Draft Plan: Slow passage through non-smooth fold bifurcations: deterministic and stochastic

Those interested in either ramped or stochastic non-smooth bifurcations

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Abstract

1 Introduction

2 Development of the model(s)

Notes: Introduce "typical" ODE model, in simplest form(s) of non-smooth fold bifurcations. Reference to other work on this: [1]. Relevant for case where there are only stable/unstable equilibria, vs. other cases where other types of dynamics are possible. Reference other work on identifying these basic models with other possible behaviours. [2].
Notes:Connect to Stommel-type models studied in applications. Identify appropriate ramped bifurcation variables and possible noise sources. Complete model accordingly.

In the following we consider a reduced model that has both a smooth saddle node and a non-smooth saddle-type (?) bifurcation, of the form

$$\frac{dx}{dt} = -\mu + 2|x| - x|x| \qquad \mu = \mu_0 - \epsilon t \tag{2.1}$$

The non-dimensionalized Stommel model is [3]

$$\frac{dS}{dt} = \eta_2 - \eta_3 S - S|T - S| \tag{2.2}$$

$$\frac{dT}{dt} = \eta_1 - T - T|T - S| \tag{2.3}$$

for S salinity and T temperature.

We find it convenient to write in terms of $V = T - S$

$$\frac{dV}{dt} = (\eta_1 - \eta_2) - V|V| - T + \eta_3(T - V) \tag{2.4}$$

$$\frac{dT}{dt} = \eta_1 - T(1 + |V|) \tag{2.5}$$

For this model there is a non-smooth bifurcation at $\eta_2 = \eta_{2\mathrm{ns}} = 0$ and $V = V_{\mathrm{ns}} = 0$, and a saddle node bifurcation at $\eta_2 = \eta_{2_c} > \eta_{2\mathrm{ns}}$.

2.1 Notes on simple models?

*** NEW - SEPT 2016 ***

A alternate 1D model was studied by Budd + Dick:

$$\frac{dx}{dt} = x(x - 1 + \mu) \tag{2.6}$$

To be compared with results below !

For historical significance, we note that the reduced (minimal) 2D model identified at CRM workshop was:

$$\begin{aligned} V' &= \eta_1 - T(1 + |V|) \\ T' &= \eta_1 - \eta_2 - T \end{aligned} \tag{2.7}$$

Note that we need to include $|V|V$ in the 2D model to get smooth saddle node as well. This is one possible reduction from the non-dimensionalized Stommel model. It overlaps with other reductions that set $\eta_3 = 0$. Such a reduced model may not admit the case where the smooth and saddle node bifurcations are in close proximity, as is the case for η_3 taking on different values.

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As indicated from Chris's notes and book (**??**) the parameter η_3 plays a critical role: Specifically, as η_3 increases, there is a critical value at which there is no longer a non-smooth saddle at $\eta_2 = \eta_{2\mathrm{ns}}$, but rather, just a change of slope of the equilibrium branch. That is, as η_3 increases, the location of the smooth saddle decreases in V and η_2, until there is a collision of smooth and non-smooth saddle node points.

Questions related to this -

(How) Does tipping change for parameter values near this collision? Does it matter? with or without randomness?

Obvious differences with 1-D problem? Is there a 1-D analog?

Rederive "basic" model.

2.2 Eigenvalues for 2D problem - contrast with 1D

3 Deterministic case: slowly varying ramp through a bifurcation

We begin with a multiple-scale analysis of (2.1) (1D- stationary points only) in the case where μ is a slowly varying bifurcation parameter, analogous to slowly varying η_2 in (2.4).

$$\frac{dx}{dt} = -\mu + 2|x| - x|x| \qquad \mu = \mu_0 - \epsilon t \tag{3.1}$$

$$x(0) < 0, \mu(0) = \mu_0 > 0 \tag{3.2}$$

As noted above, for fixed μ, there is a non-smooth bifurcation at $\mu = \mu_{\mathrm{ns}} = 0$ and $x = x_{\mathrm{ns}} = 0$. With these initial conditions, the solution x of (3.1 is attracted to the lower branch $x < 0$, and as t increases, x follows this branch until $x = 0$ is reached. For $\mu \ll 1$, and $\mu < 0$, $x \sim -\mu/2$.

Analogous to the example of a smooth saddle node bifurcation, as μ crosses μ_{ns}, x follows $-\mu/2$ for some time, eventually making a rapid transition to the upper stable branch for $\mu < 0$. The value of μ at which x makes this transition is generally called the tipping point, and we approximate its value μ_{tip} asymptotically using an approach similar to that in [4]. For $\mu < 0$ and $x > 0$, we introduce $x = \epsilon z$, $\mu = \epsilon m$ in (3.1). With the initial condition $\mu_0 = O(1)$, it is straightforward to show that $z(0) \sim 0$, since x follows the branch $\mu = -2x + x^2$ until $x = 0$ is reached. Then, linearizing about $z \ll 1$, we have

$$z_m = m - 2z \tag{3.3}$$

Solving for z yields the behaviour for x,

$$\frac{x}{\epsilon} \sim Ce^{-2\mu/\epsilon} + \frac{\mu}{2\epsilon} - 1/4\,, \tag{3.4}$$

for C a constant. This approximation breaks down for values of μ where x is not small, that is, where $e^{-2\mu/\epsilon} = O(\epsilon^{-1})$ yielding

$$2\mu_{\text{tip}} \sim \epsilon \log \epsilon \tag{3.5}$$

Note that the model (3.1) can be compared to the local behaviour of the Stommel model near the nonsmooth bifurcation. As a rough approximation, we consider the behavior of the V equation in (2.4) near the critical value $\eta_{2\text{ns}}$. Near $V = 0$, we set $\eta_2 = \eta_{2\text{ns}} + q$, and the equation for V has the form

$$\frac{dV}{dt} = -q + (\eta_1(1 - \eta_3) - \eta_3)V + O(V^2) \tag{3.6}$$

for $q < 0$ and $V > 0$. Here we have used the behaviour $T \sim \eta_1(1 - V + O(V^2))$ in this range. Figure 1 illustrates the comparison of the asymptotic approximation for $\eta_{2\text{tip}} - \eta_{2\text{ns}}$ obtained from the full simulation of (2.4) and the asymptotic approximation based on (3.5). We expect an improved approximation can be obtained through a complete analysis of (2.4).

4 Ramp on case with complex dynamics - see Chris's notes

In this section we consider the same model in the case where there are more complicated dynamics near the non-smooth bifurcation.

5 Other notes

Multiple-scale analysis of other cases and comparison with other cases (e.g. 2D, other?)

Comparison with smooth saddle-node bifurcation results (Haberman, etc)

Is the 2008 Kuehn reference relevant?

6 Stochastic case: fixed bifurcation parameter

Simple case - stationary points only in deterministic case -

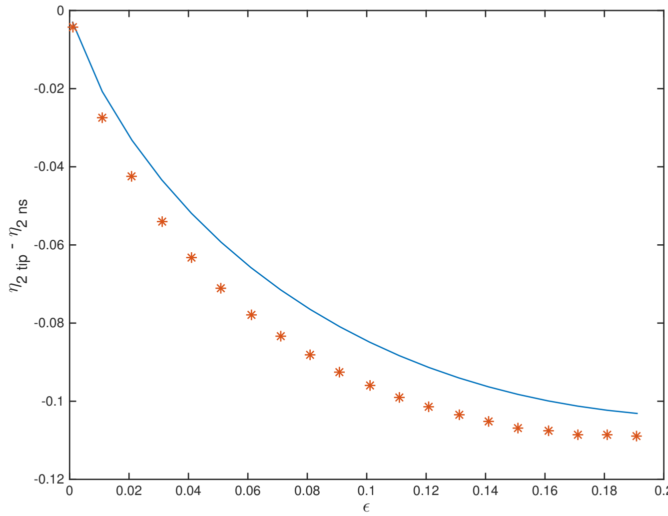

Figure 1: Comparison of $\eta_{2\mathrm{tip}} - \eta_{2\mathrm{ns}}$ as a function of the ramp speed ϵ obtained from the full Stommel model (2.4) ($*$) vs. asymptotic approximation $\epsilon \log \epsilon/(\eta_1 - \eta_1\eta_3 - \eta_3) + O(\epsilon)$ (solid line) from the analysis used to obtain (3.5). Parameters for (2.4): $\eta_1 = 4$, $\eta_3 = .375$, $\eta_2(0) = 4$, x_0

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Analyze local stochastic behavior near equilibrium and switching boundary for fixed parameter values - calculate/approximate quantities such as probability density, probability of tipping and/or crossing switching manifold.

Identify conditions for coherence resonance, heavy tails distributions, or other largish deviations.

Includes both additive and multiplicative noise.

7 Stochastic case with ramp?

Generalize approaches used in smooth case to non-smooth case? Needs to include analysis of crossing the switching manifold, different from smooth case.

References

[1] Generalize Stommel Models, CRM Climate Workshop group, Paul G. leader.

[2] Yellow book on non-smooth

[3] H. Dijkstra, Nonlinear Climate Dynamics.

[4] Haberman R. (1979), Slowly varying jump and transition phenomena associated with algebraic bifurcation, SIAM J. Appl. Math., 37(1), 69-106.