2D Osc: Outer equations

$$\dot V = \Omega_1 - \Omega_2 + \Omega_3(T-V) - T - V|V| + A\sin(\Omega t)$$
$$\dot T = \Omega_1 - T(1+|V|) + B\sin(\Omega t)$$

For $A=B=0$, $T_0 = \dfrac{\Omega_1}{1+|V_0|}$, $0 = \Omega_1 - \Omega_2 + \Omega_3(T_0 - V_0) - T_0 - V_0|V_0|$

Multi-scale: $\tau = t$, $R = \Omega t$, assume $V<0$ and T, V are far from axis

$$V_R + \Omega^{-1} V_\tau = \Omega^{-1}(\Omega_1 - \Omega_2 + \Omega_3(T-V) - T + V^2 + A\sin R)$$
$$T_R + \Omega^{-1} T_\tau = \Omega^{-1}(\Omega_1 - T(1-V) + B\sin R)$$
$$V \sim V_0 + \Omega^{-1} V_1 + \ldots, \quad T \sim T_0 + \Omega^{-1} T_1$$

$O(1)$: $V_{0R} = T_{0R} = 0 \rightarrow V_0 = V_0(\tau)$, $T_0 = T_0(\tau)$

$O(\Omega^{-1})$: $V_{1R} + V_{0\tau} = \Omega_1 - \Omega_2 + \Omega_3(T_0 - V_0) - T_0 + V_0^2 + A\sin R$
$T_{1R} + T_{0\tau} = \Omega_1 - T_0(1-V_0) + B\sin R$

Fredholm: $V_{0\tau} = \Omega_1 - \Omega_2 + \Omega_3(T_0 - V_0) - T_0 + V_0^2$
$T_{0\tau} = \Omega_1 - T_0(1-V_0)$ > Same as $A=B=0$
$V_{1R} = A\sin R$, $T_{1R} = B\sin R \rightarrow V_1 = X_1(\tau) - A\cos R$
$T_1 = y_1(\tau) - B\cos R$

$O(\Omega^{-2})$: $V_{2R} + V_{1\tau} = \Omega_3(T_1 - V_1) - T_1 + 2V_0V_1$
$T_{2R} + T_{1\tau} = -T_1 + T_1V_0 + T_0V_1$

Fredholm: $V_{1\tau} = \Omega_3(y_1 - X_1) - y_1 + 2V_0X_1$
$T_{1\tau} = -y_1 + V_0y_1 + T_0X_1$ > equil: $y_1 = \dfrac{T_0X_1}{-V_0+1} \rightarrow X_1 = 0$
$\hookrightarrow y_1 = 0$
$V_{2R} = (1-\Omega_3)B\cos R + (\Omega_3 - 2V_0)A\cos R$
$T_{2R} = (1-V_0)B\cos R - T_0A\cos R$

$\Rightarrow \cancel{V_2 = X_2(\tau) + (1-\Omega_3)B\sin R,\ X_2 = y_2(\tau)}$
$V_2 = X_2(\tau) + [(1-\Omega_3)B + (\Omega_3 - 2V_0)A]\cancel{\cos R}\cdot\sin R$
$T_2 = y_2(\tau) + [(1-V_0)B - T_0A]\cancel{\cos R}\cdot\sin R$

2D Osc: Outer equations (cont.)

$$O(\Omega^{-3}):\ V_{3R} + V_{2\tau} = \eta_3(T_2 - V_2) - T_2 + 2V_0V_2 + V_1^2$$

$$T_{3R} + T_{2\tau} = -T_2 + T_2V_0 + T_0V_2 + T_1V_1$$

Fredholm: Note $\frac{1}{2\pi}\int_0^{2\pi}\cos^2 R\,\partial R = \frac{1}{2}$

$$V_{2\tau} = \eta_3(y_2 - X_2) - y_2 + 2V_0X_2 + \frac{A^2}{2}$$

$$T_{2\tau} = -y_2 + V_0y_2 + T_0X_2 + \frac{AB}{2}$$

equil: $$y_2 = \frac{T_0X_2 + \frac{AB}{2}}{1 - V_0} = K_0X_2 + \frac{AB}{2(1-V_0)} = K_0X_2 + C$$

$$0 = \eta_3(K_0X_2 + C - X_2) - K_0X_2 - C + 2V_0X_2 + \frac{A^2}{2}$$

$$\Rightarrow X_2 = \frac{\frac{AB}{2(1-V_0)} - \eta_3\frac{AB}{2(1-V_0)} - \frac{A^2}{2}}{\eta_3K_0 - \eta_3 - K_0 + 2V_0} = \frac{(1-V_0)\left(\frac{AB}{2} - \eta_3\frac{AB}{2} - \frac{A^2}{2}(1-V_0)\right)}{\eta_3\eta_1 - \eta_1 + 2V_0(1-V_0)^2 - \eta_3(1-V_0)^2}$$

$$V \sim V_0 - \Omega^{-1}A\cos R + \Omega^{-2}\left(\frac{(1-V_0)\left(\frac{AB}{2} - \eta_3\frac{AB}{2} - \frac{A^2}{2}(1-V_0)\right)}{\eta_3\eta_1 - \eta_1 + (2V_0 - \eta_3)(1-V_0)^2} + \left[(1-\eta_3)B + (\eta_3 - 2V_0)A\right]\sin R\right)$$

$$T \sim T_0 - \Omega^{-1}B\cos R + \Omega^{-2}\left(\frac{\eta_1\left(\frac{AB}{2} - \eta_3\frac{AB}{2} - \frac{A^2}{2}(1-V_0)\right)}{(\eta_3\eta_1 - \eta_1)(1-V_0) + (2V_0 - \eta_3)(1-V_0)^3} + \frac{AB}{2(1-V_0)} + \left[(1-V_0)B - T_0A\right]\sin R\right)$$

Need to perform a local analysis but first need to find the scales of the local problem.

Recall, $(V, T, \eta_2) = (0, \eta_1, \eta_1, \eta_3)$ is the non-smooth Bif.

TWO D Scale Analysis

$V=\varepsilon x, \quad T=\eta_1+\gamma y, \quad \eta_2=\eta_1\cdot\eta_3+\xi m$

$$\varepsilon\dot{x}=-\xi m+\eta_3(\gamma y-\varepsilon x)-\gamma y-\varepsilon^2 x|x|+A\sin(\Omega t)$$
$$\gamma\dot{y}=-\eta_1\varepsilon|x|-\varepsilon\gamma|x|y-\gamma y+B\sin(\Omega t)$$

Multiple Scales: $\tau=t$, $R=\Omega t$

$$\varepsilon\Omega x_R+\varepsilon x_\tau=-\xi m+\eta_3(\gamma y-\varepsilon x)-\gamma y-\varepsilon^2 x|x|+A\sin R$$
$$\gamma\Omega y_R+\gamma y_\tau=-\eta_1\varepsilon|x|-\gamma y-\varepsilon\gamma|x|y+B\sin R$$

To see the dynamics communicate, $\gamma=\varepsilon$:

$$x_R+\Omega^{-1}x_\tau=\Omega^{-1}\left(-\frac{\xi m}{\varepsilon}+\eta_3(y-x)-y-\varepsilon x|x|+\frac{A}{\varepsilon}\sin R\right)$$

$$y_R+\Omega^{-1}y_\tau=\Omega^{-1}\left(-\eta_1|x|-y-\varepsilon|x|y+\frac{B}{\varepsilon}\sin R\right)$$

To get Sin terms to give $x_R\sim\sin R$, $\varepsilon=\Omega^{-1}$:

$$x_R+\Omega^{-1}x_\tau=\Omega^{-1}\left(-\xi\Omega m+\eta_3(y-x)-y-\Omega^{-1}x|x|\right)+A\sin R$$
$$y_R+\Omega^{-1}y_\tau=\Omega^{-1}\left(-\eta_1|x|-y-\Omega^{-1}|x|y\right)+B\sin R$$

The parameter should also communicate $\Rightarrow \xi=\Omega^{-1}$

$$x_R+\Omega^{-1}x_\tau=\Omega^{-1}\left(-m+\eta_3(y-x)-y-\Omega^{-1}x|x|\right)+A\sin R$$
$$y_R+\Omega^{-1}y_\tau=\Omega^{-1}\left(-\eta_1|x|-y-\Omega^{-1}|x|y\right)+B\sin R$$

$V=\Omega^{-1}x, \quad T=\eta_1+\Omega^{-1}y, \quad \eta_2=\eta_1\eta_3+\Omega^{-1}m$

Two D inner equations

$$\dot{x} = -m + \mathcal{R}_3(y-x) - \cancel{x}\,y - \Omega^{-1}x|x| + \Omega A\sin(\Omega t)$$
$$\dot{y} = -\mathcal{R}_1|x| - y - \Omega^{-1}|x|y + \Omega A\sin(\Omega t)$$

Multiple Scales: $t = \tau$, $R = \Omega t$

$$x_R + \Omega^{-1}x_\tau = \Omega^{-1}(-m + \mathcal{R}_3(y-x) - y - \Omega^{-1}x|x|) + A\sin R$$
$$y_R + \Omega^{-1}y_\tau = \Omega^{-1}(-\mathcal{R}_1|x| - y - \Omega^{-1}|x|y) + B\sin R$$

Expansions: $x \sim x_0 + \Omega^{-1}x_1 + \ldots$, $y \sim y_0 + \Omega^{-1}y_1 + \ldots$

$O(1)$: $x_{0R} = A\sin R \quad\longrightarrow\quad x_0 = P_0(\tau) - A\cos R$
$\qquad y_{0R} = B\sin R \qquad\qquad\;\; y_0 = Q_0(\tau) - B\cos R$

$O(\Omega^{-1})$ $x_{1R} + x_{0\tau} = -m + (\mathcal{R}_3 - 1)y_0 - \cancel{y_0}\,\mathcal{R}_3 x_0$
$\qquad y_{1R} + y_{0\tau} = -\mathcal{R}_1|x_0| - y_0$

Fredholm: $x_{0\tau} = -m + (\mathcal{R}_3 - 1)Q_0 - \cancel{P_0}\,\mathcal{R}_3 P_0$ ①
$\qquad y_{0\tau} = -\dfrac{\mathcal{R}_1}{2\pi}\displaystyle\int_0^{2\pi}|P_0(\tau) - A\cos R|\,dR - Q_0$ ②

Equil:

1D osc.

$$\overset{②}{\Rightarrow} Q_0 = -\frac{\mathcal{R}_1}{2\pi}\int_0^{2\pi}|P_0(\tau) - A\cos R|\,dR = -\frac{2\mathcal{R}_1}{\pi}\left(\sin^{-1}\left(\frac{P_0}{A}\right) + \sqrt{A^2 + P_0^2}\right)$$

Taylor approx: $Q_0 \approx -\dfrac{2\mathcal{R}_1|A|}{\pi} - \dfrac{\mathcal{R}_1}{\pi|A|}P_0^2 = -c - aP_0^2$

$$\xrightarrow{\text{Equil}} ① \quad 0 = a(1-\mathcal{R}_3)P_0^2 - \mathcal{R}_3 P_0 + c(1-\mathcal{R}_3) - m$$

$$\Rightarrow P_0 = \frac{\mathcal{R}_3}{2a(1-\mathcal{R}_3)} \pm \frac{1}{2a(1-\mathcal{R}_3)}\sqrt{\mathcal{R}_3^2 - 4a(1-\mathcal{R}_3)[c(1-\mathcal{R}_3) - m]}$$

$$\rightarrow M_{osc} = \frac{2\mathcal{R}_1(1-\mathcal{R}_3)}{\pi|A|} - \frac{\pi|A|\mathcal{R}_3^2}{4\mathcal{R}_1(1-\mathcal{R}_3)} \Rightarrow \mu_{osc} = \mathcal{R}_1\mathcal{R}_3 + \Omega^{-1}M_{osc}$$