Non Smooth 1D w/ osc.

Starting with $\dot{x} = -\mu + 2|x| - x|x| + A\sin\Omega t$, fixed μ, $A \sim O(1)$, $\Omega >> 1$

Take a multiple scales approach with "slow" and "fast" time:

$\tau = t,\ T = \Omega t \Rightarrow x_T = \Omega^{-1}(-x_\tau - \mu + 2|x| - x|x| + A\sin T)$

Next, introduce the asymptotic expansion: $x \sim x_0 + \Omega^{-1}x_1 + \Omega^{-2}x_2 + \dots$

$\Rightarrow O(1): x_{0T} = 0 \longrightarrow x_0 = x_0(\tau)$

$O(\Omega^{-1}): x_{1T} = -x_{0\tau} - \mu + 2|x_0| - x_0|x_0| + A\sin T = R_1(\tau, T)$

Using the Fredholm alternative: $\lim_{T\to\infty} \frac{1}{T}\int_0^T R_1(\tau, u)\,du = 0 \rightarrow x_{0\tau} = -\mu + 2|x_0| - x_0|x_0|$

Choosing to follow along the lower stable branch $(\mu > 0, x < 0)$,

$x_{0\tau} = -\mu - 2x_0 + x_0^2$, which has stable equilibrium solution $x_0 = 1 - \sqrt{1+\mu}$

$\Rightarrow x_{1T} = A\sin T \rightarrow x_1 = -A\cos T + V_1(\tau)$

$O(\Omega^{-2}): x_{2T} = -x_{1\tau} + 2|x_1| - x_0|x_1| - x_1|x_0|$

Again, given the order of the terms, we still have $x < 0$,

$$x_{2T} = -x_{1\tau} + 2x_1(x_0 - 1)$$
$$= -V_{1\tau} + 2V_1(x_0 - 1) - 2A\cos T(x_0 - 1)$$

Fredholm once more gives, $V_{1\tau} = 2V_1(x_0 - 1) = -2\sqrt{1+\mu}\,V_1$

which has stable equilibrium $V_1 = 0$.

Thus, $x \sim 1 - \sqrt{1+\mu} + \Omega^{-1}(-A\cos T) + O(\Omega^{-2})$ $\boxed{\text{outer solution}}$

We see this solution fails when terms reorder from $\mu \sim O(\Omega^{-1})$,

It then is natural to rescale the problem,

$$\mu = \Omega^{-1}m, \quad x = \Omega^{-1}y$$

②

Nonsmooth 1D w/ osc.

This rescaling induces,

$$\Omega^{-1} y = -\Omega^{-1} m + \Omega^{-1} 2|y| - \Omega^{-2} y|y| + A\sin \Omega t$$

A similar multiple scales approach gives,

$$y_T = \Omega^{-1}(-y_\tau - m + 2|y|) - \Omega^{-2} y|y| + A\sin T$$

which suggests the expansion, $y \sim y_0 + \Omega^{-1} y_1 + \ldots$

$$\Rightarrow O(1): y_{0T} = A\sin T \longrightarrow y_0 = -A\cos T + V_0(\tau)$$

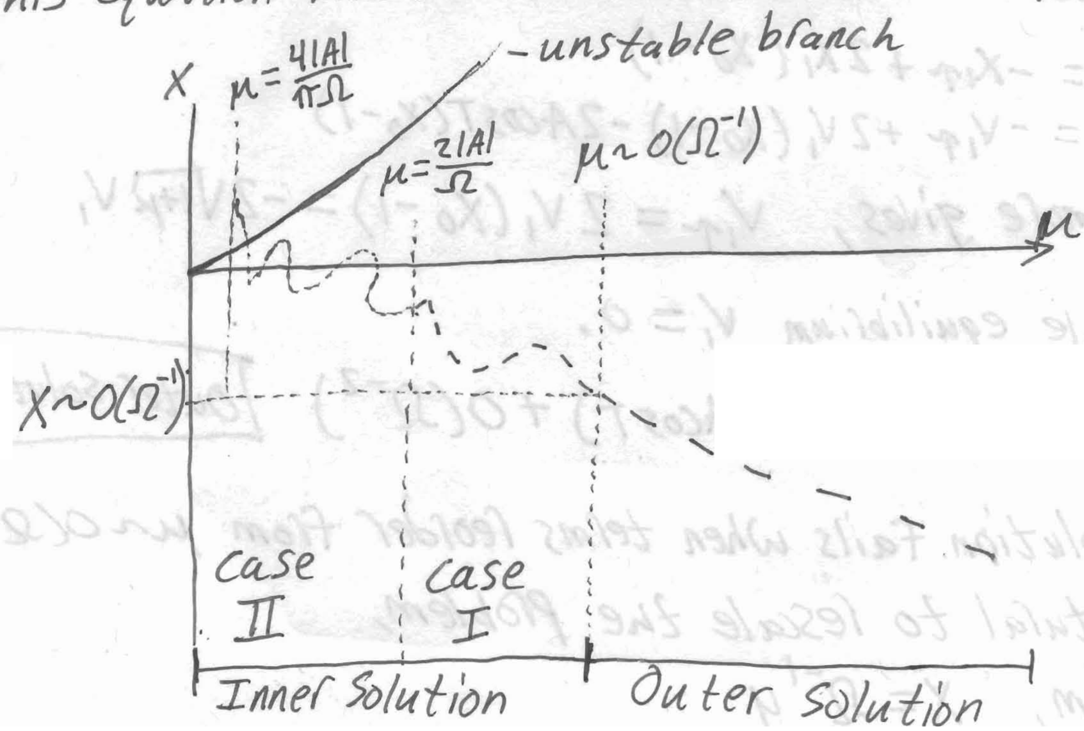
$$O(\Omega^{-1}): y_{1T} = -y_{0\tau} - m + 2|y_0|$$

Here we again impose the Fredholm alternative,

$$\Rightarrow \lim_{T\to\infty} \frac{1}{T}\int_0^T -y_{0\tau} - m + 2|y_0|\, du = 0 \Rightarrow y_{0\tau} = -m + 2\lim_{T\to\infty}\frac{1}{T}\int_0^T |-A\cos u + V_0(\tau)|\, du$$

which gives, $V_{0\tau} = -m + 2\lim_{T\to\infty}\frac{1}{T}\int_0^T |-A\cos u + V_0(\tau)|\, du$

This equation must be considered in two cases, first a picture:



③

$\boxed{\text{Non Smooth 1D w/ osc.}}$

$$V_{0T} = -m + 2\lim_{T\to\infty}\frac{1}{T}\int_0^T |-A\cos u + V_0(\tau)|\,du$$

Case I: $|V_0| \geq |A| \to$ Integral is simple: $V_{0T} = -m + 2|V_0|$

Searching for equilibrium, $|V_0| = \frac{m}{2} \Rightarrow V_0 = \frac{-m}{2}$ stable.

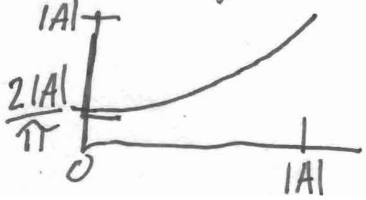
$\Rightarrow y \sim \frac{-m}{2} - A\cos T + O(\Omega^{-1}) \to x \sim \left(\frac{-\mu}{2} - A\cos T\,\Omega^{-1}\right) + O(\Omega^{-2})$

where $|V_0| \geq |A| \Leftrightarrow m \geq 2|A| \Leftrightarrow \mu \geq \frac{2|A|}{\Omega}$ (agrees with where outer soln fails)

Case II: $|V_0| < |A| \to$ Integral is nontrivial and has no analytic solution.

Approach:
- Use $m < 2|A|$ region.
- Obtain numeric results for integral. (How?)
  - $\lim_{T\to\infty}\frac{1}{T}\int_0^T |-A\cos u + V_0(\tau)|\,du = \frac{1}{2\pi}\int_0^{2\pi} |-A\cos u + V_0(\tau)|\,du$
- Find equilibrium in this region
  - The integral is bounded from above & below
  - $-m + \frac{4|A|}{\pi} \leq V_{0T} < -m + 2|A| \longrightarrow m = \frac{4|A|}{\pi}$ gives pos. V_{0T}
- Determine when equilibrium becomes unstable.
- Should see this around when the oscilations jump above the unstable branch.

Issue: The numeric integrand is tricky. I've interpolated to get:



which indicates the integral is parabolic in V_0.

$V_0(\tau)\ \frac{1}{2\pi}\int du \approx \frac{2|A|}{\pi} + \frac{1}{|A|}\left(1 - \frac{2}{\pi}\right)V_0(\tau)^2$

Non Smooth 1D OSC.

If we assume $\lim_{T\to\infty} \frac{1}{T}\int_0^T |-A\cos u + V_0(\tau)|\,du$ is quadratic in $V_0(\tau)$, certain values are known. $f(0) = \frac{2|A|}{\pi}$

$f(|A|) = |A| = f(-|A|)$

$$\Rightarrow f(V_0(\tau)) = \frac{2|A|}{\pi} + \frac{1}{|A|}\left(1-\frac{2}{\pi}\right)V_0(\tau)^2$$

$$\Rightarrow V_{0\tau} = -m + 2\lim_{T\to\infty}\frac{1}{T}\int_0^T |-A\cos u + V_0(\tau)|\,du \simeq -m + \frac{4|A|}{\pi} + \frac{2}{|A|}\left(1-\frac{2}{\pi}\right)V_0(\tau)^2$$

This approximation has maximum error of .01 for $A \sim O(1)$

(.4 for $A = 30$)

Searching for an equilibrium with this approximation,

$$V_0(\tau) \simeq \sqrt{\frac{m - \frac{4|A|}{\pi}}{\frac{2}{|A|}\left(1-\frac{2}{\pi}\right)}} = K\sqrt{m - \frac{4|A|}{\pi}} \cancel{\neq K\sqrt{\mu}}$$

which now tells us that ~~tipping~~ a bifurcation occurs $m = \frac{4|A|}{\pi}$

$$\rightarrow \mu_p = \frac{4|A|}{\pi\Omega}$$

$$\rightarrow y \sim K\sqrt{m - \frac{4|A|}{\pi}} - A\cos T + O(\Omega^{-1})$$

$$x \sim \left(K\sqrt{\Omega^{-1}\left(\mu - \frac{4|A|}{\pi\Omega}\right)} - \Omega^{-1}A\cos T\right) + O(\Omega^{-2})$$

$$f(0) = \frac{2|A|}{\pi}, \quad f(|A|) = |A| = f(-A)$$

$$f\left(\pm\frac{|A|}{2}\right) = |A|\left(\frac{1}{6} + \frac{\sqrt{3}}{\pi}\right)$$

<u>4th order</u> approx.

$$f(x) = \frac{2\cdot|A|}{\pi} + bx^2 + cx^4$$

$$\begin{cases} |A| - \frac{2|A|}{\pi} = b|A|^2 + c|A|^4 \\ |A|\left(\frac{1}{6} + \frac{\sqrt{3}}{\pi} - \frac{2}{\pi}\right) = b\frac{|A|^2}{4} + c\frac{|A|^4}{16} \end{cases}$$

$$\rightarrow \begin{cases} \cancel{|A|}\left(1 - \frac{2}{\pi}\right) = b|A| + c|A|^3 \\ 16\left(\frac{1}{6} + \frac{\sqrt{3}}{\pi} - \frac{2}{\pi}\right) = 4b|A| + c|A|^3 \end{cases} \quad \Bigg| \quad \begin{array}{l} \left(1 - \frac{2}{\pi}\right) = b|A| + c|A|^3 \\ 4\left(\frac{1}{6} + \frac{\sqrt{3}}{\pi} - \frac{2}{\pi}\right) = b|A| + \frac{1}{4}c|A|^3 \end{array}$$

$$\rightarrow b = \frac{1}{-3|A|}\left(16\left(\frac{1}{6} + \frac{\sqrt{3}}{\pi}\right) - 15\left(\frac{2}{\pi}\right) - 1\right)$$

$$c = \frac{-4}{3|A|^3}\left(4\left(\frac{1}{6} + \frac{\sqrt{3}}{\pi}\right) - 3\left(\frac{2}{\pi}\right) - 1\right)$$

$$f(V_0) = \frac{2|A|}{\pi} + \frac{1}{3|A|}\left(16\left(\frac{1}{6} + \frac{\sqrt{3}}{\pi}\right) - \frac{30}{\pi} - 1\right)V_0^2 - \frac{4}{3|A|^3}\left(4\left(\frac{1}{6} + \frac{\sqrt{3}}{\pi}\right) - \frac{6}{\pi} - 1\right)V_0^4$$

$$V_{0r} = -m + 2f(V_0) \iff 0 = \frac{-m}{2} + f(V_0)$$

Roots of $\frac{-m}{2} + f(V_0)$: $\pm\sqrt{\dfrac{-b \pm \sqrt{b^2 + 4\left(\frac{2|A|}{\pi} - \frac{m}{2}\right)c}}{2\left(\frac{2|A|}{\pi} - \frac{m}{2}\right)}}$

Issues:

$$-b^2 = \left(\frac{8|A|}{\pi} - 2m\right)c \iff m = \frac{4|A|}{\pi} + \frac{b^2}{2c} \Leftarrow$$ $c < 0$ so this is smaller

$$m = \frac{4|A|}{\pi} \longrightarrow m = \frac{4|A|}{\pi}$$ happens first

$$V_{0T} = -m + 2 \lim_{T \to \infty} \frac{1}{T} \int_0^T |-A\cos u + V_0(\tau)|\, du$$

General

Generally, $V_{0T} = f(V_0, m, A)$, search for root given V_0

$\Rightarrow$ Method: Choose a spread of $V_0(\tau)$ in an allowed range

Pick an A, search for m s.t

$$0 = f(V_0, m, A)$$

Known Seek chosen

Newton's:

$$m_{n+1} = m_n - \frac{f(V_0, m_n, A)}{f'(V_0, m_n, A)}$$

Repeat for range of A.

Result: For each V_0, will have a range of A with a corresponding m s.t $f(V_0, A, m) = 0$

or - $\forall V_0,\ \exists A = \{A_0, A_1, \cdots\}$ s.t each A_i has a $\hat{m}_i$ that solves $0 = f(V_0, A_i, \hat{m}_i)$