TD: "Images et géométrie discrète"
Extended Euclid's Algorithm

Let us consider the following Euclidean division algorithm.

Procedure `Convergents(` *(a,b), (p,q), (p',q'), i* `)`

Input: (a, b), (p, q), (p', q'), i

Output: (p', q')

1 Let r be the remainder of the Euclidean division b/a;

2 Let u be the quotient of the Euclidean division b/a;

3 $p'' \leftarrow up' + p$;

4 $q'' \leftarrow uq' + q$;

5 **if** $r > 0$ **then**

6   **return** $Convergents((r, a), (p', q'), (p'', q''), i + 1)$;

7 **else**

8   **return** (p', q')

Question 1 *Let $(p_{-1}, q_{-1}) = (1, 0)$ and $(p_0, q_0) = (0, 1)$, what is the output of* `Convergents(`$(5, 8)$, (p_{-1}, q_{-1}), (p_0, q_0), 0) *?*

Question 2 *Let us consider* `Convergents(`(a, b), (p_{-1}, q_{-1}), (p_0, q_0), 0) *(with $0 \leq a < b$ and $\gcd(a, b) = 1$). We index the recursive calls by $i = 1 \ldots n$. Show that*

$$\forall i = 1 \ldots n \,, p_i = u_i p_{i-1} + p_{i-2} \text{ and } q_i = u_i q_{i-1} + q_{i-2} \,.$$

Question 3 *Similarly, with $r_{-1} = b$ and $r_0 = a$, show that*

$$\forall i = 1 \ldots n \,, r_i = r_{i-2} - u_i r_{i-1}$$

Question 4 *Following previous results, prove the following statements:*

1. $\forall i = 1 \ldots n \,, p_{i-1} q_i - q_{i-1} p_i = \pm 1$

2. $\forall i = -1 \ldots n \,, p_i b - q_i a = \pm r_i$

Since $r_n = gcd(a, b) = 1$, what is $p_n b - q_n a$?

Question 5 *Give the definition of uni-modularity. What is the geometrical interpretation of this definition ?*

Question 6 *In the domain in appendix, draw the Euclidean segment $[(0, 0) - (b, a)]$ and all convergents (q_i, p_i) for the input given in Question **??**. With respect to the parity of i, can you say something on the position of convergents with respect to the segment ? If we construct a polygonal curve with only convergents (q_i, p_i) with even index i (plus a last point (b, a)). What kind of geometrical object I have constructed ?*

Question 7 *Let L_{odd} (resp. L_{even}) be the polygonal curve of convergents with odd index (resp. even index). Furthermore, we add the point (b, a) to the end of each list. For the input given in Question **??**, are there integer points between L_{odd} and L_{even} ? Why ?*

Question 8 *For a general setting* `Convergents(`(a, b), (p_{-1}, q_{-1}), (p_0, q_0), 0), *can you prove the statement of the previous question ?*

Question 9 *What is the complexity of* `Convergents` *with respect to a and b ?*