Optimization Models and Algorithms

Applications - Capacitated Vehicle Routing Problem

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ALMA MATER STUDIORUM
UNIVERSITÀ DI BOLOGNA

Plan

CVRP - Definition

Capacitated Vehicle Routing Problem (CVRP):

- **Determining routes** for a fleet of identical vehicles, which have the same capacity and are originally located at a central depot, in order **to serve a set of clients** having different demands.
- Routes are such that:
  - each route **starts and ends at the depot**;
  - each client is **served by exactly one vehicle**;
  - the total demand served by each vehicle **does not exceed its capacity**.
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CVRP - Illustration

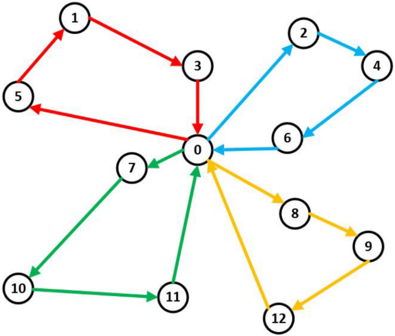

Figure: Illustration of the CVRP.
(Source: `https://ingenieriaindustrialonline.com`)

CVRP - Notations and assumptions

Notations:

- $G = (V, A)$: directed graph, with nodes $V = \{0, 1, \ldots, n\}$, such that:
  - 0 represents the **depot**;
  - $\{1, \ldots, n\}$ represent the **clients**.
- d_j: **demand** of client j (for $j \in V \setminus \{0\}$).
- k: **number** of identical vehicles.
- b: vehicle **capacity**.
- c_a: **cost** associated with arc a (for $a \in A$).

Assumptions:

- G is assumed to be **complete** (without loss of generality).
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CVRP - Various formulations

- In this course, we study **three formulations** of the CVRP:
  1. **Two-index** formulation;
  2. **Three-index** formulation;
  3. **Set-partitioning** formulation.

- One formulation might be preferred to another depending on the context / problem:
  - basic CVRP,
  - CVRP with heterogeneous capacities,
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Two-index formulation - Principle

Principle:

- This model uses, for each arc $a \in A$, a binary variable x_a to indicate whether a belongs to any vehicle route in the solution, **without specifying to which one**.
- In the literature, this model is often written using as variable indices **nodes instead of arcs**, i.e. using x_{ij}, with $(i,j) \in V^2$ such that $(i,j) \in A$, instead of x_a.
  ↪ Hence the name "**two-index**" formulation.

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Two-index formulation - Decision variables and objective

Decision variables:

$$x_a = \begin{cases} 1 & \text{if arc } a \text{ belongs to one of the } k \text{ routes,} \\ 0 & \text{otherwise.} \end{cases}$$

Objective:

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Constraints:

$$\sum_{a \in \delta^-(i)} x_a = 1, \quad i \in V \setminus \{0\}$$

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(Flow at client nodes & single client visit)

$$\sum_{a \in \delta^-(0)} x_a = k,$$

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(Flow at depot & fixing nb of vehicles to k)

??? ??? (Subtour elimination)

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$$x_a \in \{0, 1\}, \quad a \in A$$

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Two-index formulation - Challenging constraints

Two ways for modeling these challenging constraints
(one single set of constraints for both considerations).

1. **Stronger** model, but **hard-to-use** in practice, based on: variants of ATSP (alternative) **subtour elimination constraints** and the **optimal value** of a BPP;
2. **Weaker** but sufficient model, **easy-to-use** in practice, based on: variants of ATSP (alternative) **subtour elimination constraints** and the **trivial lower bound** of the same BPP.

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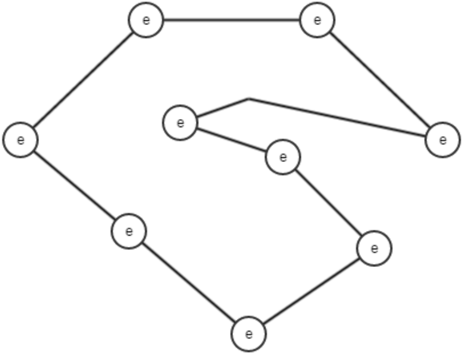

Figure: Illustration of the ATSP.
(Source: `https://en.wikipedia.org/`)

Recalls about ATSP - Subtour elimination constraints

Subtour elimination constraints:
Given a graph $G = (V, A)$ and binary decision variables $\{x_a\}$, s.t. for each $a \in A$, $x_a = 1$ if a belongs to the solution, 0 otherwise, **subtour elimination constraints** are:

$$\sum_{a \in A(S)} x_a \leq |S| - 1, \quad \forall S \subseteq V,\ 2 \leq |S| \leq |V| - 2$$

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$$\sum_{a \in \delta^+(S)} x_a \geq 1, \quad \forall S \subseteq V,\ 2 \leq |S| \leq |V| - 2.$$

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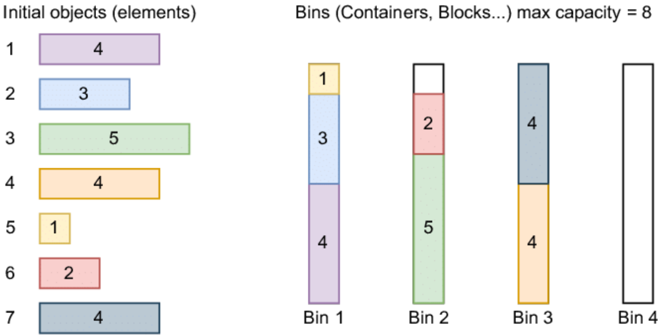

Figure: Illustration of the BPP.
(Source: Chraibi et al. (2022))

Recalls about BPP - Trivial lower bound

Trivial lower bound:
Given n items of weights $d_1, \dots, d_n$ and identical bins of capacity b, a **trivial lower bound** on the nb of bins required to pack all items is:

$$\frac{\sum_{i=1}^{n} d_i}{b}.$$

Plan

Strong model - Principle and constraints

Idea:

- Given $S \subseteq V \setminus \{0\}$, let $\sigma(S)$ be the **optimal value** of the **BPP** obtained by considering as:
  - set of **objects** $\equiv$ subset of **clients** S,
  - objects' **weights** $\equiv$ clients' **demands** $\{d_j\}_{j \in S}$,
  - container **capacity** $\equiv$ vehicle **capacity** b.
- "Given a set of clients $S \subseteq V \setminus \{0\}$, **at least $\sigma(S)$ vehicles** are needed to serve these clients."

Challenging constraints:

$$\sum_{a \in \delta^+(S)} x_a \geq \sigma(S), \quad S \subseteq V \setminus \{0\}$$

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Strong model - Separation

Consider the CVRP **LP-relaxation** (initialized with a "small" subset of challenging constraints $\tilde{\mathcal{S}}$). Let x^* be its **optimal solution**.

Separation:

- Identify **if there exists** a subset $S^* \subseteq V \setminus \{0\}$ such that:

$$\sum_{a \in \delta^+(S^*)} x_a^* < \sigma(S^*)$$

- Use **(M)ILP** to model this problem into a **separation problem**?

⚠ **Cannot be formulated** as a reasonable (M)ILP because the model would have to both select vertices to form $S^\star$ and compute the BPP optimal value $\sigma(S^*)$.

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1. Introduction to CVRP
2. Two-index formulation (Section 3.1)
   - Formulation generalities
   - Strong model of challenging constraints
   - Weak model of challenging constraints
3. Three-index formulation (Section 3.2)
4. Conclusion

Weak model - Principle and constraints

Principle:

- Same BPP problem as before, but rather than making use of the optimal value $\sigma(S)$, we use its **trivial lower bound**:

$$\sigma(S) \geq \frac{\sum_{j \in S} d_j}{b}$$

- "Given a set of clients $S \subseteq V \setminus \{0\}$, **at least** $\boldsymbol{\frac{\sum_{j \in S} d_j}{b}}$ **vehicles** are needed to serve these clients."

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Weak model - Within CVRP two-index formulation

CVRP two-index formulation:

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\begin{aligned}
\min \quad & \sum_{a \in A} c_a x_a \\
\text{s.t.} \quad & \sum_{a \in \delta^-(i)} x_a = 1, && i \in V \setminus \{0\} && \text{(Flow at client nodes \&} \\
& \sum_{a \in \delta^+(i)} x_a = 1, && i \in V \setminus \{0\} && \text{single client visit)} \\
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⚠ **Exponential nb** of constraints → **separation** with $\tilde{\mathcal{S}} \subseteq 2^{V \setminus \{0\}}$.

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$$\exists\ S^* \subseteq V \setminus \{0\}, \quad \sum_{j \in S^*} d_j - b \sum_{a \in \delta^+(S^*)} x_a^* > 0 \quad ?$$

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↪ If the optimal value is positive,
add $\boldsymbol{S^*} = \{i \in V \setminus \{0\},\ y_i = 1\}$ to $\tilde{\mathcal{S}}$.

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Plan

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Three-index formulation - Principle

Principle:

- This model uses, for each arc $a \in A$ and each vehicle $h \in \{1, \ldots, k\}$, a binary variable x_a^h to indicate whether arc a belongs to **vehicle *h*'s route**.
- From a resolution perspective, this three-index formulation does not offer any advantages over the two-index one.
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Three-index formulation - Decision variables and objective

Decision variables:

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↪ Variables $\{y_i^h\}$ are introduced for readability purpose: they correspond to sum over part of variables $\{x_a\}$.

Objective:

$$\min \quad \sum_{h=1}^{k} \sum_{a \in A} c_a x_a^h$$

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$$x_a^h \in \{0, 1\}, \quad h \in \{1, \dots, k\},\ a \in A$$

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Three-index formulation - Constraints

Constraints:

$$
\begin{array}{lll}
\displaystyle\sum_{a\in\delta^-(i)} x_a^h = y_i^h, & h \in \{1,\dots,k\},\ i \in V \setminus \{0\} & \text{(Flow at client node)} \\
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Three-index formulation - Constraints

Constraints of LP-relaxation:

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Three-index formulation - Constraints

Constraints of LP-relaxation with separation:

$$
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x_a^h \in [0,1], & h \in \{1,\dots,k\},\ a \in A & \\
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\end{array}
$$

Three-index formulation - Separation

Consider the CVRP **LP-relaxation** (initialized with a “small” subset of challenging constraints $\tilde{\mathcal{S}}$). Let $\bar{x}$, $\bar{y}$ be its **optimal solution**.

Separation:

- Identify **if there exist** a route h, a subset $\bar{S} \subseteq V \setminus \{0\}$ and a client $i \in \bar{S}$ such that:

$$\sum_{a \in \delta^+(\bar{S})} \bar{x}_a^h < \bar{y}_i^h$$

- Use **ILP** to model this as $\mathcal{O}(kn)$ **separation problems**:
  - by considering each route $h \in \{1, \dots, k\}$ and each client $i \in V \setminus \{0\}$ such that $\bar{y}_i^h > 0$,
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Three-index formulation - Separation problem

Let $h \in \{1, \dots, k\}$ be a route and $i \in V \setminus \{0\}$ a client such that $\bar{y}_i^h > 0$.

Decision variables:

- $$w_j = \begin{cases} 1 & \text{if client } i \text{ belongs to } \bar{S}, \\ 0 & \text{otherwise.} \end{cases}$$
- $$z_a = \begin{cases} 1 & \text{if arc } a \text{ belongs to } \delta^+(\bar{S}), \\ 0 & \text{otherwise.} \end{cases}$$

Objective:

$$\exists\, S^* \subseteq V \setminus \{0\}, \quad \sum_{a \in \delta^+(\bar{S})} \bar{x}_a^h < \bar{y}_i^h \quad ?$$

$$\downarrow$$

$$\max \quad \sum_{a \in A} \bar{x}_a^h z_a < \bar{y}_i^h \quad ?$$

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Three-index formulation - Separation problem

Separation problem:

$$
\begin{aligned}
\max \quad & \sum_{a \in A} \bar{x}_a^h z_a \\
\text{s.t.} \quad & w_0 = 0 \\
& w_i = 1 \\
& z_{(i,j)} \leq w_i, && (i,j) \in A \\
& z_{(i,j)} \leq 1 - w_j, && (i,j) \in A \\
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↪ If the optimal value is smaller than $\bar{y}_i^h$, **add** $S^* = \{i \in V \setminus \{0\},\ w_i = 1\}$ to $\tilde{\mathcal{S}}$.

Three-index formulation - Separation problem

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Plan

CVRP with heterog. cap. - Definition and notations

Same as for the basic CVRP with **a few changes**

Definition:

- **Determining routes** for a fleet of ~~identical~~ heterogeneous vehicles, which have ~~the same capacity~~ different capacities and are originally located at a central depot, in order **to serve a set of clients** having different demands.
- ...

Notations:

- $G = (V, A)$, $\{d_j\}$, ... ;
- ~~b: vehicle **capacity**~~ b_h: **capacity** of vehicle h.

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CVRP with heterog. cap. - Decision var. and objective

Decision variables:

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Objective:

$$\min \quad \sum_{h=1}^{k} \sum_{a \in A} c_a x_a^h$$

CVRP with heterog. cap. - Constraints

Constraints:

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\begin{array}{lll}
\sum_{a \in \delta^-(i)} x_a^h = y_i^h, & h \in \{1,\dots,k\},\ i \in V \setminus \{0\} & \text{(Flow at client node)} \\
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Plan

CVRP with time windows - Definition

Same as for the basic CVRP with **additional considerations**

Definition:

- **Determining routes** for a fleet of identical vehicles, which have the same capacity and are originally located at a central depot, in order **to serve a set of clients** having different demands and different visiting time windows.
- Routes are such that:
  - each route **starts at the depot** **at time 0**, goes from client to client **without idle time**, and **ends at the depot**;
  - each client is **served by exactly one vehicle**, during their **time window**, given that the service time is null;
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CVRP with time windows - Notations and assumptions

Notations:

- $G = (V, A)$, $\{d_j\}$, ... ;
- $[e_i, l_i]$: **time window** of client i.

Assumptions:

- G complete graph and triangle inequalities hold.
- c_a is not only the cost of the arc a but also the **travel time** (and the time spent between two nodes must be exactly c_a).

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CVRP with time windows - Decision var. and objective

Decision variables:

- $\{x_a^h\}$ and $\{y_i^h\}$
- $s_i^h \in \mathbb{R}^+$: time at which client i is served, if served by vehicle h, (s_i^h equals 0 if not served by vehicle h)

Objective:

$$\min \quad \sum_{h=1}^{k} \sum_{a \in A} c_a x_a^h$$

CVRP with time windows - Decision var. and objective

Decision variables:

- $\{x_a^h\}$ and $\{y_i^h\}$
- $s_i^h \in \mathbb{R}^+$: time at which client i is served, if served by vehicle h, (s_i^h equals 0 if not served by vehicle h)

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$$\min \quad \sum_{h=1}^{k} \sum_{a \in A} c_a x_a^h$$

CVRP with time windows - Decision var. and objective

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CVRP with time windows - Travel time constraints

Travel time constraints:

- **Logical** (**non-linear**) formulation:

$$x^h_{(i,j)} = 1 \Rightarrow s^h_j = s^h_i + c_{(i,j)}$$

- **Linear** formulation using "**big-M**" techniques:

$$s^h_j \geq s^h_i + c_{(i,j)} - M(1 - x^h_{(i,j)})$$
$$s^h_j \leq s^h_i + c_{(i,j)} + M(1 - x^h_{(i,j)})$$

with M chosen "sufficiently" large.

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CVRP with time windows - Constraints

Constraints:

$$
\begin{array}{lll}
\sum_{a \in \delta^-(i)} x_a^h = y_i^h, & h \in \{1, \dots, k\},\ i \in V \setminus \{0\} & \text{(Flow at client)} \\
\vdots & \vdots & \vdots \\
\sum_{i \in V \setminus \{0\}} d_i y_i^h \leq b, & h \in \{1, \dots, k\} & \text{(Capacity)} \\
s_0^h = 0, & h \in \{1, \dots, k\} & \text{(Time 0)} \\
s_i^h \geq e_i y_i^h, & h \in \{1, \dots, k\},\ i \in V \setminus \{0\} & \text{(Time window)} \\
s_i^h \leq l_i y_i^h, & h \in \{1, \dots, k\},\ i \in V \setminus \{0\} & \\
s_j^h \geq s_i^h + c_{(i,j)} - M(1 - x_{(i,j)}^h), & h \in \{1, \dots, k\},\ i \in V,\ j \in V \setminus \{0\} & \text{(Travel time)} \\
s_j^h \leq s_i^h + c_{(i,j)} + M(1 - x_{(i,j)}^h), & h \in \{1, \dots, k\},\ i \in V,\ j \in V \setminus \{0\} & \\
x_a^h \in \{0, 1\}, & h \in \{1, \dots, k\},\ a \in A & \\
y_i^h \in \{0, 1\}, & h \in \{1, \dots, k\},\ i \in V \setminus \{0\} & \\
s_j^h \geq 0, & h \in \{1, \dots, k\},\ i \in V &
\end{array}
$$

Plan

Conclusion

- The **CVRP** serves as a **core model** for routing problems, commonly applied to freight transportation. Despite being a basic problem, it already presents the **combined complexities of standard routing (TSP) and loading (BPP) problems**, leading to the use of **separation** techniques, or other sophisticated techniques.
- We studied two different formulations:
  - the **two-index formulation** that is the **simplest** one;
  - the **three-index formulation** that is **easier to adapt** for incorporating additional characteristics (e.g. time windows or heterogeneous capacities).
- Another formulation to study: the **set partitioning formulation** that leverages **column generation** techniques.

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