Perfect forms and perfect Delaunay polytopes

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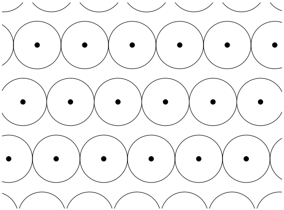
I. Lattices, packings and coverings

Lattice packings

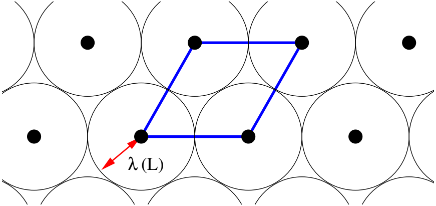
- A lattice $L \subset \mathbb{R}^n$ is a set of the form $L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$.
- A packing is a family of balls $B_n(x_i, r)$, $i \in I$ of the same radius r and center x_i such that their interiors are disjoint.



- If L is a lattice, the lattice packing is the packing defined by taking the maximal value of $\alpha > 0$ such that $L + B_n(0, \alpha)$ is a packing.

Density of lattice packings

- Take the lattice packing defined by a lattice L:



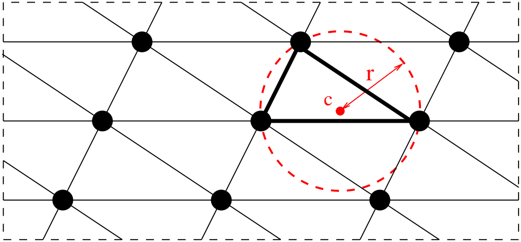
- The packing density has the expression

$$\delta(L) = \frac{\lambda(L)^n \,\mathrm{vol}(B_n(0,1))}{\det\, L} \quad \text{with} \quad \lambda(L) = \frac{1}{2} \min_{v \in L - \{0\}} ||v||,$$

$\mathrm{vol}(B_n(0,1))$ the volume of the unit ball $B_n(0,1)$ and $\det\, L$ the volume of an unit cell.

Empty sphere and Delaunay polytopes

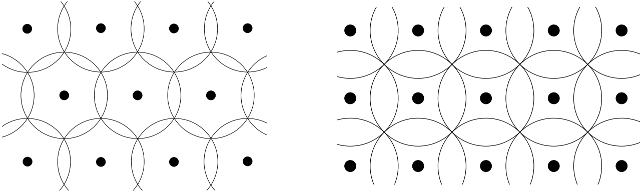
- Definition: A sphere $S(c, r)$ of center c and radius r in an n-dimensional lattice L is said to be an empty sphere if:
  (i) $\|v - c\| \geq r$ for all $v \in L$,
  (ii) the set $S(c, r) \cap L$ contains $n + 1$ affinely independent points.
- Definition: A Delaunay polytope P in a lattice L is a polytope, whose vertex-set is $L \cap S(c, r)$.



- Delaunay polytopes define a tesselation of the Euclidean space $\mathbb{R}^n$

Lattice covering

- For a lattice L we define the covering radius $\mu(L)$ to be the smallest r such that the family of balls $v + B_n(0, r)$ for $v \in L$ cover $\mathbb{R}^n$.



- The covering density has the expression

$$\Theta(L) = \frac{\mu(L)^n \operatorname{vol}(B_n(0,1))}{\det(L)} \geq 1$$

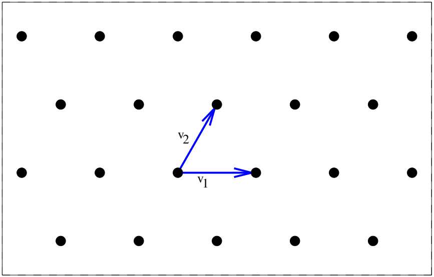
with $\mu(L)$ being the largest radius of Delaunay polytopes

- The only general method for computing $\Theta(L)$ is to compute all Delaunay polytopes of L.

II. Gram matrix formalism

Gram matrix and lattices

- Denote by S^n the vector space of real symmetric $n \times n$ matrices and $S^n_{>0}$ the convex cone of real symmetric positive definite $n \times n$ matrices.
- Take a basis $(v_1, \ldots, v_n)$ of a lattice L and associate to it the Gram matrix $G_{\mathbf{v}} = (\langle v_i, v_j \rangle)_{1 \le i,j \le n} \in S^n_{>0}$.
- Example: take the hexagonal lattice generated by $v_1 = (1, 0)$ and $v_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$



$$G_{\mathbf{v}} = \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Isometric lattices

- Take a basis $(v_1, \ldots, v_n)$ of a lattice L with $v_i = (v_{i,1}, \ldots, v_{i,n}) \in \mathbb{R}^n$ and write the matrix

$$V = \begin{pmatrix} v_{1,1} & \ldots & v_{n,1} \\ \vdots & \ddots & \vdots \\ v_{1,n} & \ldots & v_{n,n} \end{pmatrix}$$

  and $G_{\mathbf{v}} = V^T\ V$.
  The matrix $G_{\mathbf{v}}$ is defined by $\frac{n(n+1)}{2}$ variables as opposed to n^2 for the basis V.
- If $M \in S^n_{>0}$, then there exists V such that $M = V^T\ V$ (Gram Schmidt orthonormalization)
- If $M = V_1^T\ V_1 = V_2^T\ V_2$, then $V_1 = OV_2$ with $O^T\ O = I_n$ (i.e. O corresponds to an isometry of $\mathbb{R}^n$).
- Also if L is a lattice of $\mathbb{R}^n$ with basis $\mathbf{v}$ and u an isometry of $\mathbb{R}^n$, then $G_{\mathbf{v}} = G_{u(\mathbf{v})}$.

Arithmetic minimum

- The arithmetic minimum of $A \in S^n_{>0}$ is

$$\min(A) = \min_{x \in \mathbb{Z}^n - \{0\}} x^T A x$$

- The minimal vector set of $A \in S^n_{>0}$ is

$$\mathrm{Min}(A) = \left\{ x \in \mathbb{Z}^n \mid x^T A x = \min(A) \right\}$$

- Both $\min(A)$ and $\mathrm{Min}(A)$ can be computed using some programs (for example sv by Vallentin)
- The matrix $A_{hex} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ has

$$\mathrm{Min}(A_{hex}) = \{\pm(1,0), \pm(0,1), \pm(1,-1)\}.$$

Reexpression of previous definitions

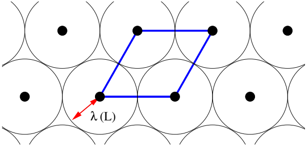
- Take a lattice $L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$. If $x \in L$,

$$x = x_1 v_1 + \cdots + x_n v_n \quad \text{with} \quad x_i \in \mathbb{Z}$$

we associate to it the column vector $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

- We get $||x||^2 = X^T G_{\mathbf{v}} X$ and

$$\det L = \sqrt{\det G_{\mathbf{v}}} \quad \text{and} \quad \lambda(L) = \frac{1}{2}\sqrt{\min(G_{\mathbf{v}})}$$



- For $A_{hex} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $\det A_{hex} = 3$ and $\min(A_{hex}) = 2$

Changing basis

- If $\mathbf{v}$ and $\mathbf{v}'$ are two basis of a lattice L then $V' = VP$ with $P \in GL_n(\mathbb{Z})$. This implies

$$G_{\mathbf{v}'} = V'^T V' = (VP)^T VP = P^T \{V^T V\} P = P^T G_{\mathbf{v}} P$$

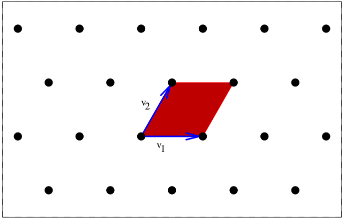
- If $A, B \in S^n_{>0}$, they are called arithmetically equivalent if there is at least one $P \in GL_n(\mathbb{Z})$ such that

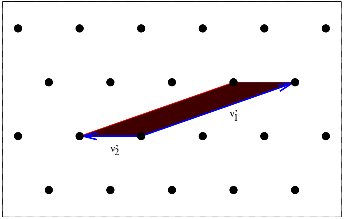
$$A = P^T BP$$

- Lattices up to isometric equivalence correspond to $S^n_{>0}$ up to arithmetic equivalence.
- In practice, Plesken/Souvignier wrote a program isom for testing arithmetic equivalence and a program autom for computing automorphism group of lattices.
  All such programs take Gram matrices as input.

An example of equivalence

- Take the hexagonal lattice and two basis in it.





$v_1 = (1,0)$ and $v_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ $\quad$ $v_1' = \left(\frac{5}{2}, \frac{\sqrt{3}}{2}\right)$ and $v_2' = (-1,0)$

- One has $v_1' = 2v_1 + v_2,\ v_2' = -v_1$ and $P = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$

$$G_{\mathbf{v}} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \text{ and } G_{\mathbf{v}'} = \begin{pmatrix} 7 & -\frac{5}{2} \\ -\frac{5}{2} & 1 \end{pmatrix} = P^T G_{\mathbf{v}} P$$

Root lattices

- Let us take the lattice
$$\mathrm{A}_n = \left\{ x \in \mathbb{Z}^{n+1} \text{ s.t. } \sum_{i=1}^{n+1} x_i = 0 \right\}$$
If we take the basis $v_i = e_{i+1} - e_i$ then we get the Gram matrix $A = (a_{ij})_{1 \le i,j \le n}$ with $a_{i,i} = 2$, $a_{i,i+1} = a_{i+1,i} = -1$ and $a_{i,j} = 0$ otherwise.
- Let us take the lattice
$$\mathrm{D}_n = \left\{ x \in \mathbb{Z}^{n} \text{ s.t. } \sum_{i=1}^{n} x_i \equiv 0 \pmod 2 \right\}$$
For the basis $v_1 = e_1 + e_2$, $v_2 = e_1 - e_2$, $v_i = e_i - e_{i-1}$ we get
$$G_v = \begin{pmatrix} 2 & 0 & -1 & 0 & \dots & 0 \\ 0 & 2 & -1 & 0 & \dots & 0 \\ -1 & -1 & 2 & -1 & \dots & 0 \\ 0 & 0 & -1 & 2 & \ddots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & -1 \\ 0 & \dots & \dots & \dots & -1 & 2 \end{pmatrix}$$

III. Perfect and eutactic forms

Hermite function

- If $A \in S^n_{>0}$ then the arithmetic minimum is

$$\min(A) = \min_{x \in \mathbb{Z}^n - \{0\}} x^T A x$$

  and the set of minimal vectors is

$$\mathrm{Min}(A) = \left\{ x \in \mathbb{Z}^n \ : \ x^T A x = \min(A) \right\}$$

- The Hermite function on the space $S^n_{>0}$ is

$$\gamma(A) = \frac{\min(A)}{(\det A)^{1/n}}$$

- The density of the lattice packing L associated to A is

$$\delta(L) = \sqrt{\gamma(A)^n} \frac{\mathrm{vol}(B_n(0,1))}{2^n}$$

- Finding lattice packings with highest packing density is the same as maximizing the Hermite function.

Perfect forms

- A form A is extreme if there is a neighborhood V of A in $S^n_{>0}$ such that

$$\text{If } B \in V \text{ with } B \neq \lambda A \text{ then } \gamma(B) < \gamma(A)$$

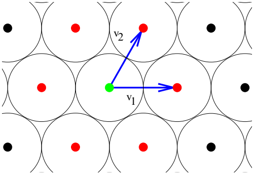
- A matrix $A \in S^n_{>0}$ is perfect (Korkine & Zolotarev, 1873) if the equation

$$B \in S^n \quad \text{and} \quad x^T B x = \min(A) \text{ for all } x \in \mathrm{Min}(A)$$

  implies $B = A$.
- Theorem: (Korkine & Zolotarev, 1873) If a form is extreme then it is perfect.
- Perfect forms are rational forms.
- If A is perfect then $\gamma(A)^n$ is rational.

A perfect form

- $A_{hex} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ corresponds to the lattice:



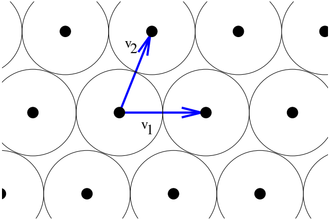
- If $B = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ satisfies to $x^T B x = \min(A_{hex})$ for $x \in \mathrm{Min}(A_{hex}) = \{\pm(1,0), \pm(0,1), \pm(1,-1)\}$, then:

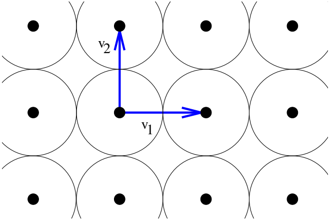
$$a = 2, \quad b = 2 \ \text{ and } \ a - 2c + b = 2$$

which implies $B = A_{hex}$. A_{hex} is perfect.

A non-perfect form

- $A_{sqr} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has $\mathrm{Min}(A_{sqr}) = \{\pm(0,1), \pm(1,0)\}$.
- See below lattices L_B, L_{sqr} associated to matrices $B, A_{sqr} \in S^2_{>0}$ with $\mathrm{Min}(B) = \mathrm{Min}(A_{sqr})$:





Eutactic forms

- A form $A \in S^n_{>0}$ is eutactic (Voronoi, 1908) if there exist $\lambda_v > 0$ such that

$$A^{-1} = \sum_{v \in \mathrm{Min}(A)} \lambda_v v v^T$$

- Theorem: (Voronoi, 1908) A form A is extreme if and only if it is perfect and eutactic.
- Theorem: (Ash, 1977)
  (i) If A is not an eutactic form then it is topologically ordinary point for γ
  (ii) If A is an eutactic form then it is a critical but topologically non-degenerate point for γ.
  (ii) γ is a topological Morse function.

Examples of perfect forms

- The root lattice are all perfect:

Name	Min	$\|Min\|$	det	$\|\text{Aut}\|$
A_n	$e_i - e_j$	$2n(n+1)$	$n+1$	$2(n+1)!$
D_n	$\pm e_i \pm e_j$	$4n(n-1)$	4	$2^n n!$
E_6	complex	72	3	103680
E_7	complex	126	2	2903040
E_8	complex	240	1	696729600

- Another remarkable lattice is the Leech lattice of dimension 24.
  - Every vector v has $\|v\|^2 \geq 4$ and $\det Leech = 1$.
  - There are 196280 shortest vectors (maximal number in dimension 24)
  - Its automorphism group quotiented by $\pm Id_{24}$ is the sporadic simple group Co_0
  - It plays a significant role in modular form theory and Lorentzian lattice theory.

Known results on lattice packing density maximization

dim.	Nr. of perfect forms	Absolute maximum of γ realized by
2	1 (Lagrange)	A_2
3	1 (Gauss)	A_3
4	2 (Korkine & Zolotarev)	D_4
5	3 (Korkine & Zolotarev)	D_5
6	7 (Barnes)	E_6 (Blichfeldt)
7	33 (Jaquet)	E_7 (Blichfeldt)
8	10916 (DSV)	E_8 (Blichfeldt)
9	$\geq$500000	Λ_9?
24	?	Leech (Cohn & Kumar)

Remarks

- The enumeration of perfect forms is done with the Voronoi algorithm.
- The solution in dimension 24 was obtained by different methods.

IV. Ryshkov polyhedron and the Voronoi algorithm

The Ryshkov polyhedron

- The Ryshkov polyhedron R_n is defined as

$$R_n = \left\{ A \in S^n \text{ s.t. } x^T A x \geq 1 \text{ for all } x \in \mathbb{Z}^n - \{0\} \right\}$$

- The cone is invariant under the action of $\mathrm{GL}_n(\mathbb{Z})$.
- The cone is locally polyhedral, i.e. for a given $A \in R_n$

$$\left\{ x \in \mathbb{Z}^n \text{ s.t. } x^T A x = 1 \right\}$$

  is finite
- Vertices of R_n correspond to perfect forms.
- For a form $A \in R_n$ we define the local cone

$$Loc(A) = \left\{ Q \in S^n \text{ s.t. } x^T Q x \geq 0 \text{ if } x^T A x = 1 \right\}$$

The Voronoi algorithm

- Find a perfect form (say A_n), insert it to the list $\mathcal{L}$ as undone.
- Iterate
  - For every undone perfect form A in $\mathcal{L}$, compute the local cone $Loc(A)$ and then its extreme rays.
  - For every extreme ray r of $Loc(A)$ realize the flipping, i.e. compute the adjacent perfect form $A' = A + \alpha r$.
  - If A' is not equivalent to a form in $\mathcal{L}$, then we insert it into $\mathcal{L}$ as undone.
- Finish when all perfect domains have been treated.

The subalgorithms are:

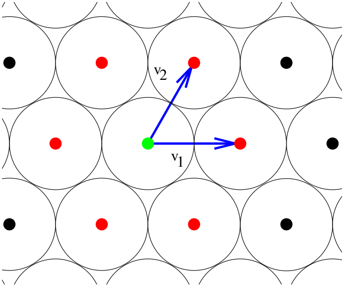
- Find the extreme rays of the local cone $Loc(A)$ (use cdd or lrs or any other program)
- For any extreme ray r of $Loc(A)$ find the adjacent perfect form A' in the Ryshkov polyhedron R_n
- Test equivalence of perfect forms using autom

Flipping on an edge I

$$\mathrm{Min}(A_{hex}) = \{\pm(1,0), \pm(0,1), \pm(1,-1)\}$$

with

$$A_{hex} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

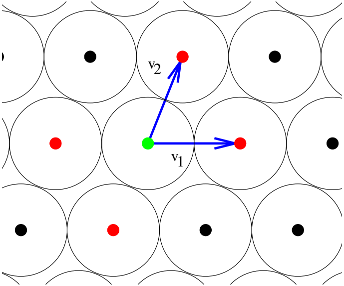


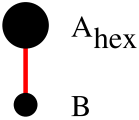
Flipping on an edge II

$$\mathrm{Min}(B) = \{\pm(1,0), \pm(0,1)\}$$

with

$$B = \begin{pmatrix} 1 & 1/4 \\ 1/4 & 1 \end{pmatrix} = A_{hex} + D/4$$



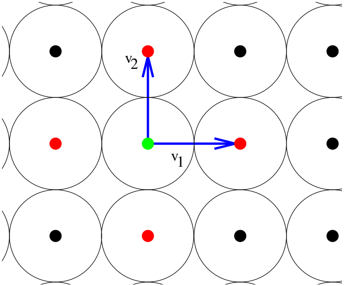


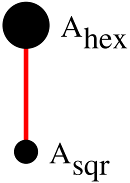
Flipping on an edge III

$$\mathrm{Min}(A_{sqr}) = \{\pm(1,0), \pm(0,1)\}$$

with

$$A_{sqr} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A_{hex} + D/2$$



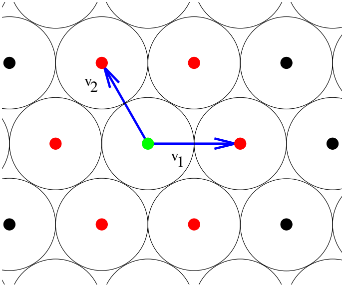


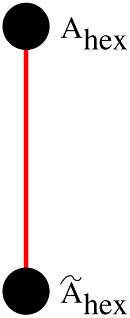
Flipping on an edge IV

$$\mathrm{Min}(\tilde{A}_{hex}) = \{\pm(1,0), \pm(0,1), \pm(1,1)\}$$

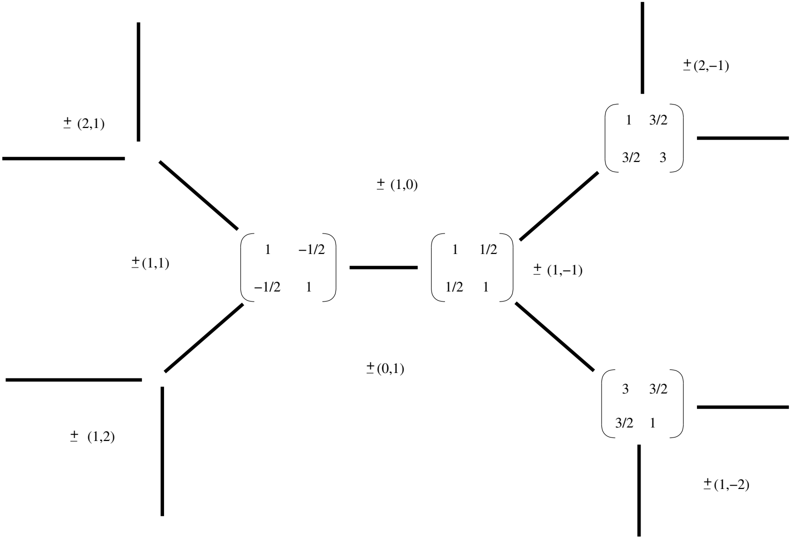
with

$$\tilde{A}_{hex} = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix} = A_{hex} + D$$





The Ryshkov polyhedron R_2



Well rounded forms and retract

- A form Q is said to be well rounded if it admits vectors v_1, ..., v_n such that
  - $(v_1, \ldots, v_n)$ form a basis of $\mathbb{R}^n$
  - $v_1, \ldots, v_n$ are shortest vectors.
  - $Q[v_1] = \cdots = Q[v_n]$.
- Well rounded forms correspond to bounded faces of R_n.
- Every form can be continuously deformed to a well rounded form and this defines a retracting homotopy of R_n onto a polyhedral complex WR_n of dimension $\frac{n(n-1)}{2}$.
- Every face of WR_n has finite stabilizer, hence we can use it for computing the homology of $\mathrm{GL}_n(\mathbb{Z})$ and other arithmetic groups.
- Actually, in term of dimension, we cannot do better:
  - A. Pettet and J. Souto, *Minimality of the well rounded retract*, Geometry and Topology, **12** (2008), 1543-1556.

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V. The lattice covering problem

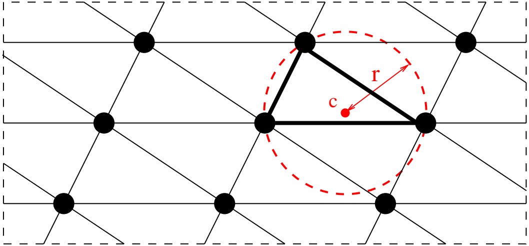
Empty sphere and Delaunay polytopes

A sphere $S(c, r)$ of radius r and center c in an n-dimensional lattice L is said to be an empty sphere if:

(i) $\|v - c\| \geq r$ for all $v \in L$,

(ii) the set $S(c, r) \cap L$ contains $n + 1$ affinely independent points.

A Delaunay polytope P in a lattice L is a polytope, whose vertex-set is $L \cap S(c, r)$.



Equalities and inequalities

- Take $M = G_v$ with $v = (v_1, \dots, v_n)$ a basis of lattice L.
- If $V = (w_1, \dots, w_N)$ with $w_i \in \mathbb{Z}^n$ are the vertices of a Delaunay polytope of empty sphere $S(c, r)$ then:

$$||w_i - c|| = r \text{ i.e. } w_i^T M w_i - 2 w_i^T M c + c^T M c = r^2$$

- Substracting one obtains

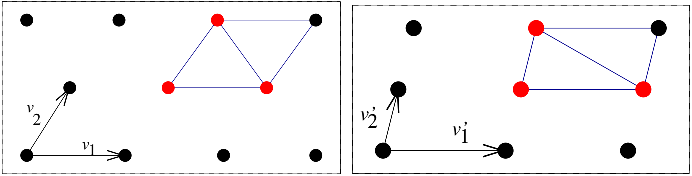
$$\{w_i^T M w_i - w_j^T M w_j\} - 2\{w_i^T - w_j^T\} M c = 0$$

- Inverting matrices, one obtains $Mc = \psi(M)$ with ψ linear and so one gets linear equalities on M.
- Similarly $||w - c|| \geq r$ translates into linear inequalities on M: Take $V = (v_0, \dots, v_n)$ a simplex ($v_i \in \mathbb{Z}^n$), $w \in \mathbb{Z}^n$. If one writes $w = \sum_{i=0}^n \lambda_i v_i$ with $1 = \sum_{i=0}^n \lambda_i$, then one has

$$||w - c|| \geq r \Leftrightarrow w^T M w - \sum_{i=0}^{n} \lambda_i v_i^T M v_i \geq 0$$

Iso-Delaunay domains

- Take a lattice L and select a basis $v_1, \ldots, v_n$.
- We want to assign the Delaunay polytopes of a lattice. Geometrically, this means that



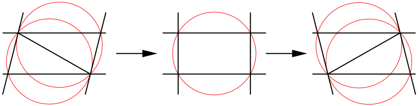
are part of the same iso-Delaunay domain.
- An iso-Delaunay domain is the assignment of Delaunay polytopes, so it is also the assignment of the Voronoi polytope of the lattice.

Primitive iso-Delaunay

- If one takes a generic matrix M in $S^n_{>0}$, then all its Delaunay are simplices and so no linear equality are implied on M.
- Hence the corresponding iso-Delaunay domain is of dimension $\frac{n(n+1)}{2}$, they are called primitive

Equivalence and enumeration

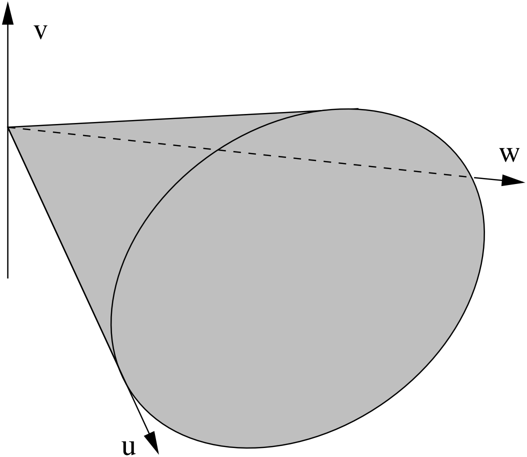
- The group $\mathrm{GL}_n(\mathbb{Z})$ acts on $S^n_{>0}$ by arithmetic equivalence and preserve the primitive iso-Delaunay domains.
- Voronoi proved that after this action, there is a finite number of primitive iso-Delaunay domains.
- Bistellar flipping creates one iso-Delaunay from a given iso-Delaunay domain and a facet of the domain. In dim. 2:



- Enumerating primitive iso-Delaunay domains is done classicaly:
  - Find one primitive iso-Delaunay domain.
  - Find the adjacent ones and reduce by arithmetic equivalence.
- This is very similar to the Voronoi algorithm for perfect forms.

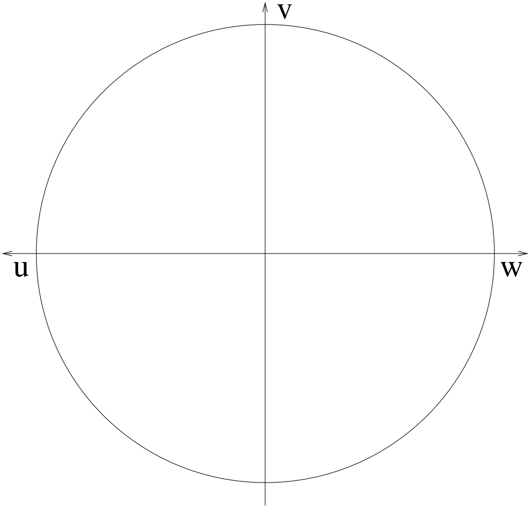
The partition of $S^2_{>0} \subset \mathbb{R}^3$ I

If $q(x, y) = \mathrm{u}x^2 + 2\mathrm{v}xy + \mathrm{w}y^2$ then $q \in S^2_{>0}$ if and only if $\mathrm{v}^2 < \mathrm{uw}$ and $\mathrm{u} > 0$.



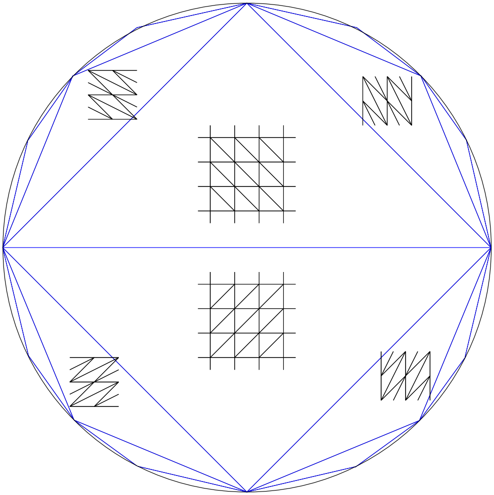
The partition of $S^2_{>0} \subset \mathbb{R}^3$ II

We cut by the plane $\mathrm{u} + \mathrm{w} = 1$ and get a circle representation.



The partition of $S^2_{>0} \subset \mathbb{R}^3$ III

Primitive iso-Delaunay domains in $S^2_{>0}$:



Optimization problem

- The lattice covering problem is to find a lattice covering of minimal density.
- Thm. Given an iso-Delaunay domain LT, there exist a unique lattice, which minimize the covering density over LT.
- The effective lattice is obtained by solving a semidefinite programming problem, so no exact solution, but approximate solutions available at any precision.
- The local maxima that are found are defined by algebraic integers.
- See for more details
  - A. Schürmann and F. Vallentin, *Computational approaches to lattice packing and covering problems*, Discrete & Compututational Geometry **35** (2006) 73–116.
  - A. Schürmann, *Computational geometry of positive definite quadratic forms*, University Lecture Notes, AMS.

Known results on covering density minimization

dim.	Best covering	nr of iso-Delaunay
2	A_2 (Kershner)	1 (Voronoi)
3	A_3^* (Bambah)	1(Voronoi)
4	A_4^* (Delone & Ryshkov)	3(Voronoi)
5	A_5^* (Ryshkov & Baranovski)	222(Engel)
6	L_6 (conj. Vallentin)?	?
7	L_7 (conj. Schürmann & Vallentin)?	?
24	Leech (conj.)?	?

- It turn out that the lattice of minimal covering density are unique for $n \leq 5$
- In general the best lattice coverings are expected to be non-rational and with low symmetry.
- But experimentations seemed to indicate that E_6 is a local covering maxima.

VI. Quadratic functions and the Erdahl cone

The Erdahl cone

- Denote by $E_2(n)$ the vector space of degree 2 polynomial functions on $\mathbb{R}^n$. We write $f \in E_2(n)$ in the form

$$f(x) = a_f + b_f \cdot x + Q_f[x]$$

  with $a_f \in \mathbb{R}$, $b_f \in \mathbb{R}^n$ and Q_f a $n \times n$ symmetric matrix
- The Erdahl cone is defined as

$$Erdahl(n) = \{f \in E_2(n) \text{ such that } f(x) \geq 0 \text{ for } x \in \mathbb{Z}^n\}$$

- It is a convex cone, which is non-polyhedral since defined by an infinity of inequalities.
- The group acting on $Erdahl(n)$ is $\mathrm{AGL}_n(\mathbb{Z})$, i.e. the group of affine integral transformations

$$x \mapsto b + Px \text{ for } b \in \mathbb{Z}^n \text{ and } P \in \mathrm{GL}_n(\mathbb{Z})$$

Scalar product

- Definition: If $f, g \in E_2(n)$, then:

$$\langle f, g\rangle = a_f a_g + \langle b_f, b_g\rangle + \langle Q_f, Q_g\rangle$$

- Definition: For $v \in \mathbb{Z}^n$, define $ev_v(x) = (1 + v \cdot x)^2$.
- We have

$$\langle f, ev_v\rangle = f(v)$$

- Thus finding the rays of $Erdahl(n)$ is a dual description problem with an infinity of inequalities and infinite group acting on it.
- If $f \in Erdahl(n)$ then Q_f is positive semidefinite.
- Definition: We also define

$$Erdahl_{>0}(n) = \{f \in Erdahl(n) \;:\; Q_f \text{ positive definite}\}$$

Relation with Delaunay polytope

- If D is a Delaunay polytope of a lattice $L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$ of empty sphere $S(c, r)$ then we define the function

$$\begin{array}{rcl} f_{D,\mathbf{v}} : \mathbb{Z}^n & \to & \mathbb{R} \\ x = (x_1, \ldots, x_n) & \mapsto & \left\| \sum_{i=1}^n x_i v_i - c \right\|^2 - r^2 \end{array}$$

  Clearly $f_{D,\mathbf{v}} \in Erdahl_{>0}(n)$.
- The perfection rank of a Delaunay polytope is the dimension of the face it defines in $Erdahl(n)$.
- Definition: If $f \in Erdahl(n)$ then

$$Z(f) = \{v \in \mathbb{Z}^n : f(v) = 0\}$$

- Theorem: If $f \in Erdahl(n)$ then there exist a lattice L_f and a lattice L' containing a Delaunay polytope D_f such that

$$Z(f) = D_f + L_f$$

- We have $\dim L' + \dim L_f \le n$. In case of equality $Z(f)$ is called a Delaunay polyhedra.

Perfect Delaunay polytopes/polyhedra

- Definition: If D is a n-dimensional Delaunay polyhedra then we define
$$\mathrm{Dom}_{\mathbf{v}}\ D = \sum_{v\mathbf{v} \in D} \mathbb{R}_+ ev_v$$
- We have $\langle f_{D,\mathbf{v}}, \mathrm{Dom}_{\mathbf{v}}\ D\rangle = 0$.
- Definition: D is perfect if Dom D is of dimension $\binom{n+2}{2} - 1$ that is if the perfection rank is 1.
- This implies that f_D is an extreme rays of $Erdahl(n)$ and f_D is rational.
- A perfect n-dimensional Delaunay polytope has at least $\binom{n+2}{2} - 1$ vertices. There is only one way to embed it as a Delaunay polytope of a lattice.
- Perfect Delaunay polytopes are remarkable and rare objects that we want to enumerate.

Perfect Delaunay polytope

- There is a finite number of them in each dimension n. Known results:

dim.	perfect Delaunay	authors
1	$[0,1]$ in $\mathbb{Z}$	
2	$\emptyset$	
3	$\emptyset$	
4	$\emptyset$	
5	$\emptyset$	$\uparrow$ (Deza, Laurent & Grishukhin)
6	2_{21} in E_6	(Deza & Dutour)
7	3_{21} in E_7	
	and ER_7 in $L(ER_7)$	
8	≥ 27	(Dutour Sikiric & Rybnikov)
9	≥ 100000	(Dutour Sikiric)

- **Theorem**: There exist perfect Delaunay polytopes D such that $\mathbb{Z}D \neq \mathbb{Z}^n$.
- **Theorem**: There exist lattices with several perfect Delaunay polytopes.
- **Theorem**: For $n \geq 6$ there exist a perfect Delaunay polytope with exactly $\binom{n+2}{2} - 1$ vertices.

Extreme rays of *Erdahl*(n)

- Definition: If $f \in Erdahl_{>0}(n)$ then we define

$$\mathrm{Dom}\ \ f = \sum_{v \in Z(f)} \mathbb{R}_+ ev_v$$

- We have $\langle f, \mathrm{Dom}\ \ f\rangle = 0$.
- Erdahl, 1992: The extreme rays of *Erdahl*(n) are
  - (a) The constant function 1.
  - (b) The functions
  $$(a_1x_1 + \dots + a_nx_n + \beta)^2$$
  with $(a_1, \dots, a_n)$ not collinear to an integral vector.
  - (c) The functions f such that $Z(f)$ is a perfect Delaunay polyhedra.
- Note that if $f \in Erdahl(n)$ with $Z(f)$ a Delaunay polyhedra, then there exist a lattice L' of dimension $k \leq n$, a Delaunay polytope D of L', a basis $\mathbf{v}'$ of L' and a function $\phi \in \mathrm{AGL}_n(\mathbb{Z})$ such that

$$f \circ \phi(x_1, \dots, x_n) = f_{D,\mathbf{v}'}(x_1, \dots, x_k)$$

Delaunay polyhedra retract

- For a function $f \in Erdahl(n)$ a proper decomposition is a pair (g, h) with $f = g + h$, $g \in Erdahl(n)$ and $h(x) \geq 0$ for $x \in \mathbb{R}^n$.
- Lemma: For a proper decomposition we have

$$Vect\, Z(f) + \text{Ker } Q_f \subset \text{Ker } Q_h$$

  and there exist a proper decomposition with equality.
- Fix an integral complement L' of $Vect\, Z(f) + \text{Ker } Q_f$. A proper decomposition is called extremal if $\det Q_h|_{L'}$ is maximal among all proper decompositions.
- Theorem: For $f \in Erdahl(n)$, there exist a unique extremal decomposition. For it we have that $Z(g)$ is a delaunay polyhedra.
- Conjecture: The decomposition depends continuously on $f \in Erdahl(n)$.
- On the other hand in a neighborhood of $f \in Erdahl(n)$ we can have an infinity of Delaunay polyhedra.

Voronoi algorithm on the Erdahl cone

- From a given n-dimensional perfect Delaunay polytope Q of form f we can define the local cone

$$Loc(f) = \{g \in E_2(n) \text{ s.t. } g(x) \geq 0 \text{ for } x \in Z(f)\}$$

- The flipping algorithm finds the adjacent quadratic perfect form g from a given perfect form f.
- The problem is $Erdahl(n)$ is not locally polyhedral, i.e. the rank of g can be lower than n.
- The technique is to use a recursive algorithm for realizing the enumeration. We start form $[0,1] \times \mathbb{R}^{n-1}$ and by subdivizion reach $[0,1]^n$ (its local cone is the cut cone CUT_{n+1} occuring in combinatorial optimization).

VII. Covering maxima, pessima and their characterization

Eutacticity

- If $f \in Erdahl_{>0}(n)$ then define μ_f and c_f such that

$$f(x) = Q_f[x - c_f] - \mu_f$$

  Then define

$$u_f(x) = (1 + c_f \cdot x)^2 + \frac{\mu_f}{n} Q_f^{-1}[x]$$

- Definition: $f \in Erdahl_{>0}(n)$ is eutactic if u_f is in the relative interior of Dom f.
- Definition: Take a Delaunay polytope P for a quadratic form Q of center c_P and square radius μ_P. P is called eutactic if there are $\alpha_v > 0$ so that

$$\begin{cases} 1 &= \sum_{v \in \text{vert } P} \alpha_v, \\ 0 &= \sum_{v \in \text{vert } P} \alpha_v (v - c_P), \\ \frac{\mu_P}{n} Q^{-1} &= \sum_{v \in \text{vert } P} \alpha_v (v - c_P)(v - c_P)^T. \end{cases}$$

Covering maxima

- A given lattice L is called a covering maxima if for any lattice L' near L we have $\Theta(L') < \Theta(L)$.
- Theorem: The following are equivalent:
  - L is a covering maxima
  - Every Delaunay polytope of maximal circumradius is perfect and eutactic.
- The following are perfect Delaunay polytope:

name	# vertices	# orbits Delaunay polytopes
E_6	27	1
E_7	56	2
ER_7	35	4
O_{10}	160	6
BW_{16}	512	4
O_{23}	94208	5
Λ_{23}	47104	709

- Theorem: For any $n \geq 6$ there exist one lattice $L(DS_n)$ which is a covering maxima.
  There is only one perfect Delaunay polytope $P(DS_n)$ of maximal radius in $L(DS_n)$.

The infinite series

- For n even $P(DS_n)$ is defined as the lamination over D_{n-1} of
  - one vertex
  - the half cube $\frac{1}{2}H_{n-1}$
  - the cross polytope CP_{n-1}

  For $n = 6$, it is E_6.
- For n odd as the lamination over D_{n-1} of
  - the cross polytope CP_{n-1}
  - the half cube $\frac{1}{2}H_{n-1}$
  - the cross polytope CP_{n-1}

  For $n = 7$, it is E_7.
- Conjecture: The lattice DS_n has the following properties:
  - $L(DS_n)$ has the maximum covering density among all covering maxima
  - Among all perfect Delaunay polytopes, $P(DS_n)$ has
    - maximum number of vertices
    - maximum volume

  If true this would imply Minkovski conjecture.

Pessimum and Morse function property

- For a lattice L let us denote $D_{crit}(L)$ the space of direction d of deformation of L such that Θ increases in the direction d.
- Definition: A lattice L is said to be a covering pessimum if the space D_{crit} is of measures 0.
- Theorem: If a lattice L has all its Delaunay polytopes of maximum circumradius are eutactic and are not simplices then Q is a pessimum.

name	# vertices	# orbits Delaunay polytopes
$\mathbb{Z}^n$	2^n	1
D_4	8	1
D_n ($n \geq 5$)	2^{n-1}	2
E_6^*	9	1
E_7^*	16	1
E_8	16	2
K_{12}	81	4

- Theorem: The covering density function $Q \mapsto \Theta(Q)$ is a topological Morse function if and only if $n \leq 3$.

THANK

YOU