The BDD Domain and Applications to Constraint Satisfaction

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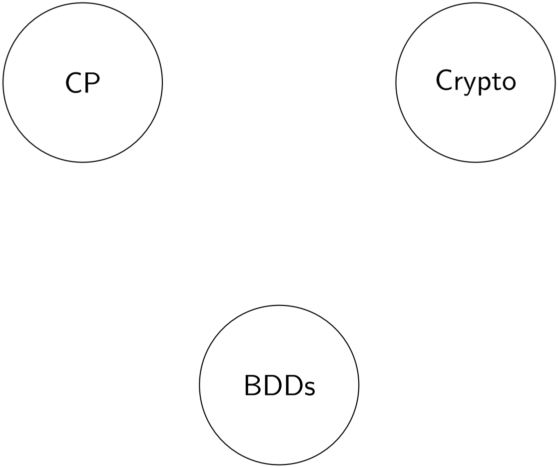
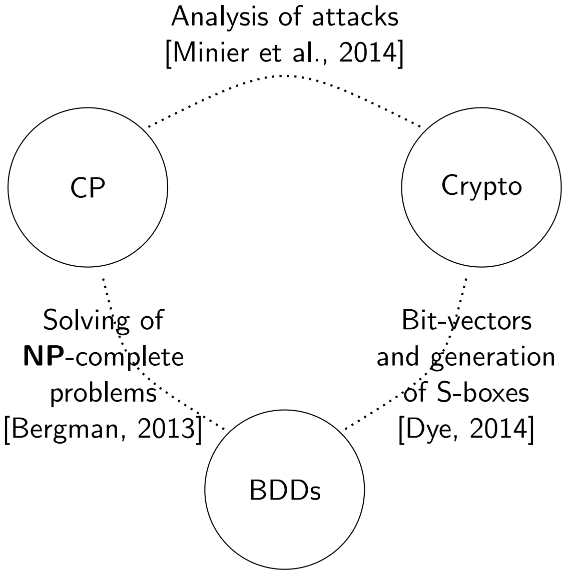

CP
Crypto
BDDs


Analysis of attacks
[Minier et al., 2014]
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Bit-vectors
and generation
of S-boxes
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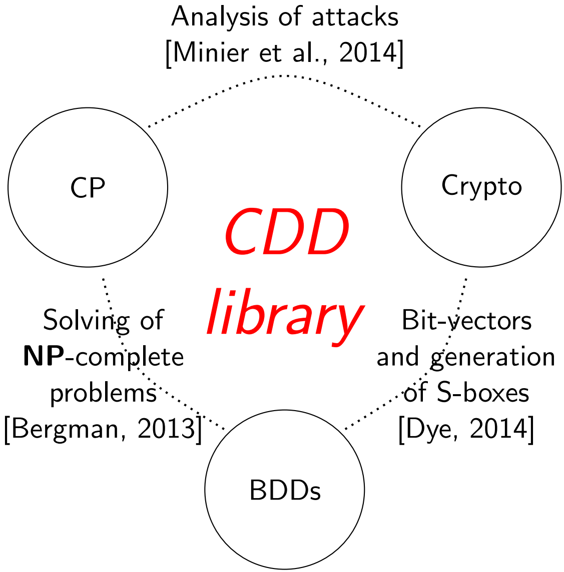
BDDs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BDDs

Constraint programming

- Define a problem declaratively with variables and constraints, and find a solution.
- A field between Artificial Intelligence and Operations Research.
- Many applications on solving NP-hard problems in planning, logistics, arts, computational sustainability and, very recently, cryptography.

Constraint satisfaction problem (CSP)

A CSP is the data $(X, \mathcal{D}, C)$ where X is a set of variables, $\mathcal{D}$ is a function from variables to their domains (values they can take), C is a set of constraints defined in:

- intension: high level language to express the constraint, or,
- extension: list of allowed solutions (called table constraints).

Constraint solving

Algorithm: propagate and search

Find the solutions in two steps:

- propagation: delete values from the domains of some variables that are inconsistent with respect to some constraint
- search: enumerate the domains to find the solutions.

Example of propagation:

$$x, y, z \in \{1, 2, 3\}$$
$$x + y \leq z$$

We find that the value $z = 1$ is not consistent (not possible).

Context

A limitation of CP solvers

- Constraint with divisions or modulo
- Constraints on bit-vectors (vectors of bits)

Goal

Investigate the use of another structure on these constraints: BDDs

Existing domains

- BDD [Bryant, 1986]: a data structure initially used for boolean functions
- MDD [Andersen et al., 2007]: the consistency
- bit-vector domain [Michel and Van Hentenryck, 2012]: the computations

Internship contribution [1]

Contribution : CDD library

A library based on a BDD structure to represent domains in a constraint-friendly way,

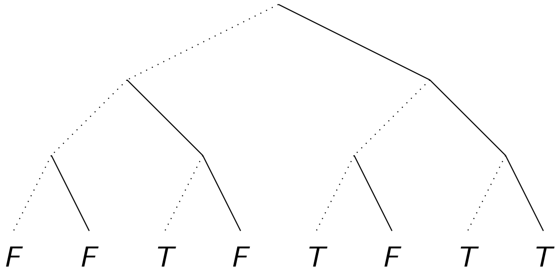
- with to type of BDDs: unrestricted and limited-width,
- implemented in OCaml within the AbSolute constraint solver,
- experimented on a cryptography problem (AES).

[1] `https://github.com/MathieuVavrille/cdd`

```
module type CDD = sig
  type bdd
  val bdd_of : bdd -> bdd -> bdd
  (*** Set operations ***)
  val intersection : bdd -> bdd -> bdd
  (* and others: difference, union, etc *)
  (*** Bitwise operations ***)
  val xor : bdd -> bdd -> bdd
  (* and others: and, or, not, etc *)
  (*** BDD manipulations ***)
  val prefix : bdd -> int -> bdd
  val suffix : bdd -> int -> Bddset.t
  val concat : bdd -> bdd -> bdd
  (*** Solving techniques ***)
  val multiple_mdd_consistency : bdd -> Bddset.t -> bdd
  val refined_consistency : bdd -> bdd -> int -> bdd
  val split: bdd -> (bdd * bdd)
```

Representation of bit-vectors

- Bit-vectors are represented by the paths from the root to the leaves T.
- The set represented is $\{010, 100, 110, 111\} \Leftrightarrow \{2, 4, 6, 7\}$



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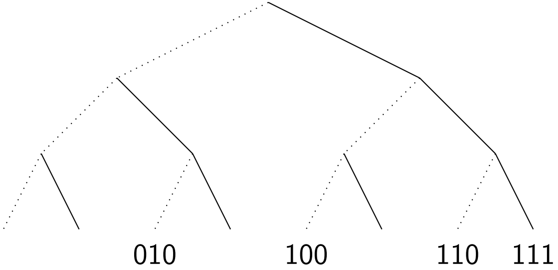
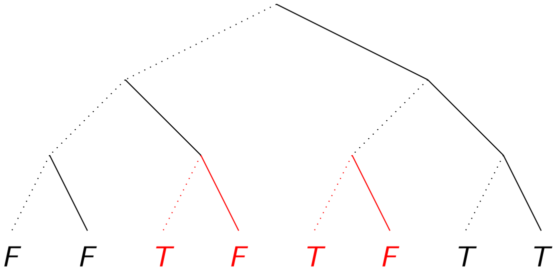


BDD: Reduction of binary tree

We now want a smaller representation



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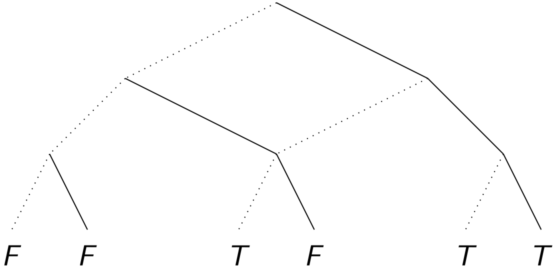

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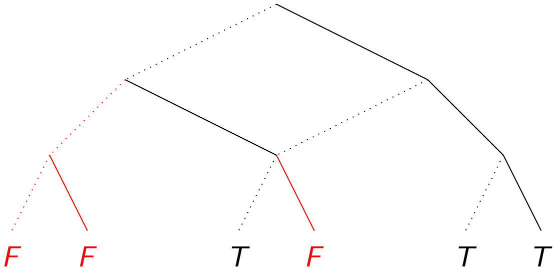

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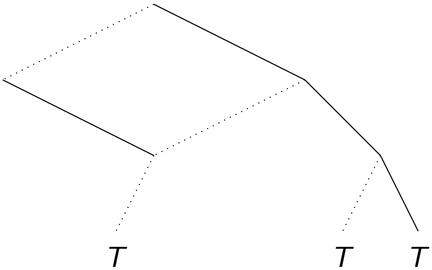

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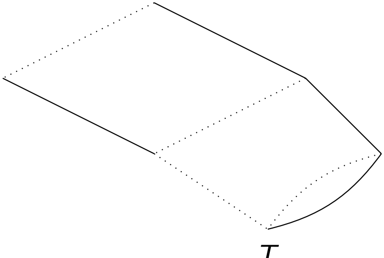
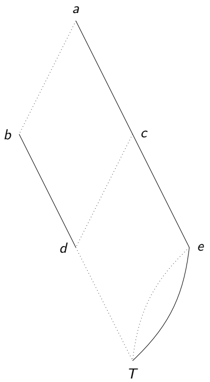


Implementation

Global hashtable

- keys are (0-child, 1-child), value is the BDD represented
- one entry (F, F) with value F.



name in the tree	ref in the table	key in the table	value in the table
-	h_0	F, F	F
d	h_1	T, F	T
b	h_2	F, h_1	h_1
e	h_3	T, T	T
c	h_4	h_1, h_3	h_1 h_3
a	h_5	h_2, h_4	h_2 h_4

Implementation

Function $\text{BDD\_OF}(a, b)$ **returns** *the BDD* $Node(a, b)$ *where we shared equivalent subtrees*

    if $(a, b) \in global\_hash$ **then**

        return $\text{FIND}(global\_hash, (a, b))$

    else

        $new\_ref \leftarrow Node(a, b)$

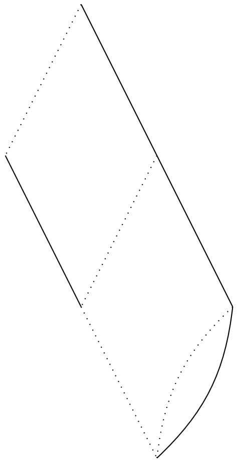
        $\text{ADD\_TO\_HASH}(global\_hash, (a, b), new\_ref)$

        return new_ref

    end

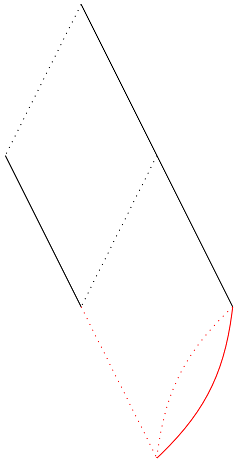
If we always use this algorithm, all the BDDs will be reduced. This is the core of the implementation.

Extraction of prefixes



- Prefixes of size k are the first k bits of the BDD B.

Extraction of prefixes



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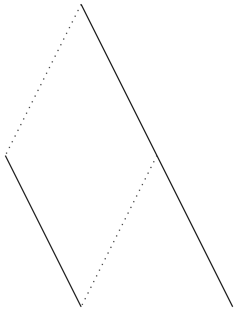


- Prefixes of size k are the first k bits of the BDD B.
- The set $Pref(B,k)$ is $\{01, 10, 11\}$
- We have that $\{2,4,6,7\}//2 = \{1,2,3\}$

General propagation

And constraint

Let the constraint $X \wedge Y = Z$. We compute

$$And^{-1}(\mathcal{D}(Z), \mathcal{D}(X)) = \{y | x \wedge y = z, x \in \mathcal{D}(X), y \in \mathcal{D}(Y), z \in \mathcal{D}(Z)\}$$

Then the propagation on Y is simply

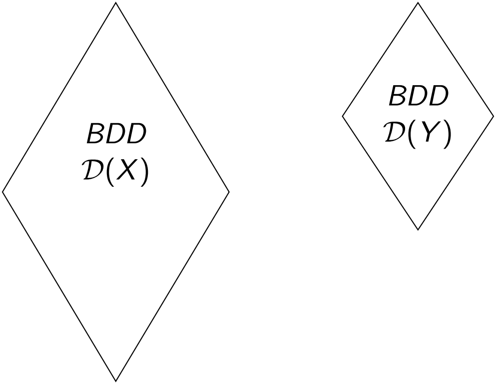
$$\mathcal{D}(Y) \leftarrow \mathcal{D}(Y) \cap And^{-1}(\mathcal{D}(Z), \mathcal{D}(X))$$

What do we need ?

- Bitwise operations
- Intersection

Div constraint: $X//2^k = Y$

Condition on the depth: $depth(X) = k + depth(Y)$
Propagation on Y: $\mathcal{D}(Y) \leftarrow \mathcal{D}(Y) \cap Pref(\mathcal{D}(X), depth(X) - k)$
Propagation on X: $\mathcal{D}(X) \leftarrow \mathcal{D}(X) \cap \mathcal{D}(Y) \cdot \textsc{complete}(k)$



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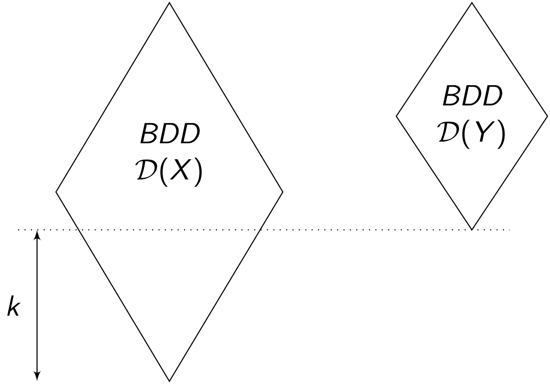

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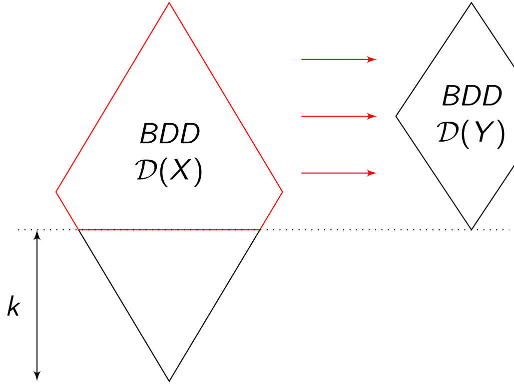

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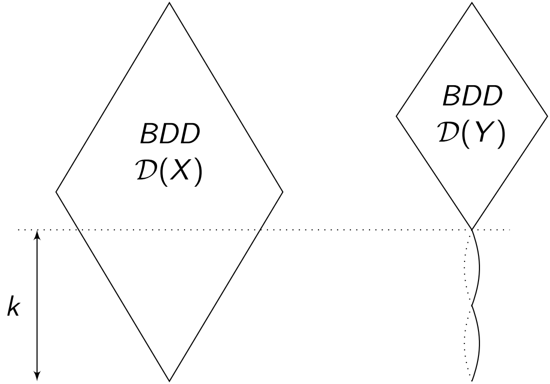

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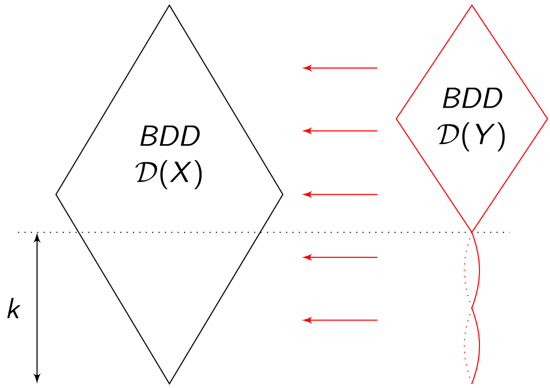


Table constraint

Table constraint

A *table constraint* is a constraint defined on n variables in extension by the set of allowed n-uplet.

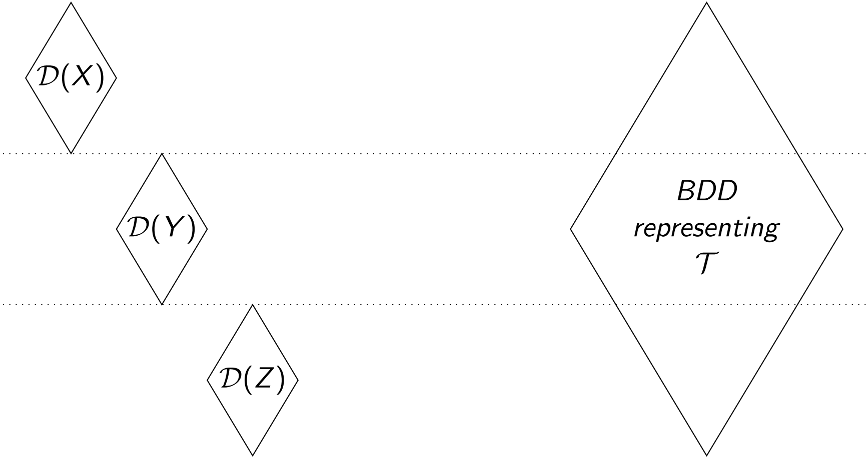
Transformation to BDD

Let $\mathcal{T}$ the set representing a table constraint on 3 variables. Let

$$S = \{b^1 \cdot b^2 \cdot b^3 | (b^1, b^2, b^3) \in \mathcal{T}\}$$

S is a set of bit-vectors, thus it can be represented by a BDD.

Table constraint: $Table(X, Y, Z, \mathcal{T})$



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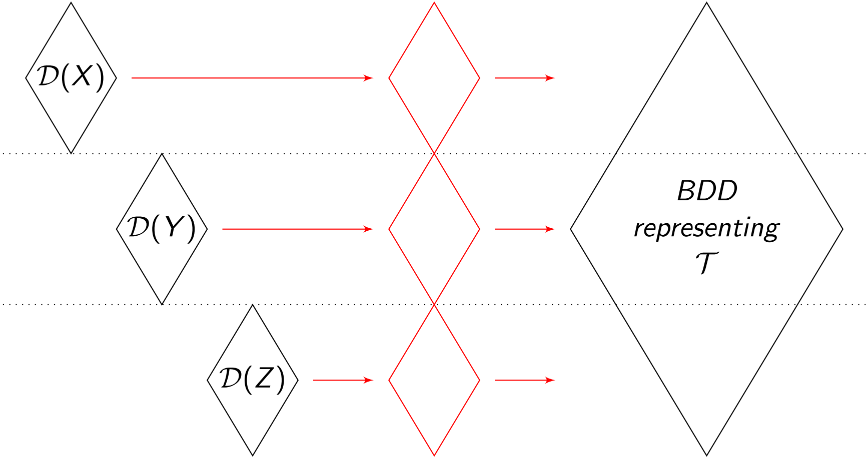

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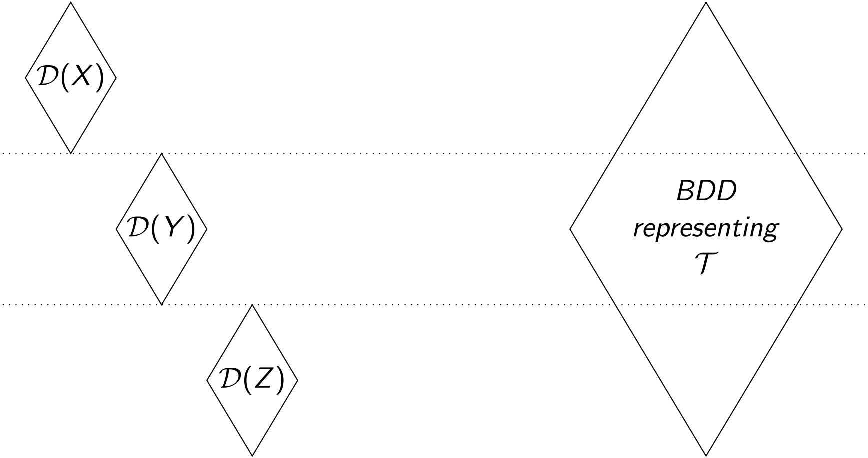

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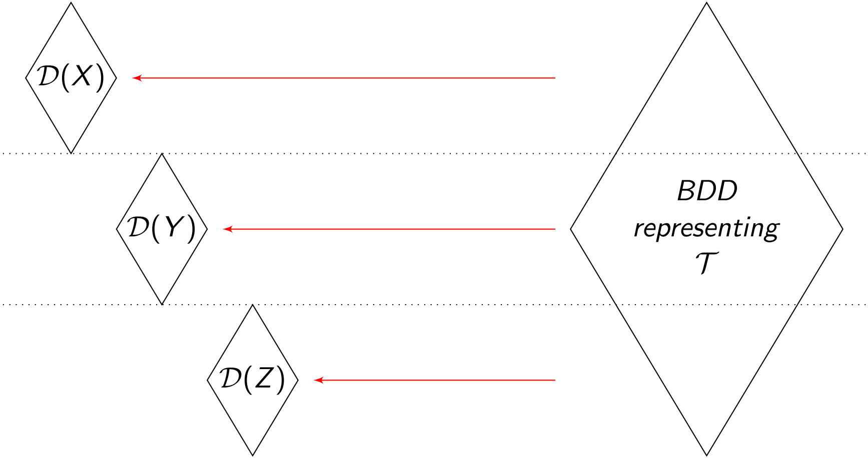
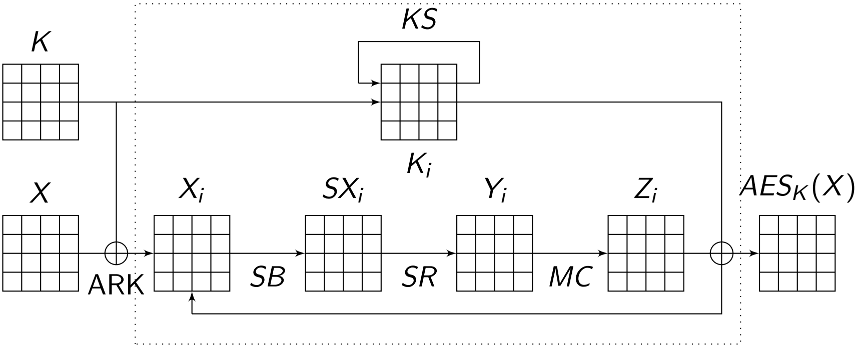


AES



Differential analysis

Start with messages X, X', and keys K, K'. Compute $E_K(X)$ and $E_{K'}(X')$. What is the probability to get $E_K(X) \oplus E_{K'}(X')$ knowing $X \oplus X'$ and $K \oplus K'$?

Constraint program solving

Translation

- Variables: $\delta X = X \oplus X', \delta K = K \oplus K'$.
- Constraints: representation of the computations (Xor, mix column, S-box, ...)

Representation from [Minier et al., 2014, Gerault et al., 2016, Gerault et al., 2017, Gerault et al., 2018]

Phase 1 (SAT solver)

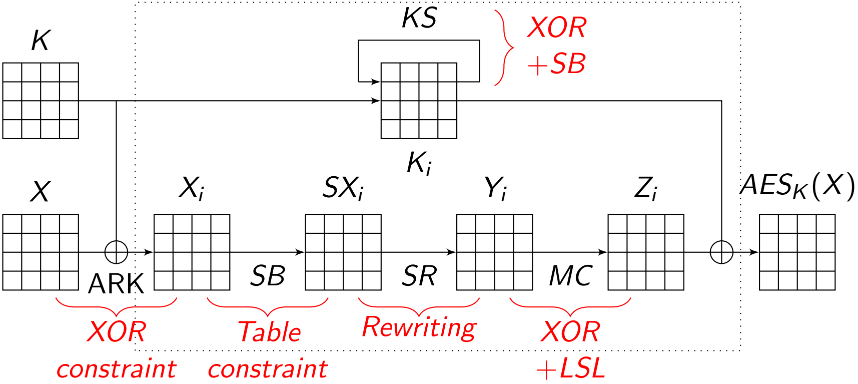
Abstract the bytes with variables

$$\Delta X = 0 \Leftrightarrow \delta X = 0^8$$
$$\Delta X = 1 \Leftrightarrow \delta X \in [1, 255]$$

Phase 2 (Constraint solver)

Find the real values of the bytes δX knowing ΔX

Can we solve it with BDDs ?



Conclusion, what I didn't talk about

Design and implementation of a BDD library for CP.

- Limited-width-BDDs,
- Bitwise propagators,
- Set operations

Theoretical work on BDDs

- Definition of merge value
- Analysis of the width
- Equivalence between domains

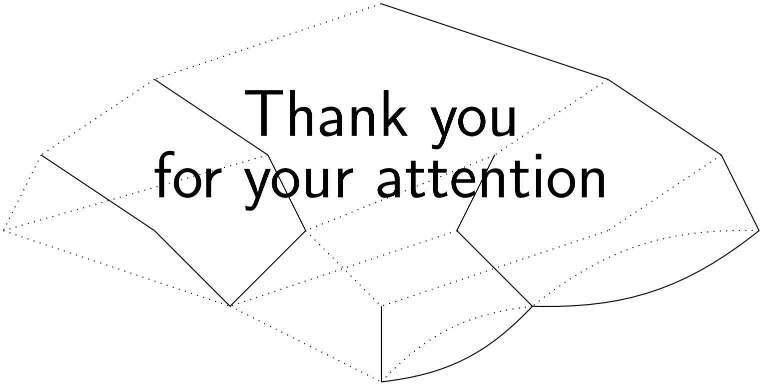
Future work

On BDDs:

- Integer domain: algebraic computations
- Global constraints: `all_different`, `global_cardinality`, `regular`
- Comparison with other domains (bit-vector domain)
- Improvement of heuristics
- General constraint $X \equiv Y \mod c$ (not $c = 2^k$)

About cryptography: project ANR Decrypt about CP and cryptography

- Analysis of other protocols
- Usage of BDDs to design protocols



Thank you for your attention

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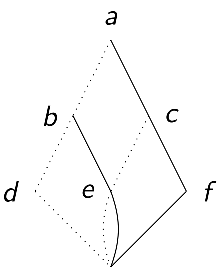
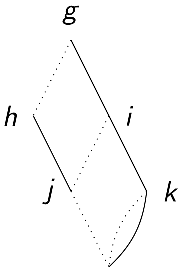
Example: intersection

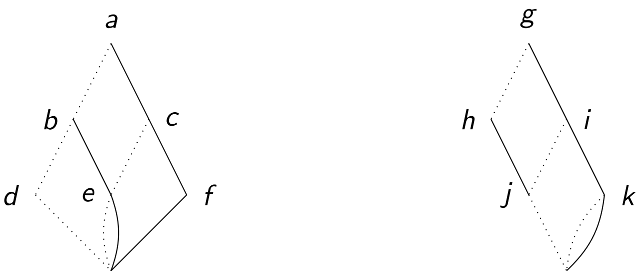




Intersections computed recursively: $M \cap M' = a \cap g$

- $b \cap h$
  - $d \cap F$
  - $e \cap j$
- $c \cap i$
  - $e \cap j$
  - $f \cap k$

Example: intersection



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- $c \cap i$
  - $e \cap j$
  - $f \cap k$

Implementation of intersection

Idea

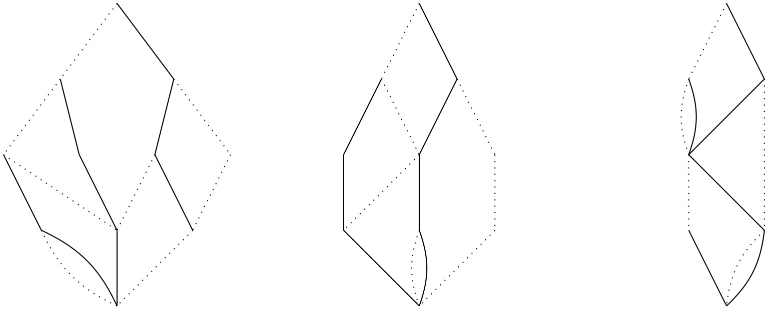
Use caching to compute only once the functions

```
Function INTER(B1, B2) returns B1 ∩ B2
    if (B1, B2) ∈ inter_hash then
        return FIND(inter_hash, (B1, B2))
    else
        if B1 = F or B2 = F then
            result ← F
        else if B1 = T or B2 = T then
            result ← T
        else
            Let a1, b1 such that B1 = Node(a1, b1)
            Let a2, b2 such that B2 = Node(a2, b2)
            result ← BDD_OF(INTER(a1, a2), INTER(b1, b2))
        end
        ADD_TO_HASH(inter_hash, (B1, B2), result)
        return result
    end
```

Limited-width-BDDs

Goal

Having a polynomial size representation (depending on the depth)



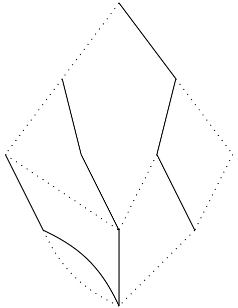
(a)
4-limited-width-BDD representing S

(b)
3-limited-width-BDD: $S \cup \{15\}$

(c)
2-limited-width-BDD: $S \cup \{5, 6, 9, 15\}$

Figure 3: Representation of the set
$S = \{1, 2, 3, 7, 8, 13, 14\} = \{0001, 0010, 0011, 0111, 1000, 1101, 1110\}$

How to generate them ?



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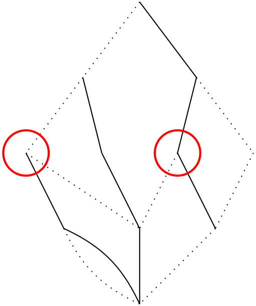

- Merging paths $00 \cdot \{01, 10, 11\}$ and $11 \cdot \{01, 10\}$

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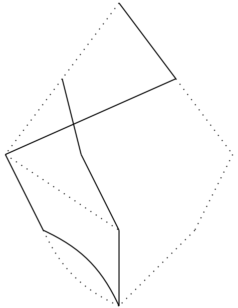

- Merging paths $00 \cdot \{01, 10, 11\}$ and $11 \cdot \{01, 10\}$
  - Adds the bit-vector 1111

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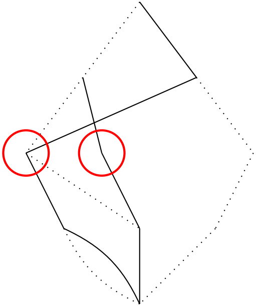
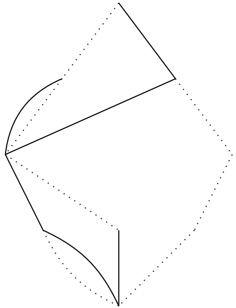


- Merging paths $00 \cdot \{01, 10, 11\}$ and $11 \cdot \{01, 10\}$
  - Adds the bit-vector 1111
- Merging paths $00 \cdot \{01, 10, 11\}$ and $01 \cdot 11$

How to generate them ?



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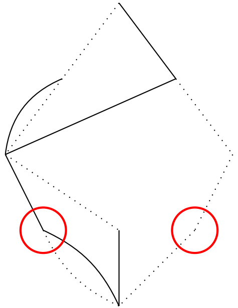

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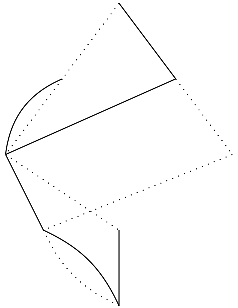
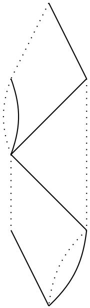
- Merging paths $00 \cdot \{01, 10, 11\}$ and $01 \cdot 11$
  - Adds the bit-vectors $01 \cdot \{01, 10\}$
- Merging paths $001 \cdot \{0, 1\}$ and $100 \cdot 0$

How to generate them ?



- Merging paths $00 \cdot \{01, 10, 11\}$ and $11 \cdot \{01, 10\}$
  - Adds the bit-vector 1111
- Merging paths $00 \cdot \{01, 10, 11\}$ and $01 \cdot 11$
  - Adds the bit-vectors $01 \cdot \{01, 10\}$
- Merging paths $001 \cdot \{0, 1\}$ and $100 \cdot 0$
  - Adds the bit-vector 1001

How to generate them ?



- Merging paths $00 \cdot \{01, 10, 11\}$ and $11 \cdot \{01, 10\}$
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  - Adds the bit-vectors $01 \cdot \{01, 10\}$
- Merging paths $001 \cdot \{0, 1\}$ and $100 \cdot 0$
  - Adds the bit-vector 1001

What is the over-approximation

Definition (Merge Value)

The *merge value* of two nodes ($mv(u, v)$) is the number of bit-vectors added when merging the two nodes u and v.

Theorem

Let B be a BDD of root r, and u and v two nodes of the same layer. Let p_u (resp p_v) the number of paths that go from r to u (resp. v). The merge value is then

$$\begin{aligned} mv(u, v) &= p_u|\gamma(v)\backslash\gamma(u)| + p_v|\gamma(u)\backslash\gamma(v)| \\ &= p_u|\gamma(v)| + p_v|\gamma(u)| - (p_u + p_v)|\gamma(u) \cap \gamma(v)| \end{aligned}$$

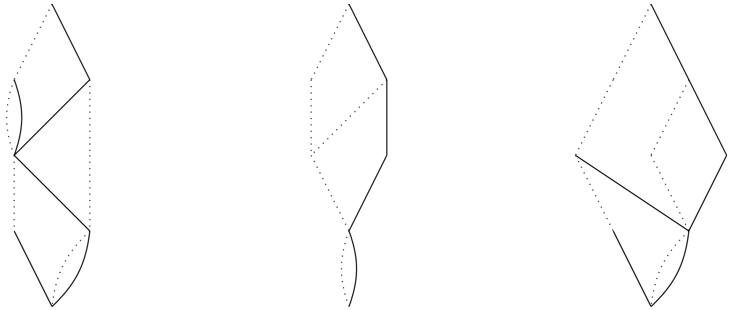
Consistency for limited BDDs

Issue

The intersection may increases the width

Solution from MDD-consistency

Consistency of M with respect to M': delete edges of M that are not in paths of M'.



(a) BDD M (b) BDD M' (c) BDD $M \cap M'$

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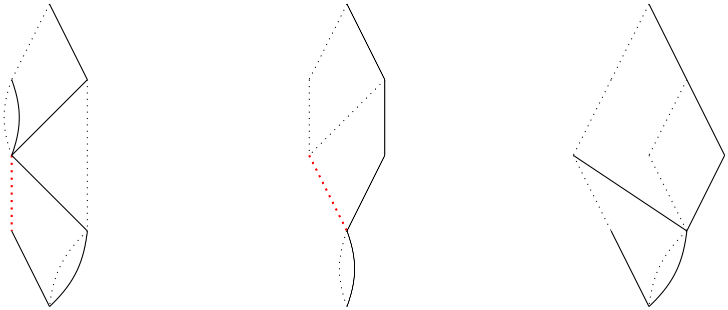

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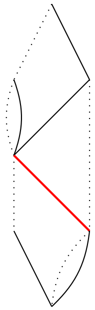
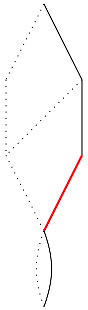
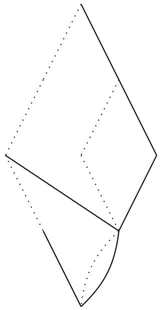


(a) BDD M



(b) BDD M'



(c) BDD $M \cap M'$

Consistency for limited BDDs

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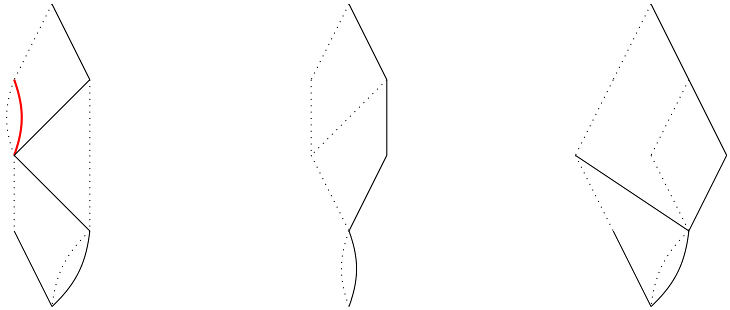

(a) BDD M (b) BDD M' (c) BDD $M \cap M'$

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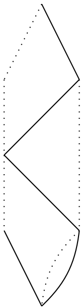

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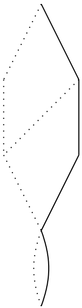
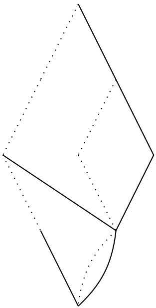


(b) BDD M'



(c) BDD $M \cap M'$

Multiple consistency

Improvement

We often need to do the consistency of M w.r.t $\bigcup_i M_i$

Algorithm

Simply check if the edges are in paths of at least one BDD in the set

Cost

If the BDDs come from the same original BDD, we share a lot of computations.

Refining

Improvement

The width will only decrease with this consistency. This limits the expressiveness.

Solution 1

Choose nodes to split: have twice the same node in the BDD, an then do a propagation that will change one and not the other.
$\rightarrow$ Not really great with a functional implementation.

Solution 2

Increase the width during consistency.
$\rightarrow$ Great compromise between computing the intersection and doing the consistency