ETH Zürich

Quantum Simulations
of Gauge Theories

High Energy Physics Master Program

Mathieu FEREY

Pure $\mathbb{Z}_2$ Gauge Theory
and the Transverse Field Ising Model

Herbstsemester 2023

Lecturers

Prof. Dr. Marina Krstic Marinkovic

Dr. Joao Carlos Pinto Barros

Institut für Theoretische Physik

ETH Zürich

1 Introduction

The Ising gauge theory, theory with a discrete $\mathbb{Z}_2$ gauge symmetry in $2+1$-dimensions

$$H_{\mathbb{Z}_2} = -g \sum_{\vec{x},j} \sigma_j^x(\vec{x}) - \frac{1}{g} \sum_{\vec{x}} \sigma_1^z(\vec{x})\sigma_2^z(\vec{x}+e_1)\sigma_1^z(\vec{x}+e_2)\sigma_2^z(\vec{x}), \tag{1}$$

where $\vec{x}$ refers to a position on the lattice, $j = 1, 2$ the two possible directions of a link, $\sigma_j^{x/z}(\vec{x})$ are the Pauli matrices, living on the links of the lattice. The local operator

$$Q(\vec{x}) \equiv \sigma_1^x(\vec{x})\sigma_1^x(\vec{x}-e_1)\sigma_2^x(\vec{x})\sigma_2^x(\vec{x}-e_2) \tag{2}$$

commutes with $H_{\mathbb{Z}_2}$. It generates local gauge transformations. One can check that $Q^2 = \mathbb{1}$, so that the local symmetry of our problem is indeed $\mathbb{Z}_2$. The operator

$$\tau^z(\vec{r}) = \prod_{(\vec{x},j) \text{ pierced by } \gamma(\vec{r})} \sigma_j^x(\vec{x}), \tag{3}$$

called the magnetic charge, is a gauge invariant quantity. γ is an open path on the dual lattice. Since

$$\left\{W_p^2(\vec{r}), \tau_z^2(\vec{r})\right\} = 0 \text{ and } W_p^2(\vec{r}) = \tau_z^2(\vec{r}) = \mathbb{1}, \tag{4}$$

one can identify W_p with the Pauli matrice τ^x on the dual lattice.

$$H = -\sum_{i=1}^{N} \tau_i^z \tau_{i+1}^z - g \sum \tau_j^x. \tag{5}$$

where i, j run over the dual lattice sites.
Map it to a $d+1$-dim classical anisotropic Ising Model

$$H_{\text{class}} = -\frac{N_y \gamma}{\beta} \sum_{i=1}^{N} \sum_{j=1}^{N_y} \sigma_z^{(i,j)} \sigma_z^{(i,j+1)} - \sum_{i=1}^{N} \sum_{j=1}^{N_y} \sigma_z^{(i,j)} \sigma_z^{(i+1,j)}, \tag{6}$$

with $\gamma = -\frac{1}{2}\log\tanh a$, $a = \frac{-\beta g}{N_y}$.
In general

$$H = -J \sum_{i,j} S_i^z S_j^z - \Gamma \sum_i S_i^x, \tag{7}$$

can be mapped to

$$H_{\text{eff}}(M) = -\sum_{k=1}^{M} \left[\frac{K}{M} \sum_{i,j} S_{ik} S_{jk} + K_M \sum_i S_{ik} S_{ik+1} \right], \tag{8}$$

with $K_M = \dfrac{1}{2}\ln\left(\coth\left(\beta\Gamma/M\right)\right)$ and $K = J\beta$.

2 MCMC-MH

Metropolis-Hasting algorithm

1. Select initial value θ_0.
2. For $i \in \{1, \cdots, N_{\text{sample}}\}$:
   - Draw candidate θ^* from proposal distribution $q(\theta^*|\theta_{i-1})$.
   - Compute $\alpha = \dfrac{g(\theta^*)}{g(\theta_{i-1})} \dfrac{q(\theta_{i-1}|\theta^*)}{q(\theta^*|\theta_{i-1})}$.
   - If $\alpha \geq 1$ accept θ^* by setting $\theta_i \leftarrow \theta^*$,
     If $0 < \alpha < 1$ accept θ^* with probability α.