Classification of eight dimensional perfect forms

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I. Introduction

Perfect forms and domains

Take a form $A \in PSD_n$ and define $\lambda(A) = \min_{v \in \mathbb{Z}^n \neq 0} vA^t v$ and $SV(A)$ the set of shortest vectors.

- A is perfect if $vB^t v = \lambda(A)$ with $v \in SV(A)$ implies $B = A$.
- If A is perfect, its perfect domain is

$$\{ \sum_{v \in SV(A)} \lambda_v {}^t v v \text{ with } \lambda_v \geq 0 \}$$

- All possible perfect domain form a polyhedral tesselation of PSD_n
- There is a finite number of perfect domains up to $GL_n(\mathbb{Z})$ equivalence.

Voronoi algorithm

- Find a perfect form, insert it to the list as unmarked.
- Iterate
  - For every undone perfect domain, compute its facets.
  - For every facet realize the flipping, i.e. compute the adjacent perfect domain.
    If it is new, then add to the list as undone.
- Finish when all perfect domains have been treated.

If the number of perfect forms is not too large, then the key computationnal step is finding the facets.

II. Polyhedral computation techniques

Computing dual description

- cdd and lrs are general purpose programs for finding dual descriptions, which do not work if the polyhedral cone is too complex.
- The idea is to use symmetries of the polytope to compute the orbits of facets (usually what is needed).
- The key idea of the Adjacency Decomposition Method is:
  - compute some initial facet (by linear programming) and insert the orbit into the list of orbits.
  - compute the adjacent facets to this facet (this is a dual description computation) and insert them into the list of orbits if they are new.
  - the algorithm finish when all orbits are done.

Computing dual description

- The algorithm provides an improvement over a straightforward application of cdd and lrs
- We represent the group as permutation group on the extreme rays and the facets by their incidence, i.e. sets.
- Then, we use two following functions in GAP

Stabilizer(GroupExt, ListInc, OnSets);

RepresentativeAction(GroupExt, ListInc1, ListInc2, OnSets);

The important thing is to use the action OnSets, which is extremely efficient and uses backtrack search, i.e. in practice we never build the full orbit.

Further strategies

- Using the Adjacency Decomposition method we can usually find a conjecturally complete list of facets. However in many cases, there remain a few orbits of facets that are particularly difficult to compute.
- If the number of untreated facets is lower than $n-1$, then we can use following theorem and finish.
  Balinski theorem The skeleton of a n-dimensional polytope is n-connected, i.e. the removal of any set of $n-1$ vertices leaves it connected.
- Otherwise, we can apply the Adjacency decomposition method to the remaining orbits of facets. This strategy is especially recommended since usually difficult facets are remarkable ones, i.e. have more symmetries than ordinary ones.

Banking methods

- When one applies the Recursive Adjacency decomposition method, one needs to compute the dual description of faces.
- Those faces can appear several times.
- The key point is that this computation is intrinsic, i.e. independent over what face or polyhedral cone the face F belongs to.
- The idea is to store the dual description of facets in a bank and when a computation happen to make call to that bank to see if it is already done.

Symmetry group of polytopes

- Suppose that we have a rank n family of vector $(v_i)_{1\leq i\leq N}$.
- Define the form
$$Q=\sum_{i=1}^{N} {}^t v_i v_i$$
- Define the edge colored graph on N vertices with edge color
$$c_{ij}=v_i Q^{-1\,t} v_j$$
- The automorphism group of this edge colored graph gives the group of matrices $A\in GL_n(\mathbb{R})$ such that
$$v_i\mapsto v_i A$$
realize a permutation of the vector family.

Symmetry group of polytopes

- Testing isomorphism of polytopes is done similarly.
- PROBLEM: The projective symmetries of a polyhedral cone are the matrices A such that

$$v_i A = \alpha_i v_{\sigma(i)}$$

with $\alpha_i > 0$ and σ a permutation of N elements, i.e. A permutes the extreme rays.
We do not know how to compute this group efficiently.

Automorphism of edge colored graph

- The program nauty by MacKay can compute efficiently automorphism groups of graphs. We can also restrict the automorphism by fixing an initial partition of the vertex set.
- We take an edge colored graph with k colors C_j and vertex set $(v_i)_{1\leq i\leq N}$.
- We can form the graph with vertex-set (v_i, C_j) and
  - edges between (v_i, C_j) and $(v_i, C_{j'})$
  - edges between (v_i, C_j) and $(v_{i'}, C_j)$ if the color between v_i and $v_{i'}$ is C_j

  This makes a graph with kN vertices.

Automorphism of edge colored graph

- Another model is to take as vertex-set $(v_i, v_{i'})$ with $i < i'$ and
  - edges between $(v_i, v_{i'})$ and $(v_k, v_{k'})$ if they share a vertex
  - k vertex-partition formed by all $(v_i, v_{i'})$ having color C_j

  This model works if $N > 4$ and has $\frac{N(N-1)}{2}$ vertices.
- We choose the model with the smallest number of vertices.

Double cosets

- Suppose that we have a set of orbits for a group G

$$\mathcal{F} = x_1 G \cup \cdots \cup x_n G$$

  and we want to represent $\mathcal{F}$ as list of orbits for a subgroup H of G.
- For every x_i do a double coset decomposition

$$G = G_{x_i} g_1 H \cup \ldots \cup G_{x_i} g_p H$$

  with G_{x_i} the stabilizer of x_i in G.
- So, $x_i G = \cup_j x_i g_j H$
- Then we have

$$\mathcal{F} = \cup_{i,j} x_i g_j H$$

Heuristics

The realization of the preceding program is based on many different heuristics:

- Choosing between cdd, lrs, etc.
- When to go for another iteration of adjacency method.
- When to save the description in the bank.
- When to use stabilizer of a face or its inner symmetry group.
- How to select invariants for the test of equivalence of faces.

III. 8-dimensional case

8-dimensional case

- There are 10916 perfect forms in dimension 8.
- Finding dual description is easy for most of them except for a few. E_8 is the most difficult.
- We computed by Adjacency Decomposition method 80000 orbits of facets. Along most difficult ones:
  - One facet with incidence 66 solved by applying again the ADM.
  - One facet with incidence 70 very easy.
  - The main facet of incidence 75, stabilizer has size 23040 but inner symmetry group has size 1474560 and this solves the problem.

As a consequence: any 8-dim perfect form with kissing number greater than 150 is E_8.

Enumeration of Perfect forms

dim	Nr of forms	Best form	Authors
1	1	A_1	
2	1	A_2	Lagrange
3	1	A_3	Gauss
4	2	D_4	Korkine-Zolotareff
5	3	D_5	Korkine-Zolotareff
6	7	E_6	Barnes
7	33	E_7	Jaquet
8	10916	E_8	Dutour, Schurmann, Vallentin

Other approaches

- Mordell inequality, knowing that the Hermite constant in dimension 7, and knowing that E_8 exists gives the correct value of the Hermite constant in dimension 8.
- Blichfeldt(1931) and Vetchinkin(1983) have proved, using Korkine Zolotarev methods, that E_8 is the unique best packing in dimension 8.
- Cohn proved that E_8 is the unique best lattice packing in dimension 8 using the method used in dimension 24 for Leech lattice.

Conjecture E_8 is a densest sphere packing in dimension 8.