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ACCURATE CALCULATION OF PROLATE SPHEROIDAL RADIAL FUNCTIONS OF THE FIRST KIND AND THEIR FIRST DERIVATIVES

By

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Abstract. Alternative expressions for calculating the prolate spheroidal radial functions of the first kind $R_{ml}^{(1)}(c,\xi)$ and their first derivatives with respect to ξ are shown to provide accurate values, even for low values of $l-m$ where the traditional expressions provide increasingly inaccurate results as the size parameter c increases to large values. These expressions also converge in fewer terms than the traditional ones. They are obtained from the expansion of the product of $R_{ml}^{(1)}(c,\xi)$ and the prolate spheroidal angular function of the first kind $S_{ml}^{(1)}(c,\eta)$ in a series of products of the corresponding spherical functions. King and Van Buren [12] had used this expansion previously in the derivation of a general addition theorem for spheroidal wave functions. The improvement in accuracy and convergence using the alternative expressions is quantified and discussed. Also, a method is described that avoids computer overflow and underflow problems in calculating $R_{ml}^{(1)}(c,\xi)$ and its first derivative.

1. Introduction. The scalar Helmholtz wave equation for steady waves, $(\nabla^2+k^2)\Psi=0$, where $k=2\pi/\lambda$ and λ is the wavelength, is separable in the prolate spheroidal coordinates (ξ,η,φ), with $1\le\xi\le\infty$, $-1\le\eta\le1$, and $0\le\phi\le2\pi$. The factored solution is given by $\Psi_{ml}(\xi,\eta,\varphi)=R_{ml}(c,\xi)S_{ml}(c,\eta)\Phi_m(\varphi)$, where $R_{ml}(c,\xi)$ is the radial function, $S_{ml}(c,\eta)$ is the angular function, and $\Phi_m(\varphi)$ is the azimuthal function. Here $c=ka/2$, where a is the interfocal distance of the elliptic cross section of the spheroid. The radial function of the first kind $R_{ml}^{(1)}(c,\xi)$ and the radial function of the second kind $R_{ml}^{(2)}(c,\xi)$ are the two independent solutions to the second-order radial differential equation resulting from the separation of variables. These solutions are dependent on four parameters (m,l,c,ξ) and an eigenvalue (separation constant) $A_{ml}(c)$. Similarly, $S_{ml}^{(1)}(c,\eta)$ and $S_{ml}^{(2)}(c,\eta)$ are the two independent solutions to the second-order angular differential equation resulting from the separation of variables. In the following discussion, we assume that m is either zero or a positive integer, with l equal to $m, m+1, m+2,\ldots$.

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Numerous investigators have developed computer codes to calculate both the prolate angular and radial functions. For example, the Naval Research Laboratory (NRL) published Fortran computer programs in 1970 for calculating both the first and second kind radial functions and their first derivatives [1] and the angular function of the first kind [2]. These codes were used to generate three volumes of tables of numerical values for the prolate radial functions and their first derivatives [3]. The volumes contain entries for the following range of parameters: $m = 0$ (Volume 1), $m = 1$ (Volume 2), $m = 2$ (Volume 3), $l = m, m+1, \ldots, m+49$; $\xi = 1.00000001, 1.0000001, \ldots, 1.01, 1.02(0.02)1.2, 1.4(0.2)2.0, 4.0(2.0)10.0$; $c = 0.1(0.1)1.0, 2.0(1.0)10.0, 12.0(2.0)30.0, 35.0, 40.0$. (The notation $1.02(0.02)1.2$ indicates $1.02, 1.04, 1.06, \ldots, 1.2$.) A single volume of tables of numerical values for the prolate angular functions and the linear prolate eigenvalues was published in 1975 [4]. This volume covers the ranges: $m = 0$; $l = 0(1)49$; $\theta = 0°(1°)90°$, where $\eta = \cos\theta$; $c = 0.1(0.1)1.0, 2.0(1.0)10.0, 12.0(2.0)30.0, 35.0, 40.0$.

NRL published a companion Fortran computer program for calculating the oblate spheroidal radial functions and their first derivatives [5]. The program for calculating the prolate angular functions [2] can also calculate the oblate angular functions. NRL published three volumes of oblate radial functions [6] and one volume of oblate angular functions [7].

The 1970 computer programs utilized the exponent size (± 307) and the arithmetical precision (26 decimal digits in double precision) available on the CDC 3800 computer at NRL. These programs are not easily modified to run on computers with a significantly smaller exponent range or with less precision. We developed a new "universal" computer program in 1981 [8] for calculating the prolate angular functions of the first kind. This program, written in standard FORTRAN77, is designed for optimum performance with any specified exponent range and arithmetical precision. [A related universal computer program [9] calculates the linear prolate functions and eigenvalues. These functions, which are useful in the representation of band-limited and time-limited physical processes, are constructed from the prolate angular functions with m set equal to zero.]

We are presently developing a companion universal computer program for calculating the prolate radial functions and their first derivatives that will replace the earlier 1970 program. During this effort we have identified an expression for calculating the prolate radial functions of the first kind that is a significant improvement over the traditional expression used for this purpose. Meixner and Schäfke [10, p. 309] derived this alternative expression nearly 50 years ago, but we believe that we are the first to use it to calculate $R^{(1)}_{ml}(c,\xi)$. The expression follows directly from the expansion of the product of the radial and angular functions of the first kind in a series of products of the corresponding spherical Bessel and associated Legendre functions. Meixner and Schäfke [10, p. 307] provided the product expansion in a form that is also applicable to $R^{(2)}_{ml}(c,\xi)$. The product expansion for $R^{(1)}_{ml}(c,\xi)$ is also given by Flammer [11], but with a typographical error. King and Van Buren [12] used the product expansion in 1973 in the derivation of a general addition theorem for the spheroidal functions that includes both translation and rotation. We note that the addition theorem was rediscovered and published by others in the 1980s, first for translation alone [13] and then with rotation included [14].

We can use the product expansion to calculate $R_{ml}^{(1)}(c,\xi)$ by choosing a value for the angular coordinate η and evaluating the resulting expression. When the angular coordinate η is chosen equal to unity, the product expansion reduces to the traditional expression for $R_{ml}^{(1)}(c,\xi)$ [see, e.g., Ref. 11, p. 31]. We show that this expression suffers from loss of accuracy for low values of $l-m$ due to subtraction errors that increase as c increases. It also suffers subtraction errors at higher values of $l-m$, the error increasing as ξ approaches unity. For example, over 15 decimal digits of accuracy are lost when m and l are equal to zero and c is equal to 40. We show that when η is set equal to zero instead, one obtains an expression for $R_{ml}^{(1)}(c,\xi)$ that provides values with nearly full accuracy. Calculations show that this alternative expression requires no more terms for convergence than the traditional expression. In fact, it requires fewer terms for low values of $l-m$. We describe an approach to the calculation of $R_{ml}^{(1)}(c,\xi)$ that avoids the overflow and underflow problems that can arise from the large dynamic ranges involved in components of the expressions, and even in $R_{ml}^{(1)}(c,\xi)$ itself. We conclude the paper with a summary.

2. Angular functions of the first kind. The prolate angular function of the first kind $S_{ml}^{(1)}(c,\eta)$ is expressed [see, for example, ref. 11, p. 16] in terms of the corresponding associated Legendre functions of the first kind by

$$S_{ml}^{(1)}(c,\eta) = \sum_{n=0,1}^{\infty}{}' d_n(c|ml) P_{m+n}^m(\eta), \tag{1}$$

where the prime sign on the summation indicates that $n = 0, 2, 4, \ldots$ if $l-m$ is even or $n = 1, 3, 5, \ldots$ if $l-m$ is odd. A three-term recursion formula relates successive expansion coefficients d_n, d_{n+2}, and d_{n+4} for given values of l, m, and c. Use of this formula to calculate the expansion coefficients requires a value for the separation constant or eigenvalue $A_{ml}(c)$, which is chosen to ensure nontrivial convergent solutions for $S_{ml}^{(1)}(c,\eta)$. An efficient algorithm given by Bouwkamp [15] provides accurate values for both the eigenvalue and the ratios of successive coefficients d_{n+2}/d_n.

The coefficients are then normalized to ensure that the angular functions reduce exactly to the corresponding associated Legendre function when c becomes zero. One way to do this is to match their behavior at a particular value of η. Flammer [11] and Chu and Stratton [16] normalize at $\eta = 0$ while Morse and Feshbach [17] and Page [18] normalize at $\eta = 1$. We choose instead the scheme used by Meixner and Schäfke [10] where $S_{ml}^{(1)}(c,\eta)$ has the same normalization factor as $P_l^m(\eta)$, i.e.,

$$\int_{-1}^{1} [S_{ml}^{(1)}(c,\eta)]^2 \, d\eta = \int_{-1}^{1} [P_l^m(\eta)]^2 \, d\eta = \frac{2(l+m)!}{(2l+1)(l-m)!}. \tag{2}$$

Substituting the expansion for $S_{ml}^{(1)}(c,\eta)$ given in (1) and using the known orthogonality properties of $P_l^m(\eta)$, we obtain the normalizing relation for d_n:

$$\sum_{n=0,1}^{\infty}{}' \frac{2(n+2m)!}{[2(n+m)+1]n!} [d_n(c|ml)]^2 = \frac{2(l+m)!}{(2l+1)(l-m)!}. \tag{3}$$

This normalization has the practical advantage of eliminating the need to numerically evaluate the normalization factor that is often encountered in problems involving expansions in spheroidal angular functions. Use of (3) also avoids introducing inaccuracies from subtraction errors into the normalized d_n since all terms in the series are positive. Use of the Morse and Feshbach normalization for the prolate case introduces subtraction errors that result in a loss of accuracy for d_n at low l that increases as c increases. The Flammer normalization scheme is numerically robust and does not introduce subtraction errors for the prolate case.

3. Expansion of the product of the radial and angular functions. The expansion of the product of $R_{ml}^{(1)}(c,\xi)$ and $S_{ml}^{(1)}(c,\eta)$ in terms of the corresponding spherical functions is given by

$$R_{ml}^{(1)}(c,\xi)S_{ml}^{(1)}(c,\eta) = \sum_{n=0,1}^{\infty}{}' i^{n+m-l} d_n(c|ml) j_{n+m}(kr) P_{n+m}^m(\cos\theta). \tag{4}$$

This is a special case of the more general expansion given by Meixner and Schäfke [10, p. 307]. It is also found in Flammer [11, p. 48] with the unfortunate omission of the factor i^{n+m-l}. Using the relationship between the spherical coordinates r and θ and spheroidal coordinates (about the same origin and with $\eta = 1$ coincident with $\theta = 0$) we obtain $kr = c\sqrt{\xi^2+\eta^2-1}$ and $\cos\theta = \xi\eta/\sqrt{\xi^2+\eta^2-1}$. Substituting for $S_{ml}^{(1)}(c,\eta)$ from (1) and solving for $R_{ml}^{(1)}(c,\xi)$ produces

$$R_{ml}^{(1)}(c,\xi) = \frac{\sum_{n=0,1}^{\infty}{}' i^{n+m-l} d_n(c|ml) j_{n+m}(c\sqrt{\xi^2+\eta^2-1}) P_{n+m}^m(\eta\xi/\sqrt{\xi^2+\eta^2-1})}{\sum_{n=0,1}^{\infty}{}' d_n(c|ml) P_{n+m}^m(\eta)}. \tag{5}$$

The significance of this general expression is that it allows us to choose the value for η that provides the maximum accuracy for calculated values of $R_{ml}^{(1)}(c,\xi)$.

We consider first the case when $\eta = 1$. The argument of P_{n+m}^m in both the numerator and the denominator approaches unity as η approaches unity. Although P_{n+m}^m approaches zero in this case for $m \neq 0$, the limit of the right-hand side of (5) exists and is obtained using the following behavior of $P_{n+m}^m(x)$ near unity:

$$P_{n+m}^m(x) \underset{x\to 1}{\longrightarrow} (1-x^2)^{m/2}\frac{(n+2m)!}{2^m m! n!}. \tag{6}$$

Taking the limit of the right-hand side of (3) and using (6), we obtain

$$R_{ml}^{(1)}(c,\xi) = \lim_{\eta\to 1}\left\{\frac{1-[\eta^2\xi^2/(\xi^2+\eta^2-1)]}{1-\eta^2}\right\}^{m/2} \frac{\sum_{n=0,1}^{\infty}{}' i^{n+m-l} d_n(c|ml) j_{n+m}(c\xi)\frac{(n+2m)!}{n!}}{\sum_{n=0,1}^{\infty}{}' d_n(c|ml)\frac{(n+2m)!}{n!}}. \tag{7}$$

The limit of the term in braces is easily shown to be equal to $[(\xi^2-1)/\xi^2]^{m/2}$, resulting in the following expression traditionally given for $R_{ml}^{(1)}(c,\xi)$:

$$R_{ml}^{(1)}(c,\xi)=\left(\frac{\xi^2-1}{\xi^2}\right)^{m/2}\frac{\sum\limits_{n=0,1}^{\infty}{}' \, i^{n+m-l}d_n(c|ml)j_{n+m}(c\xi)\frac{(n+2m)!}{n!}}{\sum\limits_{n=0,1}^{\infty}{}' \, d_n(c|ml)\frac{(n+2m)!}{n!}}. \tag{8}$$

Flammer [11, p. 31] derives (8) using integral representations of the spheroidal wave functions. The corresponding expression for the first derivative of $R_{ml}^{(1)}(c,\xi)$ with respect to ξ is obtained by taking the first derivative of the right-hand side of (8).

Equation (8) is the expression normally used to calculate numerical values for $R_{ml}^{(1)}(c,\xi)$. It provides reasonably accurate results when neither c nor $l-m$ is large. However, one or both of the series in (8) can suffer subtraction errors outside these ranges, i.e., the sum of the positive terms in the series becomes close in magnitude to the sum of the negative terms in the series. The subtraction error is defined to be the number of accurate decimal digits that are lost in calculating the sum of the series. It is equal to the number of leading decimal digits that are the same in the positive and negative sums.

For the series in the denominator of the right-hand side of (8), the subtraction error increases as c increases. At a given value of c and for $m=0$, the error is greatest at $l=0$, decreases as l increases, and becomes negligible for l somewhat larger than $2c/\pi$. As m increases for a given value of c, the error decreases at $l=m$ and falls off more slowly with increasing $l-m$. Figure 1 shows examples of this behavior. We calculated the subtraction errors shown here by keeping track of both positive and negative terms while summing the series and comparing the two subsums. Use of quadruple precision arithmetic (128 bits) allowed us to calculate the subtraction error up to 30 digits.

We note that the series in the denominator of the right-hand side of (8) can be replaced by $(l+m)!/(l-m)!$ if one normalizes the d_n coefficients according to Morse and Feshbach [17]. Unfortunately, this does not remove the subtraction error. The d_n coefficients in the numerator series now become inaccurate to the same degree since their normalization requires evaluation of the same series. Of course, we can calculate $R_{ml}^{(1)}(c,\xi)$ from (8) without normalizing the d_n, since the radial functions, unlike the angular functions, do not depend on the choice of normalization. This is evident from the form of (8).

For the series in the numerator of the right-hand side of (8), there is a subtraction error that increases as $l-m$ increases beyond values somewhat larger than $2c/\pi$. For given c, m, and $l-m$, the error increases as ξ approaches unity. For given ξ, c, and $l-m$, the error decreases as m increases. The numerator series also suffers subtraction errors for all values of ξ at smaller values of $l-m$. This subtraction error is comparable to that of the denominator series for the same values of c, m, and $l-m$. Figure 2 shows the subtraction error incurred in the numerator series for selected parameter choices.

The inaccuracy introduced into $R_{ml}^{(1)}(c,\xi)$ from subtraction errors is the larger of the subtraction errors in the numerator and denominator series. We note that a subtraction error larger than that shown in Fig. 2 is encountered in the numerator series when ξ is near one of the roots of $R_{ml}^{(1)}(c,\xi)$. The subtraction error in calculating the first derivative of $R_{ml}^{(1)}(c,\xi)$ with respect to ξ using the traditional expression is comparable to

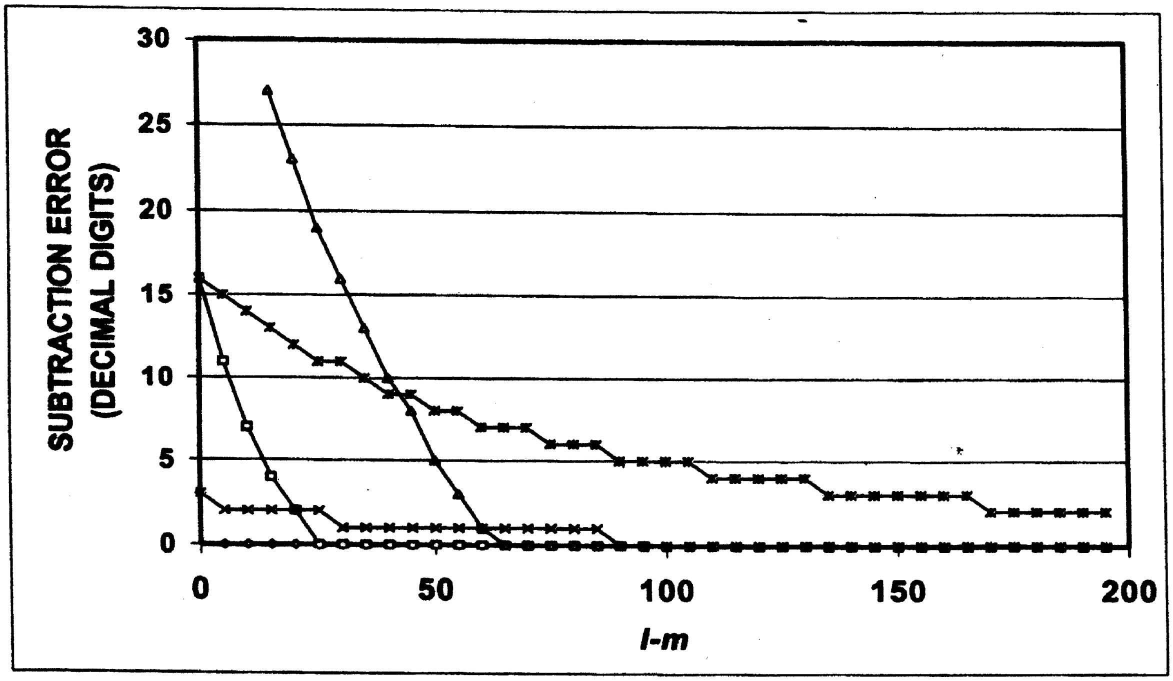

FIG. 1. Subtraction error in decimal digits obtained in the denominator of the traditional expression (8) plotted versus $l-m$ for selected parameters (c, m) : $\diamond(10,0)$; $\square(40,0)$; $\times(40,100)$; $\Delta(100,0)$; $\star(100,100)$

that shown in Fig. 2 for the numerator and identical to that of Fig. 1 for the denominator. Calculated values for $R_{ml}^{(1)}(c,\xi)$ and its first derivative have additional inaccuracy due to a lack of full accuracy in the d_n coefficients and the spherical Bessel functions used in their calculation. This inaccuracy can be limited to approximately two decimal digits with careful numerical technique. We have confirmed this by comparison of the theoretical value for the Wronskian with its value computed from numerical values for the radial functions of the first and second kinds and their first derivatives. We find that the number of decimal digits of agreement is consistent with accuracy estimates for the functions obtained by subtracting their subtraction error plus two digits from the number of decimal digits used in the calculations.

4. An alternative expression for calculating $R_{ml}^{(1)}(c,\xi)$. We now return to (5) and take the limit of the right-hand side as η approaches zero. For $l-m$ even, i.e., for n even, $P_{n+m}^{m}(0)$ is finite and nonzero. We have the following expression for $R_{ml}^{(1)}(c,\xi)$:

$$R_{ml}^{(1)}(c,\xi) = \frac{\sum\limits_{n=0,1}^{\infty}{}' \, i^{n+m-l} d_n(c|ml) j_{n+m}(c\sqrt{\xi^2-1}) P_{n+m}^{m}(0)}{\sum\limits_{n=0,1}^{\infty}{}' \, d_n(c|ml) P_{n+m}^{m}(0)}, \quad l-m \text{ even.} \tag{9}$$

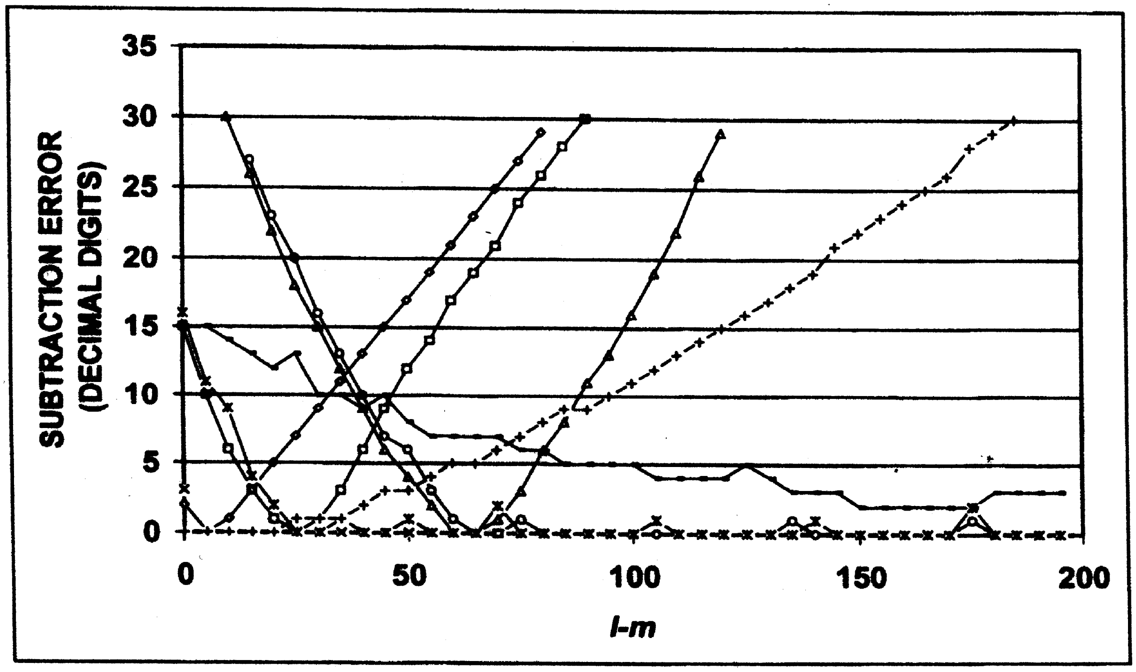

FIG. 2. Subtraction error in decimal digits obtained in the numerator of the traditional expression (8) plotted versus $l - m$ for selected parameters (c, m, ξ): $\diamond(10, 0, 1.00000001)$; $\times(10, 0, 10)$; $+(10, 100, 1.01)$; $\square(40, 0, 1.00000001)$; $\star(40, 0, 10)$; $\Delta(100, 0, 1.00000001)$; $\circ(100, 0, 10)$; $-(100, 100, 10)$

We can evaluate (9) using the expression for $P^m_{n+m}(0)$ found, e.g., in [11, p. 20]:

$$P^m_{n+m}(0) = \frac{(-1)^{n/2}(n+2m)!}{2^{n+m}(n/2)![(n+2m)/2]!}, \quad l-m \text{ even.} \tag{10}$$

When $l - m$ is odd, i.e., n is odd, $P^m_{n+m}(0)$ is equal to zero and both the numerator and denominator vanish. However, the limit of the right-hand side of (5) as η approaches zero exists and is found by L'Hospital's rule. We obtain

$$R^{(1)}_{ml}(c,\xi) = \left(\frac{\xi}{\sqrt{\xi^2-1}}\right) \frac{\sum\limits_{n=0,1}^{\infty}{}' \, i^{n+m-l} d_n(c|ml) j_{n+m}(c\sqrt{\xi^2-1}) \left[\frac{dP^m_{n+m}(\eta)}{d\eta}\right]_{\eta=0}}{\sum\limits_{n=0,1}^{\infty}{}' \, d_n(c|ml) \left[\frac{dP^m_{n+m}(\eta)}{d\eta}\right]_{\eta=0}}, \quad l-m \text{ odd.} \tag{11}$$

The leading factor in the right-hand side results from the limit of $\frac{d}{d\eta}(n\xi/\sqrt{\xi^2+\eta^2-1})$. The first derivative of P^m_{n+m} at $\eta = 0$ is given by the expression [11, p. 20]:

$$\left[\frac{dP^m_{n+m}(\eta)}{d\eta}\right]_{\eta=0} = \frac{(-1)^{(n-1)/2}(n+2m+1)!}{2^{n+m}[(n-1)/2)]![(n+2m+1)/2]!}, \quad l-m \text{ odd.} \tag{12}$$

We note that (6), (10), and (12) are based on Ferrer's definition of P^m_{n+m} [19]. A factor of $(-1)^m$ must be added if one uses instead the definition found in Abramowitz and Stegun

[20, p. 334]. Since this factor appears in both the numerator and denominator of the expressions for $R_{ml}^{(1)}(c,\xi)$, it cancels with no effect on the calculated value.

Both (9) and (11) are numerically robust, providing values for $R_{ml}^{(1)}(c,\xi)$ with no subtraction error unless ξ is near a root of $R_{ml}^{(1)}(c,\xi)$. We demonstrated this by calculating the subtraction errors for a wide range of parameters including values of c as large as 10,000, l as large as 3,000, and m as large as 500.

In order to calculate $R_{ml}^{(1)}(c,\xi)$ from (9) and (11), we first obtain the ratios d_{n+2}/d_n by use of the Bouwkamp procedure [15]. We calculate ratios of successive spherical Bessel functions $j_{n+1}(c\sqrt{\xi^2-1})/j_n(c\sqrt{\xi^2-1})$ using the standard backward recursion. Individual Bessel functions are then calculated by forward multiplication of the ratios starting with j_0 or with j_1 if $\sin(c\sqrt{\xi^2-1})$ is less than 0.5. To avoid computer underflow, we strip out and store separately the integer exponent p of each Bessel function (base 10) as we proceed. The remaining characteristic $j_n(c\sqrt{\xi^2-1}) \times 10^{-p}$ has a magnitude between 1 and 10.

We start the calculation with the $n = l - m$ term in both the numerator and denominator series. This tends to be the largest term (in magnitude) in both series unless ξ is near unity, where the largest term in the numerator series can occur at smaller values of n. The coefficient i^{n+m-l} is unity for the $n = l - m$ term. We factor and cancel both $d_{l-m}(c|ml)$ and either $P_l^m(0)$ or $[dP_l^m(\eta)/d\eta]_{\eta=0}$ from both the numerator and denominator series in (9) and (11). We also factor $j_l(c\sqrt{\xi^2-1})$ from the numerator series, leaving unity as the $n = l - m$ term in both series. The ratios $P_{n+m+2}^m(0)/P_{n+m}^m(0)$ and $(dP_{n+m+2}^m(\eta)/d\eta)_{\eta=0}/(dP_{n+m}^m(\eta)/d\eta)_{\eta=0}$ are both expressed by $-(n+2m+2-\gamma)/(n+1+\gamma)$, where γ is equal to unity if $l-m$ is even and zero otherwise.

Each series is now summed separately forward and backward. We compute each term in the series by multiplication (forward series) or division (backward series) of the previous term by the ratios described above. For example, the $n = l-m+2$ term in the forward numerator series of both (9) and (11) is equal to $[d_{l-m+2}/d_{l-m}][j_{l+1}/j_l][j_{l+2}/j_{l+1}][(l+m+2-\gamma)/(l-m+1+\gamma)]$, while the corresponding denominator term is $-[d_{l-m+2}/d_{l-m}][(l+m+2-\gamma)/(l-m+1+\gamma)]$. Each of the series is summed to the desired convergence, forward and backward sums are added, the numerator sum is divided by the denominator sum, and the ratio is multiplied by the stored value for $j_n(c\sqrt{\xi^2-1}) \times 10^{-p}$ and then multiplied by $\xi/\sqrt{\xi^2-1}$ when $l-m$ is odd. We then strip the integer exponent q out of the result, leaving the characteristic for $R_{ml}^{(1)}(c,\xi)$. The exponent for $R_{ml}^{(1)}(c,\xi)$ is equal to $p+q$. If the value for $R_{ml}^{(1)}(c,\xi)$ falls within the dynamic range of the highest precision available for real numbers (usually quadruple precision), then we can take the product of its characteristic times 10^{p+q}. Otherwise, we can leave it separated into its characteristic and exponent.

When ξ is near unity, we must take special care to avoid overflow in the backward summation for the numerator series. Here the sum of the series increases in magnitude as n decreases until it is approximately equal to the ratio of $R_{ml}^{(1)}(c,\xi)$ to $j_l(c\sqrt{\xi^2-1})$ for $l-m$ even, and approximately equal to this ratio multiplied by $\sqrt{\xi^2-1}/\xi$ for $l-m$ odd. We reduce the possibility of overflow by replacing unity for the first term in the backward sum by 10^{-r}, where $r = \text{nex} - \text{ndec} - 10$, with nex and ndec being the maximum exponent and number of decimal digits used in the calculation. However, overflow can still occur at very large values of $l-m$. In this case, we use the procedure described above for stripping the integer exponent out of the series. Our approach to calculating $R_{ml}^{(1)}(c,\xi)$ avoids the computer overflow and underflow problems that can arise from the large dynamic ranges involved in the component terms in (9) and (11), and even in $R_{ml}^{(1)}(c,\xi)$ itself.

We compared the convergence of (9) and (11) to the traditional expression (8) for a wide range of values for m, l, c, ξ. Both the numerator and the denominator series of (9) and (11) converged in the same or fewer terms than the corresponding series in (8). The reduction in the number of terms in both the numerator and the denominator series is largest for $l-m$ equal to 0, decreases with increasing $l-m$, and becomes insignificant for $l-m$ much greater than $2c/\pi$. The reduction in the number of terms in the numerator series also increases as ξ approaches unity.

For example, consider the situation where the convergence threshold is set to 10^{-31} in quadruple precision arithmetic. When (m,l,c,ξ) is equal to $(0,0,10,1.00000001)$, only 5 terms are required in the numerator of (9) while the numerator of (8) requires 18 terms. When l is increased to 50, the numerator of (9) now requires 53 terms while (8) requires 60. When l is equal to 99, we require 102 and 107 terms, respectively. For $(0,0,10,1.01)$, we require 11 and 18 terms, respectively. When $l = 50$, we require 56 and 60 terms and when $l = 99$, we require 105 and 107 terms. For $(0,l,10,10)$, nearly the same number of terms is required in the numerators. In all of the above cases, there are no significant differences in the number of terms required in the denominators. Similar results were obtained for $m = 100$.

Corresponding expressions for the first derivative of $R_{ml}^{(1)}(c,\xi)$ with respect to ξ are readily obtained from (9) and (11). We have

$$\frac{dR_{ml}^{(1)}(c,\xi)}{d\xi} = \left(\frac{c\xi}{\sqrt{\xi^2-1}}\right) \frac{\sum\limits_{n=0,1}^{\infty}{}' \, i^{n+m-l} d_n(c|ml) j'_{n+m}(c\sqrt{\xi^2-1}) P_{n+m}^m(0)}{\sum\limits_{n=0,1}^{\infty}{}' \, d_n(c|ml) P_{n+m}^m(0)}, \quad l-m \text{ even}, \tag{13}$$

$$\begin{aligned} \frac{dR_{ml}^{(1)}(c,\xi)}{d\xi} &= -\left(\frac{1}{\xi(\xi^2-1)}\right) R_{ml}^{(1)}(c,\xi) \\ &+ \left(\frac{c\xi^2}{\xi^2-1}\right) \frac{\sum\limits_{n=0,1}^{\infty}{}' \, i^{n+m-l} d_n(c|ml) j'_{n+m}(c\sqrt{\xi^2-1}) \left[\frac{dP_{n+m}^m(\eta)}{d\eta}\right]_{\eta=0}}{\sum\limits_{n=0,1}^{\infty}{}' \, d_n(c|ml) \left[\frac{dP_{n+m}^m(\eta)}{d\eta}\right]_{\eta=0}}, \quad l-m \text{ odd}. \end{aligned} \tag{14}$$

Here $j'_{n+m}(x)$ is the first derivative with respect to x. One can calculate the right-hand side of (13) and (14) using values for $j'_{n+m}(x)$ obtained from backward recursion on an easily derived three-term recurrence relation for j'. Or one can substitute for $j'_{n+m}(x)$ its equivalent $(n/x)j_{n+m} - j_{n+m+1}$ and use values for the j's obtained from backward recursion on the three-term recurrence relation for j.

There is no subtraction error involved in evaluating the right-hand side of (13), but a modest subtraction error occurs in (14) for the limited region where ξ is near unity, m is equal to zero, and l is odd and small. This error results from a subtraction of the two terms on the right-hand side of (14). It is maximum for $l = 1$ and small c, decreasing with increasing l and c. We can approximate the maximum subtraction error in decimal digits by $\log_{10}(\xi - 1)$. For example, when $\xi = 1.00000001$, 8 digits are lost for $l = 1$ and $c = 0.1$. This still leaves about 6 digits of accuracy when the calculations are carried out in 64 bit double precision arithmetic (assuming two additional digits of accuracy are lost elsewhere in the calculation). At $l = 49$, the error for $c = 0.1$ has decreased to 4 digits. At $c = 40$ and 500, the error for $l = 1$ has decreased to 5 digits and 2 digits, respectively.

5. Summary. We have identified expressions for calculating the prolate spheroidal radial functions of the first kind $R_{ml}^{(1)}(c,\xi)$ and their first derivatives with respect to ξ that are numerically robust. These expressions do not suffer the subtraction errors that limit the use of the traditional expressions for calculating $R_{ml}^{(1)}(c,\xi)$ to non-large values of c and $l-m$. Moreover, these expressions tend to converge in fewer terms. We also described an approach to calculating $R_{ml}^{(1)}(c,\xi)$ that avoids the overflow and underflow problems that can arise from the large dynamic ranges involved in components of the expressions, and even in $R_{ml}^{(1)}(c,\xi)$ itself. Finally, we note that we are presently exploring the use of similar expressions for improved calculation of the radial functions of the second kind $R_{ml}^{(2)}(c,\xi)$ and their first derivatives.

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REFERENCES

[1] B. J. King, R. V. Baier, and S. Hanish, *A Fortran computer program for calculating the prolate spheroidal radial functions of the first and second kind and their first derivatives*, Naval Research Lab. Rpt. 7012 (1970)

[2] B. J. King and A. L. Van Buren, *A Fortran computer program for calculating the prolate and oblate angle functions of the first kind and their first and second derivatives*, Naval Research Lab. Rpt. 7161 (1970)

[3] S. Hanish, R. V. Baier, A. L. Van Buren, and B. J. King, *Tables of radial spheroidal wave functions, volumes* 1, 2, 3, *prolate*, $m = 0, 1, 2$, Naval Research Lab. Rpts. 7088, 7089, and 7090 (1970)

[4] A. L. Van Buren, B. J. King, R. V. Baier, and S. Hanish, *Tables of angular spheroidal wave functions, vol. I, prolate*, $m = 0$, Naval Research Lab. Publication, U.S. Govt. Printing Office (1975)

[5] A. L. Van Buren, R. V. Baier, and S. Hanish, *A Fortran computer program for calculating the oblate spheroidal radial functions of the first and second kind and their first derivatives*, Naval Research Lab. Rpt. 6959 (1969)

[6] S. Hanish, R. V. Baier, A. L. Van Buren, and B. J. King, *Tables of radial spheroidal wave functions, volumes* 4, 5, 6, *oblate*, $m = 0, 1, 2$, Naval Research Lab. Rpts. 7091, 7092, and 7093 (1970)

[7] A. L. Van Buren, B. J. King, R. V. Baier, and S. Hanish, *Tables of angular spheroidal wave functions, vol. II, oblate*, $m = 0$, Naval Research Lab. Publication, U.S. Govt. Printing Office (1975)

[8] B. J. Patz and A. L. Van Buren, *A Fortran computer program for calculating the prolate spheroidal angular functions of the first kind*, Naval Research Lab. Memo. Rpt. 4414 (1981)

[9] A. L. Van Buren, *A Fortran computer program for calculating the linear prolate functions*, Naval Research Lab. Rpt. 7994 (1976)

[10] J. Meixner and F. W. Schäfke, *Mathieusche Funktionen und Sphäroidfunktionen, mit Anwendungen auf physikalische und technische Probleme*, Die Grundlehren der mathematischen Wissenschaften, Vol. 71, Springer-Verlag, Berlin, 1954

[11] C. Flammer, *Spheroidal Wave Functions*, Stanford Univ. Press, Stanford, CA, 1957

[12] B. J. King and A. L. Van Buren, *A general addition theorem for spheroidal wave functions*, SIAM J. Math. Anal. **4**, 149–160 (1973)

[13] B. P. Sinha and R. H. Macphie, *Translational addition theorems for spheroidal scalar and vector wave functions*, Quart. Appl. Math. **38**, 143–158 (1980)

[14] R. H. Macphie, J. Dalmas, and R. Deleuil, *Rotational-translational addition theorems for scalar spheroidal wave functions*, Quart. Appl. Math. **44**, 737–749 (1987)

[15] C. J. Bouwkamp, *Theoretical and numerical treatment of diffraction through a circular aperture*, IEEE Trans. Antennas and Propagation **AP-18**, 152–176 (1970)

[16] L. J. Chu and J. A. Stratton, *Elliptic and spheroidal wave functions*, J. Appl. Phys. **20**, 259–309 (1941)

[17] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, Parts I and II, McGraw-Hill, New York, 1953

[18] L. Page, *The electrical oscillations of a prolate spheroid* II, III, Phys. Rev. **65**, 98–117 (1944)

[19] E. Whittaker and G. Watson, *A Course in Modern Analysis*, 4th edition, Cambridge, 1952

[20] M. Abramowitz and C. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th printing, Dover, NY, 1972