# Formal Proof of a Gathering Algorithm in the Pactole Framework


Mathis Bouverot-Dupuis

Boss : Pierre Courtieu (CNAM)


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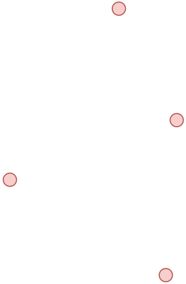
# Talk outline

- Suzuki & Yamashita's Model
- Weber Point Properties
- Alignment
- Gathering

# Suzuki & Yamashita's Model

Very simple (dumb) robots :

- Points in $\mathbb{R}^2$ (can overlap)



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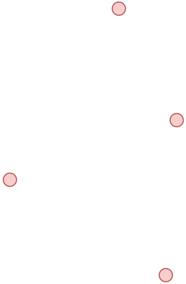

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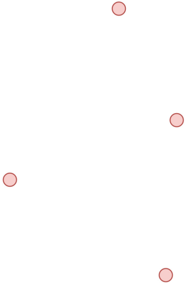

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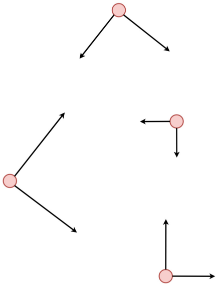

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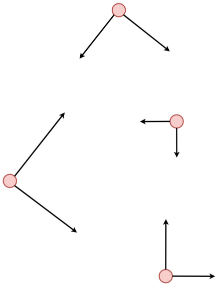

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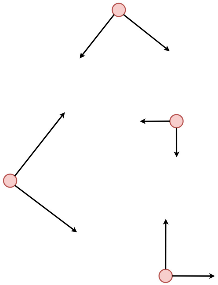

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- Similar to barycenter (sum of distances v.s. sum of distances squared).

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Why use the weber point ?

# Contraction Lemma

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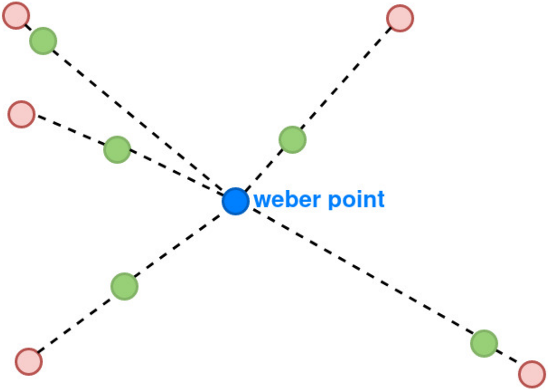

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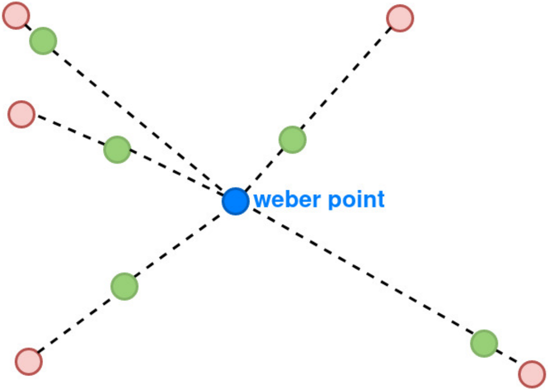

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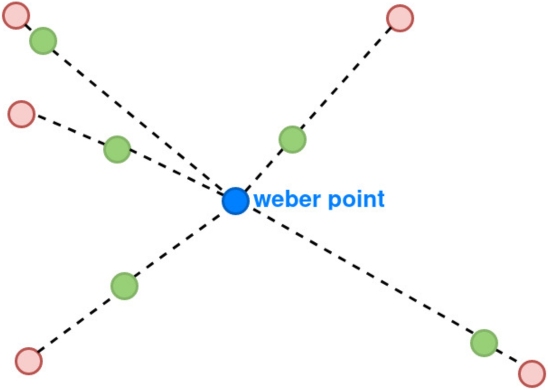

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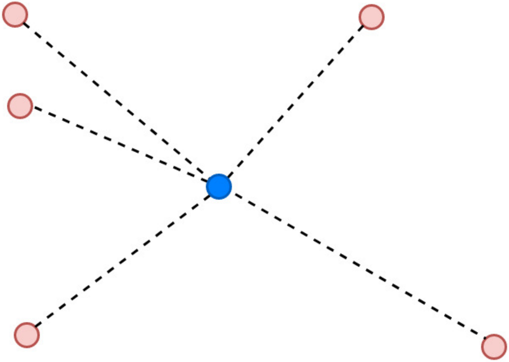

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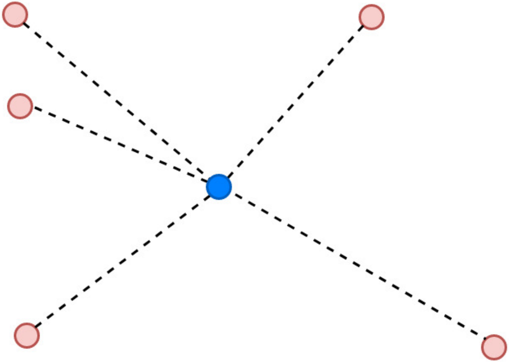

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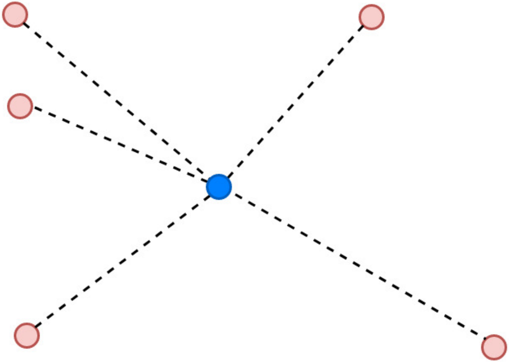

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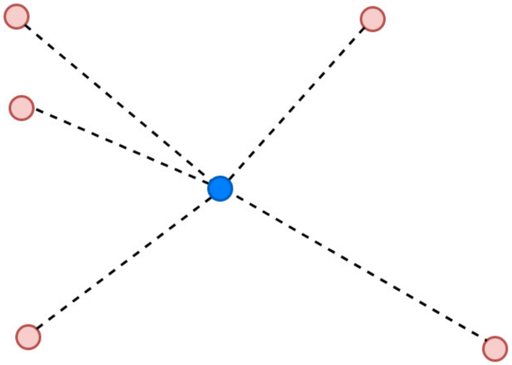

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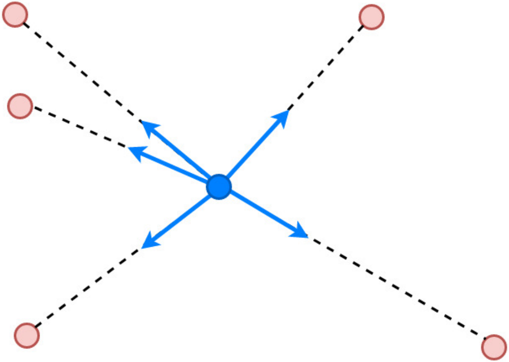
- Gradient : $\nabla D_X(p) = \sum_{x \in X} \frac{p-x}{\|p-x\|}$



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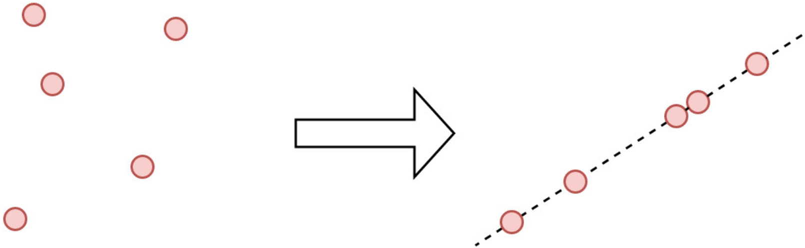

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Theorem alignment_correct :=
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Definition observation := multiset R2.
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- ASYNC (first time in Pactole) & flexible.

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- Other robots :

```
new_ratio <- ratio + demon_ratio
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Flexible :

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Definition flex_da_prop (da:demonic_action) (δ:R) :=
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    get_location (c id) ≡ get_destination (c id) ∨
    δ ≤ dist (get_start (c id)) (get_location (c id)).
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Lemma round_decreases_measure :
  ∀ (c:config) (da:demonic_action),
    (* Hypotheses on da *) →
    (∃ id, da.(activate) id = true  ∧
      get_location (c id) =/= weber_calc c) →
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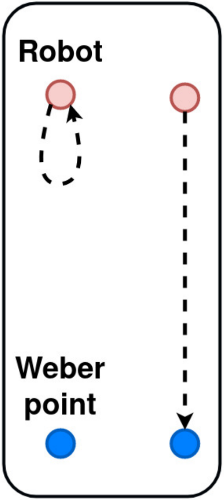

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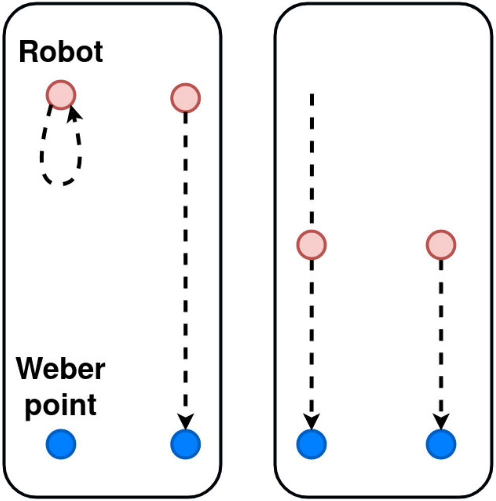

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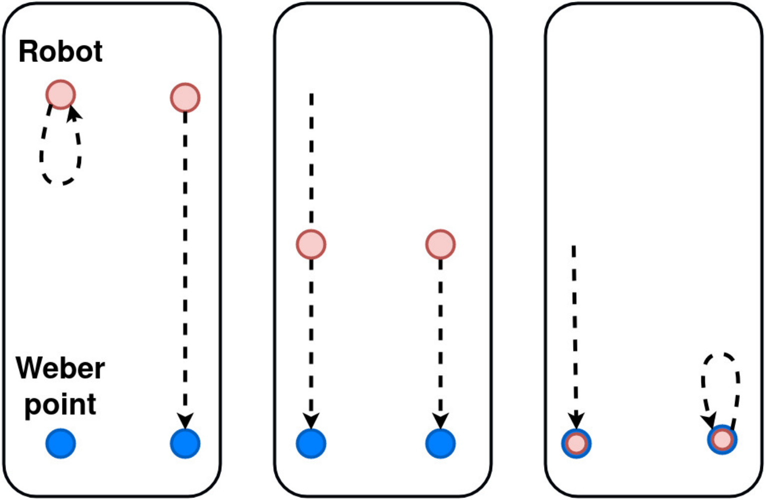

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Lemma lt_config_wf : well_founded lt_config.
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For which initial configurations does the weber algorithm work ?

- When the weber point is unique (initially).

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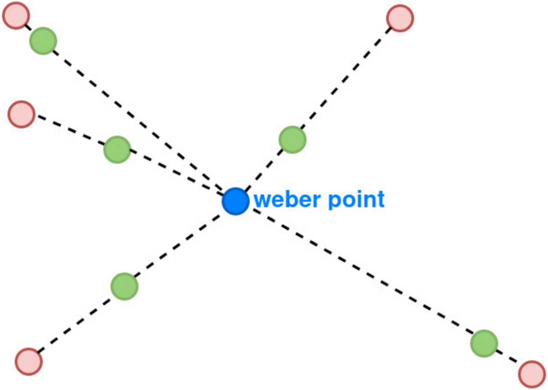
- Let X, Y : multiset of points (cf. figure).



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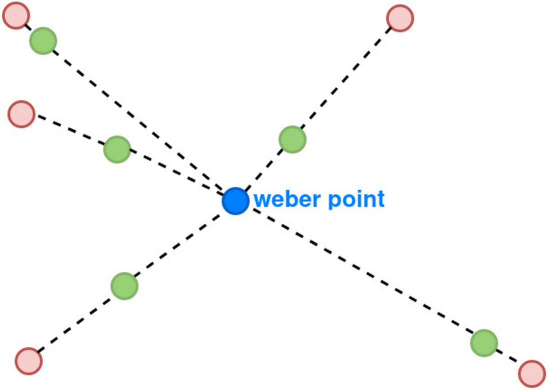

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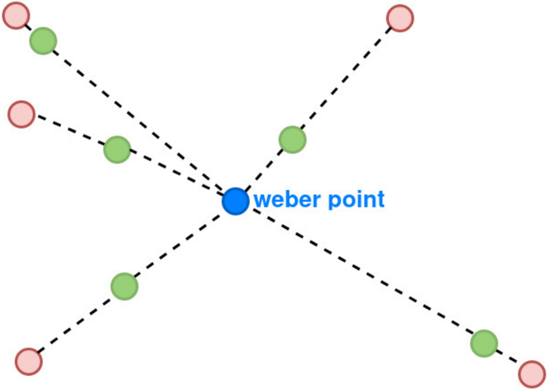

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- Formal proof in Coq of the properties of the weber point.

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Thank you !