Geometry of numbers, L-types and hypermetrics

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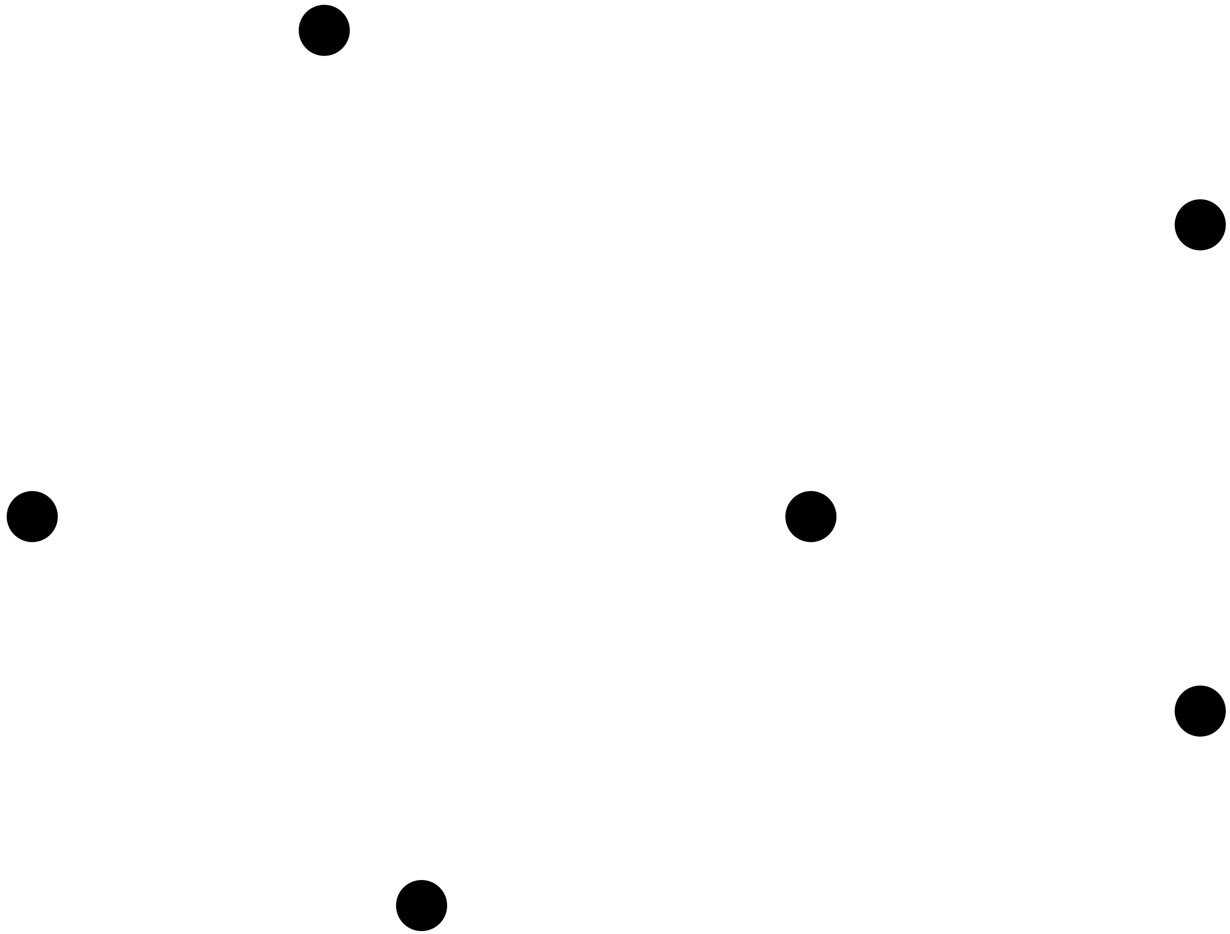
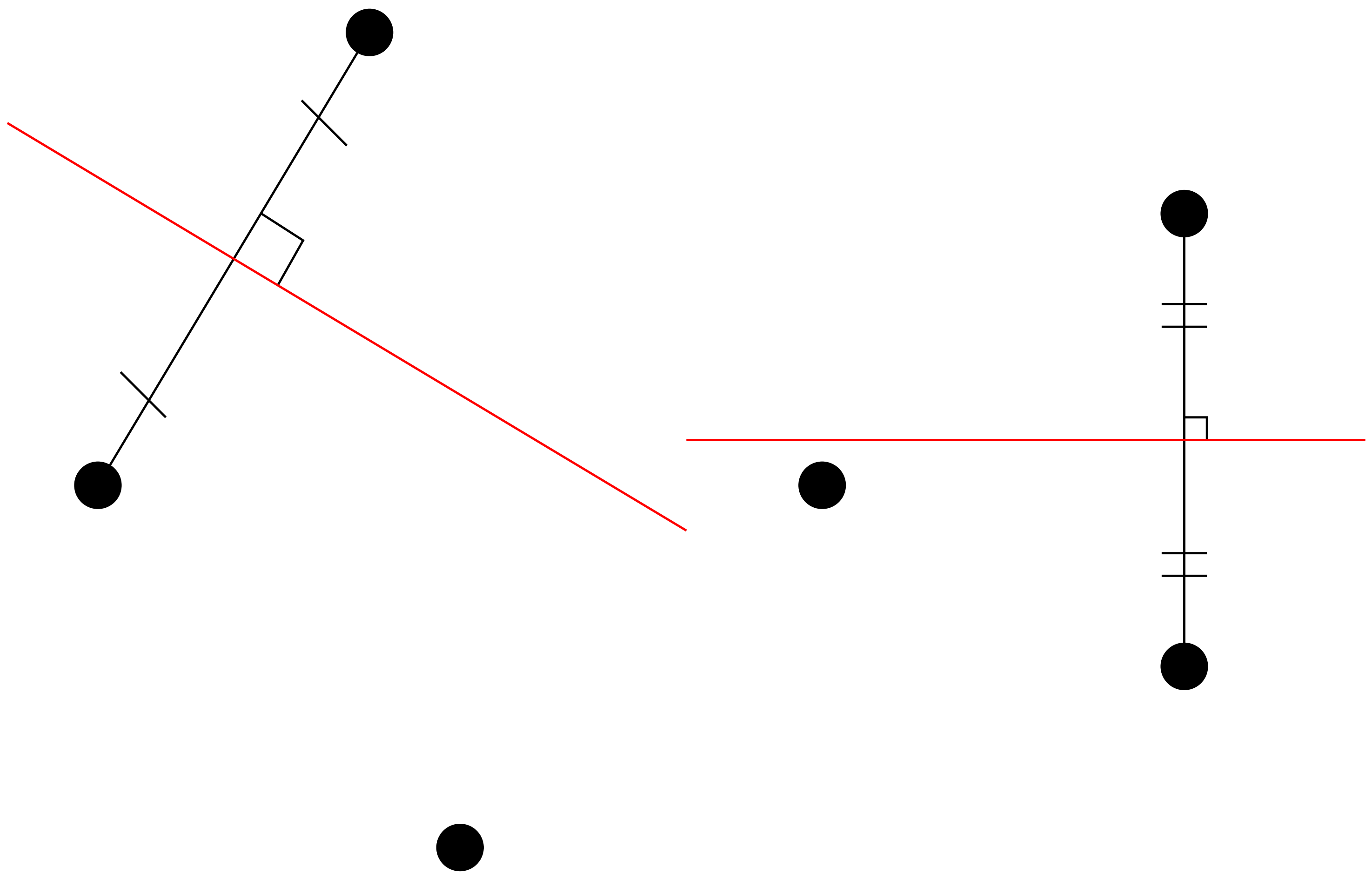
Voronoi and Delaunay polytopes

A finite set of points



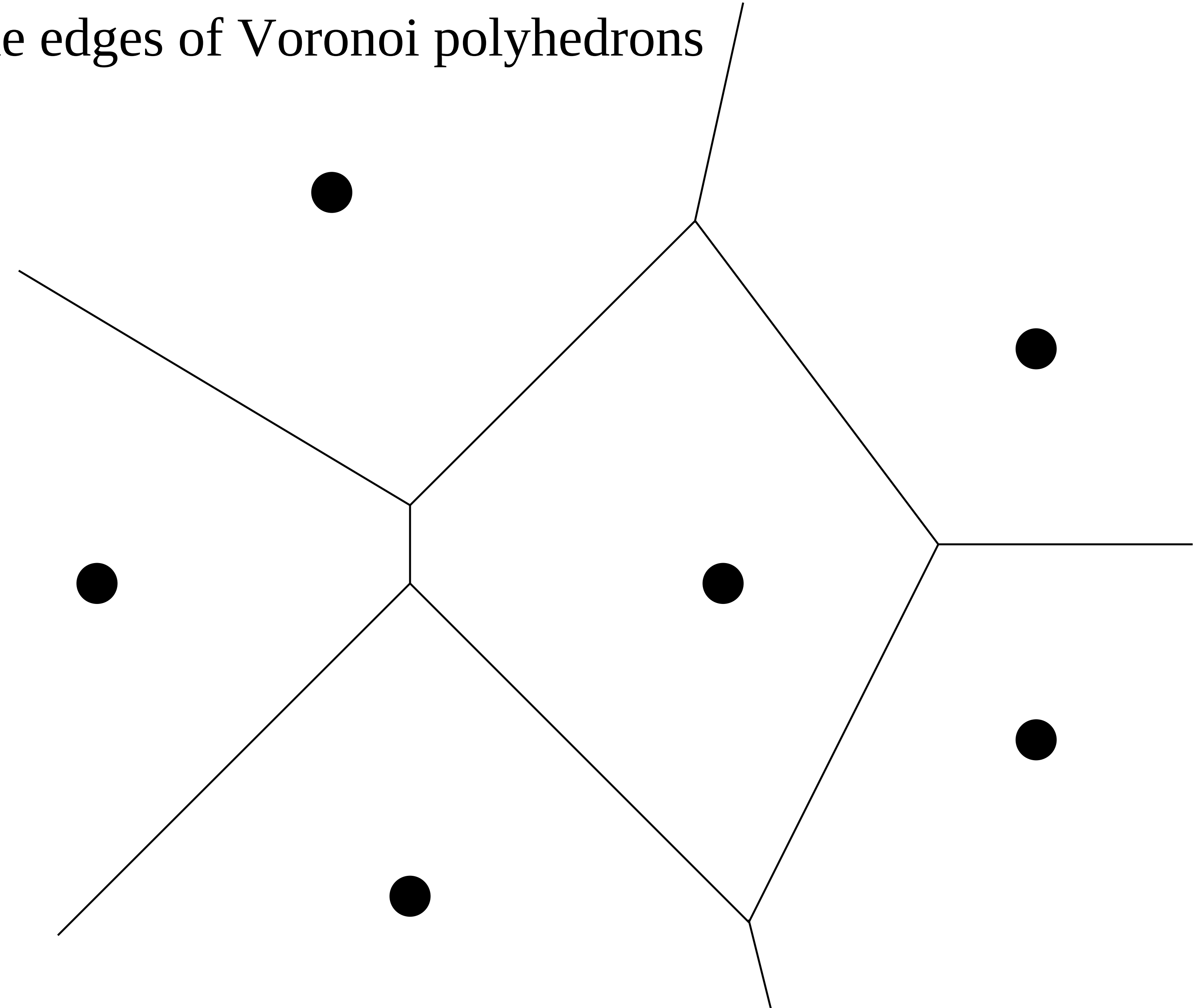
Voronoi and Delaunay polytopes

Some relevant perpendicular bisectors



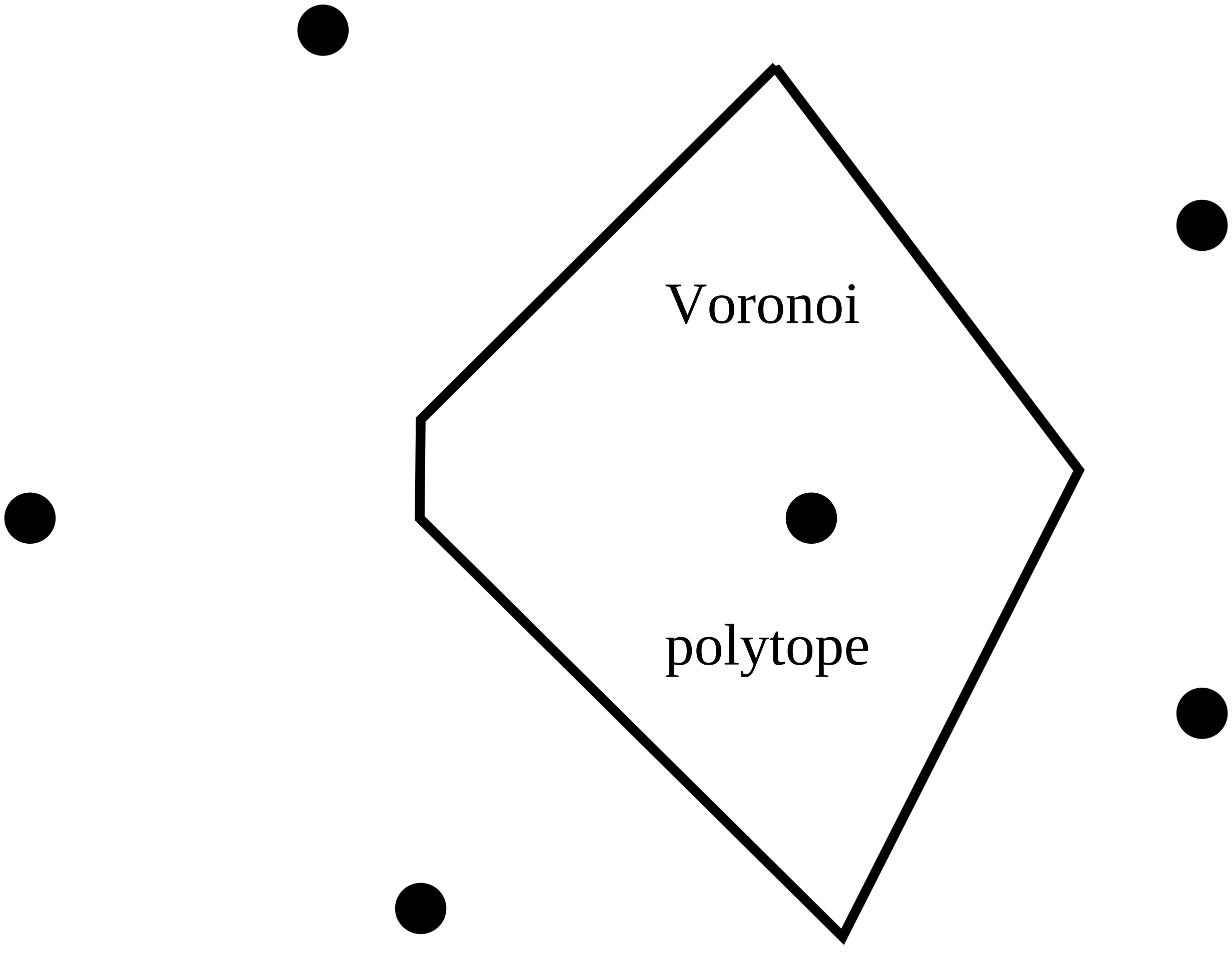
Voronoi and Delaunay polytopes

The edges of Voronoi polyhedrons



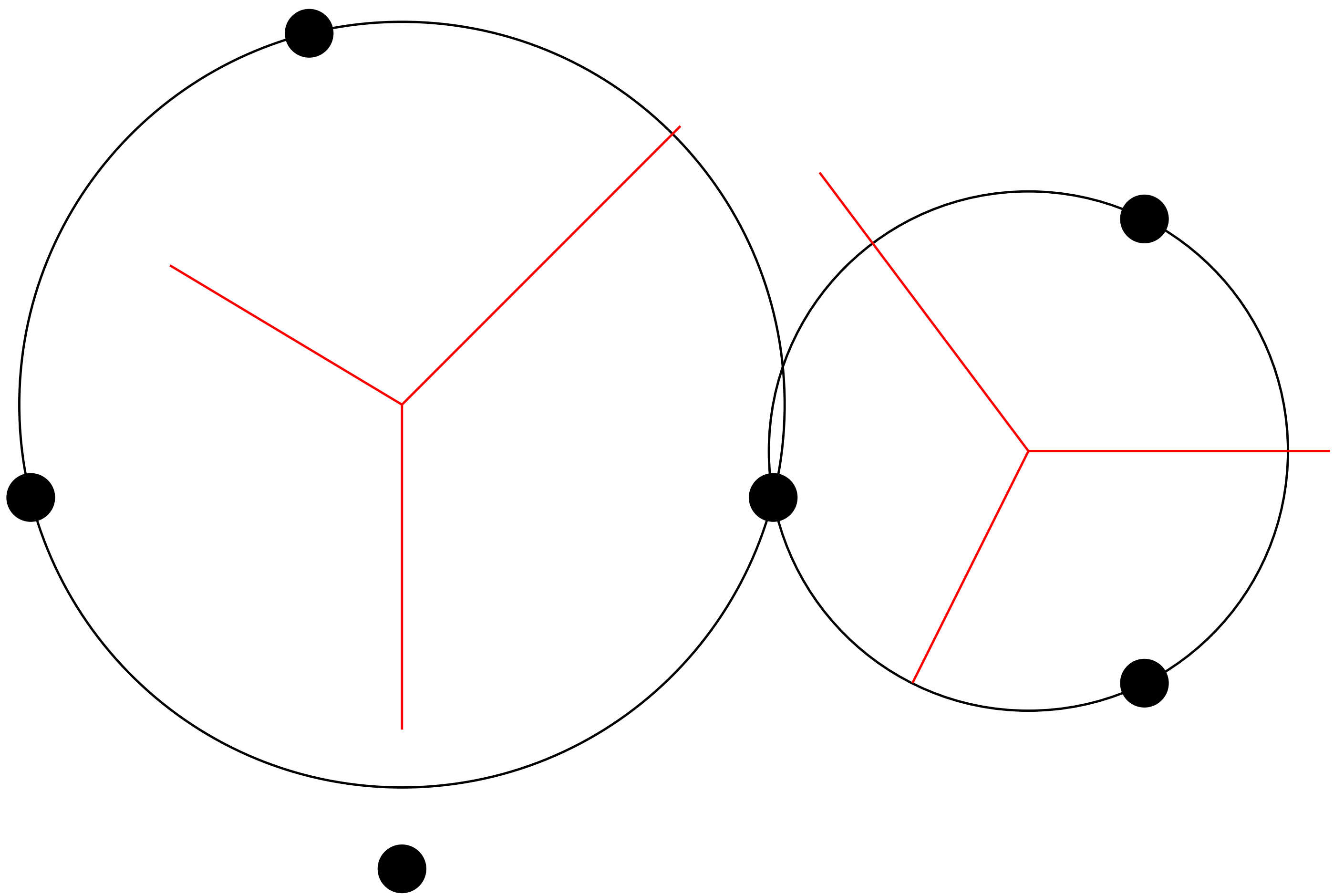
Voronoi and Delaunay polytopes

Voronoi polytope



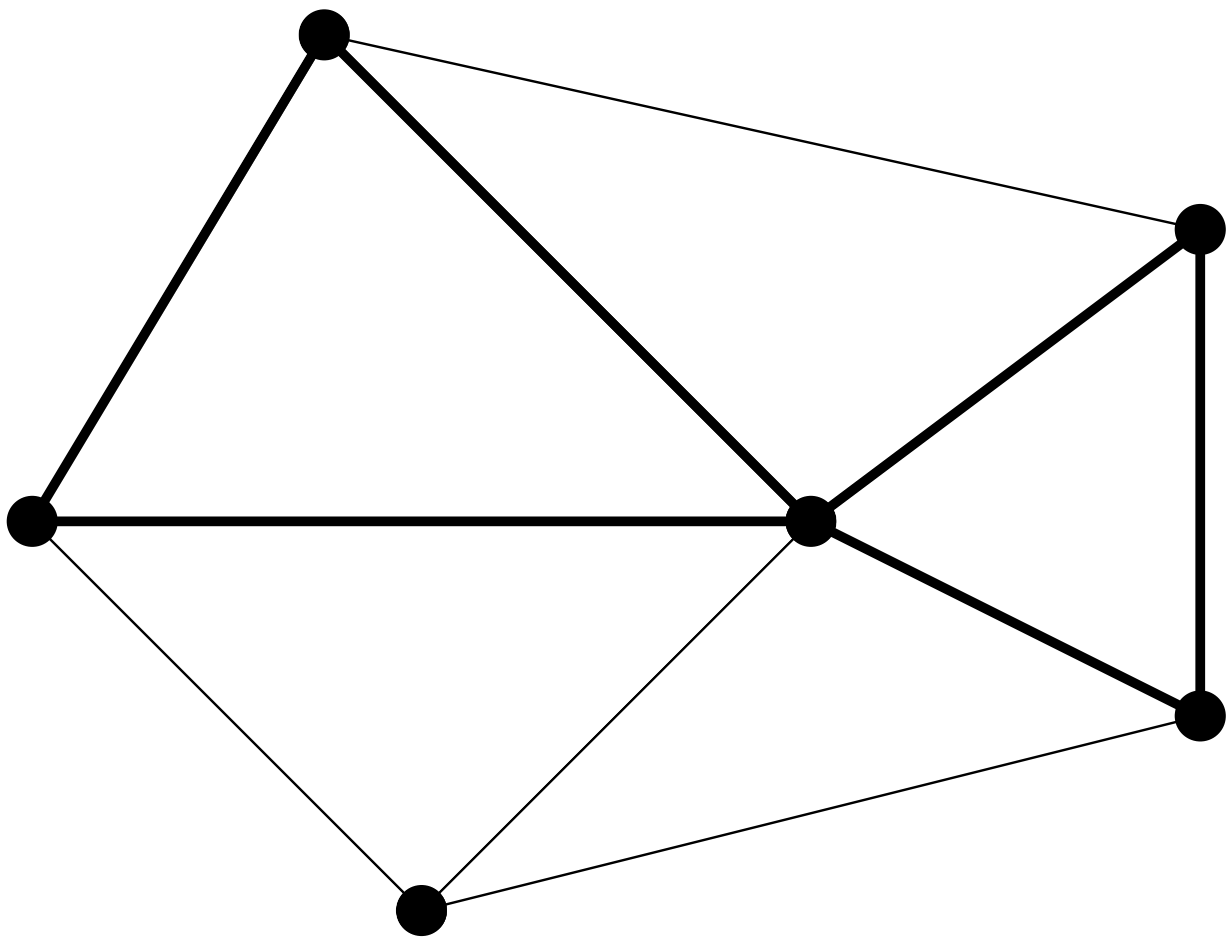
Voronoi and Delaunay polytopes

Empty spheres



Voronoi and Delaunay polytopes

Delaunay polytopes



Synonyms

Voronoi polytope synonyms:

- Dirichlet domains (lattice theory, 2-dimensional case)
- Voronoi polytope (n-dimensional lattice, computational geometry)
- Thiessen polygons (geography)
- Wigner-Seitz cell (solid state physic, crystallography)
- first Brillouin zone (solid state physic, momentum space)
- domain of influence (politics)

Delaunay polytopes synonyms:

- L-polytope (Voronoi in “Second mémoire”)
- hole (in Conway-Sloane), they are deep if of maximal radius and shallow otherwise.

I. Voronoi polytopes

in

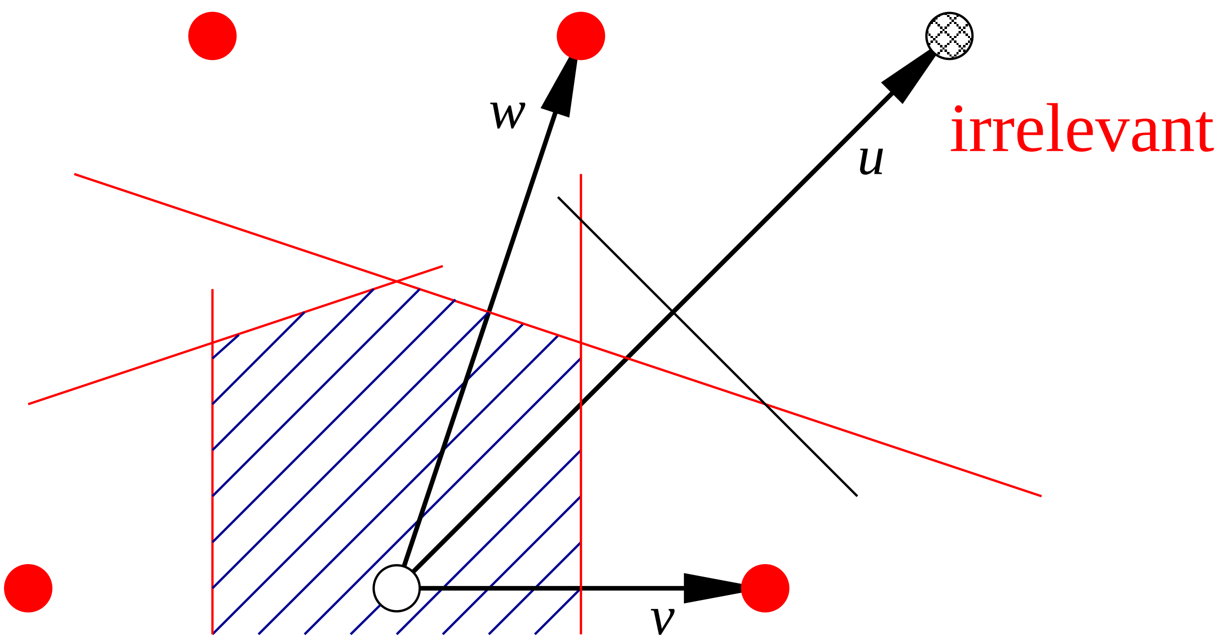
lattices

The Voronoi polytope of a lattice

- Polytope $\mathcal{V}$ defined by inequalities $\langle x, v\rangle \leq \frac{1}{2}||v||^2$.

➠ $\mathcal{V}$ is polyhedral, vector v_0 such that $\langle x, v_0\rangle = \frac{1}{2}||v_0||^2$ is a facet are called relevant.

➠ Voronoi Theorem: A vector u is relevant if and only if it can not be written as $u = v + w$ with $\langle v, w\rangle \geq 0$.



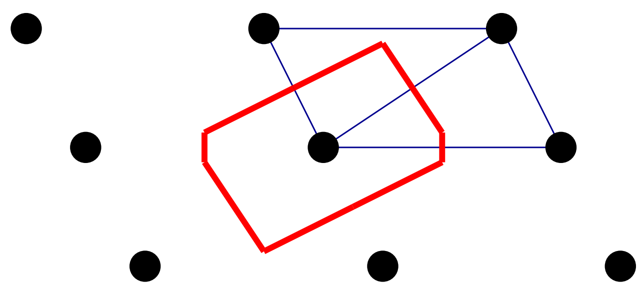
- The translates $v + \mathcal{V}$ with $v \in L$ tiles $\mathbb{R}^n$.
- Shortest vector in L are relevant.
- Only for root lattice shortest vector are all relevant vectors.

Name	Nr. facets	Nr. Vertices	Nr. Orbit
A_n	$n(n+1)$	$2^{n+1}-2$	$\lfloor \frac{n+1}{2} \rfloor$
D_n	$2n(n-1)$	$2^n + 2n$	2
E_6	72	54	1
E_7	126	632	2
E_8	240	19440	2

- At most $2(2^n - 1)$ facets and at most $(n+1)!$ vertices.

Voronoi and Delaunay in lattices

- Vertices of Voronoi polytope are center of empty spheres which defines Delaunay polytopes.
- Voronoi and Delaunay polytopes define dual tesselations of the space $\mathbb{R}^n$ by polytopes.
- Every k-dimensional face of a Delaunay polytope is orthogonal to a $(n-k)$-dimensional face of a Voronoi polytope.



- Given a lattice L, it has a finite number of orbits of Delaunay polytopes under translation.

Lattices with two Delaunay polytopes

- Take $L = \mathbb{Z}^n$; Delaunay:

Name	Center	Nr. vertices	Radius
Cube	$(\frac{1}{2})^n$	2^n	$\frac{1}{2}\sqrt{n}$

- Take $D_n = \{x \in \mathbb{Z}^n \mid \sum_{i=1}^n x_i$ is even$\}$; Delaunay:

Name	Center	Nr. vertices	Radius
Half-Cube	$(\frac{1}{2})^n$	$\frac{1}{2}2^n$	$\frac{1}{2}\sqrt{n}$
Cross-polytope	$(1, 0^{n-1})$	$2n$	1

- Take $E_8 = \{x \in \mathbb{Z}^8 \cup (\frac{1}{2} + \mathbb{Z})^8 \mid \sum_{i=1}^n x_i$ is even$\}$; Delaunay:

Name	Center	Nr. vertices	Radius
Simplex	$(\frac{5}{6}, \frac{1}{6}^7)$	9	$\sqrt{\frac{8}{9}}$
Cross-polytope	$(1, 0^7)$	16	1

Geometry of numbers by Minkowski

PSD_n=Cone of real symmetric positive definite $n \times n$ matrices.

- Lattice L spanned by $v_1, \ldots, v_n$ corresponds to

$$M_v = (\langle v_i, v_j \rangle)_{1 \le i,j \le n} \in PSD_n \ .$$

- If L spanned by v'_1,...,v'_n then $(v'_1, \ldots, v'_n) = P(v_1, \ldots, v_n)$ with $P \in GL_n(\mathbb{Z})$ and

$$M_{v'} = P M_v {}^t P \ .$$

- Lattices up to isometric equivalence correspond $GL_n(\mathbb{Z})$ equivalence classes in PSD_n.

Think in terms of lattices and compute in terms of quadratic forms!

L-type domains by Voronoi

A L-type domain is the set of quadratic forms having the same combinatorial type of Voronoi polytope.

- L-types are convex polyhedral cones, which form a tesselation of PSD_n.
- the partition is invariant with respect to $GL_n(\mathbb{Z})$.
- there are finitely many orbits, which correspond to nonisomorphic combinatorial type of Voronoi polytopes.

Two lattices in the same L-type $\mathcal{LT}$ domain can be continuously deformed without changing the structure.

- If $dim(\mathcal{LT}) = \binom{n+1}{2}$ then $\mathcal{LT}$ is called primitive $\Leftrightarrow$ its Delaunay polytope are simplices.
- If $dim(\mathcal{LT}) = 1$ then $\mathcal{LT}$ is called rigid.

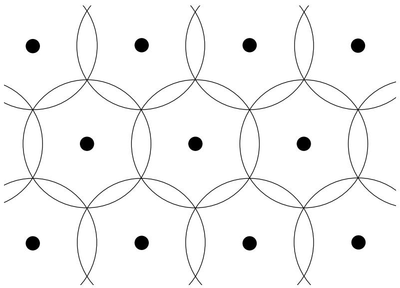
Enumeration of L-types

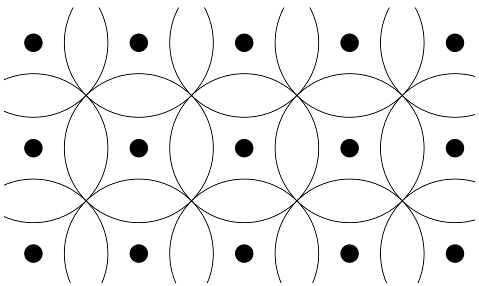
Dimension	Nr. Voronoi polytopes	Nr. primitive	Nr rigid lattices
1	1	1	1
2	2	1	0
3	5 Fedorov	1 Fedorov	0
4	52 DeSh	3 Delaunay	1
5	179377 Engel	222 BaRy, Engel	7 $\uparrow$ BaGr
6	?	$\geq 2.5.10^6$ Engel, Va	$\geq 2.10^4$ DuVa
7	?	?	?

Covering density and analogs

L is a n-dimensional lattice.

- Packing: See “premier mémoire” by Voronoi (1908).
- Covering: Denote $\Theta(L)$ its covering density by spheres centered on vectors of L:





One has $\Theta(L) = \frac{\mu(L)^n}{det(L)}\kappa_n$, with $\mu(L)$ being the largest radius of Delaunay polytopes and κ_n the volume of the unit ball B^n.

- Packing-covering: The quotient of the covering constant by the packing constant is $(\mu(L)/\lambda(L))^n$ with $\lambda(L)$ being the length of shortest vectors in L.

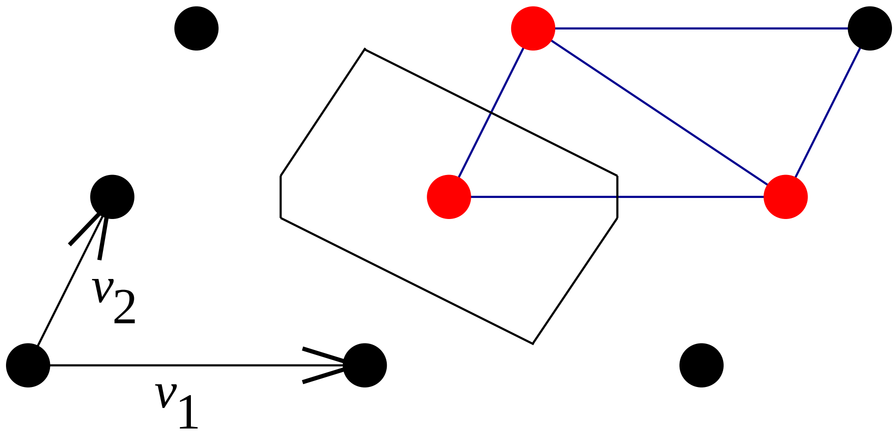
Optimization

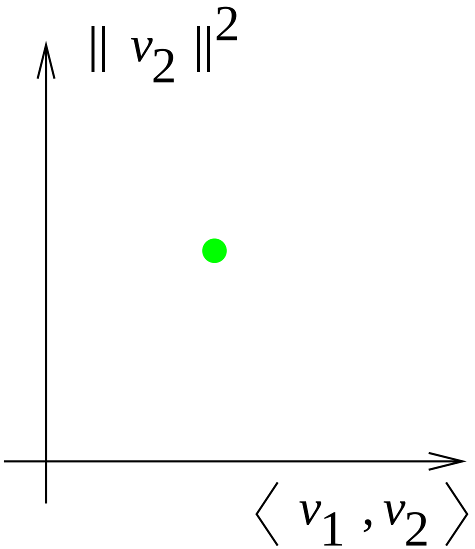
The objective is to find the lattice minimizing the covering or the packing-covering densities.

- Those functions are non-linear.
- If $\mathcal{D}$ is an L-type domain, $\{q \in \mathcal{D} \quad f(q) \leq C\}$ is convex.
- The optimization over a given primitive L-type domain is a semidefinite programming problem.
- The solution to those problem is known for $n \leq 5$.

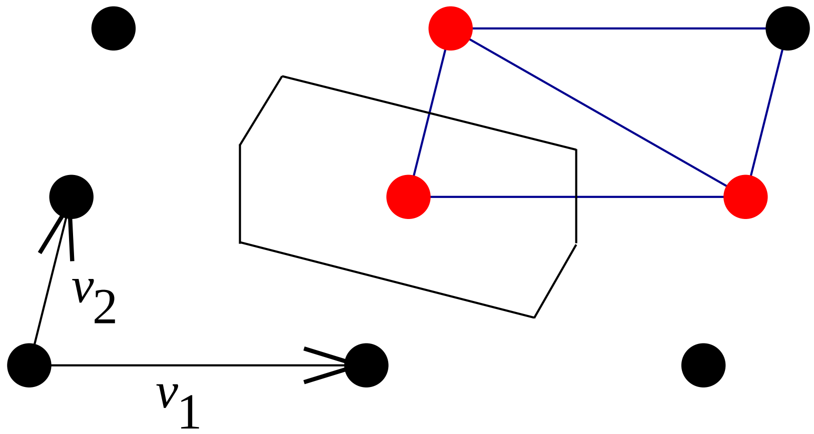
See thesis “*Sphere coverings, lattices, and tilings*” by Vallentin and the paper “*Semidefinite programming approaches to lattice packing and covering problems*” by Schürmann & Vallentin.

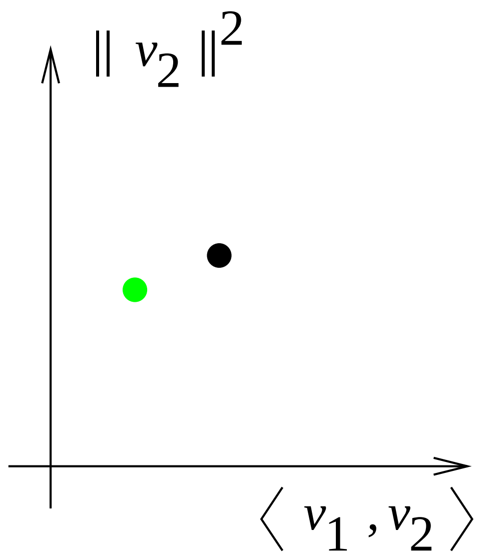
Lattices in dimension 2



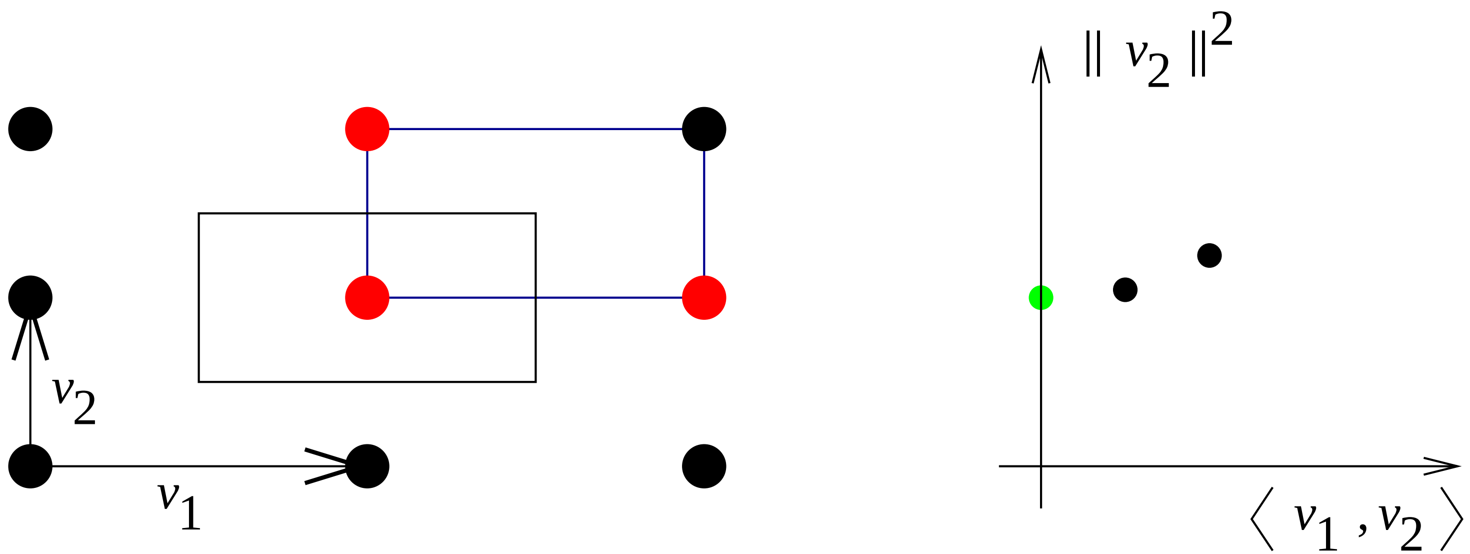


Lattices in dimension 2

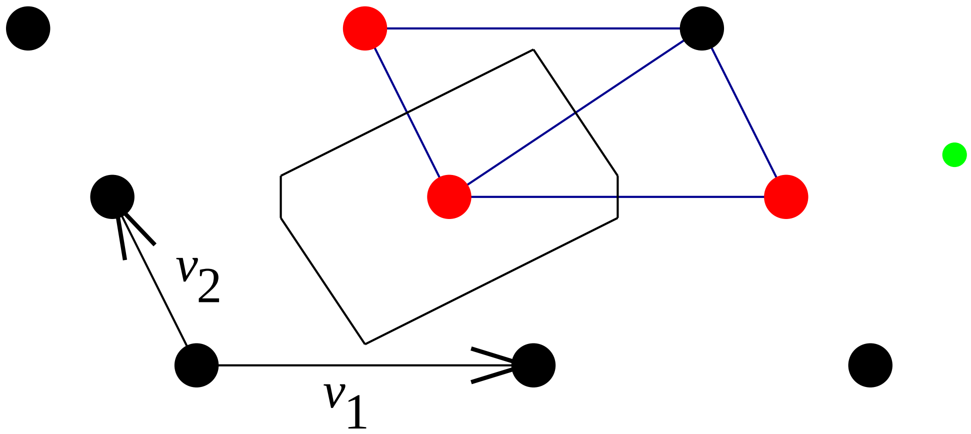


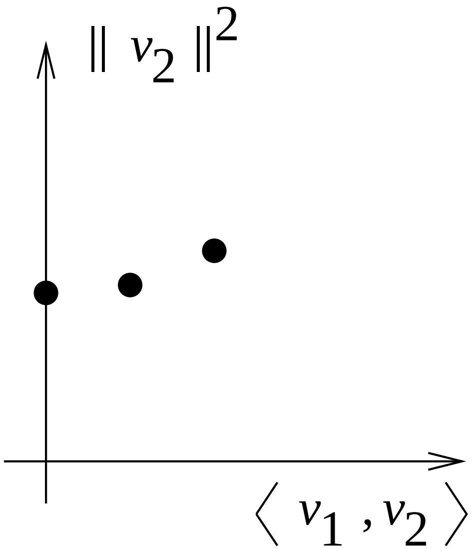


Lattices in dimension 2



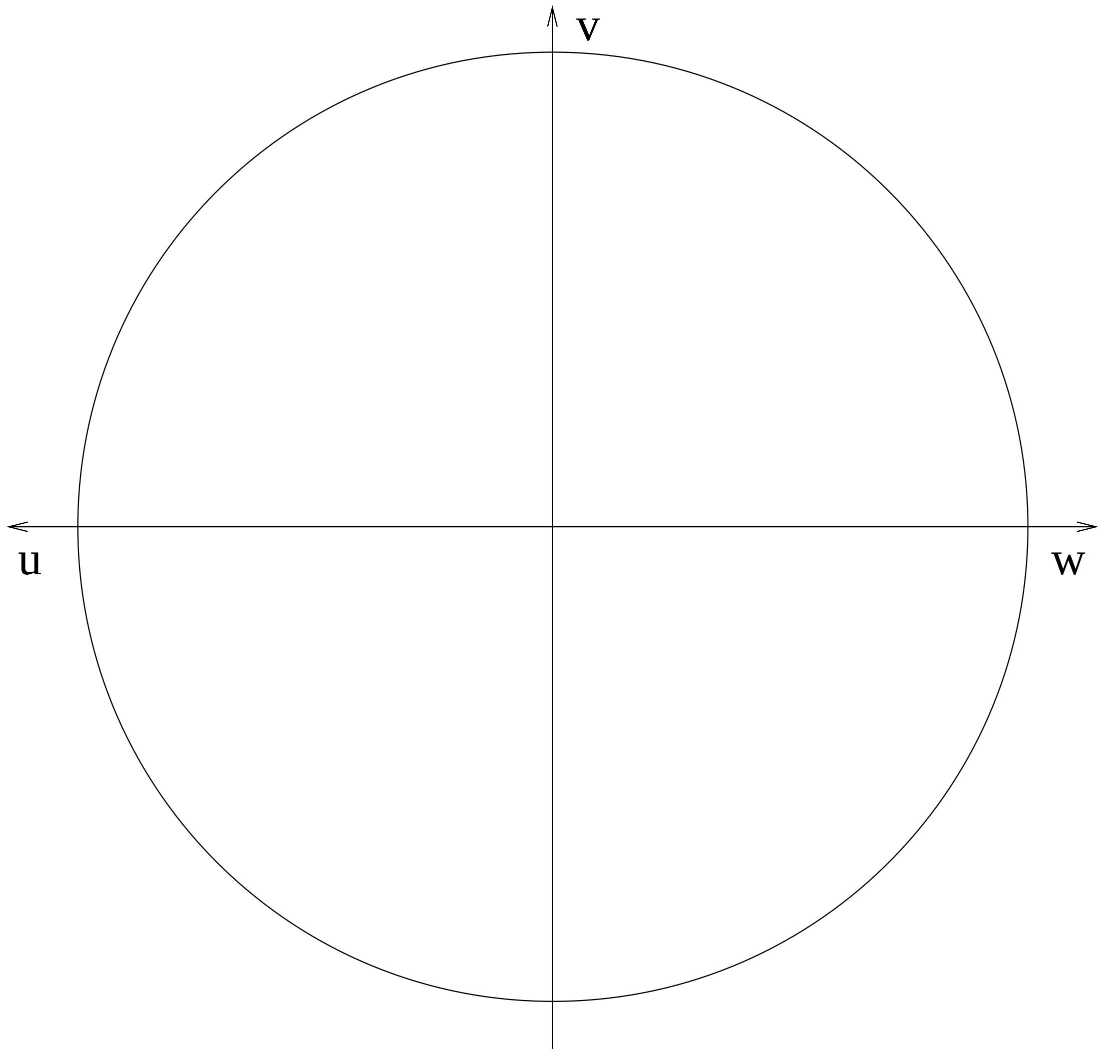
Lattices in dimension 2





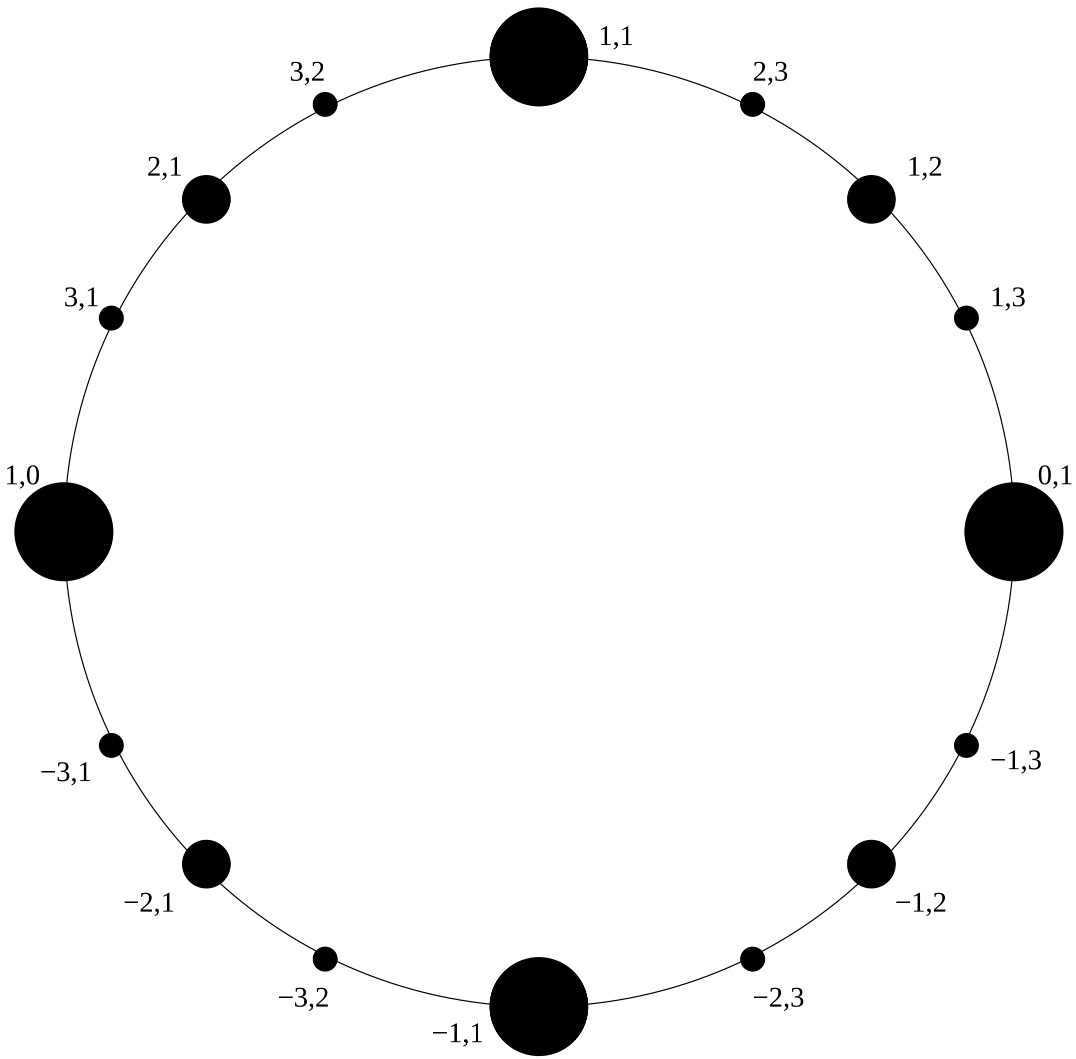
The partition of $PSD_2 \subset \mathbb{R}^3$

If $q(x,y) = \mathrm{u}x^2 + 2\mathrm{v}xy + \mathrm{w}y^2$ then $q \in PSD_2$ if and only if $\mathrm{v}^2 < \mathrm{uw}$ and $\mathrm{u} > 0$; we cut by the plane $\mathrm{u} + \mathrm{w} = 1$.



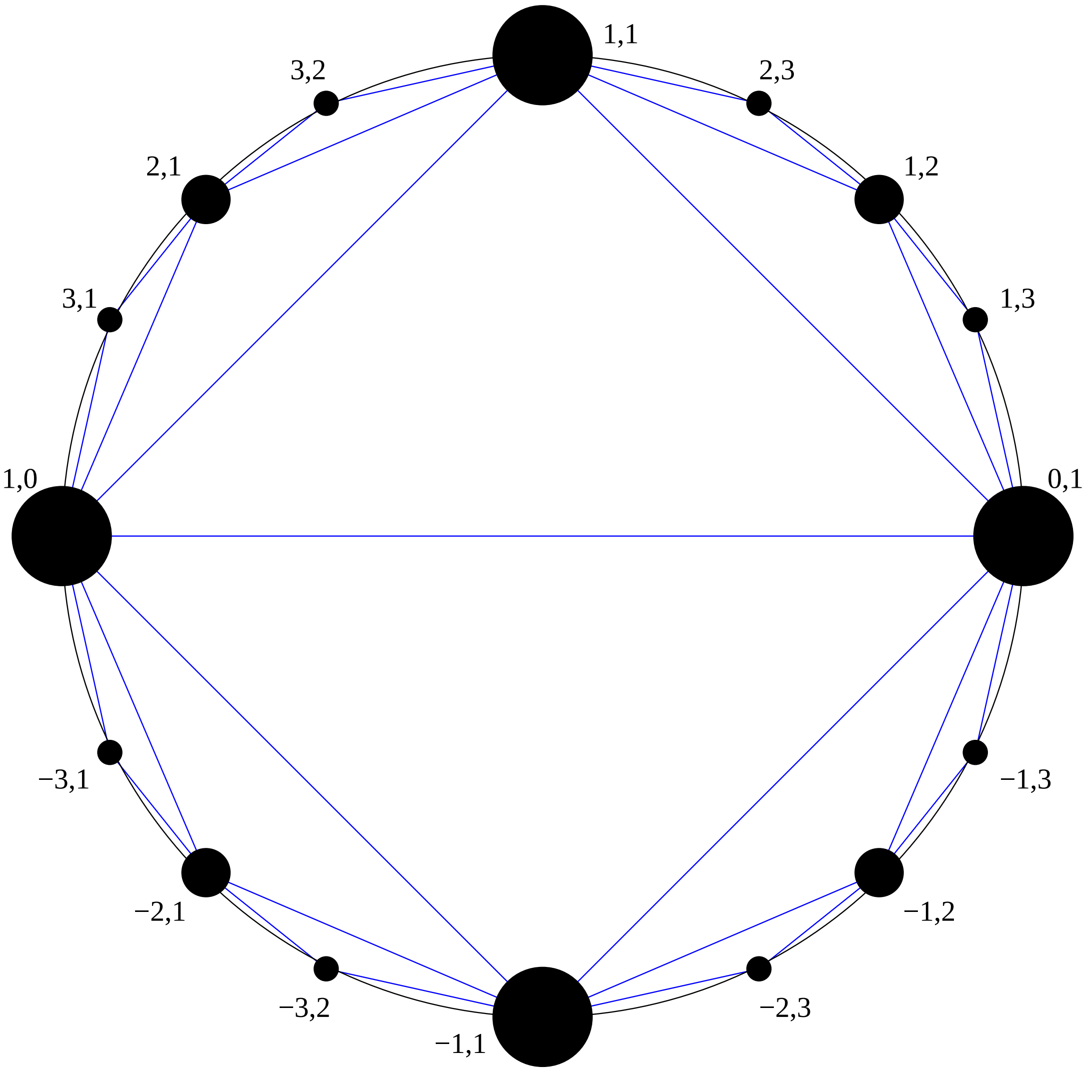
The partition of $PSD_2 \subset \mathbb{R}^3$

The group $GL_2(\mathbb{Z})$ transform the limit form x^2 into the forms $(ax+by)^2$ with $a, b \in \mathbb{Z}$.

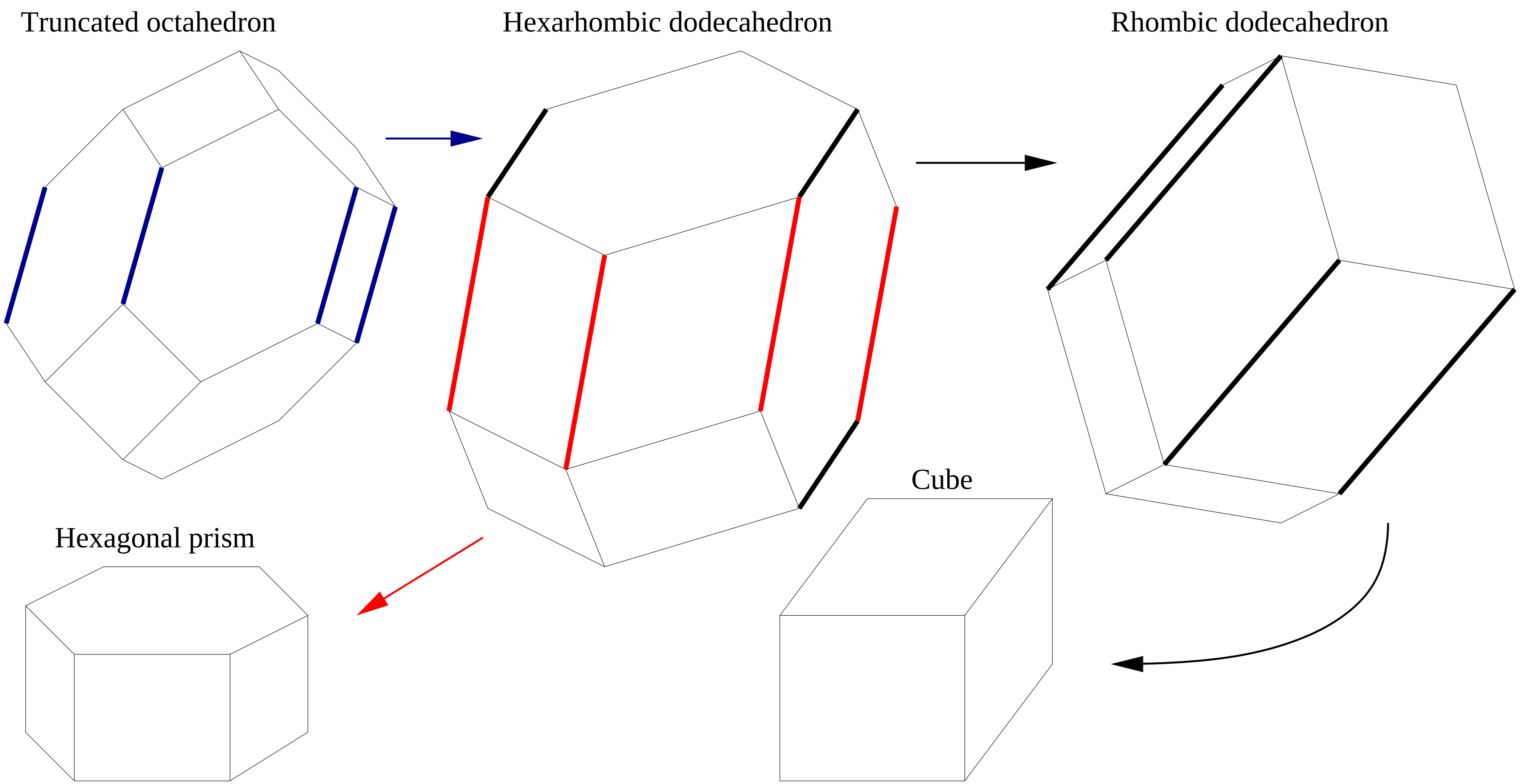


The partition of $PSD_2 \subset \mathbb{R}^3$

PSD_2 partition: Line: Voronoi polytope is rectangular.
Triangle: Voronoi polytope is hexagonal.



3-dimensional Voronoi polytopes



Rigid lattices

- One rigid in dimension 1: $\mathbb{Z}$.
- First rigid lattice appear in dimension 4: it is D_4.
- There are 7 rigid lattices in dimension 5.
- Some rigid lattices in dimension 6: we take two primitive non-simplex L-type domains with symmetry group of size 120 and 1920 (among possibly billions of primitive L-type domains) and computed the extreme rays of them.

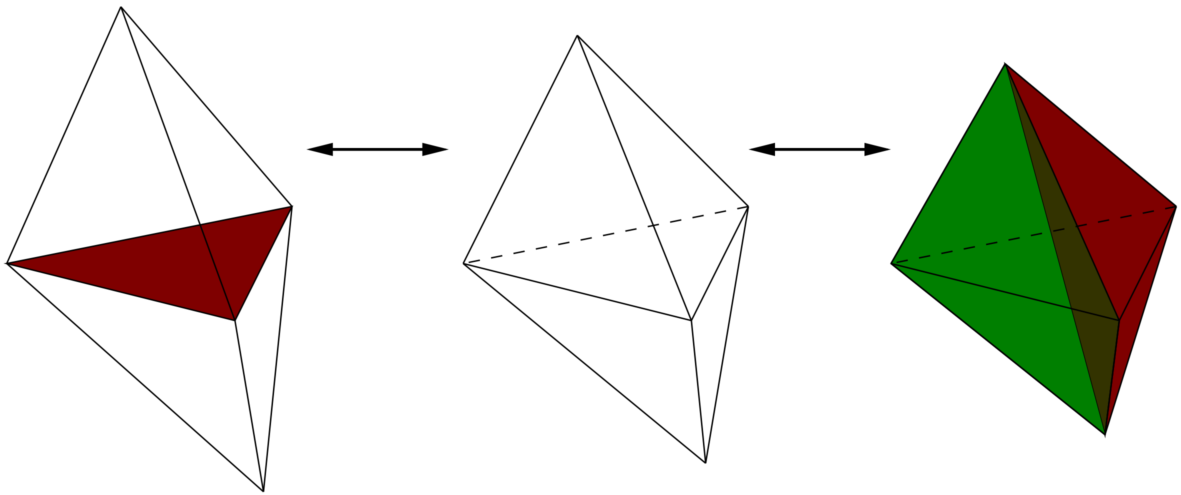
We found 25263 rigid forms in those two cones.

Remark: The infinite graph of primitive L-type domain is a tree if $n \leq 5$.

Flipping

Given a primitive L-type domain, how describe the structure of adjacent L-type domains?

- Its Delaunay tesselation consists of simplices.
- A facet of the L-type correspond to some Delaunay simplices merging into a Delaunay polytope with $d+2$ vertices, called repartitioning polytopes.
- Polytopes with $d+2$ vertices admit exactly two triangulations, the other one yield the adjacent L-type.



Equivariant L-type domains

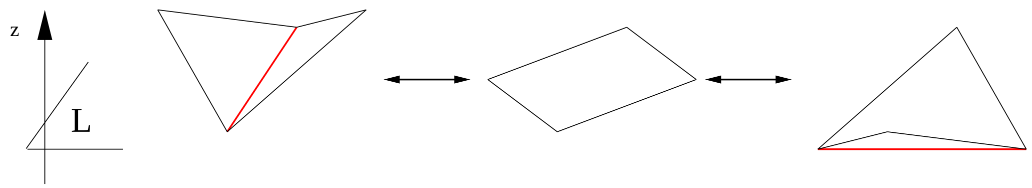
- The enumeration of L-types is done up to dimension 5, perhaps possible for dimension 6 but certainly not for higher dimension.
- We want to consider L-type domains with a given symmetry group, we will work in the space of invariant quadratic forms.
- Conjugacy classes of maximal irreducible matrix groups are known up to dimension 24 (Plesken & Nebe).
- Primitive equivariant L-type are of dimension s. Their Delaunay tesselation is not necessarily simplicial.

General flipping

- The Delaunay polytopes of a lattice L correspond to the facets of the convex cone $\mathcal{C}(L)$ with vertices:

$$\{(x, ||x||^2) \text{ with } x \in L\} \subset \mathbb{R}^{d+1} .$$

- Flipping can be interpreted on $\mathcal{C}(L)$:



- Combinatorially flipping correspond to switching from the lower facets to the higher facets of the lifted merging of Delaunay polytopes.

➠ One can do flipping of equivariant L-type domain.

II. Delaunay polytopes

and

hypermetrics

Hypermetric inequalities

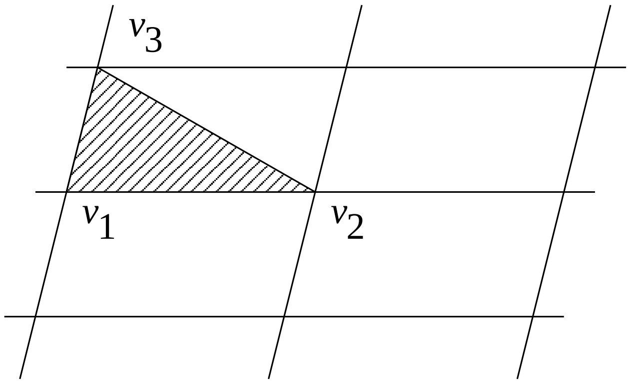
- If $b \in \mathbb{Z}^{n+1}$, $\sum_{i=0}^{n} b_i = 1$ then the hypermetric inequality is
$$H(b)d = \sum_{0 \le i < j \le n} b_i b_j d(i,j) \le 0 \; .$$
- If $b = (1, 1, -1, 0, \dots, 0)$ then $H(b)$=triangular inequality.
- The hypermetric cone HYP_{n+1} is the set of all d such that $H(b)d \le 0$ for all b.
- $dim\, HYP_{n+1} = \binom{n+1}{2}$
- HYP_{n+1} is defined by an infinite set of inequalities.

Delaunay polytopes

If $\mathcal{D}$ is an n dimensional Delaunay polytope with center c, radius r and vertices $\{v_0, \dots, v_N\}$ then $d(i,j) = \|v_i - v_j\|^2$ satisfies

$$\sum_{i,j} b_i b_j d(i,j) = 2(r^2 - \|\sum_i b_i v_i - c\|^2) \leq 0$$

i.e. Delaunay polytope $\Leftrightarrow$ hypermetrics.
Moreover $\sum_i b_i v_i$ is a vertex of $\mathcal{D}$ if and only if $H(b)d = 0$.



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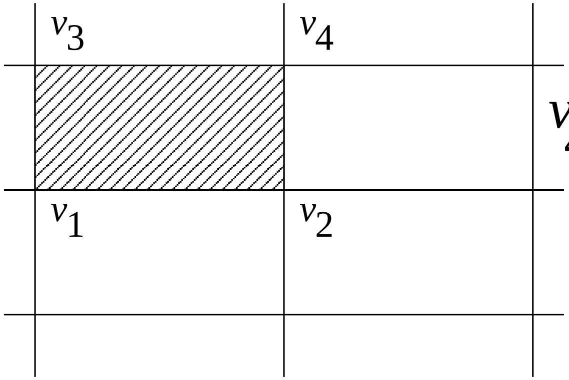
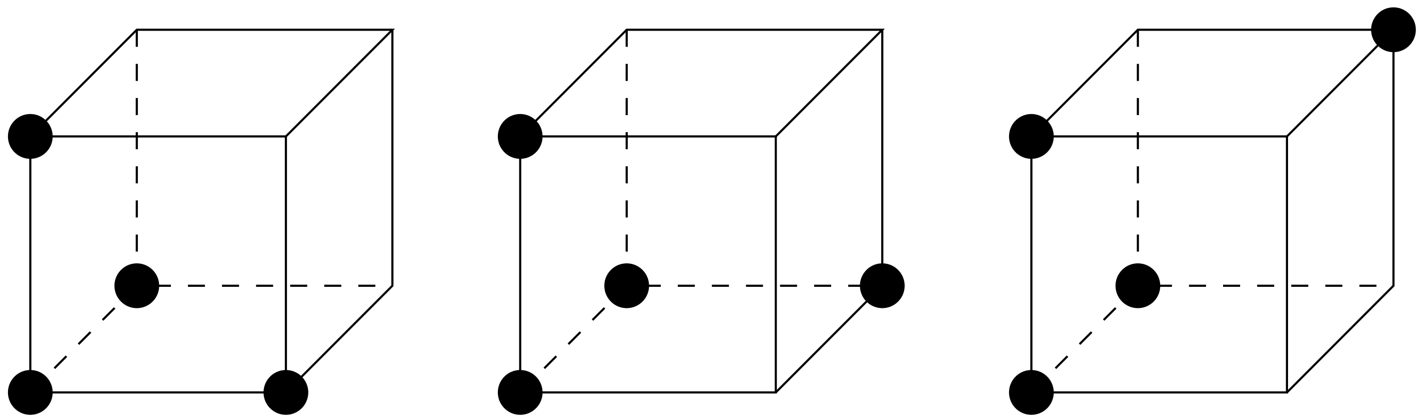


$v_4 = v_2 + v_3 - v_1$

Affine basis

An affine basis of an n-dimensional polytope P is $\{v_0, \dots, v_n\}$ such that for every vertex v of P, there is

$$b_i \in \mathbb{Z}, \text{such that } \ b_0 + \cdots + b_n = 1$$
$$\text{and } \ b_0 v_0 + b_1 v_1 + \cdots + b_n v_n = v \ .$$



Baranovski & Ryshkov: every Delaunay polytope of dimension ≤ 6 has an affine basis.
There exist Delaunay polytopes without affine basis.

Polyhedrality of HYP_n

- HYP_n is polyhedral as union of L-type domain.
- Lovasz if $H(b)$ defines a facet then $|b_i| \leq \frac{2^n}{\binom{2n}{n}} n!$.

Combinatorial types of n-dimensional Delaunay polytope P (with affine basis) correspond to faces F of HYP_{n+1}.

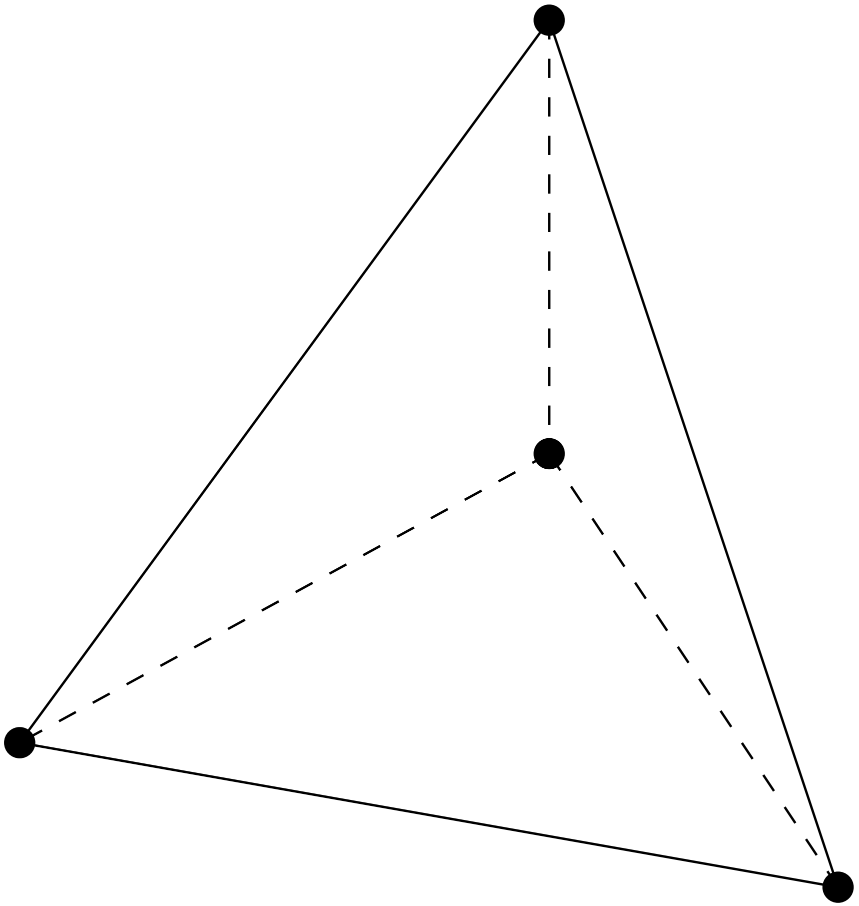
One defines $rank(P) = dim\, F$.

$rank(P)$ is the number of degrees of freedom.

- $rank\,(P) = \binom{n+1}{2}$, then P is a simplex.
- $rank\,(P) = 1$, then P is an extreme Delaunay polytope.

We are interested in extreme Delaunay polytopes (their only degree of freedom is homotheties and rotations).

3-dimensional case

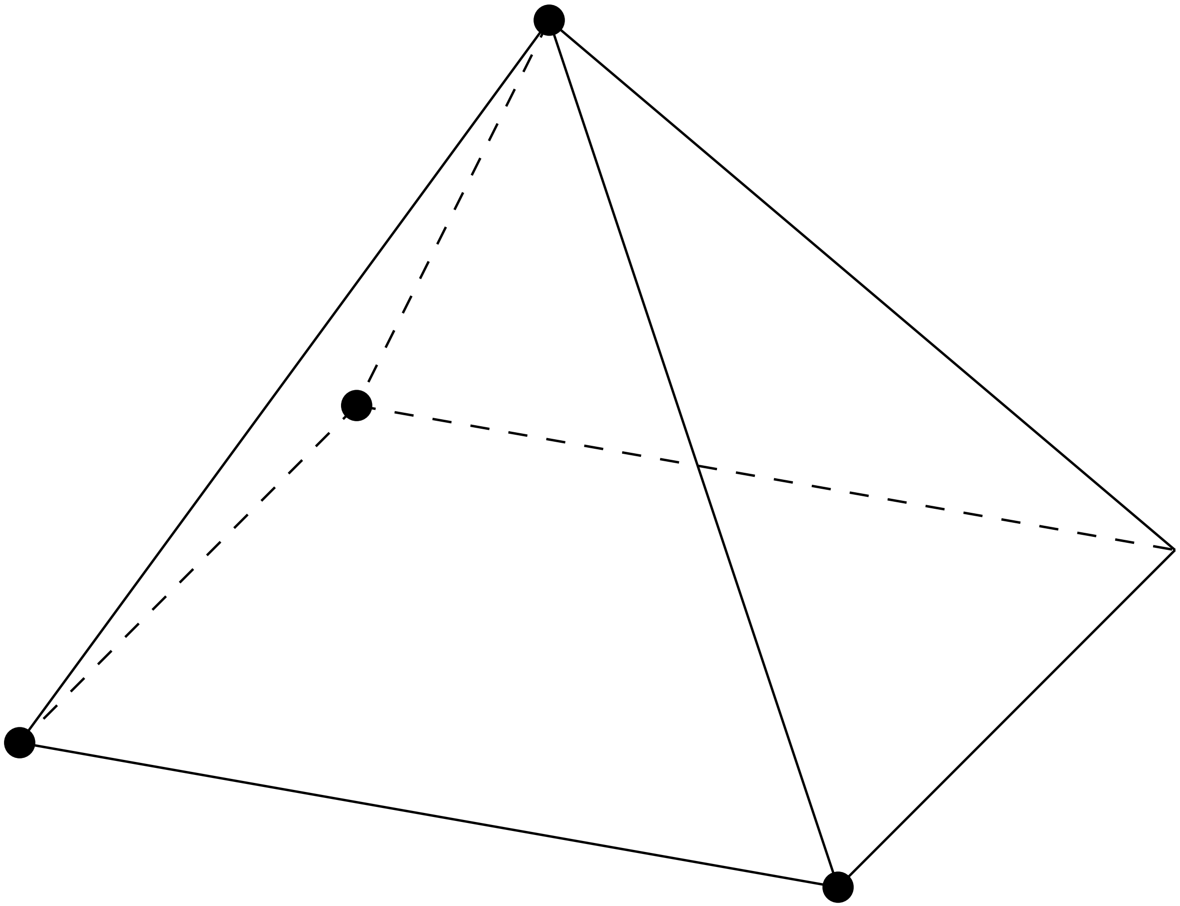


3−simplex

Hypermetric Vectors

Rank: 6

3-dimensional case



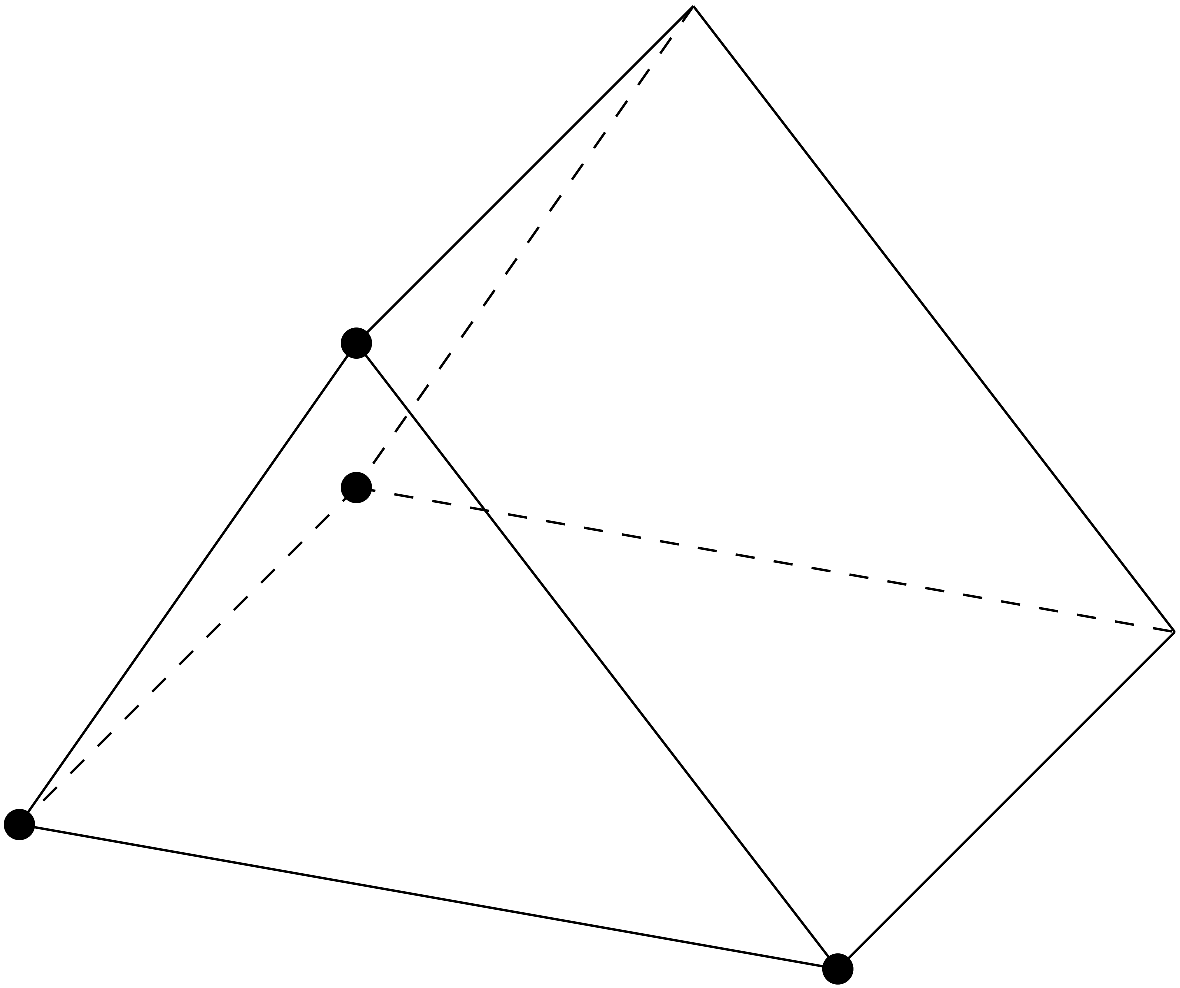
Pyramid

Hypermetric Vectors

$(-1\ ,\ 0\ ,\ 1\ ,\ 1)$

Rank: 5

3-dimensional case



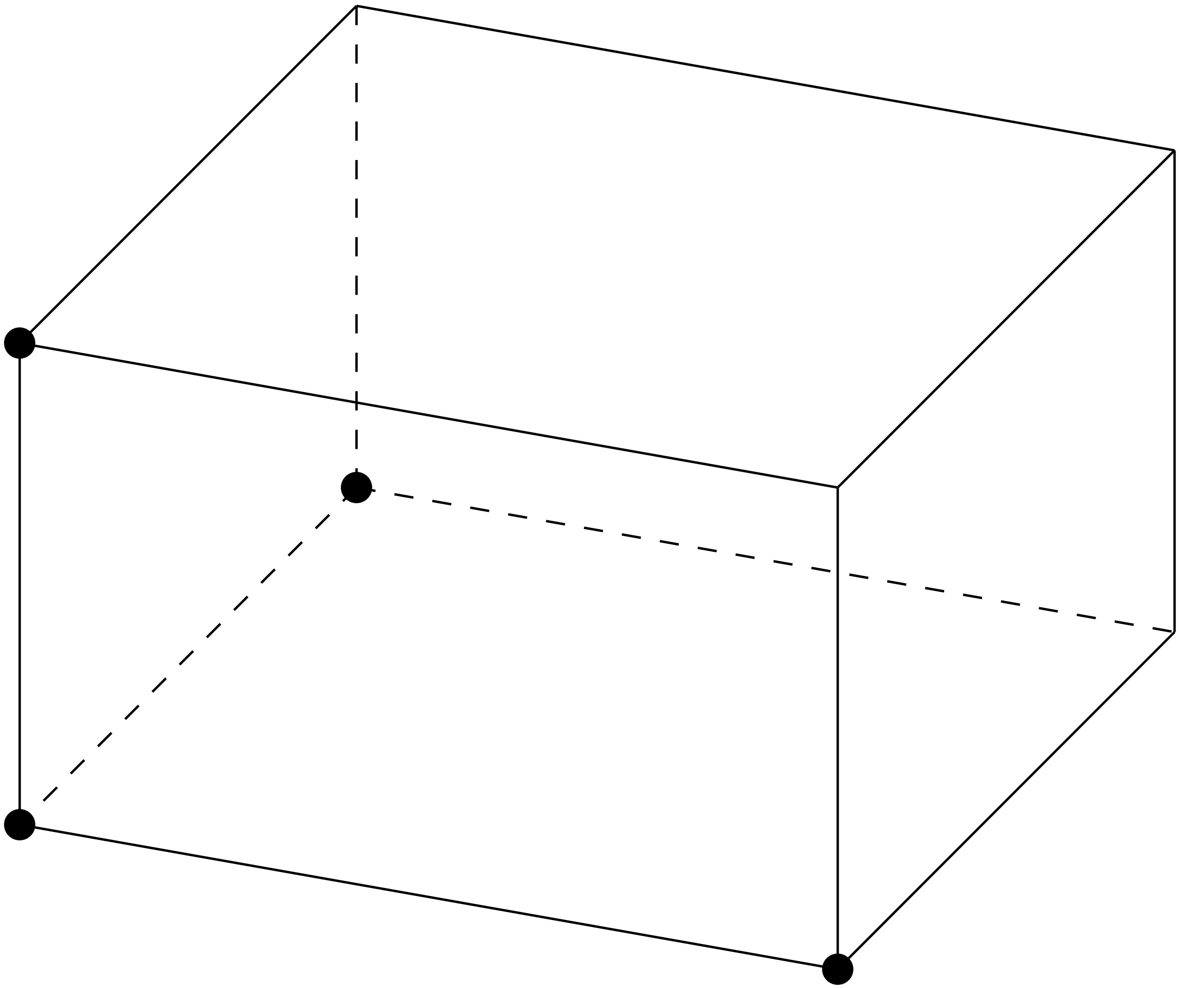
3−Prism

Hypermetric Vectors

(−1 , 0 , 1 , 1)
(−1 , 1 , 0 , 1)

Rank: 4

3-dimensional case



Cube

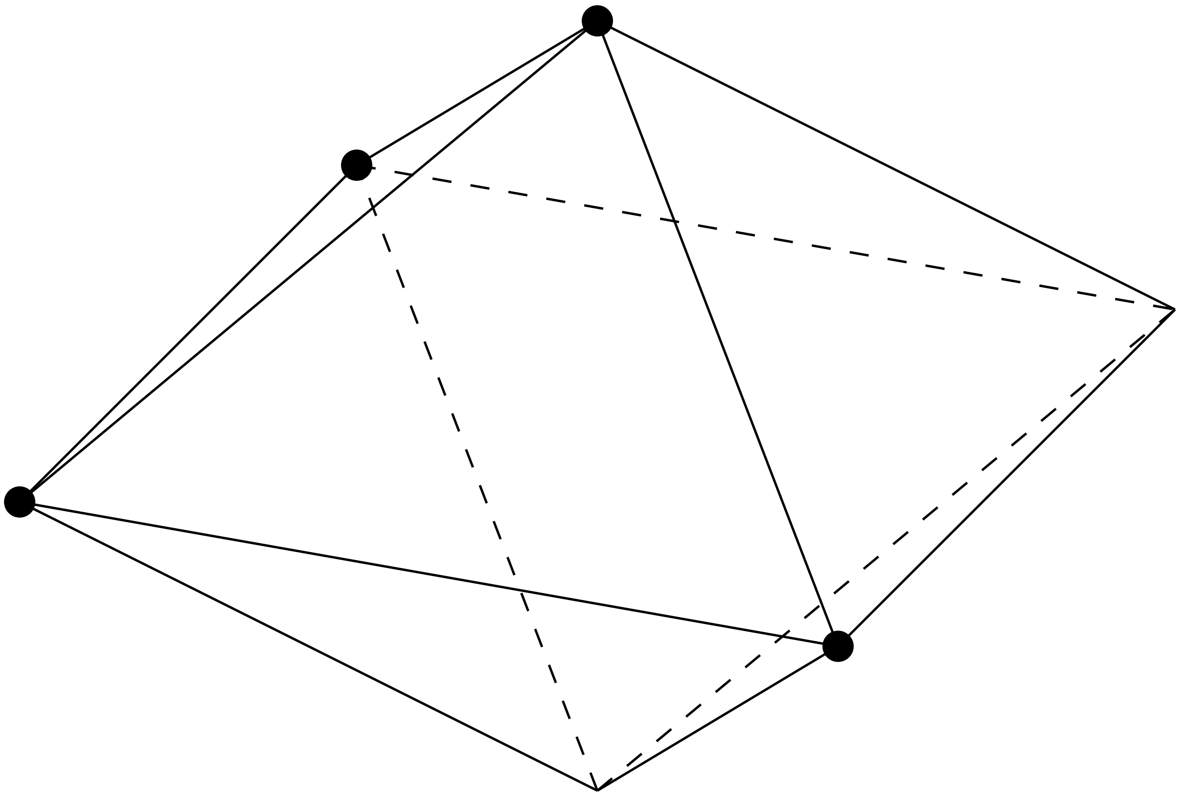
Hypermetric Vectors

$(-1\ ,\ 0\ ,\ 1\ ,\ 1)$
$(-1\ ,\ 1\ ,\ 0\ ,\ 1)$
$(-1\ ,\ 1\ ,\ 1\ ,\ 0)$
$(-2\ ,\ 1\ ,\ 1\ ,\ 1)$

Rank: 3

$$H(-2, 1, 1, 1) = H(-1, 0, 1, 1) + H(-1, 1, 0, 1) + H(-1, 1, 1, 0)$$

3-dimensional case



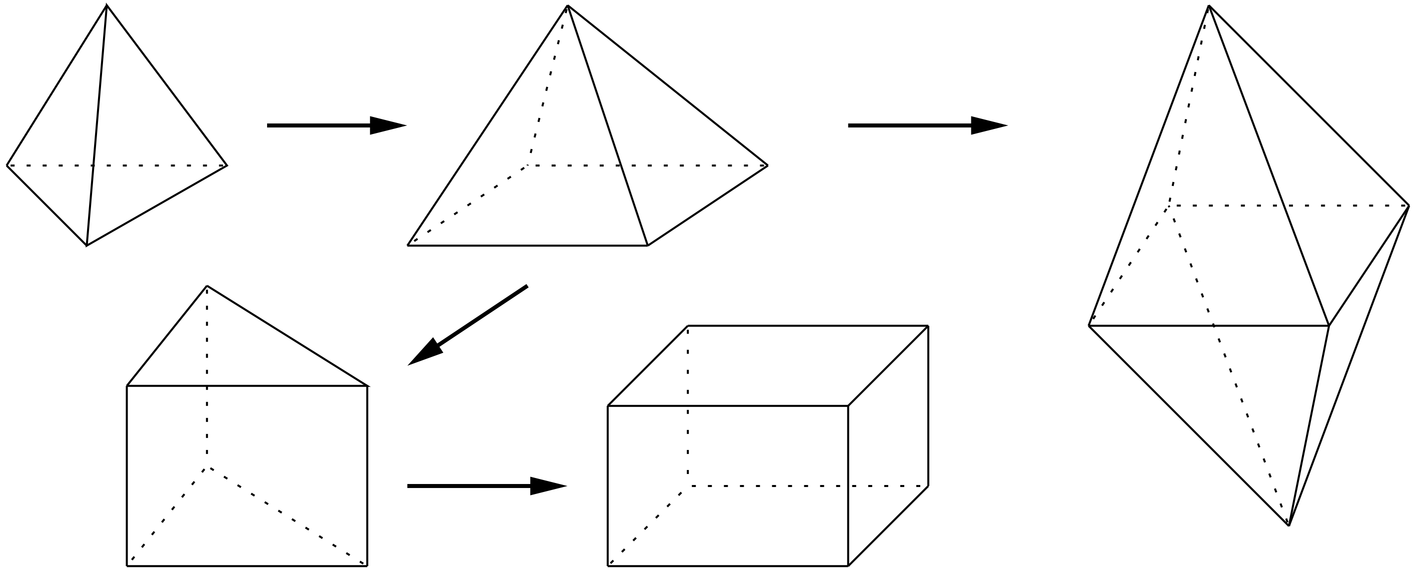
Octahedron

Hypermetric Vectors

$(-1\ ,\ 0\ ,\ 1\ ,\ 1)$
$(\ \ 0\ ,-1\ ,\ 1,\ 1)$

Rank: 4

Combinatorial types



dim	Nr of types	Authors	Computing time
2	2	Fedorov (1885)	
3	5	Fedorov (1885)	23s
4	19	Erdahl & Ryshkov (1987)	52s
5	138	Kononenko (1997)	5m
6	6241	Dutour (2002)	50h

Extremal problems

Take L an n-dimensionnal lattice.

- The number of vertices of a Delaunay polytope is at most 2^n and at least $n+1$.
- The volume of a Delaunay polytope is an integral multiple, say α of $\frac{det(L)}{n!}$.

What could be the volume of a Delaunay simplex?

- Voronoi: if $n=4$, then $\alpha=1$.
- Baranovski: if $n=5$, then $\alpha=1$ or 2.
- Ryshkov: if $n=6$, then $\alpha=1$, 2 or 3.
- Santos, Schürmann & Vallentin: there exist Delaunay simplices in dimension divisible by 24 with $\alpha\geq 1.5^n$.
- Lovasz $\alpha\leq\frac{2^n}{\binom{2n}{n}}n!$.

Lower bound

- Every incidence $H(b)d = 0$ correspond to a vertex $b_0 v_0 + \cdots + b_n v_n$ of a Delaunay polytope P.
- The number N of vertices satisfies

$$\begin{array}{rcl} N & \geq & n + 1 + corank(P) \\ & \geq & n + 1 + \binom{n+1}{2} - rank(P) \end{array}$$

- Extreme Delaunay polytopes have at least $\binom{n+2}{2} - 1$ vertices.
- If they have exactly $\binom{n+2}{2} - 1$ vertices, then the corresponding extreme ray of HYP_{n+1} are simple for which adjacency computation is easy.

Extreme Delaunay polytopes

- The interval $[0, 1]$ is the only extreme Delaunay polytope in dimension lower than 6, since $HYP_n = CUT_n$ if $n \leq 6$.
- Deza & Dutour: There is an unique extreme Delaunay polytope in dimension 6, the Schläfli polytope.
- Deza, Grishukhin & Laurent: some extreme polytopes

Name	dimension	Nr. vertices	Equality	section of
Schläfli	6	27	yes	E_8
Gosset	7	56	no	E_8
	16	512	no	BarnesWall
B_{15}	15	135	yes	BarnesWall
	22	275	yes	Leech
	23	552	no	Leech

Computing methods

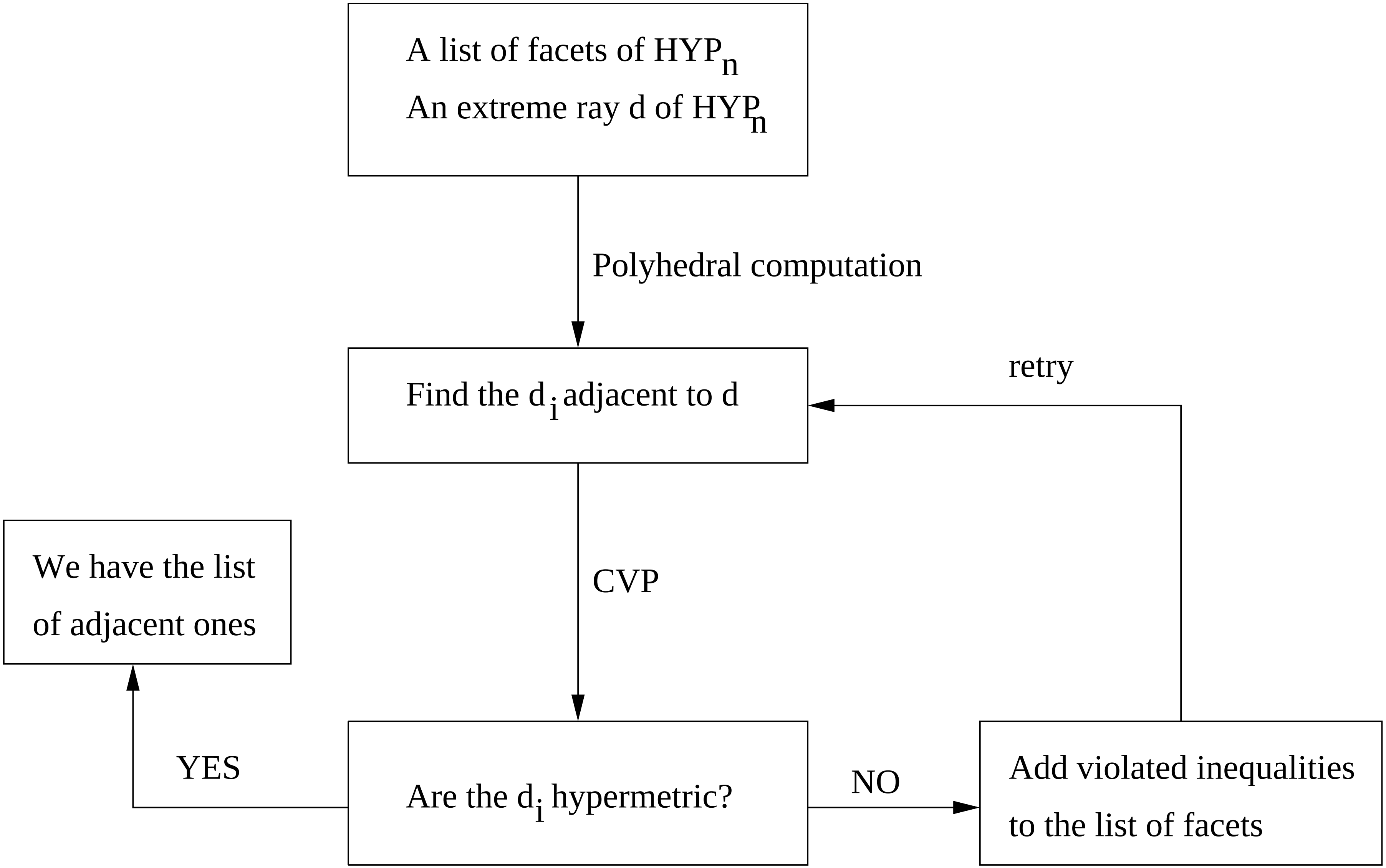
Given $d_{ij} = \|v_i - v_j\|^2$ a distance vector,

- one can compute the Gram matrix $\langle (v_i - v_0), (v_j - v_0) \rangle$,
- test if d is non-degenerate,
- compute the sphere $S(c, R)$ around the v_i.
- $d \in HYP_{n+1}$ if and only if there is no b such that

$$\|b_0 v_0 + \cdots + b_n v_n - c\| < R \quad .$$

  (i.e. Closest Vector Problem)
- Find the b such that $H(b)d = 0$ is also a CVP.

Bounding method



8-dimensional extreme Delaunay

B_{15} satisfies the equality bound.
We computed its adjacent extreme rays: they correspond to a 8-dimensional extreme Delaunay polytope with f-vector:

$$(79, 1268, 7896, 23520, 36456, 29876, 11364, 1131)$$

It has a symmetry group of size 322560 not transitive on vertices.
There are three orbits of vertices:

- a vertex
- 64-vertices: the 7-half-cube
- 14-vertices: the 7-cross-polytope

Infinite sequence of extreme Delaunay

- If n even, $n \geq 6$, there is a n-dimensional extreme Delaunay ED_n formed with 3 layers of D_{n-1} lattice:
  - a vertex
  - the $n-1$ half-cube
  - the $n-1$ cross-polytope

  $n = 6$: Schläfli polytope; $n = 8$: the 8-dimensional one
- If n odd, $n \geq 7$, there is a n-dimensional extreme Delaunay ED_n formed with 4 layers of previous lattice:
  - a vertex
  - the ED_{n-1} extreme Delaunay
  - the ED_{n-1} extreme Delaunay
  - a vertex

  $n = 7$: Gosset polytope

Remarks on infinite series

- Conjecture: the polytopes ED_n have the highest number of vertices among all extreme Delaunay polytopes of dimension n.
- Conjecture: the polytopes ED_n for n odd have two orbits of vertices.
- Erdahl & Rybnikov created another infinite serie of extreme Delaunay polytopes by lamination of A_n lattices (it starts with Schläfli polytope). Their number of vertices is
$$\binom{n+2}{2} - 1,$$
i.e. the lowest possible number.

Other extreme Delaunay

Idea is to apply the bounding method to the extreme Delaunay ED_8, obtain new extreme Delaunay, reapply the method, test by isomorphy, etc.

- Conjectured list in dimension 7:

$\#vertices$	$\#facets$	$\|Sym\|$
35	228	1440 (Erdahl & Rybnikov)
56	702	2903040 (Gosset)

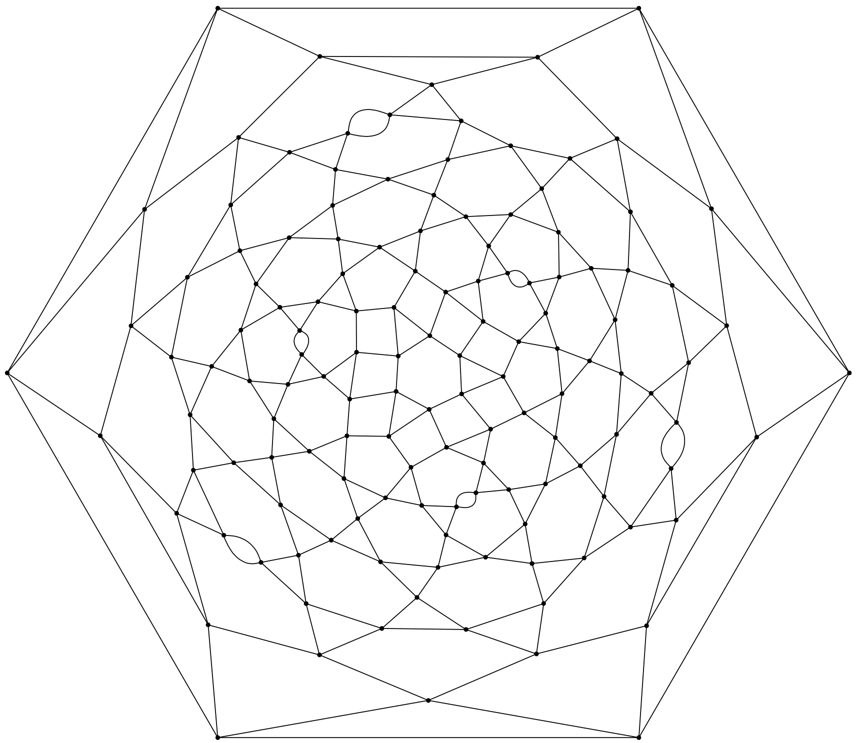
- 27 extreme Delaunay in dimension 8.
- ≥ 1500 in dimension 9. The initial extreme Delaunay was the one of Erdahl & Rybnikov.

Result in dimension 8

$\#vert.$	$\#facets$	$\|Sym\|$
79	1131	$322560 ED_8$
72	1798	80640
72	354	80640
58	664	1440
55	355	288
54	375	864
54	539	384
52	634	192
49	546	288
49	535	960
47	534	48
47	474	24
47	395	48

46	523	36
46	476	288
45	571	192
45	559	48
45	582	144
45	414	1296
44	559	48
44	559	240
44	504	2880
44	599	144
44	529	10080
44	538	72
44	480	288
44	559	72

The End



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