# Chapter 1

# Linear Models with Fixed Effects Only

The basic linear model with fixed effects only is:

$$\begin{aligned} \boldsymbol{y} &= \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} \\ \boldsymbol{\varepsilon} &\sim \mathrm{N}(\boldsymbol{0}, \sigma^2 \boldsymbol{I}_n) \end{aligned}$$

$\boldsymbol{y}, \boldsymbol{\varepsilon} \in \mathbb{R}^n$, $\boldsymbol{\beta} \in \mathbb{R}^p$, $\boldsymbol{X} \in \mathbb{R}^{n\times p}$ with rank $p$.

**Example 1.1.** Simple linear regression model.

$$\boldsymbol{X} = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \qquad \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$$

Single-observation representation:

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i \qquad \varepsilon_i \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0, \sigma^2) \qquad \triangle$$

**Example 1.2.** One-way ANOVA model. Cell means form:

$$\boldsymbol{X} = \left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\ \vdots\\ 1\end{array}}\right\} n_1 \text{ rows} \\ \left.\vphantom{\begin{array}{c}1\\ \vdots\\ 1\end{array}}\right\} n_2 \text{ rows} \\ \vphantom{\vdots} \\ \left.\vphantom{\begin{array}{c}1\\ \vdots\\ 1\end{array}}\right\} n_M \text{ rows} \end{array} \qquad \boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_M \end{bmatrix}$$

Single-observation representation using a double index:

$$y_{ij} = \beta_i + \varepsilon_{ij} \qquad \varepsilon_{ij} \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0, \sigma^2)$$

Alternative parametrization, using the first group as the reference (as in R's "treatment contrasts"):

$$\boldsymbol{X} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ 1 & 0 & 0 & & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & & 1 \end{bmatrix} \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} n_1 \text{ rows} \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} n_2 \text{ rows} \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} n_3 \text{ rows} \\ \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} n_M \text{ rows} \end{array} \qquad \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_M \end{bmatrix}$$

Single-observation representation using a double index:

$$y_{ij} = \beta_0 + \beta_i + \varepsilon_{ij} \qquad \varepsilon_{ij} \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0, \sigma^2)$$

with $\beta_1 = 0$. △

**Example 1.3.** Rail data (P&B p. 4ff) – see Figure 1.1: Travel times of ultrasonic waves in six railway rails (in nanoseconds, 36 100 subtracted). Three measurements per rail were taken; we have a one-way classification.

The fixed-effects model in cell means form, $y_{ij} = \beta_i + \varepsilon_{ij}$ (where $i$ indicates the rail), can be fitted in R as follows:

```
> rail.lm.1 <- lm(travel ~ Rail - 1, data = rail.df)
> rail.lm.1

Call:
lm(formula = travel ~ Rail - 1, data = rail.df)

Coefficients:
Rail1  Rail2  Rail3  Rail4  Rail5  Rail6
54.00  31.67  84.67  96.00  50.00  82.67
```

We compare this model to the null model, $y_{ij} = \beta + \varepsilon_{ij}$:

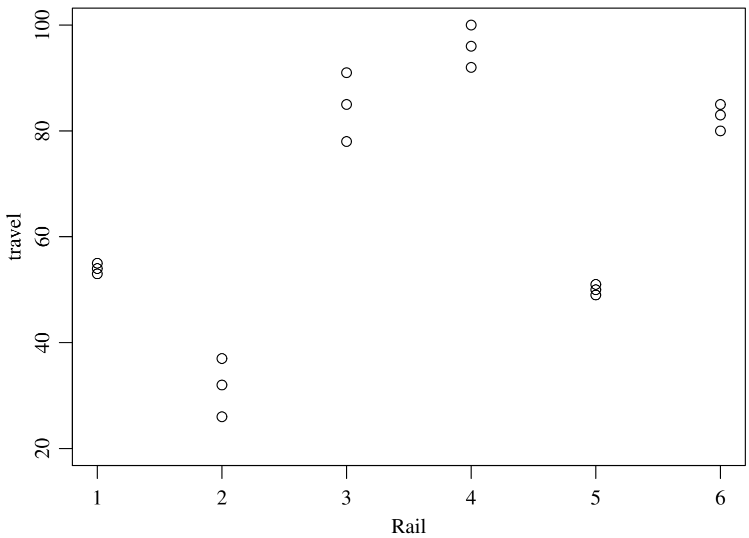


Figure 1.1: The rail data.

```
> rail.lm.0 <- lm(travel ~ 1, data = rail.df)
> anova(rail.lm.0, rail.lm.1)
Analysis of Variance Table

Model 1: travel ~ 1
Model 2: travel ~ Rail - 1
  Res.Df    RSS Df Sum of Sq      F    Pr(>F)
1     17 9504.5
2     12  194.0  5    9310.5 115.18 1.033e-09 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

The rail effect is highly significant. The same $F$ test can be obtained from the output of:

```
> summary(lm(travel ~ Rail, data = rail.df))
[...]

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   54.000      2.321  23.262 2.37e-11 ***
Rail2        -22.333      3.283  -6.803 1.90e-05 ***
Rail3         30.667      3.283   9.341 7.44e-07 ***
Rail4         42.000      3.283  12.793 2.36e-08 ***
Rail5         -4.000      3.283  -1.218    0.246
Rail6         28.667      3.283   8.732 1.52e-06 ***
[...]

Residual standard error: 4.021 on 12 degrees of freedom
Multiple R-squared:  0.9796,    Adjusted R-squared:  0.9711
F-statistic: 115.2 on 5 and 12 DF,  p-value: 1.033e-09
```

△