Non-Smooth Dynamics in the Stommel Model

With a Subtitle

by

Cody Griffith & Rachel Kuske

B.Sc., Metropolitan State University of Denver, 2016

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in

The Faculty of Graduate Studies

(Mathematics)

THE UNIVERSITY OF BRITISH COLUMBIA

(Vancouver)

July 2017

Abstract

Abstract

Introduction

MAIN RESULT: Non-smooth bifurcations are a topic that arise in special systems and for how frequent they appear, they have not been analyzed nearly as much as their smooth counter parts. This paper will discuss pinpointing the tipping behavior in the classic Stommel model around the non-smooth bifurcation as well as generalize the canonical system. First an analysis on a simpler one dimensional topologically equivalent system provides insight into approaching the full two dimensional Stommel model.

Background

Stommel Model

([1] [5] [6] [7] are all good references on the climate change focus of the THC)
In an effort to better understand the oceanic patterns around the mixing of two bodies of differing temperature and salinity, Henry Stommel proposed the two box model in 1961, see [8]. In this paper, Stommel suggests that there are actually two different stability regimes in the system that he has suggested and concluded that the oceanic dynamics behave very similarly about these equilibria. These type of systems have since been a heavily studied area for both climatology due to the wide ranging applications and dynamical systems for its generalization into dual stability.
In the non-dimensionalized Stommel Model, we consider the system

$$\begin{aligned}\frac{\mathrm{d}T}{\mathrm{d}t} &= \eta_1 - T(1 + |T - S|)\\ \frac{\mathrm{d}S}{\mathrm{d}t} &= \eta_2 - S(\eta_3 + |T - S|)\end{aligned} \tag{1}$$

where the variables T, S are the temperature and salinity respectively. The parameters η_1, η_2, η_3 all have physical interpretation to the relaxation times and volumes of the box. Where η_1 can be thought of as the thermal variance (ASK RACHEL), η_2 as the saline variance, and η_3 as the ratio of relaxation times of temperature and salinity. (Also maybe talk about the range of these parameters as well as the significance of $\eta_3 <=> 1$.)
The parameter η_2 turns out to be much more interesting as different values can cause major qualitative and quantitative changes in the dynamics of the system. This behavior has been discovered to cause sudden changes at two different points in the system, each being either a smooth or a non-smooth saddle-node bifurcation.

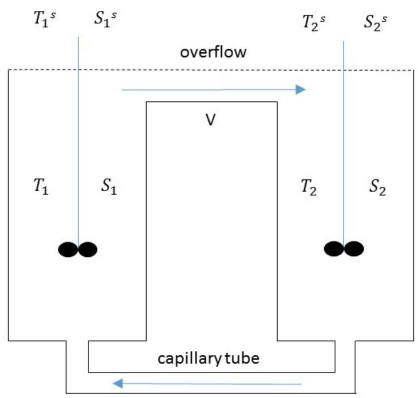
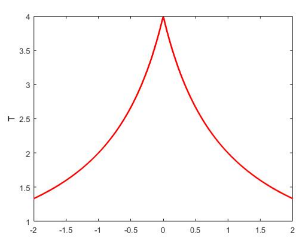


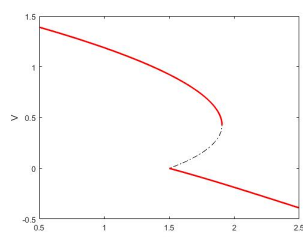
Figure 1: The Stommel Two Box Model: Differing volume boxes with a temperature and salinity T_i and S_i. The boxes are connected by an overflow and capillary tube that has a flow V. There is also a surface temperature and salinity for each box $T_i{}^s$ and $S_i{}^s$. We also assume well mixing occurs.

It is convenient to view this system in terms of the variable $V = T - S$, which leads to the system:

$$\begin{aligned} \frac{\mathrm{d}T}{\mathrm{d}t} &= \eta_1 - T(1 + |T - S|) \\ \frac{\mathrm{d}V}{\mathrm{d}t} &= (\eta_1 - \eta_2) - V|V| - T + \eta_3(T - V) \end{aligned} \tag{2}$$



(a) The equilibrium solution for T

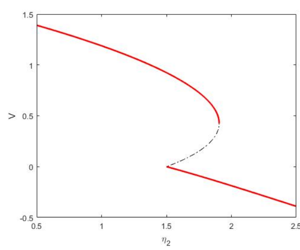


(b) The equilibrium solution for V

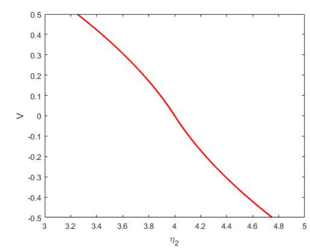
Figure 2: The equilibria of the non-dimensionalized system with $\eta_1 = 4$ and $\eta_3 = .375$.

As shown in Fig 2, the equilibrium curves reveal much about the dynamics.

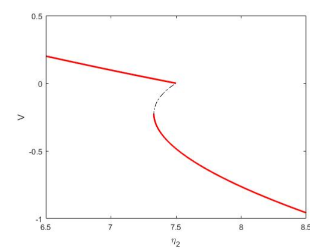
In Fig. 2a non-smooth behavior occurs at $V = 0$ and in 2b the two types of bifurcation appear clearly in the graph of equilibria for V vs. η_2. The location of the non-smooth bifurcation can be found analytically, $\eta_2 = \eta_1\eta_3$, and the smooth bifurcation is the solution only real solution to a cubic polynomial. In this plot, both the upper and lower branches of the equilibrium are stable with the middle branch being unstable. The smoothness of each bifurcation is apparent and arises from the absolute value term in the defining dynamics of (2).



(a) $\eta_3 = .375$



(b) $eta_3 = 1$



(c) $\eta_3 = 1.875$

Figure 3: The choice in η_3 turns out to dictate the orientation of the problem, in each plot we have fixed $\eta_1 = 4$.

Much is known about the Stommel model in the case where η_2 is fixed to be a constant value throughout the analysis but realistically this is not the case. (In [6] the parameter is described as the influx of freshwater into the Atlantic and the change in this parameter is justified by a positive feedback loop for salinity that drives the THC to move high-salinity water towards deep oceans. This loop causes the abrupt smooth bifurcation but then afterwards a salinity deficit causing the parameter to decrease back towards the non-smooth bifurcation). This value will change very slowly and here is where the focus of this paper lies. A system with a parameter known to cause a bifurcation, varying the parameter no longer allows the bifurcation to occur in the standard sense. Instead, these conditions give rise to a smooth but rapid change in the system and where this occurs is called a tipping point.

Slowly Varying Tipping

A tipping point is a point that causes an abrupt transition in dynamical behavior as the system moves into a qualitatively different state. These are known to be caused by small changes in one or more parameters in the system. An analysis that considers a multitude of mechanisms interacting with tipping can be found in [9]. Tipping points have been discovered to occur in a wide variety of systems and have become a big staple in catastrophe theory . They can aid in predicting the future of a system and even be a warning for irreversible change. A tipping point thus share similar characteristics of a bifurcation and may or may not occur close to the standard bifurcation location
(EXPLAIN WHEN TIPPING OCCURS)

One Dimensional Model

We consider a system that is topologically equivalent to the more complex two dimensional Stommel model,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\mu + 2|x| - x|x| + A\sin(\Omega t), \quad \frac{\mathrm{d}\mu}{\mathrm{d}t} = -\epsilon, \tag{3}$$
$$x(0) = x_i, \quad \mu(0) = \mu_i,$$

where the constants used are the drift rate, $0 < \epsilon \ll 1$, A is the amplitude of oscillation and Ω is the frequency.
The system (3) is generalized from a basic model that contains both a smooth and non-smooth saddle-node bifurcation. This type of behavior gives the topological equivalence to the Stommel model and hence good reason to test generalizations on. In each case, emphasis is put on the non-smooth component of the model give further insight

Static μ and Bifurcations

Consider the system (3) with $A = 0$ and $\epsilon = 0$, which is our basic system with a static μ and no forcing. Setting the system equal to zero, we find there are two stable equilibrium branches. Denote these x_l and x_u for lower and upper respectfully,

$$x_l = 1 - \sqrt{1+\mu}, \quad x_u = 1 + \sqrt{1-\mu}$$

and a single unstable branch,

$$x_{un} = 1 - \sqrt{1 - \mu},$$

where x_l is valid for $\mu \geq 0$ and x_u for $\mu \leq 1$. Thus this system always has a stable equilibrium for every choice in the parameter, but has a region of bi-stability for $0 \leq \mu \leq 1$. This indicates that both edges of this region are bifurcations, $\mu_{ns} = 0$ and $\mu_s = 1$, as they cause shifts in stability. Upon further inspection, the points $(x, \mu) = (0, 0)$ and $(x, \mu) = (1, 1)$ are non-smooth and smooth saddle-node bifurcations respectfully.

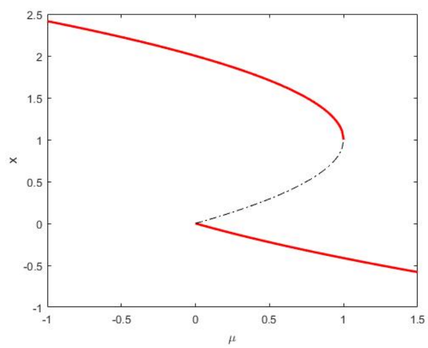


Figure 4: This is the one-dimensional bifurcation diagram and we see the upper and lower equilibrium branches as well as the unstable branch. The non-smooth bifurcation occurs at (0,0) and the smooth bifurcation occurs at (1,1). Both of which are saddle-nodes due to the annihilating equilibria.

Slowly varying μ

Consider (3) with $A = 0$ and $\epsilon > 0$. Here the parameter is allowed to change and thus a bifurcation no longer occurs. Instead, it should be expected that a tipping point occurs nearby the previous bifurcation points. The smooth saddle-node is well understood (see Zhu & Kuske [9]), so let us consider an approach towards discovering the non-smooth tipping point. To do so we begin by allowing the initial conditions $x_i = 1 - \sqrt{1 + \mu_i}$ and $\mu_i > 0$. Now our calculations will be centered around the stable lower branch and have emphasis on $x < 0$. Now we consider a rescaling approach where its useful to introduce the notion of slow time, $t = \epsilon\tau$, with $x < 0$, the system (3) becomes

$$\epsilon \frac{\mathrm{d}x}{\mathrm{d}\tau} = -\mu - 2x + x^2, \quad \frac{\mathrm{d}\mu}{\mathrm{d}\tau} = -1. \tag{4}$$

Next, we use an asymptotic expansion in terms of the present small quantity ϵ,

$$x \sim x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots \tag{5}$$

(SEE OTHER PAPERS FOR BETTER PHRASING) This expansion captures the slow variable's dynamics and separate them by magnitude. Thus, substituting (5) into (4), we get the following system of equations.

$$\begin{aligned} O(1) : 0 = -\mu - 2x_0 + x_0^2 \Rightarrow &\quad x_0 = 1 - \sqrt{1+\mu} \\ O(\epsilon) : x_{0_\tau} = -2x_1 + 2x_1 x_0 \Rightarrow &\quad x_1 = \frac{1}{4(1+\mu)} \\ O(\epsilon^2) : x_{1_\tau} = -2x_2 + 2x_0 x_2 + x_1^2 \Rightarrow &\quad x_2 = \frac{-3}{32(1+\mu)^{5/2}} \end{aligned}$$

Thus, (5) becomes

$$x \sim 1 - \sqrt{1+\mu} + \epsilon \frac{1}{4(1+\mu)} - \epsilon^2 \frac{3}{32(1+\mu)^{5/2}} + \dots \tag{6}$$

Since the dynamics of x in (3) change at $x = 0$, this solution is valid only for $x < 0$ and $\mu > 0$. This gives rise to a critical point at $(x_c, \mu_c) = (0, 0)$ which is the non-smooth bifurcation, and a local analysis about this point is necessary.

Time had already been scaled but now scaling our spatial variable is a must to conduct the local analysis. A systematic search for this scaling leads to both the spatial variable and the parameter being scaled by ϵ, thus let $x = \epsilon z$ and $\mu = \epsilon m$. From this our system (3) becomes,

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -m + 2z - \epsilon z^2, \quad \frac{\mathrm{d}m}{\mathrm{d}t} = -1. \tag{7}$$

An important method to capture the interacting dynamics between the system and the parameter is to change the variable to be in terms of the parameter. Introducing this into our approach, we get a system that we can find it's leading order solution to,

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\mathrm{d}z}{\mathrm{d}m}\frac{\mathrm{d}m}{\mathrm{d}t}$$

$$-\frac{\mathrm{d}z}{\mathrm{d}m} = -m + 2z - \epsilon z^2.$$

$$z = Ce^{-2m} + \frac{m}{2} - \frac{1}{4}$$

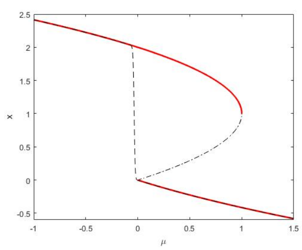
Finally writing this in terms of the original coordinates gives,

$$\frac{x}{\epsilon} \sim Ce^{-2\mu/\epsilon} + \frac{\mu}{2\epsilon} - \frac{1}{4}. \tag{8}$$

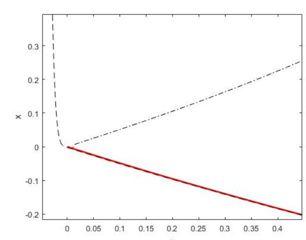
The tipping point can be found when this solution (8) begins growing exponentially, so we focus on when the exponential term becomes large ($O(1/\epsilon)$),

$$e^{-2\mu/\epsilon} \sim O(1/\epsilon) \Rightarrow 2\mu \sim \epsilon \log(\epsilon). \tag{9}$$

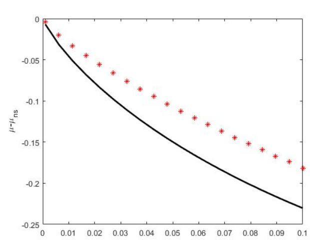
With this result, we compare our estimate to numerical results to evaluate it's performance for a varying size of ϵ.



(a) The slowly varying numerical solution (black dashed line) with the bifurcation plot.



(b) A zoom about the non-smooth region



(c) A plot of the tipping (red stars) versus the asymptotic approximation (black line).

Figure 5: The numerical solution (black dotted line) to (3) with $A = 0$ and $\epsilon = .01$. The tipping occurs slightly after where the bifurcation would have occurred, here at $\mu = -.0114$ where our prediction is at $\mu = -.0230$.

In Fig. (5) we have an example of the tipping occurring for a choice in ϵ but it is Fig. (5c) that demonstrates the tipping approximation across a range of ϵ. Qualities that show our approximation is doing well is the concavity is well represented and that as $\epsilon \to 0$ the asymptotics converge with the dynamics.

High Frequency Oscillatory Forcing

Consider the system (3) with $A = O(1)$ and $\epsilon = 0$, where we have oscillatory forcing but no drifting parameter; also assume that $\Omega \gg 1$ as to have a high frequency. Where the previous section required rescaling the problem due to the slowly varying dynamics, here we have dynamics occurring on both a regular time scale and a 'fast' scale. This naturally suggests a multiple scales approach where we will call our regular time $\tau = t$ and our 'fast' time $T = \Omega t$, then we search for a solution that is dependent on these scales, $x(t) = x(\tau, T)$. Introducing this method with the appropriate variable change,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} + \frac{\mathrm{d}x}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}\tau} + \Omega\frac{\mathrm{d}x}{\mathrm{d}T}$$

$$\frac{\mathrm{d}x}{\mathrm{d}T} = \Omega^{-1}\left(-\frac{\mathrm{d}x}{\mathrm{d}\tau} - \mu + 2|x| - x|x| + A\sin(T)\right). \tag{10}$$

From this system (10), we see this small quantity, Ω^{-1}, appear which suggests the asymptotic expansion be in powers of this quantity,

$$x \sim x_0 + \Omega^{-1}x_1 + \Omega^{-2}x_2 + \dots \tag{11}$$

Using (11) in our multiple scales system (10), we separate by orders of Ω to get,

$$O(1) : x_{0T} = 0 = R_0(\tau, T) \tag{12}$$
$$O(\Omega^{-1}) : x_{1T} = -x_{0\tau} - \mu + 2|x_0| - x_0|x_0| + A\sin(T) = R_1(\tau, T) \tag{13}$$
$$O(\Omega^{-2}) : x_{2T} = -x_{1\tau} - \mu + 2|x_1| - x_0|x_1| - x_1|x_0| = R_2(\tau, T) \tag{14}$$

Now that we have an equation on each order, we must be able to solve each one but further restrict our solution from having resonant or linearly growing terms. This will assure that the terms in the asymptotic expansion are compatible with one another and we have a robust solution. A common method to guarantee a solution on each order can be found with less than linearly growing terms is the Fredholm alternative which gives a solvability condition on each order,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T R_i(\tau,u)\,du = 0, \quad \forall i.$$

We also must recall that our focus is on the non-smooth behavior and thus we restrict the solution to follow along the lower stable equilibrium branch, $x < 0$. (SEE APPENDIX?) Applying these conditions to each order we can find the terms in (11),

$$x \sim 1 - \sqrt{1+\mu} - \Omega^{-1} A\cos(T) + O(\Omega^{-2}). \tag{15}$$

With this expansion, we probe it to see when it may become invalid (i.e the terms in the series begin reordering). Upon a closer look, we see that when $\mu \sim O(\Omega^{-1})$, the leading order term is now $x_0 \sim O(\Omega^{-1})$ and thus we have the scaling for an inner expansion,

$$\mu = \Omega^{-1} m, \quad x = \Omega^{-1} y.$$

Using this scaling along with the same approach as the outer solution, we have an inner multiple scales system which also suggests an expansion,

$$\frac{\mathrm{d}y}{\mathrm{d}T} = \Omega^{-1}\left(-\frac{\mathrm{d}y}{\mathrm{d}\tau} - m + 2|y|\right) - \Omega^{-2} y|y| + A\sin(T), \tag{16}$$

$$y \sim y_0 + \Omega^{-1} y_1 + \Omega^{-2} y_2 + \dots \tag{17}$$

Like before, we use (17) in (16) to collect by orders of Ω and then apply the Fredholm alternative to each equation,

$$O(1) : \frac{\mathrm{d}y_0}{\mathrm{d}T} = A\sin(T) \tag{18}$$

$$O(\Omega^{-1}) : \frac{\mathrm{d}y_1}{\mathrm{d}T} = -\frac{\mathrm{d}y_0}{\mathrm{d}\tau} - m + 2|y_0| \tag{19}$$

$$y_0 = -A\cos(T) + v_0(\tau) \tag{20}$$

$$\frac{\mathrm{d}v_0}{\mathrm{d}\tau} = -m + 2\lim_{T\to\infty}\int_0^T |A\cos(u) + v_0(\tau)|\,du \tag{21}$$

Here we must consider two cases of $v_0(\tau)$ that determine the difficulty of the integrand, case I: if the unknown function is large enough to keep the interior from ever changing signs and case II: if it is too small and the interior can change sign. In Fig. (6) we can see the behavior of each case where the solution on the right is following under case I, the first vertical line defining the boundary criteria between the cases, the middle region following under case II and the second vertical line giving the bifurcation.

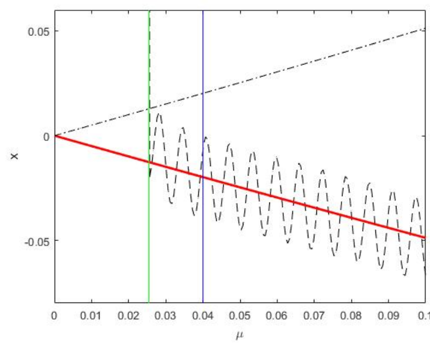


Figure 6: The asymptotic solution (black dashed line) to (3) with $A = 2$ and $\Omega = 100$. The boundary criteria here is $\mu = \frac{2|A|}{\Omega} = .04$ and the bifurcation occurs at $\mu = \frac{4|A|}{\pi\Omega} = .0255$. For reference, the original bifurcation diagram is overlayed.

Case I: $|v_0| \geq |A|$

In this case, our integral equation simplifies very nicely to a simple equation that we can use to find a stable equilibrium.

$$\frac{\mathrm{d}v_0}{\mathrm{d}\tau} = -m + 2|v_0| \Rightarrow v_0 = \pm\frac{m}{2} \tag{22}$$

Using these solutions, we can determine form a stability check that $\frac{m}{2}$ is unstable and $-\frac{m}{2}$ is stable, thus we have the leading order stable solution to our inner equation for this case,

$$y \sim -\frac{m}{2} - A\cos(T) + O(\Omega^{-1}) \Rightarrow x \sim -\frac{\mu}{2} - \Omega^{-1}A\cos(T) + O(\Omega^{-2}) \tag{23}$$

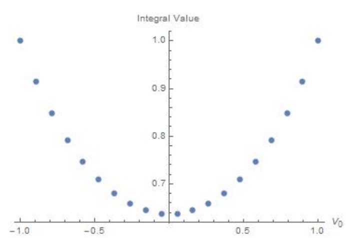
Where the condition $|v_0| \geq |A|$ leads us to establish when this solution fails, $\mu \geq \frac{2|A|}{\Omega}$.

Case II: $|v_0| < |A|$

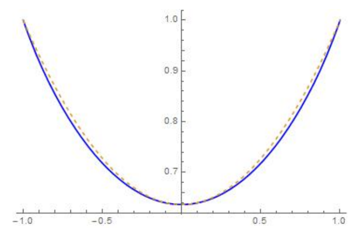
From case I, we have a range for μ that this case applies, $\mu < \frac{2|A|}{\Omega}$. But the integrand in (21) is non-trivial when $|v_0| < |A|$ and from initial numerical evaluations in Fig. (7a), it is appropriate to use either quadratic or quintic interpolation to estimate the integral. We choose to use quadratic interpolation although quintic is shown in the appendix.
For quadratic interpolation, we assume that there is a function $f(v_0) = a + bv_0 + c{v_0}^2$ and we know a few special values, $f(0) = \frac{2|A|}{\pi}$ and $f(|A|) = |A| = f(-|A|)$. From this, we get

$$f(v_0) = \frac{2|A|}{\pi} + \frac{1}{|A|}\left(1 - \frac{2}{\pi}\right){v_0}^2 \tag{24}$$

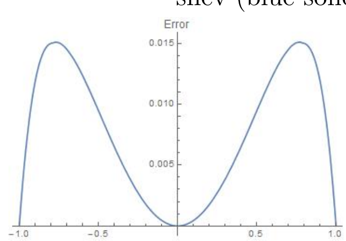
In Fig. (7) we take a range of values in A that fall into case II, compare (24) to a Chebyshev interpolation and conclude that the approximation has sufficiently small error on $A \sim O(1)$ and hence appropriate to use in place of the integral equation (21).



(a) Interior points for V_0



(b) Quadric (orange dotted) and Chebyshev (blue solid) interpolation.



(c) The resulting error.

Figure 7: Here is a comparison of quadratic interpolation to Chebyshev interpolation given a range of 20 interior points.

Thus, (21) becomes

$$\frac{\mathrm{d}v_0}{\mathrm{d}\tau} \approx -m + \frac{4|A|}{\pi} + \frac{2}{|A|}\left(1 - \frac{2}{|A|}\right){v_0}^2. \tag{25}$$

Here we search for the equilibrium as before and find,

$$v_0 \approx \pm\sqrt{\frac{m-\frac{4|A|}{\pi}}{\frac{2}{|A|}(1-\frac{2}{\pi})}} = \pm C\sqrt{m-\frac{4|A|}{\pi}}. \tag{26}$$

Another stability analysis leads us to choose the negative solution and this naturally leads to the inner solution

$$y \sim -C\sqrt{m-\frac{4|A|}{\pi}} - A\cos(T) + O(\Omega^{-1}) \tag{27}$$

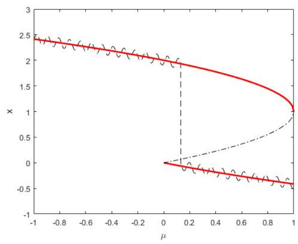
$$x \sim -C\sqrt{\frac{1}{\Omega}\left(\mu-\frac{4|A|}{\pi\Omega}\right)} - \Omega^{-1}A\cos(T) + O(\Omega^{-2}) \tag{28}$$

It then is clear that this equilibrium fails when $m = \frac{4|A|}{\pi}$ or back in original coordinates, $\mu = \frac{4|A|}{\pi\Omega}$. From this, we can gather that the periodic forcing in the system causes the bifurcation to occur sooner.
In Fig. (8) we have an example of the effect oscillatory forcing has for a choice of A and Ω but again, in Fig. (8c) we see the bifurcation approximation across a range of Ω. There is an allowed range of Ω from our assumption, we cannot allow $\Omega \sim O(1)$ as our approximation assumed that $\Omega >> 1$. Also, the tail behavior, $\Omega \to \infty$ has resonant behavior and higher order terms must be considered to capture this behavior.

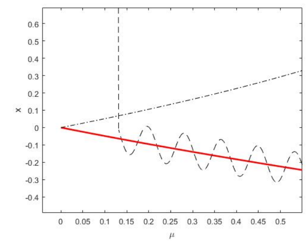
Two Dimensional Model

With the methods and approaches developed in the one dimensional model, we have at least an expectation of the behavior of the two dimensional Stommel model around the non-smooth bifurcation for differing cases. Consider the canonical model,

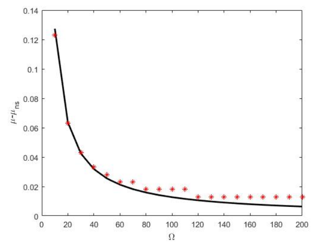
$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= \eta_1 - \eta_2 - V|V| - T + \eta_3(T-V) \\
\frac{\mathrm{d}T}{\mathrm{d}t} &= \eta_1 - T(1+|V|) \\
\frac{\mathrm{d}\eta_2}{\mathrm{d}t} &= -\epsilon \\
T(0) = T_i, \quad V(0) &= V_i, \quad \eta_2(0) = \eta_{2i} > \eta_1\eta_3
\end{aligned} \tag{29}$$



(a) The oscilating numerical solution (black dashed line) with the bifurcation plot.



(b) A zoom about the non-smooth region.



(c) A plot of the bifurcation (red stars) versus the asymptotic approximation (black line).

Figure 8: The numerical solution to (3) with $A = 1$, $\Omega = 10$ and $\epsilon = .01$. The bifurcation occurs slightly before the basic model's occurred, here at $\mu = .1314$ where our prediction is at $\mu = .1273$.

where $\epsilon \ll 1$, η_1 and η_3 are fixed positive constants. This is the Stommel model with the bifurcation parameter allowed to slowly vary towards the non-smooth bifurcation location. For the remainder of this section, we make the assumption that $\eta_3 < 1$ which causes (29) to admit the smooth bifurcation in the positive V region. Although not necessary, this will align our focus and give us a case to analyze in depth, the case of $\eta_3 > 1$ follows similarly.

Slowly varying η_2

With the choice of η_3, the lower branch with $V < 0$ is the branch we focus on in order to approach the non-smooth behavior, thus (29) becomes

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= \eta_1 - \eta_2 + V^2 - T + \eta_3(T - V)\\ \frac{\mathrm{d}T}{\mathrm{d}t} &= \eta_1 - T(1 - V)\\ \frac{\mathrm{d}\eta_2}{\mathrm{d}t} &= -\epsilon\end{aligned} \tag{30}$$

Here, we immediately solve for the equilibrium of T in terms of V and write down an equation for the leading order solution for V,

$$T(V) = \frac{\eta_1}{1 - V}, \quad 0 = \eta_1 - \eta_2 + V_0^2 - T(V_0) + \eta_3(T(V_0) - V_0)$$

Since we know our non-smooth bifurcation to occur at $\eta_2 = \eta_1\eta_3$ when $V = 0$ and $T = \eta_1$, it makes the most sense to rescale (**??**) around these values. After an analysis to determine the appropriate scaling, it is determined that the change in variable is,

$$\begin{aligned}\eta_2 =& \eta_1\eta_3 + \epsilon m\\ V =& -\epsilon x\\ T =& \eta_1 - \epsilon y\end{aligned} \tag{31}$$

Introducing (31) into (30) and shifting the focus onto the parameter we arrive at the following inner system with the following leading order problem,

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}m} &= -m + \eta_3 x + (1 - \eta_3)y + \epsilon x^2\\ \frac{\mathrm{d}y}{\mathrm{d}m} &= -\eta_1 x + y + \epsilon xy\\ \frac{\mathrm{d}m}{\mathrm{d}t} &= -1\end{aligned} \tag{32}$$

$$\begin{pmatrix} x_m \\ y_m \end{pmatrix} = \begin{pmatrix} \eta_3 & 1 - \eta_3 \\ -\eta_1 & 1 \end{pmatrix} - \begin{pmatrix} m \\ 0 \end{pmatrix}.$$

This leading order system has complex eigenvalues,

$$\lambda = \frac{\eta_3 + 1}{2} \pm \frac{i}{2}\sqrt{(1 - \eta_3)(4\eta_1 + \eta_3 - 1)}$$

We expect the tipping to occur just after the standard bifurcation from the results of the one dimensional case. But we can expect that the scaling will hold as we enter the $V > 0$ region, thus we arrive at the system and leading order problem,

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}m} &= -m + \eta_3 x + (1-\eta_3)y + \epsilon x^2 \\ \frac{\mathrm{d}y}{\mathrm{d}m} &= \eta_1 x + y + \epsilon xy \\ \frac{\mathrm{d}m}{\mathrm{d}t} &= -1\end{aligned} \tag{33}$$

$$\begin{pmatrix} x_m \\ y_m \end{pmatrix} = \begin{pmatrix} \eta_3 & 1-\eta_3 \\ \eta_1 & 1 \end{pmatrix} - \begin{pmatrix} m \\ 0 \end{pmatrix}.$$

Which now has real eigenvalues

$$\lambda_{1,2} = \frac{\eta_3 + 1}{2} \pm \frac{1}{2}\sqrt{(1-\eta_3)(4\eta_1 - \eta_3 + 1)}$$

With real eigenvalues the solution in the $V > 0$ region takes exponential form with constants $K_{i,j}$ being the eigenvector component corresponding to the ith eigenvalue,

$$\begin{aligned} x &\sim K_{1,1}e^{\lambda_1 m} + K_{2,1}e^{\lambda_2 m} + C_1 m + C_2 \\ y &\sim K_{1,2}e^{\lambda_1 m} + K_{2,2}e^{\lambda_2 m} + C_3 m + C_4 \end{aligned} \tag{34}$$

Translating back to our original coordinates from (34) and recalling that V contains the dynamics we are interested in,

$$\frac{V}{\epsilon} \sim K_{1,1}e^{\lambda_1(\eta_2 - \eta_1\eta_3)/\epsilon} + K_{2,1}e^{\lambda_1(\eta_2-\eta_1\eta_3)/\epsilon} + C_1\frac{\eta_2 - \eta_1\eta_3}{\epsilon} + C_2 \tag{35}$$

From here we have access to determining when the system is likely to tip, when one of these exponentials becomes large the system will diverge away towards the upper stable branch,

$$e^{\lambda_i(\eta_2-\eta_1\eta_3)/\epsilon} \sim O(\epsilon^{-1}) \Rightarrow \eta_2 \sim \min\{\eta_1\eta_3 - \epsilon\ln\epsilon/\lambda_i\}, \quad i = 1, 2$$

Note: for the above solution to be valid, one of the eigenvalues must be negative which introduces the restriction on the parameters,

$$\eta_1 > \frac{1}{\eta_3} - 1$$

If this restriction is not met, then the exponentials will always decay, and the analysis will then rely on the linear term.
Interestingly enough, when

Preface

You must include a preface if any part of your research was partly or wholly published in articles, was part of a collaboration, or required the approval of UBC Research Ethics Boards.
The Preface must include the following:

- A statement indicating the relative contributions of all collaborators and co-authors of publications (if any), emphasizing details of your contribution, and stating the proportion of research and writing conducted by you.
- A list of any publications arising from work presented in the dissertation, and the chapter(s) in which the work is located.
- The name of the particular UBC Research Ethics Board, and the Certificate Number(s) of the Ethics Certificate(s) obtained, if ethics approval was required for the research.

Examples

Chapter 2 is based on work conducted in UBC's Maple Syrup Laboratory by Dr. A. Apple, Professor B. Boat, and Michael McNeil Forbes. I was responsible for tapping the trees in forests X and Z, conducted and supervised all boiling operations, and performed frequent quality control tests on the product.
A version of chapter 2 has been published [?]. I conducted all the testing and wrote most of the manuscript. The section on "Testing Implements" was originally drafted by Boat, B. Check the first pages of this chapter to see footnotes with similar information.
Note that this preface must come before the table of contents. Note also that this section "Examples" should not be listed in the table of contents, so we have used the starred form: `\section*{Example}`.

Table of Contents

Appendices

List of Tables

List of Figures

List of Programs

Acknowledgements

This is the place to thank professional colleagues and people who have given you the most help during the course of your graduate work.

Dedication

The dedication is usually quite short, and is a personal rather than an academic recognition. The *Dedication* does not have to be titled, but it must appear in the table of contents. If you want to skip the chapter title but still enter it into the Table of Contents, use this command `\chapter[Dedication]{}`.
Note that this section is the last of the preliminary pages (with lowercase Roman numeral page numbers). It must be placed *before* the `\mainmatter` command. After that, Arabic numbered pages will begin.

Chapter 1

This is a Chapter

1.1 A Section

Here is a section with some text. Equations look like this $y = x$.[1]
This is an example of a second paragraph in a section so you can see how much it is indented by.

1.1.1 This is a Subsection

Here is an example of a citation: [**?**]. The actual form of the citation is governed by the bibliographystyle. These citations are maintained in a BIBTeX file `sample.bib`. You could type these directly into the file. For an example of the format to use look at the file `ubcsample.bbl` after you compile this file.[2]
This is an example of a second paragraph in a subsection so you can see how much it is indented by.

This is a Subsubsection

Here are some more citations [**?**, **?**, **?**]. If you use the `natbib` package with the `sort&compress` option, then the following citation will look the same as the first citation in this section: [**?**, **?**, **?**].
This is an example of a second paragraph in a subsubsection so you can see how much it is indented by.

This is a Paragraph Paragraphs and subparagraphs are the smallest units of text. There is no subsubsubsection etc.

[1] Here is a footnote.
[2] Here is another footnote.

Phoenix	\$960.35
Calgary	\$250.00

Table 1.1: Here is the caption for this wonderful table. It has not been centered and the positioning has been specified to be at the top of the page. Thus it appears above the babble rather than below where it is defined in the source file.

This is a Subparagraph This is the last level of organisation. If you need more than this, you should consider reorganizing your work...

$$\mathrm{f}(x) = \int_{-\infty}^{\int_{-\infty}^{x} e^{-\frac{y^2}{2}} \mathrm{d}y} e^{-z^2} \mathrm{d}z \tag{1.1}$$

In order to show you what a separate page would look like (i.e. without a chapter heading) I must type some more text. Thus I will babble a bit and keep babbling for at least one more page... What you should notice is that the chapter titles appear substantially lower than the continuing text. Babble babble.

Babble babble.

1.2 Quote

Here is a quote:

> This is a small poem,
> a little poem, a Haiku,
> to show you how to.
> —Michael M^c^Neil Forbes.

This small poem shows several features:

- The use of the `quote` and `center` environments.
- The `\newpage` command has been used to force a page break. (Sections do not usually start on a new page.)
- The pagestyle has been set to suppress the headers using the command `\thispagestyle{plain}`. Note that using `\pagestyle{plain}` would have affected all of the subsequent pages.

1.3 Programs

Here we give an example of a new float as defined using the `float` package. In the preamble we have used the commands

```
\floatstyle{ruled}
\newfloat{Program}{htbp}{lop}[chapter]
```

This creates a “Program” environment that may be used for program fragments. A sample `python` program is shown in Program 1.1. (Note that Python places a fairly restrictive limit on recursion so trying to call this with a large n before building up the cache is likely to fail unless you increase the recursion depth.) Instead of using a `verbatim` environment for your program chunks, you might like to `include` them within an `alltt` envrironment by including the `\usepackage{alltt}` package (see page 187 of the LaTeX book). Another useful package is the `\usepackage{listings}` which can pretty-print many different types of source code.

Program 1.1 Python program that computes the n^{th} Fibonacci number using memoization.

```
def fib(n,_cache={}):
    if n < 2:
        return 1
    if n in _cache:
        return _cache[n]
    else:
        result = fib(n-1)+fib(n-2)
        _cache[n] = result
        return result
```

Chapter 2

Another Chapter with a Very Long Chapter-name that will Probably Cause Problems

This chapter name is very long and does not display properly in the running headers or in the table of contents. To deal with this, we provide a shorter version of the title as the optional argument to the `\chapter[]{}` command. For example, this chapter's title and associated table of contents heading and running header was created with
`\chapter[Another Chapter\ldots]{Another Chapter with a Very Long Chapter-name that will Probably Cause Problems}`.
Note that, according to the thesis regulations, the heading included in the table of contents must be a truncation of the actual heading.
This Chapter was used as a demonstration in the Preface for how to attribute contribution from collaborators. If there are any such contributions, details must be included in the Preface. If you wish, you may additionally use a footnote such as this.[3]

2.1 Another Section

Another bunch of text to demonstrate what this file does. You might want a list for example:[4]

- An item in a list.
- Another item in a list.

[3] This chapter is based on work conducted in UBC's Maple Syrup Laboratory by Dr. A. Apple, Professor B. Boat, and C. Cat.

[4] Here is a footnote in a different chapter. Footnotes should come after punctuation.

An Unnumbered Section That is Not Included in the Table of Contents

Here is an example of a figure environment. Perhaps I should say that the example of a figure can be seen in Figure **??**. Figure placement can be tricky with LaTeX because figures and tables are treated as “floats”: text can flow around them, but if there is not enough space, they will appear later. To prevent figures from going too far, the `\afterpage{\clearpage}` command can be used. This makes sure that the figure are typeset at the end of the page (possibly appear on their own on the following pages) and before any subsequent text.

The `\clearpage` forces a page break so that the figure can be placed, but without the the `\afterpage{}` command, the page would be broken too early (at the `\clearpage` statement). The `\afterpage{}` command tells LaTeX to issue the command after the present page has been rendered.

2.2 Tables

We have already included one table: 1.1. Another table is plopped right here. Well, actually, as with Figures, tables do not necessarily appear right

	Singular		Plural	
	English	**Gaeilge**	English	**Gaeilge**
1st Person	at me	**agam**	at us	**againn**
2nd Person	at you	**agat**	at you	**agaibh**
3rd Person	at him	**aige**	at them	**acu**
	at her	**aici**		

Table 2.1: Another table.

“here” because tables are also “floats”. LaTeX puts them where it can. Because of this, one should refer to floats by their labels rather than by their location. This example is demonstrated by Table 2.1. This one is pretty close, however. (Note: you should generally not put tables or figures in the middle of a paragraph. This example is for demonstration purposes only.)

Another useful package is `\usepackage{longtable}` which provides the `longtable` environment. This is nice because it allows tables to span multiple pages. Table 2.2 has been formatted this way.

Table 2.2: Feasible triples for highly variable Grid

Time (s)	Triple chosen	Other feasible triples
0	(1, 11, 13725)	(1, 12, 10980), (1, 13, 8235), (2, 2, 0), (3, 1, 0)
274	(1, 12, 10980)	(1, 13, 8235), (2, 2, 0), (2, 3, 0), (3, 1, 0)
5490	(1, 12, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
8235	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
10980	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
13725	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
16470	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
19215	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
21960	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
24705	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
27450	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
30195	(2, 2, 2745)	(2, 3, 0), (3, 1, 0)
32940	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
35685	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
38430	(1, 13, 10980)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
41175	(1, 12, 13725)	(1, 13, 10980), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
43920	(1, 13, 10980)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
46665	(2, 2, 2745)	(2, 3, 0), (3, 1, 0)
49410	(2, 2, 2745)	(2, 3, 0), (3, 1, 0)
52155	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
54900	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
57645	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
60390	(1, 12, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
63135	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
65880	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
68625	(2, 2, 2745)	(2, 3, 0), (3, 1, 0)
71370	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
74115	(1, 12, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
76860	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
79605	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
82350	(1, 12, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
85095	(1, 12, 13725)	(1, 13, 10980), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
87840	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
90585	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
93330	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)

Continued on next page

Table 2.2 – continued from previous page

Time (s)	Triple chosen	Other feasible triples
96075	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
98820	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
101565	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
104310	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
107055	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
109800	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
112545	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
115290	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
118035	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
120780	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
123525	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
126270	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
129015	(2, 2, 2745)	(2, 3, 0), (3, 1, 0)
131760	(2, 2, 2745)	(2, 3, 0), (3, 1, 0)
134505	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
137250	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
139995	(2, 2, 2745)	(2, 3, 0), (3, 1, 0)
142740	(2, 2, 2745)	(2, 3, 0), (3, 1, 0)
145485	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
148230	(2, 2, 2745)	(2, 3, 0), (3, 1, 0)
150975	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
153720	(1, 12, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
156465	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
159210	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
161955	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
164700	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)

An Unnumbered Subsection

Note that if you use subsections or further divisions under an unnumbered section, then you should make them unnumbered as well otherwise you will end up with zeros in the section numbering.

Chapter 3

Landscape Mode

The landscape mode allows you to rotate a page through 90 degrees. It is generally not a good idea to make the chapter heading landscape, but it can be useful for long tables etc.

This text should appear rotated, allowing for formatting of very wide tables etc. Note that this might only work after you convert the `dvi` file to a postscript (`ps`) or `pdf` file using `dvips` or `dvipdf` etc. This feature is provided by the `lscape` and the `pdflscape` packages. The latter is preferred if it works as it also rotates the pages in the pdf file for easier viewing.

[8] [3] [4] [9] [2] [1] [5] [6] [7]

Bibliography

[1] Richard B Alley, Jochem Marotzke, William D Nordhaus, Jonathan T Overpeck, Dorothy M Peteet, Roger A Pielke, RT Pierrehumbert, PB Rhines, TF Stocker, LD Talley, et al. Abrupt climate change. *science*, 299(5615):2005–2010, 2003.

[2] Mario Bernardo, Chris Budd, Alan Richard Champneys, and Piotr Kowalczyk. *Piecewise-smooth dynamical systems: theory and applications*, volume 163. Springer Science & Business Media, 2008.

[3] Henk A Dijkstra. *Nonlinear climate dynamics*. Cambridge University Press, 2013.

[4] Richard Haberman. Slowly varying jump and transition phenomena associated with algebraic bifurcation problems. *SIAM Journal on Applied Mathematics*, 37(1):69–106, 1979.

[5] Jochem Marotzke. Abrupt climate change and thermohaline circulation: Mechanisms and predictability. *Proceedings of the National Academy of Sciences*, 97(4):1347–1350, 2000.

[6] Stefan Rahmstorf. The thermohaline ocean circulation: A system with dangerous thresholds? *Climatic Change*, 46(3):247–256, 2000.

[7] Stefan Rahmstorf. Ocean circulation and climate during the past 120,000 years. *Nature*, 419(6903):207–214, 2002.

[8] Henry Stommel. Thermohaline convection with two stable regimes of flow. *Tellus*, 13(2):224–230, 1961.

[9] Jielin Zhu, Rachel Kuske, and Thomas Erneux. Tipping points near a delayed saddle node bifurcation with periodic forcing. *SIAM journal on applied dynamical systems*, 14(4):2030–2068, 2015.

Appendix A

First Appendix

Here you can have your appendices. Note that if you only have a single appendix, you should issue `\renewcommand{\appendicesname}{Appendix}` before calling `\appendix` to display the singular "Appendix" rather than the default plural "Appendices".

Appendix B

Second Appendix

Here is the second appendix.

Additional Information

This chapter shows you how to include additional information in your thesis, the removal of which will not affect the submission. Such material should be removed before the thesis is actually submitted.

First, the chapter is unnumbered and not included in the Table of Contents. Second, it is the last section of the thesis, so its removal will not alter any of the page numbering etc. for the previous sections. Do not include any floats, however, as these will appear in the initial lists.

The `ubcthesis` LaTeX class has been designed to aid you in producing a thesis that conforms to the requirements of The University of British Columbia Faculty of Graduate Studies (FoGS).

Proper use of this class and sample is highly recommended—and should produce a well formatted document that meets the FoGS requirement. Notwithstanding, complex theses may require additional formatting that may conflict with some of the requirements. We therefore *highly recommend* that you consult one of the FoGS staff for assistance and an assessment of potential problems *before* starting final draft.

While we have attemped to address most of the thesis formatting requirements in these files, they do not constitute an official set of thesis requirements. The official requirements are available at the following section of the FoGS web site:

`http://www.grad.ubc.ca/current-students/dissertation-thesis-preparation`

We recommend that you review these instructions carefully.