Stochastic climate dynamics: does non-smoothness really matter?

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Stochastic climate dynamics: does non-smoothness really matter?

Sometimes …

Noise driven order: Stabilized / ”facilitated” transients

Specific areas of (stochastic) facilitation:

Tipping, Escapes/Exits,

Stochastic/Coherence Resonance: optimal noise level to drive tipping and/or sustained transients

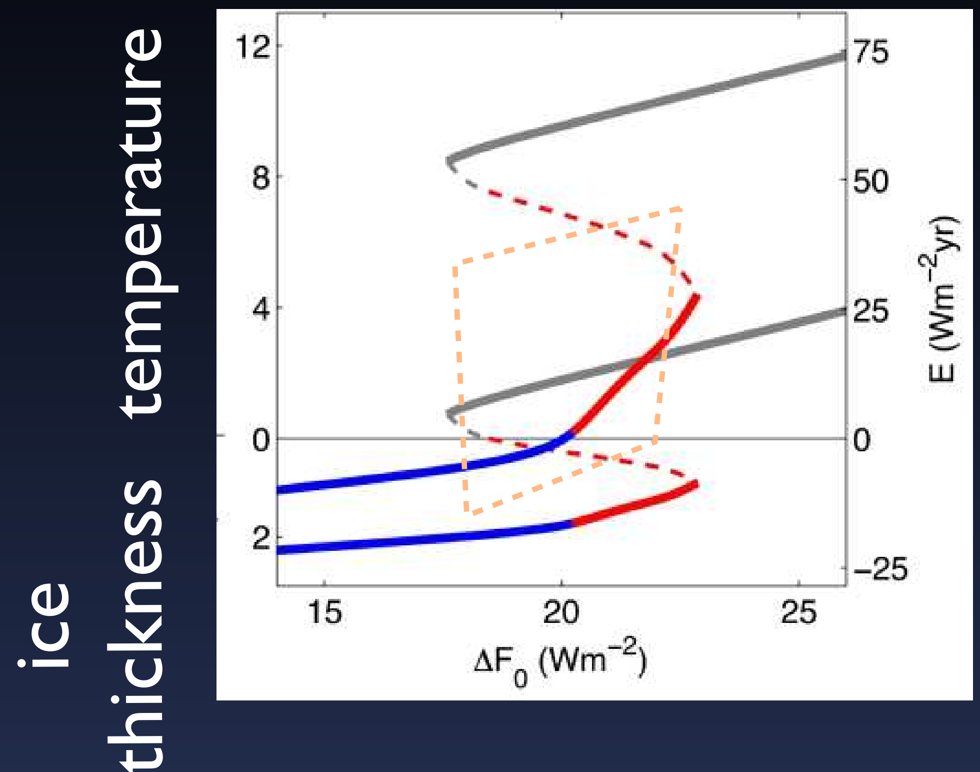
Reduced systems - approximations of fluctuations: averaged, local approximations

Non-autonomous systems and oscillatory forcing

Scale - what is large? what is small?

Reversibility: Reduced models vs. full GCM

Is there hysteresis, or is the ice-free state reversible?

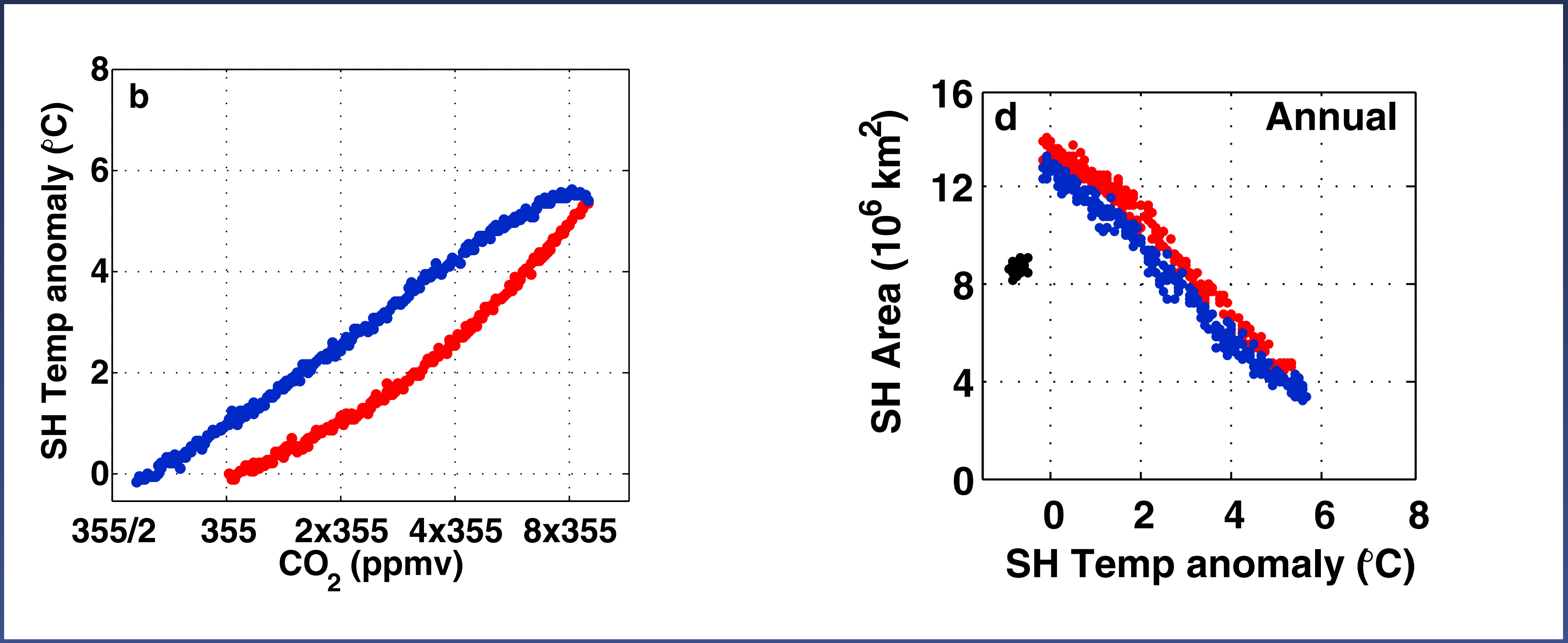


Reduced energy-type model for arctic sea ice
Bistability = potential for hysteresis ?

Eisenman, Wettlaufer, 2009

Noise, oscillatory parameters, non-smooth models- influence or limit hysteresis?

Effects of time lags results in larger hysteresis loops, even when reversible



Amour, Eisenman, et al 2011

Suggests some areas for consideration:

Oscillations

Hysteresis (or not)

Coherence Resonance

Other types of noise - e.g. heavy tails

Scales

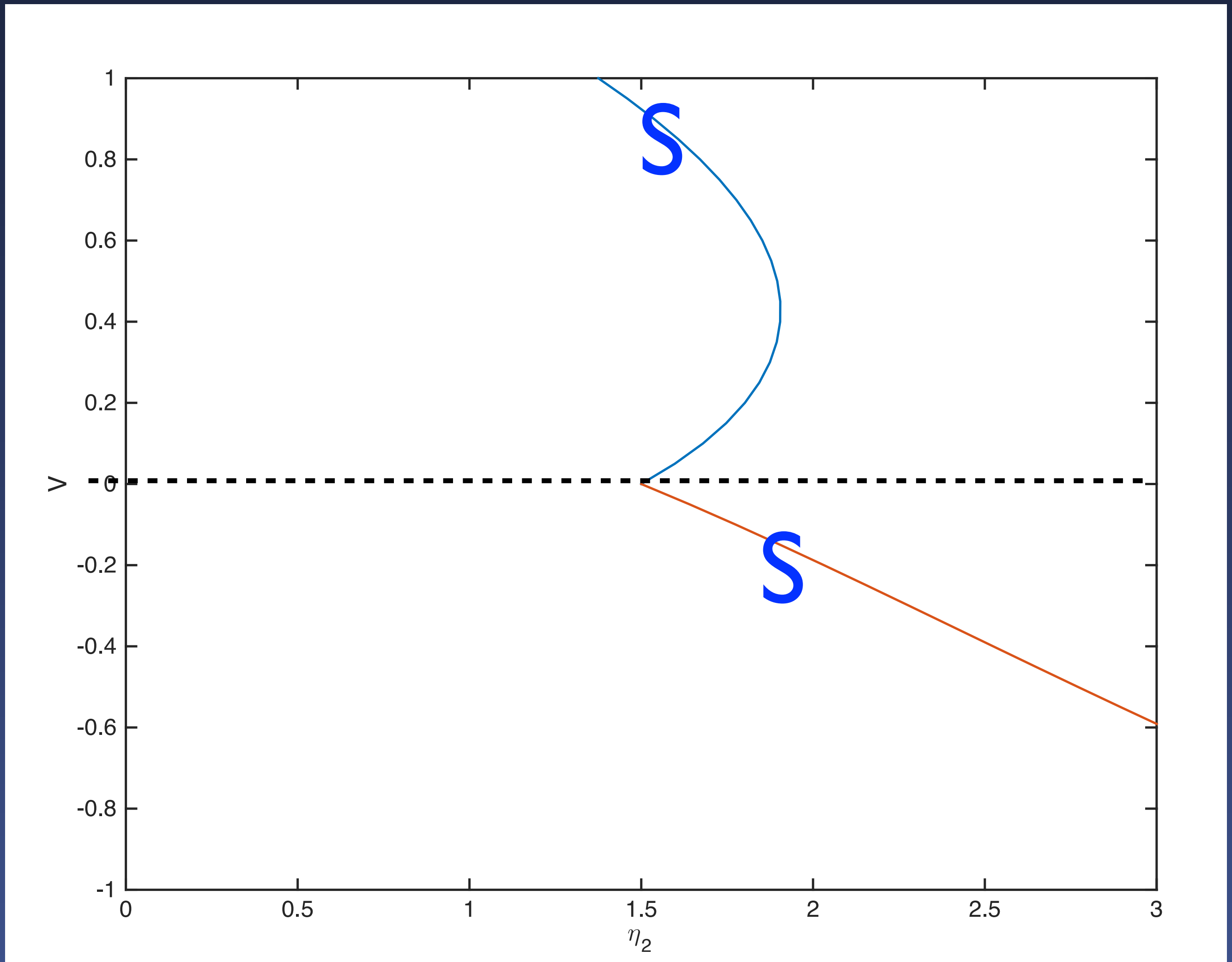
Coherence resonance near bifurcations (discts)

Stommel 2-box model: Potential for hysteresis with ramped parameter for (freshwater) forcing

$$\frac{dT}{dt} = \eta_1 - T(1+ \mid T - S \mid),$$
$$\frac{dS}{dt} = \eta_2 - S(\eta_3+ \mid T - S \mid).$$

Dijkstra, 2014

V = T-S

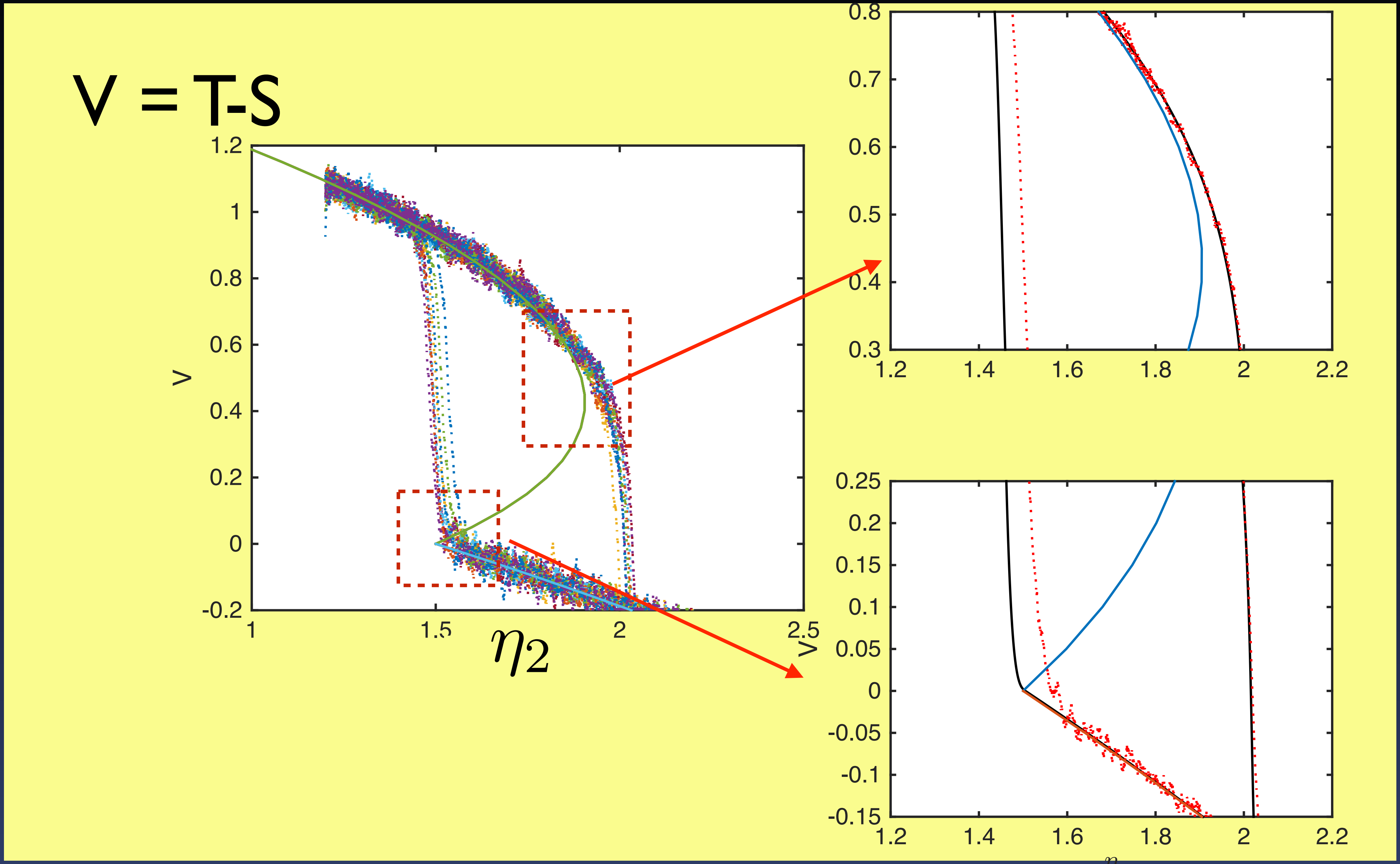


η_2

Bi-stability of thermally and salinity dominated states

T and S temperature/ salinity differences, (equatorial vs polar)

Transitions/Tipping, including noise + oscillations



Noise does not advance transition/tipping from thermally-dominated near fold

Noise advances the transition/tipping from the salinity dominated state (near switching)

Results for ramps through slow bifurcation + noise

Genz and Berglund: book + many papers
drift = (noise)^2

Sieber + Thompson: comparison with climate papers

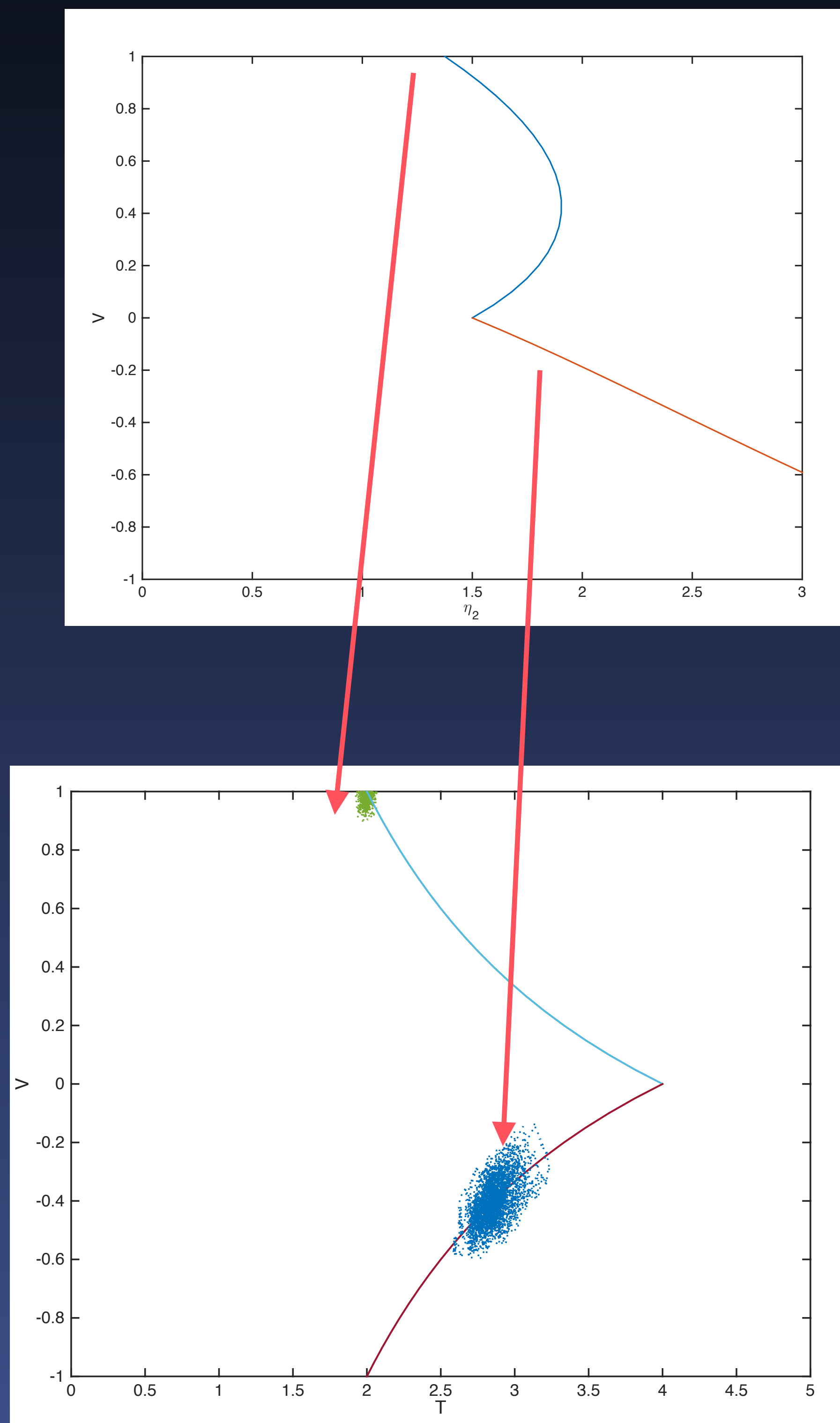
Hopf, pitchfork, : Kuske, et al, pdf

neural/bio dynamics: higher dimensional models, embedded canards

Fat tails vs. oscillations? An experiment

Stommel 2-box model: Potential parameter ranges where local behaviour near equilibrium has fat tails? CAM noise?

V = T-S



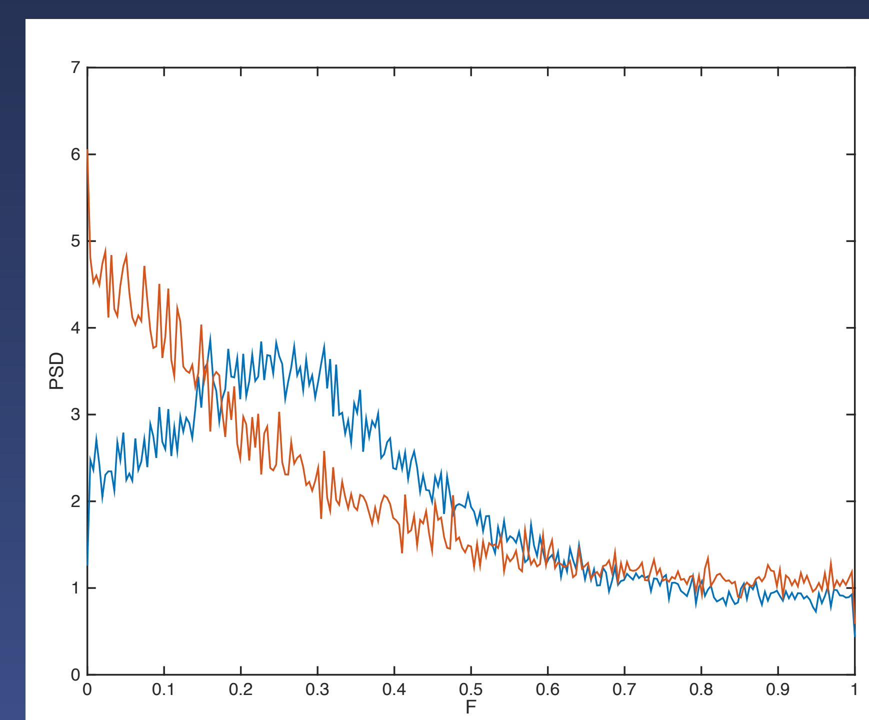
V = T-S

η_2

T

$$\frac{dT}{dt} = \eta_1 - T(1+ \mid T - S \mid),$$
$$\frac{dS}{dt} = \eta_2 - S(\eta_3+ \mid T - S \mid).$$

PSD: amplified noise via coherence resonance w/ additive noise



Tail behaviour of Correlated Additive and Multiplicative (CAM) noise:

$$\frac{dx}{dt} = Ax + b\eta_1 + (Ex + g)\eta_2 - \frac{1}{2}Eg,$$

$$p(x) \sim x^{2\left(\frac{1}{\alpha}-1\right)} \exp\left[-\frac{2g}{\alpha b}\arctan\left(\frac{Ex}{b}\right)\right]$$

$$M = A + 0.5E^2 < 0 \qquad \alpha = E^2/M < 0$$

Linear stochastically forced - both multiplicative and additive noise, part of it uncorrelated

Appears in generic slow-fast systems w/ quadratic nonlinearities

Sadeshmukh and Sura, J. Climate, 2009

SST variability, Sura et al, J. Phys. Ocean. 2008

Penland, various reference since mid 1990's

Expansion around mean, projection on slow components

Other characteristics: Skewness, kurtosis, asymmetries

In the context of larger systems: How might fat tail behaviour show up in models or data?

$$\begin{cases} d\mathbf{x} &= \epsilon \mathbf{F}(\mathbf{x})\,dt + \epsilon^{\gamma+1}\mathbf{y}\,dt \\ d\mathbf{y} &= \left(\mathbf{f}^{(1)} + L^{(11)}\mathbf{y} + L^{(12)}\mathbf{z} + B^{(111)}(\mathbf{y},\mathbf{y}) + B^{(112)}(\mathbf{y},\mathbf{z}) + B^{(122)}(\mathbf{z},\mathbf{z})\right)\,dt \\ \epsilon\,d\mathbf{z} &= \left(\mathbf{f}^{(2)} + L^{(21)}\mathbf{y} + L^{(22)}\mathbf{z} + B^{(211)}(\mathbf{y},\mathbf{y}) + B^{(212)}(\mathbf{y},\mathbf{z}) + B^{(222)}(\mathbf{z},\mathbf{z})\right)\,dt \end{cases}$$

$\epsilon \ll 1$

Reduction: fluctuations, nonlinearities on faster time scale replaced with white and correlated, linearize in y

$$dy = \left(L + \frac{E^2}{2}\right) y dt + (Ey + g) dW_1 + b dW_2$$

(Stratonovich) $dy = \left(Ly - \frac{Eg}{2}\right) dt + (Ey + g) \circ dW_1 + b \circ dW_2$ (b =const)

Correlated Additive and Multiplicative (CAM) noise: heavy tails, not necessarily Levy

Penland, et al 2009, 2012

Coherence resonance near bifurcations (discts)

Stommel 2-box model: Potential parameter ranges where local behaviour near equilibrium has fat tails?

Correlated Additive + Multiplicative (CAM) noise?

$$\frac{dT}{dt} = \eta_1 - T(1+ \mid T - S \mid),$$
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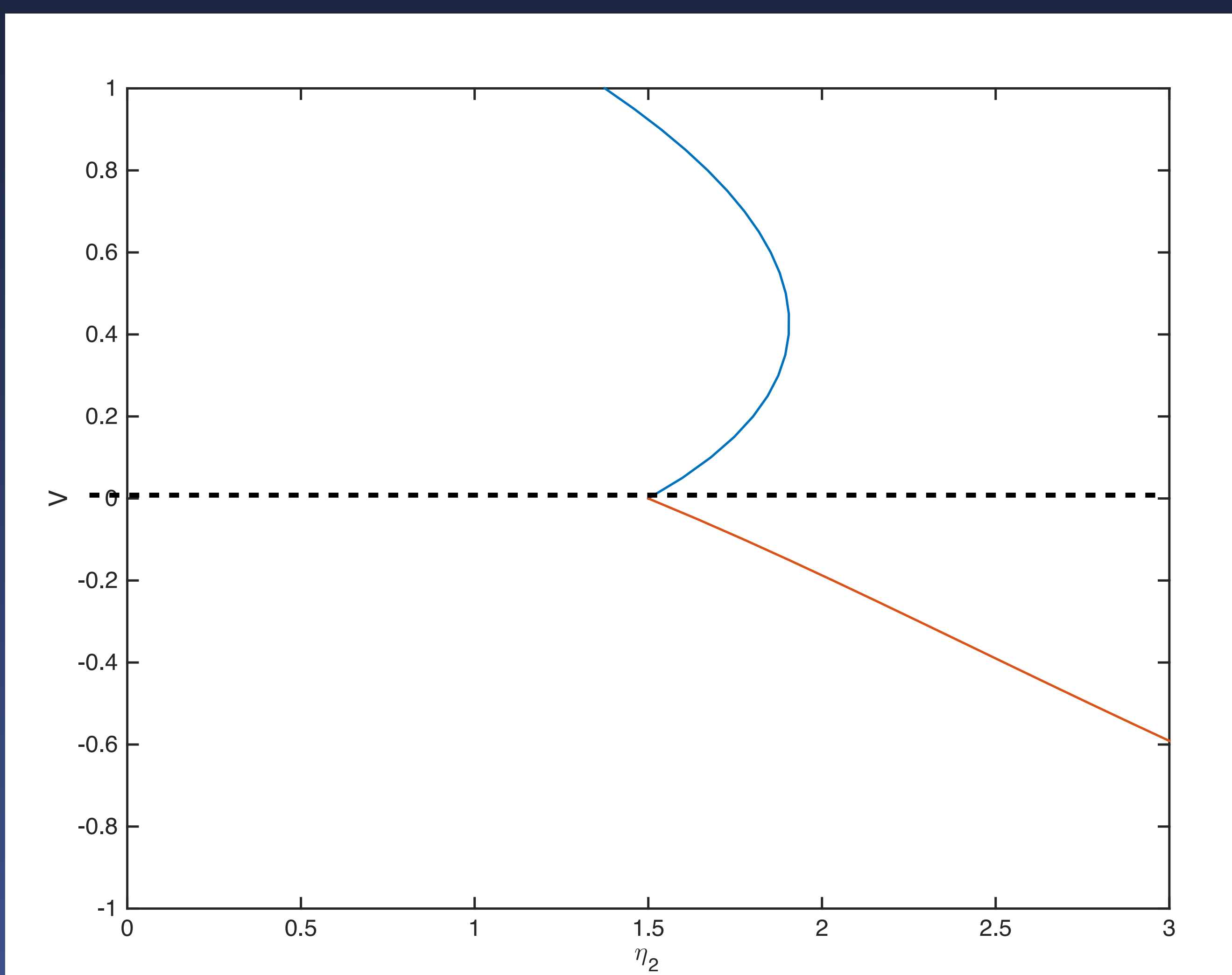
Dijkstra, 2014

V = T-S



η_2

Bi-stability of thermally and salinity dominated states

Hysteresis within a larger system with slowly varying parameters

Larger model + reduction:

$$V_p \frac{dT_p}{dt} = C_p^T (T_p^a - T_p) + \mid \Psi \mid (T_e - T_p),$$

$$V_e \frac{dT_e}{dt} = C_e^T (T_e^a - T_e) + \mid \Psi \mid (T_p - T_e),$$

$$V_p \frac{dS_p}{dt} = C_p^S (S_p^a - S_p) + \mid \Psi \mid (S_e - S_p),$$

$$V_e \frac{dS_e}{dt} = C_e^S (S_e^a - S_e) + \mid \Psi \mid (S_p - S_e).$$

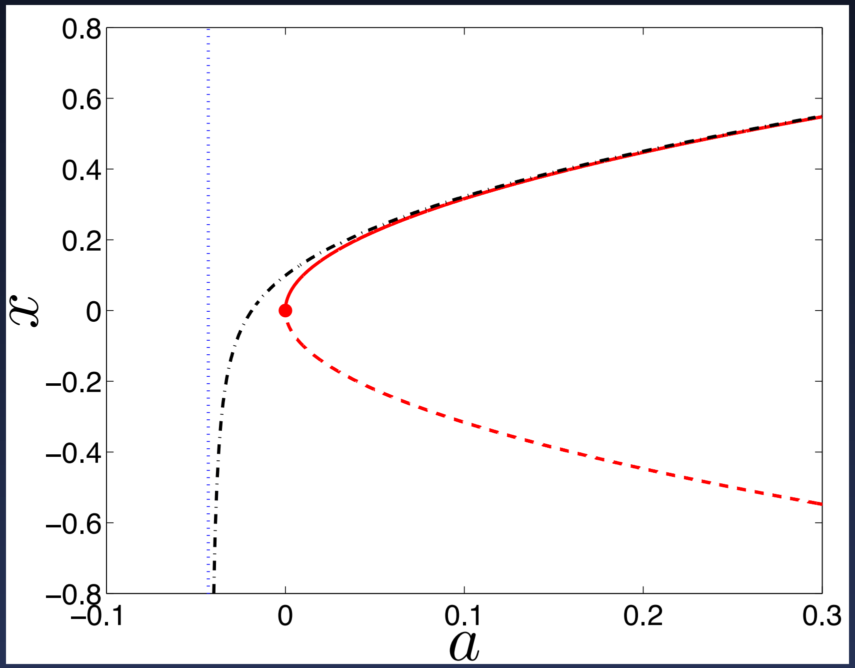
When time, temperature, salinity and flow rate are scaled with $1/R_T$, $V_e V_p R_T/(\gamma \alpha_T (V_e + V_p))$, $V_e V_p R_T/(\gamma \alpha_S (V_e + V_p))$ and $V_e V_p R_T/((V_e + V_p)$, respectively, the dimensionless equations become

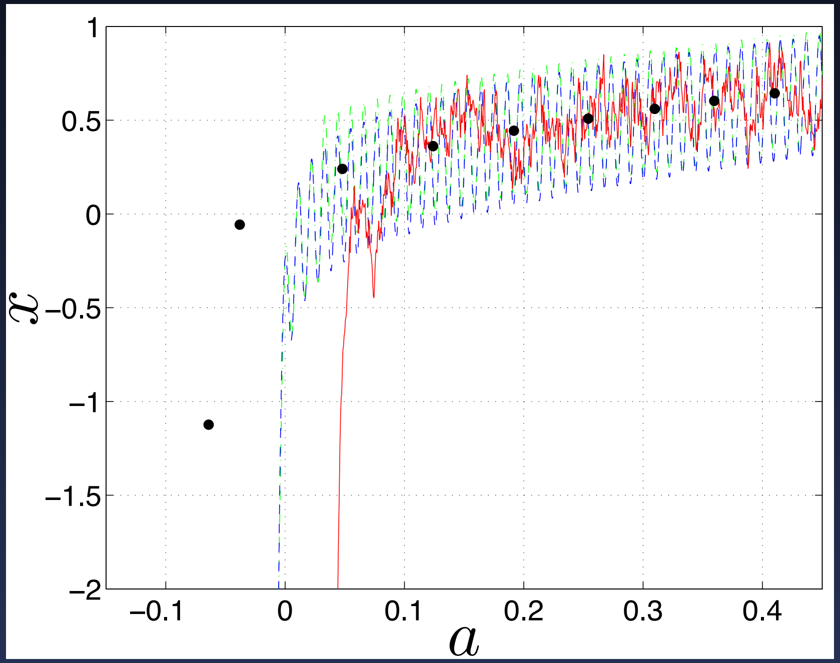
Fluctuations/uncertainty in parameters - leading to multiplicative + additive noise/oscillations, correlated

Similarly about equilibria

Saddle node normal form: oscillatory forcing

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a - x^2 + A\sin(\Omega t), \quad \frac{\mathrm{d}a}{\mathrm{d}t} = -\mu,$$





A = amplitude of oscillation, Ω = frequency

Previous results: Haberman (1979), slow drifting only

Stochastic models: Berglund, Genz, (2010) Sieber, Thompson (2011)

High Frequency Periodic Forcing

Relationship between drift rate μ and $\Omega = \mu^{-\lambda}$

Amplitude is O(1) (or smaller)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a - x^2 + A\sin(\Omega t), \quad \frac{\mathrm{d}a}{\mathrm{d}t} = -\mu,$$

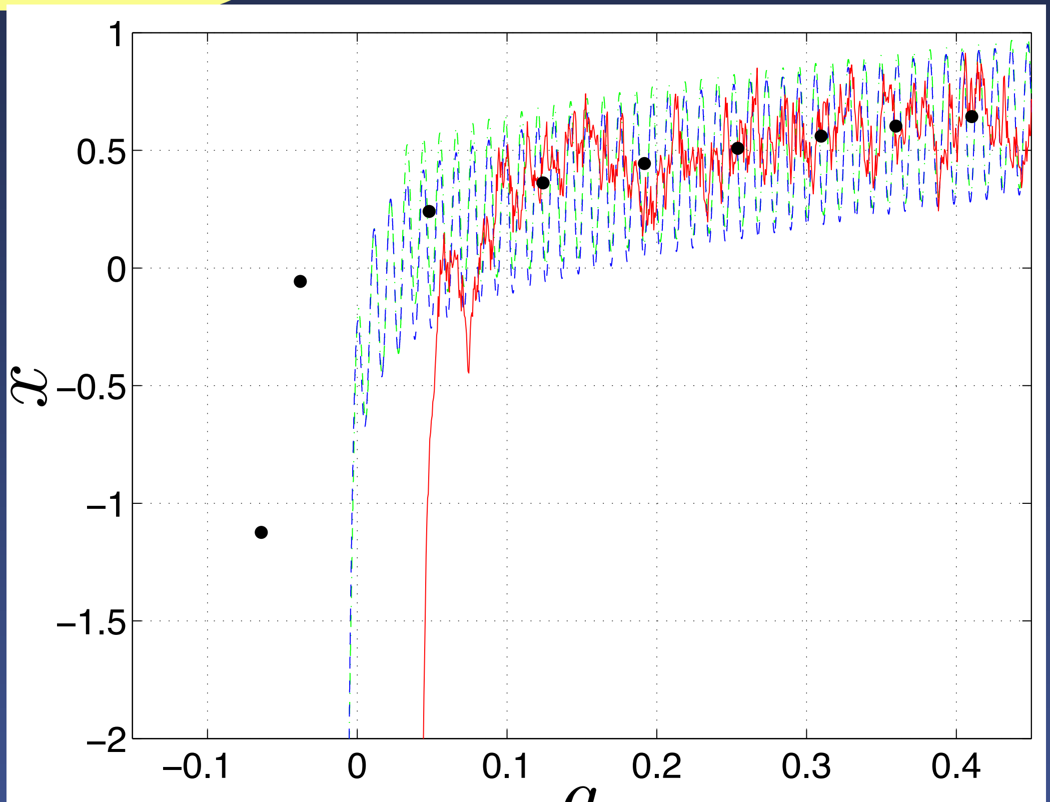
$$x = -\mu^{\lambda} A\cos(T) + \mu^{1/3} y(T, s)$$

$$y_{0s} = \alpha - y_0^2 - \mu^{2\lambda-2/3}\frac{A^2}{2}, \quad \Rightarrow \quad y_0 = -\frac{\mathrm{Ai}'(\alpha - \mu^{2\lambda-2/3}\cdot\frac{A^2}{2})}{\mathrm{Ai}(\alpha - \mu^{2\lambda-2/3}\cdot\frac{A^2}{2})}.$$

Cases for different λ summarized

Method of Multiple Scales: slow = t, fast = $T = \Omega t$

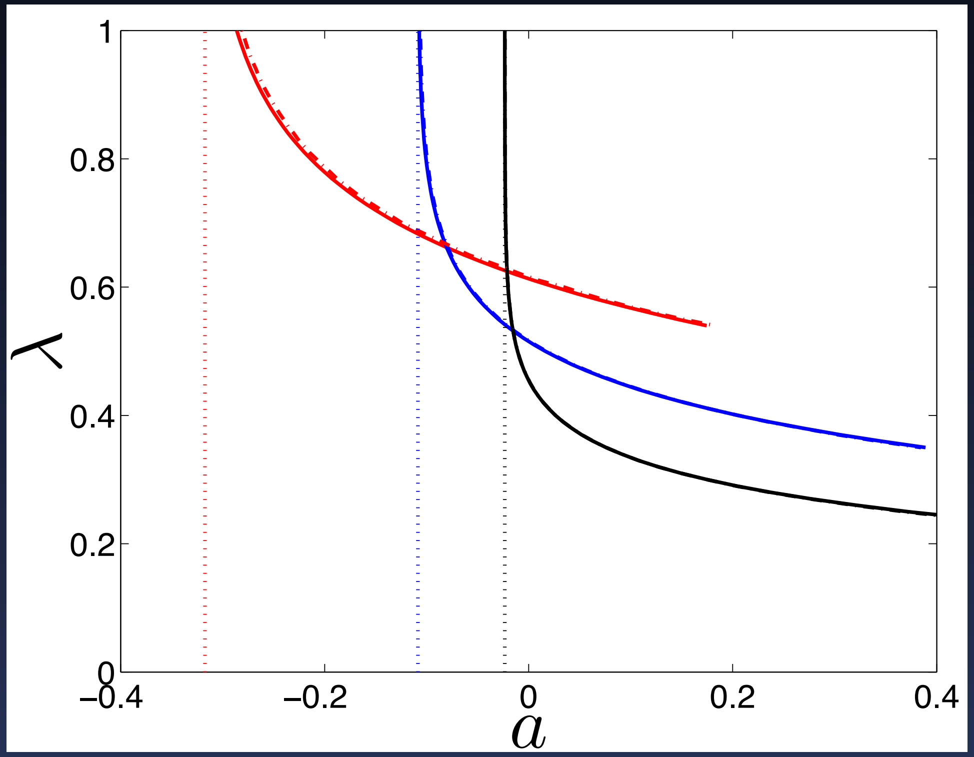
Tipping points: singularity of the Airy function



High Frequency Periodic Forcing

Tipping points: blow-up of Airy function:

decreasing values of the drift



advance due to periodic forcing

$$a_{\mathrm{hf}} \sim a_d + a_p = \mu^{2/3} K + \frac{A^2}{2\Omega^2},$$

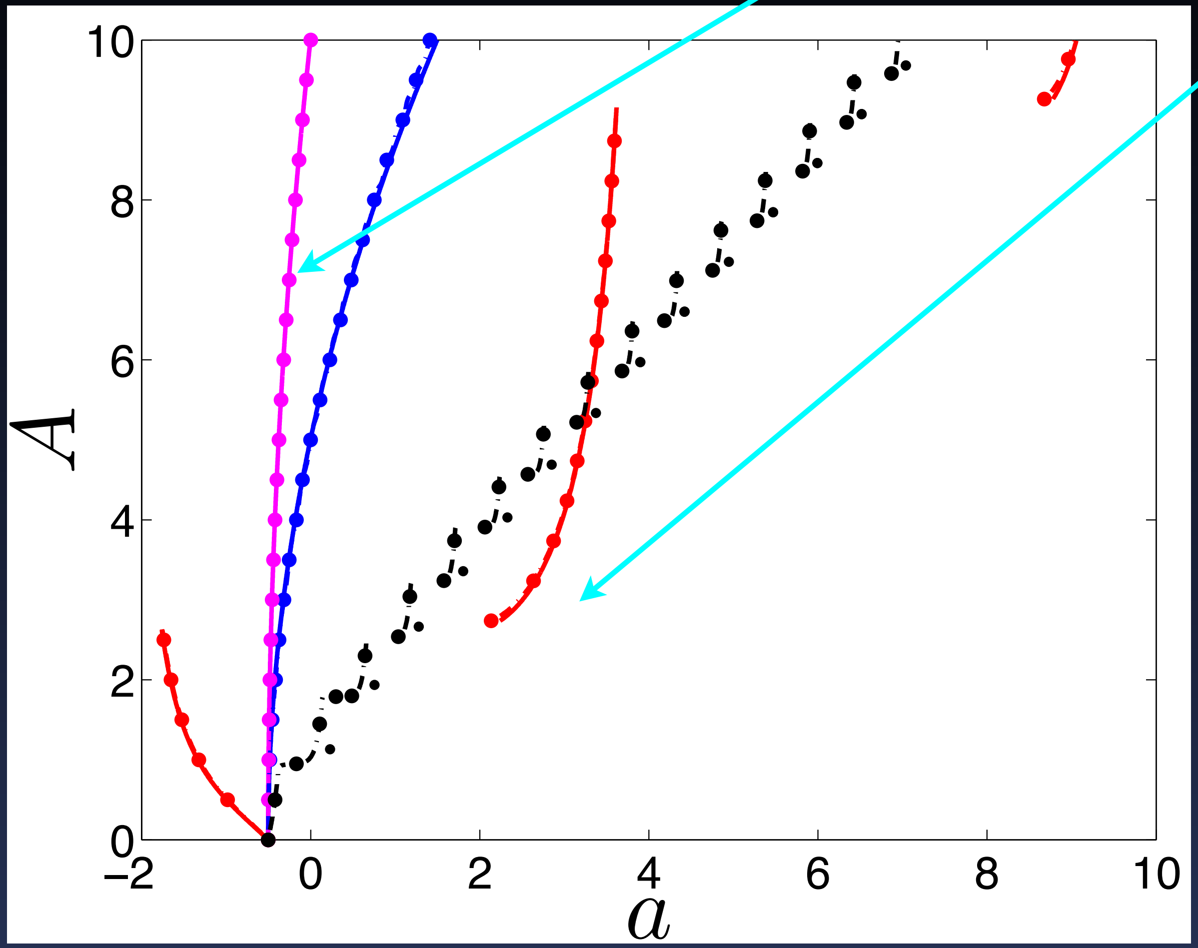
delay due to slow drift

Zhu, K, Erneux, 2015

What about large amplitude: $A = \Omega^P, P \geq 1$

Rescale to be in high or low frequency for certain ranges of P

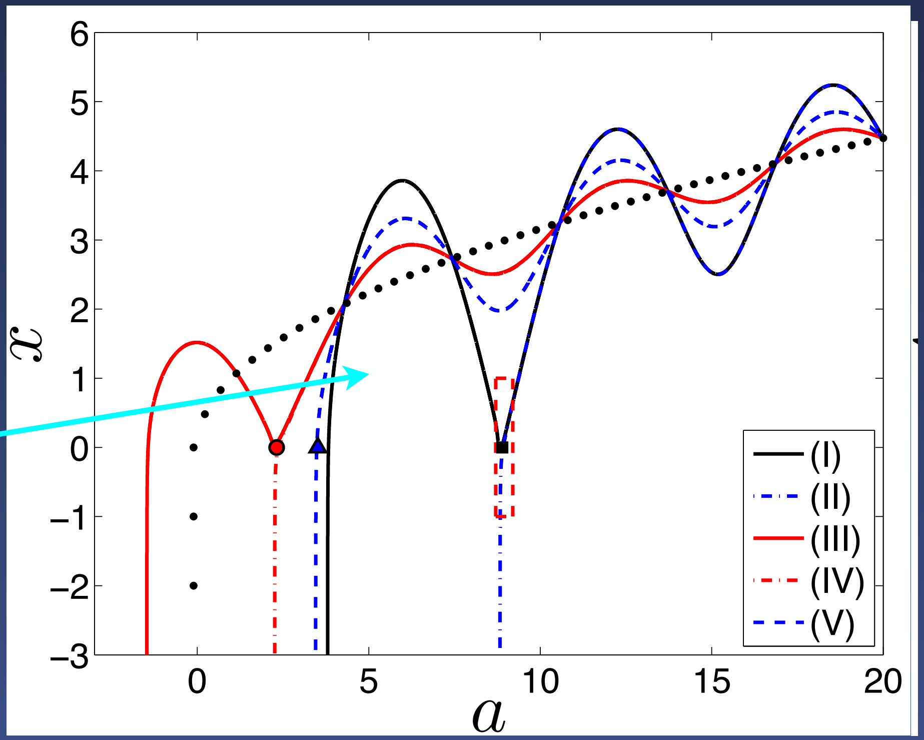
Comparison of High and Low Frequency Periodic Forcing



Zhu, K, Erneux, 2015

Low frequency: discontinuity in dependence of tipping points on amplitude + phase

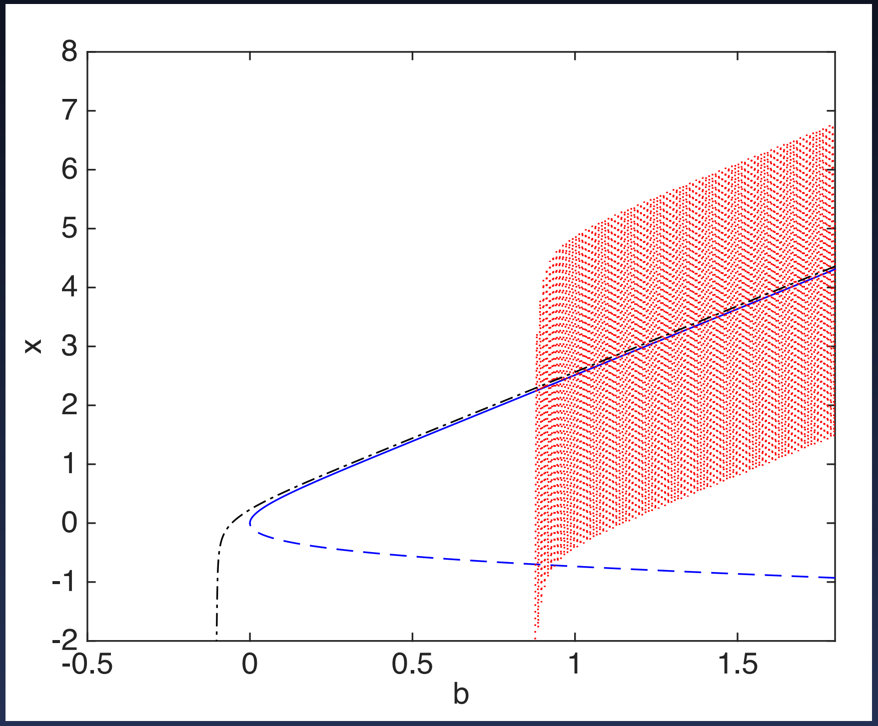
Local analysis near values where x approaches zero, change in concavity corresponds to advance or delay



Sea ice model:

Eisenman, Wettlaufer, 2009

$$\frac{dE}{dt} = [1-\alpha(E)]F_S(t) - F_0(t) + \Delta F_0 - F_T(t)\,T(t,E) + F_B + \nu_0 \Re(-E),$$



Rescale and normalize to identify appropriate asymptotics & relationships of critical parameters - drift, amplitude, frequency

$$\frac{dx}{dt} = b + H(x) + q(T),$$
$$\frac{db}{dt} = -\mu.$$
$$H(x) = G_1 + G_2 \tanh(g_3(x+1)) + G_4 x, \quad q(T) = g_1(t) - G_1 + g_2(t) - G_2\,,$$

Large amplitude, in between high/low frequency (upper branch)

Sea ice model:

$$\frac{dE}{dt} = [1-\alpha(E)]F_S(t) - F_0(t) + \Delta F_0 - F_T(t)T(t,E) + F_B + \nu_0 \Re(-E),$$

$$\frac{dx}{dt} = b + H(x) + q(T),$$
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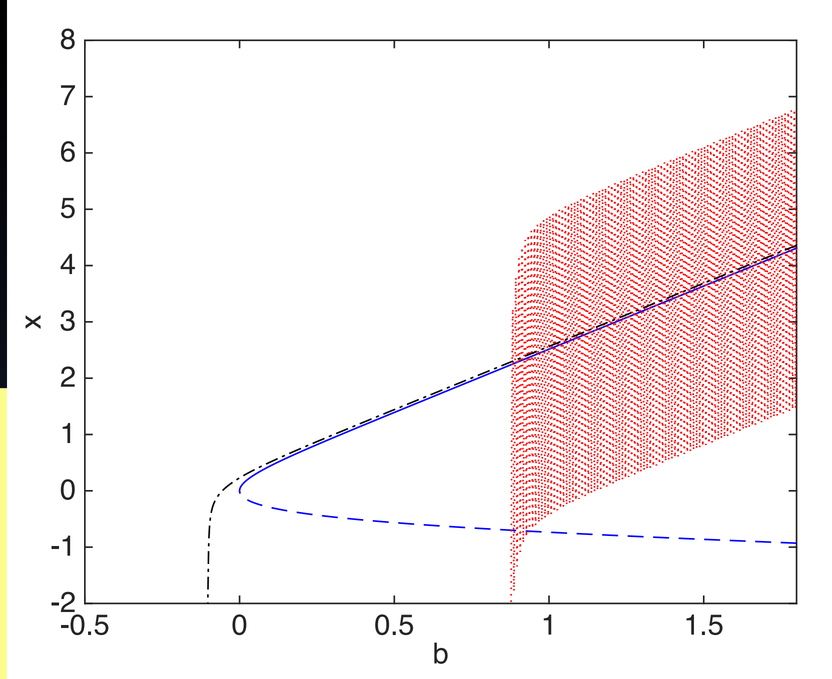


High frequency-type approximation, adapted to large amplitudes:

Identify a critical value $x^* \neq 0$

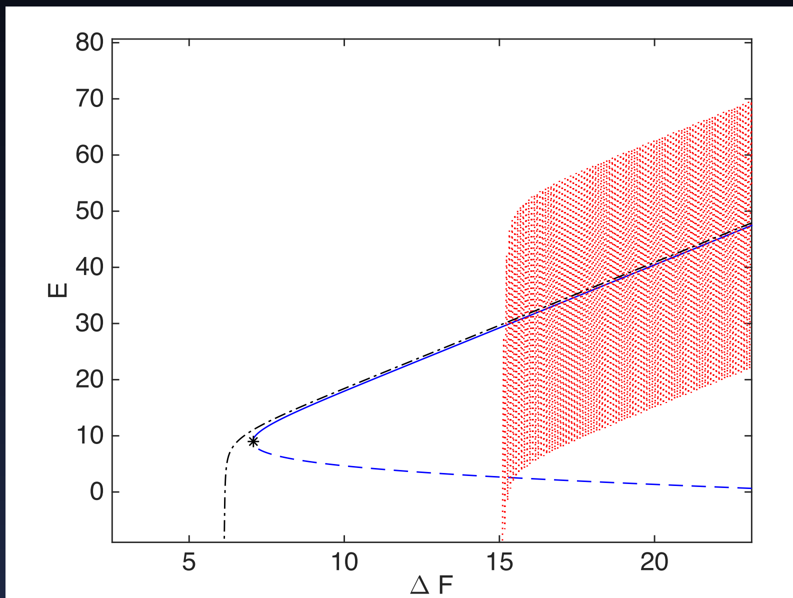
$$x = x^* + \Omega^{-1} Q(T) + Y(T,t).$$

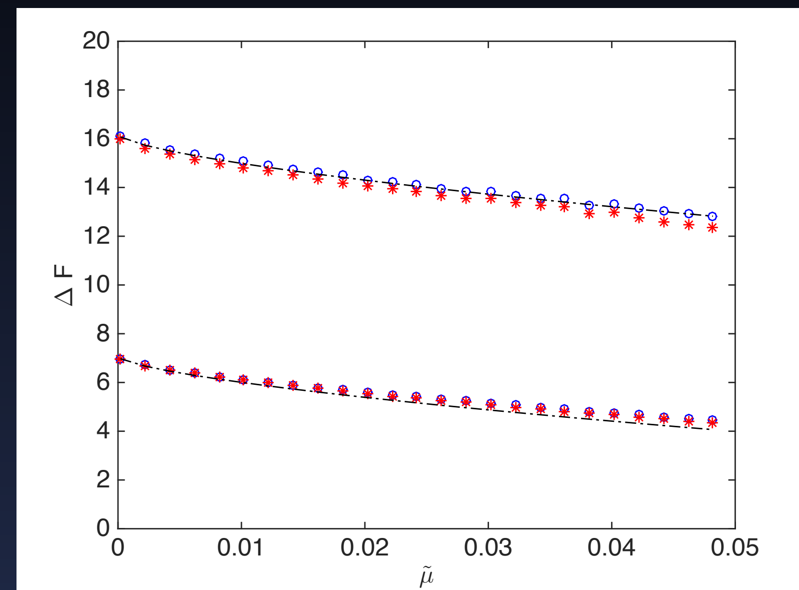
$$\begin{aligned} Y_t &= b + \left(\lim_{L\to\infty} \frac{1}{L}\int_0^L H(x^* + \Omega^{-1}Q(T))dT\right) + \left(\lim_{L\to\infty} \frac{1}{L}\int_0^L H'(x^* + \Omega^{-1}Q(T))dT\right) Y \\ &+ \left(\lim_{L\to\infty} \frac{1}{L}\int_0^L \frac{H''(x^* + \Omega^{-1}Q(T))}{2}dT\right) Y^2 = b + \mathcal{H}_0 + \mathcal{H}_1 Y + \mathcal{H}_2 Y^2 \end{aligned}$$

$$b_{\text{tip}} = |\mathcal{H}_2|^{-1/3} a_d - b_Q \quad \text{for} \quad b_Q = \mathcal{H}_0 + \frac{\mathcal{H}_1^2}{4|\mathcal{H}_2|} \cdot$$ singularity in eqn for Y

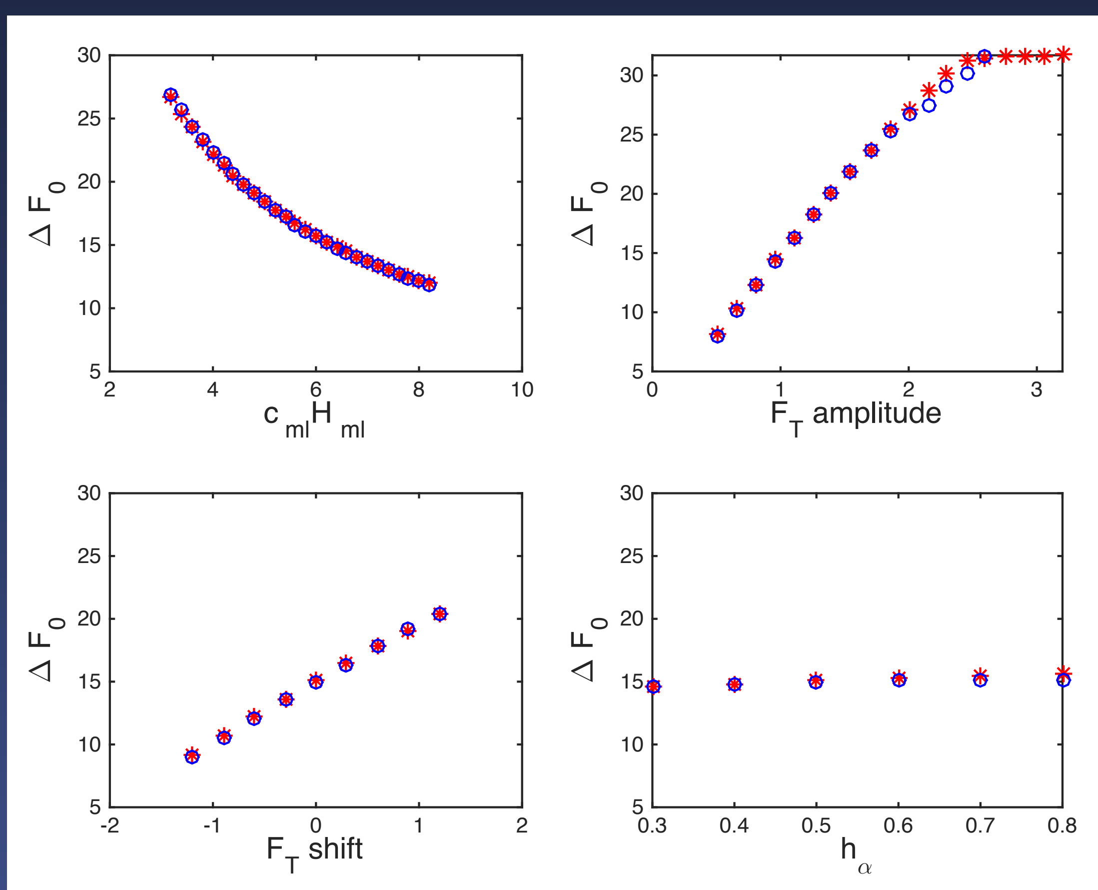
Parametric dependence

$$\frac{dE}{dt} = [1-\alpha(E)]F_S(t) - F_0(t) + \Delta F_0 - F_T(t)T(t,E) + F_B + \nu_0 \Re(-E),$$





Eisenman, Wettlaufer, 2009



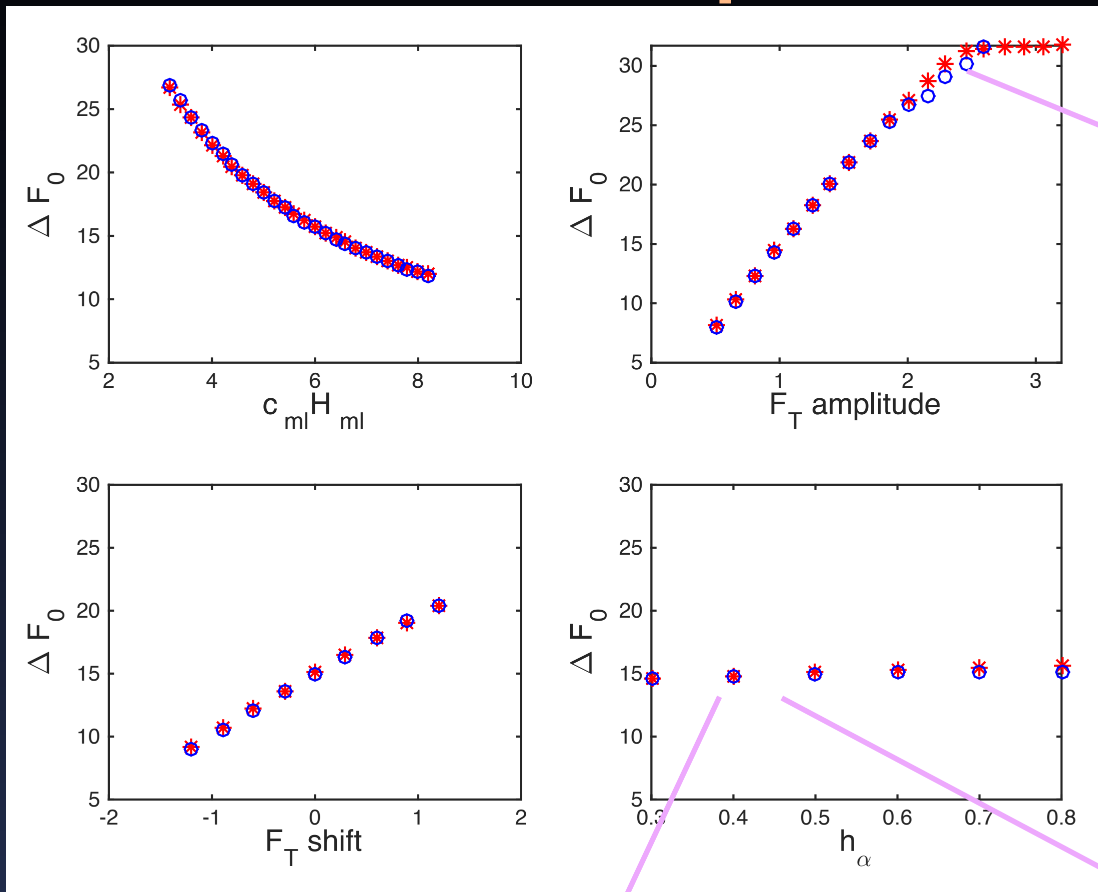
Variability of tipping point with different parameters: amplitude magnitude, shift in forcing, sharpness of albedo transition

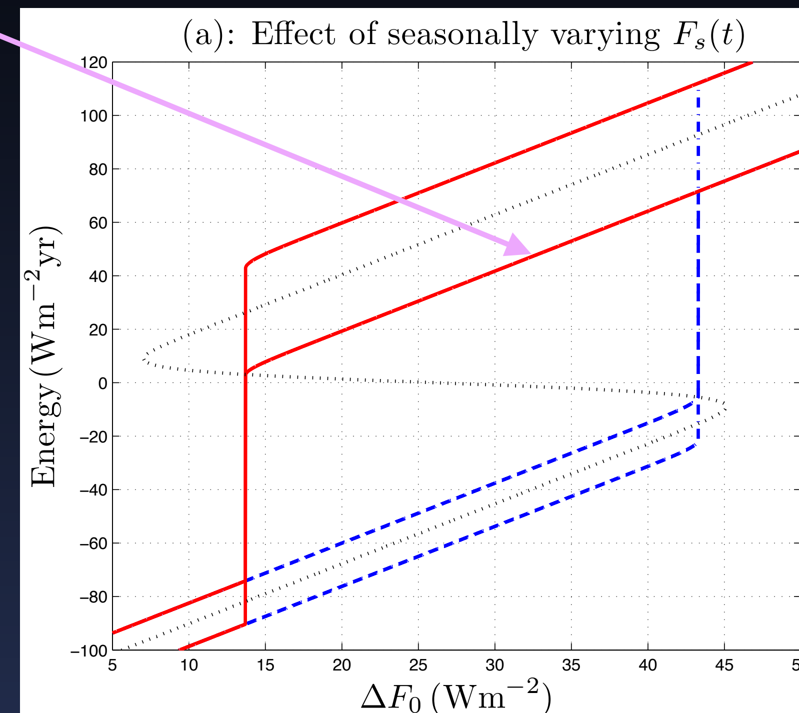
Compare analytical + numerical results, also to Eisenman, 2012

Parametric dependence

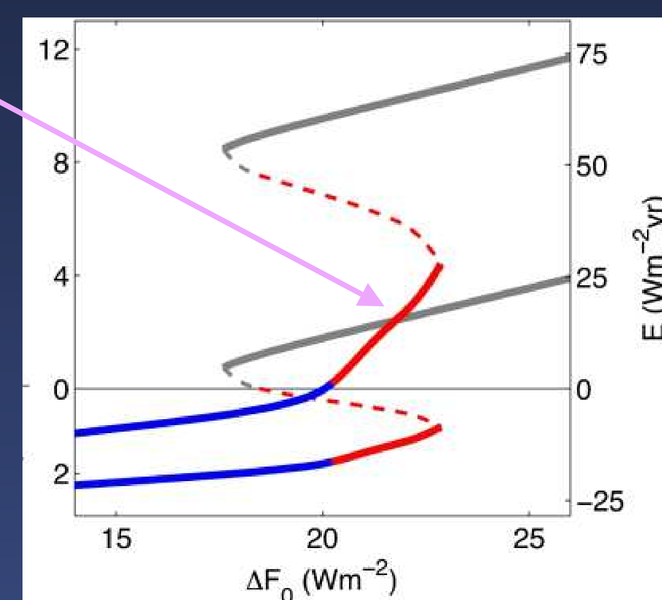
Eisenman, Wettlaufer, 2009

$$\frac{dE}{dt} = [1-\alpha(E)]F_S(t) - F_0(t) + \Delta F_0 - F_T(t)T(t,E) + F_B + \nu_0\Re(-E),$$





Loss of hysteresis for shifted tipping point - (linearized model)



Other effects: loss of branch for reduced smoothing in albedo parameter, change in bifurcation structure

Eisenman 2012, Hill et al 2015

Oscillations + ramped bifurcations (discts)

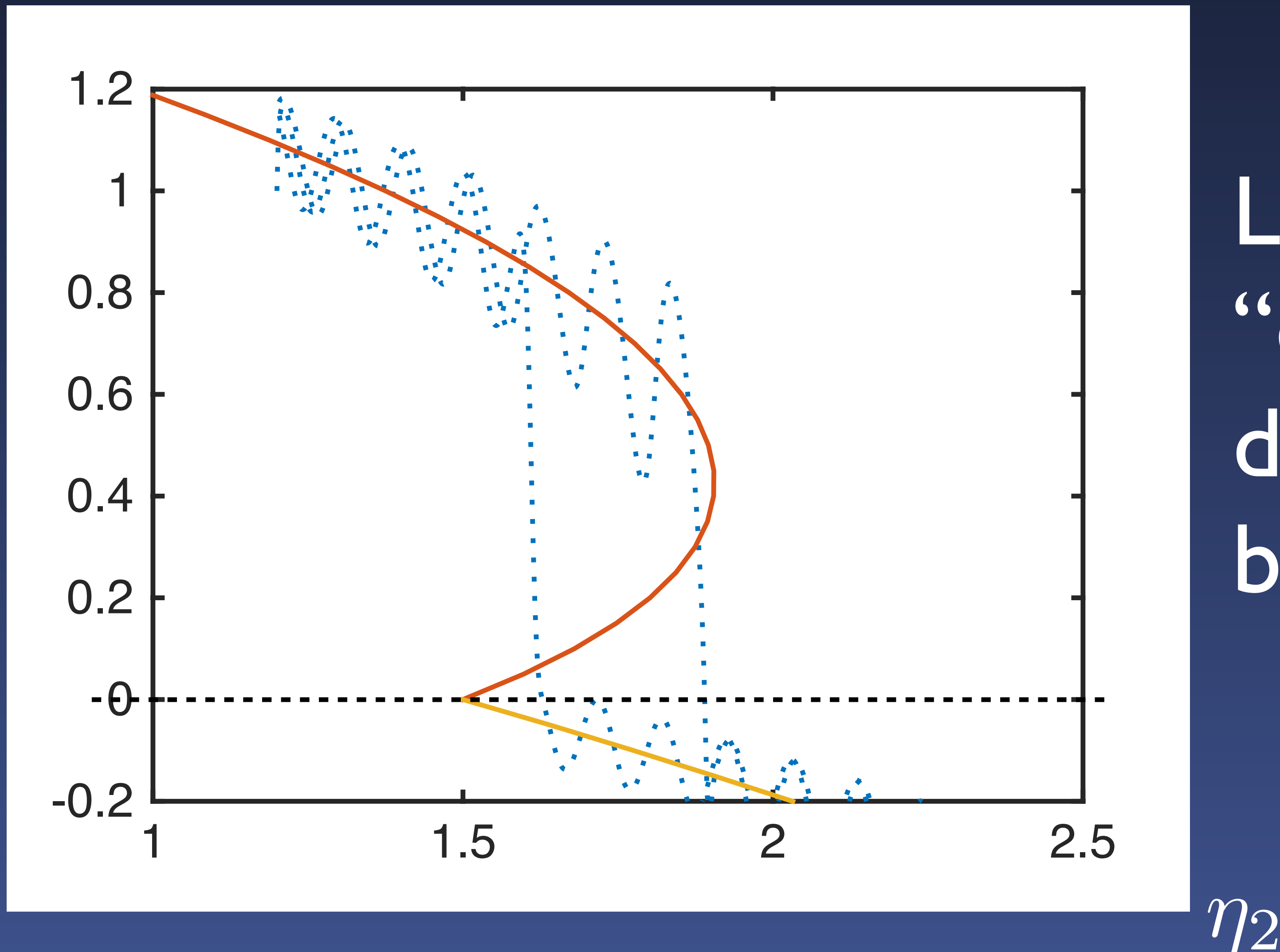
Stommel 2-box model:

High frequency: Analysis across discontinuity - averaging including switch

V = T-S

$$\frac{dT}{dt} = \eta_1 - T(1+ \mid T - S \mid),$$
$$\frac{dS}{dt} = \eta_2 - S(\eta_3+ \mid T - S \mid).$$

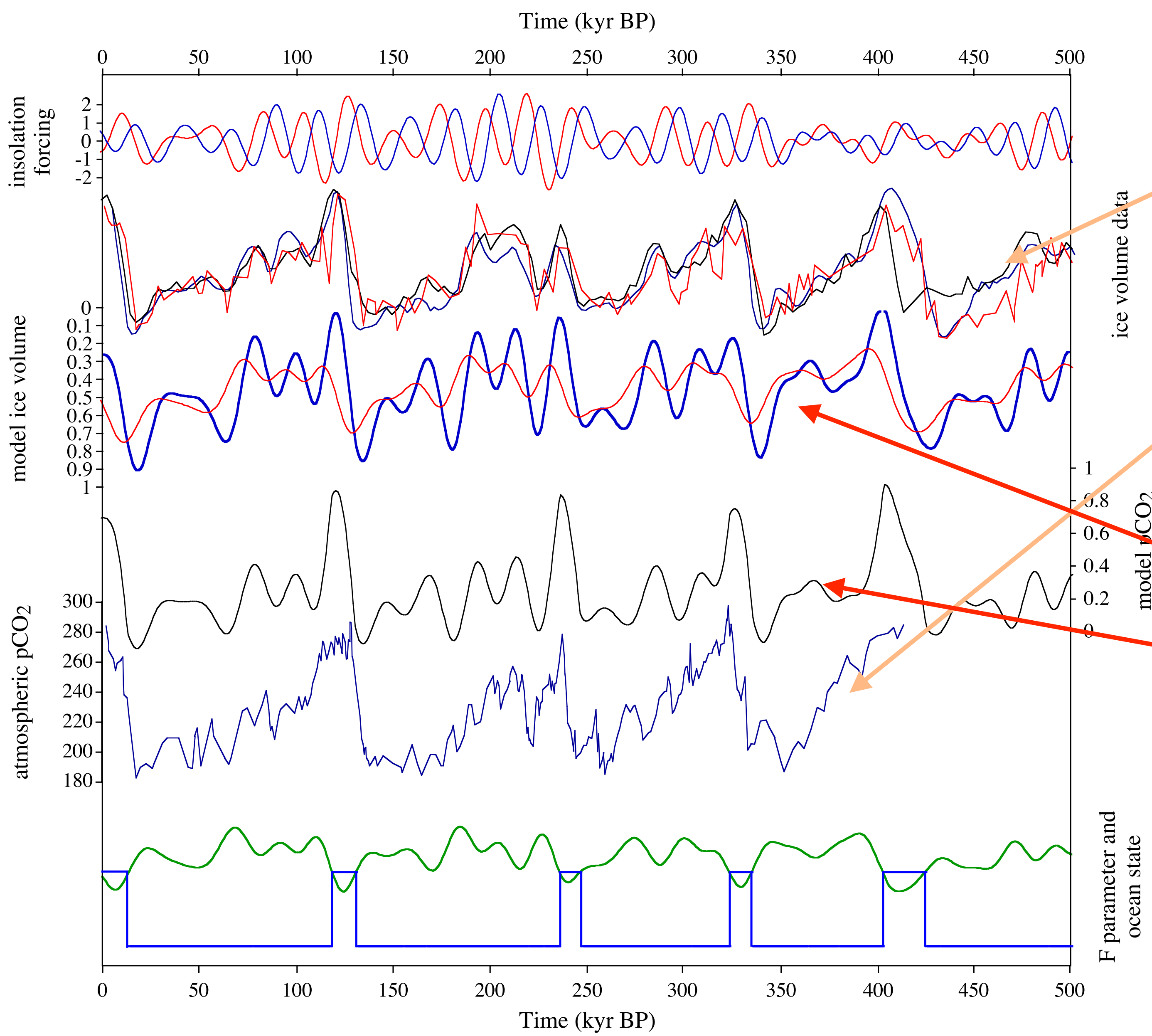
Dijkstra, 2014



Low frequency: Local "damping" of oscillations differs from saddle node bifurcation

Exits/Escapes:

- (Paleo)-Climate dynamics: MMOs in glacial/ interglacial periods



Paillard & Parrenin, 2004

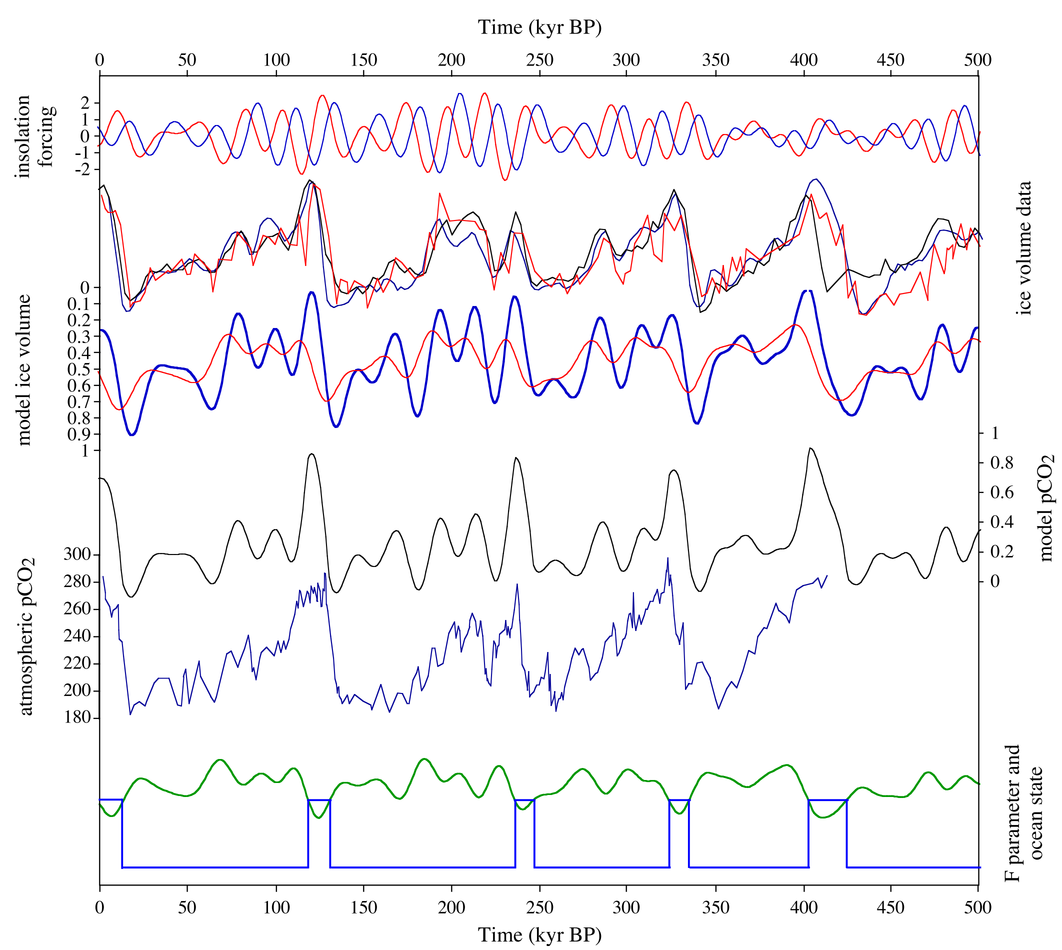
Data from ice core proxies

Single time series

3 “species” model with non-smooth switching, oscillatory forcing

Exits/Escapes:

- (Paleo)-Climate dynamics: MMOs in glacial/ interglacial periods



Paillard & Parrenin, 2004

$$dV/dt = (V_{\mathrm{R}} - V)/\tau_{\mathrm{V}}$$

$$dA/dt = (V - A)/\tau_{\mathrm{A}}$$

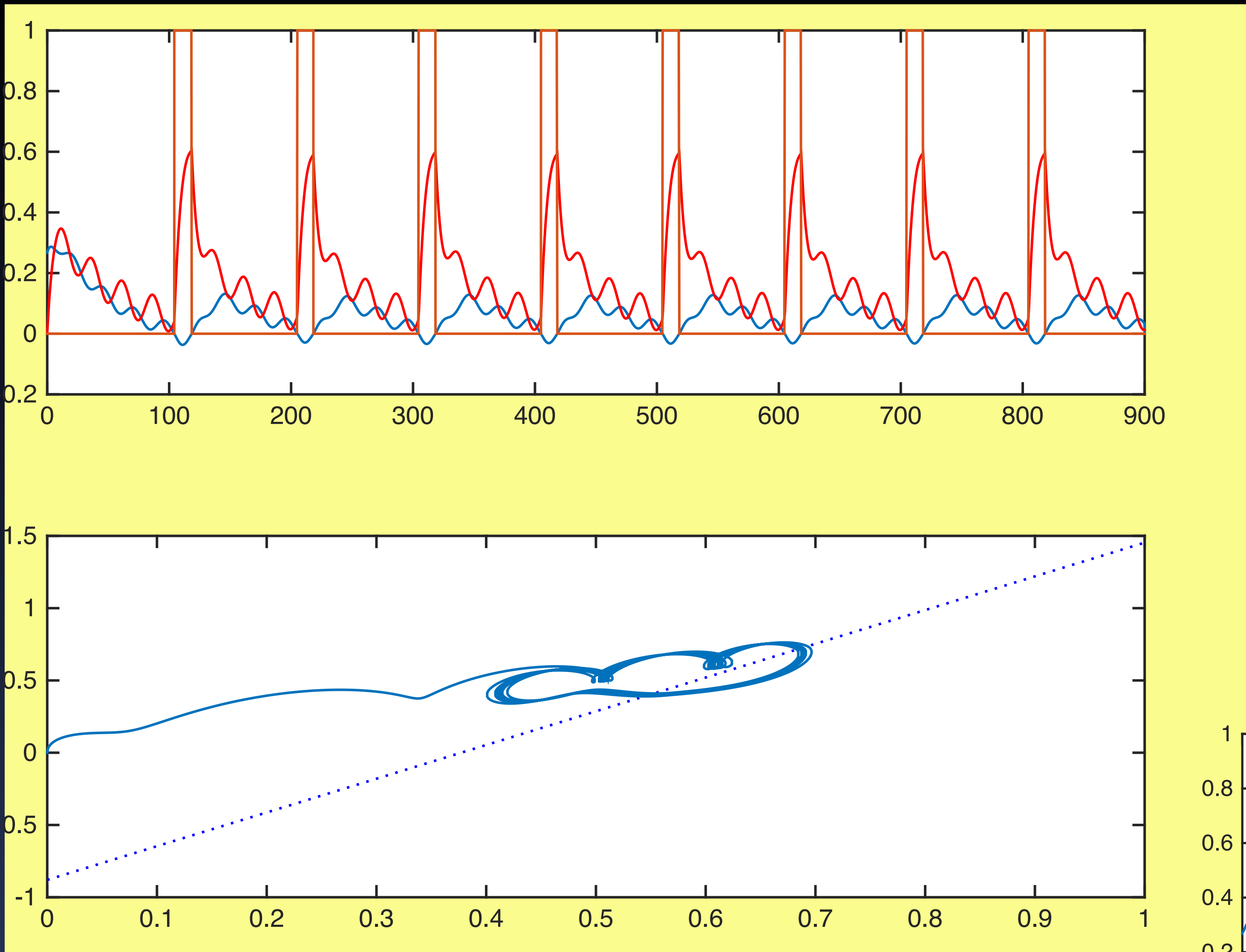
$$dC/dt = (C_{\mathrm{R}} - C)/\tau_{\mathrm{C}}$$

Switch

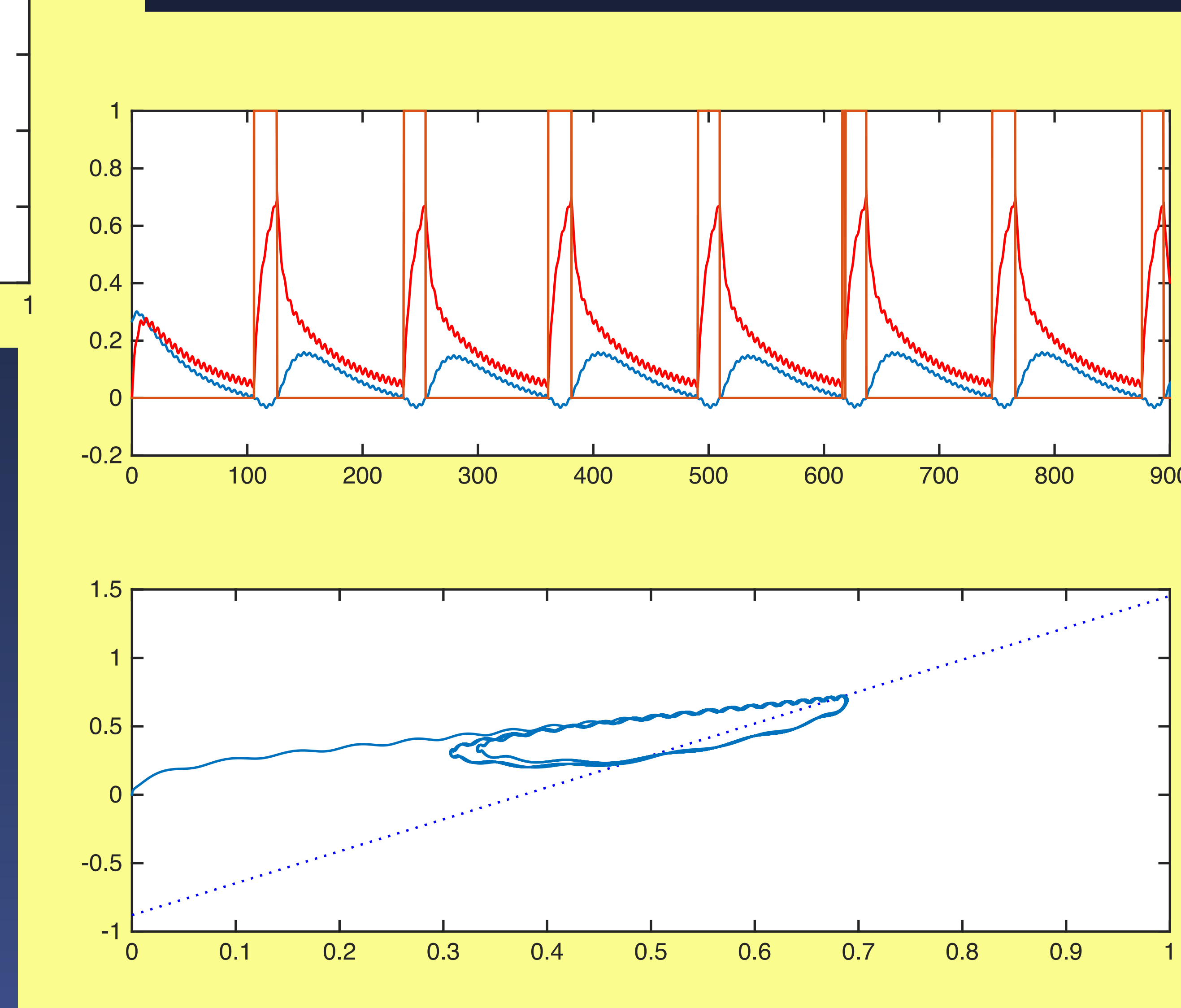
$$C_{\mathrm{R}} = \alpha I_{65} - \beta V + \gamma \boxed{H(-F)} + \delta$$

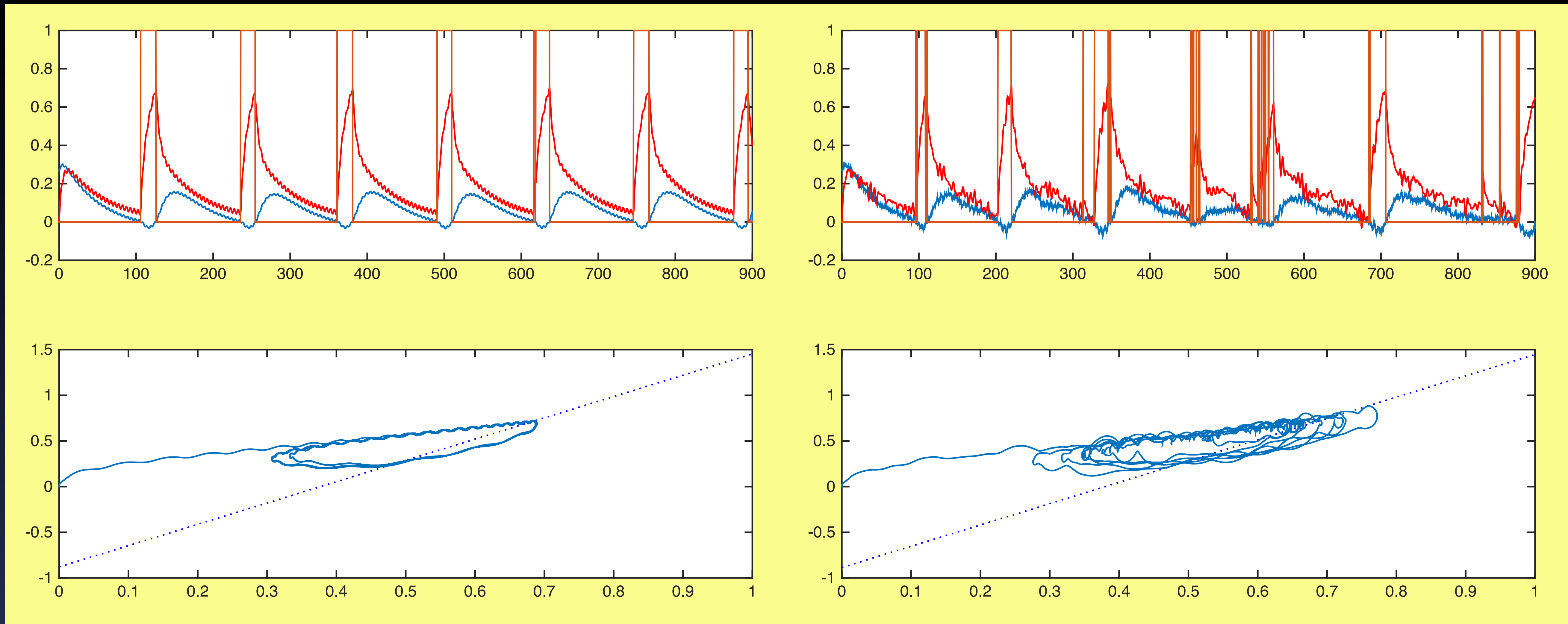
$$F = aV - bA - cI_{60} + d$$

View of model with switch and oscillatory forcing - 2-D projection



Single frequency oscillatory forcing - single crossing of switching surface:



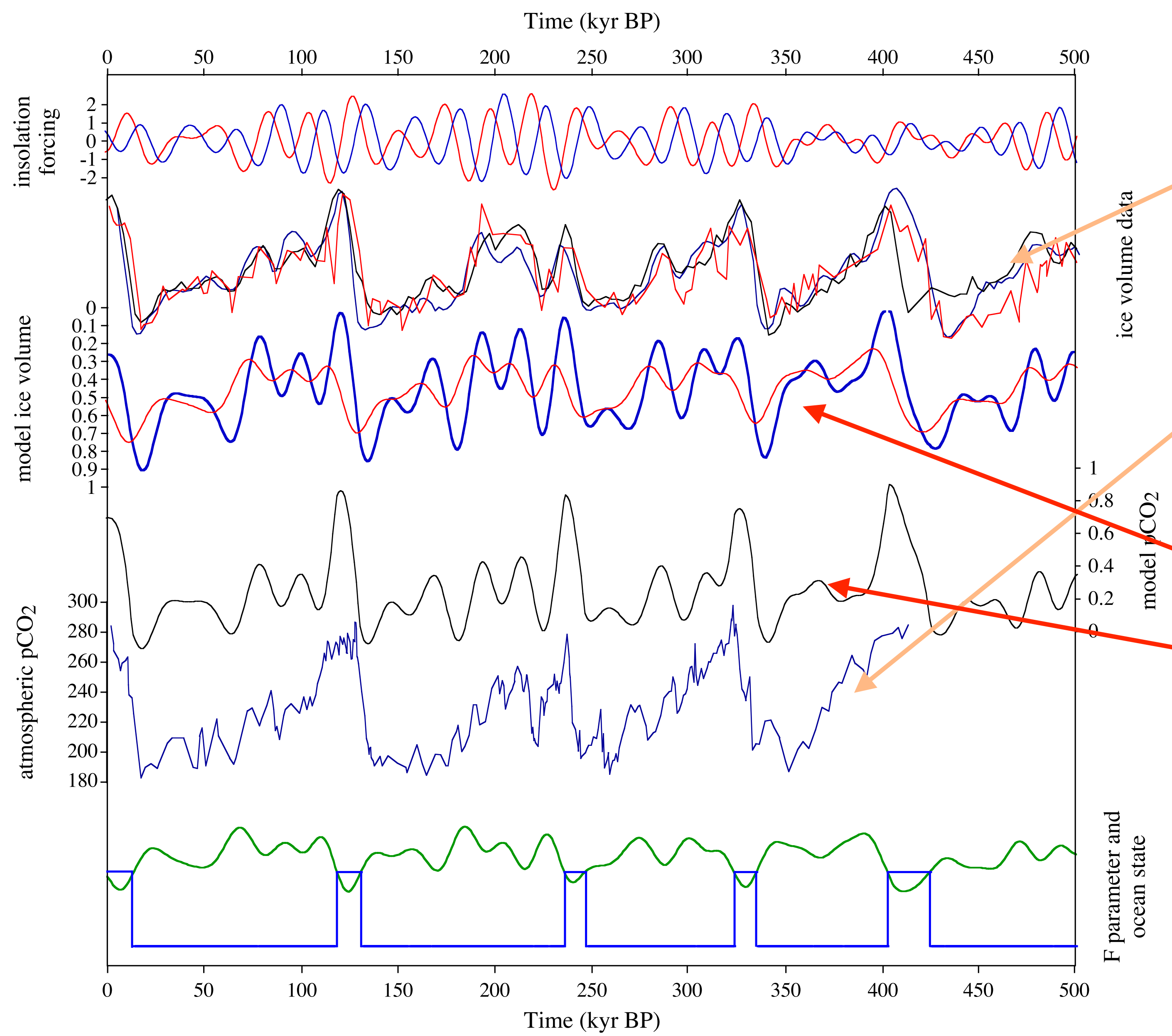


Clear change of state with simple periodic forcing

With stochastically modulated forcing: Multiple crossings - switching back and forth between glacial and non-glacial periods

Exits/Escapes:

- (Paleo)-Climate dynamics: MMOs in glacial/interglacial periods



Paillard & Parrenin, 2004

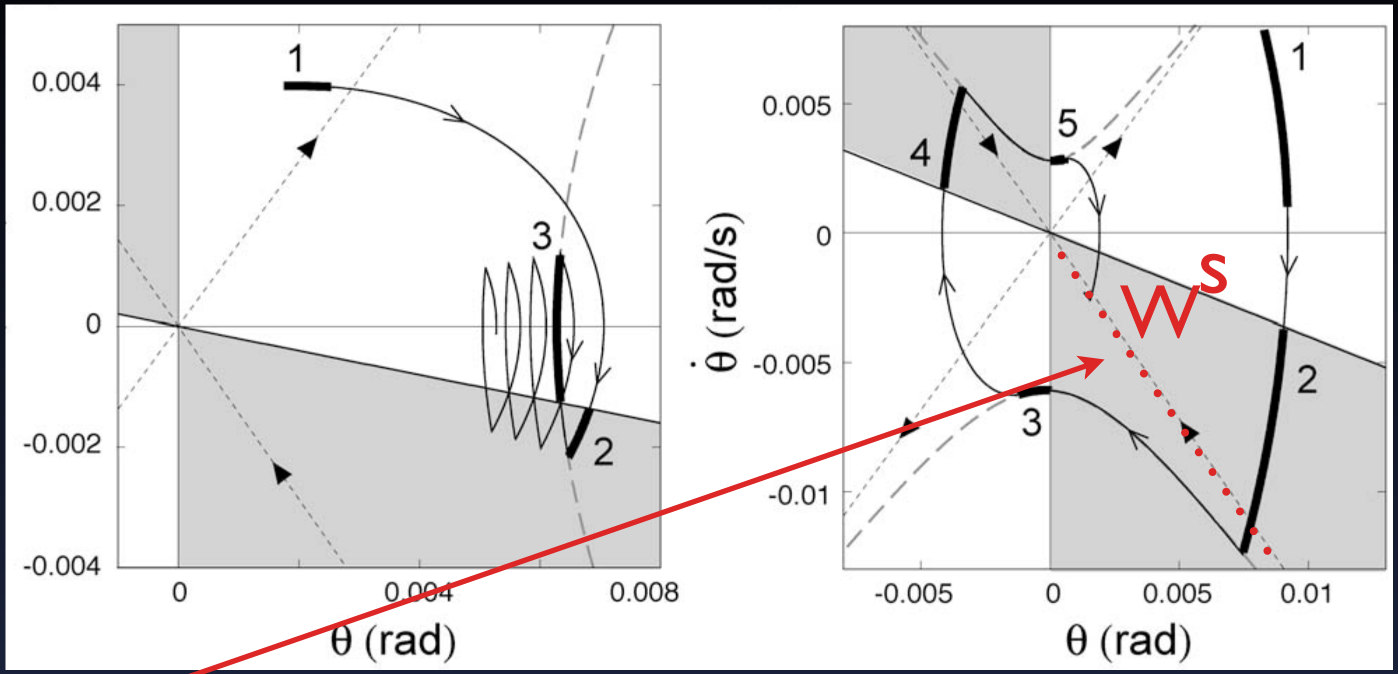
Data from ice core proxies

Single time series

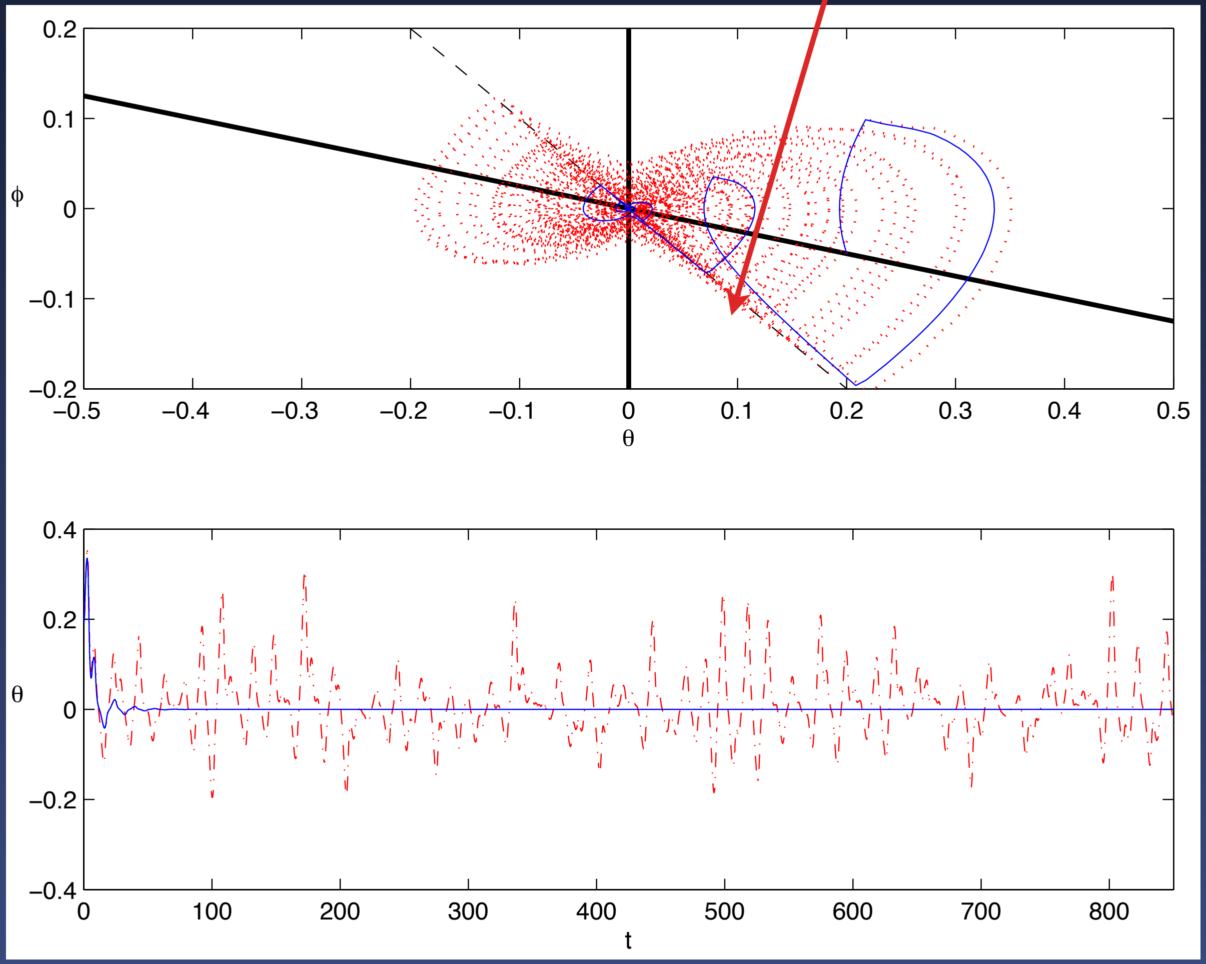
3 “species” model with non-smooth switching, oscillatory forcing

Delay + nonsmooth - potential for recurring transients

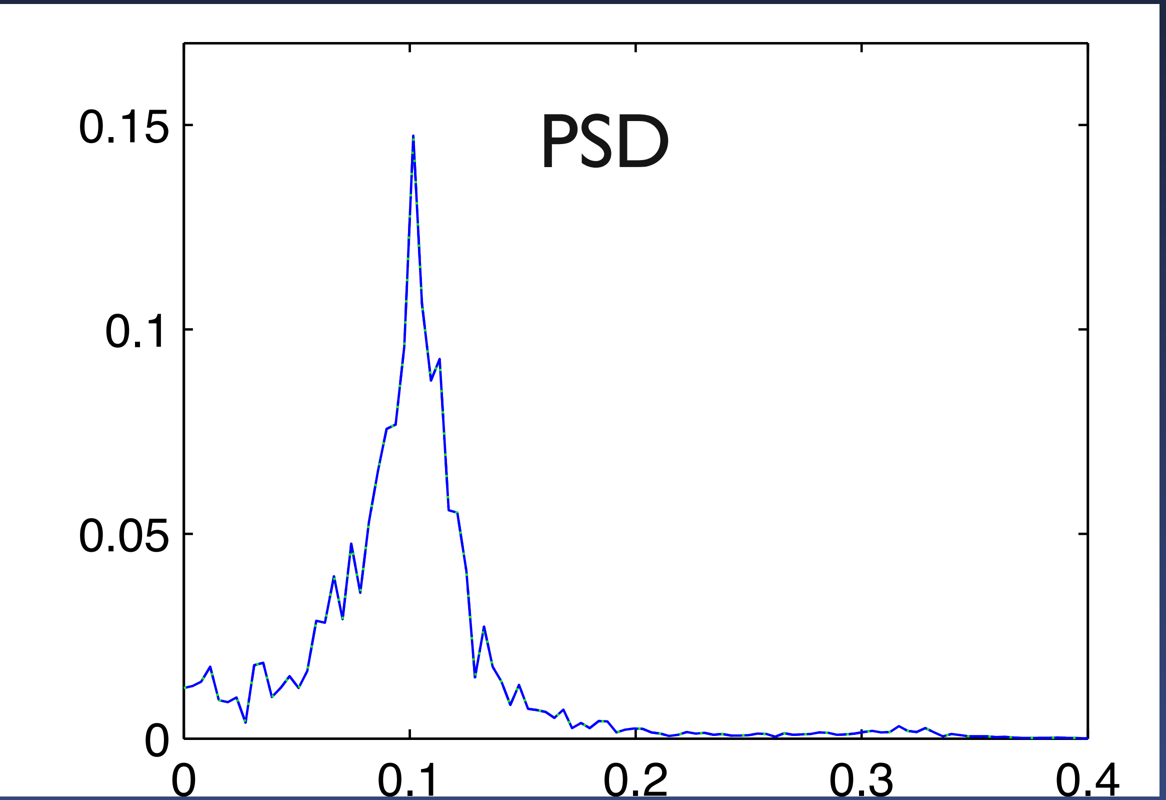
New mechanism for **coherence resonance:** Noise driven/amplified spirals, sustained transients



Stable manifold for OFF system



OFF

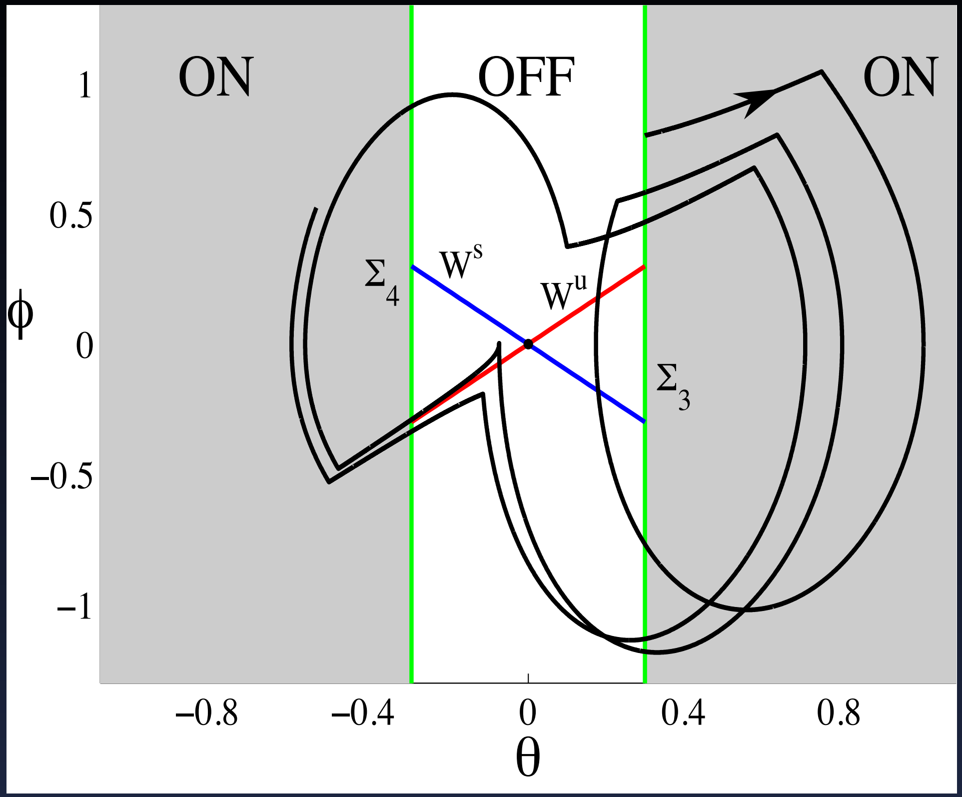


damped ZZ sustained as spiral via noise

w/o noise - attracting to steady state
w/ noise - damped oscillations revisited

Non-smooth systems as mechanism for coherence resonance

ON



Other on-off + delayed feedback:

(Transient) oscillations or unstable cycles sustained by stochastic fluctuations?

Potential for larger systems or networks?

Already used in Markov chain approximations of collection of meta-stable equilibria?

Already used in communication/queueing networks (fluid limits with non-smooth dynamics)?

Lots of mathematical and modeling challenges:

- Interactions with bifurcations or transitions
- Piecewise continuous nonlinear systems: new bifurcations/nonlinear states + sustained or disrupted by randomness
- Stochastic modeling for systems with (delay and)discontinuities: open problems analytically and computationally, new approaches needed
- Robustness of different on/off control strategies
- Recent work: extension of results for continuous cases to discontinuous drift cases with "nice" noise (Mohammed, et al, 2013, preprint)

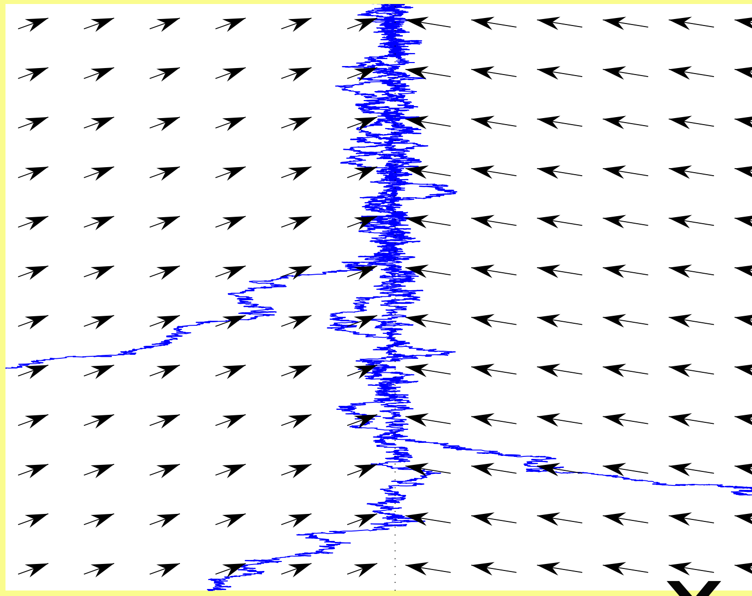
Different types of stochastic discontinuous dynamics: need a variety of ideas

- Mixed mode oscillations: Semi-analytical time dependent probability density functions
- Discontinuity induced bifurcations - underlying sources of noise-sensitivity
- Positive occupation times: sliding, other non-smooth transitions?
- Switching layers and non-standard scaling limits: transitions near switching

Distributions, moments: near sliding

Quasi-steady state density (in x)

$$p_{\mathrm{qss},\varepsilon}(x) = \begin{cases} \frac{K_\varepsilon}{\varepsilon} \mathrm{e}^{\frac{1}{\varepsilon}(2a_L x + c_L x^2 + o(x^2)) + O(\varepsilon^M)}, & x \le 0 \\ \frac{K_\varepsilon}{\varepsilon} \mathrm{e}^{\frac{1}{\varepsilon}(-2a_R x + c_R x^2 + o(x^2)) + O(\varepsilon^M)}, & x \ge 0 \end{cases}$$



Note: obtain Fillipov dynamics with $O(\epsilon)$ correction

$$\langle y(t) \rangle = y_{\mathrm{slide}}(t) + \frac{(a_L^2 d_R - a_R^2 d_L)(a_L + a_R) - (a_L^2 c_R - a_R^2 c_L)(b_L - b_R)}{2a_L a_R (a_L + a_R)^2} \varepsilon t + o(\varepsilon)$$

Variability in y: how long in sliding state?

Positive occupation time of the process affects the dynamics near sliding