Formal Proof of a Gathering Algorithm in the Pactole Framework

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June - July 2022

Talk outline

Suzuki & Yamashita's Model

Weber Point Properties

Alignment

Gathering

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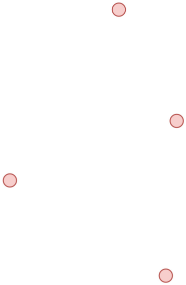
Very simple (dumb) robots :

- Points in $\mathbb{R}^2$ (can overlap)



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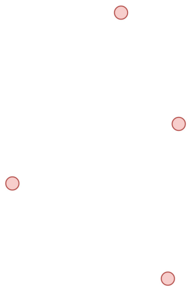

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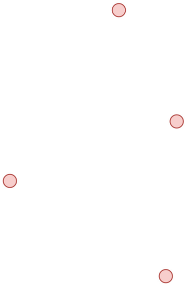

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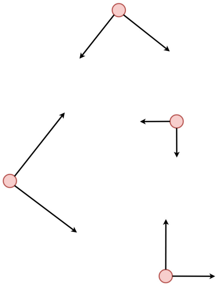

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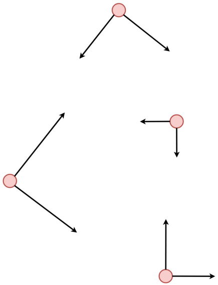

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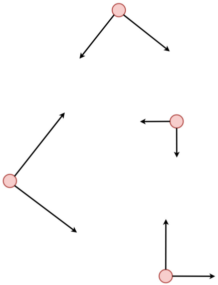

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- Similar to barycenter (sum of distances v.s. sum of distances squared).

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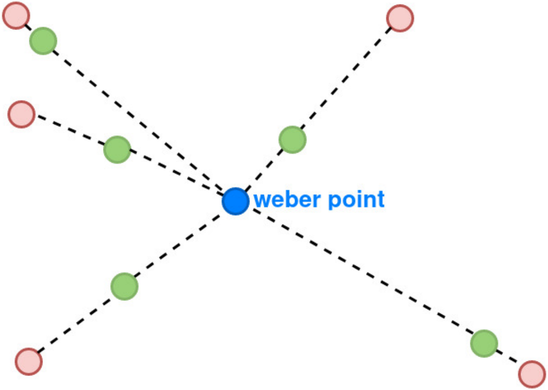
Why use the weber point ?

Contraction Lemma

- Let $\mathbf{X}$, $\mathbf{Y}$: multiset of points (cf. figure).



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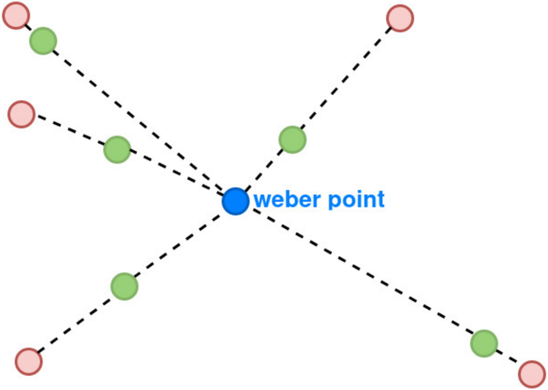

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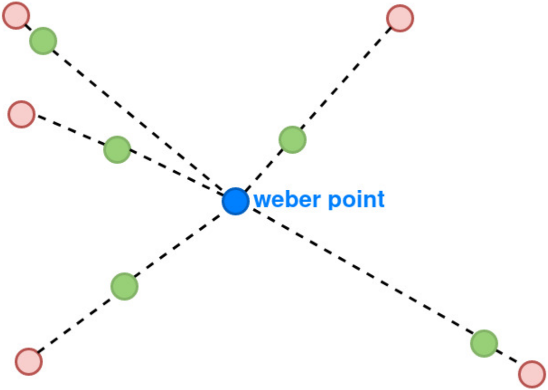

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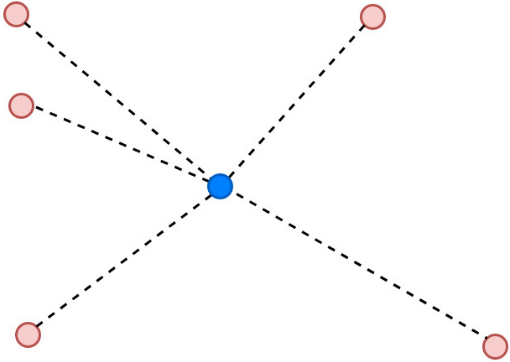

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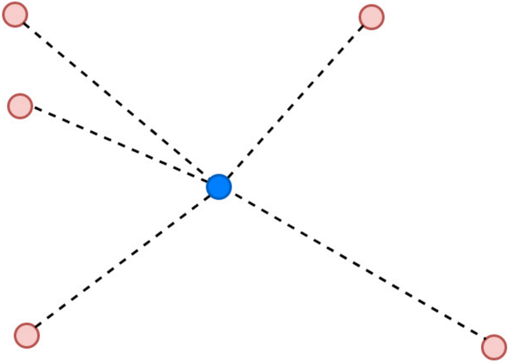

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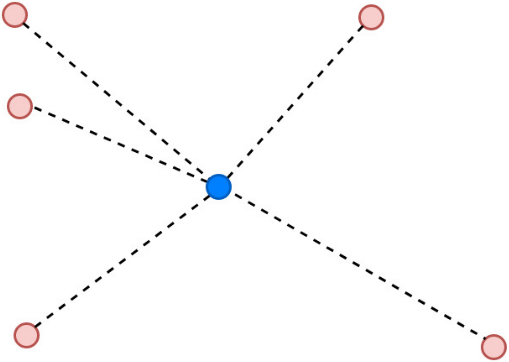

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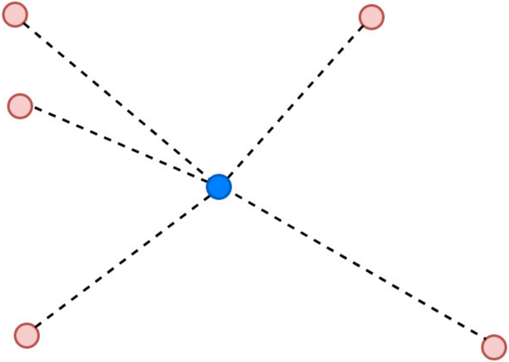

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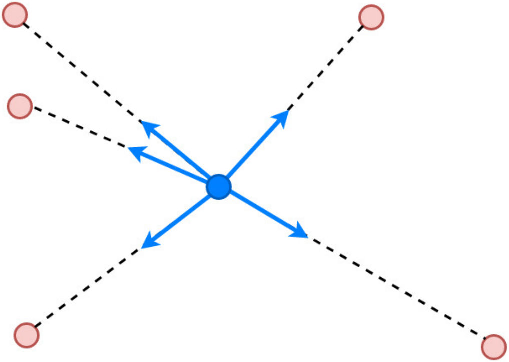
- Gradient : $\nabla D_X(p) = \sum_{x \in X} \frac{p-x}{\|p-x\|}$



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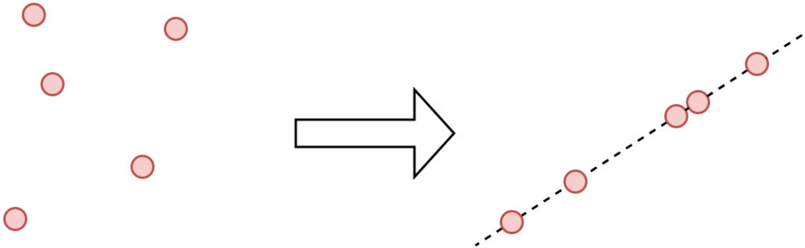

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Theorem alignment_correct :=
  ∀ (c:configuration) (d:demon),
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    eventually_aligned c d align_robogram.
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```coq
Definition observation := multiset R2.
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- ASYNC (first time in Pactole) & flexible.

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Instance St : State info :=
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new_start <- straight_path start dest ratio
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```
new_start <- straight_path start dest ratio
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- Other robots :

```
new_ratio <- ratio + demon_ratio
```

ASYNC model : rigid & flexible

How to define rigid & flexible demons in ASYNC ?

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Coq
Definition rigid_da_prop (da:demonic_action) :=
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Flexible :

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Definition flex_da_prop (da:demonic_action) (δ:R) :=
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    δ ≤ dist (get_start (c id)) (get_location (c id)).
```

Decreasing measure

We define a measure that decreases each time a robot that isn't on the weber point is activated.

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Lemma round_decreases_measure :
  ∀ (c:config) (da:demonic_action),
    (* Hypotheses on da *) →
    (∃ id, da.(activate) id = true  ∧
      get_location (c id) =/= weber_calc c) →
   aligned (round gatherW da c).
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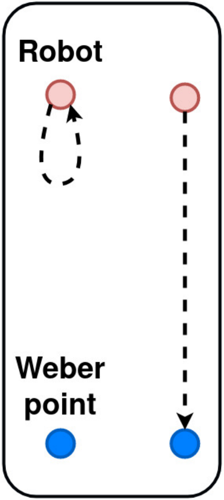

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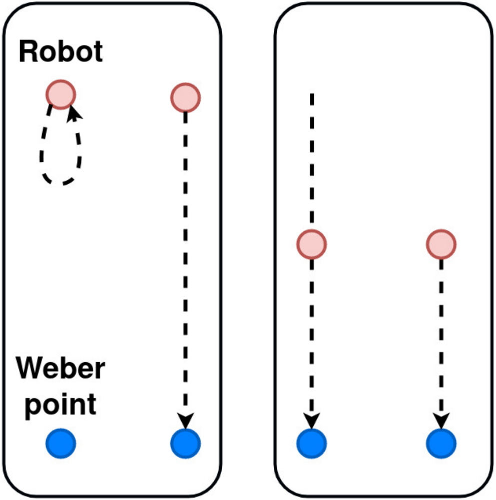

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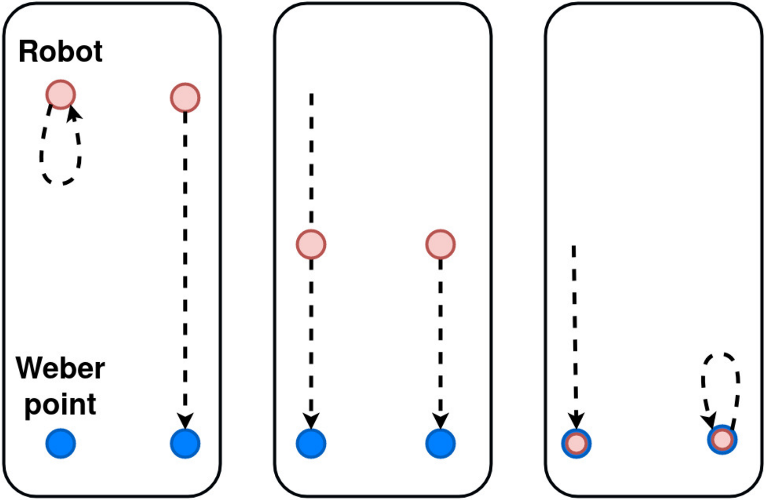

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Lemma lt_config_wf : well_founded lt_config.
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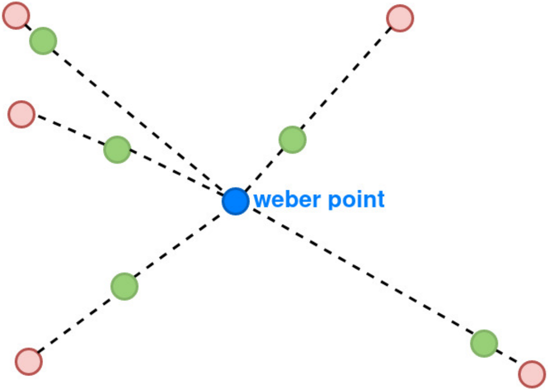
For which initial configurations does the weber algorithm work ?

- When the weber point is unique (initially).

Strong Contraction Lemma (Zohir Bouzid)

- Let **X**, **Y** : multiset of points (cf. figure).



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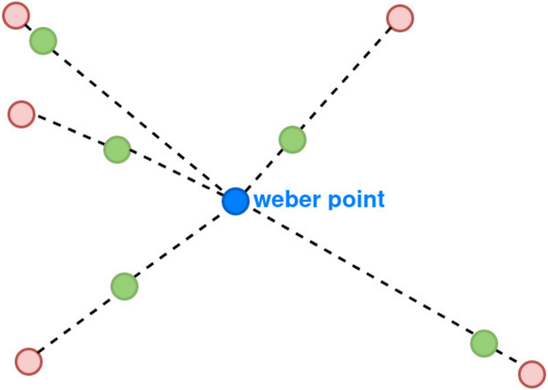

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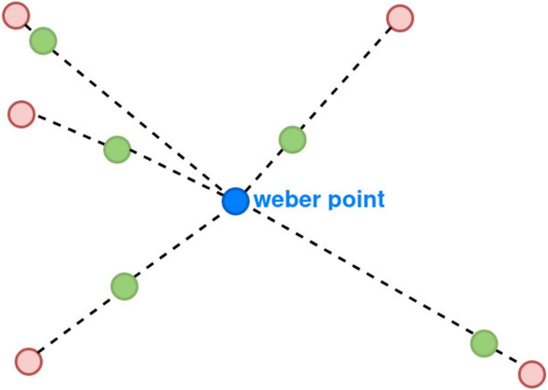

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- Formal proof in Coq of the properties of the weber point.
- Assumes the weber point is computable.
- Asynchronous and flexible setting.

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Thank you !