Slowly varying... eq. (14)-(23)

Starting w/ $\dot{x} = a - x^2 + A\sin\Omega t,\ \dot{a} = -\mu,\ \Omega >> 1$

Let $\Omega = \mu^{-\lambda},\ T = \mu^{-\lambda} t,\ \tau = \mu t$

$$\Rightarrow \frac{d}{dt} = \frac{\partial}{\partial T}\cdot\mu^{-\lambda} + \frac{\partial}{\partial\tau}\cdot\mu,\ \frac{d}{dt} = \frac{d}{d\tau}\mu$$

$$\Rightarrow \quad X_T + \mu^{\lambda+1} x_\tau = \mu^{\lambda}[a - x^2 + A\sin T],\ a_\tau = -1 \quad (14)$$

choosing the expansion $\quad x \sim x_0 + \mu^{\lambda} x_1 + \mu^{2\lambda} x_2 + \dots \quad (15)$

we get the equations:

$O(1)$: $X_{0T} = 0 \longrightarrow X_0 = X_0(\tau)$

$O(\mu^{\lambda})$: $X_{1T} = a - X_0^2 + A\sin T$

using Fredholm, $X_0^2 = a \longrightarrow X_0 = \sqrt{a}$ (equ. Soln)

$$X_{1T} = A\sin T \longrightarrow X_1 = -A\cos T + V_1(\tau)$$

Next order depends on λ:

$0 \leq \lambda \leq 1 \quad O(\mu^{2\lambda})$: $X_{2T} = -\mu^{1-\lambda} X_{0\tau} - 2X_0X_1$

$\lambda > 1 \quad O(\mu^{1+\lambda})$: $\mu^{\lambda-1} X_{2T} = -X_{0\tau} - \mu^{\lambda-1}(2X_0X_1)$

Q: How is μ in these eq.?

A: This puts extra special terms into the same eq., otherwise a $\mu^{1-\lambda}$ term would be needed in (15)

Note: see pg. 3 for alt. calculation

For both cases, Fredholm $\Rightarrow X_{0\tau} = -\mu^{\lambda-1}(2X_0X_1)$

since $X_0 = \sqrt{a} \longrightarrow V_1(\tau) = \mu^{1-\lambda}\cdot\frac{1}{4a}$

$$\rightarrow x \sim \sqrt{a} + \frac{\mu}{4a} + \mu^{\lambda}(-A\cos T) + \dots$$

But, as before, if $a \sim O(\mu^{2/3})$, we see term reordering.

Thus a local expansion is needed with the rescales $s = \mu^{1/3} t$,

$a(t) = \mu^{2/3}\alpha(s),\ T = \mu^{-\lambda} t$

$$\Rightarrow \frac{d}{dt} = \frac{\partial}{\partial s}\cdot\mu^{1/3} + \frac{\partial}{\partial T}\cdot\mu^{-\lambda},\ \frac{d}{dt} = \frac{d}{ds}\cdot\mu^{1/3}$$

These rescalings give the system as,

$$(17)\ \begin{cases} X_T + \mu^{1/3+\lambda} X_S = \mu^{\lambda}[\mu^{2/3}\alpha - X^2 + A\sin T] \\ a_S = -1 \end{cases}$$

Choosing the substitution $X = -\mu^{\lambda}(A\overset{\cos T}{\cancel{\sin T}}) + \mu^{1/3} y(S,T)$

{Q: Why this sub? to elim. μ^{λ} term? A: The term $-\mu^{\lambda}\cos T$ persists no matter the choice of λ. $\boxed{\text{See pg. 4 for detail}}$

$$\rightarrow X_T = \mu^{\lambda}(A\sin T) + \mu^{1/3} y_T,\quad X_S = \mu^{1/3} y_S$$

Resulting in,

$$\mu^{1/3}[y_T + \mu^{1/3+\lambda} \cdot y_S] = \mu^{\lambda}[\mu^{2/3}\cdot\alpha - \mu^{2\lambda}\cdot(A^2[\cos^2 T]) + \mu^{1/3+\lambda}(2Ay\cos T) - \mu^{2/3}\cdot y^2]$$

$$(18) \hookrightarrow y_T + \mu^{1/3+\lambda}\cdot y_S = \mu^{\lambda}[\mu^{1/3}\cdot\alpha - \mu^{2\lambda-1/3}(A^2[1+\cos 2T]/2) + \mu^{\lambda}(2Ay\cos T) - \mu^{1/3}\cdot y^2]$$

Next, using the expansion $y \sim y_0 + \mu^{1/3+\lambda}\cdot y_1 + \ldots$ (19)

$$(20)\ \begin{aligned} &O(1):\ y_{0T} = 0 \rightarrow y_0 = y_0(S) \\ &O(\mu^{1/3+\lambda}):\ y_{1T} = -y_{0S} + \alpha - \mu^{2\lambda-2/3}\cdot\frac{A^2}{2}(1+\cos 2T) + \mu^{-1/3+\lambda}\cdot(2Ay_0\cos T) - y_0^2 \end{aligned}$$

Using Fredholm, $y_{0S} = \alpha - y_0^2 - \mu^{2\lambda-2/3}\cdot\frac{A^2}{2} = (\alpha - \mu^{2\lambda-2/3}\cdot\frac{A^2}{2}) - y_0^2$

Which has form similar to the Airy solution from before,

$$(21) \rightarrow y_0 = -\frac{Ai'(\alpha - \mu^{2\lambda-2/3}\cdot\frac{A^2}{2})}{Ai(\alpha - \mu^{2\lambda-2/3}\cdot\frac{A^2}{2})}$$

$$\rightarrow X \sim \mu^{\lambda}\cdot(-A\cos T) + \mu^{1/3}\cdot\left(-\frac{Ai'(\alpha-\mu^{2\lambda-2/3}\cdot\frac{A^2}{2})}{Ai(\alpha-\mu^{2\lambda-2/3}\cdot\frac{A^2}{2})}\right) + \ldots \quad (22)$$

Which, like before, sees a divergence at $Ai(\alpha - \mu^{2\lambda-2/3}\cdot\frac{A^2}{2}) = 0$

$$\Rightarrow \alpha - \mu^{2\lambda-2/3}\cdot\frac{A^2}{2} \approx -2.33810 \rightarrow a - \mu^{2\lambda}\cdot\frac{A^2}{2} \approx \mu^{2/3}\cdot(-2.33810)$$

$$(23) \hookrightarrow a_{hf} \approx \mu^{2/3}\cdot(-2.33810) + \mu^{2\lambda}\cdot\frac{A^2}{2} = \mu^{2/3}\cdot(-2.33810) + \frac{A^2}{2\Omega^2} = a_d + a_p$$

Note: This is the case for $\frac{1}{3} \le \lambda < 1$ (due to substitution), but a similar result will follow for $\frac{1}{6} < \lambda < \frac{1}{3}$. {Q: Why $\frac{1}{6}$? $\cancel{A}$

$\lambda \le \frac{1}{6} \rightarrow \Omega \sim O(1)$.

{A: For the sizes of μ we deal with ($\mu \approx 10^{-7}$), Not too small $\mu^{-1/6} \rightarrow$ Rule of thumb.

Choose $\lambda = \frac{1}{2} \rightarrow x \sim x_0 + \sqrt{\mu}\, x_1 + \mu x_2 + \cdots$

$$x_T + \mu^{3/2} x_\tau = \mu^{1/2}[a - x^2 + A\sin T], \quad a_\tau = -1$$

$O(1)$: $x_{0T} = 0 \rightarrow x_0 = x_0(\tau)$

$O(\mu^{1/2})$: $x_{1T} = a - x_0^2 + A\sin T$

$O(\mu)$: $x_{2T} = -2x_0 x_1$

$O(\mu^{3/2})$: $x_{3T} + x_{0\tau} = -2x_0 x_2 - x_1^2$

using Fredholm, $x_0^2 = a \rightarrow x_0 = \sqrt{a}$

$x_{1T} = A\sin T \rightarrow x_1 = -A\cos T + V_1(\tau)$

$\longrightarrow x_{2T} = -2\sqrt{a}\,(-A\cos T + V_1(\tau))$

$\hookrightarrow x_2 = 2\sqrt{a}\, A\sin T - 2\sqrt{a}\, V_1(\tau) T + V_2(\tau)$

But Fredholm requires $V_1 = 0$

$\longrightarrow x_{3T} + x_{0\tau} = -4a A\sin T - 2\sqrt{a}\, V_2(\tau) - (A\cos T)^2$

Fredholm once more gives,

$x_{0\tau} = -2\sqrt{a}\, V_2(\tau) \longrightarrow \frac{1}{2\sqrt{a}} a_\tau = -2\sqrt{a}\, V_2(\tau)$

$\Rightarrow V_2(\tau) = \frac{1}{4a}$

$$\rightarrow x \sim \sqrt{a} + \mu^{1/2}(-A\cos T) + \mu\left(\frac{1}{4a} + 2\sqrt{a}\, A\sin T\right) + \cdots$$

Rescaling w/ ~~a(t)~~ $\mu^{2/3}\alpha(s)$, $T=\mu^{-1/2}t$, $s=\mu^{1/3}t$

$$\rightarrow \begin{cases} x_T+\mu^{5/6}x_s=\mu^{1/2}[\mu^{2/3}\alpha - x^2 + A\sin T] \\ a_s=-1 \end{cases}$$

It is clear that there is only 1 term of $\mu^{1/2}$,

thus $x \sim \mu^{1/2}(-A\cos T)+\ldots$

The next issue is that the α and x^2 terms ~~t~~ don't communicate, thus a rescale of $x^2=\mu^{2/3}y^2=(\mu^{1/3}y)^2$ is needed.

Thus, we form the substitution,

$$x=\mu^{1/2}(-A\cos T)+\mu^{1/3}y$$