Parameter Space of Delaunay tessellations: L-types

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Gram matrix and lattices

- What really matters for lattice is their isometry class, i.e., if u is an isometry of $\mathbb{R}^n$ then the lattices L and uL have the same geometry.
- Denote $S^n_{>0}$ the cone of real symmetric positive definite $n \times n$ matrices and $S^n_{\geq 0}$ the positive semidefinite ones.
- Lattice L spanned by $v_1, \dots, v_n$ corresponds to

$$G_v = (\langle v_i, v_j \rangle)_{1 \leq i,j \leq n} \in S^n_{>0} \ .$$

  G_v depends only on the isometry class of L.
- Given $M \in S^n_{>0}$, one can find vectors $v_1, \dots, v_n$ such that $M = G_v$.

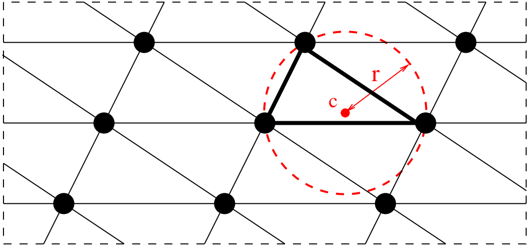
Empty sphere and Delaunay polytopes

A sphere $S(c, r)$ of radius r and center c in an n-dimensional lattice L is said to be an empty sphere if:

(i) $\|v - c\| \geq r$ for all $v \in L$,

(ii) the set $S(c, r) \cap L$ contains $n + 1$ affinely independent points.

A Delaunay polytope P in a lattice L is a polytope, whose vertex-set is $L \cap S(c, r)$.



Equalities and inequalities

- Take $M = G_v$ with $v = (v_1, \dots, v_n)$ a basis of lattice L.
- If $V = (w_1, \dots, w_N)$ with $w_i \in \mathbb{Z}^n$ are the vertices of a Delaunay polytope of empty sphere $S(c, r)$ then:

$$||w_i - c|| = r \quad \text{i.e.} \quad {w_i}^T M w_i - 2{w_i}^T Mc + c^T Mc = r^2$$

- Substracting one obtains

$$\{{w_i}^T M w_i - {w_j}^T M w_j\} - 2\{{w_i}^T - {w_j}^T\} Mc = 0$$

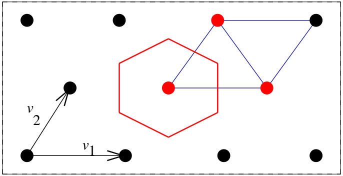
- Inverting matrices, one obtains $Mc = \psi(M)$ with ψ linear and so one gets linear equalities on M.
- Similarly $||w - c|| \geq r$ translates into linear inequalities on M: Take $V = (v_0, \dots, v_n)$ a simplex ($v_i \in \mathbb{Z}^n$), $w \in \mathbb{Z}^n$. If one writes $w = \sum_{i=0}^{n} \lambda_i v_i$ with $1 = \sum_{i=0}^{n} \lambda_i$, then one has

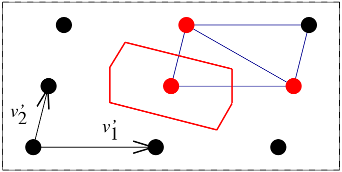
$$||w - c|| \geq r \Leftrightarrow w^T M w - \sum_{i=0}^{n} \lambda_i {v_i}^T M v_i \geq 0$$

III. L-type domain

L-type domains

- Take a lattice L and select a basis $v_1, \dots, v_n$.
- We want to assign the Delaunay polytopes of a lattice. Geometrically, this means that





are part of the same L-type domain.

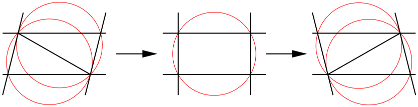
- A L-type domain is the assignment of Delaunay polytopes, so it is also the assignment of the Voronoi polytope of the lattice.

Primitive L-types

- If one takes a generic matrix M in $S^n_{>0}$, then all its Delaunay are simplices and so no linear equality are implied on M.
- Hence the corresponding L-type is of dimension $\frac{n(n+1)}{2}$, they are called primitive

Equivalence and enumeration

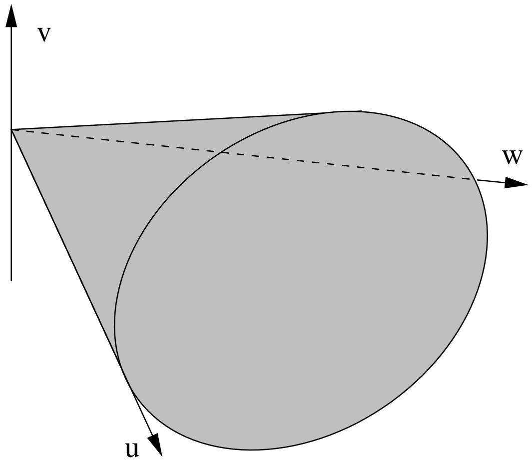
- Voronoi's theorem The inequalities obtained by taking adjacent simplices suffice to describe all inequalities.
- The group $\mathrm{GL}_n(\mathbb{Z})$ acts on $S^n_{>0}$ by arithmetic equivalence and preserve the primitive L-type domains.
- Voronoi proved that after this action, there is a finite number of primitive L-type domains.
- Bistellar flipping creates new triangulation. In dim. 2:



- Enumerating primitive L-types is done classicaly:
  - Find one primitive L-type domain.
  - Find the adjacent ones and reduce by arithmetic equivalence.

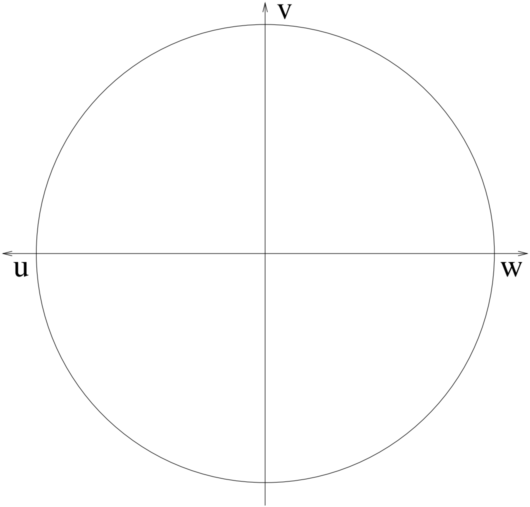
The partition of $S^2_{>0} \subset \mathbb{R}^3$

If $q(x, y) = \mathrm{u}x^2 + 2\mathrm{v}xy + \mathrm{w}y^2$ then $q \in S^2_{>0}$ if and only if $\mathrm{v}^2 < \mathrm{uw}$ and $\mathrm{u} > 0$.



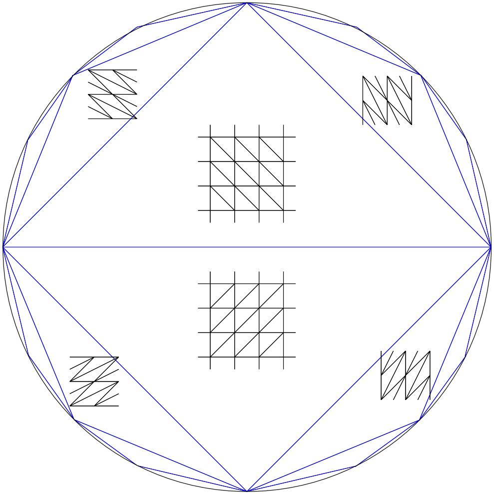
The partition of $S^2_{>0} \subset \mathbb{R}^3$

We cut by the plane $\mathrm{u} + \mathrm{w} = 1$ and get a circle representation.



The partition of $S^2_{>0} \subset \mathbb{R}^3$

Primitive L-types in $S^2_{>0}$:



Rigid lattices

A lattice is rigid (notion introduced by Baranovski & Grishukhin) if its L-type domain has dimension 1.

- One rigid in dimension 1: $\mathbb{Z}$.
- No rigid lattices in dimension 2 and 3.
- one rigid lattice in dimension 4: it is D_4.
- 7 rigid lattices in dimension 5.
  - ➠ E. Baranovskii, V. Grishukhin, *Non-rigidity degree of a lattice and rigid lattices*, European J. Combin. **22-7** (2001) 921–935.
- In dimension 6, we obtained 25263 rigid lattices. Probably many more.
  - ➠ M. Dutour and F. Vallentin, *Some six-dimensional rigid lattices*, Proceedings of "Third Voronoï Conference of the Number Theory and Spatial Tesselations", 102–108.

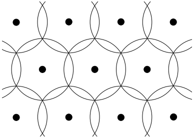
Enumeration of L-types

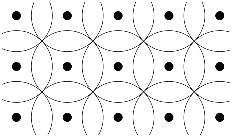
Dimension	Nr. L-type	Nr. primitive	Nr rigid lattices
1	1	1	1
2	2	1	0
3	5 Fedorov	1 Fedorov	0
4	52 DeSh	3 Voronoi	1
5	179377 Engel	222 BaRy, Engel & Gr	7 $\uparrow$ BaGr
6	?	$\geq 2.5.10^6$ Engel, Va	$\geq 2.10^4$ DuVa
7	?	?	?

IV. Covering and optimization

Lattice covering

- We consider covering of $\mathbb{R}^n$ by n-dimensional balls of the same radius, whose center belong to a lattice L.





- The covering density has the expression

$$\Theta(L) = \frac{\mu(L)^n \kappa_n}{det(L)} \geq 1$$

with $\mu(L)$ being the largest radius of Delaunay polytopes and κ_n the volume of the unit ball B^n.

- Objective is to minimize $\Theta(L)$. Solution for $n \leq 5$: A_n^*.

Lattice packing-covering

L is a n-dimensional lattice.

- We want a lattice, such that the sphere packing (resp, covering) obtained by taking spheres centered in L with maximal (resp, minimal) radius are both good.
- The quantity of interest is

$$\frac{\Theta(L)}{\alpha(L)} = \{\frac{\mu(L)}{\lambda(L)}\}^n \geq 1$$

- Lattice packing-covering problem: minimize $\frac{\Theta(L)}{\alpha(L)}$.

Dimension	Solution		
2	A_2^*	4	H_4 (Horváth lattice)
3	A_3^*	5	H_5 (Horváth lattice)

➠ J. Horváth, *PhD thesis: Several problems of n-dimensional geometry*, Steklov Inst. Math., 1986.

Optimization problem

We want to find the best covering, packing-covering.

- Thm. Given a L-type domain LT, there exist a unique lattice, which minimize the covering density over LT.
- Thm. Given a L-type domain LT, there exist a lattice (possibly several), which minimize the packing-covering density over LT.
- See for more details
  - A. Schürmann and F. Vallentin, *Computational approaches to lattice packing and covering problems*, Discrete Comput. Geom.**35-1** (2006) 73–116.

Radius of Delaunay polytope

- Fix a primitive L-type domain, i.e. a collection of simplexes as Delaunay polytopes D_1, ..., D_m.
- Thm. For every $D_i = Conv(0, v_1, \dots, v_n)$, the radius of the Delaunay polytope is at most 1 if and only if

$$\begin{pmatrix} 4 & \langle v_1, v_1\rangle & \langle v_2, v_2\rangle & \dots & \langle v_n, v_n\rangle \\ \langle v_1, v_1\rangle & \langle v_1, v_1\rangle & \langle v_1, v_2\rangle & \dots & \langle v_1, v_n\rangle \\ \langle v_2, v_2\rangle & \langle v_2, v_1\rangle & \langle v_2, v_2\rangle & \dots & \langle v_2, v_n\rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle v_n, v_n\rangle & \langle v_n, v_1\rangle & \langle v_n, v_2\rangle & \dots & \langle v_n, v_n\rangle \end{pmatrix} \in S^{n+1}_{\geq 0}$$

It is a semidefinite condition.

➠ B.N. Delone, N.P. Dolbilin, S.S. Ryškov and M.I. Stogrin, *A new construction of the theory of lattice coverings of an n-dimensional space by congruent balls*, Izv. Akad. Nauk SSSR Ser. Mat. **34** (1970) 289–298.

Convex programming problems

- A convex programming problem is of the following type

  Maximize g over $\mathcal{C}$ with g convex and $\mathcal{C}$ convex.
- For many subproblems, there are efficient procedures:
  - quadratically constrained quadratic programming,
  - geometrical programming,
  - approximation in l_p-norms,
  - optimization over the cone of positive semidefinite symmetric matrices,
  - finding extremal ellipsoids, etc.
- See for more details:
  - Y. Nesterov, A. Nemirovskii, *Interior-point polynomial algorithms in convex programming*, (1994) SIAM Studies in Applied Mathematics, 13.

Covering problem

- Fix a primitive L-type domain, i.e. a collection of simplexes as Delaunay polytopes $D_1, \dots, D_m$.
- Minkowski The function $-\log\det(M)$ is strictly convex on $S^n_{>0}$.
- Solve the problem
  - M in the L-type (linear condition),
  - the Delaunay D_i have radius at most 1 (semidefinite condition),
  - minimize $-\log\det(M)$ (strictly convex).
- The above problem is solved by the interior point methods implemented in MAXDET by Vandenberghe, Boyd & Wu.
- Unicity comes from the strict convexity of the objective function.

Packing covering problem

- We fix a primitive L-type domain.
- A shortest vector is an edge of a Delaunay. So, from the Delaunay decomposition, we know which vectors $v_1, \dots, v_p$ can be shortest.
- We consider the problem on $(M, m) \in S^n_{>0} \times \mathbb{R}$
  - M belong to the L-type domain (linear constraint)
  - all Delaunay have radius at most 1 (semidefinite condition)
  - $m \leq ||v_j||^2 = v_j^t M v_j$ for all i (linear constraint)
  - maximize m.
- The maximal value of m gives the maximal length of shortest vector and so the best packing-covering over a specific primitive L-type domain.
- A priori no unicity.

V. L-types of $S^n_{>0}$-spaces

$S^n_{>0}$-spaces

- A $S^n_{>0}$-space is a vector space $\mathcal{SP}$ of S^n, which intersect $S^n_{>0}$.
- We want to describe the Delaunay decomposition of matrices $M \in S^n_{>0} \cap \mathcal{SP}$.
- Motivations:
  - The enumeration of L-types is done up to dimension 5, perhaps possible for dimension 6 but certainly not for higher dimension.
  - We hope to find some good covering, and packing-covering by selecting judicious $\mathcal{SP}$. This is a search for best but unproven to be optimal coverings.
- A L-type in $\mathcal{SP}$ is an open convex polyhedral set included in $S^n_{>0} \cap \mathcal{SP}$, for which every element has the same Delaunay decomposition.

Rigidity and primitivity

- $(\mathcal{SP}, L)$-types form a polyhedral tessellation of the space $\mathcal{SP} \cap S^n_{>0}$.
- If $M \in \mathcal{SP} \cap S^n_{>0}$, then the rigidity degree of M is the dimension of the smallest L-type containing M, it is computed using the Delaunay decomposition of M.
- A $(\mathcal{SP}, L)$-type is primitive if it is full-dimensional in $\mathcal{SP}$.
- A $(\mathcal{SP}, L)$-type is rigid if it is one dimensional.
- Las Vegas algorithm: Algorithm for finding a primitive $(\mathcal{SP}, L)$-type domain
  - Generate a random element in $S^n_{>0} \cap \mathcal{SP}$.
  - Compute its Delaunay decomposition.
    - If the dimension of the $(\mathcal{SP}, L)$-type is maximal, then return it.
    - Otherwise, rerun.

Testing equivalence of $(\mathcal{SP}, L)$-type

- Given a primitive $(\mathcal{SP}, L)$-type domain $\mathcal{LT}$, find its extreme rays e_i and normalize the corresponding matrices by imposing that they have integer coefficients with $gcd = 1$.
- We associate to the $(\mathcal{SP}, L)$-type $\mathcal{LT}$ the matrix in $S^n_{>0} \cap \mathcal{SP}$: $M_{\mathcal{LT}} = \sum_i e_i$
- Two primitive $(\mathcal{SP}, L)$-type domains $\mathcal{LT}_1$ and $\mathcal{LT}_2$ are isomorphic if there a matrix P such that

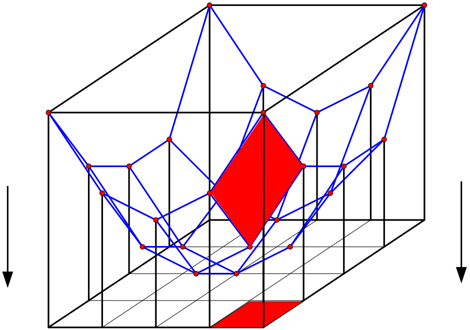
$$PM_{\mathcal{LT}_1}{}^tP = M_{\mathcal{LT}_2} \text{ and } P\mathcal{SP}^tP = \mathcal{SP}$$

First equation is solved by program Isom and we iterate over the possible solutions for testing the second.

Lifted Delaunay decomposition

- The Delaunay polytopes of a lattice L correspond to the facets of the convex cone $\mathcal{C}(L)$ with vertex-set:

$$\{(x, ||x||^2) \text{ with } x \in L\} \subset \mathbb{R}^{d+1} \ .$$

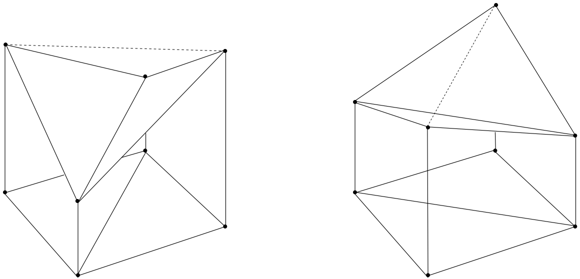


Flipping

- Take a primitive $(\mathcal{SP}, L)$-type domain with Delaunay polytopes $D_1, \ldots, D_p$.
- If F is a facet of D_i and D' is the other Delaunay polytope, then it defines an inequality $f_{D',D_i}(M) \geq 0$. This form a finite set of defining inequalities of the $(\mathcal{SP}, L)$-type.
- We can extract relevant inequalities, which correspond to facets of the $(\mathcal{SP}, L)$-type. Select a relevant ineq. $f(M) \geq 0$.
- One has $f(M) = \alpha_1 f_{D_{j(1)},D'_1}(M) = \cdots = \alpha_r f_{D_{j(r)},D'_r}(M)$ for some $\alpha_i > 0$ and some Delaunay D'_i adjacent to $D_{j(i)}$ on a facet F_i.
- If one moves to $f(M) = 0$, then all F_i disappear and the corresponding Delaunays merge.

Geometrical expression

- The “glued” Delaunay form a Delaunay decomposition for a matrix M in the $(\mathcal{SP}, L)$-type satisfying to $f(M) = 0$.
- The flipping break those Delaunays in a different way.
- Two triangulations of $\mathbb{Z}^2$ correpond in the lifting to:



- The polytope represented is called the repartitioning polytope.

Flipping of a repartitioning polytope

- Given a Delaunay decomposition D, the graph $G(D)$ is formed of all Delaunays with two Delaunay d_i, d_j adjacent if:
  - d_i and d_j share a facet
  - the inequality $f_{d_i,d_j}(M) = \alpha f(M)$ for $\alpha > 0$
- For every connected component C of this graph, the repartitioning polytope $R(P)$ is the polytope with vertex set

$$\{(v, {}^t vMv) \text{ with } v \text{ a vertex of a Delaunay of } C\}$$

- Combinatorially flipping correspond to switching from the lower facets to the higher facets of the lifted merging of Delaunay polytopes.

Enumeration technique

- Find a primitive $(\mathcal{SP}, L)$-type domain, insert it to the list as undone.
- Iterate
  - For every undone primitive $(\mathcal{SP}, L)$-type domain, compute the facets.
  - Eliminate redundant inequalities.
  - For every non-redundant inequality realize the flipping, i.e. compute the adjacent primitive $(\mathcal{SP}, L)$-type domain. If it is new, then add to the list as undone.

VI. Applications

Space of invariant forms

- Given a subgroup G of $\mathrm{GL}_n(\mathbb{Z})$, define

  $$\mathcal{SP}(G) = \{ \; X \in S^n \;\; \text{such} \;\; \text{that} \;\; gX^t g = X \;\; \text{for} \;\; \text{all} \;\; g \in G \; \}$$

- Given a $S^n_{>0}$-space $\mathcal{SP}$, define

  $$Aut(\mathcal{SP}) = \left\{ \begin{array}{c} g \in \mathrm{GL}_n(\mathbb{Z}) \;\; \text{such} \;\; \text{that} \\ gX^t g = X \;\; \text{for} \;\; \text{all} \;\; X \in \mathcal{SP} \end{array} \right\}$$

- A Bravais group satisfies to $Aut(\mathcal{SP}(G)) = G$.

Equivariant L-type domains

- Equivariant L-type domains are L-types of a $S^n_{>0}$-space $\mathcal{SP}(G)$ for G Bravais.
- Thm. (Zassenhaus) One has the equality

$$\{g \in \mathrm{GL}_n(\mathbb{Z}) \mid g\mathcal{SP}(G)^t g = \mathcal{SP}(G)\} = N_{\mathrm{GL}_n(\mathbb{Z})}(G)$$

- Thm. For a given finite group $G \in \mathrm{GL}_n(\mathbb{Z})$, there are a finite number of equivariant L-types under the action of $N_{\mathrm{GL}_n(\mathbb{Z})}(G)$.
- Note that if a T-space $\mathcal{SP}$ is defined by rational equations, then it does not necessarily have a finite number of L-types under $Aut(\mathcal{SP})$.
  Example (courtesy of Yves Benoist):

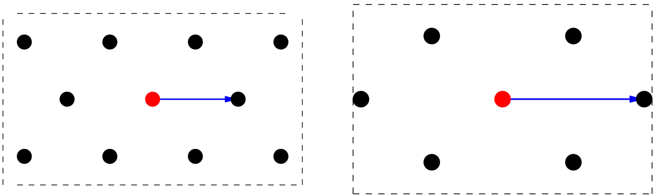
$$\mathcal{SP} = \mathbb{R}(x^2 + 2y^2 + z^2) + \mathbb{R}(xy)$$

Small dimensions

- Dimension 6:
  - Vallentin found a better lattice covering than A_6^* in the vicinity of E_6^*.
  - No better in Bravais groups of rank 4.
- Dimension 7:
  - Vallentin & Schürmann found a better lattice covering than A_7^* in the vicinity of E_7^*.
  - No better in Bravais groups of rank 4.
- Dimension 8:
  - Vallentin & Schürmann proved that E_8 is not a local optimum of the covering density.
    - ➠ A. Schürmann and F. Vallentin, *Local covering optimality of lattices: Leech lattice versus root lattice* E_8, Int. Math. Res. Not. 2005, no. 32, 1937–1955.
  - Conjecture (Zong) E_8 is the best lattice packing-covering in dimension 8.
    - ➠ C. Zong, *From deep holes to free planes*, Bull. Amer. Math. Soc. (N.S.) **39-4** (2002) 533–555.

Extension of Coxeter lattices

- Anzin & Baranovski computed the Delaunay decompositions of the lattices A_9^5, A_{11}^4, A_{13}^7, A_{14}^5, A_{15}^8 and found them to be better coverings than A_n^*.
- We do extension along short vectors

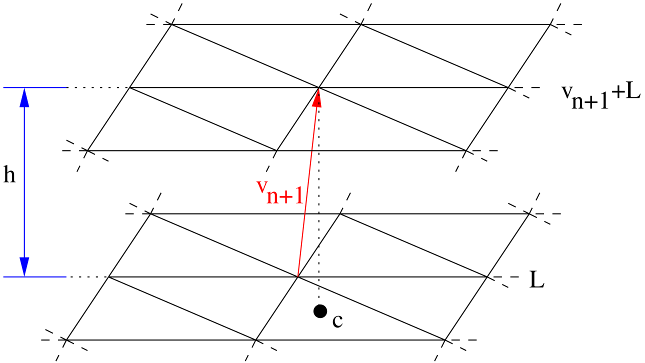


or compute in the Bravais space of short vectors.

- We manage to find record coverings in dimension 9, 11, 13, 14 and 15.

Lamination

- Given a n-dim. lattice L, create a $n+1$-dim. lattice L':



- The point c is fixed and we adjust the value of h.
- This defines a T-space.

Lamination

- In terms of Gram matrices
$$Gram(L) = A \text{ and } Gram(L') = \begin{pmatrix} A & A^t c \\ cA & \alpha \end{pmatrix}$$
c is the projection of the vector $(0, \dots, 0, 1)$ on the lattice L.
- The symmetries of L' are the symmetries of L preserving the center c and if $2c \in \mathbb{Z}^n$ the othogonal symmetry
$$\begin{pmatrix} I_n & 0 \\ 2c & -1 \end{pmatrix}$$
- c can be chosen as center of a Delaunay.
- For the covering problem things are not so simple.
  - One cannot solve the general problem with c unspecified, since it has no symmetry and too much parameters
  - One restriction is to assume the value of c, this makes a rank 2 problem.
- Doing lamination over A_9^5 and A_{11}^4 one gets a record covering in dimension 10 and 12.

Best known lattice coverings

d	lattice	covering density Θ			
1	$\mathbb{Z}^1$	1	13	L_{13}^c	7.762108
2	A_2^*	1.209199	14	L_{14}^c	8.825210
3	A_3^*	1.463505	15	L_{15}^c	11.004951
4	A_4^*	1.765529	16	A_{16}^*	15.310927
5	A_5^*	2.124286	17	A_{17}^9	12.357468
6	L_6^c	2.464801	18	A_{18}^*	21.840949
7	L_7^c	2.900024	19	A_{19}^{10}	21.229200
8	L_8^c	3.142202	20	A_{20}^7	20.366828
9	L_9^c	4.268575	21	A_{21}^{11}	27.773140
10	L_{10}^c	5.154463	22	Λ_{22}^*	$\leq$ 27.8839
11	L_{11}^c	5.505591	23	Λ_{23}^*	$\leq$ 15.3218
12	L_{12}^c	7.465518	24	*Leech*	7.903536

THANK

YOU