Proof of Post-Augmentation Positive-Definiteness

Let $x \in \mathbb{R}^{n+k}$ be an arbitrary vector. In addition, $x^T = (x_1^T | x_2^T)$, $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^k$.
We want to prove that $\forall x \neq 0_{n+k}$, $x^T \cdot A \cdot x > 0$.

$$x^T \cdot A \cdot x = (x_1^T | x_2^T) \cdot \left(\begin{array}{c|c} C + V_\alpha \cdot V_\alpha^T & V_\alpha \\ \hline V_\alpha^T & I_k \end{array} \right) \cdot x$$

$$= (x_1^T \cdot C + x_1^T \cdot V_\alpha \cdot V_\alpha^T + x_2^T \cdot V_\alpha^T | x_1^T \cdot V_\alpha + x_2^T \cdot I_k) \cdot \left(\frac{x_1}{x_2} \right)$$

$$= x_1^T \cdot C \cdot x_1 + x_1^T \cdot V_\alpha \cdot V_\alpha^T \cdot x_1 + x_2^T \cdot V_\alpha^T \cdot x_1 + x_1^T \cdot V_\alpha \cdot x_2 + x_2^T \cdot I_k \cdot x_2$$

$$= x_1^T \cdot C \cdot x_1 + (x_1^T \cdot V_\alpha) \cdot (x_1^T \cdot V_\alpha)^T + (x_1^T \cdot V_\alpha) \cdot x_2 + x_2^T \cdot (x_1^T \cdot V_\alpha)^T + x_2^T \cdot x_2$$

Therefore:

$$x^T \cdot A \cdot x = x_1^T \cdot C \cdot x_1 + (x_1^T \cdot V_\alpha + x_2^T) \cdot (x_1^T \cdot V_\alpha + x_2^T)^T$$

with $x_1^T \cdot V_\alpha$ and x_2^T line vectors of length k.

$\rightarrow (x_1^T \cdot V_\alpha + x_2^T) \cdot (x_1^T \cdot V_\alpha + x_2^T)^T \geq 0$

However:

- If $x_1 \neq 0_n$, $x_1^T \cdot C \cdot x_1 > 0$ (by definition, as $C \in SPD(n)$)
- If $x_1 = 0_n$ but $x_2 \neq 0_k$, $(x_1^T \cdot V_\alpha + x_2^T) \cdot (x_1^T \cdot V_\alpha + x_2^T)^T = x_2^T \cdot x_2 > 0$

In conclusion:

$$x^T \cdot A \cdot x = 0 \Leftrightarrow x = 0_{n+k}$$

$$\implies A \in SPD(n+k)$$