Covering maxima and Delaunay polytopes in lattices

Mathieu Dutour Sikirić

Rudjer Bošković Institute, Croatia

Konstantin Rybnikov

Massachussets, Lowell, USA

Achill Schuermann

TU Delft, Netherland

Frank Vallentin

TU Delft, Netherland

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I. Quadratic functions and the Erdahl cone

The Erdahl cone

- Denote by $E_2(n)$ the vector space of degree 2 polynomial functions on $\mathbb{R}^n$. We write $f \in E_2(n)$ in the form

$$f(x) = a_f + b_f \cdot x + Q_f[x]$$

  with $a_f \in \mathbb{R}$, $b_f \in \mathbb{R}^n$ and Q_f a $n \times n$ symmetric matrix
- The Erdahl cone is defined as

$$\mathit{Erdahl}(n) = \{f \in E_2(n) \text{ such that } f(x) \geq 0 \text{ for } x \in \mathbb{Z}^n\}$$

- It is a convex cone, which is non-polyhedral since defined by an infinity of inequalities.
- The group acting on $\mathit{Erdahl}(n)$ is $\mathrm{AGL}_n(\mathbb{Z})$, i.e. the group of affine integral transformations

$$x \mapsto b + xP \text{ for } b \in \mathbb{Z}^n \text{ and } P \in \mathrm{GL}_n(\mathbb{Z})$$

Scalar product

- Definition: If $f, g \in E_2(n)$, then:

$$\langle f, g\rangle = a_f a_g + \langle b_f, b_g\rangle + \langle Q_f, Q_g\rangle$$

- Definition: For $v \in \mathbb{Z}^n$, define $ev_v(x) = (1 + v \cdot x)^2$.
- We have

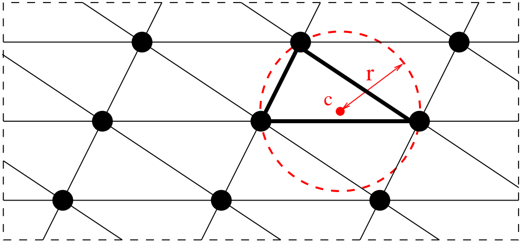
$$\langle f, ev_v\rangle = f(v)$$

- Thus finding the rays of $Erdahl(n)$ is a dual description problem with an infinity of inequalities and infinite group acting on it.
- Definition: We also define

$$Erdahl_{>0}(n) = \{f \in Erdahl(n) \ : \ Q_f \text{ positive definite}\}$$

Empty sphere and Delaunay polytopes

- Definition: A sphere $S(c, r)$ of center c and radius r in an n-dimensional lattice L is said to be an empty sphere if:
  (i) $\|v - c\| \geq r$ for all $v \in L$,
  (ii) the set $S(c, r) \cap L$ contains $n+1$ affinely independent points.
- Definition: A Delaunay polytope P in a lattice L is a polytope, whose vertex-set is $L \cap S(c, r)$.



- Delaunay polytopes define a tesselation of the Euclidean space.

Relation with Delaunay polytope

- If D is a Delaunay polytope of a lattice L of empty sphere $S(c, r)$ then for every basis $v_1, \dots, v_n$ of L we define the function

$$\begin{array}{rcl} f_D : \mathbb{Z}^n & \rightarrow & \mathbb{R} \\ x = (x_1, \dots, x_n) & \mapsto & \| \sum_{i=1}^n x_i v_i - c \|^2 - r^2 \end{array}$$

  Clearly $f_D \in Erdahl_{>0}(n)$.
- On the other hand if $f \in Erdahl(n)$ then there exist a lattice L' of dimension $k \leq n$, a Delaunay polytope D of L', a function $\phi \in \mathrm{AGL}_n(\mathbb{Z})$ such that

$$f \circ \phi(x_1, \dots, x_n) = f_D(x_1, \dots, x_k)$$

- Thus the faces of $Erdahl(n)$ correspond to the Delaunay polytope of dimension at most n.
- The rank of a Delaunay polytope is the dimension of the face it defines in $Erdahl(n)$.

Perfect quadratic function

- Definition: If $f \in Erdahl(n)$ then

$$Z(f) = \{v \in \mathbb{Z}^n : f(v) = 0\}$$

  and

$$\text{Dom } f = \sum_{v \in Z(f)} \mathbb{R}_+ ev_v$$

- We have $\langle f, \text{Dom } f \rangle = 0$.
- Definition: $f \in Erdahl(n)$ is perfect if Dom f is of dimension $\binom{n+2}{2} - 1$.
- This is equivalent to say that f defines an extreme ray in $Erdahl(n)$.
- The interval $[0, 1]$ is a perfect Delaunay polytope and for every $n \geq 6$ there exist an n-dimensional perfect Delaunay polytope.
- Geometrically, this means that the only affine transformations that preserve the Delaunay property are isometries and homotheties.

Eutacticity

- If $f \in Erdahl_{>0}(n)$ then define μ_f and c_f such that

$$f(x) = Q_f[x - c_f] - \mu_f$$

Then define

$$u_f(x) = (1 + c_f \cdot x)^2 + \frac{\mu_f}{n} Q_f^{-1}[x]$$

- Definition: $f \in Erdahl_{>0}(n)$ is eutactic if $u_f \in relint(\mathrm{Dom}\ f)$ and weakly eutactic if $u_f \in \mathrm{Dom}\ f$.
- Definition: Take a Delaunay polytope P for a quadratic form Q of center c_P and square radius μ_P. P is called eutactic if there are $\alpha_v > 0$ so that

$$\left\{\begin{array}{rcl} 1 & = & \sum_{v \in \mathrm{vert}\, P} \alpha_v, \\ 0 & = & \sum_{v \in \mathrm{vert}\, P} \alpha_v (v - c_P), \\ \frac{\mu_P}{n} Q^{-1} & = & \sum_{v \in \mathrm{vert}\, P} \alpha_v (v - c_P)(v - c_P)^t. \end{array}\right.$$

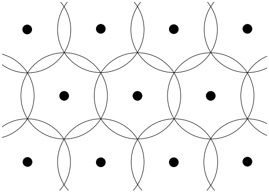
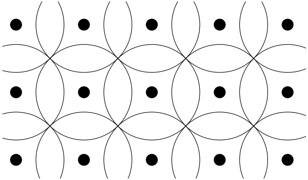
Strongly perfect Delaunay polytopes

- Definition: We say that a finite, nonempty subset X in $\mathbb{R}^n$ carries a spherical t-design if there is a similarity transformation mapping X to points on the unit sphere S^{n-1} so that
$$\frac{1}{|Y|}\sum_{y\in Y} f(y) = \frac{1}{\omega_n}\int_{S^{n-1}} f(y)d\omega(y).$$
holds, for all polynomials $f \in \mathbb{R}[x_1, \dots, x_n]$ up to degree t.
- If the vertices of P form a 2-design, then they define an eutactic form.
- Definition: A Delaunay polytope P is called strongly perfect if its vertex set form a 4-design.
- Theorem: A strongly perfect Delaunay polytope P is perfect.
- Proof: If $f(v) = 0$ for v a vertex of P then
$$\frac{1}{\omega_n}\int_{S^{n-1}} f(x)^2 dx = \frac{1}{|\operatorname{vert} P|}\sum_{v\in \operatorname{vert} P} f(v)^2 = 0$$

II. Covering maxima and characterization

Lattice covering

- We consider covering of $\mathbb{R}^n$ by n-dimensional balls of the same radius, whose center belong to a lattice L.

- The covering density has the expression

$$\Theta(L) = \frac{\mu(L)^n \kappa_n}{\det(L)} \geq 1$$

  with $\mu(L)$ being the largest radius of Delaunay polytopes and κ_n the volume of the unit ball B^n.
- The only general method for computing $\mu(L)$ is to compute all Delaunay polytopes of L.

Covering maxima and pessima

- Definition: If $f \in Erdahl_{>0}(n)$ then we define the inhomogeneous Hermite minimum by

$$H(f) = \mu_f (\det\ Q_f)^{-1/n}$$

- We then have if Q is a quadratic form for L

$$\begin{array}{rcl} \mu(L)^2 & = & \max\{\mu_f : f \in Erdahl(n) \text{ and } Q_f = Q\} \\ \Theta(L) & = & \kappa_n \max\{H(f)^{n/2} : f \in Erdahl(n) \text{ with } Q_f = Q\} \end{array}$$

- Definition: A lattice L is said to be a covering maximum if for every lattice L', which is near L we have $\Theta(L) > \Theta(L')$.
- Definition: A lattice L is said to be a covering pessimum if for every lattice L', which is near L we have $\Theta(L) > \Theta(L')$ except in a set of measure 0.
- Theorem: We have equivalence between:
  - L is a covering maximum (resp. pessimum).
  - For every $f \in Erdahl(n)$ with $\mu_f = Q$ and $\mu(L)^2 = \mu_f$ the function $H(f)$ is a covering maximum (resp. pessimum)

Convexity results

- Theorem: The function $f \mapsto \mu_f$ is convex on $Erdahl(n)$.
- Proof: Write f as

$$f(x) = Q_f[x - c_f] - \mu_f$$

  and we have

$$-\mu_f = \min_{x \in \mathbb{R}^n} f(x)$$

  which implies the result.
- Theorem (Minkovski): The function $f \mapsto d_f = (\det Q_f)^{-1/n}$ is convex on $Erdahl(n)$.
- But the product $f \mapsto H(f) = \mu_f d_f$ is not convex.

Characterization of covering maxima

- Theorem: Take $f \in Erdahl_{>0}(n)$. The following are equivalent:
  - f is a covering maximum for H.
  - f is perfect and eutactic.
- Proof: Suppose that f is perfect and eutactic. We have the differentiation formula:

$$H(f + \delta f) = H(f) - \frac{1}{(\det\ Q_f)^{-1/n}} \langle u_f, \delta f \rangle + o(\delta f)$$

  We also have

$$\langle u_f, \delta f \rangle = \sum_{v \in Z(f)} \alpha_v (\delta f)(v)$$

  with $\alpha_v > 0$. If δf is not collinear to f then there exists $v \in Z(f)$ with $(\delta f)(v) > 0$. Thus H decreases in direction δf.
- The reverse implication follows by preceding convexity results.

Characterization of covering pessima

- Theorem: Take $f \in Erdahl_{>0}(n)$. The following are equivalent:
  - f is a covering pessimum for H
  - f is weakly eutactic
- Proof: Suppose that f is weakly eutactic. We have the differentiation formula:

$$H(f+\delta f) = H(f) - \frac{1}{(\det\ Q_f)^{-1/n}} \langle u_f, \delta f \rangle + o(\delta f)$$

  We also have

$$\langle u_f, \delta f \rangle = \sum_{v \in Z(f)} \alpha_v (\delta f)(v)$$

  with $\alpha_v \geq 0$. If δf is not collinear to f and chosen outside of a set of measure 0 then $(\delta f)(v) > 0$ for all v and $\langle u_f, \delta f \rangle > 0$.
- The reverse implication follows by preceding convexity results.

Topological Morse function property

- Call $Symm(n)$ the space of $n \times n$ symmetric matrices.
- Theorem: Take a lattice L and assume that all the Delaunay polytopes of maximal circumradius are eutactic. The following are equivalent:
  - L is a non-degenerate topological Morse point for Θ
  - There exist a proper subspace SP_L of $Symm(n)$ such that for every $f_0 \in Erdahl(n)$ with $Q_{f_0} = Q_0$ and $\mu(L)^2 = \mu_{f_0}$ we have

$$\left\{ \begin{array}{c} Q \in Symm(n) : \exists f \in Erdahl(n), \\ Q_f = Q \text{ and } Z(f) = Z(f_0) \end{array} \right\} \subset SP_L$$

- Theorem: The covering density function $Q \mapsto \Theta(Q)$ is a topological Morse function if and only if $n \leq 3$.
- Proof: The lattice D_4 has three translation classes of Delaunay polytopes with the subspaces having zero intersection and their sum spanning $Symm(4)$.

III. Examples

Known perfect Delaunay polytopes

- In dimension 6, there is only Gosset's 2_{21}, which defines the Delaunay polytopes of E_6.
- No classification in higher dimension.
  - For $n = 7$, conjectured only two cases:

Lattice	# vertices	# facets	$\vert$ Sym $\vert$
E_7	56	702	2903040 (Gosset's 3_{21})
$\mathsf{E}R_7$	35	228	1440

  - For $n = 8$, a conjectured list of 27 possibilities.
  - For $n = 9$, at least 100000 perfect Delaunay polytopes.
- Some infinite series were built in
  - M. Dutour, *Infinite serie of extreme Delaunay polytopes*, European J. Combin. **26** (2005) 129–132.
  - V.P. Grishukhin, *Infinite series of extreme Delaunay polytopes*, European J. Combin. **27** (2006) 481–495.
  - R. Erdahl and K. Rybnikov, *An infinite series of perfect quadratic forms and big Delaunay simplices in $\mathbb{Z}^n$*, Proc. Steklov Inst. Math. **239** (2002) 159–167.

Example of covering maxima and minima

- The following lattices are covering maxima:

name	design strength	# vertices	# orbits Delaunay polytopes
E_6	4	27	1
E_7	5	56	2
ER_7	0	35	4
O_{10}	3	160	6
BW_{16}	5	512	4
O_{23}	7	94208	5
Λ_{23}	7	47104	709

- The following lattices are covering pessima:

name	design strength	# vertices	# orbits Delaunay polytopes
$\mathbb{Z}^n$	3	2^n	1
D_n	3	2^{n-1}	2 (or 1)
E_6^*	2	9	1
E_7^*	3	16	1
E_8	3	16	2
K_{12}	3	81	4

- Proof method: Take L a lattice
  - Enumerate all Delaunay polytopes of L up to symmetry
  - Select the ones of maximum circumradius.
  - Check for perfection (linear system problem)
  - check for eutacticity, resp. weakly eutacticity (linear programming problem)

Infinite sequence of covering maxima

- We need to find explicit lattice.
- If n even, $n \geq 6$, there is a n-dimensional extreme Delaunay ED_n formed with 3 layers of D_{n-1} lattice
  - a vertex
  - the $n-1$ half-cube
  - the $n-1$ cross-polytope
- If n odd, $n \geq 7$, there is a n-dimensional extreme Delaunay ED_n formed with 3 layers of D_{n-1} lattice
  - the $n-1$ cross polytope
  - the $n-1$ half-cube
  - the $n-1$ cross polytope
- ED_6 is Gosset's 2_{21} and ED_7 is Gosset's 3_{21}.
- For a Delaunay polytope P, we define $L(P)$ to be the smallest lattice containing P.
- Theorem: The lattice $L(ED_n)$ spanned by ED_n is a covering maxima.
- We need the full list of orbits of Delaunay of $L(ED_n)$.

Proof method

- Suppose that we have p orbits $O_1, \dots, O_p$ of Delaunay polytopes $P_1, \dots, P_p$ then

$$1 \geq \sum_{i=1}^{p} |O_i| \operatorname{vol} P_i.$$

  If equality then the list $O_1, \dots, O_p$ is complete.

- Take P is a polytope $c \in P$ a point invariant under Aut P. If $F_1, \dots, F_r$ are r orbits $Orb_1, \dots, Orb_r$ of facets of P then we have

$$\operatorname{vol} P \geq \sum_{i=1}^{r} |Orb_i| \operatorname{vol}(\operatorname{conv}(F_i, c))$$

  If equality then the list $F_1, \dots F_r$ is complete.

- We get a conjectured list of orbits of Delaunay polytope of $L(ED_n)$, of facets of ED_n and prove their completude by the volume formula.

Non generating perfect Delaunay polytopes

- Theorem: There exist perfect Delaunay polytopes P in lattices L with $L(P) \neq L$.
- Proof:
  - Take $2 \leq s$ and $4s \leq n$ and define the n-polytope

$$J(n,s) = \{x \in \{0,1\}^{n+1} \text{ such that } \sum_{i=1}^{n+1} x_i = s\}$$

  - Define the vector $v_{n,s} = \left(\left(\frac{1}{4}\right)^{4s}, 0^{n+1-4s}\right)$.
  - $P(n,s) = J(n,s) \cup 2v_{n,s} - J(n,s)$ is a perfect n-dimensional Delaunay polytope in $L_{n,s} = L(P(n,s))$.
  - Fact: If

$$6s < \begin{cases} n+1 & \text{if } n \text{ is odd,} \\ n & \text{if } n \text{ is even.} \end{cases}$$

    then there exist a superlattice $L'_{n,s}$ of $L_{n,s}$ in which $P_{n,s}$ is still a Delaunay polytope.

IV. A Voronoi algorithm?

Non local-polyhedrality

- In the theory of perfect forms, when one applies the Voronoi algorithm we deal only with the quadratic form of dimension n and not the ones of lower dimension.
- A perfect quadratic form has a finite number of neighbors. Enumerating them is a dual description problem.
- But if $f \in Erdahl(n)$ we can only conclude that Q_f is positive semidefinite. We can have

  $$Z(f) = \phi(\text{vert}\, D \times \mathbb{Z}^{n-k}) \text{ with } \phi \in \text{AGL}_n(\mathbb{Z})$$

  with D a k-dimensional Delaunay polytope and $k \leq n$. $Erdahl(n)$ contains the description of all the Delaunay polytopes of dimension at most n.
- Thus $Erdahl(n)$ is not locally polyhedral.

Cellular structure

- Suppose that D_1, D_2, D_3 are Delaunay polyhedra with
  - the inclusions $\operatorname{vert} D_1 \subset \operatorname{vert} D_2 \subset \operatorname{vert} D_3$
  - The rank conditions

  $$\operatorname{rank} D_1 = 2 + \operatorname{rank} D_3 \text{ and } \operatorname{rank} D_2 = 1 + \operatorname{rank} D_3$$

  Then there exist a unique Delaunay polyhedra D_2' with

  $$\operatorname{vert} D_1 \subset \operatorname{vert} D_2' \subset \operatorname{vert} D_3 \text{ and } D_2 \neq D_2'$$

- For every $1 \leq k \leq \frac{(n+1)(n+2)}{2}$ there are faces of dimension k of *Erdahl*(n).
- So, the Erdahl cone behaves like a polytope.

Recursive adjacency decomposition

We adapt the recursive adjacency decomposition for polytopes.

- For a Delaunay polyhedron D, we define $Sub(D)$ the orbits of Delaunay polytopes contained in D and of rank $1 + \text{rank } D$.
- The method is iterative
  - If $\text{vert } D$ is finite (i.e. D a polytope) then we can compute directly $Sub(D)$.
  - If D is not finite, then we find a subpolytope D' with $\text{vert } D' \subset \text{vert } D$ then by computing $Sub(D')$ we can find the Delaunay adjacent to D' in $Sub(D)$

Implementation done but:

- Insufficient speed
- So far works in dimension 6.
- Need for Balinski type theorem to avoid some computation.