2D Stommel model
outer soln

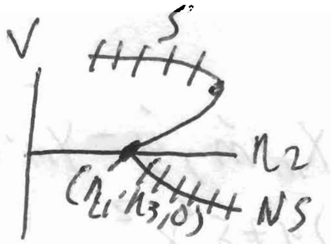
Given the system,

$$\begin{cases} \dot{V} = (\eta_1 - \eta_2) - V|V| - T + \eta_3(T - V) \\ \dot{T} = \eta_1 - T(1+|V|) \\ \dot{\eta}_2 = -\varepsilon \quad , \varepsilon \ll 1 \end{cases}$$

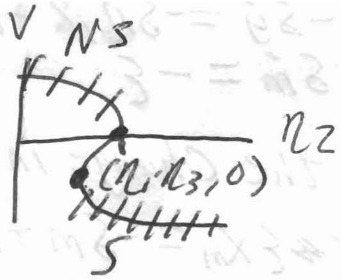
Immediately, we see that the sign of V will induce different dynamics, thus there is a critical point at $V=0$. Also, η_3 plays a significant role as a fixed parameter, namely $\eta_3 \,(<=>)\, 1$.

To begin our analysis, we will fix (wlog) $\eta_3 < 1$ and consider $V<0$, the nonsmooth branch exists in this region.

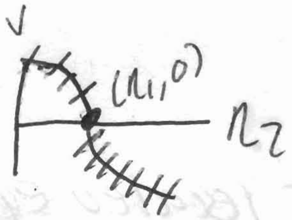
Note: $\eta_3 < 1$



$\eta_3 > 1$



$\eta_3 = 1$



For $V<0$,

$$\begin{cases} \dot{V} = (\eta_1 - \eta_2) + V^2 - T + \eta_3(T - V) \\ \dot{T} = \eta_1 - T(1-V) \\ \dot{\eta}_2 = -\varepsilon \end{cases} \Rightarrow \quad \begin{aligned} T_0 &= \frac{\eta_1}{1-V_0} \\ 0 &= (\eta_1 - \eta_2) + V_0^2 - \frac{\eta_1}{1-V_0} + \eta_3\left(\frac{\eta_1}{1-V_0} - V_0\right) \end{aligned}$$

With $T_0(\eta_2)$, $V_0(\eta_2)$ being the numerical leading order solutions for our lower stable branch.

Scaling Factors

So with V_0 being a solution that holds in $V<0$, we established $V=0$ is a critical point, which in turn gives $T=\eta_1$ and $\eta_2=\eta_1\cdot\eta_3$. The last, $\eta_2=\eta_1\eta_3$ is the NS bif. our analyis is about.

Since $V<0$ and we concern ourselves w/ $V=0$, a rescaling of $V=-\delta x$, $x>0$ makes sense.

Recall $T_0=\dfrac{\eta_1}{1-V_0} \longrightarrow \eta_1$ from below as $V_0 \to 0 \Rightarrow T=\eta_1-\delta y$

$$\Rightarrow \begin{cases} -\delta\dot{x} = -\eta_2+\eta_3\eta_1+\delta\eta_3 x+\delta(1-\eta_3)y+\delta^2x^2 \\ -\delta\dot{y} = -\eta_1\delta x+\delta y+\delta^2 xy \\ \dot{\eta}_2=-\varepsilon \end{cases}$$

To keep the parameter on the same order as the dynamics,

$$\eta_2=\eta_1\eta_3+\delta m$$

$$\Rightarrow \begin{cases} -\delta\dot{x} = -\delta m+\delta\eta_3 x+\delta(1-\eta_3)y+\delta^2x^2 \\ -\delta\dot{y} = -\delta\eta_1 x+\delta y+\delta^2 xy \\ \delta\dot{m}=-\varepsilon \end{cases}$$

Allow the change in variable, $\dot{x}=x_m\cdot\dot{m}$, $\dot{y}=y_m\cdot\dot{m}$

$$\Rightarrow \begin{cases} \varepsilon x_m = -\delta m+\delta\eta_3 x+\delta(1-\eta_3)y+\delta^2x^2 \\ \varepsilon y_m = -\delta\eta_1 x+\delta y+\delta^2 xy \\ \delta\dot{m}=-\varepsilon \end{cases}$$

Now, it is clear $\delta=\varepsilon$ and our rescaled equations are,

$$\begin{cases} x_m = -m+\eta_3 x+(1-\eta_3)y+\varepsilon x^2 \\ y_m = -\eta_1 x+\cancel{\delta} y+\varepsilon xy \\ \dot{m}=-1 \end{cases} \Rightarrow A=\begin{pmatrix} \eta_3 & 1-\eta_3 \\ -\eta_1 & 1 \end{pmatrix}$$

$$\lambda=\frac{(\eta_3+1)}{2} \pm \frac{\sqrt{(\eta_3-1)(4\eta_1+\eta_3-1)}}{2}$$

$\rightarrow$ complex for $\eta_3<1$

Local Analysis

Using the scaling found, $V = -\varepsilon x$, $T = \eta_1 - \varepsilon y$, $\eta_2 = \eta_1 \eta_3 + \varepsilon m$ we ended with $(V < 0)$

$$\begin{cases} x_m = -m + \eta_3 x + (1-\eta_3) y + \varepsilon x^2 \\ y_m = -\eta_1 x + y + \varepsilon x y \\ \dot{m} = 1 \end{cases}$$

Thus, our leading order problem is

$$\begin{pmatrix} x_m \\ y_m \end{pmatrix} = \begin{pmatrix} \eta_3 & 1-\eta_3 \\ -\eta_1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} m \\ 0 \end{pmatrix}$$

Our matrix here has complex eigenvalues, $(\eta_3 < 1)$

$$\lambda = \frac{\eta_3 + 1}{2} \pm i \sqrt{\frac{(1-\eta_3)(4\eta_1 + \eta_3 - 1)}{2}}$$

Once we enter the $V > 0$ region, our same scaling will hold, resulting in

$$\begin{cases} x_m = -m + \eta_3 x + (1-\eta_3) y - \varepsilon x^2 \\ y_m = \eta_1 x + y - \varepsilon x y \\ \dot{m} = -1 \end{cases}$$

With the leading order problem,

$$\begin{pmatrix} x_m \\ y_m \end{pmatrix} = \begin{pmatrix} \eta_3 & 1-\eta_3 \\ \eta_1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} m \\ 0 \end{pmatrix}$$

$$\lambda = \frac{\eta_3 + 1}{2} \pm \frac{1}{2} \sqrt{(1-\eta_3)(4\eta_1 + 1 - \eta_3)} \quad (\text{For } \eta_3 < 1, \text{ real})$$

Tipping Point

The leading order ~~solution~~ Problem in our local analysis is:

$$\begin{pmatrix} x_m \\ y_m \end{pmatrix} = \begin{pmatrix} R_3 & 1-R_3 \\ R_1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -m \\ 0 \end{pmatrix}$$

Since in the $V > 0$ region the eigenvalues are ~~posi~~ real, we have a solution of the form:

$$x \sim c_1 e^{\lambda_1 m} + c_2 e^{\lambda_2 m} + c_3 m + c_4$$

$$y \sim c_5 e^{\lambda_1 m} + c_6 e^{\lambda_2 m} + c_7 m + c_8$$

where for V,

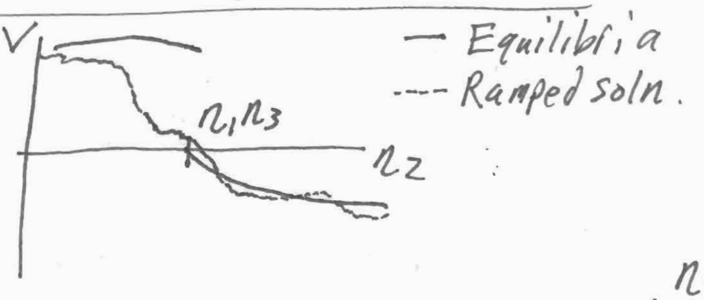
$$\frac{V}{\varepsilon} \sim c_1 e^{\lambda_1 \left(\frac{R_2 - R_1 R_3}{\varepsilon}\right)} + c_2 e^{\lambda_2 \left(\frac{R_2 - R_1 R_3}{\varepsilon}\right)} + c_3 \left(\frac{R_2 - R_1 R_3}{\varepsilon}\right) + c_4$$

To determine the tipping point, we search for which exponential first becomes large,

$$e^{\lambda_i \left(\frac{R_2 - R_1 R_3}{\varepsilon}\right)} \sim O(\varepsilon^{-1}) \iff R_2 - R_1 R_3 \sim -\varepsilon \ln(\varepsilon)/\lambda_i$$

Thus $R_{2\,tip} \sim \min\{R_1 R_3 - \varepsilon \ln(\varepsilon)/\lambda_i\}$ for $i = 1, 2$

Ideas for tipping cut off



$$\eta_2 = \eta_1\eta_3 + \varepsilon m$$

- When Scaling the equations, $V = -\varepsilon x$, $T = \eta_1 - \varepsilon y$, were chosen as $(V, T, \eta_2) = (0, \eta_1, \eta_1\eta_3)$ is where the NS bif occurs. In the $V > 0$ region, we find the leading order system

$$\begin{pmatrix} x_m \\ y_m \end{pmatrix} = \begin{pmatrix} \eta_3 & 1-\eta_3 \\ \eta_1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} m \\ 0 \end{pmatrix} \rightarrow \lambda = \frac{\eta_3 + 1}{2} \pm \frac{1}{2}\sqrt{(1-\eta_3)(4\eta_1 + 1 - \eta_3)}$$

Where $\lambda_1 < 0 < \lambda_2$.

The x solution is $x \sim c_1 e^{\lambda_1 m} + c_2 e^{\lambda_2 m} + c_3 m + c_4$

$$\Rightarrow -\frac{V}{\varepsilon} \sim c_1 e^{\lambda_1\left(\frac{\eta_2 - \eta_1\eta_3}{\varepsilon}\right)} + c_2 e^{\lambda_2\left(\frac{\eta_2 - \eta_1\eta_3}{\varepsilon}\right)} + c_3\left(\frac{\eta_2 - \eta_1\eta_3}{\varepsilon}\right) + c_4$$

where in $V > 0$, $\eta_2 - \eta_1\eta_3 < 0 \Rightarrow \eta_2 - \eta_1\eta_3 \sim -\frac{\varepsilon \ln \varepsilon}{\lambda_1}$ $(\lambda_1 < 0)$

$$\Rightarrow -\frac{V}{\varepsilon} \sim \frac{c_1}{\varepsilon} + c_2 \varepsilon^{\cancel{3} - \lambda_2/\lambda_1} + \frac{\varepsilon \ln \varepsilon\, c_3}{\lambda_1} + c_4 \quad (\cancel{3} - \lambda_2/\lambda_1 > 0)$$

$$\rightarrow V \sim -c_1 - c_2 \varepsilon^{1 - \lambda_2/\lambda_1} + \frac{\varepsilon^2 \ln \varepsilon \cdot c_3}{\lambda_1} + c_4 \varepsilon$$

Clearly the first term contributes the most so we search for $V \approx -c_1$ to be our tipping point.

c_1 is the V-component of the eigVec for λ_1