ETH Zürich

Quantum Simulations
of Gauge Theories

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Pure $\mathbb{Z}_2$ Gauge Theory
and the Transverse Field Ising Model

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1 Introduction

2 Theoretical background

2.1 The $\mathbb{Z}_2$ gauge theory and its Quantum Ising dual

The Ising gauge theory, theory with a discrete $\mathbb{Z}_2$ gauge symmetry in $2+1$-dimensions [1]

$$H_{\mathbb{Z}_2} = -g \sum_{\vec{x},j} \sigma_j^x(\vec{x}) - \frac{1}{g} \sum_{\vec{x}} \sigma_1^z(\vec{x}) \sigma_2^z(\vec{x}+e_1) \sigma_1^z(\vec{x}+e_2) \sigma_2^z(\vec{x}), \tag{1}$$

where $\vec{x}$ refers to a position on the lattice, $j = 1, 2$ the two possible directions of a link, $\sigma_j^{x/z}(\vec{x})$ are the Pauli matrices, living on the links of the lattice. The local operator

$$Q(\vec{x}) \equiv \sigma_1^x(\vec{x}) \sigma_1^x(\vec{x}-e_1) \sigma_2^x(\vec{x}) \sigma_2^x(\vec{x}-e_2) \tag{2}$$

commutes with $H_{\mathbb{Z}_2}$. It generates local gauge transformations. One can check that $Q^2 = \mathbb{1}$, so that the local symmetry of our problem is indeed $\mathbb{Z}_2$. The operator

$$\tau^z(\vec{r}) = \prod_{(\vec{x},j) \text{ pierced by } \gamma(\vec{r})} \sigma_j^x(\vec{x}), \tag{3}$$

called the magnetic charge, is a gauge invariant quantity. γ is an open path on the dual lattice. Since

$$\left\{W_p^2(\vec{r}), \tau_z^2(\vec{r})\right\} = 0 \text{ and } W_p^2(\vec{r}) = \tau_z^2(\vec{r}) = \mathbb{1}, \tag{4}$$

one can identify W_p with the Pauli matrice τ^x on the dual lattice.

$$H = -\sum_{i=1}^{N} \tau_i^z \tau_{i+1}^z - g \sum \tau_j^x. \tag{5}$$

where i, j run over the dual lattice sites.

2.2 The classical mapping of the Quantum Ising Model

Our transverse Ising Model is not diagonal in the eigen-basis of S^z and requires a quantum treatment. Luckily for us, a combination of clever tricks allows us to map the transverse field Ising Hamiltonian in d dimensions to a classical anisotropic Ising Hamiltonian in $d+1$ dimensions [2]. Let us demonstrate this in the case of the $1-D$ Quantum Ising model for simplicity. Its Hamiltonian simply reads

$$H = -J \sum_{i=1}^{N} S_i^z S_{i+1}^z - \Gamma \sum_{i=1}^{N} S_i^x. \tag{6}$$

The thermodynamical properties of this system can all be derived through its partition function $Z = \operatorname{Tr} \exp(-\beta H)$, with $\beta = 1/k_B T$. Everything starts with the Trotter formula [3]:

$$\exp(A_1 + A_2) = \lim_{M \to \infty} \left[\exp(A_1/M) \exp(A_2/M)\right]^M. \tag{7}$$

Writing our Hamiltonian as $H = H_0 + V$, where H_0 is the spin-spin interaction and V the action of the magnetic field, one can expand the partition function as follows (keeping the large M limit implicit for neatness):

$$
\begin{aligned}
Z &= \mathrm{Tr} e^{-\beta(H_0+V)} \\
&= \mathrm{Tr}\left[e^{-\beta H_0/M} e^{-\beta V/M}\right]^M \\
&= \sum_{\{S_1^1,\cdots,S_N^1\}} \langle S_1^1,\cdots,S_N^1| \, e^{-\beta H_0/M} e^{-\beta V/M} \cdots \\
&\qquad\qquad\qquad \times e^{-\beta H_0/M} e^{-\beta V/M} \, |S_1^1,\cdots,S_N^1\rangle \,,
\end{aligned}
$$

where $|S_i^1\rangle = |\pm 1\rangle_i$ is an eigenstates of S_i^z. The sum runs over all possible configurations for the lattice. Now, between each pair of exponential we can insert the identity in the form

$$
\mathbb{1} = \sum_{\{S_1^k,\cdots,S_N^k\}} |S_1^k,\cdots,S_N^k\rangle \langle S_1^k,\cdots,S_N^k| \,. \tag{8}
$$

The k index, which simply labels one complete set of eigenstates of S_z, can be regarded as an additional dimension to our lattice (often referred to as the Trotter dimension). The spin S_i^k can be understood the spin living on the (i,k) site of a 2 dimensional lattice. Then

$$
Z = \sum_{\{S\}} \prod_{k=1}^{M} \langle S_1^k,\cdots,S_N^k| \, e^{-\beta H_0/M} e^{-\beta V/M} \, |S_1^{k+1},\cdots,S_N^{k+1}\rangle \,, \tag{9}
$$

where we introduced periodic boundary conditions in the Trotter dimension, $|S_1^{M+1},\cdots,S_N^{M+1}\rangle = |S_1^1,\cdots,S_N^1\rangle$ so as to match the braket sandwiching of the trace. We have use the shorthand $\sum_{\{S\}} = \sum_{\{S_1^1,\cdots,S_N^1\}} \cdots \sum_{\{S_1^M,\cdots,S_N^M\}}$, which is just a sum over all possible configurations of our 2D lattice. Now, the H_0 exponential is diagonal in the $|S_1^k,\cdots,S_N^k\rangle$ basis, so that, using its hermiticity

$$
\langle S_1^k,\cdots,S_N^k| \, e^{-\beta H_0/M} = \exp\left[\frac{\beta J}{M}\sum_{i=1}^{N} S_i^k S_{i+1}^k\right] \langle S_1^k,\cdots,S_N^k| \,. \tag{10}
$$

The second exponential now reads

$$
\begin{aligned}
\langle S_1^k,\cdots,S_N^k| \, e^{-\beta V/M} \, |S_1^{k+1},\cdots,S_N^{k+1}\rangle &= \langle S_1^k,\cdots,S_N^k| \exp\left[\frac{\beta\Gamma}{M}\sum_{i=1}^{N} S_i^x\right] |S_1^{k+1},\cdots,S_N^{k+1}\rangle \\
&= \prod_{i=1}^{N} \langle S_i^k| \, e^{\beta\Gamma S_i^x/M} \, |S_i^{k+1}\rangle \,.
\end{aligned}
$$

Here comes a trick, keeping in mind that S^x just flips the spin and that $(S^x)^2 = \mathbb{1}$:

$$
\begin{aligned}
\langle S| \, e^{aS^x} \, |S'\rangle &= \langle S| \left(\sum_{n=0}^{\infty} \frac{a^{2n}(S^x)^{2n}}{n!} + \sum_{n=0}^{\infty} \frac{a^{2n+1}(S^x)^{2n+1}}{n!}\right) |S'\rangle \\
&= \cosh a \, \langle S, S'\rangle + \sinh a \, \langle S| \, S^x \, |S'\rangle \\
&= \cosh a \, \langle S, S'\rangle + \sinh a \, \langle S, -S'\rangle \\
&= \begin{cases} \cosh a \text{ if } S = S', \\ \sinh a \text{ if } S = -S'. \end{cases}
\end{aligned}
$$

Now, simply note that

$$\left(\frac{1}{2}\sinh 2a\right)^{1/2}\exp\left[\frac{SS'}{2}\ln\coth a\right] = \sqrt{\sinh a\cosh a}\times\begin{cases}\exp\left[\frac{1}{2}\ln\coth a\right] & \text{if } S=S'\\ \exp\left[-\frac{1}{2}\ln\coth a\right] & \text{if } S=-S'\end{cases}$$
$$= \sqrt{\sinh a\cosh a}\times\begin{cases}\sqrt{\cosh a/\sinh a} \text{ if } S=S'\\ \sqrt{\sinh a/\cosh a} \text{ if } S=-S'\end{cases}$$
$$= \begin{cases}\cosh a \text{ if } S=S'\\ \sinh a \text{ if } S=-S'\end{cases}$$
$$= \langle S|\, e^{aS^x}\, |S'\rangle$$

This was a long road, but we can finally put everything together.

$$\begin{aligned}Z &= \sum_{\{S\}}\prod_{k=1}^{M}\prod_{i=1}^{N}\exp\left[\frac{\beta J}{M}S_i^kS_{i+1}^k\right]\left(\frac{1}{2}\sinh 2\beta\Gamma/M\right)^{1/2}\exp\left[\frac{S_i^kS_i^{k+1}}{2}\ln\coth(\beta\Gamma/M)\right]\\ &= \left(\frac{1}{2}\sinh(2\beta\Gamma/M)\right)^{MN/2}\sum_{\{S\}}\exp\left[-\frac{\beta}{M}\left(-J\sum_{i=1}^{N}\sum_{k=1}^{M}S_i^kS_{i+1}^k-\frac{M}{2\beta}\ln\coth\left(\frac{\beta\Gamma}{M}\right)\sum_{i=1}^{N}\sum_{k=1}^{M}S_i^kS_i^{k+1}\right)\right]\\ &= C\ \mathrm{Tr}\left[e^{-\beta_{\mathrm{cl}}H_{\mathrm{eff}}}\right],\end{aligned}$$

with the classical temperature $\beta_{\mathrm{cl}} = \beta/M$. Let us also abbreviate $K_M = \frac{1}{2\beta}\ln\coth\left(\frac{\beta\Gamma}{M}\right)$. In the large M limit, the prefactor vanishes, so that it drops out of any physical observables (obtained by differentiating the logarithm of the partition function). One recognizes in H_{eff} the Hamiltonian of a classical (since the S_i^k are just numbers) anisotropic Ising model without any magnetic field. The weird thing is that the coupling constant of the spin-spin interactions depend on the lattice size in the Trotter dimension and even on the temperature for the interactions along the Trotter dimension. The quantum Ising model dual to our $\mathbb{Z}_2$ gauge Hamiltonian is two dimensional, the above result therefore needs to be generalized to higher dimensions. The quantum Ising model in 2D

$$H = -J\sum_{i,j}S_{i,j}\left(S^z_{i+1,j}+S^z_{i,j+1}\right)-\Gamma\sum_{i,j}S^x_{i,j} \tag{11}$$

can be mapped to the 3D classical anisotropic Ising model

$$H_{\mathrm{eff}} = -\sum_{k=1}^{M}\sum_{i,j}\left[KS_{i,j,k}\left(S_{i+1,j,k}+S_{i,j+1,k}\right)+K_MS_{i,j,k}S_{i,j,k+1}\right] \tag{12}$$

3 MCMC-MH

Metropolis-Hasting algorithm

1. Select initial value θ_0.
2. For $i \in \{1, \cdots, N_{\text{sample}}\}$:
   - Draw candidate θ^* from proposal distribution $q(\theta^*|\theta_{i-1})$.
   - Compute $\alpha = \dfrac{g(\theta^*)}{g(\theta_{i-1})}\dfrac{q(\theta_{i-1}|\theta^*)}{q(\theta^*|\theta_{i-1})}$.
   - If $\alpha \geq 1$ accept θ^* by setting $\theta_i \leftarrow \theta^*$,
     If $0 < \alpha < 1$ accept θ^* with probability α.

References

[1] Eduardo Fradkin. *Field Theories of Condensed Matter Physics*. 2nd ed. Cambridge University Press, 2013. DOI: 10.1017/CBO9781139015509.

[2] *Quantum Ising Phases and Transitions in Transverse Ising Models*. Springer Berlin Heidelberg, 1996. ISBN: 978-3-540-49865-0. DOI: 10.1007/978-3-540-49865-0_3. URL: https://doi.org/10.1007/978-3-540-49865-0_3.

[3] H.F. Trotter. “On the product of semi-groups of operators”. In: *Proc. Amer. Math. Soc. 10* (1959), pp. 545–551.