“Images et géométrie discrète”
Final Exam (3h)

All exercises are independent. All material (lecture notes, slides) are allowed. Questions with $\star$ symbols may be more difficult and thus may give you extra points.

1 “But for the chicken, where are the ears ?”

Jesse P. and Walter W. are cookers in Gustavo F.’s restaurant “Los Pollos Hermanos”. They have to prepare a new recipe based on fried chicken ears. Walter, who is quite skilled in sciences, suggests the following thing: “Let’s model the chicken as a simple polygon $P = (v_1, \ldots, v_n)$ (ordered clockwise, no self-intersection, single connected component) and let’s define an *ear* of a polygon at v_i by three consecutive vertices (v_{i-1}, v_i, v_{i+1}) such that the segment (*chord*) $[v_{i-1}v_{i+1}]$ lies in the interior of P” (since P is simple, we can always distinguish interior from exterior).

Jesse: “Okay... but where are the ears ?”

Walter: “I’ll show you that any polygon has at least two ears”

Jesse: “Yo great.. Let’s cook!”

Question 1 *Let’s start by a warm-up: first prove the following statement:*

v is an ear of P $\Leftrightarrow$ it’s interior angle is less than π and there is no vertex z of P inside the triangle (v_{i-1}, v_i, v_{i+1}).

Question 2 *At a given vertex v of P, give the pseudo-code of the function which returns true if a vertex z belongs to the triangle (v_{i-1}, v_i, v_{i+1}) (false otherwise). Please make sure that you use the exact geometrical predicate `Orientation`(a, b, c).*

Give the pseudo-code of the function which returns true if v is an ear of P. What is its complexity ?

In the following question, we prove the following statement: For any simple polygon P with at least 4 vertices, P has at least two **non-overlapping** ears (non-overlapping is important in proofs). The proof will be done by induction on the number of vertices of P. Let’s denote by $H(k)$ the hypothesis that the statement is true for any P with at most k vertices.

Question 3 *First prove that we have $H(4)$.*

We assume that $H(n-1)$ is true and that P has n vertices. We want to prove $H(n)$.

Question 4 *Let us consider a convex vertex v of P (vertex with interior angle less that π). We call v^- (respectively v^+) the previous vertex (resp. next vertex) of v on P. If v is an ear, prove that P as at least two non-overlapping ears.*

Question 5 *If v is not an ear, there exists a vertex z of P inside the triangle (v^-, v, v^+). Let L be the line parallel to (v^-, v^+) going through z (if there are several points inside the triangle (v^-, v, v^+), z will be such that it gives a line L closest to v), see Figure 1.*

Show that the chord $[vz]$ is inside P.

Use this chord to split P and complete the proof.

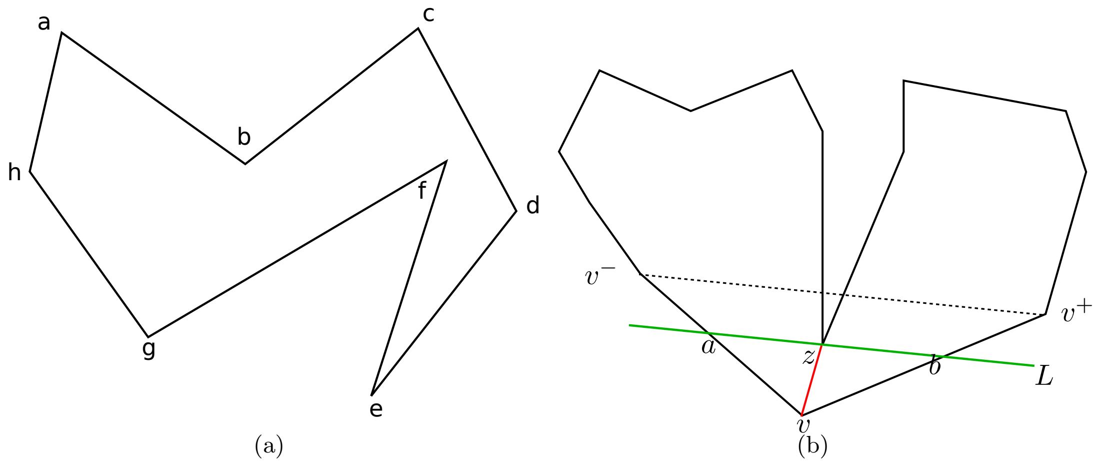


Figure 1: Illustration for Exercise 1. In (a), only vertices $\{a, e, g, h\}$ induce ears.

Question 6 *Show how this property can be used to triangulate any simple polygon. What would be the complexity of such process ? How many ears Walter and Jesse will get from a polygonal chicken with n vertices ? Is it a good deal for Gustavo ?*

> The original proof has been given by G.H. Meisters in "*Polygons Have Ears*", The American Mathematical Monthly, 82(6):648–651, 1975. The questions follows Meisters's workflow.

2 CarWash

Skyler W. is running a car-wash business and she wants to set-up a license plate recognition system in a do-it-yourself mode. Walter W. (her husband) has suggested her to consider mathematical morphology tools for that.

Let us first recall some definitions from the lecture on mathematical morphology on a set E:

- A, B, X are subsets of E;
- B_x with $x \in E$ is $\{z + x \,|\, z \in B\}$ (translation of B from x);
- Dilation of X by a structuring element B:

$$\delta_B(X) = X \oplus B = \bigcup_{x \in X} B_x = \bigcup_{b \in B} X_b\,;$$

- Erosion of X by a structuring element B:

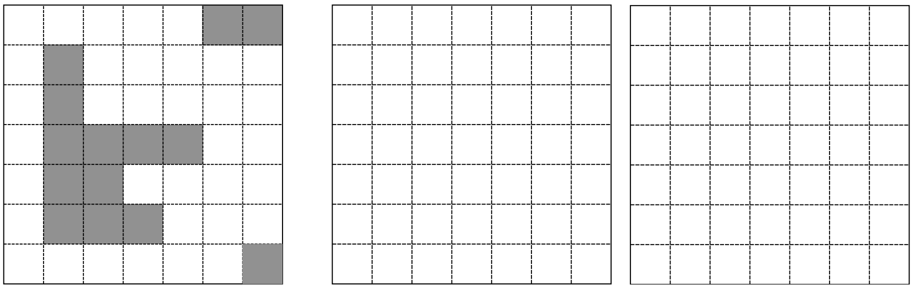
$$\epsilon_B(X) = X \ominus B = \{z \in E | B_z \subseteq X\} = \bigcap_{b \in B} X_{-b}\,;$$

- Opening: $A \circ B = (A \ominus B) \oplus B$;
- Closing: $A \bullet B = (A \oplus B) \ominus B$.

For grayscale/scalar images, the erosion and dilation are given as follows:

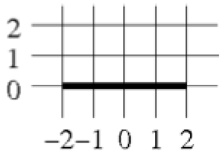
$$(F \oplus G)(x) = \sup_{y \in E}\{F(y) + G(x-y)\}\,,$$
$$(F \ominus G)(x) = \inf_{y \in E}\{F(y) - G(x-y)\}\,.$$

Question 7 *Consider the following set $X \subset \mathbb{R}^2$ given by black unit squares (left).*

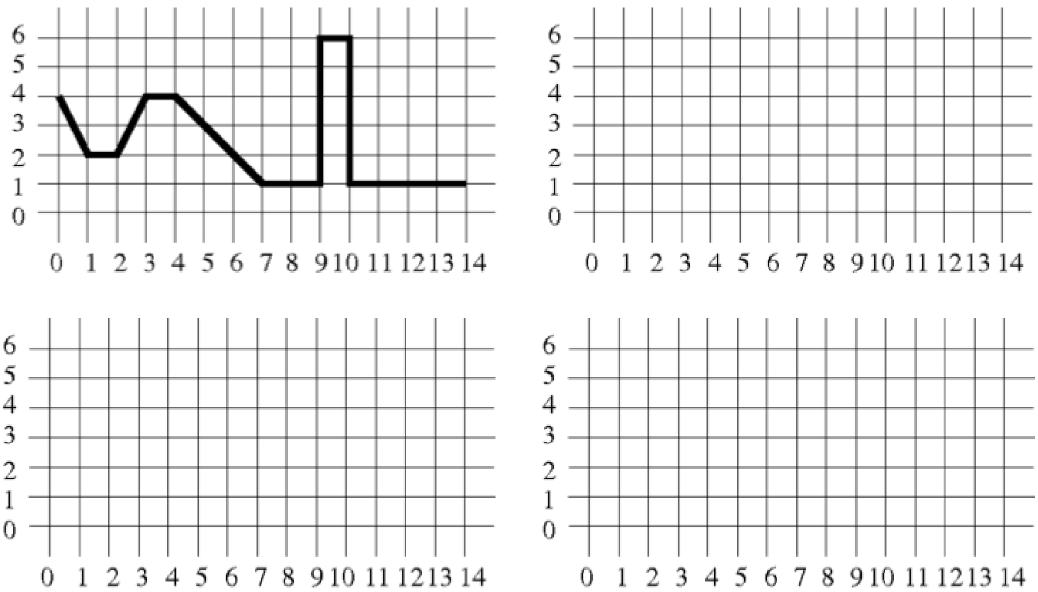


Could you please draw the result of the dilation (middle) and the result of the erosion (right) of X by a structuring element B which is an open Euclidean disk with radius 1.

Question 8 *Let us consider this simple structuring element on scalar functions:*



We consider a simple 1D grayscale image as depicted below (please, only consider the discrete values only). Using the above mentioned structuring element, use the empty grids to draw the result of the erosion and dilation operators of this 1D image.



Question 9 *Skyler have tested some preliminary operators (see Fig 2). On the input image Fig 2–(a), she got the two images Fig 2–(b) and Fig 2–(c). What morphological operators did she use ? Why has she considered them ?*

(a)

(b) (c)

Figure 2: License plate example: (a) Skyler's input and $(b) - (c)$ her results.

3 Pinkman's Theorem

Before breaking bad as a cooker in Gus's restaurant, Jesse was a nice college student. In fact, he proved a great result on lattice polygon, the Pinkman's theorem. Actually, several typos later, this result is now known as *Pick's theorem* (very famous in digital geometry).

For short, given a simple polygon P with integer coordinate vertices (lattice polygon), this theorem links the (Euclidean) area of P $\mathcal{A}(P)$ with its number of (digital) interior points I_P and the number of (digital) points on its boundary B_P:

$$\mathcal{A}(P) = I_P + \frac{B_P}{2} - 1 \tag{1}$$

Question 10 *For shapes depicted in Figure 3, give the $\mathcal{A}$, I and B quantities and show that Eq. (1) is true for these lattice polygons.*

To prove this theorem, let us first consider elementary shapes:

Question 11 *For a rectangle with width m and height n (in Fig. 3, we have $m = 3$ and $n = 4$ for the rectangle), prove that the result holds.*

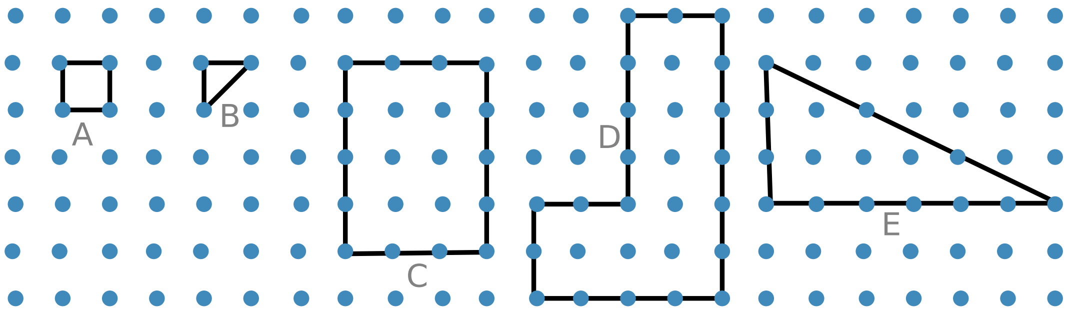


Figure 3: Examples for Pinkman's theorem.

Question 12 *Consider now a* axis-aligned *right triangle T (rightmost shape in Fig. 3), prove the theorem for this object.*

Hint: use the result of previous question with a variable k which is the –unknown– number of digital points on the triangle diagonal

To get the final result, we need a last ingredient: the general triangle.

Question 13 *Using previous results (or assuming that the theorem is true for rectangles and axis-aligned right triangles), prove that the result also holds for triangles with interior angles less than $\frac{pi}{2}$.*

Hint: consider the bounding box of the triangle and decompose it into T and three right triangles.

If one of the triangle angle is greater than $\frac{pi}{2}$, we assume that Eq. (1) is true.

Question 14 *Consider now two lattice polygons P_1 and P_2, we assume that P is given by gluing together P_1 and P_2 along an edge e (the edge e belongs to both lattice polygon P_1 and P_2).*

If Eq. (1) is true for both P_1 and P_2, prove that this result is also true for P.

Hint: consider the number of points on e.

Question 15 *Using the results of Exercise 1, can you conclude to prove the theorem for general simple lattice polygon ?*

Now, let's play a bit with arithmetic of lattice polygons (again, all points and vectors have integer coordinates).

Question 16 *For a straight segment $[(0,0)-(u,v)]$ what are the properties on u and v to have only two digital points on it, the extremities ?*

Question 17 *Let P be a parallelogram defined by the two vectors $(37,7)^T$ and $(9,2)^T$ and origin point $(0,0)$. How many boundary points have P ? How many interior points ?*

Question 18 *Applying Eq. (1), what is the area of the parallelogram defined by $(37,7)^T$ and $(9,2)^T$?*

and more generally:

Question 19 *For two vectors* $(u,v)^T$ *and* $(w,t)^T$*, what are the properties on* u, v, w *and* t *so that there is no interior point in the parallelogram defined by these vectors and* $(0,0)$ *?*

Question ⋆ 20 *Let's apply the following transformation to each vertex* $(i,j) \in \mathbb{Z}^2$ *of a lattice polygon* P*:*

$$(x,y) = \begin{pmatrix} u & w \\ v & t \end{pmatrix} \cdot \begin{pmatrix} i \\ j \end{pmatrix} \tag{2}$$

using u, v, w, t *has described in Question 19. Let* P' *by the resulting polygon from the transformation of* P *vertices. Is the obtained polygon* P' *still a lattice polygon ? What is its area and number of boundary points ?*

Question ⋆ 21 *If we suppose that the lattice polygon has* n *holes (each hole boundary being defined by a simple lattice polygon), a similar result for Eq. (1) exists. Can you guess the general formula ? (Hints: draw some examples with 1, 2 and 3 holes...)*

Let's be honest, Jesse was not really interested in mathematics, original result is due to Georg Alexander Pick in 1899.