TD: “Images et géométrie discrète”
Extended Euclid’s Algorithm

Let us consider the following Euclidean division algorithm.

Procedure `Convergents(` *(a,b), (p,q), (p’,q’), i*)

Input: (a, b), (p, q), (p', q'), i
Output: (p', q')

Let r be the remainder of the Euclidean division b/a;
Let u be the quotient of the Euclidean division b/a;
$p'' \leftarrow up' + p$;
$q'' \leftarrow uq' + q$;
if $r > 0$ **then**
  return *Convergents(*(r, a), (p', q'), $(p'', q''), i + 1$*)*;
else
  return (p', q')

Question 1 *Let* $(p_{-1}, q_{-1}) = (1, 0)$ *and* $(p_0, q_0) = (0, 1)$, *what is the output of* `Convergents`*(*$(5, 8)$, (p_{-1}, q_{-1}), (p_0, q_0), *0) ?*

Question 2 *Let us consider* `Convergents`*(*(a, b), (p_{-1}, q_{-1}), (p_0, q_0), *0) (with* $0 \leq a < b$ *and* $\gcd(a, b) = 1$*). We index the recursive calls by* $i = 1 \ldots n$. *Show that*

$$\forall i = 1 \ldots n \,, p_i = u_i p_{i-1} + p_{i-2} \text{ and } q_i = u_i q_{i-1} + q_{i-2} \,.$$

Question 3 *Similarly, with* $r_{-1} = b$ *and* $r_0 = a$, *show that*

$$\forall i = 1 \ldots n \,, r_i = r_{i-2} - u_i r_{i-1}$$

Question 4 *Following previous results, prove the following statements:*

1. $\forall i = 1 \ldots n \,, p_{i-1} q_i - q_{i-1} p_i = \pm 1$
2. $\forall i = -1 \ldots n \,, p_i b - q_i a = \pm r_i$

Since $r_n = gcd(a, b) = 1$, *what is* $p_n b - q_n a$ *?*

Question 5 *Give the definition of uni-modularity. What is the geometrical interpretation of this definition ?*

Question 6 *In the domain in appendix, draw the Euclidean segment* $[(0, 0) - (b, a)]$ *and all convergents* (q_i, p_i) *for the input given in Question 1. With respect to the parity of* i, *can you say something on the position of convergents with respect to the segment ? If we construct a polygonal curve with only convergents* (q_i, p_i) *with even index* i *(plus a last point* (b, a)*). What kind of geometrical object I have constructed ?*

Question 7 *Let* L_{odd} *(resp.* L_{even}*) be the polygonal curve of convergents with odd index (resp. even index). Furthermore, we add the point* (b, a) *to the end of each list. For the input given in Question 1, are there integer points between* L_{odd} *and* L_{even} *? Why ?*

Question 8 *For a general setting* `Convergents`*(*(a, b), (p_{-1}, q_{-1}), (p_0, q_0), *0), can you prove the statement of the previous question ?*

Question 9 *What is the complexity of* `Convergents` *with respect to* a *and* b *?*