Slow passage
outer soln

Given $\dot{x} = -\mu + 2|x| - x|x|$, $\dot{\mu} = -\varepsilon$

we will perform an analysis on $x < 0$, $\mu > 0$.

$\Rightarrow \dot{x} = -\mu - 2x + x^2$, $\dot{\mu} = -\varepsilon \rightarrow \mu = -2x + x^2$

$\hookrightarrow x_0 = 1 - \sqrt{1+\mu}$

Next, introduce slowtime $t = \varepsilon\tau \rightarrow \dot{x} = \varepsilon x_\tau$, $\dot{\mu} = \varepsilon \mu_\tau$

$\Rightarrow \varepsilon x_\tau = -\mu - 2x + x^2$, $\mu_\tau = -1$

Let $x \sim x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots$

$O(\varepsilon)$: $x_{0\tau} = -2x_1 + 2x_1 x_0 \rightarrow x_1 = \dfrac{x_{0\tau}}{2\sqrt{1+\mu}} = \dfrac{1}{4(1+\mu)}$

$O(\varepsilon^2)$: $x_{1\tau} = -2x_2 + 2x_0 x_2 + x_1^2 \rightarrow x_2 = -\dfrac{3}{32(1+\mu)^{5/2}}$

$$\rightarrow x \sim 1 - \sqrt{1+\mu} + \frac{\varepsilon}{4(1+\mu)} - \frac{3\varepsilon^2}{32(1+\mu)^{5/2}} + \ldots$$

Since the dynamics of the problem change at $x=0$, this solution is valid only for $x<0$, $\mu>0$.

We have a critical point at $(\mu, x) = (0,0)$ and hence a local analysis is needed.

Local analysis

We've established $x=0$ is a critical point, so allow
$x<0,\ x<<1 \Rightarrow \dot{x} = -\mu - 2x \rightarrow x_0 = \frac{-\mu}{2}$

Again, rescaling time, $\varepsilon x_\tau = -\mu - 2x,\ \mu_\tau = -1$

Let $x \sim x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots$

$\Rightarrow O(\varepsilon):\ x_{0\tau} = -2x_1 \longrightarrow x_1 = -\frac{1}{4}$

$n \geq 2,\ O(\varepsilon^n):\ 0 = -2x_n \rightarrow x_n = 0$

$$\Rightarrow x \sim \frac{-\mu}{2} - \frac{\varepsilon}{4} \rightarrow x = 0 \text{ when } \mu = -\frac{\varepsilon}{2}$$

Thus, we have extended our region of validity for μ.

Scaling Factor

Entering the $(x>0, \mu<0)$ region will require a rescaled equation. Here we perform a systematic search for the appropriate scaling:

Let $x = \delta(\varepsilon) z \Rightarrow \dot{x} = \delta \dot{z}\ \ (x>0, \mu<0)$

$\rightarrow \dot{x} = -\mu + 2x - x^2 \rightarrow \delta \dot{z} = -\mu + 2\delta z - \delta^2 z^2$

This suggests μ must also be rescaled to assure it communicates between the other terms:

Let $\mu = \gamma(\varepsilon) m \rightarrow \dot{\mu} = \gamma \dot{m} = -\varepsilon$

$\rightarrow \delta \dot{z} = -\gamma m + 2\delta z - \delta^2 z^2$

Since we have just entered $(x>0, \mu<0)$, $z<<1$ and thus we seek communication between the parameter and linear term; $\Rightarrow \gamma = \delta$.

$$\Rightarrow \delta \dot{z} = -\delta m + 2\delta z - \delta^2 z^2$$

Next, allow a change in variables to shift focus onto the slowly varying aspect of μ,

$$\dot{z} = z_m \cdot \dot{m} = -\frac{\varepsilon}{\delta} z_m$$

$$\Rightarrow -\varepsilon z_m = -\delta m + 2\delta z - \delta^2 z^2$$

Now the scale choice becomes clear, $\delta = \gamma = \varepsilon$

$\boxed{\text{Inner Solution}}$

With $x = \varepsilon z$, $\mu = \varepsilon m$, $(x > 0, \mu < 0)$, picking up from the previous equation:

$$-\varepsilon z_m = -\varepsilon m + 2\varepsilon z - \varepsilon^2 z^2$$

We linearize,

$$z_m = m - 2z$$

which has solution, $z \sim Ce^{-2m} + \frac{m}{2} - \frac{1}{4}$

In our original coordinates,

$$\frac{x}{\varepsilon} \sim Ce^{-2\mu/\varepsilon} + \frac{\mu}{2\varepsilon} - \frac{1}{4}$$

where tipping will occur when the exponential becomes large

$$e^{-2\mu/\varepsilon} \sim O(1/\varepsilon) \Rightarrow 2\mu_{tip} \sim \varepsilon \ln(\varepsilon)$$