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NonSmooth 1D w/ osc.

Starting with $\dot{x} = -\mu + 2|x| - x|x| + A\sin\Omega t$, fixed μ, $A \sim O(1)$, $\Omega \gg 1$

Take a multiple scales approach with "slow" and "fast" time:

$\tau = t,\ T = \Omega t \Rightarrow x_T = \Omega^{-1}(-x_\tau - \mu + 2|x| - x|x| + A\sin T)$

Next, introduce the asymptotic expansion: $x \sim x_0 + \Omega^{-1}x_1 + \Omega^{-2}x_2 + \ldots$

$\Rightarrow O(1):\ x_{0T} = 0 \longrightarrow x_0 = x_0(\tau)$

$O(\Omega^{-1}):\ x_{1T} = -x_{0\tau} - \mu + 2|x_0| - x_0|x_0| + A\sin T = R_1(\tau, T)$

Using the Fredholm alternative: $\lim_{T\to\infty} \frac{1}{T}\int_0^T R_1(\tau, u)\,du = 0 \Rightarrow x_{0\tau} = -\mu + 2|x_0| - x_0|x_0|$

Choosing to follow along the lower stable branch ($\mu > 0$, $x < 0$),

$x_{0\tau} = -\mu - 2x_0 + x_0^2$, which has stable equilibrium solution $x_0 = 1 - \sqrt{1+\mu}$

$\Rightarrow x_{1T} = A\sin T \rightarrow x_1 = -A\cos T + V_1(\tau)$

$O(\Omega^{-2}):\ x_{2T} = -x_{1\tau} + 2|x_1| - x_0|x_1| - x_1|x_0|$

Again, given the order of the terms, we still have $x < 0$,

$$x_{2T} = -x_{1\tau} + 2x_1(x_0 - 1)$$
$$= -V_{1\tau} + 2V_1(x_0 - 1) - 2A\cos T(x_0 - 1)$$

Fredholm once more gives, $V_{1\tau} = 2V_1(x_0 - 1) = -2\sqrt{1+\mu}\,V_1$

which has stable equilibrium $V_1 = 0$.

Thus, $x \sim 1 - \sqrt{1+\mu} + \Omega^{-1}(-A\cos T) + O(\Omega^{-2})$ $\boxed{\text{outer solution}}$

We see this solution fails when terms reorder from $\mu \sim O(\Omega^{-1})$,
It then is natural to rescale the problem,

$$\mu = \Omega^{-1}m, \quad x = \Omega^{-1}y$$

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This rescaling induces,

$$\Omega^{-1}\dot{y} = -\Omega^{-1}m + \Omega^{-1}2|y| - \Omega^{-2}y|y| + A\sin\Omega t$$

A similar multiple scales approach gives,

$$y_T = \Omega^{-1}(-y_\tau - m + 2|y|) - \Omega^{-2}y|y| + A\sin T$$

which suggests the expansion, $y \sim y_0 + \Omega^{-1}y_1 + \dots$

$\rightarrow$ $O(1)$: $y_{0T} = A\sin T \longrightarrow y_0 = -A\cos T + V_0(\tau)$

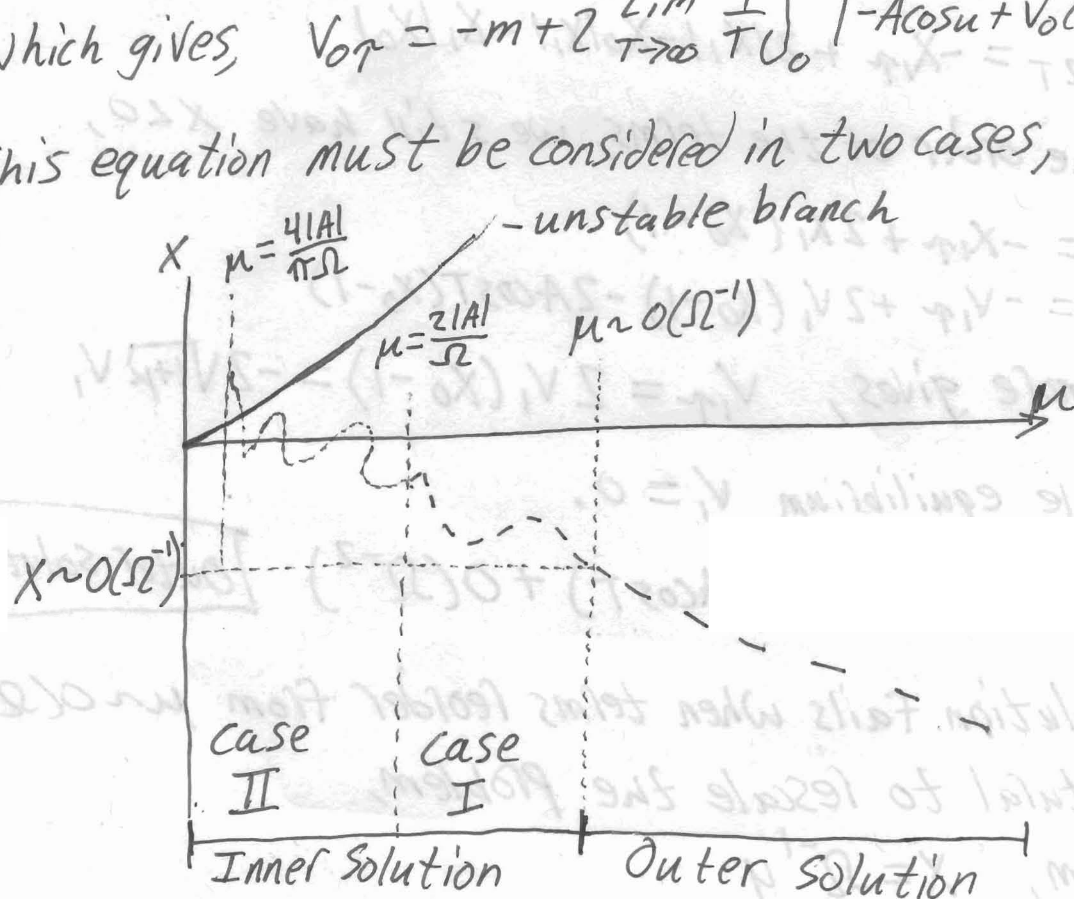
$O(\Omega^{-1})$: $y_{1T} = -y_{0\tau} - m + 2|y_0|$

Here we again impose the Fredholm alternative,

$$\rightarrow \lim_{T\to\infty}\frac{1}{T}\int_0^T -y_{0\tau} - m + 2|y_0|\,du = 0 \rightarrow y_{0\tau} = -m + 2\lim_{T\to\infty}\frac{1}{T}\int_0^T |-A\cos u + V_0(\tau)|\,du$$

which gives, $V_{0\tau} = -m + 2\lim_{T\to\infty}\frac{1}{T}\int_0^T |-A\cos u + V_0(\tau)|\,du$

This equation must be considered in two cases, first a picture:



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$\boxed{\text{Non Smooth 1D w/ osc.}}$

$$V_{0\tau} = -m + 2\lim_{T\to\infty}\frac{1}{T}\int_0^T |-A\cos u + V_0(\tau)|\,du$$

Case I: $|V_0| \geq |A| \to$ Integral is simple: $V_{0\tau} = -m + 2|V_0|$

Searching for equilibrium, $|V_0| = \frac{m}{2} \Rightarrow V_0 = \frac{-m}{2}$ stable.

$\Rightarrow y \sim -\frac{m}{2} - A\cos T + O(\Omega^{-1}) \to x \sim \left(-\frac{\mu}{2} - A\cos T\,\Omega^{-1}\right) + O(\Omega^{-2})$

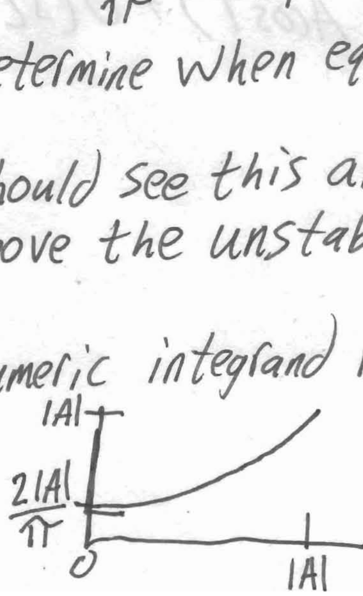
where $|V_0| \geq |A| \Leftrightarrow m \geq 2|A| \Leftrightarrow \mu \geq \frac{2|A|}{\Omega}$ (agrees with where outer soln fails)

Case II: $|V_0| < |A| \to$ Integral is nontrivial and has no analytic solution.

Approach:
- Use $m < 2|A|$ region.
- Obtain numeric results for integral. (How?)
  - $\lim_{T\to\infty}\frac{1}{T}\int_0^T |-A\cos u + V_0(\tau)|\,du = \frac{1}{2\pi}\int_0^{2\pi} |-A\cos u + V_0(\tau)|\,du$
- Find equilibrium in this region
  - The integral is bounded from above & below

  $-m + \frac{4|A|}{\pi} \leq V_{0\tau} < -m + 2|A| \to m = \frac{4|A|}{\pi}$ gives pos. $V_{0\tau}$
- Determine when equilibrium becomes unstable.
- Should see this around when the oscilations jump above the unstable branch.

Issue: The numeric integrand is tricky. I've interpolated to get:



which indicates the integral is parabolic in V_0.

$\frac{1}{2\pi}\int du \approx \frac{2|A|}{\pi} + \frac{1}{|A|}\left(1 - \frac{2}{\pi}\right)V_0(\tau)^2$

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If we assume $\lim_{T\to\infty}\frac{1}{T}\int_0^T |-A\cos u+V_0(\tau)|du$ is quadratic in $V_0(\tau)$, certain values are known. $f(0)=\frac{2|A|}{\pi}$

$f(|A|)=|A|=f(-|A|)$

$$\Rightarrow f(V_0(\tau))=\frac{2|A|}{\pi}+\frac{1}{|A|}\left(1-\frac{2}{\pi}\right)V_0(\tau)^2$$

$$\Rightarrow V_{0\tau}=-m+2\lim_{T\to\infty}\frac{1}{T}\int_0^T|-A\cos u+V_0(\tau)|du \simeq -m+\frac{4|A|}{\pi}+\frac{2}{|A|}\left(1-\frac{2}{\pi}\right)V_0(\tau)^2$$

This approximation has maximum error of .01 for $A\sim O(1)$
(.4 for $A=30$)

Searching for an equilibrium with this approximation,

$$V_0(\tau)\simeq\sqrt{\frac{m-\frac{4|A|}{\pi}}{\frac{2}{|A|}\left(1-\frac{2}{\pi}\right)}}=K\sqrt{m-\frac{4|A|}{\pi}}$$

which now tells us that a bifurcation ~~tipping~~ occurs $m=\frac{4|A|}{\pi}$

$$\to \mu_p=\frac{4|A|}{\pi\Omega}$$

$$\to y\sim K\sqrt{m-\frac{4|A|}{\pi}}-A\cos T+O(\Omega^{-1})$$

$$x\sim\left(K\sqrt{\Omega^{-1}\left(\mu-\frac{4|A|}{\pi\Omega}\right)}-\Omega^{-1}A\cos T\right)+O(\Omega^{-2})$$