Slowly varying.... eq. (5) – (13)

Starting w/ $\dot{x} = a - x^2 + A\sin(\Omega t)$, $\Omega \gg 1$.

Using a multiple Scales approach to get asymptotical series for x, Let $x = x(\tau, T)$, $t = \tau$ and $T = \Omega t$

$$\Rightarrow \frac{dx}{dt} = \frac{\partial x}{\partial \tau}\cdot\frac{\partial \tau}{\partial t} + \frac{\partial x}{\partial T}\cdot\frac{\partial T}{\partial t} = x_\tau + \Omega\cdot x_T$$

$$\hookrightarrow x_T + \Omega^{-1}\cdot x_\tau = \Omega^{-1}\cdot a - \Omega^{-1}\cdot x^2 + \Omega^{-1}\cdot A\sin T \quad (5)$$

Expand,
$$x = x_0 + \Omega^{-1}x_1 + \Omega^{-2}x_2 + \dots$$

$$O(1): \Rightarrow x_{0T} = 0 \quad , \quad O(\Omega^{-1}): \; x_{1T} = \underbrace{a - x_0^2 - x_{0\tau} + A\sin T}_{R_1(\tau, T)}$$

$O(1)$ implies that $x_0(\tau, T) = x_0(\tau)$ and $O(\Omega^{-1})$ must have a solvability condition. We impose

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T R_1(\tau, u)\,du = 0$$

$$\Rightarrow x_{0\tau} = a - x_0^2 \xrightarrow{\text{fixed } a} x_0 = \sqrt{a} \quad (11)$$

$$\rightarrow x_{1T} = A\sin T \longrightarrow x_1 = -A\cos T + V_1(\tau)$$

where $V_1(\tau)$ needs $R_2(\tau, T)$ to solve.

$$\Rightarrow x \sim \sqrt{a} + \Omega^{-1}(-A\cos T + V_1(\tau)) + O(\Omega^{-2}) \quad (12)$$

But, when $a \sim O(\Omega^{-2})$, the terms reorder, telling us the scaling and location of an inner solution.

Letting $a = \Omega^{-2} b$, (5) becomes

$$X_T + \Omega^{-1} X_\tau = \Omega^{-3} b - \Omega^{-1} X^2 + \Omega^{-1} A \sin T$$

w/ expansion $X \sim X_0 + \Omega^{-1} X_1 + \dots$

$\Rightarrow$ $O(1)$: $X_{0T} = 0$, $O(\Omega^{-1})$: $X_{1T} = -X_0^2 - X_{0\tau} + A\sin T$

$O(\Omega^{-2})$: $X_{2T} = -X_{1\tau} - 2X_0X_1$

$O(\Omega^{-3})$: $X_{3T} = -X_{2\tau} + b - X_1^2 - 2X_0X_2$

The $O(1) \Rightarrow$ $X_0 = X_0(\tau)$, using Fredholm alternative the $O(\Omega^{-1}) \Rightarrow X_{0\tau} = -X_0^2$, $X_{1T} = A\sin T$

$X_1 = -A\cos T + V_1(t)$ and $X_0 = 0$ (equil. solution)

$O(\Omega^{-2}) \Rightarrow X_{2T} = -V_{1t}(t)$, using Fredholm $\to V_{1t}(t) = 0$

$\Rightarrow V_1(t) = d \to X_{2T} = 0 \Rightarrow X_2 = X_2(t)$

$O(\Omega^{-3}) \Rightarrow$ Again, Fredholm $X_{2\tau} = b - \frac{A^2}{2} - d^2$

which searching for the eq. soln, $d^2 = b - \frac{A^2}{2}$

Thus, $X \sim \left(\sqrt{\Omega^2 a - \frac{A^2}{2}} - A\cos T\right)\Omega^{-1} + \dots$

$$\sim \sqrt{a - \frac{A^2}{2\Omega^2}} - \frac{A}{\Omega}\cos T + \dots \quad (13)$$

Thus, we see the sol. breaks at $a_p = \frac{A^2}{2\Omega^2}$, in the suspected $O(\Omega^{-2})$.