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# SLOWLY VARYING JUMP AND TRANSITION PHENOMENA ASSOCIATED WITH ALGEBRAIC BIFURCATION PROBLEMS*

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**Abstract.** Parameter-dependent equilibrium solutions are analyzed as the parameter slowly varies through critical values corresponding to a bifurcation or to a jump phenomena. At these critical times, interior nonlinear transition layers are necessary. Depending on the particular situation, local scaling analysis yields the first and a second Painlevé transcendent among other generic equations. In specific cases the resulting boundary layer solutions either increase algebraically or explode (via a singularity). The algebraic growth corresponds to a smooth transition to a bifurcated equilibrium. When a jump phenomena is expected, an explosion can occur. In this case, the solution of first-order differential equations approaches the equilibrium, describing the slow evolution through such a jump. However, second-order differential equations have finite amplitude oscillations around the new equilibrium.

**1. Introduction.** We consider first and second order nonlinear differential equations dependent on a parameter $\lambda$:

$$\frac{du}{dt}=F(u,\lambda), \tag{1.1a}$$

$$\frac{d^2u}{dt^2}=F(u,\lambda). \tag{1.1b}$$

If $\lambda$ is independent of $t$, then there are equilibrium solutions which satisfy an algebraic equation

$$F(u,\lambda)=0. \tag{1.2}$$

Often in physical problems there is an equilibrium solution independent of $\lambda$; i.e., $u=u_0$ if $F(u_0,\lambda)\equiv 0$ (for example, $u=0$ if $F(u,\lambda)=uG(u,\lambda)$). However, here we will not necessarily assume that such a simple equilibrium exists. We will analyze certain critical values of $\lambda$ where the number of equilibrium solutions changes. We will discuss the manner in which smooth transitions or abrupt jumps take place between two different solutions as $\lambda$ slowly varies in time (i.e., $\lambda(\varepsilon t)$, where $0<\varepsilon \ll 1$).

We begin by describing some well-known phenomena which occurs if $\lambda$ is constant in time. The linear stability of an equilibrium solution $u_E(\lambda)$ is determined by the linearization of (1.1):

$$\left.\begin{array}{l}\dfrac{d}{dt}(u-u_E)\\ \dfrac{d^2}{dt^2}(u-u_E)\end{array}\right\}=F_u(u_E,\lambda)(u-u_E). \tag{1.3}$$

On the basis of linear theory, $u_E$ is stable if $F_u(u_E,\lambda)<0$ and $u_E$ is unstable if $F_u(u_E,\lambda)>0$. The stability or instability of an equilibrium solution is easily illustrated in a $u-\lambda$ diagram. We will use an argument due to Poincaré (for example, see Andronov, Chaikin, and Witt [5]). The equilibrium solution corresponds to $F(u,\lambda)=0$, while stability is determined by $F_u(u_E,\lambda)$. Thus we want to know if $F$ is increasing or

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decreasing as $u$ is increased across $F=0$. Simply knowing the sign of $F(u, \lambda)$ on either side of the equilibrium curve suffices. We will indicate the sign of $F(u, \lambda)$ on a $u-\lambda$ diagram with $+$(shaded) and $-$ (clear). We also introduce arrows downward if $F<0$ and upward if $F>0$. For first-order equations (1.1a), these arrows tell the direction of change. For second-order equations (1.1b), $F(u, \lambda)$ represents a conservative force, and thus the arrows give the direction of the force. The arrows confirm our notion of stability. We are able to say the solution is stable if arrows converge on an equilibrium curve and vice versa. It is noted that the portion of an equilibrium curve above a shaded region is stable.

A critical value $\lambda_c$ of $\lambda$ occurs if $F_u(u_E(\lambda_c), \lambda_c)=0$. This separates stable from unstable solutions if $F_{u\lambda}(u_E(\lambda_c), \lambda_c)\neq 0$. Since from (1.2) $du_E/d\lambda = -F_\lambda/F_u$, at a critical value either $F_\lambda(u_E(\lambda_c), \lambda_c)=0$ or $du_E/d\lambda(\lambda_c)=\infty$. Before describing these two cases, we introduce the notation used throughout this paper for the Taylor expansion of $F(u, \lambda)$ around $u=u_E(\lambda_c)\equiv u_c$ and $\lambda=\lambda_c$:

$$F(u, \lambda)=\alpha_{01}(\lambda-\lambda_c)+\alpha_{20}(u-u_c)^2+\alpha_{11}(u-u_c)(\lambda-\lambda_c)+\alpha_{02}(\lambda-\lambda_c)^2+\cdots, \tag{1.4}$$

where

$$\alpha_{nm}=\frac{1}{n!m!}\left(\frac{\partial}{\partial u}\right)^n\left(\frac{\partial}{\partial \lambda}\right)^m F(u_c, \lambda_c).$$

If $du_E/d\lambda\neq\infty$, then for a critical value of $\lambda_c$, $\alpha_{01}=F_\lambda(u_c, \lambda_c)=0$. By considering the Taylor expansion of $F(u, \lambda)$ near $u=u_c$ and $\lambda=\lambda_c$, it follows from (1.4) that locally the quadratic terms are most important. Bifurcation occurs if there is more than one solution that coalesces. Locally there will be two solutions, for example, if $\alpha_{20}\neq 0$ and $\alpha_{11}^2-4\alpha_{20}\alpha_{02}>0$. This analysis could correspond to primary or secondary bifurcation. Typical bifurcation diagrams are given in Fig. 1, where $u_E$ is sketched as a function of $\lambda$. These diagrams are the sketch of the equilibrium solution in the neighborhood of $\lambda=\lambda_c$ and $u=u_c$. Near $\lambda=\lambda_c$, we also assume $u_E=u_{E_1}$ is stable for $\lambda<\lambda_c$ and unstable for $\lambda>\lambda_c$. We mark on the diagrams $s$ for stable and $u$ for unstable and sketch arrows as previously described.

Figure 1a illustrates the crossing of two equilibrium solutions. Their stabilities exchange at $\lambda=\lambda_c$. We will call this case "straight-straight" bifurcation to indicate its geometric property. In the neighborhood of $u=u_c$, $\lambda=\lambda_c$

$$F(u, \lambda)=\alpha_{20}(\tilde{u}-\beta_1\tilde{\lambda}+\cdots)(\tilde{u}-\beta_2\tilde{\lambda}+\cdots)=\alpha_{20}\tilde{u}^2+\alpha_{11}\tilde{u}\tilde{\lambda}+\alpha_{02}\tilde{\lambda}^2+\cdots, \tag{1.5}$$

where $\tilde{u}\equiv u-u_c$ and $\tilde{\lambda}\equiv\lambda-\lambda_c$. We define $\beta_2>\beta_1$ and note that $\alpha_{11}^2-4\alpha_{02}\alpha_{20}>0$ and $\alpha_{20}<0$ (in order for $F(u, \lambda)<0$ for sufficiently large $u$).

On the other hand Fig. 1b illustrates what Lebovitz and Schaar [15] call "vertical" bifurcation and we call "parabolic" bifurcation. In the simplest case, in the neighborhood of $u=u_c$, $\lambda=\lambda_c$

$$\begin{aligned}F(u, \lambda)&=\alpha_{30}(\tilde{u}-\beta_1\tilde{\lambda}+\cdots)(\tilde{u}^2-\sigma^2\tilde{\lambda}+\cdots)\\&=\alpha_{11}\tilde{u}\tilde{\lambda}+\alpha_{02}\tilde{\lambda}^2+\alpha_{30}\tilde{u}^3+\alpha_{21}\tilde{u}^2\tilde{\lambda}+\alpha_{12}\tilde{u}\tilde{\lambda}^2+\alpha_{03}\tilde{\lambda}^3+\cdots.\end{aligned} \tag{1.6}$$

This case is characterized by $\alpha_{20}=0$. Note that $\alpha_{30}<0$ (in order for $F(u, \lambda)<0$ for $u$ sufficiently large) and $\alpha_{11}>0$ (in order for the parabolic curve to exist for $\lambda>\lambda_c$). In this case $\alpha_{11}=-\sigma^2\alpha_{30}>0$ and $\alpha_{21}=-\beta_1\alpha_{30}$. We will see that two different cases must be analyzed; the transition from the parabolic arc to the straight line curve as $\lambda$ decreases through $\lambda_c$ and the opposite case which occurs as $\lambda$ is increased through $\lambda_c$.

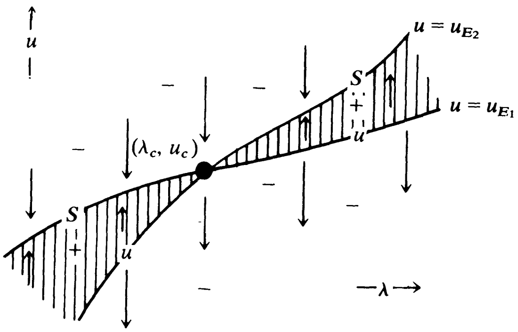



FIG. 1a. *Straight-straight bifurcation.*

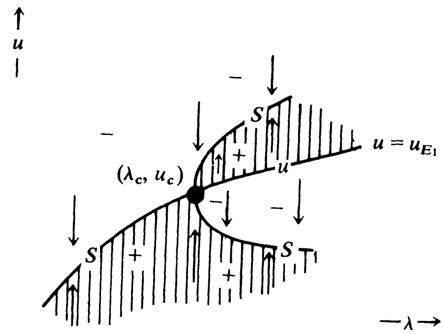
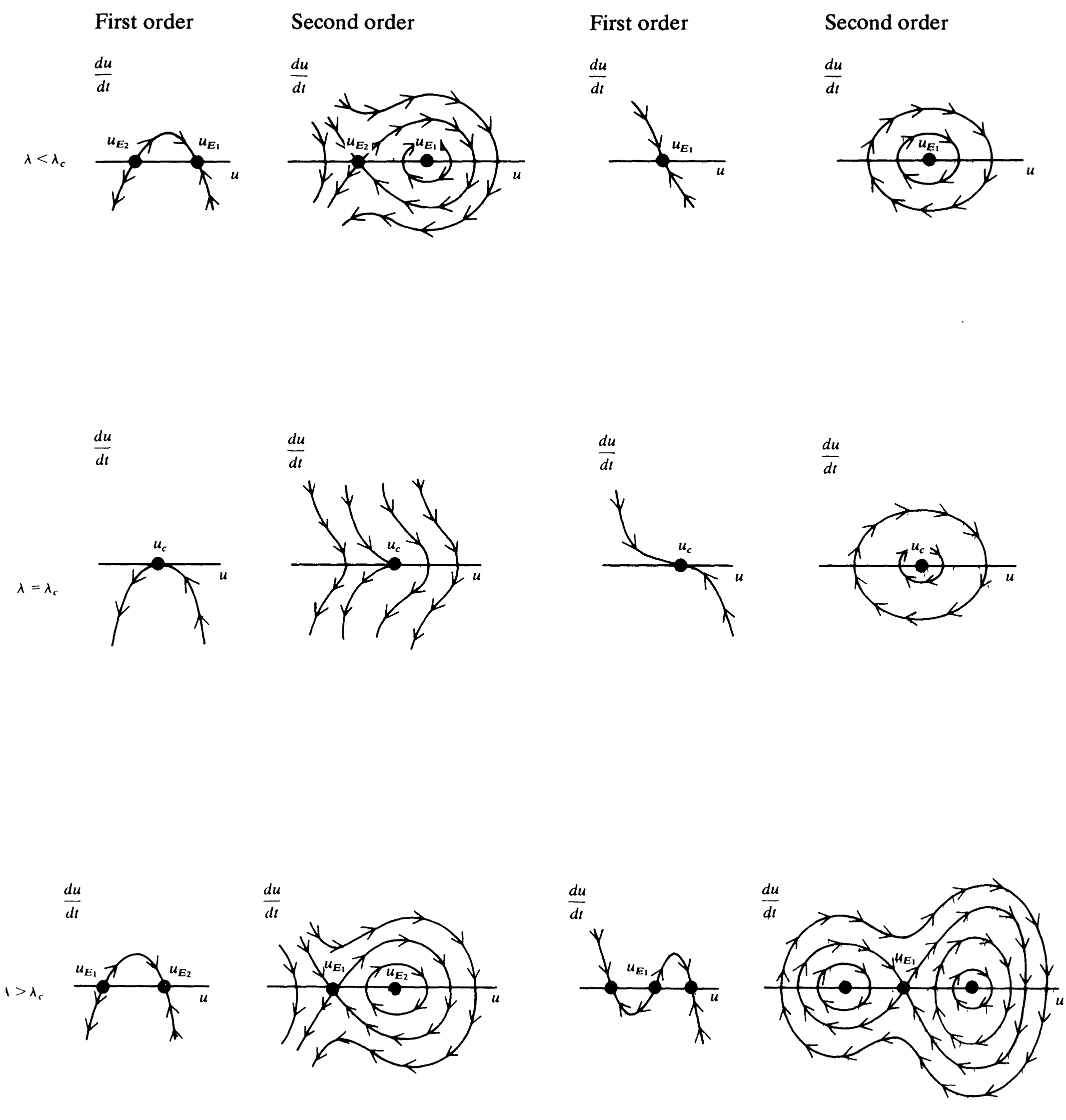



FIG. 1b. *Parabolic bifurcation.*



The other critical case occurs when $du_E/d\lambda=\infty$. We only consider the case in which the equilibrium solution turns back as illustrated at point $A$ in the bifurcation diagrams of Figs. 2a or 2b. In the simplest case, in the neighborhood of $u=u_c$, $\lambda=\lambda_c$,

$$(1.7)\qquad F(u,\lambda)=\alpha_{20}(\tilde{u}^2-\sigma^2\tilde{\lambda}+\cdots)=\alpha_{01}\tilde{\lambda}+\alpha_{20}\tilde{u}^2+\alpha_{11}\tilde{u}\tilde{\lambda}+\alpha_{02}\tilde{\lambda}^2+\cdots$$

Here $\alpha_{20}<0$ (in order that $\tilde{u}\sim\pm\sigma\tilde{\lambda}^{1/2}$ be stable) and $\alpha_{01}>0$ (so that the parabolic equilibrium exists for $\lambda\geqq\lambda_c$ as illustrated in Fig. 2). Note that $\sigma=(-\alpha_{01}/\alpha_{20})^{1/2}$.

Figure 1a shows what is called a "subcritical" instability of $u_{E_1}$. If $\lambda$ is less than the critical value for the linear instability of $u=u_{E_1}$, then $u=u_{E_1}$ is linearly stable, but unstable to finite amplitude perturbations. Figure 1b illustrates that for $\lambda>\lambda_c$, although $u=u_{E_1}$ is now unstable, there exists another "supercritical" stable equilibrium. Figure 2a (with dashes) shows the existence of a subcritical instability such that there also exists a critical value of $\lambda$ at which it is first possible for the solution to evolve into a subcritical $u=u_{E_1}$ equilibrium solution. These cases are of interest in certain fluid dynamic contexts (where the the equations of motion are entirely different and more complex than (1.1)). However, perturbation theories based on the analysis of Stuart [20] and Watson [21] yield first-order equations of the form (1.1a). It is for these reasons that we wish to study (1.1a) and (1.1b) in the cases corresponding to Figs. 1a, 1b, and 2a.

As $\lambda$ varies (but not dynamically) the stable equilibrium illustrated in Figs. 1a and 1b gradually changes from $u_{E_1}$ to $u_{E_2}$ (what is called an exchange of stabilities). To understand this, we have also sketched in Fig. 1 the phase planes for $\lambda<\lambda_c$, $\lambda=\lambda_c$ and $\lambda>\lambda_c$ for these two types of bifurcation problems for both first and second order systems. In particular note that at $\lambda=\lambda_c$, the unique equilibrium $u=u_c$ is unstable for straight-straight bifurcation and stable for parabolic bifurcation.

Figure 2 shows how an equilibrium may suddenly disappear as $\lambda$ varies. If the solution is on the branch $AB$ and if the value of $\lambda$ is sufficiently decreased, then it might be expected that $u$ will jump onto a different segment of the equilibrium curve (if one exists for $\lambda<\lambda_c$ as in Fig. 2a, but not as in Fig. 2b). The various phase planes are also sketched in Fig. 2. In Fig. 2a there are three equilibria for $\lambda>\lambda_c$ and only one for $\lambda<\lambda_c$; two of the equilibria coalesce at $\lambda=\lambda_c$ and then both disappear for $\lambda<\lambda_c$.

Here we wish to investigate first and second order systems for these two classes of problems, corresponding to a gradual bifurcation as in Figs. 1a and 1b and corresponding to a sudden jump as in Fig. 2a. In particular we wish to know precisely how (if at all) these new equilibria are approached. We ask what happens if the parameter $\lambda(\varepsilon t)$ slowly varies dynamically, where $\varepsilon$ is a small parameter, $0<\varepsilon\ll 1$. Introducing the slow time $T$, $T=\varepsilon t$, equations (1.1) may be rewritten as

$$(1.8)\qquad \begin{aligned}\frac{du}{dt}&=\varepsilon\frac{du}{dT}=F(u,\lambda(T)),\\ \frac{d^2u}{dt^2}&=\varepsilon^2\frac{d^2u}{dT^2}=F(u,\lambda(T)).\end{aligned}$$

Thus there are two equivalent standard singular perturbation problems: differential equations with slowly varying coefficients and differential equations with a small parameter multiplying the highest derivative.

We will discuss both first and second order systems. First-order systems have been analyzed by Lebovitz and Schaar [14], [15] for bifurcation problems corresponding to Figs. 1a and 1b. They, however, discuss autonomous systems by assuming the parameter

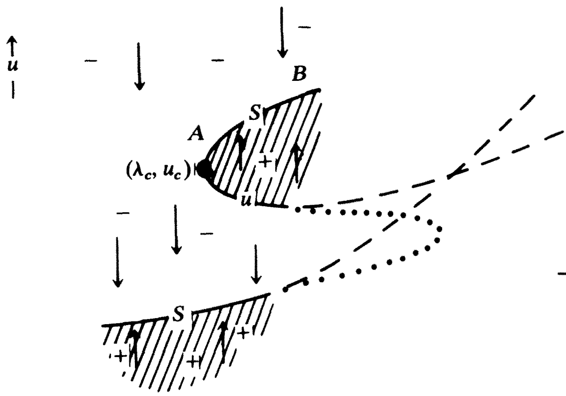



FIG 2a. *Different cases with a jump phenomena.*

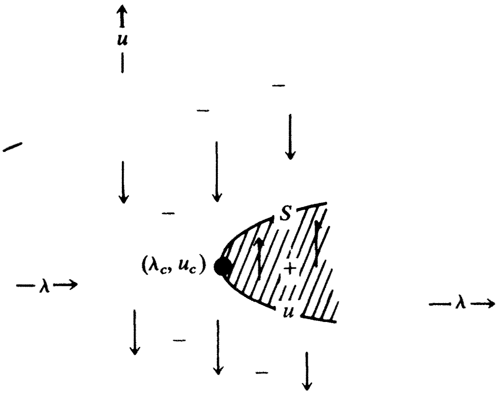
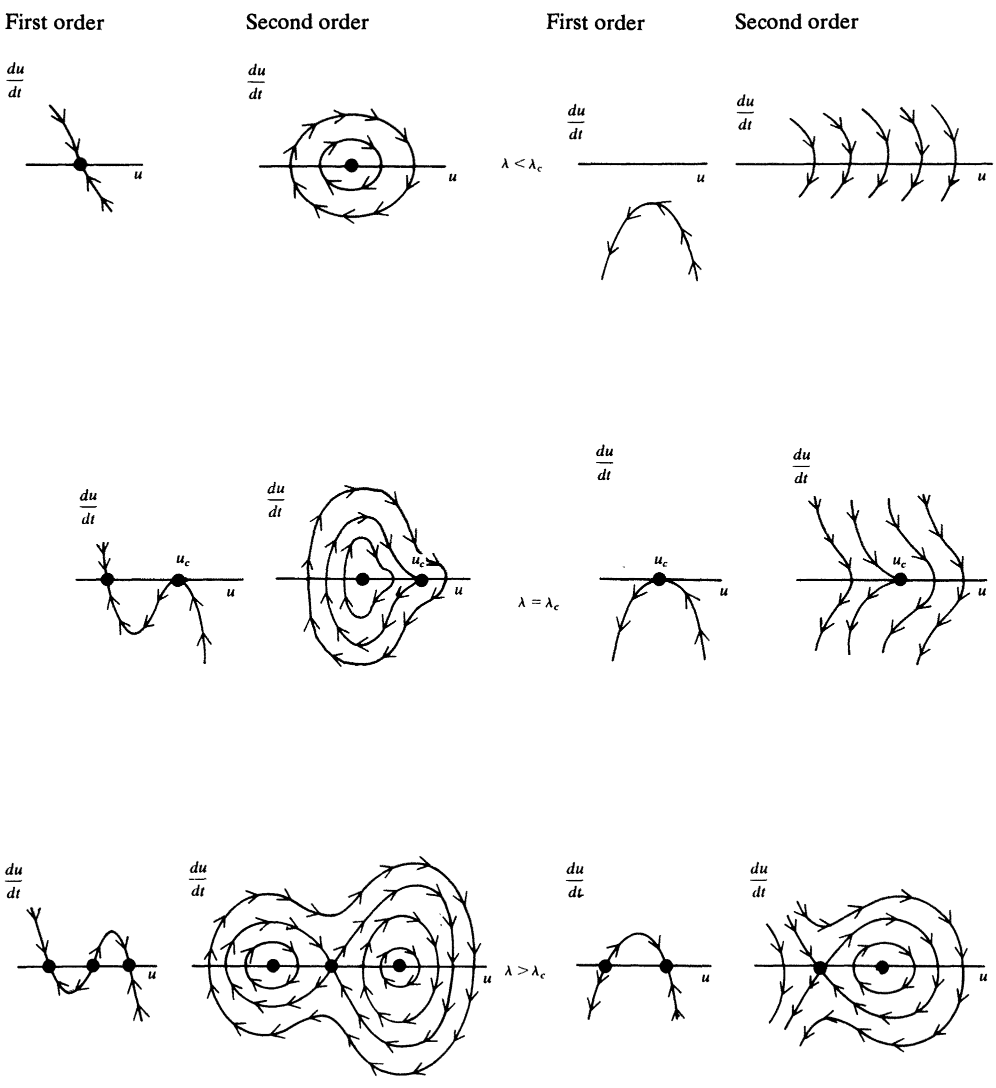



FIG 2b.



$\lambda(T)$ is controlled by the solution $u$:

$$\frac{du}{dt}=\varepsilon\frac{du}{dT}=F(u,\lambda),\qquad \frac{d\lambda}{dt}=\varepsilon f(u,\lambda).$$

We instead prefer to discuss variable coefficient (nonautonomous) systems as though the critical parameter $\lambda$ (the bifurcation parameter) is externally controlled and slowly evolves through the critical condition $\lambda=\lambda_c$. However, many of the ideas of Lebovitz and Schaar [14], [15] are still valid for our first-order systems.

Letting $\varepsilon=0$ in (1.8) yields the *reduced* problem (algebraic in nature)

$$0=F(u,\lambda(T)). \tag{1.9}$$

We denote solutions of (1.9) as $u_r(\lambda)$. If $\lambda$ were constant, then $u_r(\lambda)$ is an equilibrium solution $u_E(\lambda)$. However, we note at the onset that these are no longer equilibrium solutions of (1.1) or (1.8) since $\lambda$ slowly varies in time. Nevertheless, we expect there to exist solutions which are nearly constant, obtained in § 2 by an expansion of (1.8) in powers of $\varepsilon$. Such solutions which are nearly $u_r(\lambda)$ we call *slowly varying equilibrium solutions*, $u_{\text{sve}}(T)$. Different expansions of the slowly varying equilibrium solutions exist for the various types of equilibrium curves. In all cases the asymptotic expansion becomes invalid as the critical time is approached.

In § 3 we consider initial conditions near to those corresponding to the slowly varying equilibrium solutions. For first-order systems, perturbations to the slowly varying stable equilibria exponentially decay, while for second-order systems oscillations occur around these equilibria. For second-order systems the technique of multiple-time scales (see Kevorkian [10], Cole [6] or Nayfeh [17]) is utilized for the resulting weakly nonlinear slowly varying oscillators. This is done in the manner the present author previously used to analyze nonlinear transition layers [7].

In the remaining sections (§§ 4–7), the various specific problems are discussed. The outer asymptotic expansion (valid away from the critical time where $\lambda=\lambda_c$) is nearly an equilibrium solution; it slowly varies, but is not valid in some neighborhood of the critical time. We determine the inner expansion in the neighborhood of transition and employ the method of matched asymptotic expansions. The leading order transition layer equations follow from local analyses of (1.1). We then describe the solution as the parameter $\lambda$ slowly evolves through its critical value. In §§ 4–7 there is some repetition in the analysis and discussion so that attention may be restricted to those cases of particular interest to the reader.

In § 4 we consider second-order jump phenomena. The leading order transition layer equation[1] is

$$\frac{d^2w}{dz^2}=-z-w^2, \tag{1.10}$$

where the right-hand side corresponds to the normalized local description of $F(u,\lambda)$ near a turning of the parabolic curve and where $z$ and $w$ are the local scalings of time and $u$ respectively. We note that (1.10) is an example of the first Painlevé transcendent (Ince [8]). Equation (1.10) matches as $z\to-\infty$ to the stable oscillations around the slowly varying equilibrium. Equation (1.10) develops an explosive singularity (at a finite value of $z$) enabling the inner solution to match to a different outer solution corresponding to the expected amplitude jump. However, the new outer solution is not a slowly varying equilibrium solution. Instead, oscillations around this equilibrium occur,

[1] In this introduction we write normalized versions of the transition equations derived in §§ 4–7.

whose amplitude is $O(1)$. These oscillations are similar to those described by Ablowitz, Funk and Newell [1] in their study of the slow variations through resonance for the jump phenomena associated with the response curve for the periodically forced Duffing equation, originally studied by Kevorkian [11], [12] and more recently by Rubenfeld [18]. In our case the new outer solution is a slowly varying fully nonlinear oscillator which is analyzed by a method due to Kuzmak [13]. However, matching to (1.10) occurs only if the infinite-period singular limit of the results of Kuzmak are considered, as has been done in other contexts first by Johnson [9] and later by Miles [16] and Ablowitz and Segur [2]. We illustrate the resulting transition in Fig. 3.

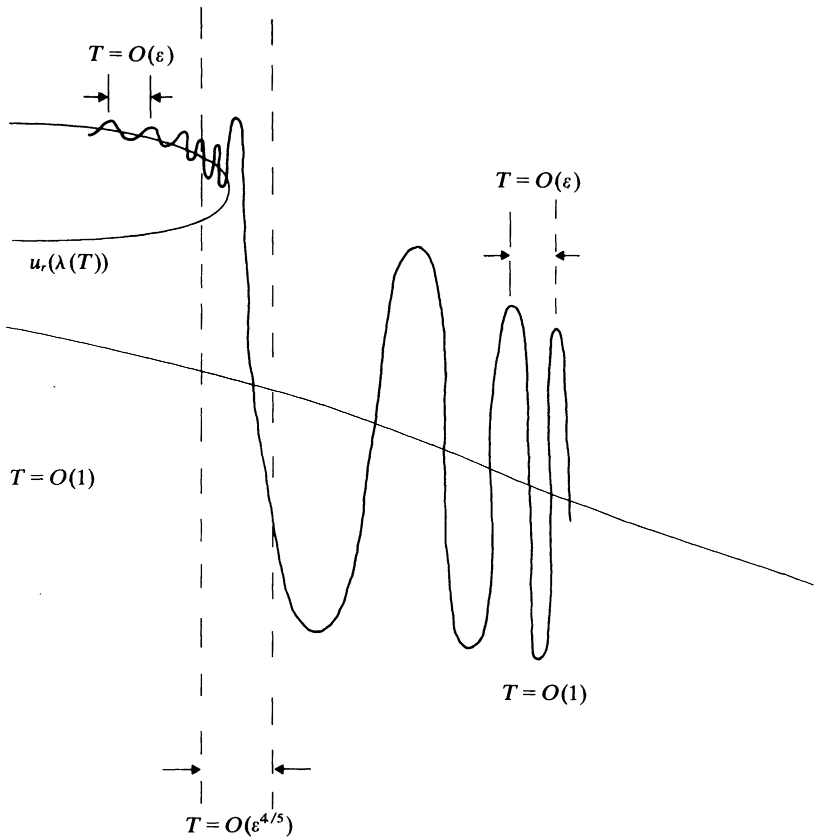
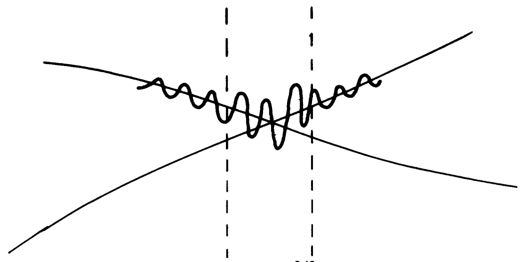


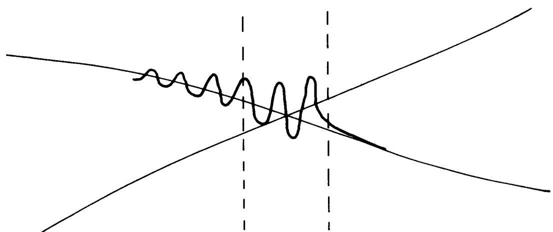

FIG 3. *Second-order jump phenomena—the first Painlevé transcendent provides the jump transition between the slowly varying parabolic equilibrium and the slowly varying nonlinear oscillation.*

In § 5, we discuss jump phenomena governed by first-order differential equations. The leading order transition layer equation is obtained by local analysis of (1.1) using (1.7):

$$\frac{dw}{dz} = -z - w^2. \tag{1.11}$$

Equation (1.11) is a Riccati equation, and it also arises in describing the transition for the van der Pol relaxation oscillator (see Cole [6]). The exact solution of (1.11) develops an explosive singularity at a finite value of $z$. This is shown to match to a fully nonlinear



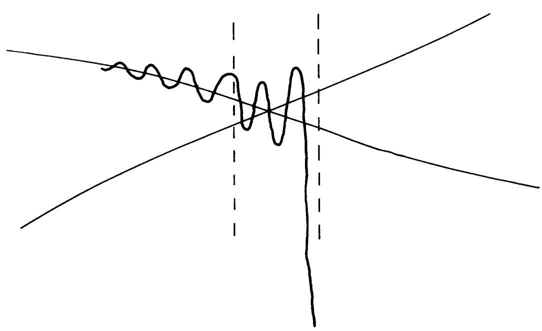




FIG. 4. *Second-order straight-straight bifurcation—after slow evolution through the critical conditions, the solution may approach the stable equilibrium* (*Fig.* 4a), *the unstable equilibrium* (*Fig.* 4b), *or explode* (*Fig.* 4c).

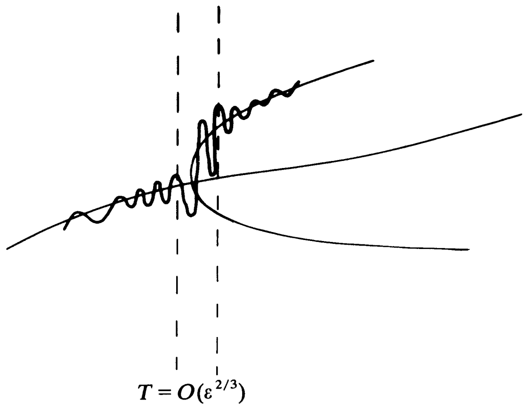
outer solution which exponentially decays to the slowly varying equilibrium showing how the transition between two equilibria occur.

In §§ 6–7 bifurcation problems are analyzed. A fundamental difference exists when $\lambda$ is near the critical value between the bifurcations illustrated by Figs. 1a and 1b. The arrows indicate a type of global stability for Fig. 1b, but for Fig. 1a it is possible for the solution to approach $-\infty$. We will demonstrate that the slow time dynamics reflects this difference. Furthermore we will show certain similarities between Figs. 2 and 1a. These exist because for both figures arrows are pointing downwards in the vicinity of the critical point. This indicates the possibility of exposion towards $-\infty$.

In § 6 second-order bifurcating systems are analyzed. The leading order transition equation is derived from the local analysis of (1.1) which depends on whether we are discussing the case of straight-straight or parabolic bifurcation. For straight-straight bifurcation the equation describing transition is

$$\frac{d^2w}{dz^2} = -(w-\beta_1 z)(w-\beta_2 z), \tag{1.12}$$

where $w$ and $z$ are the inner dependent and independent variables. Depending on the



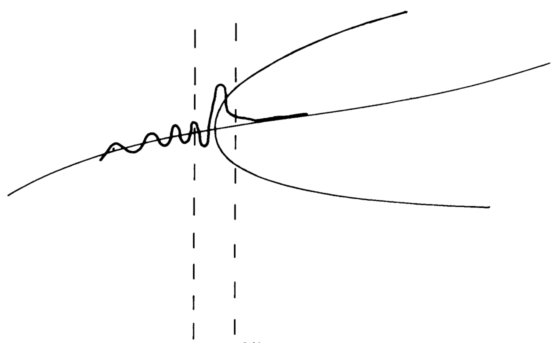




FIG. 5. *Second-order parabolic bifurcation—the second Painlevé transcendent provides the transition between two stable slowly varying equilibria* (*Fig.* 5a) *or the rare transition to the unstable state* (*Fig.* 5b).

matching conditions as $z \to -\infty$, the solution of (1.12) either algebraically grows as $z \to +\infty$, exponentially decays as $z \to +\infty$, or explodes towards $w = -\infty$. The case of algebraic growth matches as $z \to +\infty$ to a slowly varying stable equilibrium solution. This corresponds to the transition from one stable equilibrium to another. An exponentially decaying solution corresponds to the transition from a stable equilibrium to an unstable equilibrium. If the solution of (1.12) explodes, then this corresponds to an actual explosion as Fig. 1a indicates is possible. Figure 4 illustrates the possible outcomes as the parameter $\lambda$ slowly varies through a straight-straight bifurcation point.

For parabolic bifurcation, the leading order transition layer equation is shown in § 6 to be

$$\frac{d^2w}{dz^2} = -zw - w^3, \tag{1.13}$$

which is an example of the second Painlevé transcendent (see Ince [8]). Solutions of (1.13) either algebraically grow, corresponding to the transition from the straight equilibrium to one or the other of the two stable branches of the parabolic curve, or exponentially decay, corresponding to the transition to the unstable slowly varying equilibrium. These transitions are illustrated in Fig. 5. It is also possible for (1.13) to represent the transition from a parabolic arc to the straight equilibrium curve.

In § 7 first-order systems are analyzed. For straight-straight bifurcation the leading order transition equation is the local analysis of (1.1)

$$\frac{dw}{dz} = -(w - \beta_1 z)(w - \beta_2 z). \tag{1.14}$$

Lebovitz and Schaar [14] proved that (1.14) describes the transition from one stable, locally-straight, and slowly varying equilibrium to another only if $\beta_1 > 0$ (assuming $\beta_2 > \beta_1$). If $\beta_1 < 0$ the solution explodes. The important case, corresponding to $u = 0$ being an exact equilibrium solution, in which $\beta_1 = 0$ is analyzed in Appendix A.

For parabolic bifurcation two different cases are described in § 7. For the transition from a parabolic curve to a straight curve, local analysis yields the leading order transition equation:

$$\frac{dw}{dz} = -zw - w^3, \tag{1.15}$$

a Bernoulli equation. Its exact solution (Ince [8]) explicitly shows the expected transition.

The analysis for straight to parabolic transition is more complex. Lebovitz and Schaar [15] give bounds on the solution. We show the boundary layer equation, obtained by the method of matched asymptotic expansions, is to leading order

$$\frac{dw}{dz} = z(w - z). \tag{1.16}$$

Equation (1.16) is linear since in the transition from a straight line solution, the amplitude is at first rather small. However for $z > 0$ the exact solution exponentially grows. The nonlinear terms are important in a secondary transition layer (described in § 7). This secondary layer is shown to match to the slowly varying parabolic equilibrium curve. Except for this last case, all the problems studied have a single nonlinear transition region describing the evolution between the two outer solutions.

**2. Slowly varying equilibrium solutions.** As described in the Introduction, solutions $u_r(\lambda)$ of the reduced problem $F(u_r, \lambda) = 0$ are *usually* not solutions of (1.1). Instead $u_r(\lambda)$ forms the first term of an asymptotic expansion, which we call the *slowly varying equilibrium solution*. We will calculate this perturbation expansion in eight cases: straight-straight bifurcation and the two directions of parabolic bifurcation, as well as jump transitions, all for first *and* second order systems.

To simultaneously analyze in all cases both first and second order systems, it is convenient to write (1.1) as

$$\frac{d^m u}{dt^m} = \varepsilon^m \frac{d^m u}{dT^m} = F(u, \lambda), \tag{2.1}$$

where $m$ equals 1 or 2. If the derivative term in (2.1) is small, then a valid slowly varying equilibrium solution may be obtained by perturbing $u$ around the reduced solution, $u_r(\lambda)$. Using the Taylor series of $F(u, \lambda)$ around $u = u_r(\lambda)$ (where $F(u_r, \lambda) = 0$), (2.1) becomes

$$\varepsilon^m \frac{d^m u}{dT^m} = (u - u_r)\frac{\partial F}{\partial u}\bigg|_{u_r} + \frac{(u-u_r)^2}{2!}\frac{\partial^2 F}{\partial u^2}\bigg|_{u_r} + \frac{(u-u_r)^3}{3!}\frac{\partial^3 F}{\partial u^3}\bigg|_{u_r} + \cdots. \tag{2.2}$$

Introducing the asymptotic expansion for this slowly varying equilibrium solution, $u_{\text{sve}}$

(2.3) $$u_{\text{sve}} = u_r(\lambda) + \varepsilon^m u_1 + \varepsilon^{2m} u_2 + \cdots,$$

yields

(2.4a) $$u_1 = (d^m u_r/dT^m) \Big/ \frac{\partial F}{\partial u}\bigg|_{u_r},$$

(2.4b) $$u_2 = \left( d^m u_1/dT^m - \frac{u_1^2}{2} \frac{\partial^2 F}{\partial u^2}\bigg|_{u_r} \right) \Big/ \frac{\partial F}{\partial u}\bigg|_{u_r},$$

(2.4c) $$u_3 = \left( d^m u_2/dT^m - u_1 u_2 \frac{\partial^2 F}{\partial u^2}\bigg|_{u_r} - \frac{u_1^3}{3!} \frac{\partial^3 F}{\partial u^3}\bigg|_{u_r} \right) \Big/ \frac{\partial F}{\partial u}\bigg|_{u_r}.$$

Expansion (2.3) is valid in an outer region away from the zeros of the denominators of (2.4), where $\partial F/\partial u|_{u_r} = 0$. This occurs at what we have called a critical point, $\lambda = \lambda_c$.

To determine what happens to the solution represented by (2.3) as $\lambda$ approaches $\lambda_c$, the method of matched asymptotic expansions is employed. We analyze the behavior of $u_r(\lambda)$ and $u_1$, $u_2$, $u_3$, etc. in the neighborhood of a critical point. Near $\lambda = \lambda_c$, the reduced solution $u_r(\lambda)$ is nearly $u_r(\lambda_c) \equiv u_c$. We define

(2.5) $$\tilde{\lambda} \equiv \lambda - \lambda_c \quad \text{and} \quad \tilde{u} = u - u_c.$$

We assume that $\lambda$ evolves slowly (and monotonically) through $\lambda_c$ such that $\lambda - \lambda_c$ has a simple zero ($\lambda_c' \neq 0$) at $T = 0$:

(2.6) $$\tilde{\lambda} = \lambda - \lambda_c = \lambda_c' T + \lambda_c'' T^2/2! + \cdots,$$

where $\lambda_c^{(n)} \equiv (d^n/dT^n)\lambda(T)$ evaluated at the critical time, $T = 0$. For second-order systems we also assume $\lambda_c'' \neq 0$.

At first, let us restrict our attention to one case, that of a jump phenomena for second-order differential equations. After we complete the analysis of the slowly varying equilibrium in this case, we will tabulate the analogous results for the other cases. For the jump phenomena illustrated by Fig. 2 and (1.7), we assume that near $\lambda = \lambda_c$ and $u = u_c$,

(2.7) $$F = \alpha_{20}(\tilde{u} - \sigma\tilde{\lambda}^{1/2} + \cdots)(\tilde{u} + \sigma\tilde{\lambda}^{1/2} + \cdots),$$

where again $\tilde{\lambda} = \lambda - \lambda_c$ and $\tilde{u} = u - u_c$. In this case, the stable reduced solution is

(2.8) $$u_r(\lambda) = u_c + \sigma(\lambda - \lambda_c)^{1/2} + \cdots,$$

where $\sigma > 0$ corresponds to the upper branch. Note from Fig. 2 that this reduced solution exists only if $\lambda \geqq \lambda_c$. We steadily decrease $\lambda$ through criticality (assuming $\lambda_c' < 0$). To determine the outer expansion (away from $\lambda = \lambda_c$) in the vicinity of the critical time, we evaluate $u_1$, $u_2$, $u_3$, etc. as $T \to 0$. To do so we need $\partial F/\partial u|_{u_r}$ (as well as higher partial derivatives):

(2.9) $$\frac{\partial F}{\partial u}\bigg|_{u_r} = 2\alpha_{20}\sigma(\lambda_c' T)^{1/2} + \cdots$$

and hence it follows that

(2.10) $$u_1 = -\frac{1}{8\alpha_{20}}(-T)^{-2} + \cdots.$$

Thus, the outer expansion in the vicinity of the critical time becomes

(2.11) $$u = u_c + \sigma(\lambda_c' T)^{1/2} + \cdots \frac{-\varepsilon^2}{8\alpha_{20}}(-T)^{-2} + \cdots$$

The disordering as $T \to 0$ of the outer expansion is clearly observed. Further calculations show that the outer expansion (2.3) breaks down (as illustrated in (2.11)) as $\lambda \to \lambda_c$ when $\varepsilon^2(-T)^{-2}$ balances $(-T)^{1/2}$ rather than balances the constant $u_c$. Thus the asymptotic expansion of the slowly varying equilibrium solution becomes disordered when $T = O(\varepsilon^{4/5})$. This result is valid for the case of a jump phenomena.

In the remainder of this section, we outline and tabulate the calculation of the time at which the expansion of the slowly varying equilibrium solution becomes disordered for the other cases. In Table 1, we record our assumptions concerning the behavior of the reduced solution near criticality. In particular for parabolic bifurcation the notation $s \to p$ indicates that $\lambda$ evolves from the region of a stable straight $(s)$ equilibrium to a region with a stable parabolic $(p)$ equilibrium. Similarly $p \to s$ indicates evolution in the opposite direction. The behavior of the reduced solution as $T \to 0$ is

(2.12) $$u_r = u_c + \beta_1 \lambda_c' T + \cdots = u_c + O(T)$$

for bifurcation problems that begin to evolve from a straight equilibrium (i.e., $s \to s$ and $s \to p$). However, when the equilibrium is initially on a parabolic curve (i.e., $p \to s$ and the jump phenomena),

(2.13) $$u_r = u_c + \sigma(\lambda_c' T)^{1/2} + \cdots = u_c + O(T^{1/2}).$$

To calculate $u_1$ and $u_2$ asymptotically as $T \to 0$, we need $\partial F/\partial u|_{u_r}$ and $\partial^2 F/\partial u^2|_{u_r}$, which are given in Table 2. For all types of bifurcation, $\partial F/\partial u|_{u_r}$ has a simple zero, if $\tilde{\lambda}$

TABLE 1

*Behavior near $u = u_c$, $\lambda = \lambda_c$.*

| | | $F(u,\lambda)$ | $u_r(\lambda)$ |
|---|---|---|---|
| Straight-straight bifurcation | $(s \to s)$ | $\alpha_{20}(\tilde{u} - \beta_1\tilde{\lambda} + \cdots)(\tilde{u} - \beta_2\tilde{\lambda} + \cdots)$ | $u_c + \beta_1\tilde{\lambda} + \cdots$ |
| Parabolic bifurcation | $(s \to p)$ | $\alpha_{30}(\tilde{u} - \beta_1\tilde{\lambda} + \cdots)(\tilde{u}^2 - \sigma^2\tilde{\lambda} + \cdots)$ | $u_c + \beta_1\tilde{\lambda} + \cdots$ |
| | $(p \to s)$ | | $u_c + \sigma\tilde{\lambda}^{1/2} + \cdots$ |
| Jump phenomena | | $\alpha_{20}(\tilde{u} - \sigma\tilde{\lambda}^{1/2} + \cdots)(\tilde{u} + \sigma\tilde{\lambda}^{1/2} + \cdots)$ | $u_c + \sigma\tilde{\lambda}^{1/2} + \cdots$ |

TABLE 2

| | $\left.\frac{\partial F}{\partial u}\right\|_{u_r}$ (for small $\tilde{u}$, $\tilde{\lambda}$) | $\left.\frac{\partial^2 F}{\partial u^2}\right\|_{u_r}$ |
|---|---|---|
| $s \to s$ | $\alpha_{20}(\beta_1 - \beta_2)\tilde{\lambda}$ | $2\alpha_{20}$ |
| $s \to p$ | $-\alpha_{30}\sigma^2\tilde{\lambda}$ | $4\beta_1\alpha_{30}\tilde{\lambda}$ |
| $p \to s$ | $2\alpha_{30}\sigma^2\tilde{\lambda}$ | $6\alpha_{30}\tilde{\lambda}^{1/2}$ |
| jump | $2\alpha_{20}\sigma\tilde{\lambda}^{1/2}$ | $2\alpha_{20}$ |

TABLE 3

| | First-order ($m=1$) | Second-order ($m=2$) |
|---|---|---|
| $s \to s$<br>$s \to p$ | $u_1 = O(T^{-1})$ | $u_1 = O(T^{-1})$ |
| $p \to s$ | $u_1 = O(T^{-3/2})$ | $u_1 = O(T^{-5/2})$ |
| jump | $u_1 = O(T^{-1})$ | $u_1 = O(T^{-2})$ |

TABLE 4

| | First-order ($m=1$) | Second-order ($m=2$) |
|---|---|---|
| $s \to s$<br>$s \to p$ | $u_2 = O(T^{-3})$ | $u_2 = O(T^{-4})$ |
| $p \to s$ | $u_2 = O(T^{-7/2})$ | $u_2 = O(T^{-11/2})$ |
| jump | $u_2 = O(T^{-5/2})$ | $u_2 = O(T^{-9/2})$ |

TABLE 5
*Disordering of slowly varying equilibrium solutions.*

| | First-order ($m=1$) | Second-order ($m=2$) |
|---|---|---|
| $s \to s$<br>$s \to p$<br>$p \to s$ | $T = O(\varepsilon^{1/2})$ | $T = O(\varepsilon^{2/3})$ |
| jump | $T = O(\varepsilon^{2/3})$ | $T = O(\varepsilon^{4/5})$ |

has a simple zero. However, for the jump phenomena $\partial F/\partial u|_{u_r}$ has a square root singularity if $\tilde{\lambda}$ has a simple zero. The order of magnitude as $T \to 0$ of $u_1$ and $u_2$ is recorded in Tables 3 and 4. The time at which the expansion of the slowly varying equilibrium becomes disordered is noted in Table 5.

**3. Solutions in the neighborhood of slowly varying equilibria.** In the previous section we have done a preliminary analysis of the slowly varying equilibrium solution. This is a special solution of (1.1) satisfying a unique initial condition. None the less, we have calculated its outer expansion and indicated that this expansion breaks down in the vicinity of the critical time at which a boundary layer in time is necessary. Before proceeding to the inner expansion, it is convenient to discuss more general solutions to the outer initial value problem, corresponding to initial conditions nearly on the outer slowly varying equilibrium solution. Thus we consider perturbations of the entire expansion (2.3) of the form

$$u = u_{\text{sve}} + \mu \bar{u}, \tag{3.1}$$

where $\mu$ is a small parameter measuring the distance from $u_{\text{sve}}$. For now we consider all cases including first and second order systems. Soon we will describe important distinct properties of first and second order systems.

The slowly varying equilibrium solution $u_{sve}(T)$ is an exact solution of

$$\frac{d^m u}{dt^m} = F(u, \lambda). \tag{3.2}$$

In considering perturbations of $u_{sve}(T)$ given by (3.1), we need the Taylor expansion of $F(u, \lambda)$ around $u = u_{sve}$:

$$F(u, \lambda) = F(u_{sve}, \lambda) + (u - u_{sve}) \frac{\partial F}{\partial u}\bigg|_{u_{sve}} + \frac{(u-u_{sve})^2}{2!} \frac{\partial^2 F}{\partial u^2}\bigg|_{u_{sve}} + \cdots, \tag{3.3}$$

where all derivatives are to be evaluated at $u = u_{sve}(T)$. In this way all the coefficients of the power series in (3.3) are slowly varying functions. By substituting expansion (3.1) into (3.2) using (3.3), we are able to determine the equation for the perturbed quantity:

$$\frac{d^m \bar{u}}{dt^m} = \bar{u} \frac{\partial F}{\partial u}\bigg|_{u_{sve}} + \frac{\mu}{2} \bar{u}^2 \frac{\partial^2 F}{\partial u^2}\bigg|_{u_{sve}} + \frac{\mu^2}{6} \bar{u}^3 \frac{\partial^3 F}{\partial u^3}\bigg|_{u_{sve}} + \cdots. \tag{3.4}$$

In particular, let us briefly examine the equation for the leading order perturbation $\bar{u}_1$:

$$\frac{d^m \bar{u}_1}{dt^m} = \frac{\partial F}{\partial u}(u_{sve}, \lambda)\bar{u}_1. \tag{3.5}$$

To leading order in the outer expansion [away from a zero of $\partial F/\partial u\ (u_r, \lambda)$], $u_{sve}$ may be replaced by $u_r$. Thus in the inner limit of the asymptotic expansion of the outer solution, the coefficient $\partial F/\partial u(u_{sve}, \lambda)$ is approximately $\partial F/\partial u(u_r, \lambda)$. Recall that $\partial F/\partial u(u_r, \lambda) = 0$ is the criteria for a critical point. Thus, the inner limit of (3.5) involves the coefficient of the differential equation evolving through a zero. However, the type of zero varies for the different cases (see § 2). It is at this stage that we must distinguish between first and second order differential equations, since the behavior of these cases is drastically different, especially as the key coefficient evolves through a zero.

(a) *First-order systems.* For first-order systems ($m = 1$), we solve (3.4) as an expansion in powers of $\mu$:

$$\bar{u} = \bar{u}_1 + \mu\bar{u}_2 + \cdots. \tag{3.6}$$

The equations for $\bar{u}_1$ and $\bar{u}_2$ are

$$\frac{d\bar{u}_1}{dt} + k(T)\bar{u}_1 = 0, \tag{3.7a}$$

$$\frac{d\bar{u}_2}{dt} + k(T)\bar{u}_2 = \beta(T)\bar{u}_1^2, \tag{3.7b}$$

where for convenience we have introduced

$$k(T) = -\frac{\partial F}{\partial u}(u_{sve}(T), \lambda(T)), \tag{3.8a}$$

$$\beta(T) = \frac{1}{2}\frac{\partial^2 F}{\partial u^2}(u_{sve}(T), \lambda(T)). \tag{3.8b}$$

Equations (3.7) vary on the fast time with slowly varying coefficients. We assume initially ($t = t_0$) that $u_{sve}(T)$ is stable, and thus $k(\varepsilon t_0) > 0$. In the outer region (away from a transition point) $u_{sve}(T) = u_r(T)$ to leading order. Thus, to leading order in powers of

$\varepsilon$, $k(T)$ evolves through a zero, but $\beta(T)$ near criticality depends on the particular problem under investigation. For straight-straight bifurcation and for the jump phenomena, $\partial^2F/\partial u^2(u_r(T), \lambda(T))$ approaches $\partial^2F/\partial u^2(u_c, \lambda_c) \equiv 2\alpha_{20} \neq 0$. However, for parabolic bifurcation to leading order $\alpha_{20} = 0$, and thus $\beta(T)$ becomes small. If at $t = t_0$, $u - u_{\text{sve}} = \mu A$, then solving (3.7) yields

$$\text{(3.9a)} \qquad \bar{u}_1 = A \exp\left[-\int_{t_0}^{t} k(\varepsilon\bar{t})\, d\bar{t}\right] = A \exp\left[-\frac{1}{\varepsilon}\int_{\varepsilon t_0}^{\varepsilon t} k(s)\, ds\right],$$

$$\text{(3.9b)} \qquad \bar{u}_2 = A^2 \exp\left[-\int_{t_0}^{t} k(\varepsilon\bar{t})\, d\bar{t}\right] \int_{t_0}^{t} \beta(\varepsilon\bar{t}) \exp\left[-\int_{t_0}^{\bar{t}} k(\varepsilon\bar{\bar{t}})\, d\bar{\bar{t}}\right] d\bar{t}$$

$$= A^2 \exp\left[-\frac{1}{\varepsilon}\int_{\varepsilon t_0}^{\varepsilon t} k(s)\, ds\right] \frac{1}{\varepsilon}\int_{\varepsilon t_0}^{\varepsilon t} \beta(s) \exp\left[-\frac{1}{\varepsilon}\int_{\varepsilon t_0}^{s} k(\bar{s})\, d\bar{s}\right] ds.$$

Expansion (3.6) is not uniformly valid since eventually $k(\varepsilon t)$ becomes negative. Then $\bar{u}_1$ will grow, and in particular $\mu\bar{u}_2$ will be equally important as $\bar{u}_1$ when

$$\text{(3.10)} \qquad \frac{\mu}{\varepsilon}\int_{\varepsilon t_0}^{\varepsilon t} \beta(s) \exp\left[-\frac{1}{\varepsilon}\int_{\varepsilon t_0}^{s} k(\bar{s})\, d\bar{s}\right] ds = O(1).$$

The integrand is exponentially *small* in the *initial* stages ($\varepsilon t$ such that $k(\varepsilon t) > 0$). The breakdown (disordering) of expansion (3.6) cannot occur until $k(\varepsilon t)$ has become negative. Thus a zero of $k(\varepsilon t)$ is *not* the place at which the expansion in powers of $\mu$ breaks down; if the expansion of the slowly varying equilibrium breaks down, then this break down occurs sooner. On the time scale of this breakdown, the perturbations of the slowly varying equilibrium are not significant; in fact $\bar{u}_1$ is exponentially small in the parameter $\varepsilon$. Thus in this case the solutions nearby to $u_{\text{sve}}$ (for first-order systems) correspond to the same boundary layer scales and matching conditions as the slowly varying equilibrium itself.

Thus we can *usually* ignore the perturbations to $u_{\text{sve}}$. The only case in which we cannot neglect these perturbations is if the expansion of $u_{\text{sve}}$ around $u_r(\lambda)$ does not become disordered at all. For example, if $u = 0$ is an *exact* equilibrium for all $\lambda$, then an expansion of $u_{\text{sve}}$ is not needed. In this case solutions nearby to $u = 0$ will breakdown *after* a zero of $k$ is passed. The evolution in this case is described in Appendix A.

(b) *Second-order systems.* Again letting

$$\text{(3.11)} \qquad k(T) = -\frac{\partial F}{\partial u}(u_{\text{sve}}(T), \lambda(T))$$

we rewrite the equation for the perturbation ($\mu\bar{u} = u - u_{\text{sve}}$) to $u_{\text{sve}}(T)$:

$$\text{(3.12)} \qquad \frac{d^2\bar{u}}{dt^2} + k(T)\bar{u} = \tfrac{1}{2}\mu\bar{u}^2 \left.\frac{\partial^2 F}{\partial u^2}\right|_{u_{\text{sve}}} + \tfrac{1}{6}\mu^2\bar{u}^3 \left.\frac{\partial^3 F}{\partial u^3}\right|_{u_{\text{sve}}} + \cdots.$$

Equation (3.12) represents a slowly varying oscillator ($k > 0$) approaching a turning point as $k \to 0$ with weakly nonlinear corrections. The kind of turning point depends on the zero of $k(T)$. For all types of bifurcation problems $k(T)$ has a simple zero, but has a square root singularity for the jump transition problem. For this reason at first we analyze (3.12) for general $k(T)$.

The well-known [6], [10], [17] Liouville–Green formulas must be modified due to the nonlinear terms. The leading order term from (3.12) is a slowly varying periodic solution, where both the amplitude and phase may be slowly varying. Slow variations occur for two reasons. The coefficient $k(T)$ varies on the slow time $T = \varepsilon t$, and the nonlinearities generate secular terms causing a slow modulation of the solution on the time scale $\mu^2 t$. There are thus three cases. If the amplitude of the perturbation is sufficiently small, $\mu^2 < O(\varepsilon)$, then on the slow time $\varepsilon t = O(1)$ the nonlinear secular term has not had enough time to develop. On the other hand if $\mu^2 > O(\varepsilon)$, then the appropriate slow scale is the nonlinear one, not the time scale associated with the slowly varying coefficients. Of particular interest is the case in which $\mu^2 = O(\varepsilon)$. Then slow changes of amplitude and phase are due to both terms with variable coefficients and nonlinearities.

Any outer asymptotic expansion will become disordered as $k$ evolves through a zero. We will pick $\mu$ such that as $k$ evolves through a zero the expansion due to the nonlinear terms breaks down on the same time scale as the expansion due to the slowly varying coefficient. We will show in most cases $\mu^2 = O(\varepsilon)$. However, for straight-straight bifurcation the nonlinear terms are more important as $k \to 0$. In that case only we will show $\mu = \varepsilon^{5/6}$ is the scale at which the expansions simultaneously break down.

We use the method of multiple scales (more specifically two-timing) with the usual fast time $\eta$,

$$\eta = \frac{g(\varepsilon t)}{\varepsilon} = \frac{\int^{\varepsilon t} k^{1/2}(s)\, ds}{\varepsilon}, \tag{3.13}$$

and slow time $T$,

$$T = \varepsilon t. \tag{3.14}$$

Time derivatives are transformed by the rule $d/dt = (g')\partial/\partial\eta + (\varepsilon)\partial/\partial T$, and hence (3.12) becomes

$$k(\bar{u}_{\eta\eta} + \bar{u}) = -\varepsilon(2g'\bar{u}_{\eta T} + g''\bar{u}_\eta) - \varepsilon^2 \bar{u}_{TT} + \mu \frac{\bar{u}^2}{2} \frac{\partial^2 F}{\partial u^2}\bigg|_{u_{\text{sve}}} + \mu^2 \frac{\bar{u}^3}{6} \frac{\partial^3 F}{\partial u^3}\bigg|_{u_{\text{sve}}} + \cdots. \tag{3.15}$$

Letting $\bar{u} = \bar{u}_1 + \cdots$, we see that the leading order term is

$$\bar{u}_1 = A_0(T)\, e^{i(\eta+\phi_0)} + A_0(T)\, e^{-i(\eta+\phi_0)}. \tag{3.16}$$

As $k \to 0$ proportional to some power of $T$, the order of magnitude of variations of the phase $\eta$ may be shown to be $Tk^{1/2}(T)/\varepsilon$. This phase variation will be $O(1)$ if $k^{1/2}T = O(\varepsilon)$. Furthermore, this same scale is suggested by the disordering as $k \to 0$ of the Liouville–Green expansion based on (3.15) when $\mu = 0$ (the small amplitude limit corresponding to a linear turning point problem). We will pick $\mu$ such that the nonlinear terms also become disordered when $k^{1/2}T = O(\varepsilon)$. If $k$ has a simple zero, the correct transition layer scaling is characterized by $T = O(\varepsilon^{2/3})$. This corresponds to all types of bifurcations. However for the jump phenomena $k \sim -2\alpha_{20}\sigma(\lambda_c' T)^{1/2}$ (from Table 2) and hence the transition scaling is $T = O(\varepsilon^{4/5})$.

To determine the dependence of $\mu$ on $\varepsilon$, we will first briefly calculate nonlinear effects from (3.15) assuming $\mu$ is much smaller than any power of $\varepsilon$. This calculation is to be used only as a motivation for the choice of $\mu$. Once $\mu$ is chosen, the expansion of (3.15) must be done correctly. We let

$$\bar{u} = \bar{u}_1 + \mu\bar{u}_2 + \mu^2\bar{u}_3 + \cdots, \tag{3.17}$$

where $\bar{u}_1$ is obtained by eliminating secular terms at $O(\varepsilon)$:

$$\bar{u}_1 = ck^{-1/4}(e^{i(\eta+\phi_0)} + e^{-i(\eta+\phi_0)}), \tag{3.18}$$

where $c$ and $\phi_0$ are constants. By substituting (3.18) into the expansion of (3.15) we determine equations for the nonlinear corrections $\bar{u}_2, \bar{u}_3, \bar{u}_4$. Although additional secular terms occur at $O(\mu^2)$, these are no more important than the higher harmonics, since this expansion is analyzed at the time of disordering which we have already shown corresponds to $\eta = O(1)$, not $\eta$ large. Solving $\bar{u}_2, \bar{u}_3, \bar{u}_4$ successively, we obtain *among other terms* as $k \to 0$

$$\bar{u}_2 = c^2 e^{2i(\eta+\phi_0)} \tfrac{1}{2}k^{-3/2} \frac{\partial^2 F}{\partial u^2}\bigg|_{u_{\text{sve}}} + \cdots,$$

$$\bar{u}_3 = c^3 e^{3i(\eta+\phi_0)}\left[\tfrac{1}{2} k^{-11/4}\left(\frac{\partial^2 F}{\partial u^2}\bigg|_{u_{\text{sve}}}\right)^2 + \tfrac{1}{6} k^{-7/4} \frac{\partial^3 F}{\partial u^3}\bigg|_{u_{\text{sve}}}\right] + \cdots,$$

$$\bar{u}_4 = c^4 e^{4i(\eta+\phi_0)}\left[\frac{5}{8} k^{-4}\left(\frac{\partial^2 F}{\partial u^2}\bigg|_{u_{\text{sve}}}\right)^3 + \tfrac{5}{12}k^{-3} \frac{\partial^2 F}{\partial u^2}\bigg|_{u_{\text{sve}}}\frac{\partial^3 F}{\partial u^3}\bigg|_{u_{\text{sve}}} + \tfrac{1}{24} k^{-2} \frac{\partial^4 F}{\partial u^4}\bigg|_{u_{\text{sve}}}\right] + \cdots.$$

These terms represent higher harmonics generated by the nonlinear terms. As $k \to 0$ they are more singular than the fundamental.

The type of breakdown of the expansion in powers of $\mu$ depends on the different cases. In particular $\partial^2 F/\partial u^2|_{u_{\text{sve}}}$ is important (as it was for first-order systems). For straight-straight bifurcation and the jump phenomena $\partial^2 F/\partial u^2(u_c, \lambda_c) \equiv 2\alpha_{20} \neq 0$. However, for parabolic bifurcation $\alpha_{20} = 0$, and thus $\partial^2 F/\partial u^2(u_{\text{sve}}, \lambda)$ becomes small. More accurately for parabolic bifurcation the behavior of $\partial^2 F/\partial u^2|_{u_{\text{sve}}}$ depends on whether the transition is from a straight line equilibrium to a parabolic curve or vice versa. The results for the various cases follow.

For straight-straight bifurcation, the terms for the $u - u_c$ are of order $T$, $\mu T^{-1/4}$, $\mu^2 T^{-6/4}$, $\mu^3 T^{-11/4}$, $\mu^4 T^{-16/4}, \cdots$. Consequently breakdown occurs when $T = O(\mu^{4/5})$. Since we have already determined that $T = O(\varepsilon^{2/3})$, we choose $\mu = \varepsilon^{5/6}$ in order for the two types of expansions to breakdown simultaneously for straight-straight bifurcation.

For jump transitions, the terms for $u - u_c$ are of order $T^{1/2}$, $\mu T^{-1/8}$, $\mu^2 T^{-6/8}$, $\mu^3 T^{-11/8}, \cdots$. Consequently breakdown occurs when $T = O(\mu^{8/5})$. For the jump phenomena, $T = O(\varepsilon^{4/5})$, and thus as previously described we determine $\mu$, $\mu = \varepsilon^{1/2}$.

However, for parabolic bifurcation there are two cases. For the evolution of straight towards parabolic, the terms in the expansion (in powers of $\mu$) of $u - u_c$ are $T, \mu T^{-1/4}$, $\mu^2 T^{-2/4}$, $\mu^3 T^{-7/4}$, $\mu^4 T^{-8/4}, \cdots$. This is a type of disordering we call "leap-frog" disordering in which each successive term is more singular, but the power of increase of the singularity alternates (from $T^{-1/4}$ to $T^{-5/4}$). Scales based on $\mu T^{-1/4} = O(1)$ and $\mu T^{-5/4} = O(1)$ are *not* appropriate since in either case only two successive terms become equally important. Instead, an infinite number of terms (both the even and odd terms) simultaneously become of the same order if we balance in a leap-frog fashion, $\mu^2 T^{-6/4} = O(1)$. Thus we categorize breakdown by $T = O(\mu^{4/3})$. Here we know $T = O(\varepsilon^{2/3})$ and thus $\mu = \varepsilon^{1/2}$. This leap-frog type analysis would have been avoided if we had incorrectly taken $\partial^2 F/\partial u^2|_{u_{\text{sve}}} = 0$, rather than it being small. Then $\bar{u}_2 = 0$, but $\bar{u}_3$ exists because $\partial^3 F/\partial u^3|_{u_{\text{sve}}} = O(1)$. In this way $\bar{u}_3 = O(k^{-1}\bar{u}_1^3)$ as $k \to 0$. Using these terms to determine breakdown (with $\bar{u}_1 = O(k^{-1/4})$), it follows that $\mu^2 k^{-7/4} = O(k^{-1/4})$. Since $k = O(T)$ as $T \to 0$, we conclude as before that $T = O(\mu^{4/3})$. Thus the leap-frog effect is caused by the smallness of $\partial^2 F/\partial u^2|_{u_{\text{sve}}}$ as $T \to 0$.

For bifurcations that represent a transition from a parabolic equilibrium curve to a straight line, the order of magnitude of the terms representing $u - u_c$ are $T^{1/2}$, $\mu T^{-1/4}$, $\mu^2 T^{-1}$, $\mu^3 T^{-7/4}, \cdots$. Thus the transition layer occurs when $T = O(\mu^{4/3})$ as with the other direction of parabolic bifurcation. Here we know $T = O(\varepsilon^{2/3})$ and hence again

$\mu=\varepsilon^{1/2}$. The scalings for both directions of parabolic bifurcation are the same even though the asymptotic matching conditions will be different.

For straight-straight bifurcation, $\mu=\varepsilon^{5/6}$ and the matching will be that of a linear turning point problem,

$$\bar{u}\sim 2ck^{-1/4}\cos\left(\frac{\int^{t}k^{1/2}(\varepsilon\bar{t})\,d\bar{t}}{\varepsilon}+\theta\right), \tag{3.19}$$

where $c$ and $\theta$ are constants. Since as $T\to 0$, $k\sim-\alpha_{20}(\beta_1-\beta_2)\tilde{\lambda}$ and $\tilde{\lambda}\sim\lambda_c'T$, it follows that

$$u\sim u_c+\beta_1\lambda_c'T+\cdots+2c\varepsilon^{5/6}\gamma^{-1/4}(-T)^{-1/4}\cos\left(-\gamma^{1/2}\frac{2}{3\varepsilon}(-T)^{3/2}+\theta\right), \tag{3.20}$$

as $T\to 0$, where $\gamma=\alpha_{20}(\beta_1-\beta_2)\lambda_c'>0$.

For all other cases, $\mu=\varepsilon^{1/2}$ and we must analyze the asymptotics of weakly nonlinear transition point problems. With $\mu=\varepsilon^{1/2}$, (3.15) becomes

$$\begin{aligned} k(\bar{u}_{\eta\eta}+\bar{u})=\varepsilon^{1/2}\frac{\bar{u}^2}{2}\frac{\partial^2F}{\partial u^2}\bigg|_{u_{\text{sve}}}&+\varepsilon\left(\frac{\bar{u}^3}{6}\frac{\partial^3F}{\partial u^3}\bigg|_{u_{\text{sve}}}-2g'\bar{u}_{\eta T}-g''\bar{u}_{\eta}\right)\\ &+\varepsilon^{3/2}\frac{\bar{u}^4}{4!}\frac{\partial^4F}{\partial u^4}\bigg|_{u_{\text{sve}}}+\varepsilon^2\left(\frac{\bar{u}^5}{5!}\frac{\partial^5F}{\partial u^5}\bigg|_{u_{\text{sve}}}-\bar{u}_{TT}\right)+\cdots, \end{aligned} \tag{3.21}$$

where $g'=k^{1/2}$. Equation (3.21) is similar to one analyzed by Haberman [7]. There the quadratic nonlinear terms (and in general all the even terms) identically vanished. We look for an expansion (valid away from the transition point) of the form

$$\bar{u}=\bar{u}_1+\varepsilon^{1/2}\bar{u}_2+\varepsilon\bar{u}_3+\cdots, \tag{3.22}$$

where the $O(1)$ equation implies that

$$\bar{u}_1=A_0\,e^{i(\eta+\phi_0)}+A_0\,e^{-i(\eta+\phi_0)}. \tag{3.23}$$

It is most important to determine the slow time variation of the amplitude $2A_0$ and the phase $\phi_0$. $\bar{u}_2$, the $O(\varepsilon^{1/2})$ correction, is due to the quadratic nonlinearity. No fundamental terms are generated at this order, only a mean term and second harmonic occur:

$$u_2=k^{-1}A_0^2\frac{\partial^2F}{\partial u^2}\bigg|_{u_{\text{sve}}}\left(\frac{1}{2}-\frac{e^{2i(\eta+\phi_0)}+e^{-2i(\eta+\phi_0)}}{6}\right). \tag{3.24}$$

The slow variation of the amplitude and phase are determined by eliminating secular terms at $O(\varepsilon)$:

$$k(\bar{u}_{3_{\eta\eta}}+\bar{u}_3)=\frac{1}{3!}\frac{\partial^3F}{\partial u^3}\bigg|_{u_{\text{sve}}}\bar{u}_1^3+\frac{1}{2}\frac{\partial^2F}{\partial u^2}\bigg|_{u_{\text{sve}}}(2\bar{u}_1\bar{u}_2)-(2g'\bar{u}_{1_{\eta T}}+g''\bar{u}_{1_{\eta}}). \tag{3.25}$$

In addition to third harmonic terms, standard secular terms of the form $e^{i(\eta+\phi_0)}$ and $e^{-i(\eta+\phi_0)}$ are generated. We only need to eliminate the secular terms corresponding to $e^{i(\eta+\phi_0)}$ since the other equation is equivalent. The vanishing of the secular terms which are real multiples of $e^{i(\eta+\phi_0)}$ yields

$$-2g'A_0\phi_0'=A_0^3\left[3\frac{\partial^3F}{\partial u^3}\bigg|_{u_{\text{sve}}}+2k^{-1}\left(\frac{\partial^2F}{\partial u^2}\bigg|_{u_{\text{sve}}}\right)^2\right]\Big/6, \tag{3.26}$$

while the terms which are imaginary multiples yield

$$2g'A_0' + g''A_0 = 0. \tag{3.27}$$

The latter equation (3.27) implies that

$$A_0 = \frac{c}{k^{1/4}}, \tag{3.28}$$

where $c$ is an arbitrary constant. This is the standard linear Liouville–Green result, not influenced by the nonlinearity. The effect of the nonlinearity is only to alter the phase. The modulation of the phase follows from (3.26):

$$\phi_0' = -A_0^2 k^{-1/2}\left[3\left.\frac{\partial^3 F}{\partial u^3}\right|_{u_{\text{sve}}} + 2k^{-1}\left(\left.\frac{\partial^2 F}{\partial u^2}\right|_{u_{\text{sve}}}\right)^2\right]\bigg/12. \tag{3.29}$$

For either direction of parabolic bifurcation, as $k \to 0$ $|2k^{-1}(\partial^2 F/\partial u^2|_{u_{\text{sve}}})^2| \ll |3\partial^3 F/\partial u^3|_{u_{\text{sve}}}|$, and hence as $T \to 0$

$$\phi_0' \sim -3c^2\alpha_{30}k^{-1}/2. \tag{3.30}$$

Since as $T \to 0$, $\tilde{\lambda} \sim \lambda_c' T$ and $k \sim \delta\alpha_{30}\sigma^2\tilde{\lambda}$ (where for $s \to p$, $\delta = -1$, while for $p \to s$, $\delta = 2$), it follows that

$$\phi_0 \sim \frac{-3c^2}{2\delta\sigma^2\lambda_c'} \ln|T| + \theta, \tag{3.31}$$

where $\theta$ is a constant phase. Consequently, the matching condition for parabolic bifurcation is that as $T \to 0$

$$u \sim u_{\text{sve}} + 2c\varepsilon^{1/2}\gamma^{-1/4}(-T)^{-1/4}\cos\left(-\frac{2}{3\varepsilon}\gamma^{1/2}(-T)^{3/2} + \frac{3c^2\alpha_{30}}{2\gamma}\ln|T| + \theta\right) + \cdots, \tag{3.32}$$

where $\gamma = -\delta\alpha_{30}\sigma^2\lambda_c' = \delta\alpha_{11}\lambda_c' > 0$ and where for $s \to p$, $\delta = -1$ and $u_{\text{sve}} \sim u_c + \beta_1\lambda_c' T + \cdots$, while for $p \to s$, $\delta = 2$ and $u_{\text{sve}} \sim u_c + \sigma(\lambda_c' T)^{1/2} + \cdots$.

For the jump phenomena, as $k \to 0$, $|2k^{-1}(\partial^2 F/\partial u^2|_{u_{\text{sve}}})^2| \gg |3\partial^3 F/\partial u^3|_{u_{\text{sve}}}|$, and hence as $T \to 0$

$$\phi_0' \sim -2c^2\alpha_{20}^2 k^{-2}/3. \tag{3.33}$$

Since as $T \to 0$, $k \sim -2\sigma\alpha_{20}\tilde{\lambda}^{1/2}$ and $\tilde{\lambda} \sim \lambda_c' T$, we conclude that

$$\phi_0 \sim -\frac{c^2}{6\sigma^2\lambda_c'}\ln|T| + \theta, \tag{3.34}$$

as $T \to 0$, where $\theta$ is a constant. Consequently, the matching condition when a jump transition is expected is

$$\begin{aligned} u \sim\; & u_c + \sigma(\lambda_c' T)^{1/2} + \cdots \\ & + 2c\varepsilon^{1/2}\gamma^{-1/4}(-T)^{-1/8}\cos\left(-\gamma^{1/2}\frac{4}{5\varepsilon}(-T)^{5/4} + \frac{2c^2\alpha_{20}^2}{3\gamma^2}\ln|T| + \theta\right), \end{aligned} \tag{3.35}$$

where $\gamma = -2\sigma\alpha_{20}(-\lambda_c')^{1/2} = 2(\alpha_{01}\alpha_{20}\lambda_c')^{1/2}$.

The matching condition includes a slowly varying oscillatory term which for all cases except straight-straight bifurcation has a logarithmic phase singularity due to the weakly nonlinear interactions. In these case as $T \to 0$, the circular frequency approaches infinity instead of approaching zero as the linear terms suggest.

**4. Second-order jump phenomena.** Here we analyze

$$\frac{d^2u}{dt^2}=F(u,\lambda) \tag{4.1}$$

in the case in which an equilibrium curve turns around (as illustrated by Fig. 2a), i.e., $du_r/d\lambda=\infty$ at $\lambda=\lambda_c$. We assume the critical equilibrium curve is locally of the simplest type with $du_r/d\lambda=\infty$, a parabola, i.e., $\alpha_{01}=F_\lambda(u_c,\lambda_c)\neq 0$ and $\alpha_{20}=\frac{1}{2}F_{uu}(u_c,\lambda_c)\neq 0$:

$$F(u,\lambda)=\alpha_{20}(\tilde{u}^2-\sigma^2\tilde{\lambda}+\cdots)=\alpha_{01}\tilde{\lambda}+\alpha_{20}\tilde{u}^2+\cdots, \tag{4.2}$$

in the neighborhood of $u=u_c$ and $\lambda=\lambda_c$. $\alpha_{20}<0$ so that $u_r(\lambda)$ is stable for $u>u_c$, and $\alpha_{01}>0$ in order for the equilibrium to exist for $\lambda>\lambda_c$ (see Fig. 2a).

We assume initially $\lambda>\lambda_c$ and investigate what happens as $\lambda$ is slowly decreased. In §§ 2 and 3 we assumed that the solution $u$ is near the upper branch of $u_r(\lambda)$. In particular we calculated the slowly varying equilibrium solution $u_{\text{sve}}(\lambda)$ and considered perturbations to it which were shown to be oscillatory in nature. These outer asymptotic expansions become disordered when

$$T=\varepsilon^{4/5}z, \tag{4.3}$$

where $z$ is an inner transition variable. Breakdown occurs near the critical time based on the slow scale (since $T$ is small), but the fast time $t$ is still large (since $t=T/\varepsilon=\varepsilon^{-1/5}z$).

Matching (see (2.13) and (3.35)) then implies that the local inner equation follows from the scaling

$$u=u_c+\varepsilon^{2/5}w(z). \tag{4.4}$$

Upon making these scale changes, we find that (4.1) becomes:

$$\frac{d^2w}{dz^2}=\alpha_{01}\lambda_c'z+\alpha_{20}w^2+\varepsilon^{2/5}(\alpha_{30}w^3+\alpha_{11}zw)+O(\varepsilon^{4/5}). \tag{4.5}$$

Thus we introduce the inner expansion

$$w=w_0+\varepsilon^{2/5}w_1+\cdots. \tag{4.6}$$

To leading order, not surprisingly, the inner equation is

$$\frac{d^2w_0}{dz^2}=\alpha_{01}\lambda_c'z+\alpha_{20}w_0^2, \tag{4.7}$$

with $\alpha_{01}\lambda_c'<0$ and $\alpha_{20}<0$. Equation (4.7) is the leading order local form of the original problem (4.1). We note that (4.7) is the first Painlevé transcendent (Ince [8]). Equation (4.7) must be solved with the matching condition (following from (3.35)):

$$\begin{aligned} w_0 &\sim (-\alpha_{01}\lambda_c'z/\alpha_{20})^{1/2}+\cdots \\ &+2c\gamma^{-1/4}(-z)^{-1/8}\cos\left(-\tfrac{4}{5}\gamma^{1/2}|z|^{5/4}+\frac{2c^2\alpha_{20}^2}{3\gamma^2}(\ln|z|+\tfrac{4}{5}\ln\varepsilon)+\theta\right)+\cdots \\ &\text{as } z\to-\infty, \end{aligned} \tag{4.8}$$

where $c$, $\theta$ are the amplitude and phase (determined by initial conditions) and $\gamma=2(\alpha_{01}\alpha_{20}\lambda_c')^{1/2}$. Matching condition (4.8) is appropriate for (4.7) as can be seen by introducing weakly nonlinear modifications of the oscillatory perturbations to $w_0\sim(-\alpha_{01}\lambda_c'z/\alpha_{20})^{1/2}$ as $z\to-\infty$. Equation (4.8) represents many terms of the outer expansion expanded to one term of the inner expansion. Higher order terms (in powers of $\varepsilon$) will be necessary. We therefore note the $O(\varepsilon^{2/5})$ equation:

$$\frac{d^2 w_1}{dz^2} = 2\alpha_{20} w_0 w_1 + (\alpha_{30} w_0^3 + \alpha_{11} z w_0). \tag{4.9}$$

We have shown that near the time at which a jump phenomena may be expected, equation (4.7) governs with matching condition (as $z \to -\infty$) given by (4.8). However, all solutions to (4.7) explode towards $-\infty$ at a finite value of $z$ (before $z$ approaches $+\infty$). In particular for the first Painlevé transcendent, the type of singularity is

$$w_0 \sim \frac{6/\alpha_{20}}{(z-z_0)^2}, \tag{4.10}$$

where $z = z_0$ is the time of the singularity. $z_0$ depends on the specific asymptotic condition as $z \to -\infty$, i.e., it depends on $c$ and $\theta$ (see (4.8)). This as yet unknown information would require further analysis (most likely numerical) of the first Painlevé transcendent.

The significance of this singularity is that the inner expansion, governed to leading order by (4.7), cannot be valid as time increases near the singularity $z = z_0$. This can be seen by showing that the higher order corrections to $w_0$ are more singular as $z \to z_0$. In particular, from (4.9)

$$w_1 = O((z-z_0)^{-4}). \tag{4.11}$$

Thus, the transition layer expansion (4.6) breaks down and must match to some other solution as $z \to z_0$. We will show that in this new outer region changes in $t$ are $O(1)$. Thus the inner expansion (see equations (4.3)–(4.6)) represents an interior boundary layer in time which matches as $z \to -\infty$ and as $z \to z_0$ to outer expansions.

We follow the approach used by Haberman [7]. A local scaling of $z$ in the neighborhood of $z_0$ is necessary (see (4.10) and (4.11)):

$$z - z_0 = \varepsilon^{1/5} \tau. \tag{4.12}$$

The type of explosion (4.10) then suggests $w = O(\varepsilon^{-2/5})$, and therefore from (4.4), $u = O(1)$. Thus the explosion has transformed $u$ from being nearly $u_c$ to differing from $u_c$ by an order one quantity. Furthermore the time scale (4.12) corresponds to

$$t = \varepsilon^{-1/5} z = \varepsilon^{-1/5} z_0 + \tau. \tag{4.13}$$

Thus $\tau$ is time shifted by a large fixed amount. However, the slow variable $T = \varepsilon t = \varepsilon^{4/5} z_0 + \varepsilon\tau$ is still small (and nearly the small constant $\varepsilon^{4/5} z_0$). For this reason in this scale, $u$ satisfies (4.1) with to leading order $\lambda = \lambda_c$ (the critical value):

$$\frac{d^2 u}{d\tau^2} = F(u, \lambda_c). \tag{4.14}$$

Autonomous nonlinear system (4.14) must be solved with the matching condition as $\tau \to -\infty$ (corresponding to $z \to z_0$)

$$u \sim u_c + \frac{6/\alpha_{20}}{\tau^2} + \cdots. \tag{4.15}$$

(4.15) represents the special trajectory (separatrix) of (4.14) sketched in Fig. 2. (If there is only one other equilibrium, then Fig. 2a applies.) Illustrated is the equilibrium solution $u_c$ corresponding to the coalescence of two equilibria. Near $u = u_c$, $F(u, \lambda_c) \sim$

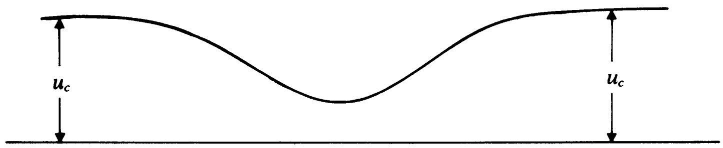



FIG 6. *Solitary wave—the infinite period limit of the nonlinear oscillator.*

$\alpha_{20}(u-u_c)^2$ ($\alpha_{20}<0$). Thus linear stability analysis as well as a linear phase plane analysis in the neighborhood of the equilibrium is not appropriate. Instead $u=u_c$ is a higher order singular point of (4.14). In general, from the phase plane the exact solution of (4.14) is thus a single-humped solitary wave which algebraically decays to $u_c$ asymptotically as $\tau\to\pm\infty$, as sketched in Fig. 6. It takes an infinite amount of time ($\tau\to\infty$) for $u\to u_c$ according to (4.14). However, the coefficients of (4.14) actually change slowly since $T=\varepsilon^{4/5}z_0+\varepsilon\tau$ implies that

$$\frac{d^2u}{d\tau^2}=F(u,\lambda(\varepsilon^{4/5}z_0+\varepsilon\tau)). \tag{4.16}$$

Thus before $u$ again reaches $u_c$, the bifurcation parameter will have become less than $\lambda_c$. In other words, as one "cycle" in the phase plane is approached along the infinite-period separatrix for (4.14) with $\lambda=\lambda_c$, the tail of the solitary wave ($\tau\to+\infty$) is also not a uniformly valid solution of (4.16). It can be shown that it matches another nearly solitary wave. Thus we may think of the solution as evolving into a series of solitary waves. Each solitary wave is one cycle of the periodic solution suggested by Fig. 2a (for $\lambda<\lambda_c$) in the limit of an extremely long period.

Instead of analyzing (4.16) as a perturbation expansion in powers of $\varepsilon^{4/5}$ and $\varepsilon$, as we have just done qualitatively, it is better to consider (4.16) directly as a slowly varying nonlinear system. The asymptotically invalid solution in the transition region will match to a slowly varying periodic solution. For $\lambda<\lambda_c$, Fig. 2a implies there is a periodic solution (if $\lambda$ is constant), and the work of Kuzmak [13] (also described in Cole [6] and Nayfeh [17]) shows how to construct the corresponding slowly varying periodic solution when $\lambda$ slowly varies. Using the ideas of Kuzmak [13] a slowly varying periodic solution (oscillating around $u=u_{E_1}(\lambda)$) of (4.16) is sought depending on two time scales, a slow time $T$ and an as yet unknown fast time $t_f$:

$$T=\varepsilon^{4/5}z_0+\varepsilon\tau, \qquad t_f=Q(T)/\varepsilon. \tag{4.17}$$

Derivatives transform according to the rule

$$\frac{\partial}{\partial\tau}=Q'(T)\frac{\partial}{\partial t_f}+\varepsilon\frac{\partial}{\partial T}. \tag{4.18}$$

A perturbation expansion is assumed

$$u=u_0+\varepsilon u_1+\varepsilon^2u_2+\cdots. \tag{4.19}$$

We introduce the slowly varying potential energy $G(u,\lambda)$,

$$F(u,\lambda(T))=-\frac{\partial G}{\partial u}, \tag{4.20}$$

and sketch potential energy curves in Fig. 7 for $\lambda<\lambda_c$ and $\lambda=\lambda_c$. To leading order, (4.16) is integrated to yield conservation of energy

$$\tfrac{1}{2}(Q')^2u_{0t_f}^2+G(u_0,\lambda(T))=E(T), \tag{4.21}$$

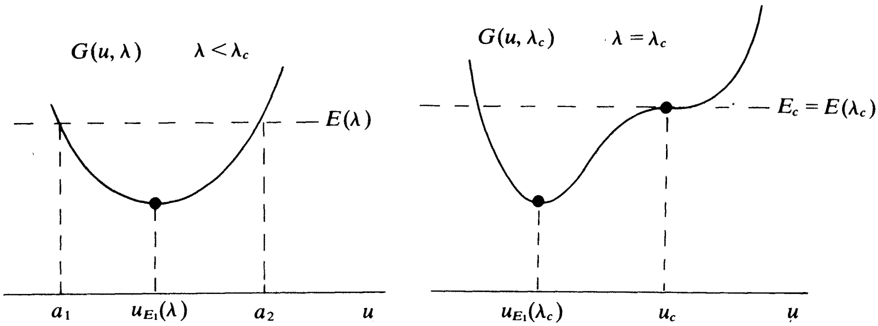



FIG. 7. *Slowly varying potential energy*: $\lambda < \lambda_c$ (*Fig.* 7a) *and* $\lambda = \lambda_c$ (*Fig.* 7b).

where $E(T)$ is the slowly varying energy level. Kuzmak [13] shows that asymptotic expansion (4.19) will not be valid unless the fast time scale is chosen so that the period of the fast time oscillation is constant (rather than slowly varying). If the period is normalized to unity, then (4.21) implies that

$$Q' = \frac{\sqrt{2}/2}{\int_{a_1(T)}^{a_2(T)} du_0/\sqrt{E-G}}, \tag{4.22}$$

where $a_1(T)$ and $a_2(T)$ are the zeros of $E - G$ (sketched in Fig. 7). $a_1(T)$ and $a_2(T)$ are determined only by $E(T)$, and they are the lower and upper limits respectively of the amplitude of the nonlinear oscillator. We are most interested in determinining how the energy level $E(T)$ slowly varies due to the slow variation of $\lambda(T)$. If we insist that perturbation expansion (4.19) is asymptotically valid and hence has no secular terms, then $u_1$ must be periodic (on the fast scale). Performing this calculation (see Kuzmak [13] or Cole [6]) shows that

$$\int_{a_1(T)}^{a_2(T)} \sqrt{E(T) - G(u_0, \lambda(T))}\, du_0 = \text{const.} \tag{4.23}$$

Equation (4.23) determines the slow variation of $E(T)$, and $u_0$ is obtained from (4.21). This analysis is valid as long as the actual period is finite (i.e., $Q' \neq 0$ which is guaranteed if $\lambda < \lambda_c$). However, the solution $u_0$ must match as $T \to 0$ to the tail of a solitary wave given by (4.15). This determines the constant in (4.23) because as $T \to 0$ the constant must be the critical one corresponding to the energy level with a nonperiodic solution as illustrated in Fig. 7. Thus

$$\int_{a_1(T)}^{a_2(T)} \sqrt{E(T) - G(u_0, \lambda(T))}\, du_0 = \int_{a_1(0)}^{a_2(0)} \sqrt{E_c - G(u_0, \lambda_c)}\, du_0, \tag{4.24}$$

since $\lambda(0) = \lambda_c$, $a_2(0) = u_c$, and $E(0) = G(u_c, \lambda_c) \equiv E_c$. Then $u_0$ given by (4.21) will match as $T \to 0$ to the solitary tail, (4.15) or (4.10) and (4.11). Johnson [9] and later Ablowitz and Segur [2] and Miles [16], all based on the work of Kuzmak [13], first presented these ideas of how a slowly varying periodic solution matches to a solitary wave solution.

In summary, the work of Kuzmak [13] assumes the solution remains periodic, in particular that the period is finite as slow variations occur. In our case, as $\lambda \to \lambda_c$, Kuzmak's solution must match to the infinite period limit suggested by (4.14). Kuzmak's expansion is not uniformly valid as $\lambda \to \lambda_c$, but matches to the exploding solution of the first Painlevé transcendent, (4.10) and (4.11). Thus we are able to sketch in Fig. 3 how *the first Painlevé transcendent provides the jump transition from a slowly varying* *equilibrium solution to a slowly varying periodic nonlinear wave oscillating around the new stable equilibrium.* The solution does not approach the new equilibrium. Only a damping phenomena (not present in these second-order models) would force $u$ to decay towards the new equilibrium.

**5. First-order jump phenomena.** Here we consider the first-order equation,

$$\frac{du}{dt}=F(u,\lambda), \tag{5.1}$$

in the case in which the reduced equilibrium turns back (as illustrated by Fig. 2a). As in § 4, we assume that the equilibrium curve is locally a parabola:

$$F(u,\lambda)=\alpha_{20}(\tilde{u}^2-\sigma^2\tilde{\lambda}+\cdots)=\alpha_{01}\tilde{\lambda}+\alpha_{20}\tilde{u}^2+\cdots, \tag{5.2}$$

where $\alpha_{20}<0$ and $\alpha_{01}>0$ for the reasons expressed in § 4.

We investigate what happens as $\lambda$ is slowly decreased. We have assumed in §§ 2 and 3 that the solution $u$ is near the upper branch of $u_r(\lambda)$. We calculated the slowly varying equilibrium solution $u_{\text{sve}}(\lambda)$ and showed that perturbations to it are exponentially small (and can be ignored in the matching). The transition variable $z$ is introduced,

$$T=\varepsilon^{2/3}z, \tag{5.3}$$

when the outer expansion becomes disordered. On this scale $T$ is small, but the fast time $t$ is still large (since $t=T/\varepsilon=\varepsilon^{-1/3}z$).

The method of matched asymptotic expansions (using (2.13)) then implies that the inner equations follow from the scaling

$$u=u_c+\varepsilon^{1/3}w(z). \tag{5.4}$$

On making these scale changes, (5.1) becomes

$$\frac{dw}{dz}=\alpha_{01}\lambda_c'z+\alpha_{20}w^2+\varepsilon^{1/3}(\alpha_{30}w^3+\alpha_{02}\lambda_c'^2z^2)+O(\varepsilon^{2/3}). \tag{5.5}$$

Because of this, we assume an inner expansion of the following form:

$$w=w_0+\varepsilon^{1/3}w_1+\cdots. \tag{5.6}$$

To leading order, the inner equation is the local form of (5.1):

$$\frac{dw_0}{dz}=\alpha_{01}\lambda_c'z+\alpha_{20}w_0^2. \tag{5.7}$$

We note that (5.7) is an example of an exactly solvable Riccati equation (Ince [8, pp. 23–25]). It must be solved with the matching condition (following from (2.13)):

$$w_0\sim(-\alpha_{01}\lambda_c'z/\alpha_{20})^{1/2},\quad \text{as } z\to-\infty. \tag{5.8}$$

Higher order terms will be necessary. We therefore calculate the $O(\varepsilon^{1/3})$ equation:

$$\frac{dw_1}{dz}=2\alpha_{20}w_0w_1+\alpha_{30}w_0^3+\alpha_{02}\lambda_c'^2z^2. \tag{5.9}$$

We have shown that near the time at which a jump phenomena may be expected, equation (5.7) governs with matching condition (as $z\to-\infty$) given by (5.8). However, all solutions to (5.7) have a singularity at a finite value of $z$; they explode before $z$

approaches $+\infty$. In particular as $z \to z_0$,

$$w_0 \sim \frac{-1/\alpha_{20}}{z - z_0}, \tag{5.10}$$

where $z = z_0$ is the as yet unknown time of the singularity. It is interesting to note that we are able to determine $z_0$ (unlike the corresponding problem for the first Painlevé transcendent, (4.10)). Under the transformation (Ince [8], also used by Cole [6] to analyze (5.7) for transitions for the van der Pol relaxation oscillator)

$$w_0 = \left(\frac{\alpha_{01}\lambda_c'}{\alpha_{20}^2}\right)^{1/3} \frac{1}{\phi}\frac{d\phi}{dx},$$

with the scaling $x = -(\alpha_{01}\alpha_{20}\lambda_c')^{1/3} z$, Riccati equation (5.7) becomes Airy's differential equation $d^2\phi/dx^2 = x\phi$, whose solution is $\phi = c_1\mathrm{Ai}(x) + c_2\mathrm{Bi}(x)$. Thus the general solution of (5.7) is

$$w_0 = \left(\frac{\alpha_{01}\lambda_c'}{\alpha_{20}^2}\right)^{1/3} \frac{c_1\mathrm{Ai}'(x) + c_2\mathrm{Bi}'(x)}{c_1\mathrm{Ai}(x) + c_2\mathrm{Bi}(x)}.$$

The matching condition (5.8) can be satisfied only if $c_2 = 0$ (using the asymptotics of the Airy functions, for example given in Abramowitz and Stegun [4]). This explicitly shows that there will be a simple pole at the first zero of the Airy function [4], $x = -2.33810\ldots$. Thus the position of the singularity is

$$z_0 = \frac{2.33810\ldots}{(\alpha_{01}\alpha_{20}\lambda_c')^{1/3}}.$$

The inner expansion, governed to leading order by (5.7), cannot be valid for times near the singularity $z = z_0$. This can be seen by showing that the higher order corrections to $w_0$ are more singular as $z \to z_0$:

$$w_1 = O\left(\frac{\ln|z - z_0|}{(z - z_0)^2}\right). \tag{5.11}$$

The singularity (5.10) that occurs for first-order systems is quite similar to that which occurs for second-order systems (see § 4). The disordering of the expansion, on the other hand, is quite different. The logarithmic term in (5.11) occurs because the equation for $w_1$ is forced by a homogeneous solution. Higher order terms continue to be dominated by logarithmic expressions. However, this is somewhat deceptive with respect to analyzing the breakdown of this inner problem. We can procede more directly to the answer by analyzing $z(w)$ in the neighborhood of $z = z_0$ rather than $w(z)$. Then instead of (5.6)

$$z = z^{(0)} + \varepsilon^{1/3} z^{(1)} + \varepsilon^{2/3} z^{(2)} + \cdots. \tag{5.12}$$

On redoing the calculations and approximations in the neighborhood of the singularity, we obtain the following for $w$ large: $z^{(0)} = z_0 - 1/(\alpha_{20}w) + \cdots$, $z^{(1)} = O(\ln|w|)$, $z$ $O(w)$, $z^{(3)} = O(w^2)$, $z^{(4)} = O(w^3)$, etc. We see the appearance of the logarithmic term as before. However, the primary breakdown of (5.12) occurs when $w = O(\varepsilon^{-1/3})$. Consequently, $u = O(1)$. We thus rescale on this basis noting that $z$ is nearly a constant, and hence (as with second-order systems) $T$ is nearly zero. In the resulting scale,

$$t = \varepsilon^{-1/3} z_0 + \tau, \tag{5.13}$$

the new outer solution to leading order is

(5.14) $$\frac{du}{d\tau}=F(u,\lambda_c),$$

quite analogous to (4.14). However, the analysis of first-order systems with the critical value $\lambda_c$ is substantially different from second-order systems. A sketch of (5.14) in the phase plane is given in Fig. 2a. Explicitly note that at $u=u_c$, $F=F_u=0$ and $F_{uu}<0$. We see $u$ monotonically approaches the nearest equilibrium (with $u<u_c$). As with second-order systems, *the transition equation* (5.7) *connects the slowly varying parabolic equilibrium with the fully nonlinear outer problem* (5.14). For both (4.14) and (5.14), the slowly varying coefficients may be considered constant on the fast scale. However, for first-order equations the solution of (5.14) exponentially decays (on the fast time) to the stable equilibrium that exists for $\lambda=\lambda_c$. It takes an infinite amount of time ($\tau\to\infty$) for $u$ to approach this equilibrium according to (5.14). During this time the coefficients of (5.14) actually change slowly. It is more accurate to state that (5.1) becomes

(5.15) $$\frac{du}{d\tau}=F(u\ \lambda(\varepsilon^{2/3}z_0+\varepsilon\tau)),$$

using the time scale (5.13). In this way *the slowly varying equilibrium is approached for first-order equations* but not for second-order equations due to the fundamental differences between (4.16) and (5.15).

**6. Second-order bifurcation problems.** In this section we analyze the case of second-order bifurcation,

(6.1) $$\frac{d^2u}{dt^2}=F(u,\lambda(\varepsilon t)).$$

We study straight-straight bifurcation in which in the neighborhood of $\lambda_c$ and $u_c$,

(6.2) $$F(u,\lambda)=\alpha_{20}(\tilde u-\beta_1\tilde\lambda+\cdots)(\tilde u-\beta_2\tilde\lambda+\cdots)=\alpha_{20}\tilde u^2+\alpha_{11}\tilde u\tilde\lambda+\alpha_{02}\tilde\lambda^2+\cdots,$$

where $\alpha_{20}<0$ and $\beta_2>\beta_1$, as well as study parabolic bifurcation in which

(6.3) $$\begin{aligned}F(u,\lambda)&=\alpha_{30}(\tilde u-\beta_1\tilde\lambda+\cdots)(\tilde u^2-\sigma^2\tilde\lambda+\cdots)\\&=\alpha_{11}\tilde\lambda\tilde u+\alpha_{02}\tilde\lambda^2+\alpha_{30}\tilde u^3+\alpha_{21}\tilde u^2\tilde\lambda+\alpha_{12}\tilde u\tilde\lambda^2+\alpha_{03}\tilde\lambda^3+\cdots,\end{aligned}$$

where $\alpha_{30}<0$ and $\alpha_{11}>0$. As a reminder $\tilde\lambda=\lambda-\lambda_c$ and $\tilde u=u-u_c$. In § 3 we noted that parabolic bifurcation had two different asymptotic conditions depending on whether the transition was from a unique straight ($s$) solution to a parabolic ($p$) solution or vice versa.

In §§ 2 and 3 we showed that perturbations oscillate around the slowly varying equilibrium. These outer asymptotic expansions become disordered when

(6.4) $$T=\varepsilon^{2/3}z,$$

where $z$ is an inner variable. This scaling is the same for all cases of bifurcation. However, the matching depends on the specific case, which we now separately describe.

(a) *Straight-straight bifurcation.* For straight-straight bifurcation the transition layer scaling is

(6.5) $$u=u_c+\varepsilon^{2/3}w.$$

The resulting leading order equation corresponds to the local scaling of the dynamical equation (6.1) as $\lambda$ varies through its critical value:

$$\frac{d^2w}{dz^2}=\alpha_{20}(w-\beta_1\lambda_c'z)(w-\beta_2\lambda_c'z). \tag{6.6}$$

The matching condition as $z\to-\infty$ for straight-straight bifurcation is

$$w\sim\beta_1\lambda_c'z+2c\gamma^{-1/4}(-z)^{-1/4}\cos\left(-\frac{2\gamma^{1/2}}{3}(-z)^{3/2}+\theta\right), \tag{6.7}$$

where $\gamma=\alpha_{20}(\beta_1-\beta_2)\lambda_c'>0$ and $c$, $\theta$ are constants determined by the initial conditions.

As motivated by (6.7), equation (6.6) may be analyzed by the following Boutroux transformation (Ince [8, p. 353]):

$$w=\beta_1\lambda_c'z+(\beta_2-\beta_1)\lambda_c'zf(s), \tag{6.8a}$$

$$s=\tfrac{2}{3}\gamma^{1/2}|z|^{3/2}, \tag{6.8b}$$

with $\gamma=\alpha_{20}(\beta_1-\beta_2)\lambda_c'>0$, in which case (6.6) becomes

$$\frac{d^2f}{ds^2}\pm f(-1+f)=-\frac{5}{3s}\frac{df}{ds}, \tag{6.9}$$

where + corresponds to $z>0$ (and vice versa). We may then easily interpret (6.9) as a mechanical system with a conservative nonlinear force and a variable damping force. The nonautonomous force is small for large $s$ (i.e., $|z|$ large). Note that exact equilibrium solutions of (6.9) are $f=0$ and $f=1$.

We are interested in the solution of (6.9), especially asymptotically for $s$ large. The qualitative solution, determined shortly by a phase plane sketch, of the l.h.s. (left-hand side) of (6.9) equaling zero,

$$\frac{d^2f}{ds^2}\pm f(-1+f)=0, \tag{6.10}$$

must only be modified for large $s$ by the small but important effect of the perturbation on the r.h.s. of (6.9). Conclusions about stability are not altered by the r.h.s. of (6.9). The equilibrium $f=0$ corresponds to the stable (for $\lambda<\lambda_c$) straight equilibrium curve, $w=\beta_1\lambda_c'z$. For (6.10), $f=0$ is stable for large $z<0$ and unstable for large $z>0$. The equilibrium $f=1$ corresponds to the unstable (for $\lambda<\lambda_c$) straight equilibrium curve, $w=\beta_2\lambda_c'z$. $f=1$ is unstable for large $z<0$ and stable for large $z>0$. These facts are illustrated in the phase plane diagrams of (6.9); $z<0$ in Fig. 8a, $z>0$ in Fig. 8b. As $z\to-\infty$, the solution matches to $f=0$ (stable). As $z\to+\infty$, with the small friction inherent in (6.9), there are three types of solutions:

(1) Solutions which eventually become near the stable equilibrium $f=1$ approach this value as $z\to+\infty$. Thus, $w\sim\beta_2\lambda_c'z$ as $z\to+\infty$, *corresponding to the transition to the newly stable equilibrium for* $\lambda>\lambda_c$. The solution approaches this equilibrium with a slow oscillatory decay corresponding to a solution near the slowly varying equilibrium. This is illustrated in Fig. 4a.

(2) Very rarely a solution will approach the unstable equilibrium $f=0$, such that $w\to\beta_1\lambda_c'z$ as $z\to+\infty$. In this case, as illustrated in Fig. 4b, the solution is approaching the unstable equilibrium and transition has not occurred.

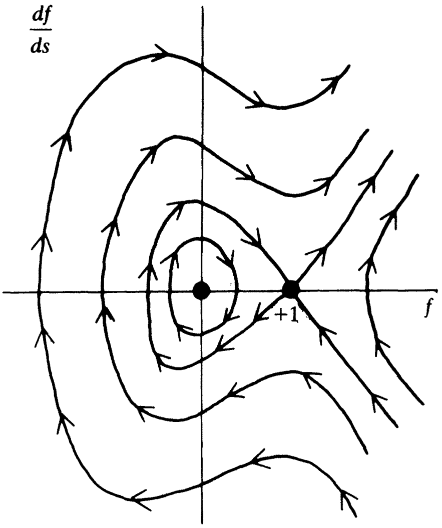



FIG. 8a. $d^2f/ds^2 - f(-1+f) = 0$.

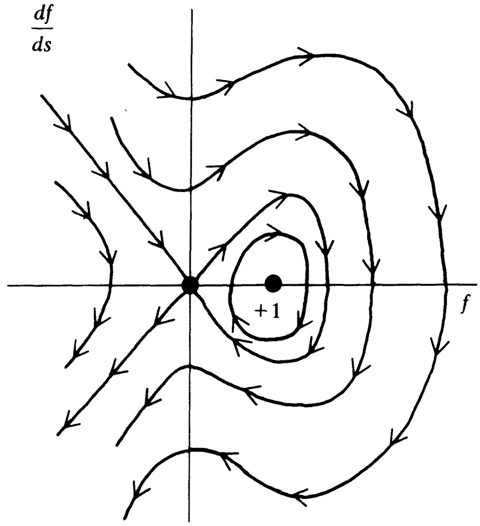



FIG. 8b. $d^2f/ds^2 + f(-1+f) = 0$.

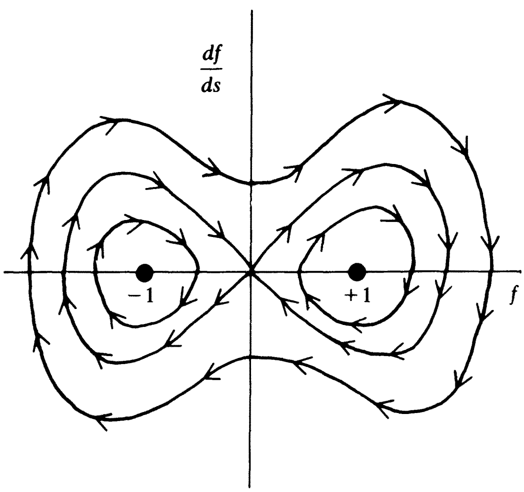



FIG. 8c. $d^2f/ds^2 + f(-1+f^2) = 0$.

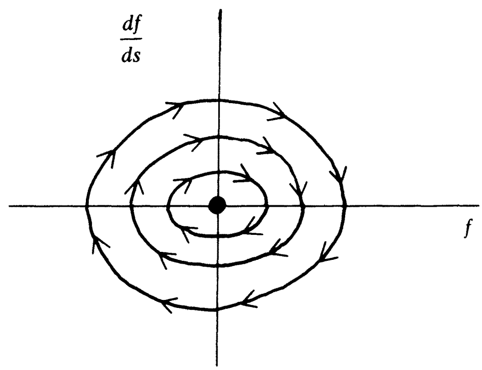



FIG. 8d. $d^2f/ds^2 + f(+1+f^2) = 0$.

(3) It is also possible for an explosion to occur in a finite time (here case (2) separates case (1) from case (3)). We illustrate this in Fig. 4c. As discussed by Haberman [7], if $F(u, \lambda_c) = 0$ only for $u = u_c$ (as illustrated in Fig. 1a), then $u$ explodes ($u \to -\infty$) as the arrows indicate. If for $\lambda < \lambda_c$ there are other equilibria (not illustrated in Fig. 1a), then this explosion could correspond to a jump transition type problem as discussed in § 4.

(b) *Parabolic bifurcation.* For either direction of parabolic bifurcation the transition layer scaling is

$$u = u_c + \varepsilon^{1/3} w. \tag{6.11}$$

The leading order transition equation follows from the local cubic scaling of $F(u, \lambda)$ as $\lambda$

varies through criticality:

$$\frac{d^2w}{dz^2}=\alpha_{11}\lambda_c'zw+\alpha_{30}w^3, \tag{6.12}$$

where $\alpha_{30}<0$ and $\alpha_{11}>0$. Note that for $s\to p$, $\lambda$ increases and thus $\alpha_{11}\lambda_c'>0$, while for $p\to s$, $\lambda$ decreases and thus $\alpha_{11}\lambda_c'<0$. This equation is an example of the second Painlevé transcendent (Ince [8]). For the second Painlevé transcendent (6.12), describing parabolic bifurcations, the asymptotic condition ($z\to-\infty$) differs for the two directions:

$$(s\to p)w\sim 2c\gamma^{-1/4}(-z)^{-1/4}\cos\left(-\frac{2\gamma^{1/2}}{3}(-z)^{3/2}+\frac{6c^2\alpha_{30}}{4\gamma}\ln|\varepsilon^{2/3}z|+\theta\right) \tag{6.13a}$$

$$\begin{aligned}(p\to s)w\sim(\alpha_{11}\lambda_c'z/\alpha_{30})^{1/2}+2c\gamma^{-1/4}(-z)^{-1/4}\cos\Big(&-\frac{2\gamma^{1/2}}{3}(-z)^{3/2}\\&+\frac{6c^2\alpha_{30}}{4\gamma}\ln|\varepsilon^{2/3}z|+\theta\Big),\end{aligned} \tag{6.13b}$$

where $\gamma=\delta\alpha_{11}\lambda_c'>0$ and $\delta=-1$ for $s\to p$, while $\delta=2$ for $p\to s$.

The second Painlevé transcendent may be discussed using the following Boutroux transformation (Ince [8] and Haberman [7]):

$$w=|\alpha_{11}\lambda_c'z/\alpha_{30}|^{1/2}f(s) \tag{6.14a}$$

$$s=\tfrac{2}{3}|\alpha_{11}\lambda_c'|^{1/2}|z|^{3/2}, \tag{6.14b}$$

in which case

$$\frac{d^2f}{ds^2}-f(-f^2+Q)=-\frac{1}{s}\frac{df}{ds}+\frac{1}{9}\frac{f}{s^2}, \tag{6.15}$$

where $Q=\operatorname{sgn}(z)\operatorname{sgn}(\lambda_c'\alpha_{11})$. Because of $Q$ there are two cases, $Q=\pm1$. There are also two directions of parabolic bifurcation, $p\to s$ and $s\to p$, but these are seen to be equivalent by reversing the sign of $z$; i.e., $z>0$ for $s\to p$ corresponds to $z<0$ for $p\to s$. As with (6.9), (6.15) may be interpreted as a conservative mechanical system with a variable damping force $(-(1/s)(df/ds))$ with $f=0$ being an exact solution. However, unlike (6.9), (6.15) also has a small variable linear destabilizing force $(\frac{1}{9}f/s^2)$. $f=\pm1$ is an asymptotic solution of (6.15) for $s$ large only if $Q=1$. It is then not an exact solution because of the $\frac{1}{9}f/s^2$ term, corresponding to the fact that $\pm(-\alpha_{11}\lambda_c'z/\alpha_{30})^{1/2}$ is not an exact solution of (6.12), while $\beta_1\lambda_c'z$ and $\beta_2\lambda_c'z$ are exact solutions of (6.6).

We are interested in the solution of (6.15), especially asymptotically for large $s$. The qualitative solution of the l.h.s. of (6.15) equalling zero,

$$\frac{d^2f}{ds^2}-f(-f^2\pm1)=0, \tag{6.16}$$

is determined by a phase plane sketch illustrated in Fig. 8c ($Q=1$) and Fig. 8d ($Q=-1$). The resulting solution of (6.16) must be modified for large $s$ by the important perturbation appearing on the r.h.s. of (6.15). The stability of the equilibrium solutions of (6.16) are not changed by the r.h.s. of (6.15). The equilibrium $f=0$ is stable for one sign of $z$ ($Q<0$) and unstable for the other ($Q>0$). This corresponds to the fact that for parabolic bifurcation the straight line equilibrium (corresponding to $f=0$) is stable for $\lambda<\lambda_c$ and unstable for $\lambda>\lambda_c$. The equilibria $f=\pm1$ (which occur only for $Q>0$) are always stable. This corresponds to the parabolic equilibrium curve always being stable. However, this curve occurs only for $z>0$ when $s\to p$ and only for $z<0$ when $p\to s$. The two equilibria $\pm 1$ correspond to the two branches of the parabolic equilibrium curve.

For $p\to s$, as $z\to-\infty$ the solution matches to $f=1$ (stable). As $z\to+\infty$, with the small variable friction in (6.15) the solution always decays in an oscillatory manner to $f=0$, *corresponding to the transition from the parabolic equilibrium curve to the straight line* as illustrated in Fig. 5a.

For $s\to p$, as $z\to-\infty$ the small friction enables the solution to match to $f=0$ (stable). As $z\to+\infty$, there are two possible types of solutions:

(1) Most solutions will approach either of the stable values $f=\pm 1$ as $z\to+\infty$. Thus $w\sim\pm|\alpha_{11}\lambda_c' z/\alpha_{30}|^{1/2}$, *corresponding to the transition from the stable straight equilibrium* ($\lambda<\lambda_c$)) *to the newly stable parabolic curve for* $\lambda>\lambda_c$. The equilibrium is approached by the already known oscillatory decay corresponding to a solution near the slowly varying nearly parabolic equilibrium. The existence of two possibilities corresponds to the two branches of the stable bifurcated solution. The branch approached depends on the as yet unknown details of the second Painlevé transcendent. This case is illustrated by Fig. 5a.

(2) As with straight-straight bifurcation it is also possible for the solution to correspond to a connection between the stable locally straight equilibrium and the unstable straight equilibrium, a situation that we do not refer to as transition. The solution will then approach the unstable equilibrium in an exponential manner (see Haberman [7]) illustrated in Fig. 5b.

In this last case only, Ablowitz and Segur [3] have recently obtained analytic[2] explicit connection formulas for the second Painlevé transcendent. Otherwise, details of the matching require numerical (or perhaps analytic) investigations of the transition layer equations.

**7. First-order bifurcation.** Here we analyze first-order bifurcation problems:

$$\frac{du}{dt}=F(u,\lambda(\varepsilon t)). \tag{7.1}$$

We study straight-straight bifurcation ($s\to s$), illustrated in Fig. 1a, in which in the neighborhood of the critical conditions $\lambda=\lambda_c$ and $u=u_c$:

$$F(u,\lambda)=\alpha_{20}(\tilde{u}-\beta_1\tilde{\lambda}+\cdots)(\tilde{u}-\beta_2\tilde{\lambda}+\cdots)=\alpha_{20}\tilde{u}^2+\alpha_{11}\tilde{u}\tilde{\lambda}+\alpha_{02}\tilde{\lambda}^2+\cdots, \tag{7.2}$$

where $\alpha_{20}<0$ and $\beta_2>\beta_1$. We also study parabolic bifurcation, in which

$$\begin{aligned} F(u,\lambda)&=\alpha_{30}(\tilde{u}-\beta_1\tilde{\lambda}+\cdots)(\tilde{u}^2-\sigma^2\tilde{\lambda}+\cdots)\\ &=\alpha_{11}\tilde{\lambda}\tilde{u}+\alpha_{02}\tilde{\lambda}^2+\alpha_{30}\tilde{u}^3+\alpha_{21}\tilde{u}^2\tilde{\lambda}+\alpha_{12}\tilde{u}\tilde{\lambda}^2+\alpha_{03}\tilde{\lambda}^3+\cdots \end{aligned} \tag{7.3}$$

corresponds to Fig. 1b. In § 3, it was shown that for parabolic bifurcation the asymptotic matching conditions for the transition from a straight equilibrium curve to a parabolic curve ($s\to p$) differ from the conditions for the opposite direction of transition ($p\to s$). These two cases must be analyzed individually unlike the same problem for second-order systems (see § 6).

(a) *Straight-straight bifurcation* ($s\to s$). In all cases perturbations to the slowly varying equilibrium are exponentially small in the transition region. In the case of

[2] The phase-amplitude dependence of oscillation necessary to produce an exponentially decaying solution has not yet been analytically determined.

straight-straight bifurcation, the asymptotic expansion of the slowly varying equilibrium becomes disordered when

$$T=\varepsilon^{1/2}z, \tag{7.4}$$

where $z$ is an inner variable. The inner dependent variable $w$ is defined by the scaling

$$u=u_c+\varepsilon^{1/2}w. \tag{7.5}$$

Then the leading order transition layer equation is

$$\frac{dw}{dz}=\alpha_{20}(w-\beta_1\lambda'_c z)(w-\beta_2\lambda'_c z), \tag{7.6}$$

analogous to the similar second-order equation (6.6). As motivated by sketches of (7.6) in the $(w, z)$ plane, Lebovitz and Schaar [14] proved that (7.6) *provides the transition between the two stable straight line solutions only if* $\beta_1<0$ (assuming $\beta_2>\beta_1$). In this case ($\beta_1<0$) as $z\to+\infty$, $w\sim\beta_2\lambda'_c z$ which matches to the newly stable slowly varying equilibrium $\lambda>\lambda_c$. Otherwise ($\beta_1\geqq 0$) $w$ explodes towards $-\infty$ in a finite time, $w\sim-[\alpha_{20}(z-z_0)]^{-1}$ as $z\to z_0$ (if $\beta_1=0$, the explosion occurs only if initially $w<0$). Fig. 1a clearly shows the possibility of an explosion.

$\beta_1=0$ is a particularly important case because it corresponds to $u=0$ being an exact solution for all $\lambda$. In this case the slowly varying equilibrium solution is exactly $u=0$; there is no expansion which becomes disordered as $\lambda\to\lambda_c$. Instead perturbations to $u=0$ must be included even though they are initially exponentially small. This has been discussed in § 2. The eventual breakdown when $\lambda>\lambda_c$ (and $u=0$ becomes unstable) is analyzed in Appendix A.

(b) *Parabolic bifurcation* ($s\to p$ *and* $p\to s$). In this case the outer asymptotic expansion becomes disordered when

$$T=\varepsilon^{1/2}z, \tag{7.7}$$

where $z$ is an inner variable. The matching condition depends on whether the expected transition is from the parabolic equilibrium curve to the stable straight curve or vice versa. We analyze each case separately.

(b1) *Transition from parabolic equilibrium to straight equilibrium* ($p\to s$). As stated in (7.7), $T=\varepsilon^{1/2}z$. The matching condition is that as $z\to-\infty$, the solution is on the vertically bifurcated parabolic curve, $\tilde{u}\sim\sigma\tilde{\lambda}^{1/2}$, and thus the appropriate scale is

$$u=u_c+\varepsilon^{1/4}w. \tag{7.8}$$

Thus the leading order matching condition,

$$w\sim(-\alpha_{11}\lambda'_c z/\alpha_{30})^{1/2}, \quad \text{as } z\to-\infty, \tag{7.9}$$

must be applied to the inner equation

$$\frac{dw}{dz}=\alpha_{11}\lambda'_c zw+\alpha_{30}w^3+\varepsilon^{1/4}(\alpha_{02}\lambda_c'^2 z^2+\alpha_{12}\lambda'_c zw+\alpha_{40}w^4)+O(\varepsilon^{1/2}). \tag{7.10}$$

On introducing an inner expansion

$$w=w_0+\varepsilon^{1/4}w_1+\varepsilon^{1/2}w_2+\varepsilon^{3/4}w_3+\cdots, \tag{7.11}$$

(7.10) becomes the leading order transition layer equation,

$$\frac{dw_0}{dz}=\alpha_{11}\lambda'_c zw_0+\alpha_{30}w_0^3, \tag{7.12}$$

where we note $\alpha_{11}>0$, $\lambda_c'<0$, and $\alpha_{30}<0$. Equation (7.12) may be solved exactly since it is an example of Bernoulli's equation. (7.12) is linearized by a transformation (for example, Ince [8, p. 22]), $v=w_0^{-2}$, in which case

$$w_0=\left(-2\alpha_{30}e^{-\alpha_{11}\lambda_c' z^2}\int_{-\infty}^{z} e^{\alpha_{11}\lambda_c'\bar z^2}\,d\bar z\right)^{-1/2}, \tag{7.13}$$

where the matching condition (7.9) has been utilized (integration by parts verifies (7.9)). Thus as $z\to+\infty$

$$w_0\sim\left(\frac{-\alpha_{11}\lambda_c'}{4\pi\alpha_{30}^2}\right)^{1/4}e^{+\alpha_{11}\lambda_c' z^2/2}, \tag{7.14}$$

i.e., $w_0$ is exponentially decaying. We show this matches to the slowly varying equilibrium solution (the unique one for $\lambda<\lambda_c$), illustrating the transition from the vertical bifurcated parabolic equilibrium to the straight equilibrium.

The matching is made clearer by the analysis of $w_1$, $w_2$, and $w_3$:

$$\text{as } z\to\infty \begin{cases} w_1\sim-\lambda_c'\alpha_{02}z/\alpha_{11}, \\ w_2=O(z), \\ w_3=O(z^2). \end{cases} \tag{7.15}$$

For large $z$, the expansion for $w$ is

$$w=\varepsilon^{1/4}(-\lambda_c'\alpha_{02}z/\alpha_{11})+O(\varepsilon^{3/4}z^2)+\cdots, \tag{7.16}$$

where the contribution from $w_0$, given by (7.14), is exponentially small for large $z$. The breakdown of expansion (7.16) suggests that the new region is characterized by $z=O(\varepsilon^{-1/2})$ and $w=O(\varepsilon^{-1/4})$. In other words $T=O(1)$ and $u=O(1)$, and thus (7.16) matches to the outer problem

$$\varepsilon\frac{du}{dT}=F(u,\lambda(T)). \tag{7.17}$$

This is the same type of result as obtained in § 5 for second-order bifurcating systems. The matching condition for (7.17) is that as $T\to 0$

$$u=-\alpha_{20}\lambda_c' T/\alpha_{11}+O(T^2)+O(\varepsilon), \tag{7.18}$$

corresponding to the slowly varying straight equilibrium.

On the fast scale $t$ the solution of (7.17) approaches the newly stable equilibrium. Since it takes an infinite time to reach an equilibrium, the slow time variation ($\lambda(T)$) in (7.17) will become important, showing how the solution actually approaches the slowly varying equilibrium solution. *This then illustrates the transition between two slowly varying equilibrium solutions, the leading order transition layer equation being the Bernoulli equation* (7.12).

(b2) *Transition from straight equilibrium to parabolic* ($s\to p$). Lebovitz and Schaar [15] derived error bounds for this case when $\lambda$ formed part of an autonomous system. They showed that there are two overlapping transition regions. We will show a similar result when $\lambda$ is externally controlled slowly, $\lambda(T)$.

The disordering of the expansion of the slowly varying equilibrium solution, which is asymptotically a straight line, again is given by

$$T=\varepsilon^{1/2}z. \tag{7.19}$$

However, the matching condition is that as $z \to -\infty$, the solution approaches the primary straight equilibrium, $\tilde{u} \sim \beta_1 \tilde{\lambda}$. Thus, the correct scale of the dependent variable is

$$u = u_c + \varepsilon^{1/2} w, \tag{7.20}$$

where as $z \to -\infty$, the leading order matching condition is

$$w \sim -\alpha_{02} \lambda_c' z / \alpha_{11} \quad \text{as } z \to -\infty. \tag{7.21}$$

Here $\lambda_c' > 0$ since $\lambda$ is increasing through criticality. Also $\alpha_{11} > 0$. Thus if $\alpha_{02} > 0$, then $u > u_c$ and vice versa. With these scales, the inner equation is

$$\frac{dw}{dz} = z(\alpha_{11}\lambda_c' w + \alpha_{02}\lambda_c'^2 z) + \varepsilon^{1/2}(\alpha_{30} w^3 + \alpha_{21}\lambda_c' z w^2 + \alpha_{12}\lambda_c'^2 z^2 w + \alpha_{03}\lambda_c'^3 z^3) + \cdots \tag{7.22}$$

Introducing the inner expansion,

$$w = w_0 + \varepsilon^{1/2} w_1 + \cdots, \tag{7.23}$$

yields the leading order inner boundary layer equation,

$$\frac{dw_0}{dz} = z(\alpha_{11}\lambda_c' w_0 + \alpha_{02}\lambda_c'^2 z). \tag{7.24}$$

Equation (7.24) is linear; the important nonlinear terms do not occur at this order. This is because $\tilde{u}$ is very small, being only $O(\varepsilon^{1/2})$; in the interior scale ($z = O(1)$) the slowly varying equilibrium is smaller than occurs in the other direction of parabolic bifurcation. The exact solution of (7.24) which matches with (7.21) is

$$w_0 = \alpha_{02}\lambda_c' e^{\alpha_{11}\lambda_c' z^2/2} \int_{-\infty}^{z} s^2 e^{-\alpha_{11}\lambda_c' s^2/2} \, ds \tag{7.25}$$

The matching condition may be verified by integration-by-parts. We have used the fact that $\alpha_{11}\lambda_c' > 0$. Thus as $z \to +\infty$

$$w_0 \sim \frac{\alpha_{02}}{\alpha_{11}} \lambda_c' \sqrt{\frac{2\pi}{\alpha_{11}\lambda_c'}} e^{\alpha_{11}\lambda_c' z^2/2}, \tag{7.26}$$

which is exponentially increasing. Clearly the nonlinear terms must become important. We will show that the inner expansion (7.23) becomes disordered as $z \to +\infty$. To see this, we calculate $w_1$ asymptotically as $z \to +\infty$:

$$w_1 \sim \frac{\alpha_{30}}{2\alpha_{11}\lambda_c'} \left(\frac{\alpha_{02}\lambda_c'}{\alpha_{11}}\right)^3 \left(\frac{2\pi}{\alpha_{11}\lambda_c'}\right)^{3/2} \frac{e^{3\alpha_{11}\lambda_c' z^2/2}}{z}. \tag{7.27}$$

In a similar manner asymptotically $w_2$ is proportional to $z^{-2} \exp(5\alpha_{11}\lambda_c' z^2/2)$. Each succeeding term is exponentially larger than the preceding (diminished only by one power of $z$).

To determine the scaling of the region to which this matches as $z \to \infty$, we introduce the nonlinear scale transformation

$$\varepsilon^{1/2} \frac{e^{\alpha_{11}\lambda_c' z^2}}{z} = \Phi, \tag{7.28}$$

where $\Phi$ is an $O(1)$ independent-type variable. The inner expansion as $z \to \infty$ suggests to leading order that

(7.29) $$w=\varepsilon^{-1/4}z^{1/2}f(\Phi),$$

where

(7.30) $$f(\Phi)\sim\frac{\alpha_{02}}{\alpha_{11}}\left(\frac{2\pi\lambda_c'}{\alpha_{11}}\right)^{1/2}\Phi^{1/2}$$

as $\Phi\to 0$ (corresponding to the matching region in which $z\to\infty$ and $\Phi\to 0$ in (7.28)). We note that

$$\frac{d\Phi}{dz}=\Phi\left(2\alpha_{11}\lambda_c'z-\frac{1}{z}\right)\sim 2\alpha_{11}\lambda_c'z\Phi$$

since $z$ is large in the region. Thus $dw/dz\sim\varepsilon^{-1/4}z^{1/2}2\alpha_{11}\lambda_c'z\Phi(df/d\Phi)$, and consequently we derive from (7.22) that the leading order equation is

(7.31) $$2\Phi\frac{df}{d\Phi}=f+\frac{\alpha_{30}}{\alpha_{11}\lambda_c'}f^3.$$

We have used the fact that in this region, where $\Phi=O(1)$, $z\ll\varepsilon^{-1/2}$. Letting

(7.32) $$s=\ln\Phi=\tfrac{1}{2}\ln\varepsilon+\alpha_{11}\lambda_c'z^2-\ln z,$$

simplifies (7.31). It follows that

(7.33) $$2\frac{df}{ds}=f+\frac{\alpha_{30}}{\alpha_{11}\lambda_c'}f^3.$$

Since $\alpha_{30}/(\alpha_{11}\lambda_c')<0$, the matching condition is

(7.34) $$f\sim\frac{\alpha_{02}}{\alpha_{11}}\left(\frac{2\pi\lambda_c'}{\alpha_{11}}\right)^{1/2}e^{s/2}\quad\text{as } s\to-\infty.$$

Although the exact solution of (7.33) satisfying (7.34) may be easily obtained, the phase plane sketch of Fig. 9 more clearly shows the fundamental behavior of (7.33):

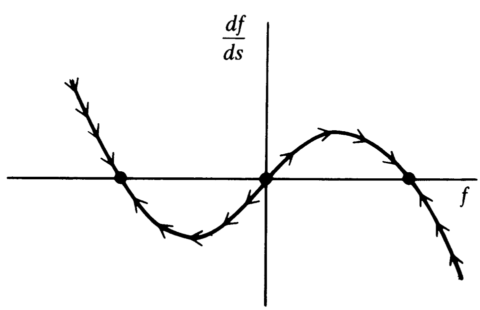



FIG. 9

Thus as $s\to\infty$ (corresponding to $z\to\infty$)

(7.35) $$f\sim(\operatorname{sgn}\alpha_{02})\left(\frac{-\alpha_{11}\lambda_c'}{\alpha_{30}}\right)^{1/2}$$

or from (7.29), as $z\to\infty$ the solution in this region is approaching

(7.36) $$w\sim\varepsilon^{-1/4}(\operatorname{sgn}\alpha_{02})\left(\frac{-\alpha_{11}\lambda_c'z}{\alpha_{30}}\right).$$

*This indicates that this second inner region matches as $z \to \infty$ to the slowly varying equilibrium*; since $\tilde{u} = \varepsilon^{1/2} w$ and $T = \varepsilon^{1/2} z$, (7.36) is the leading order slowly varying equilibrium for a parabolic equilibrium curve,

$$\tilde{u} \sim (\operatorname{sgn} \alpha_{02})(-\alpha_{11}\lambda_c' T/\alpha_{30})^{1/2}. \tag{7.37}$$

If $\alpha_{02} > 0$, then $\tilde{u}$ is asymptotic to the upper branch of the parabolic equilibrium curve (and vice versa).

This is the only case of transition in which a secondary transition layer is necessary. The first linear layer is characterized by $z = O(1)$ and hence $T = O(\varepsilon^{1/2})$, while the second nonlinear layer is characterized by $s = O(1)$ and hence from (7.32)

$$\alpha_{11}\lambda_c' T^2 - \varepsilon \ln T = -\varepsilon \ln \varepsilon + \varepsilon s. \tag{7.38}$$

$T$ is seen to be $O(-\varepsilon \ln \varepsilon)^{1/2}$ as noted by Lebovitz and Schaar [15]. However, the scaling is not $T = (-\varepsilon \ln \varepsilon)^{1/2} v$, but rather given by (7.38). It is convenient to define

$$\xi(\varepsilon) = [\varepsilon \ln(-\ln \varepsilon/\varepsilon)]^{1/2}, \tag{7.39}$$

which is small, $\xi \sim (-\varepsilon \ln \varepsilon)^{1/2}$. Some algebraic manipulations then show that the nonlinear transformation (7.38) is asymptotically equivalent to the following linear scale change

$$T = (2\alpha_{11}\lambda_c')^{-1/2}\left[\xi(\varepsilon) + \frac{\varepsilon(s - \ln(2\alpha_{11}\lambda_c')^{1/2})}{\xi(\varepsilon)}\right], \tag{7.40}$$

where $s = O(1)$. Thus $T$ is shifted by a small amount, $(2\alpha_{11}\lambda_c')^{-1/2}\xi(\varepsilon)$, which is much greater than $O(\varepsilon^{1/2})$. However, in this interior boundary layer $T$ still varies only by a small amount, since $\varepsilon\xi^{-1}(\varepsilon) \ll 1$. This explains why the differential equation (7.33) has constant coefficients; it is because the variation in $T$ is small.

**8. Discussion.** We have described how transitions and jumps occur in first and second order ordinary differential equations. Three kinds of phenomena have been obtained as $\lambda$ slowly evolves through a critical value $\lambda_c$ of an algebraic bifurcation problem:

(i) Equilibria are approached. These are usually stable slowly varying equilibria, but for second-order systems an unstable equilibria may be approached.

(ii) For the case of straight-straight bifurcation, it is possible for solutions to escape to $-\infty$ as suggested by the arrows of Fig. 1a.

(iii) For the jump phenomena for second-order systems, nonlinear oscillations around the new stable equilibrium may occur.

In all cases, jumps and transitions are locally described by transition layer equations which match to outer regions of the types discussed. A secondary interior transition layer was needed only in the case of first-order transition from a straight equilibrium to a parabolic equilibrium.

Of interest are similar transitions that should occur for nonalgebraic bifurcation problems as the parameter is slowly varied. In particular, bifurcation problems which are boundary value problems come to mind. Some work on describing how transitions take place in these cases has been initiated by Rubenfeld [19] for first-order systems in time. An ordinary differential equation is determined for the amplitude of the significant bifurcating mode. It is of the type of ordinary differential equation corresponding to transition layers for algebraic bifurcation problems. We would conjecture that many of the same ideas would hold for slowly bifurcating boundary value problems as occur in the present paper. Slowly varying equilibrium solutions and perturbations thereof should be important as well as fully nonlinear but slowly varying solutions.

**Appendix A. Straight-straight bifurcation in the case in which $u=0$ is a solution for all $\lambda$.** If there is an equilibrium solution independent of $\lambda$ (usually normalized to $u=0$) for first-order systems, then initially exponentially decaying solutions become significant after $u=0$ becomes unstable, that is after $\lambda$ evolves to be greater than $\lambda_c$. The breakdown of this expansion is important only if $u=0$ is a solution for all $\lambda$ as explained in § 2.

Here we only consider the case of straight-straight bifurcation where $\beta(T)\neq 0$. The breakdown of the expansion is characterized by condition (3.10). A zero of $k(\varepsilon t)$ is *not* the place at which the expansion breaks down. Instead $\varepsilon t$ must be near the scaled time $\varepsilon t^*$ such that

$$\int_{\varepsilon t_0}^{\varepsilon t^*} k(\bar{s})\,d\bar{s}=0. \tag{A.1}$$

At this time $t^*$ the decay ($k(\varepsilon t)>0$ for $t_0<t<0$) which occurred near $t_0$ has been overcome by the growth ($k(\varepsilon t)<0$ for $0<t<t^*$). At $t^*$, there is zero *NET* growth. To find the correct scaling, let us assume $t$ is near $t^*$:

$$\varepsilon t=\varepsilon t^*+Q, \tag{A.2}$$

where we will assume $0<Q\ll 1$. Thus for breakdown

$$\frac{\mu}{\varepsilon}\left(\int_{\varepsilon t_0}^{\varepsilon t^*}\beta(s)\exp\left[-\frac{1}{\varepsilon}\int_{\varepsilon t_0}^{s}k(\bar{s})\,d\bar{s}\right]ds+\int_{\varepsilon t^*}^{\varepsilon t^*+Q}\beta(s)\exp\left[-\frac{1}{\varepsilon}\int_{\varepsilon t_0}^{s}k(\bar{s})\,d\bar{s}\right]ds\right)=O(1).$$

The first term is exponentially small since $\int_{\varepsilon t_0}^{s} k(\bar{s})\,d\bar{s}\geqq 0$. The second term can be approximated (by letting $s=\varepsilon t^*+r$ and $\bar{s}=\varepsilon t^*+\bar{r}$) to yield the condition

$$\mu\frac{\beta(\varepsilon t^*)}{k(\varepsilon t^*)}\exp\left[-\frac{1}{\varepsilon}k(\varepsilon t^*)Q\right]=O(1)$$

for breakdown, where (A.1) has been used. From this latter expression we see $\mu\exp[-k(\varepsilon t^*)Q/\varepsilon]=O(1)$ and thus $\log\mu-(k(\varepsilon t^*)/\varepsilon)Q=\log O(1)$ or equivalently

$$Q=-\frac{\varepsilon}{k(\varepsilon t^*)}[\log O(1)-\log\mu]. \tag{A.3}$$

On this basis to understand this breakdown, we introduce from (A.2) and (A.3) the new outer scaling

$$T=\varepsilon t=\varepsilon t^*+\frac{\varepsilon\log\mu}{k(\varepsilon t^*)}+\varepsilon\tau, \tag{A.4}$$

where we assume $\varepsilon\log\mu$ is small. In this scaling all the higher order terms in (3.6) will be as important as $\bar{u}_1$. We note that using (A.4), $\bar{u}_1=O(\mu^{-1})$. Thus since $\bar{u}=O(\mu^{-1})$, from (3.1) $u=O(1)$ and scaling (A.4) must be directly applied to (3.2) to obtain to leading order

$$\frac{du}{d\tau}=F(u,\lambda(\varepsilon t^*)). \tag{A.5}$$

The matching implies that $u\to 0$ as $\tau\to-\infty$. This problem is the fully nonlinear ordinary differential equation, but with constant coefficients (for $\tau=O(1)$). Note that $t^*$ is in the region of growth away from $u=0$. The behavior of solutions to (A.5) depends on the initial conditions.

For straight-straight bifurcation if $u$ is initially negative, $u$ continues negatively and explodes. Otherwise the solution to (A.5) grows to its equilibrium value

$$u \to u_E(\lambda(\varepsilon t^*));$$

there is a smooth transition to the bifurcated equilibrium. However, note that the first bifurcated equilibrated amplitude that occurs corresponds to $\lambda(\varepsilon t^*)$ not the smooth transition to the nonzero $u_E(\lambda)$ for all values of $\lambda > 0$. The transition is delayed by the decay which occurs when $\lambda > 0$.

It is interesting to note that somewhat unexpected phenomena could occur if the bifurcation diagram were as sketched in Fig. 10. As $\lambda$ is varied slowly from negative to positive, the decay for $\lambda < 0$ could be so severe that after $\lambda$ becomes positive the amplitude would be extremely small. In that case the growth would not let the solution approach its continuous stable equilibrium (along $AB$ in Fig. 10) before $\lambda$ increases beyond the point of its existence. Instead $\lambda(\varepsilon t^*)$ might be as illustrated in Fig. 10 and hence $u$ would not be in its equilibrium until it reached its value at $D$.

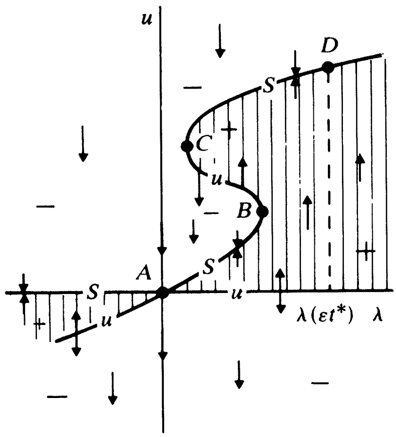



FIG. 10

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