Delaunay polytopes in lattices

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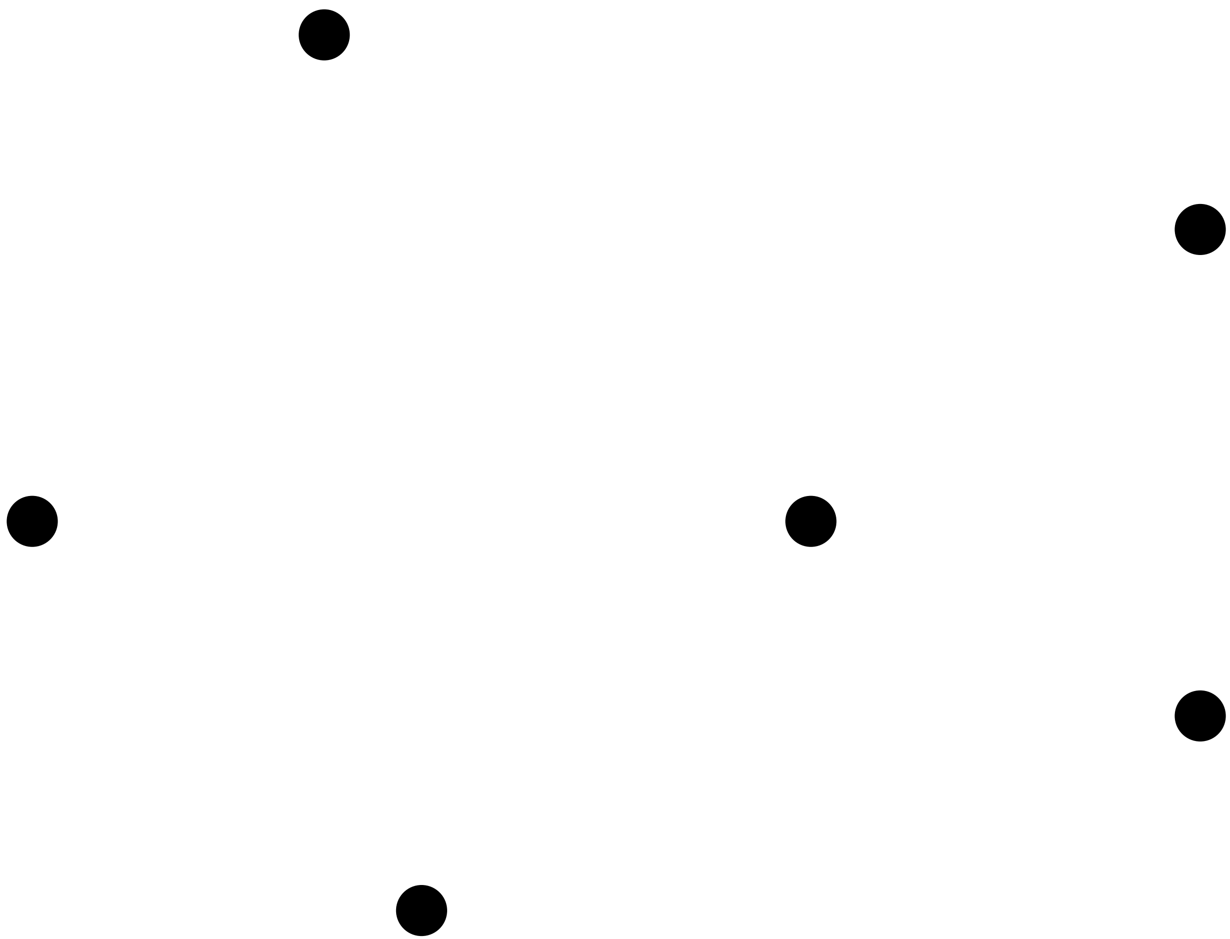
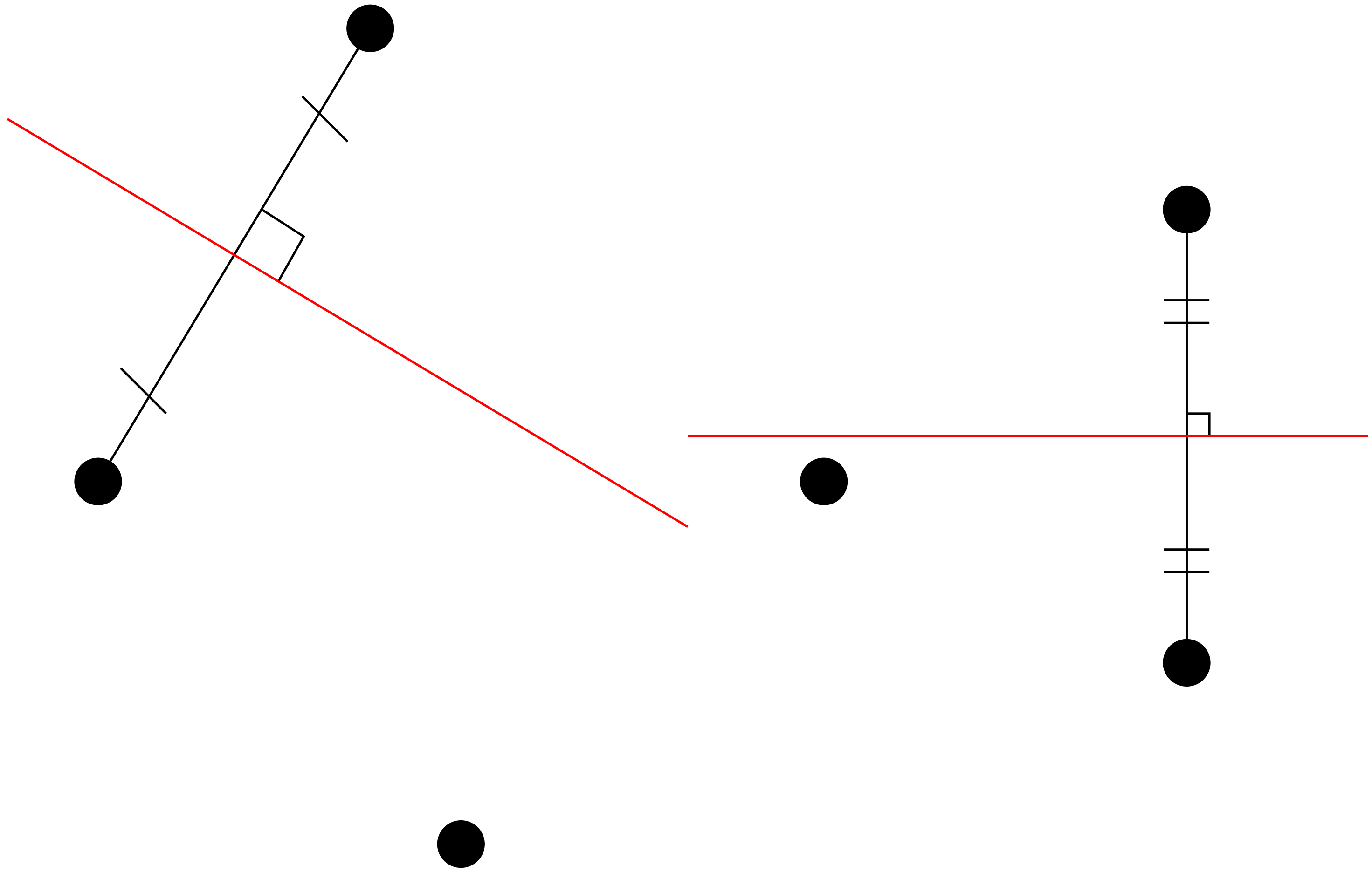
Voronoi and Delaunay polytopes

A finite set of points



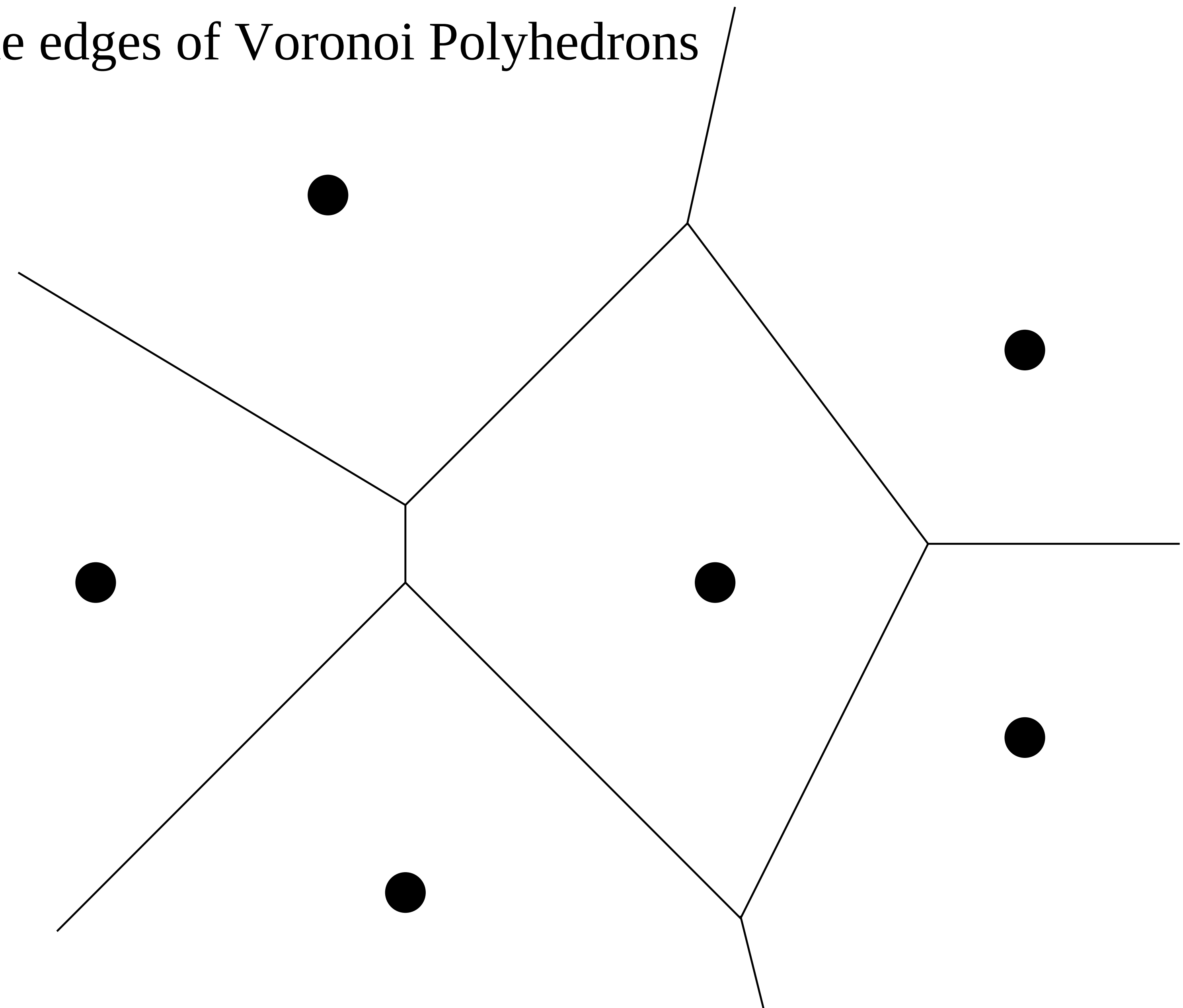
Voronoi and Delaunay polytopes

Some relevant perpendicular bisectors



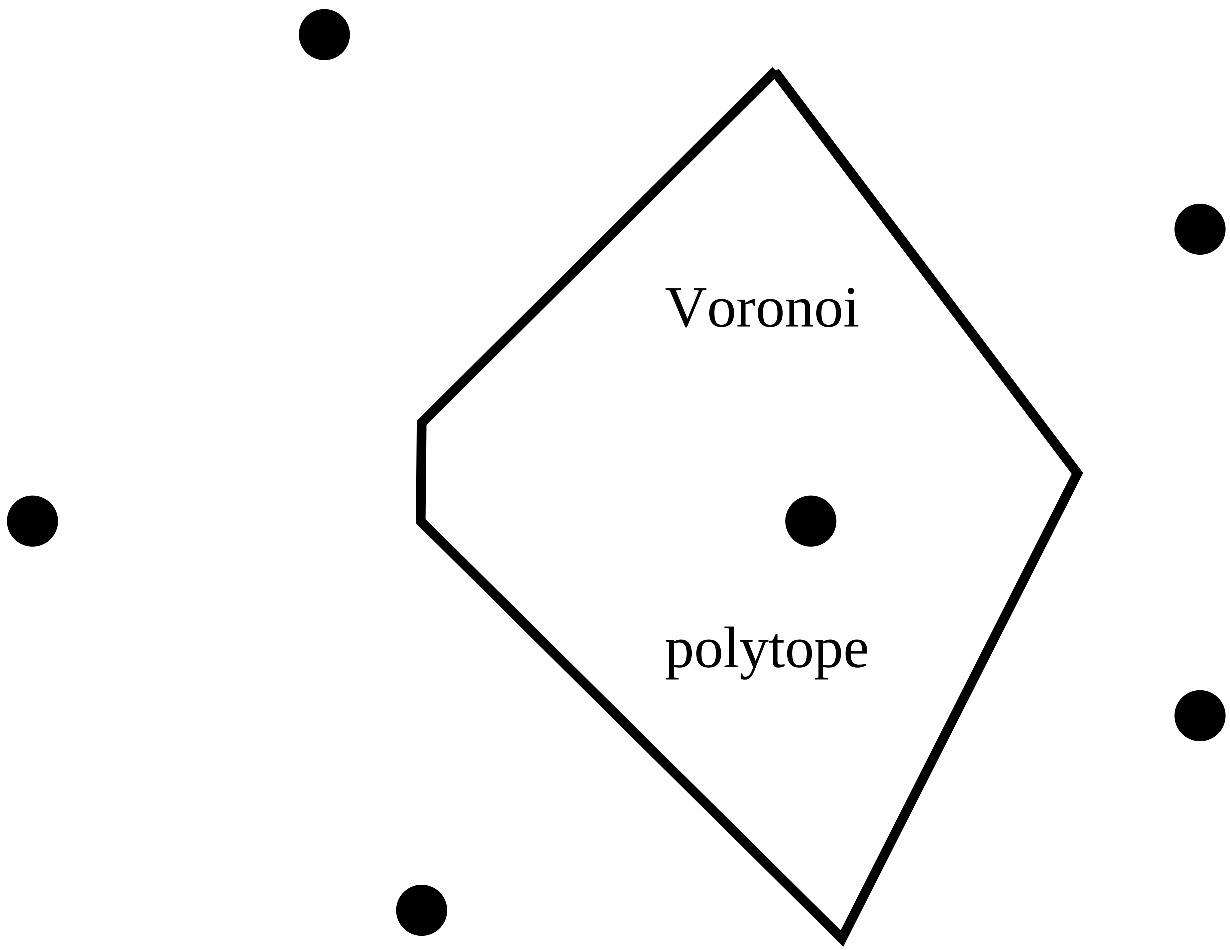
Voronoi and Delaunay polytopes

The edges of Voronoi Polyhedrons



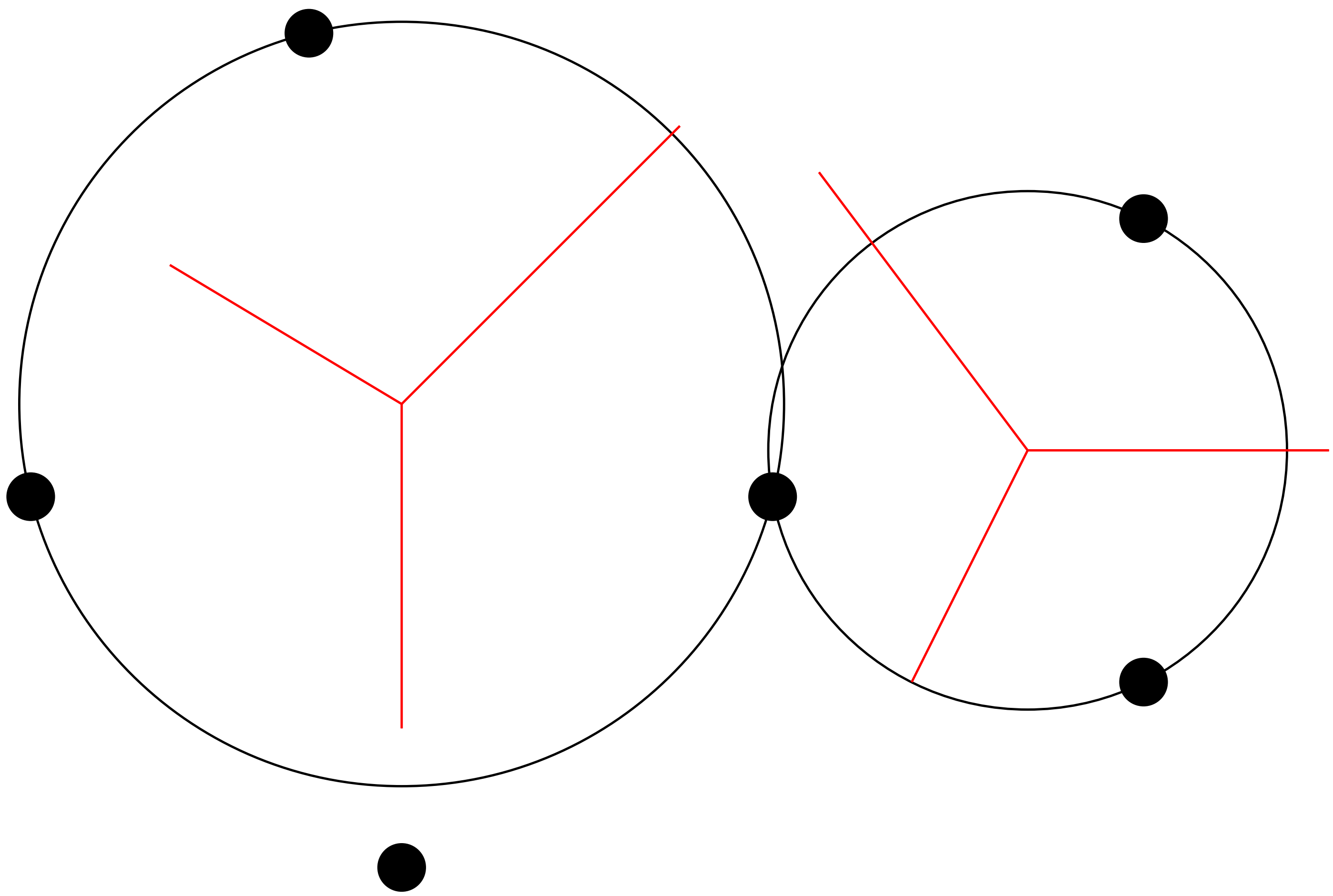
Voronoi and Delaunay polytopes

Voronoi Polytope



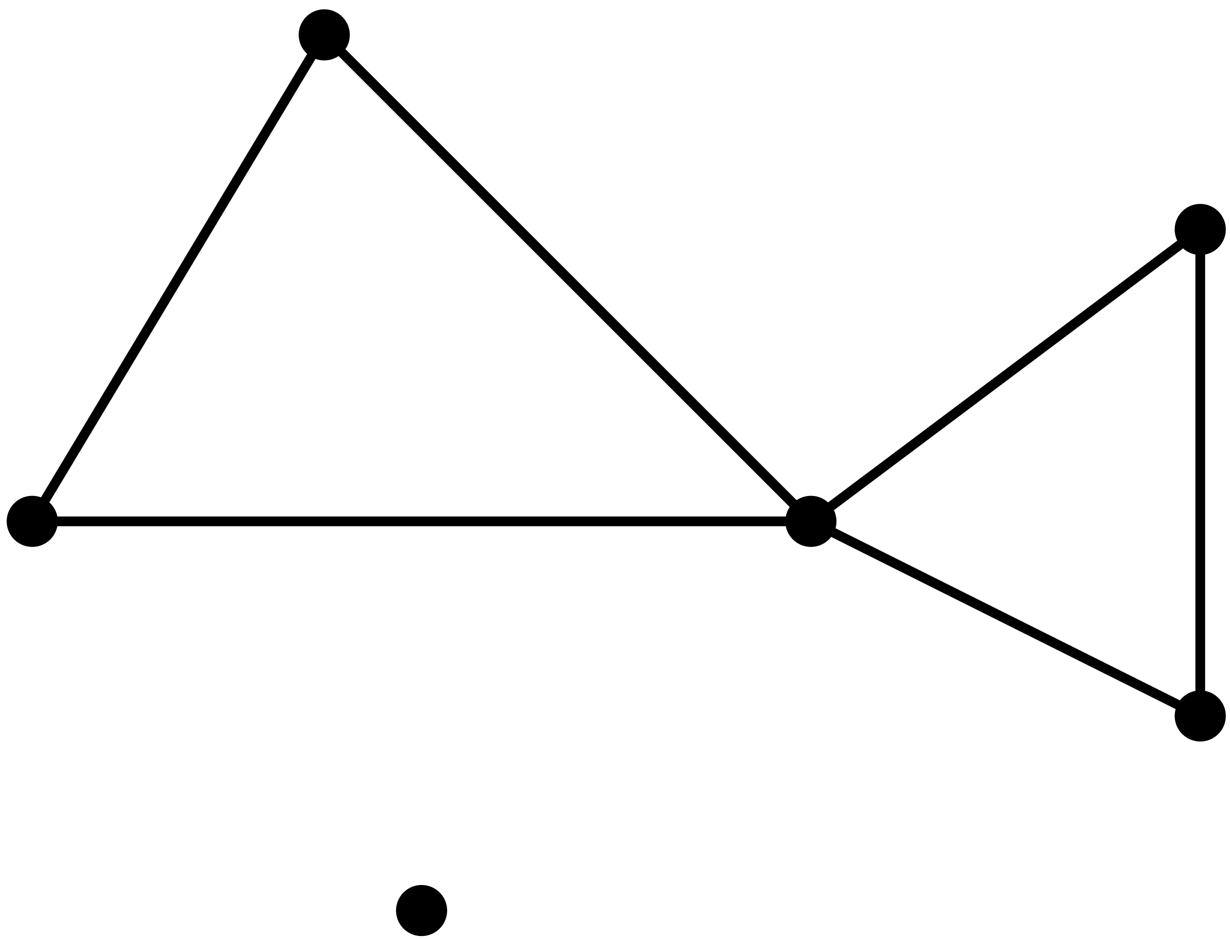
Voronoi and Delaunay polytopes

Empty spheres



Voronoi and Delaunay polytopes

Delaunay polytopes



Other names

Voronoi polytope synonyms

- Dirichlet domains (lattice theory, 2-dimensional case)
- Voronoi polytope (n-dimensional lattice, computational geometry)
- Thiessen polygons (geography)
- Wigner-Seitz cell (solid state physic, crystallography)
- first Brillouin zone (solid state physic, momentum space)
- domain of influence (politics)

Delaunay polytopes synonyms

- L-polytope (Voronoi in “Second memoire”)
- Shallow or deep hole (in Conway-Sloane)

PLAN

I. Voronoi polytopes in lattices

II. Delaunay polytopes and hypermetrics

III. The six-dimensional Delaunay polytopes

IV. Beyond dimension six

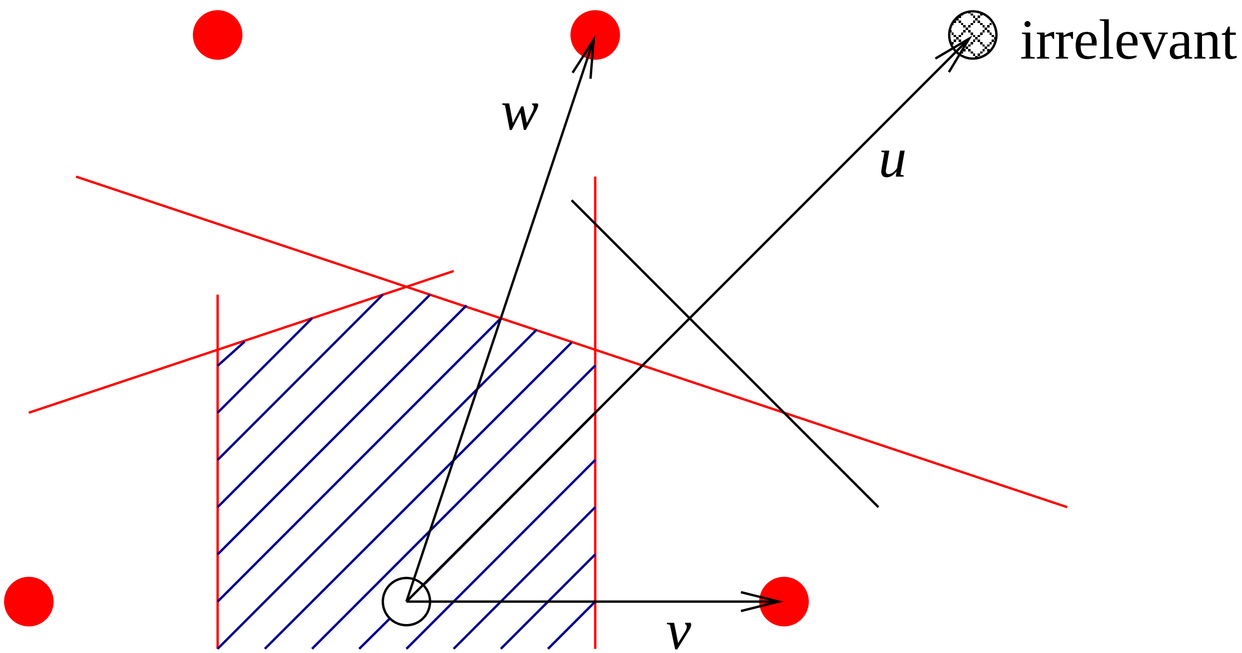
I. Voronoi polytopes

in

lattices

The Voronoi polytope of a lattice

- Polytope $\mathcal{V}$ defined by inequalities $\langle x, v\rangle \leq \frac{1}{2}||v||^2$
- ➠ $\mathcal{V}$ is polyhedral, vector v_0 such that $\langle x, v_0\rangle = \frac{1}{2}||v_0||^2$ is a facet are called relevant
- ➠ (Voronoi Theorem) A vector u is relevant if and only if it can not be written as $u = v + w$ with $\langle v, w\rangle \geq 0$



- The translates $v + \mathcal{V}$ with $v \in L$ tiles $\mathbb{R}^n$
- Vertices of Voronoi polytope are center of empty spheres which defines Delaunay polytopes
- Shortest vector in L are relevant
- Only for root lattice shortest vector are all relevant vectors

Name	Nr. facets	Nr. Vertices	Nr. Orbit
A_n	$n(n+1)$	$2^{n+1}-2$	$\lfloor \frac{n+1}{2} \rfloor$
D_n	$2n(n-1)$	2^n+2n	2
E_6	72	54	1
E_7	126	632	2
E_8	240	19440	2

A lattice with two Delaunay polytopes

- Take $L = \mathbb{Z}^n$

➠ Delaunay polytope

Name	Center	Nr. vertices	Radius
Cube	$(\frac{1}{2})^n$	2^n	$\frac{1}{2}\sqrt{n}$

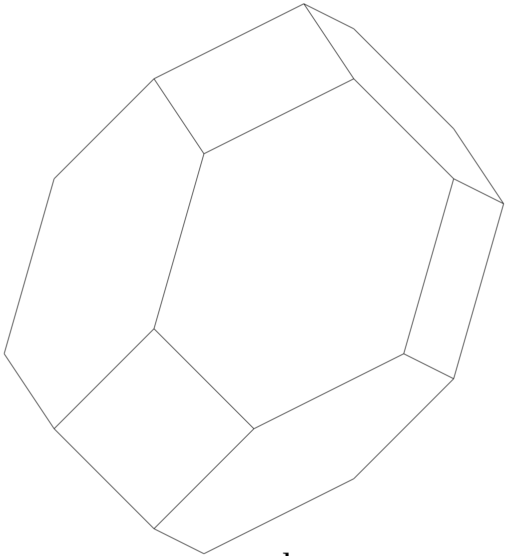
- Take $D_n = \{x \in \mathbb{Z}^n \mid \sum_{i=1}^n x_i$ is even$\}$

➠ Delaunay polytopes of D_n:

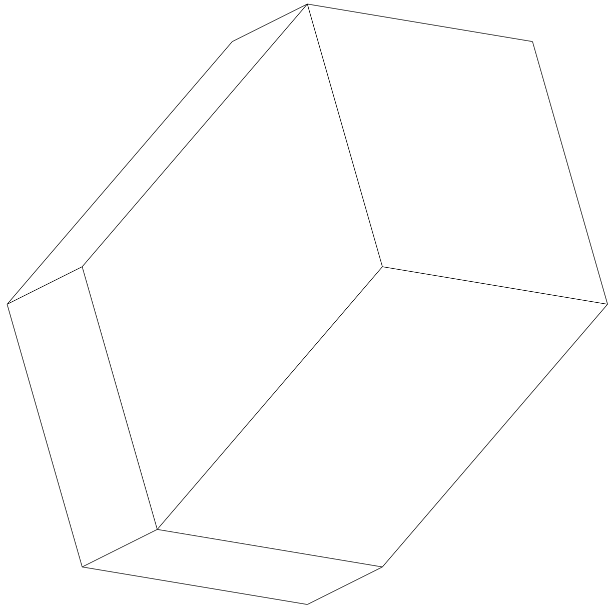
Name	Center	Nr. vertices	Radius
Half-Cube	$(\frac{1}{2})^n$	$\frac{1}{2}2^n$	$\frac{1}{2}\sqrt{n}$
Cross-polytope	$(1, 0^{n-1})$	$2n$	1

3-dimensional Voronoi polytopes

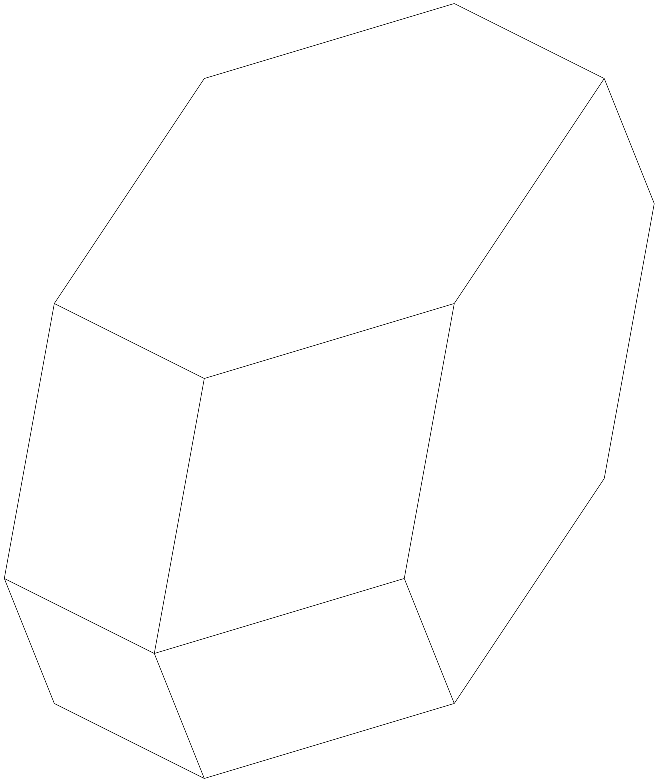
Truncated octahedron



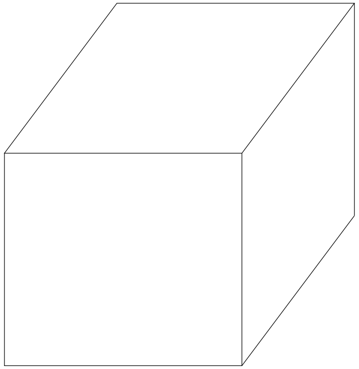
rhombic dodecahedron



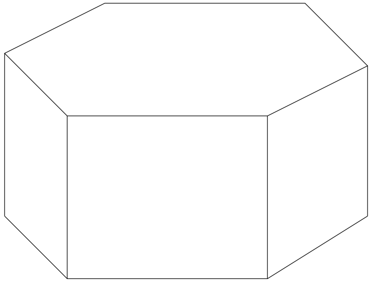
Hexarhombic dodecahedron



cube



hexagonal prism



Geometry of numbers

PSD_n=Cone of real symmetric positive definite $n \times n$ matrices
Correspondance between PSD_n and lattices:

- Lattice L spanned by $v_1, \dots, v_n$ corresponds to

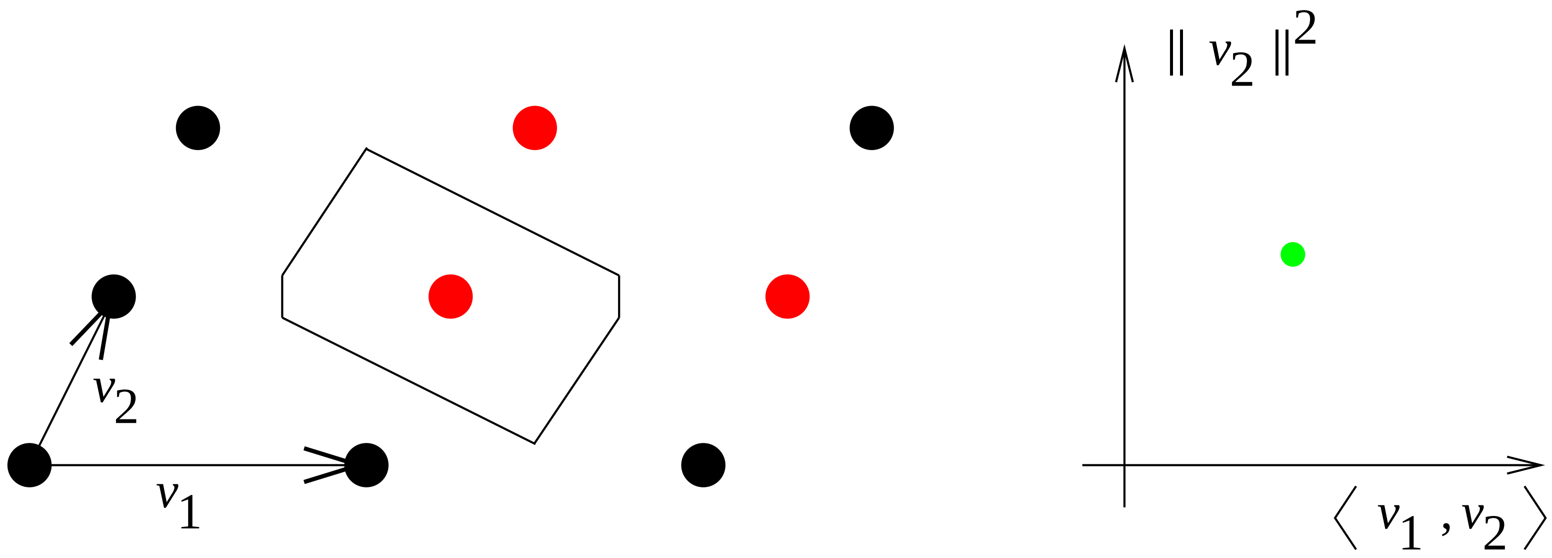
$$M_v = (\langle v_i, v_j \rangle)_{1 \le i,j \le n} \in PSD_n$$

- If L spanned by v'_1,. . . ,v'_n then $(v'_1, \dots, v'_n) = P(v_1, \dots, v_n)$ with $P \in GL_n(\mathbb{Z})$ and

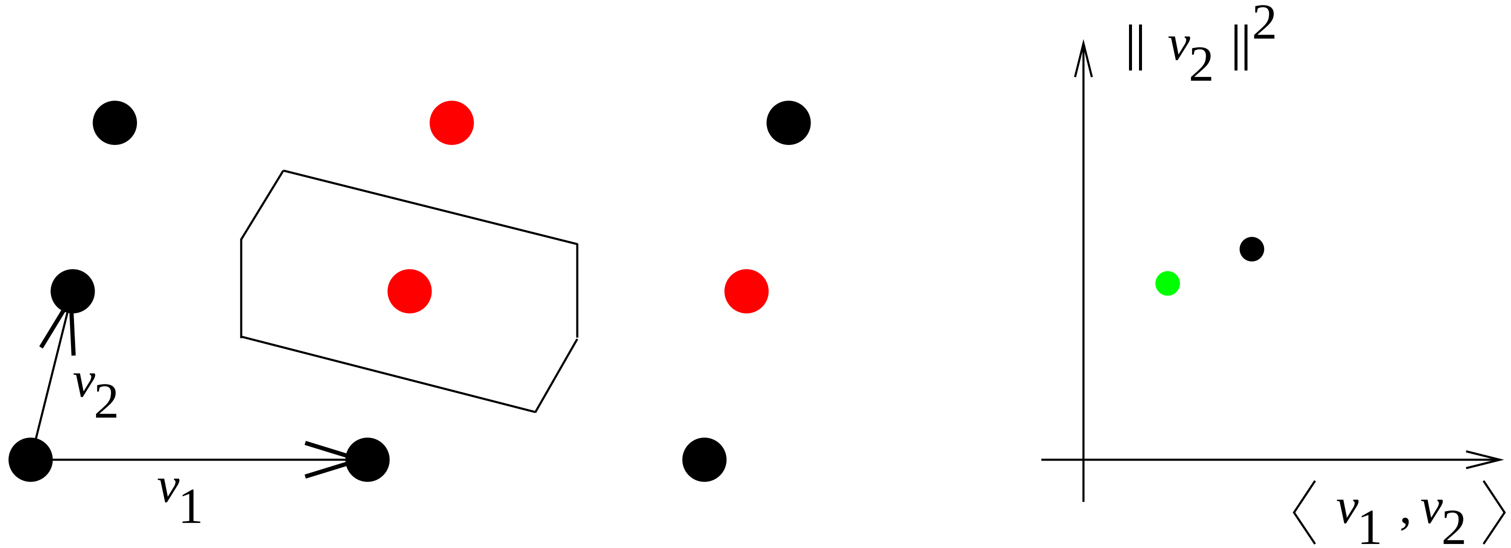
$$M_{v'} = P M_v {}^t P$$

- Lattices up to isometric equivalence correspond $GL_n(\mathbb{Z})$ equivalence classes in PSD_n

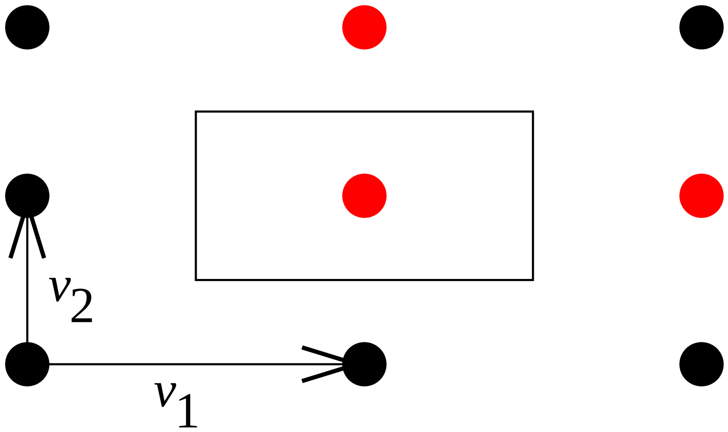
Lattices in dimension 2

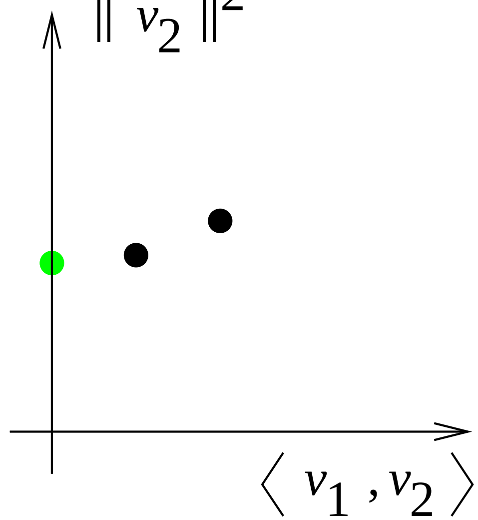


Lattices in dimension 2

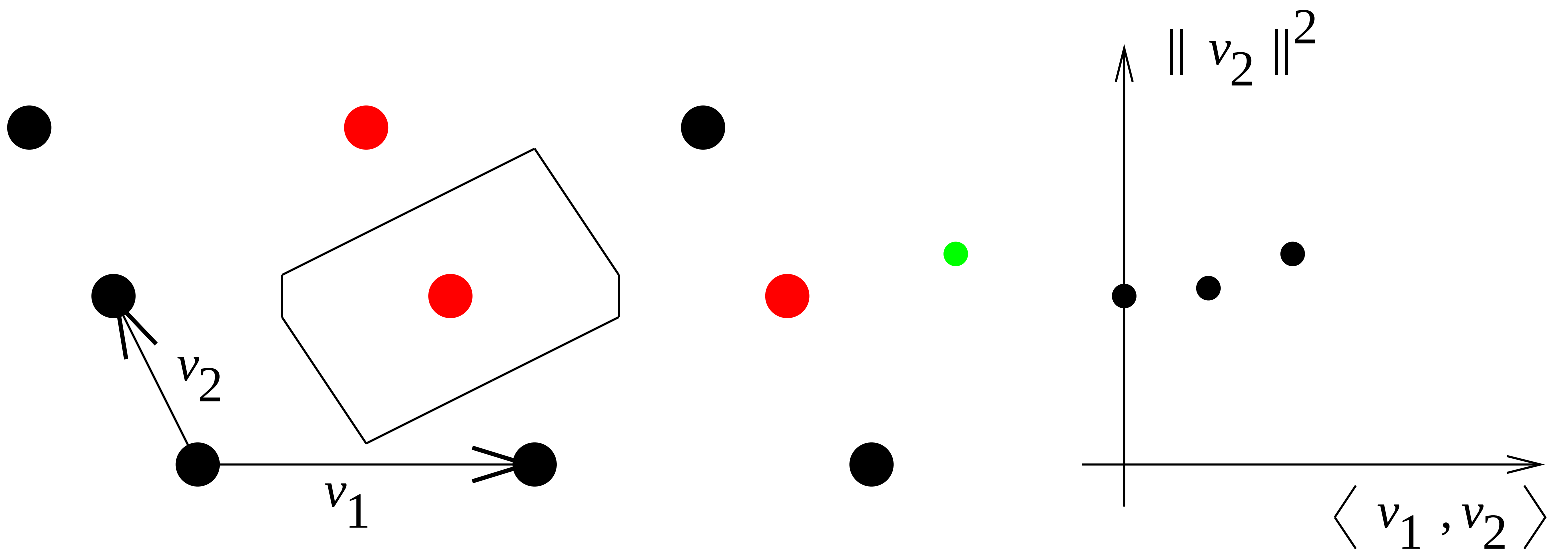


Lattices in dimension 2



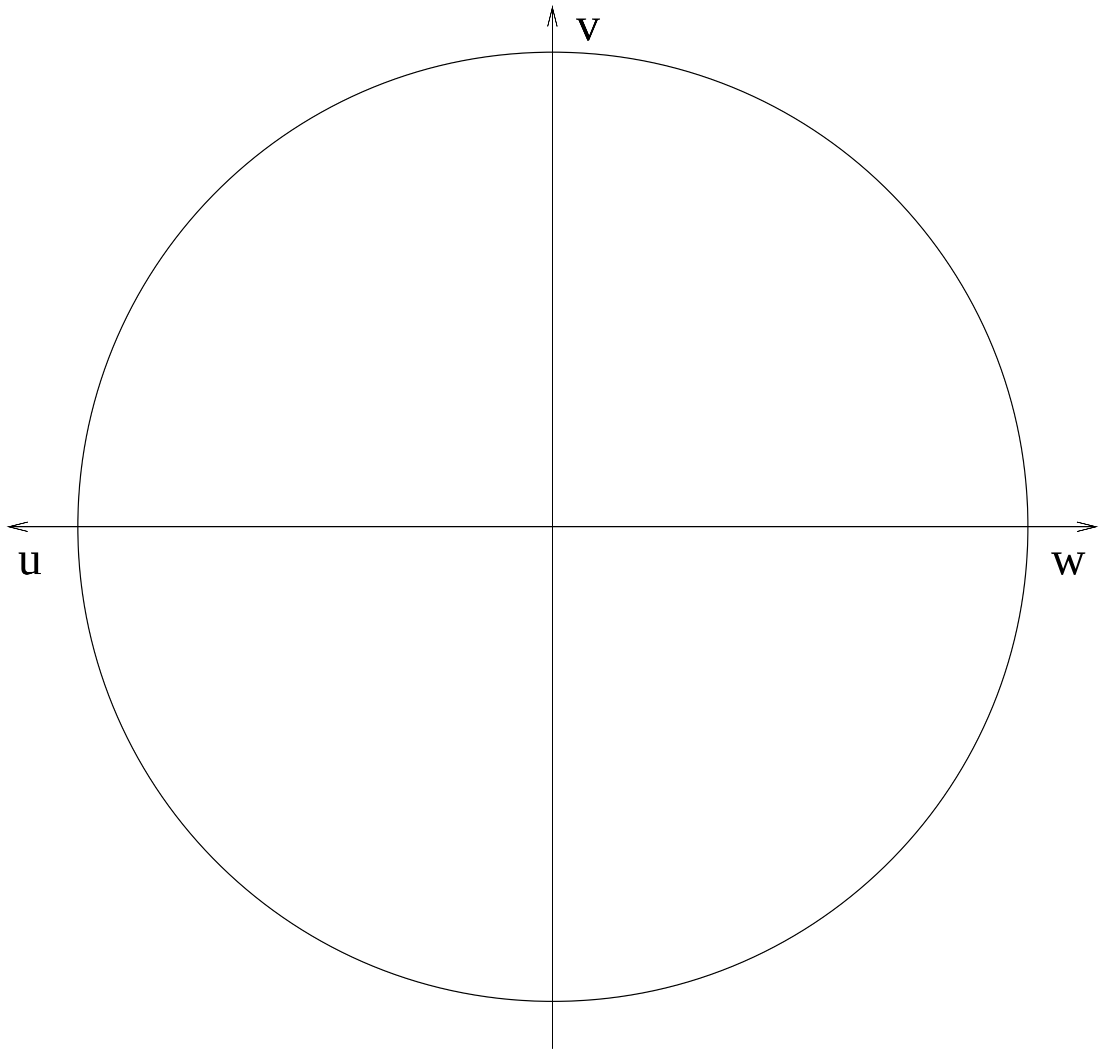


Lattices in dimension 2



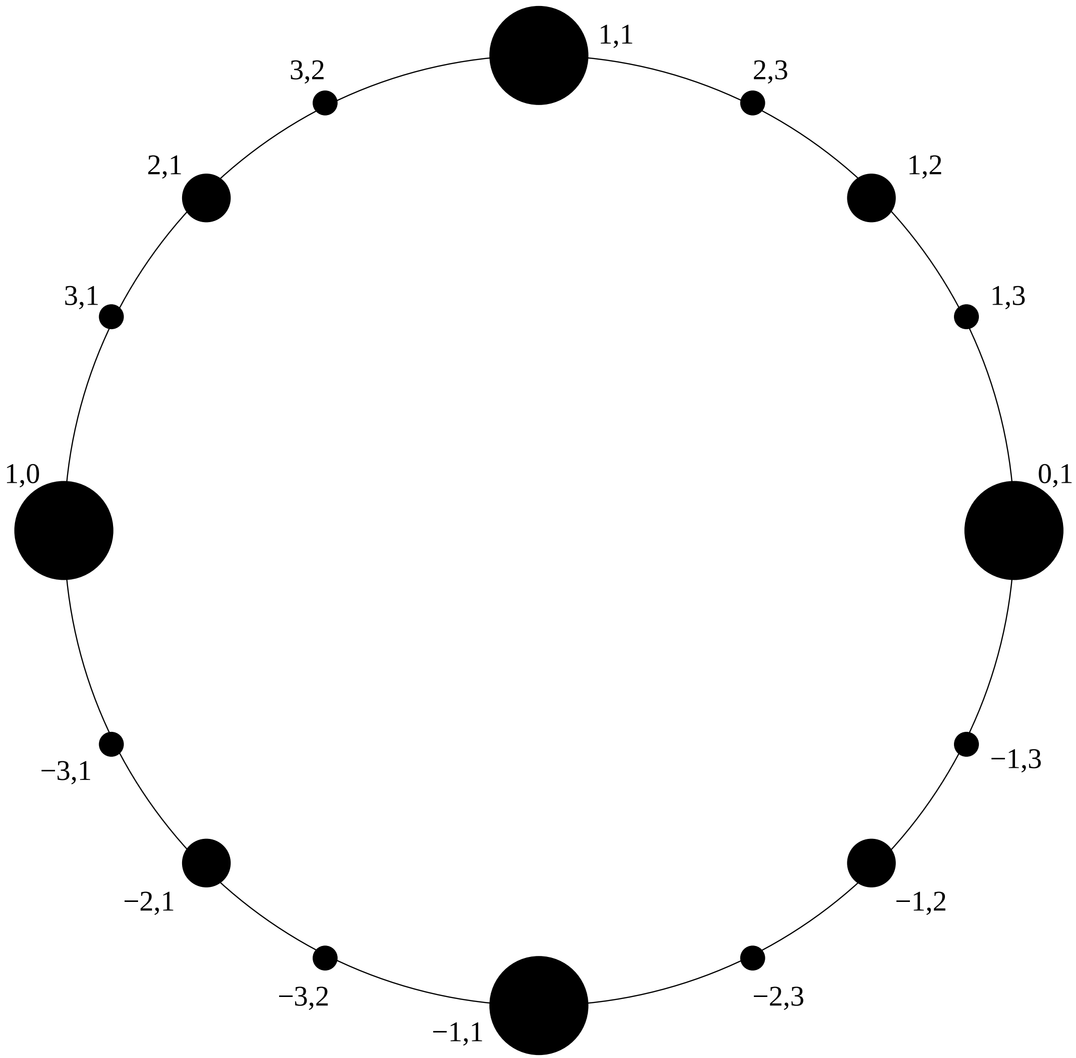
The partition of $PSD_2 \subset \mathbb{R}^3$

If $q(x, y) = \mathrm{u}x^2 + 2\mathrm{v}xy + \mathrm{w}y^2$ then $q \in PSD_2$ if and only if $\mathrm{v}^2 < \mathrm{uw}$ and $\mathrm{u} > 0$; we cut by the plane $\mathrm{u} + \mathrm{w} = 1$



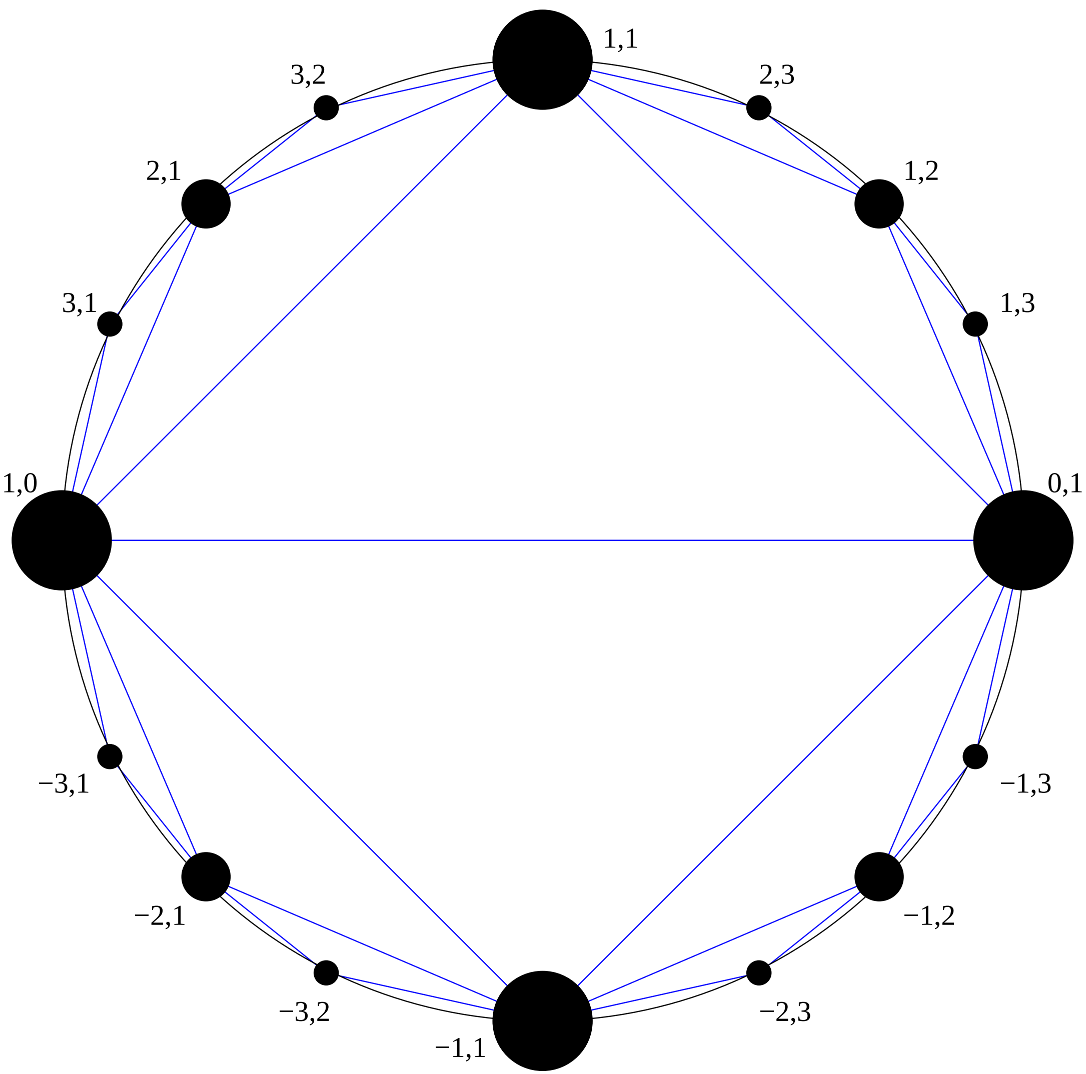
The partition of $PSD_2 \subset \mathbb{R}^3$

The group $GL_2(\mathbb{Z})$ transform the limit form x^2 into the forms $(ax+by)^2$ with $a, b \in \mathbb{Z}$



The partition of $PSD_2 \subset \mathbb{R}^3$

PSD_2 partition: Line: Voronoi polytope is rectangular.
Triangle: Voronoi polytope is hexagonal.



L-type domain

PSD_n=$\cup_i D_i$ with D_i open convex polyhedral cones called L-type domain such that

- the partition is invariant with respect to $GL_n(\mathbb{Z})$
- there are finitely many orbits (called combinatorial types)

Properties

- Two lattices in the same L-type domain can be continuously deformed without changing the combinatorial structure
- If $dim(D_i) = \binom{n+1}{2}$ then D_i is called primitive (its Delaunay polytope are simplices)
- If $dim(D_i) = 1$ then D_i is called rigid
- There exist non-simplicial L-type domain

Summary of results

dimension	1	2	3	4	5	6	7
Nr. Voronoi polytopes	1	2	5 Fedorov	52 DeSh	179377 Engel	?	?
Nr. primitive Voronoi	1	1	1 Fedorov	3 Delaunay	222 BaRy, Engel	$\geq 1.10^6$ Engel	?
Nr. rigid lattices	1	0	0	1	7 $\leftarrow$BaGr	$\geq 2.10^4$ DuVa	?
Nr. Delaunay polytopes	1	2	5 Fedorov	19 Erdahl Ryshkov	138 Kononenko	6241 Dutour	?
Nr. extreme Delaunay	1	0	0	0	0	1 DeDu	≥ 1

II. Delaunay polytopes and hypermetrics

Hypermetric inequalities

- If $b \in \mathbb{Z}^{n+1}$, $\sum_{i=0}^{n} b_i = 1$ then the hypermetric inequality is

$$H(b)d = \sum_{0 \le i < j \le n} b_i b_j d(i,j) \le 0$$

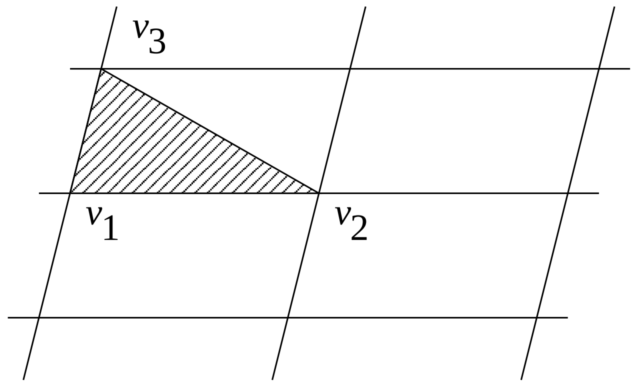
- If $b = (1, 1, -1, 0, \ldots, 0)$ then $H(b)$=triangular inequality
- The hypermetric cone HYP_{n+1} is the set of all d such that $H(b)d \le 0$ for all b
- $dim\, HYP_{n+1} = \binom{n+1}{2}$
- HYP_{n+1} is defined by an infinite set of inequalities

Delaunay polytopes

If $\mathcal{D}$ is an n dimensional Delaunay polytope with center c, radius r and vertices $\{v_0, \dots, v_N\}$ then $d(i,j) = \|v_i - v_j\|^2$ satisfies

$$\sum_{i,j} b_i b_j d(i,j) = 2(r^2 - \|\sum_i b_i v_i - c\|^2) \leq 0$$

i.e. Delaunay polytope $\Leftrightarrow$ hypermetrics
Moreover $\sum_i b_i v_i$ is a vertex of $\mathcal{D}$ if and only if $H(b)d = 0$



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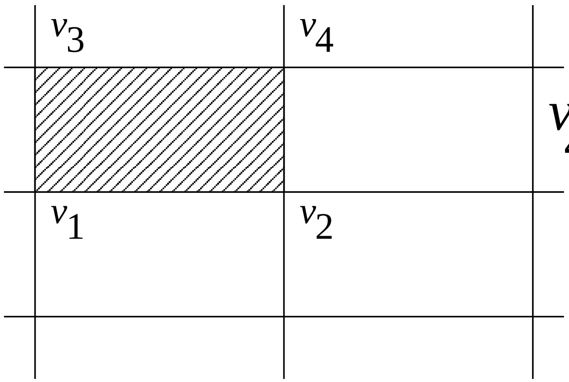
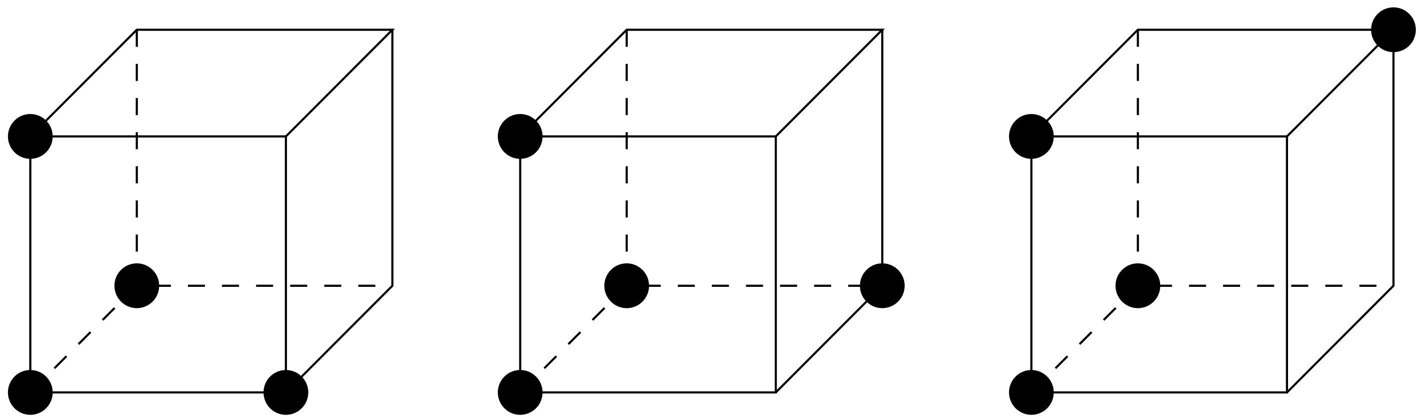


$v_4 = v_2 + v_3 - v_1$

Affine basis

An affine basis of an n-dimensional polytope P is $\{v_0, \dots, v_n\}$ such that for every vertex v of P, there is

$$b_i \in \mathbb{Z}, \text{such that } b_0 + \cdots + b_n = 1$$
$$\text{and } b_0 v_0 + b_1 v_1 + \cdots + b_n v_n = v$$



Baranovski-Ryshkov: every Delaunay polytope of dimension ≤ 6 has an affine basis

No Delaunay polytope without affine basis is known!

Polyhedrality of HYP_n

- HYP_n is polyhedral as union of L-type domain
- (Lovasz) if $H(b)$ defines a facet then $|b_i| \leq \frac{2^n}{\binom{2n}{n}} n!$

Combinatorial types of n-dimensional Delaunay polytope P correspond to faces F of HYP_{n+1}

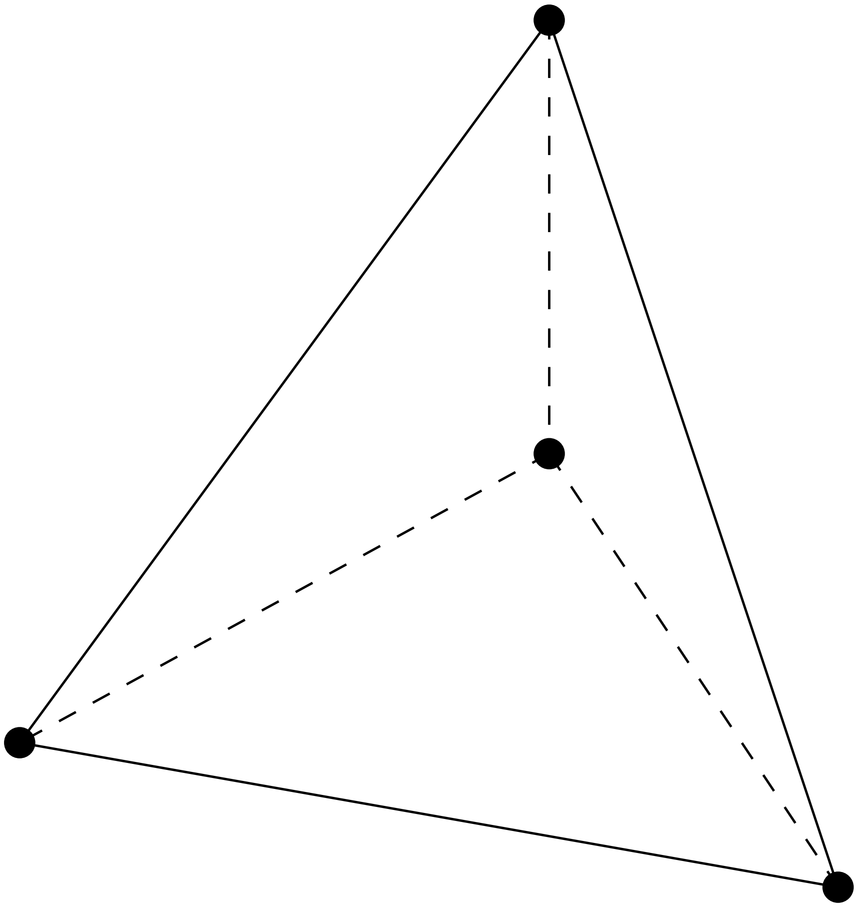
One defines $rank(P) = dim\, F$

$rank(P)$ is the number of degree of freedom

- $rank\,(P) = \binom{n+1}{2}$, then P is a simplex
- $rank\,(P) = 1$, then P is an extreme Delaunay polytope

We are interested in extreme Delaunay polytopes (their only degree of freedom is homotheties and rotations)

3-dimensional case

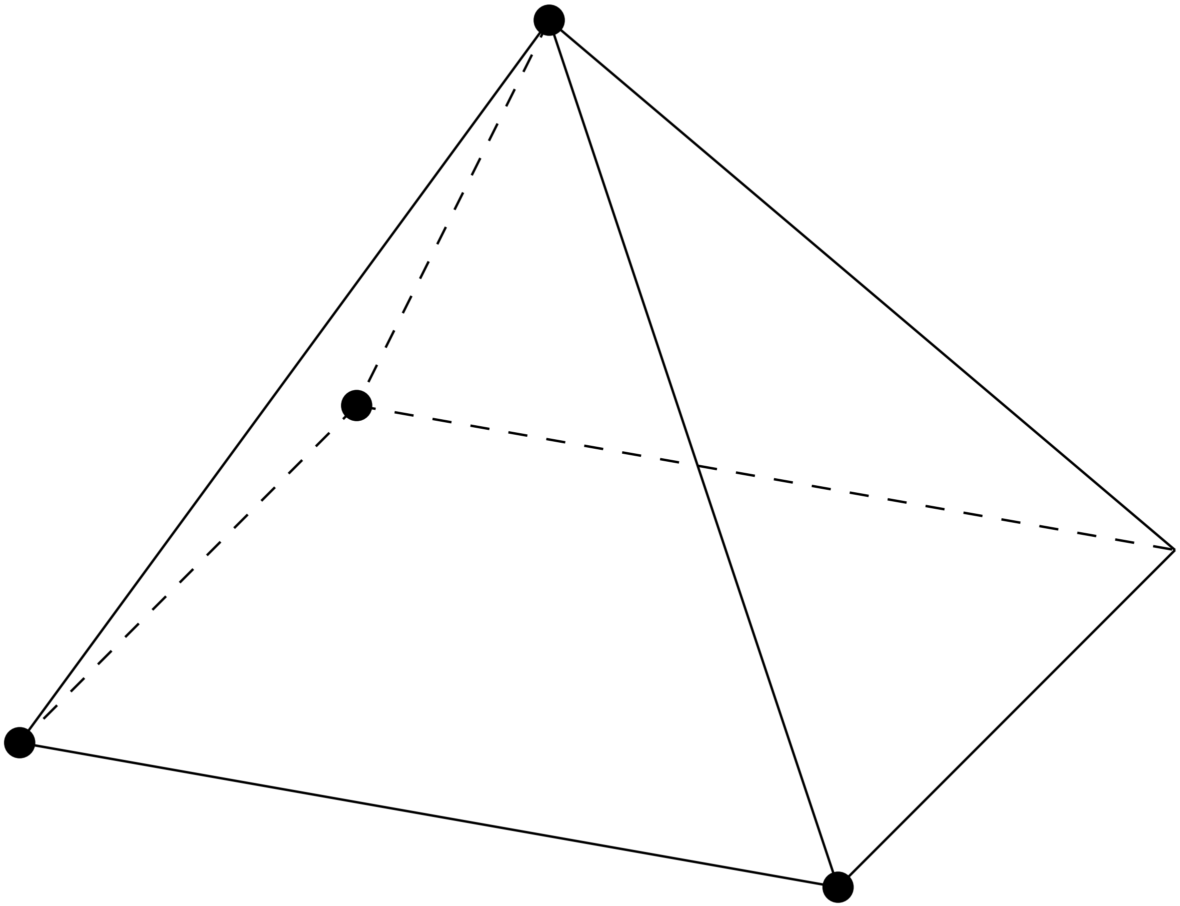


3−simplex

Hypermetric Vectors

Rank: 6

3-dimensional case



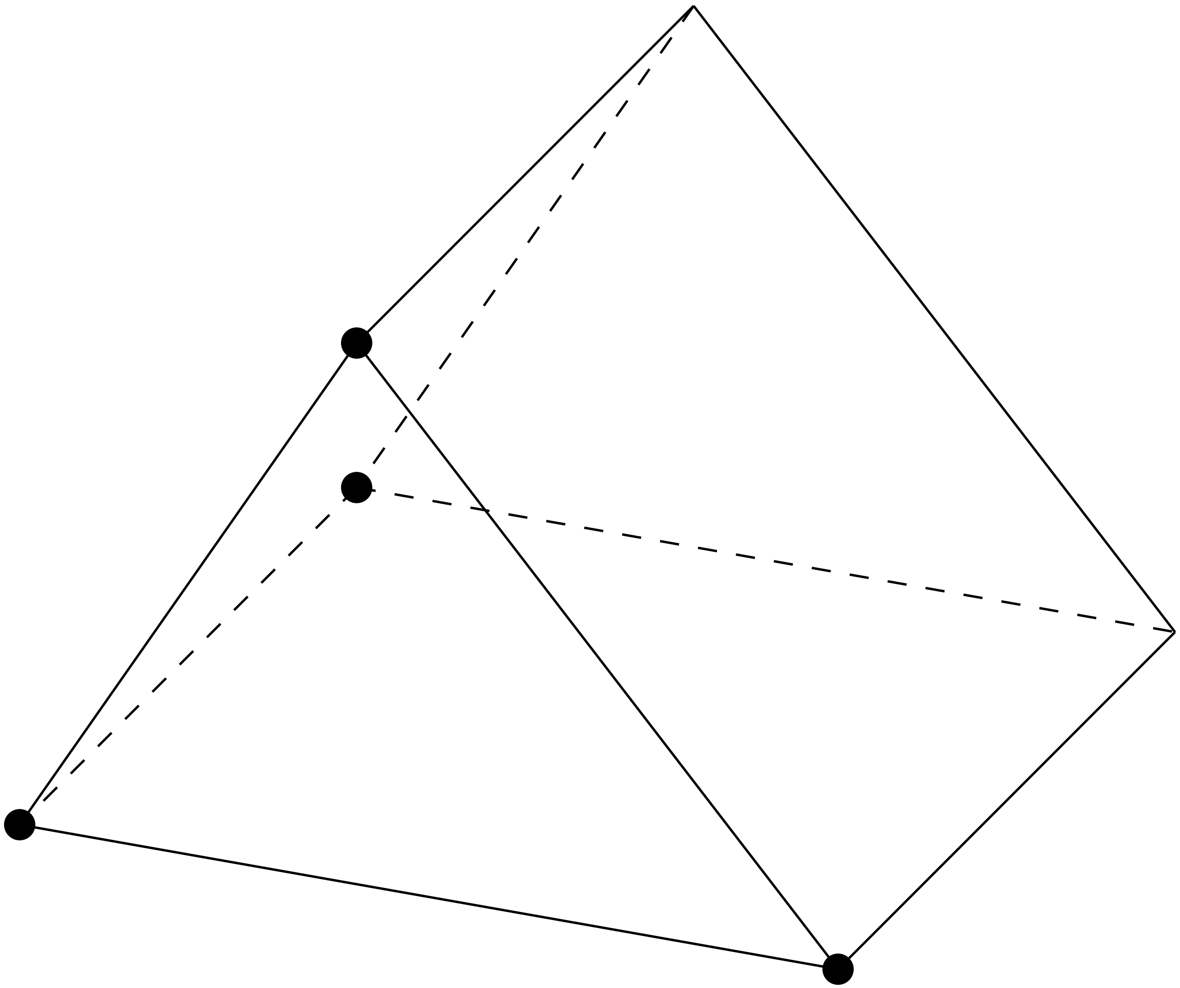
Pyramid

Hypermetric Vectors

$(-1\ ,\ 0\ ,\ 1\ ,\ 1)$

Rank: 5

3-dimensional case



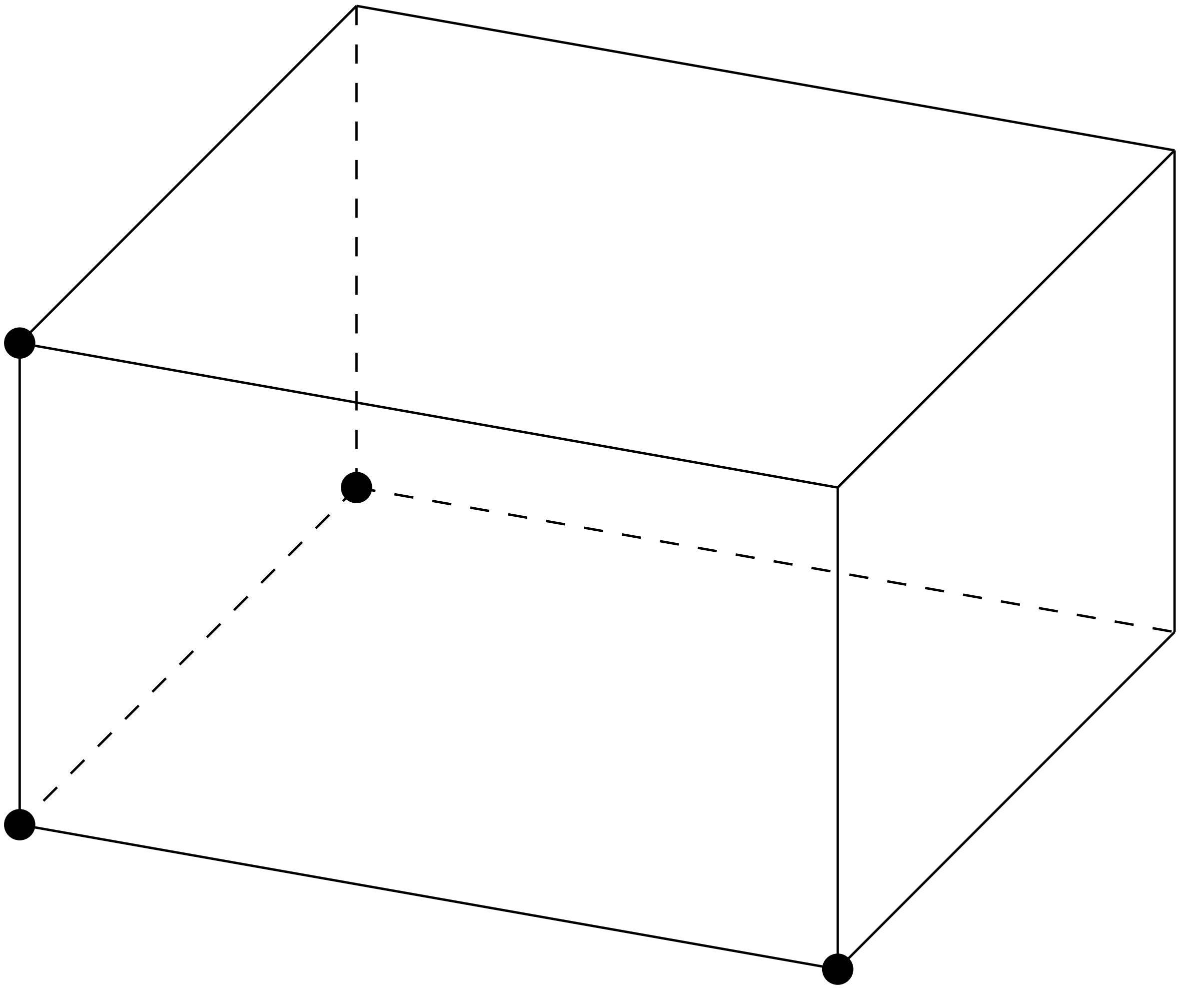
3−Prism

Hypermetric Vectors

(−1 , 0 , 1 , 1)
(−1 , 1 , 0 , 1)

Rank: 4

3-dimensional case



Cube

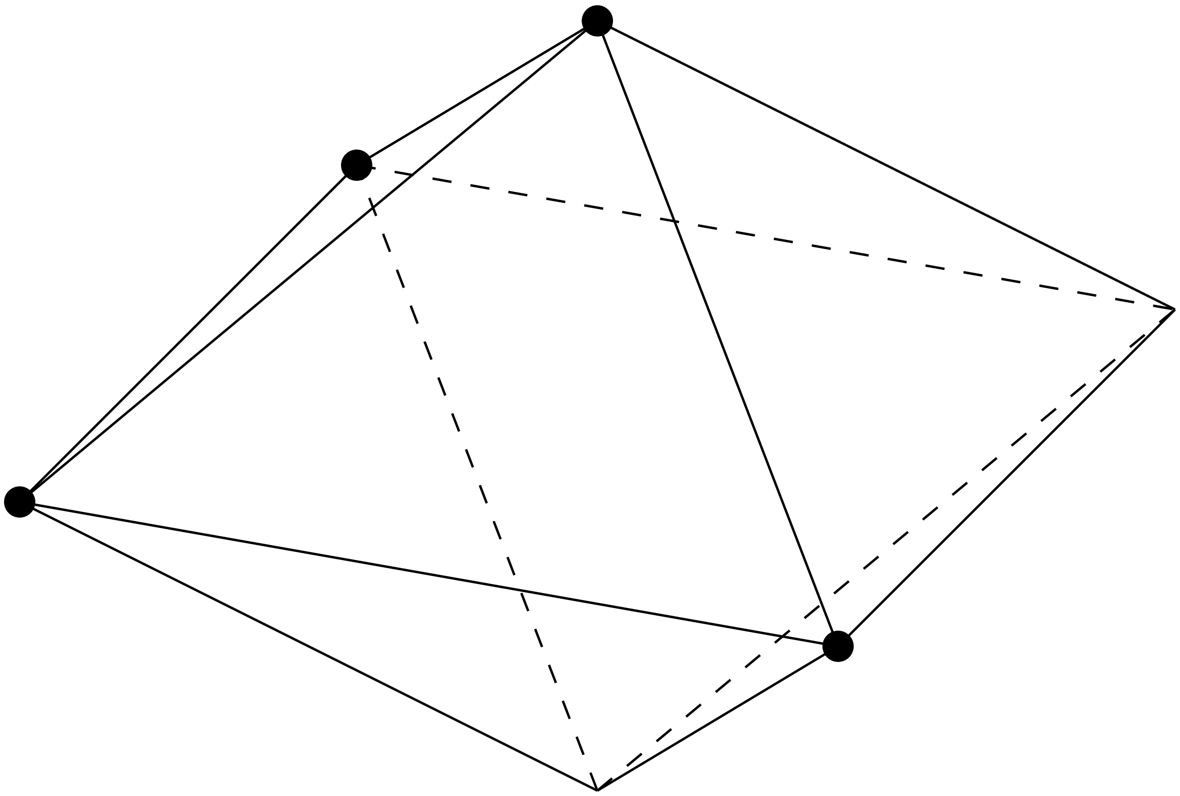
Hypermetric Vectors

$(-1\ ,\ 0\ ,\ 1\ ,\ 1)$
$(-1\ ,\ 1\ ,\ 0\ ,\ 1)$
$(-1\ ,\ 1\ ,\ 1\ ,\ 0)$
$(-2\ ,\ 1\ ,\ 1\ ,\ 1)$

Rank: 3

$$H(-2, 1, 1, 1) = H(-1, 0, 1, 1) + H(-1, 1, 0, 1) + H(-1, 1, 1, 0)$$

3-dimensional case



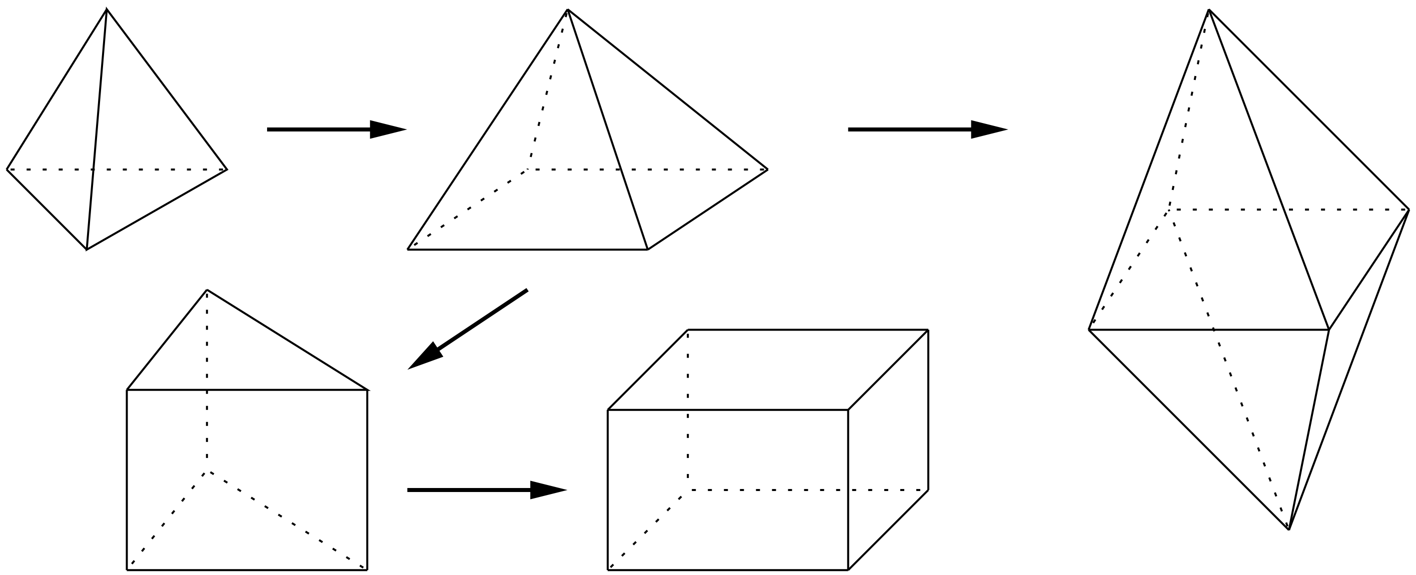
Octahedron

Hypermetric Vectors

$(-1\ ,\ 0\ ,\ 1\ ,\ 1)$
$(\ \ 0\ ,-1\ ,\ 1,\ 1)$

Rank: 4

Combinatorial types



dim	Nr of types		Computing time
2	2	Fedorov	
3	5	Fedorov	23s
4	19	Erdahl-Ryshkov	52s
5	138	Kononenko	5m
6	6241	Dutour	50h

III. The six-dimensional Delaunay polytopes

Cut cone

The cut-semi-metric δ_S on $n+1$ points can be interpreted as square distance on the one dimensional Delaunay polytope α_1 which is extreme

We denote CUT_{n+1} the cone generated by all δ_S

- $CUT_{n+1} \subset HYP_{n+1}$ for all n
- $CUT_{n+1} = HYP_{n+1}$ if $n \leq 5$
- $\Rightarrow$ no other extreme Delaunay polytope in dimension lower than 5
- But $CUT_7 \neq HYP_7 \Rightarrow$ there is an extreme six-dimensional Delaunay polytope

Facets of HYP_7 and CUT_7

Baranovski has found 14 orbits of facets of HYP_7
Method: direct proof that others are redundant
We have another proof of this result

(1, 1, -1, 0, 0, 0, 0)	(1, 1, 1, -1, -1, 0, 0)
(1, 1, 1, 1, -1, -2, 0)	(2, 1, 1, -1, -1, -1, 0)
(1, 1, 1, 1, -1, -1, -1)	(2, 2, 1, -1, -1, -1, -1)
(1, 1, 1, 1, 1, -2, -2)	(2, 1, 1, 1, -1, -1, -2)
(3, 1, 1, -1, -1, -1, -1)	(1, 1, 1, 1, 1, -1, -3)
(2, 2, 1, 1, -1, -1, -3)	(3, 1, 1, 1, -1, -2, -2)
(3, 2, 1, -1, -1, -1, -2)	(2, 1, 1, 1, 1, -2, -3)

First 10 orbits are also facet of CUT_7. CUT_7 has 36 orbits of facets, 26 of which are non-hypermetric.

The Schläfli polytope

Root lattices E_6 and E_8:

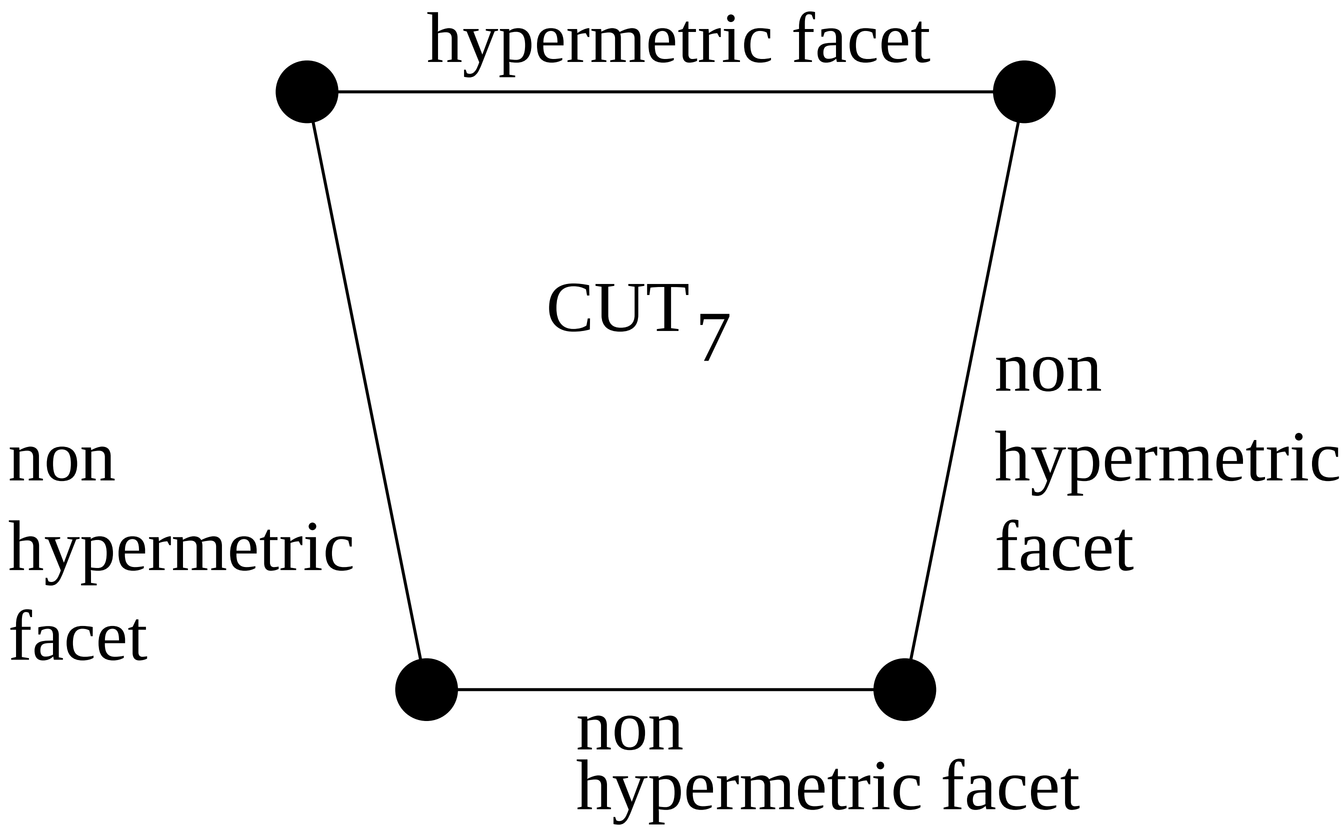
$$E_6 = \{x \in E_8 : x_1 + x_2 = x_3 + \cdots + x_8 = 0\}$$
$$E_8 = \{x \in \mathbb{Z}^8 \cup (\tfrac{1}{2} + \mathbb{Z})^8 \text{ and } \textstyle\sum_i x_i \in 2\mathbb{Z}\}$$

E_6 has unique Delaunay polytope called Schläfli polytope (which is identified to Schläfli graph)

- 27 vertices
- Symmetry group has size 51840 transitive on vertices
- Schläfli polytope is extreme
- 26 orbits of affine basis (DGL), which gives 26 orbits of extreme rays in HYP_7.

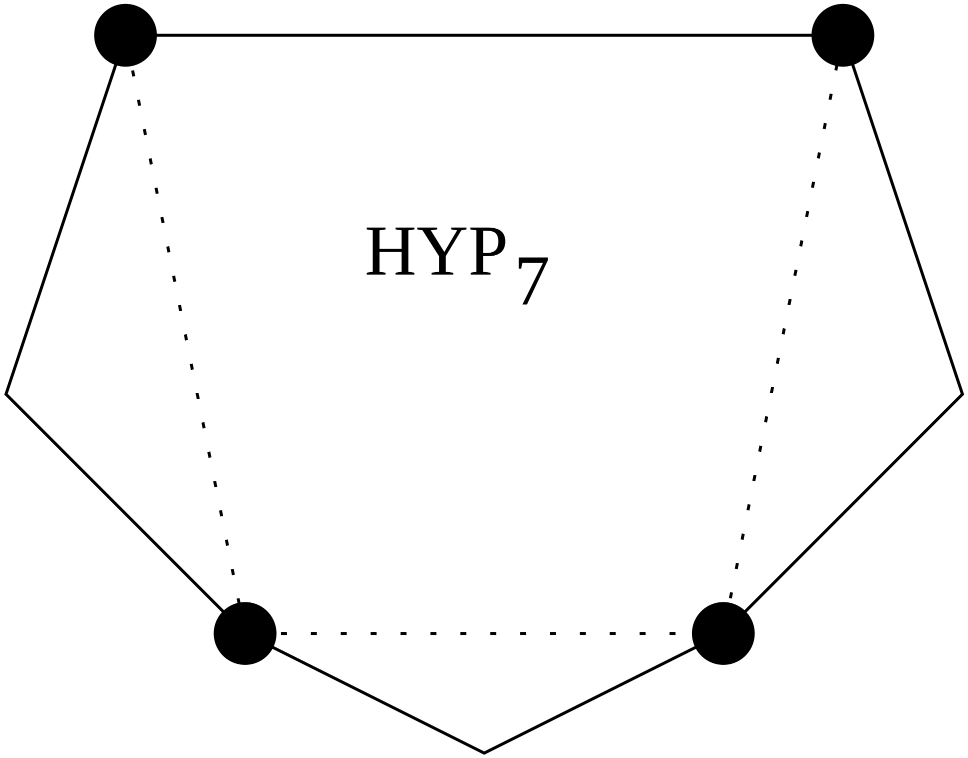
Extreme rays of HYP_7

The cone CUT_7 has hyp facet and non-hyp facet



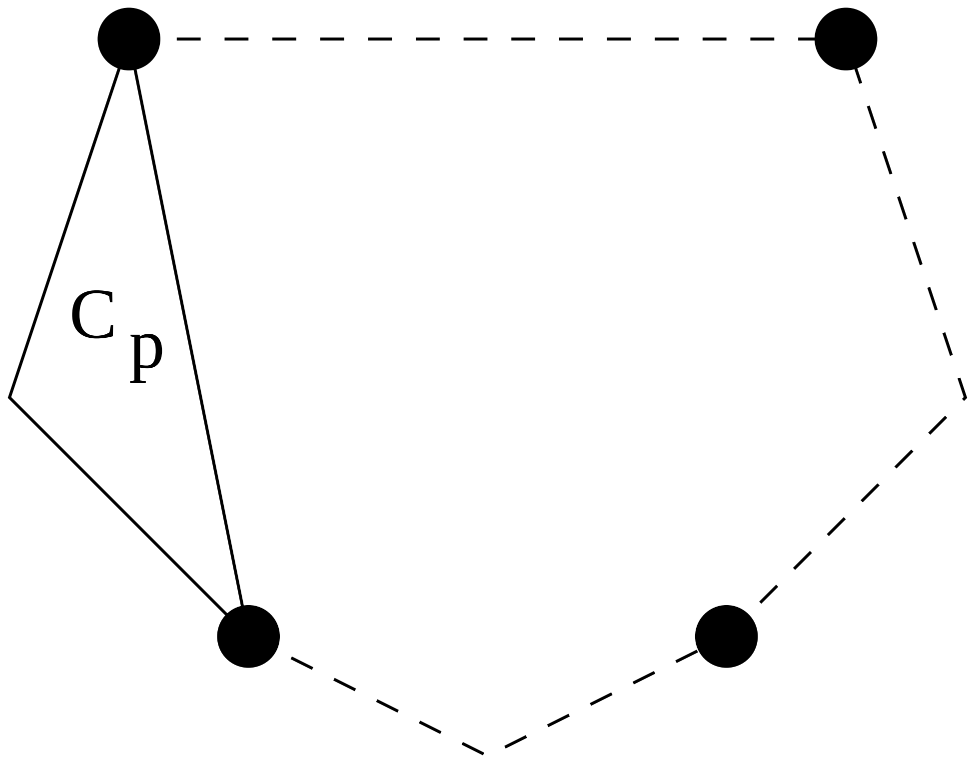
Extreme rays of HYP_7

The cone HYP_7 contains CUT_7



Extreme rays of HYP_7

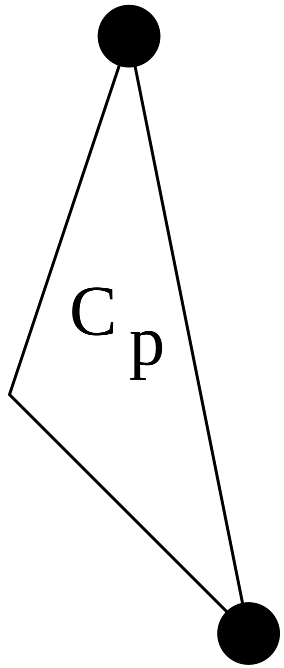
Take a non-hypermetric facet $p(x) \geq 0$ of CUT_7 and define



$$C_p = \{d \in HYP_7 \text{ such that } p(d) \leq 0\}$$

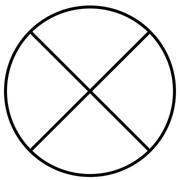
Extreme rays of HYP_7

Eliminate redundant inequalities by linear programming



Extreme rays of HYP_7

Find non-cut extreme ray (which is Schläfli)



6241 six-dimensional Delaunays

rank	Nr. in HYP_7	Nr. in CUT_7			
21	1(simplex)	0	11	686	325
20	9	1	10	417	183
19	30	2	9	218	83
18	95	8	8	108	35
17	233	28	7	52	13
16	500	95	6	21	3
15	814	241	5	8	0
14	1092	434	4	4	0
13	1145	527	3	2	0
12	984	481	2	1	0
			1	1(Schläfli)	0

Method of enumeration

Combinatorial type of rank R

↓

Intersect representatives with Facets of HYP_7

↓

Remove those which correspond to Delaunay of lower dimension

↓

Remove those that are isomorphic

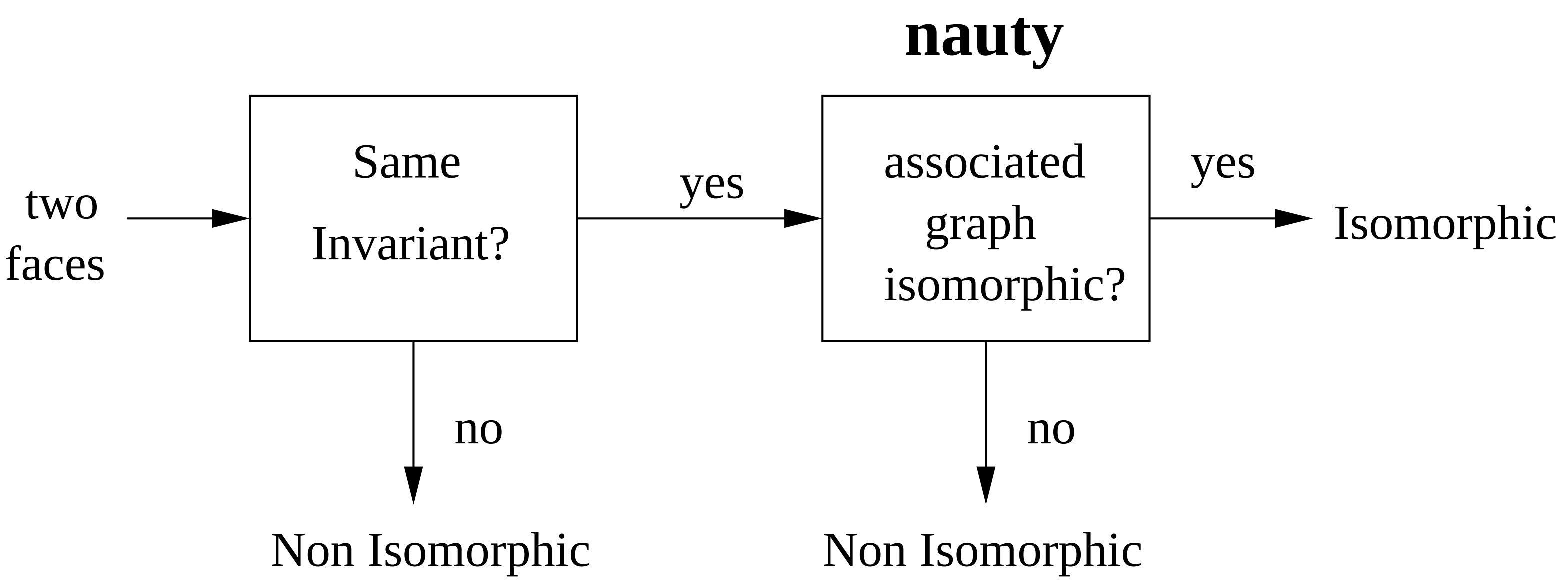
↓

Combinatorial types of rank R−1

Isomorphy test, general theory

Associate to each face

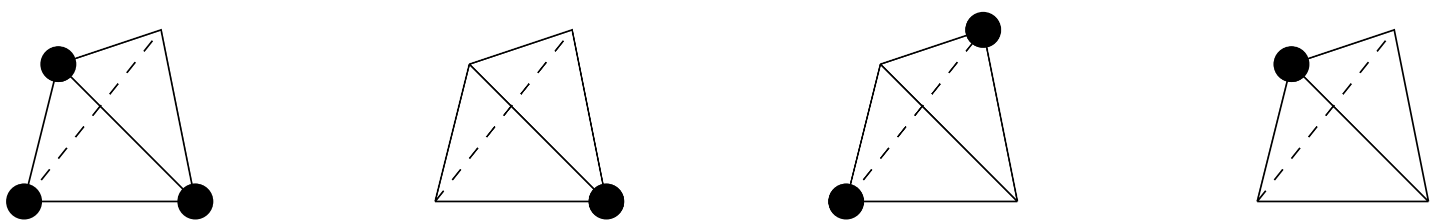
1. Some invariants
2. A graph that encode its combinatorial properties



Isomorphy test, specific methods

Let F a face of HYP_7:

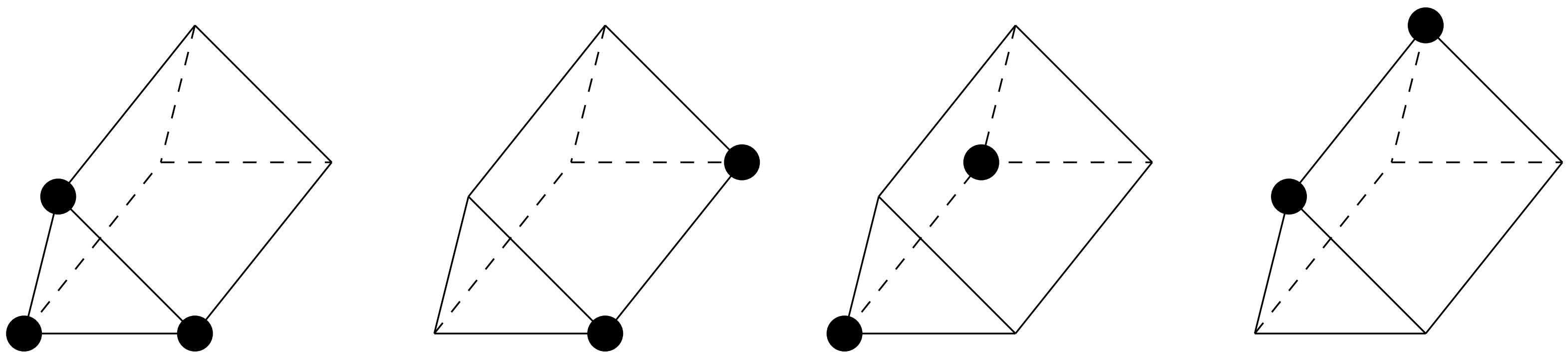
- If F contains a Schläfli extreme ray e_S.
  e_S correspond to an affine basis of Schläfli polytope and every hypermetric incidence $H(b)$ to another vertex in Schläfli polytope. $\Rightarrow F$ embedded as a subgraph of Schläfli graph.
- If F is generated by $\delta_{S_1}, \dots, \delta_{S_N}$.
  Every δ_{S_i} is extended as a cut on the set of vertices and the set of cutset is the combinatorial structure.



Isomorphy test, specific methods

Let F a face of HYP_7:

- If F contains a Schläfli extreme ray e_S.
  e_S correspond to an affine basis of Schläfli polytope and every hypermetric incidence $H(b)$ to another vertex in Schläfli polytope. $\Rightarrow F$ embedded as a subgraph of Schläfli graph.
- If F is generated by $\delta_{S_1}, \dots, \delta_{S_N}$.
  Every δ_{S_i} is extended as a cut on the set of vertices and the set of cutset is the combinatorial structure.



IV. Beyond dimension six

The known extreme Delaunay polytopes

Name	dimension	Nr. vertices	Equality	section of
Schläfli	6	27	yes	E_8
Gosset	7	56	no	E_8
	16	512	no	BarnesWall
B_{15}	15	135	yes	BarnesWall
	22	275	yes	Leech
	23	552	no	Leech

Extreme Delaunay polytope appear as section of higher dimensional lattices.

Computing methods

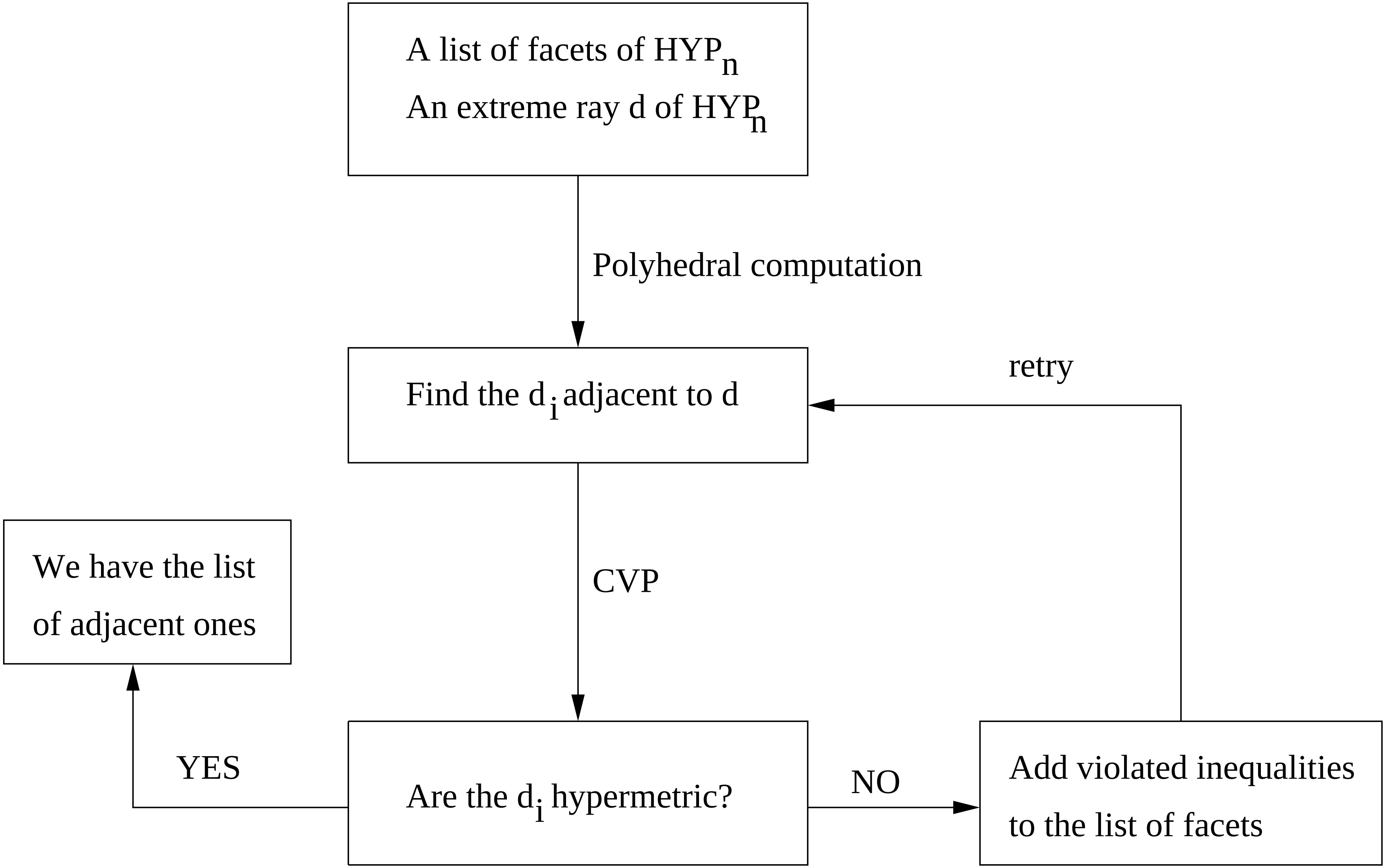
Given $d_{ij} = \|v_i - v_j\|^2$ a distance vector,

- One can compute the Gram matrix $\langle (v_i - v_0), (v_j - v_0) \rangle$
- Test if d is non-degenerate
- Compute the sphere $S(c, R)$ around the v_i
- $d \in HYP_{n+1}$ if and only if there is no b such that

$$\|b_0 v_0 + \cdots + b_n v_n - c\| < R$$

  (i.e. Closest Vector Problem)
- Find the b such that $H(b)d = 0$ is also a CVP

Bounding method



Lower bound

- Every incidence $H(b)d = 0$ correspond to a vertex $b_0 v_0 + \cdots + b_n v_n$ of a Delaunay polytope P
- The number N of vertices satisfies

$$\begin{array}{rcl} N & \geq & n + 1 + corank(P) \\ & \geq & n + 1 + \binom{n+1}{2} - rank(P) \end{array}$$

- Extreme Delaunay polytopes have at least $\binom{n+2}{2} - 1$ vertices
- If they have exactly $\binom{n+2}{2} - 1$ vertices, then the corresponding extreme ray of HYP_{n+1} are simplicial for which adjacency computation is easy

8-dimensional extreme Delaunay

B_{15} satisfies the equality bound.
We can compute its adjacent extreme rays: 77 of them correspond to a 8-dimensional extreme Delaunay polytope with f-vector

$$(79, 1268, 7896, 23520, 36456, 29876, 11364, 1131)$$

It has a symmetry group of size 322560 not transitive on vertices
There are three orbits of vertices:

- a vertex
- 64-vertices: the 7-half-cube
- 14 vertices: the 7-cross polytope

Infinite sequence of extreme Delaunay

- If n even, $n \geq 6$, there is a n-dimensional extreme Delaunay ED_n formed with 3 layers of D_{n-1} lattice
  - a vertex
  - the $n-1$ half-cube
  - the $n-1$ cross-polytope

  $n = 6$: Schläfli polytope
  $n = 8$: the 8-dimensional one
- If n odd, $n \geq 7$, there is a n-dimensional extreme Delaunay ED_n formed with 4 layers of previous lattice
  - a vertex
  - the ED_{n-1} extreme Delaunay
  - the ED_{n-1} extreme Delaunay
  - a vertex

  $n = 7$: Gosset polytope

Coordinates of M_n

Vertices of ED_n for n even in $\mathbb{R}^n$.

- a vertex

$$(\frac{1}{2}, \dots, \frac{1}{2}, \sqrt{\frac{n-2}{2}})$$

- the Half-Cube vectors

$$(x_1, \dots, x_{n-1}, 0)$$

with $\sum_{i=1}^{n-1} x_i$ even.

- the cross polytope vectors

$$(\frac{1}{2}, \dots, \frac{1}{2}, -\sqrt{\frac{n-2}{2}}) \pm e_i$$

with $1 \leq i \leq n-1$.

Thank

You