1 2PN NLO-SO

Orbital parameters

$$
\begin{aligned}
e_t^2 &= \left(1+2EL^2\right)+\frac{1}{c^2}\left(-7E^2\eta L^2+17E^2L^2-4E\eta+4E\right)-\frac{1}{c^2L^2}\left(\left(\frac{2}{3}\mathrm{EE}\left(-\eta+2\sqrt{1-4\eta}+2\right)\right)L\cdot S_1+\left(-2\mathrm{EE}\left(\eta+2\sqrt{1-4\eta}-2\right)\right)L\cdot S_2\right)+\frac{1}{c^4L^2}\left(16E^3\eta^2L^4-47E^3\eta L^4+112E^3L^4+10E^2\eta^2L^2+2E^2\eta L^2+4E^2L^2+11E\eta-17E\right)+\frac{1}{c^2}\frac{1}{2\left(2EL^2+L^2\right)}\left(\left(-90\mathrm{EE}^{5/2}(\eta-2)L^4\sqrt{\mathrm{EE}-4\mathrm{EE}\eta}+\mathrm{EE}^{5/2}(236-79\eta)L^2\sqrt{\mathrm{EE}-4\mathrm{EE}\eta}+2\mathrm{E}^3\left(32\eta^2-\left(14\sqrt{1-4\eta}+159\right)\eta+314\sqrt{1-4\eta}+124\right)L^4+\mathrm{EE}^2\left(48\eta^2-16\sqrt{1-4\eta}\,\eta-315\eta+16\sqrt{1-4\eta}+252\right)L^2+\mathrm{EE}\left(8\eta^2-78\eta+64\right)+2\sqrt{\mathrm{EE}}(32-9\eta)\sqrt{\mathrm{EE}-4\mathrm{EE}\eta}\right)L\cdot S_1+\left(90\mathrm{EE}^{5/2}(\eta-2)L^4\sqrt{\mathrm{EE}-4\mathrm{EE}\eta}+\mathrm{EE}^{5/2}(79\eta-236)L^2\sqrt{\mathrm{EE}-4\mathrm{EE}\eta}+2\mathrm{EE}^3\left(32\eta^2+\left(14\sqrt{1-4\eta}-159\right)\eta-314\sqrt{1-4\eta}+124\right)L^4+\mathrm{EE}^2\left(48\eta^2+\left(16\sqrt{1-4\eta}-315\right)\eta-16\sqrt{1-4\eta}+252\right)L^2+\mathrm{EE}\left(8\eta^2-78\eta+64\right)+2\sqrt{\mathrm{EE}}(9\eta-32)\sqrt{\mathrm{EE}-4\mathrm{EE}\eta}\right)L\cdot S_2\right)+\mathcal{O}\left(\frac{1}{c^6}\right)\\
e_r^2 &= \left(1+2EL^2\right)+\frac{1}{c^2}\left(E^2\eta L^2-15E^2L^2-12E\right)+\frac{1}{c^2L^2}\left(\left(4\mathrm{EE}\left(-\eta+2\sqrt{1-4\eta}+2\right)\left(\mathrm{EEL}^2+1\right)\right)L\cdot S_1+\left(4\mathrm{EE}\left(\eta-2\sqrt{1-4\eta}+2\right)\left(\mathrm{EEL}^2+1\right)\right)L\cdot S_2\right)+\frac{1}{c^4}\left(\frac{3}{2}E^3\eta^2L^2-15E^3\eta L^2+80E^3L^2+\frac{9E^2\eta^2}{2}+44E^2\eta-84E^2+\frac{15E\eta^2}{8L^2}+\frac{29E\eta}{L^2}-\frac{51E}{L^2}\right)+\frac{1}{c^6}\left(\left(\mathrm{EE}^2\left(-\sqrt{1-4\eta}\right)\eta+31\mathrm{EE}^2\eta+8\mathrm{EE}^2\sqrt{1-4\eta}-80\mathrm{EE}^2+\frac{4\mathrm{EE}^2\eta^2}{L^2}+\frac{68\mathrm{EE}^2\sqrt{1-4\eta}}{L^2}-\frac{30\mathrm{EE}^2\sqrt{1-4\eta}\,\eta}{L^2}-\frac{144\mathrm{EE}^2\eta}{L^2}+\frac{68\mathrm{EE}^2}{L^2}+\frac{3\mathrm{EE}\eta^2}{2L^2}+\frac{96\mathrm{EE}\sqrt{1-4\eta}}{L^2}-\frac{33\mathrm{EE}\sqrt{1-4\eta}\,\eta}{2L^2}-\frac{213\mathrm{EE}\eta}{2L^2}+\frac{96\mathrm{EE}}{L^2}\right)L\cdot S_1+\left(\mathrm{EE}^2\left(-\sqrt{1-4\eta}\right)\eta+31\mathrm{EE}^2\eta+8\mathrm{EE}^2\sqrt{1-4\eta}-80\mathrm{EE}^2+\frac{4\mathrm{EE}^2\eta^2}{L^2}+\frac{30\mathrm{EE}^2\eta\sqrt{1-4\eta}}{L^2}-\frac{144\mathrm{EE}^2\eta}{L^2}-\frac{68\mathrm{EE}^2\sqrt{1-4\eta}}{L^2}+\frac{68\mathrm{EE}^2}{L^2}+\frac{3\mathrm{EE}\eta^2}{2L^2}+\frac{33\mathrm{EE}\eta\sqrt{1-4\eta}}{2L^2}-\frac{96\mathrm{EE}\sqrt{1-4\eta}}{L^2}-\frac{213\mathrm{EE}\eta}{2L^2}+\frac{96\mathrm{EE}}{L^2}\right)L\cdot S_2\right)+\mathcal{O}\left(\frac{1}{c^8}\right)\\
e_\epsilon^2 &= \left(1+2EL^2\right)+\frac{1}{c^2}\left(5E^2\eta L^2-15E^2L^2+2E\eta-12E\right)+\frac{1}{c^2L^2}\left(\left(4\mathrm{EE}\left(-\eta+2\sqrt{1-4\eta}+2\right)\left(\mathrm{EEL}^2+1\right)\right)L\cdot S_1+\left(4\mathrm{EE}\left(-\eta-2\sqrt{1-4\eta}+2\right)\left(\mathrm{EEL}^2+1\right)\right)L\cdot S_2\right)+\frac{1}{c^4}\left(4E^3\eta^2L^2-55E^3\eta L^2+80E^3L^2+E^2\eta^2+E^2\eta+26E^2+\frac{22E\eta}{L^2}-\frac{34E}{L^2}\right)+\frac{1}{c^6}\left(\left(-6\mathrm{EE}^3\eta^2+49\mathrm{EE}^3\eta+19\mathrm{EE}^3\eta\sqrt{1-4\eta}-80\mathrm{EE}^3\sqrt{1-4\eta}-80\mathrm{EE}^3+\frac{10\mathrm{E}^3\eta^2}{L^2}+\frac{4\mathrm{EE}^2\sqrt{1-4\eta}}{L^2}-\frac{16\mathrm{EE}^2\sqrt{1-4\eta}\,\eta}{L^2}-\frac{70\mathrm{EE}^2\eta}{L^2}+\frac{4\mathrm{EE}^2}{L^2}+\frac{8\mathrm{EE}\eta^2}{L^2}+\frac{64\mathrm{EE}\sqrt{1-4\eta}}{L^2}-\frac{18\mathrm{EE}\eta\sqrt{1-4\eta}}{L^2}-\frac{78\mathrm{EE}\eta}{L^2}+\frac{64\mathrm{EE}}{L^2}\right)L\cdot S_1+\left(-6\mathrm{EE}^3\eta^2-19\mathrm{EE}^3\eta+49\mathrm{EE}^3\eta+80\mathrm{EE}^3\sqrt{1-4\eta}-80\mathrm{EE}^3+\frac{10\mathrm{E}^3\eta^2}{L^2}+\frac{16\mathrm{EE}^2\eta\sqrt{1-4\eta}}{L^2}-\frac{70\mathrm{EE}^2\eta}{L^2}-\frac{4\mathrm{EE}^2\sqrt{1-4\eta}}{L^2}+\frac{4\mathrm{EE}^2}{L^2}+\frac{8\mathrm{EE}\eta^2}{L^2}+\frac{18\mathrm{EE}\eta\sqrt{1-4\eta}}{L^2}-\frac{78\mathrm{EE}\eta}{L^2}-\frac{64\mathrm{EE}\sqrt{1-4\eta}}{L^2}+\frac{64\mathrm{EE}}{L^2}\right)L\cdot S_2\right)+\mathcal{O}\left(\frac{1}{c^8}\right)\\
a_r &= \frac{1}{2E}+\frac{7-\eta}{c^2}+\frac{1}{2c^2L^2}\left(\left(\eta-2\left(\sqrt{1-4\eta}+1\right)\right)L\cdot S_1+\left(\eta+2\sqrt{1-4\eta}-2\right)L\cdot S_2\right)+\frac{1}{8c^4L^2}\left(E\left(\eta^2+10\eta+1\right)L^2-22\eta+34\right)+\frac{1}{8c^6L^2}\left(L\cdot S_1\left(\mathrm{EE}\left(-6\eta^2+\left(5\sqrt{1-4\eta}+9\right)\eta-8\left(\sqrt{1-4\eta}+1\right)\right)L^2-8\eta^2-64\left(\sqrt{1-4\eta}+1\right)+6\eta\left(3\sqrt{1-4\eta}+13\right)\right)-L\cdot S_2\left(\mathrm{EE}\left(6\eta^2+\left(5\sqrt{1-4\eta}-19\right)\eta-8\sqrt{1-4\eta}+8\right)L^2+8\eta^2-78\eta+18\eta\sqrt{1-4\eta}-64\sqrt{1-4\eta}+64\right)\right)+\mathcal{O}\left(\frac{1}{c^8}\right)\\
n &= 2\left(\sqrt{2}E^{3/2}\right)-\frac{E^{5/2}}{\sqrt{2}c^2}\left(\eta-15\right)+\frac{E^{7/2}}{8\sqrt{2}c^4}\left(11\eta^2+30\eta+555\right)+\mathcal{O}\left(\frac{1}{c^6}\right)
\end{aligned}
$$

Kepler's equation

$$
n\left(t-t_0\right)=e_t\sinh u-u+\left(\frac{f_{\mathrm{tt}}}{c^3}+\frac{f_{\mathrm{t}\eta}}{c^5}\right)\nu+\left(\frac{g_{\mathrm{tt}}}{c^3}+\frac{g_{\mathrm{t}\eta}}{c^5}\right)\sin\nu \tag{1}
$$

$$
\begin{aligned}
f_{\mathrm{tt}} &= \frac{3\sqrt{2}E^{3/2}(5-2\eta)}{L}\\
f_{\mathrm{t}\eta} &= \frac{E^{5/2}}{2\sqrt{2}L^3}\left(\left(-14\eta^2+\left(35\sqrt{1-4\eta}+73\right)\eta-48\left(\sqrt{1-4\eta}+1\right)\right)L\cdot S_1+\left(-14\eta^2+\left(73-35\sqrt{1-4\eta}\right)\eta+48\left(\sqrt{1-4\eta}-1\right)\right)L\cdot S_2\right)\\
g_{\mathrm{tt}} &= -\frac{E^{5/2}\eta(\eta+4)\sqrt{4EL^2+2}}{4L}\\
g_{\mathrm{t}\eta} &= \frac{E^{3/2}}{2L^2}\sqrt{4EL^2+2}\left(\left(-3\eta^2+\left(9\sqrt{1-4\eta}+11\right)\eta-4\left(\sqrt{1-4\eta}+1\right)\right)L\cdot S_1+\left(-3\eta^2-9\sqrt{1-4\eta}\,\eta+11\eta+4\sqrt{1-4\eta}-4\right)L\cdot S_2\right)
\end{aligned}
$$

Angular equation

$$
\frac{\mathrm{d}\phi}{\mathrm{d}t}=\frac{d_2}{r^2}+\frac{d_3}{r^3}+\frac{d_4}{r^4}+\frac{d_5}{r^5}-\dot{\alpha}\cos\iota \tag{2}
$$

$$
\begin{aligned}
d_2 &= L+\frac{1}{c^2}\left(3E\eta L-EL\right)+\frac{1}{c^4}\left(3E^2\eta^2L-\frac{9}{2}E^2\eta L+E^2L\right)\\
d_3 &= \frac{1}{c^2}\left(2\eta L-4L\right)+\frac{1}{2Lc^2}\left(\left(-\eta+2\sqrt{1-4\eta}+2\right)L\cdot S_1+\left(-\eta-2\sqrt{1-4\eta}+2\right)L\cdot S_2\right)+\frac{1}{c^4}\left(8E\eta^2L-22E\eta L+4EL\right)+\frac{\eta E}{8Lc^2}\left(\left(-18\eta+31\sqrt{1-4\eta}+21\right)L\cdot S_1+\left(-18\eta-31\sqrt{1-4\eta}+21\right)L\cdot S_2\right)\\
d_4 &= \frac{1}{c^4}\left(5\eta^2L-11\eta L+\frac{17L}{2}\right)+\frac{1}{8Lc^2}\left(\left(-18\eta^2+23\sqrt{1-4\eta}\,\eta+21\eta-24\sqrt{1-4\eta}-24\right)L\cdot S_1+\left(-18\eta^2-23\sqrt{1-4\eta}\,\eta+21\eta+24\sqrt{1-4\eta}-24\right)L\cdot S_2\right)\\
d_5 &= \frac{1}{c^4}\left(\frac{\eta L^3}{2}-\eta^2L^3\right)+\frac{1}{c^6}\left(\left(\frac{9\eta^2L}{4}-\frac{15}{4}\sqrt{1-4\eta}\eta L-\frac{17\eta L}{4}+\sqrt{1-4\eta}L+L\right)L\cdot S_1+\left(\frac{9\eta^2L}{4}+\frac{15}{4}\sqrt{1-4\eta}\eta L-\frac{17\eta L}{4}-\sqrt{1-4\eta}L+L\right)L\cdot S_2\right)
\end{aligned}
$$

Precession equations

$$
\begin{cases}
\dot{\bar{L}} &= \left(\left(\frac{f_{\mathrm{tL}}}{c^3}+\frac{f_{\mathrm{t}L}}{c^5}\right)\bar{S}_1+\left(\frac{g_{\mathrm{tL}}}{c^3}+\frac{g_{\mathrm{t}L}}{c^5}\right)\bar{S}_2\right)\times\bar{L}\\
\dot{\bar{S}}_1 &= \left(\frac{f_{\mathrm{t}L}}{c^3}+\frac{f_{\mathrm{t}L}}{c^5}\right)\bar{L}\times\bar{S}_1\\
\dot{\bar{S}}_2 &= \left(\frac{g_{\mathrm{t}L}}{c^3}+\frac{g_{\mathrm{t}L}}{c^5}\right)\bar{L}\times\bar{S}_2
\end{cases} \tag{3}
$$

$$
\begin{aligned}
f_{\mathrm{tL}} &= \left(1+\sqrt{1-4\eta}-\frac{\eta}{2}\right)\frac{1}{r^3}\\
f_{\mathrm{tL}} &= \left(E\eta\left(-18\eta+31\sqrt{1-4\eta}+21\right)r^2+6\eta\left(\eta-\sqrt{1-4\eta}-1\right)L^2+\left(-18\eta^2+23\eta\sqrt{1-4\eta}+21\eta-24\sqrt{1-4\eta}-24\right)r\right)\frac{1}{8c^5}\\
g_{\mathrm{tL}} &= \left(1-\sqrt{1-4\eta}-\frac{\eta}{2}\right)\frac{1}{r^3}\\
g_{\mathrm{tL}} &= \left(E\eta\left(-18\eta-31\sqrt{1-4\eta}+21\right)r^2+6\eta\left(\eta+\sqrt{1-4\eta}-1\right)L^2+\left(-18\eta^2-23\eta\sqrt{1-4\eta}+21\eta+24\sqrt{1-4\eta}-24\right)r\right)\frac{1}{8c^5}
\end{aligned}
$$