z}$ polytopes to treat.
  - We need Balinski theorem in an infinite setting.

THANK

YOU