Slowly varying .... eq. (2):

$\dot{x} = a - x^2$, $f(a,x) = a - x^2$, equil: $x_0 = \sqrt{a}$

$\dot{a} = -\mu$

Using a slow time scale: $\dot{x} = \frac{dx}{dt}$, $t = \mu\tau \rightarrow \dot{x} = \mu x_\tau$

$\rightarrow \mu x_\tau = a - x^2, \quad \mu a_\tau = -\mu \rightarrow a_\tau = -1$

Taylor expanding about equil:

$$\mu x_\tau = -2\sqrt{a}(x - \sqrt{a}) - 2(x - \sqrt{a})^2$$

Choosing the Asympt. expansion:

$$x \sim \sqrt{a} + \mu x_1 + \mu^2 x_2 + \dots$$

$O(\mu)$: $x_{0\tau} = -2\sqrt{a}\, x_1 \longrightarrow x_1 = \frac{x_{0\tau}}{-2\sqrt{a}} = \frac{(\sqrt{a})_\tau}{-2\sqrt{a}} = \frac{1}{4a}$

$O(\mu^2)$: $x_{1\tau} = -2\sqrt{a}\, x_2 - 2x_1^2 \rightarrow x_2 = \frac{x_{1\tau} + 2x_1^2}{-2\sqrt{a}} = -\frac{5}{32 a^{5/2}}$

$$\longrightarrow \underset{\substack{\uparrow \\ x(a(\mu t))}}{x} \sim \sqrt{a} + \frac{\mu}{4a} - \frac{5}{32} \cdot \frac{\mu^2}{a^{3/2}} + O(\mu^3) \quad (2)$$

eq. (3):

But, if $a(\mu t) \sim O(\mu^{2/3}) \rightarrow x \sim O(\mu^{1/3})$ for each term

$\rightarrow$ Asympt. series fails.

This leads to an inner expansion from a scaling centered around the equil.:

$$x = \sqrt{a} + \mu^{1/3} w(z) \quad t = \mu^{-1/3} z = \tau/\mu \quad (\tau = \mu^{2/3} z)$$

$$\Rightarrow \dot{x} = a - x^2 \longrightarrow \frac{dw}{dz} = -\mu z - w^2 \quad \left(\frac{dx}{dt} = \mu^{1/3}\frac{dx}{dz} = \mu^{2/3}\frac{dw}{dx}\right)$$

eq. (3) (cont):

Given $\frac{\partial w}{\partial z} = -\mu z - w^2$, Let $y = \mu^{1/3} z$

$$w = -\mu^{1/3}\frac{1}{\phi}\frac{\partial \phi}{\partial y}$$

Airy's

$$\Rightarrow \frac{\partial^2 \phi}{\partial y^2} = y\phi \rightarrow \phi = c_1 Ai(y) + c_2 Bi(y)$$

$$\rightarrow w = -\mu^{1/3}\frac{c_1 Ai'(\tau/\mu^{2/3}) + c_2 Bi'(\tau/\mu^{2/3})}{c_1 Ai(\tau/\mu^{2/3}) + c_2 Bi(\tau/\mu^{2/3})}$$

using a matching condition: $w \sim (-\mu z)^{1/2}$ as $z \to -\infty$

$$c_2 = 0 \Rightarrow w = -\mu^{1/3}\frac{Ai'(\tau/\mu^{2/3})}{Ai(\tau/\mu^{2/3})} \quad (3)$$

Thus, $X(a(\mu t)) = -\mu^{1/3}\frac{Ai'(a/\mu^{2/3})}{Ai(a/\mu^{2/3})}$

would have a singularity at $Ai(a/\mu^{2/3}) = 0$

which occurs when $\frac{a}{\mu^{2/3}} = -2.33810\ldots$

or $a_d = \mu^{2/3}\cdot(-2.33810\ldots)$ (4)