2D Slowosc Outer:

Assume $V < 0$

$$\dot{V} = n_1 - n_2 + n_3(T-V) - T + V^2 + A\sin\Omega t$$
$$\dot{T} = n_1 - T(1-V) + B\sin\Omega t$$
$$\dot{n}_2 = -\varepsilon, \quad \Omega = \varepsilon^{-\lambda} >> 1$$

Multiple Scales: $\tau = \varepsilon t$, $R = \varepsilon^{-\lambda} t$

$$V_R + \varepsilon^{\lambda+1} V_\tau = \varepsilon^{\lambda}[n_1 - n_2 + n_3(T-V) - T + V^2 + A\sin R]$$

$$T_R + \varepsilon^{\lambda+1} T_\tau = \varepsilon^{\lambda}[n_1 - T(1-V) + B\sin R]$$

$$n_{2\tau} = -1$$

$$V \sim V_0 + \varepsilon^{\lambda} V_1 + \ldots, \quad T \sim T_0 + \varepsilon^{\lambda} T_1 + \ldots$$

$O(1)$: $V_{0R} = T_{0R} = 0 \;\rightarrow\; V_0 = V_0(\tau)$, $T_0 = T_0(\tau)$

$O(\varepsilon^{\lambda})$: $V_{1R} = n_1 - n_2 + n_3(T_0 - V_0) - T_0 + V_0^2 + A\sin R$
$\quad T_{1R} = n_1 - T_0(1-V_0) + B\sin R$

$\rightarrow V_{1R} = A\sin R$, $T_{1R} = B\sin R$
$\quad T_0 = n_1/(1-V_0)$, $0 = n_1 - n_2 + n_3(T_0 - V_0) - T_0 + V_0^2$
$\Rightarrow T_0, V_0$ are standard equilibria

$O(\varepsilon^{2\lambda})$: $V_{2R} + \varepsilon^{1-\lambda} V_{0\tau} = 2V_0V_1 - T_1 + n_3(T_1 - V_1)$
$\quad T_{2R} + \varepsilon^{1-\lambda} T_{0\tau} = -T_1(1-V_0) + T_0 V_1$

Fredholm
$\Rightarrow T_1 = \dfrac{T_0 V_1 - \varepsilon^{1-\lambda} T_{0\tau}}{1-V_0}$ where $T_{0\tau} = \dfrac{n_1 V_{0\tau}}{(1-V_0)^2}$

Note: $0 = R_1 - R_2 + R_3(T_0 - V_0) - T_0 + V_0^2$

$$0 = -R_{2T} + R_3(T_{0T} - V_{0T}) - T_{0T} + 2V_0 V_{0T}$$

$$\Rightarrow V_{0T} = R_{2T} / (2V_0 - T_{0T}/V_{0T} + R_3 T_{0T}/V_{0T} - R_3)$$

where $T_{0T}/V_{0T} = \dfrac{R_1}{(1-V_0)^2}$

Simplify $T_1 = \dfrac{T_0}{1-V_0} V_1 - \varepsilon^{1-\lambda} \dfrac{T_{0T}}{1-V_0} = \dfrac{T_0}{1-V_0} V_1 - \varepsilon^{1-\lambda} \dfrac{R_1 V_{0T}}{(1-V_0)^3}$

$$\Rightarrow \varepsilon^{1-\lambda} V_{0T} = 2V_0 V_1 - \frac{T_0}{1-V_0} V_1 + \varepsilon^{1-\lambda} \frac{R_1 V_{0T}}{(1-V_0)^3} + R_3\left(\frac{T_0}{1-V_0} V_1 - \varepsilon^{1-\lambda} \frac{R_1 V_{0T}}{(1-V_0)^3} - V_1\right)$$

$$\Rightarrow V_1 = \varepsilon^{1-\lambda} \frac{-V_{0T}(1-R_3)\frac{R_1}{(1-V_0)^3}}{2V_0 - (1-R_3)\frac{T_0}{1-V_0} + R_3} = \varepsilon^{1-\lambda} X_1$$

Now both $V_1 \sim \varepsilon^{1-\lambda}$ and $T_1 \sim \varepsilon^{1-\lambda}$

$$\Rightarrow V \sim V_0 + \varepsilon X_1 \mp \varepsilon^{\lambda} A \cos \Omega t, \quad T \sim V_0 + \varepsilon y_1 - \varepsilon^{\lambda} B \cos \Omega t$$

Unfortunately, this outer solution is too complex to extract when an inner equation appears, so we must rely on a separate scale analysis to find an appropriate scaling.

Determine Scaling:

$$\dot V = \eta_1 - \eta_2 + \eta_3(T-V) - T - V|V| + A\sin(\Omega t)$$

$$\dot T = \eta_1 - T(1+|V|) + B\sin(\Omega t)$$

$$\dot\eta_2 = -\varepsilon$$

$$V = \xi_1 x, \quad T = \eta_1 + \xi_2 y, \quad \eta_2 = \eta_1\eta_3 + \xi_3 m$$

$$\xi_1 \dot x = -\xi_3 m + \eta_3(\xi_2 y - \xi_1 x) - \xi_2 y - \xi_1^2 x|x| + A\sin(\Omega t)$$

$$\xi_2 \dot y = -\eta_1\xi_1|x| - \xi_2 y - \xi_1\xi_2|x|y + B\sin(\Omega t)$$

$$\xi_3 \dot m = -\varepsilon$$

Multiple scales: Since it's likely $\xi_3 = \varepsilon$, $\tau = t$, $R = \Omega t = \varepsilon^{-\lambda} t$

① $\xi_1 x_R + \xi_1 \varepsilon^\lambda x_\tau = \varepsilon^\lambda[-\xi_3 m + \eta_3(\xi_2 y - \xi_1 x) - \xi_2 y - \xi_1^2 x|x| + A\sin R]$

② $\xi_2 y_R + \xi_2 \varepsilon^\lambda y_\tau = \varepsilon^\lambda[-\eta_1\xi_1|x| - \xi_2 y - \xi_1\xi_2|x|y + B\sin R]$

③ $\xi_3 m_\tau = -\varepsilon$

③: Gives $\xi_3 = \varepsilon$, ②: gives $\xi_1 = \xi_2$, ①: Gives $\xi_3 = \xi_1$

$$\Rightarrow x_R + \varepsilon^\lambda x_\tau = \varepsilon^\lambda[-m + \eta_3(y-x) - y - \varepsilon x|x|] + \varepsilon^{\lambda-1} A\sin R$$

$$y_R + \varepsilon^\lambda y_\tau = \varepsilon^\lambda[-\eta_1|x| - y - \varepsilon|x|y] + \varepsilon^{\lambda-1} B\sin R$$

$$m_\tau = -1$$

2D Slowosc Inner:

From the Scaling analysis: $V=\varepsilon X$, $T=\Omega_1+\varepsilon y$, $\Omega_2=\Omega_1\Omega_3+\varepsilon m$

~~This way~~ The Multiple scales then would be: $\mathcal{T}=t$, $R=\varepsilon^{-\lambda}t$

$$\Rightarrow X_R+\varepsilon^{\lambda}X_{\mathcal{T}}=\varepsilon^{\lambda}[-m+\Omega_3(y-x)-y]-\varepsilon^{\lambda+1}x|x|+\varepsilon^{\lambda-1}A\sin R$$
$$y_R+\varepsilon^{\lambda}y_{\mathcal{T}}=\varepsilon^{\lambda}[-\Omega_1|x|-y]-\varepsilon^{\lambda+1}|x|y+\varepsilon^{\lambda-1}B\sin R$$

Consider $\lambda>1$, then a direct expansion makes sense

$X\sim X_0+\varepsilon^{\lambda}X_1+\dots$, $y\sim y_0+\varepsilon^{\lambda}y_1+\dots$

$O(1)$: $X_{0R}=y_{0R}=0 \Rightarrow X_0=X_0(\mathcal{T})$, $y_0=y_0(\mathcal{T})$

$O(\varepsilon^{\lambda})$: $X_{1R}+X_{0\mathcal{T}}=-m+\Omega_3(y_0-X_0)-y_0+\varepsilon^{-1}A\sin R$

$y_{1R}+y_{0\mathcal{T}}=-\Omega_1|X_0|-y_0+\varepsilon^{-1}B\sin R$

Fredholm: $X_{1R}=\varepsilon^{-1}A\sin R$, $y_{1R}=\varepsilon^{-1}B\sin R$

$$\begin{pmatrix}X_{0\mathcal{T}}\\ y_{0\mathcal{T}}\end{pmatrix}=\begin{pmatrix}-\Omega_3 & -(1-\Omega_3)\\ -\Omega_1\cdot\operatorname{Sgn}(X) & -1\end{pmatrix}\begin{pmatrix}X_0\\ y_0\end{pmatrix}-\begin{pmatrix}m\\ 0\end{pmatrix}$$

Same as purely slow: $X\sim X_0+\varepsilon^{\lambda-1}A\sin R$ $\Rightarrow V\sim\varepsilon X_0+\varepsilon^{\lambda}A\cos R$

$y\sim y_0+\varepsilon^{\lambda-1}B\sin R$ $\quad T\sim\Omega_1+\varepsilon y_0-\varepsilon^{\lambda}\cos R$

Consider $\lambda\le1$: The $\varepsilon^{\lambda-1}$ term now dominates so the same expansion now behaves as,

$O(1)$: $X_{0R}=\varepsilon^{\lambda-1}A\sin R$, $y_{0R}=\varepsilon^{\lambda-1}B\sin R$ $\Rightarrow X_0=P_0(\mathcal{T})-\varepsilon^{\lambda-1}A\cos R$

$y_0=q_0(\mathcal{T})-\varepsilon^{\lambda-1}B\cos R$

Which is enough evidence to scale and center

as $V\sim\varepsilon X=\varepsilon P_0-\varepsilon^{\lambda}A\cos R$

$T\sim\Omega_1+\varepsilon y=\Omega_1+\varepsilon q_0-\varepsilon^{\lambda}B\sin R$

2D Slowosc

With Centering

$$R = \varepsilon^{-\lambda} t$$

$$\dot{x} = -m + n_3(y - x + \varepsilon^{\lambda-1} A\cos R - \varepsilon^{\lambda-1} B\cos R) - y + \varepsilon^{\lambda-1} B\cos R - \varepsilon x|x - \varepsilon^{\lambda-1} A\cos R|$$
$$+ \varepsilon^{\lambda} A\cos R|x - \varepsilon^{\lambda-1} A\cos R|$$

$$\dot{y} = -n_1|x - \varepsilon^{\lambda-1} A\cos R| - y + \varepsilon^{\lambda-1} B\cos R + \varepsilon^{\lambda} B\cos R|x - \varepsilon^{\lambda-1} A\cos R|$$

$$\dot{m} = -1$$

Multiple Scales: $\tau = t$, $R = \varepsilon^{-\lambda} t$

$$x_R + \varepsilon^{\lambda} x_\tau = \varepsilon^{\lambda}[-m + n_3(y - x + \varepsilon^{\lambda-1}(A-B)\cos R) - y + \varepsilon^{\lambda-1} B\cos R]$$
$$-\varepsilon^{\lambda+1} x|x - \varepsilon^{\lambda-1} A\cos R| + \varepsilon^{2\lambda} A\cos R|x - \varepsilon^{\lambda-1} A\cos R|$$

$$y_R + \varepsilon^{\lambda} y_\tau = \varepsilon^{\lambda}[-n_1|x - \varepsilon^{\lambda-1} A\cos R| - y + \varepsilon^{\lambda-1} B\cos R] + \varepsilon^{2\lambda} B\cos R|x - \varepsilon^{\lambda-1} A\cos R|$$

$$m_\tau = -1$$

But we find $\varepsilon^{2\lambda-1}$ and $\varepsilon^{3\lambda-1}$ terms $\Rightarrow \lambda \in [\frac{1}{2}, 1]$

Choosing $x \sim x_0 + \varepsilon^{\lambda} x_1 + \dots$, $y \sim y_0 + \varepsilon^{\lambda} y_1 + \dots$

$O(1)$: $x_{0R} = y_{0R} = 0$

$O(\varepsilon^{\lambda})$:
$$x_{1R} + x_{0\tau} = -m + n_3(y_0 - x_0 + \varepsilon^{\lambda-1}(A-B)\cos R) - y_0 + \varepsilon^{\lambda-1} B\cos R$$
$$y_{1R} + y_{0\tau} = -n_1|x_0 - \varepsilon^{\lambda-1} A\cos R| - y_0 + \varepsilon^{\lambda-1} B\cos R$$

Fredholm:
$$x_{0\tau} = -m + n_3(y_0 - x_0) - y_0$$
$$y_{0\tau} = -\frac{n_1}{2\pi}\int_0^{2\pi} |x_0 - \varepsilon^{\lambda-1} A\cos R|\, dR - y_0$$

Note: From 1D slow+osc: $\frac{-n_1}{2\pi}\int_0^{2\pi} |x_0 - \varepsilon^{\lambda-1} A\cos R|\, dR \approx -\frac{n_1}{\pi|B|} x_0^2 - \frac{2n_1|B|}{\pi}$

$C = \varepsilon^{\lambda-1} A$

$$\Rightarrow \begin{pmatrix} x_{0\tau} \\ y_{0\tau} \end{pmatrix} = \begin{pmatrix} -n_3 & -(1-n_3) \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \begin{pmatrix} m \\ \frac{n_1}{\pi|C|} x_0^2 + \frac{2n_1|B|}{\pi} \end{pmatrix}$$

Reducing to get an estimate:

At this point, this a non-autonomous Riccati Matrix equation. There may be a result that solves this but as we are intrested in an approximate tipping, we choose to reduce this problem to a 1D.

Assume y to be in equilibrium: $y_0 = -\frac{n_1}{\pi|c|}x_0^2 + \frac{2n_1|c|}{\pi}$

$$\Rightarrow x_{0T} = -m + \frac{2n_1(1-n_3)|c|}{\pi} - n_3 x_0 + \frac{n_1(1-n_3)}{\pi|c|}x_0^2$$

$$\rightarrow x_{0m} = m - \frac{2n_1(1-n_3)|c|}{\pi} + n_3 x_0 - \frac{n_1(1-n_3)}{\pi|c|}x_0^2$$

$$= m - c_0 + n_3 x_0 - c_1 x_0^2$$

Which from the Zhu & Kuske paper gives

$$m_{tip} = -\varepsilon^{(d-1)/3}\left(\frac{\pi|A|}{n_1(1-n_3)}\right)^{1/3}\cdot(2.33810..) + \varepsilon^{d-1}\frac{n_1(1-n_3)|A|}{\pi}\left(2-\left(\frac{\pi n_3}{2n_1(1-n_3)}\right)^2\right)$$

$$n_{2tip} = n_1 n_3 - \varepsilon^{(d+2)/3}\left(\frac{\pi|A|}{n_1(1-n_3)}\right)^{1/3}\cdot(2.33810...) + \frac{n_1(1-n_3)|A|}{\pi\Omega}\left(2-\left(\frac{\pi n_3}{2n_1(1-n_3)}\right)^2\right)$$

$$= n_{2osc} + \varepsilon^{(d-1)/3}\left(\frac{\pi|A|}{n_1(1-n_3)}\right)^{1/3}\cdot\mu_{smooth}$$

Note: This is almost identical to 1D slowtosc!