Non-Smooth Dynamics in the Stommel Model

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Abstract

Your abstract.

1 Introduction

MAIN RESULT: Non-smooth bifurcations are a topic that arise in special systems and for how frequent they appear, they have not been analyzed nearly as much as their smooth counter parts. This paper will discuss pinpointing the tipping behavior in the classic Stommel model around the non-smooth bifurcation as well as generalize the canonical system. First an analysis on a simpler one dimensional topologically equivalent system provides insight into approaching the full two dimensional Stommel model.

2 Background

2.1 Stommel Model

In an effort to better understand the oceanic patterns around the mixing of two bodies of differing temperature and salinity, Henry Stommel proposed the two box model in 1961. [4] In this paper, Stommel suggests that there are actually two different stability regimes in the system that he has suggested and concluded that the oceanic dynamics behave very similarly about these equilibria. These type of systems have since been a heavily studied area for both climatology due to the wide ranging applications and dynamical systems for its generalization into dual stability.

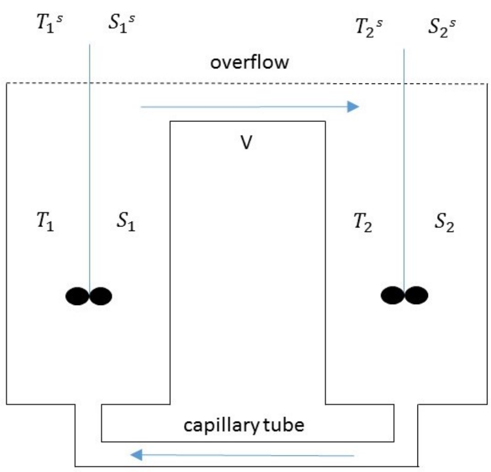


Figure 1: The Stommel Two Box Model: Differing volume boxes with a temperature and salinity T_i and S_i. The boxes are connected by an overflow and capillary tube that has a flow V. There is also a surface temperature and salinity for each box $T_i{}^s$ and $S_i{}^s$. We also assume well mixing occurs.

In the non-dimensionalized Stommel Model, we consider the system

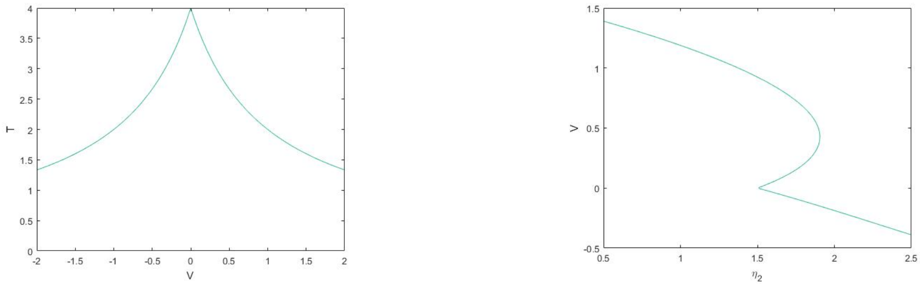
$$\frac{\mathrm{d}T}{\mathrm{d}t} = \eta_1 - T(1 + |T - S|) \tag{1}$$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \eta_2 - S(\eta_3 + |T - S|) \tag{2}$$

where the variables T, S are the temperature and salinity respectively. The constants η_1, η_3 both have physical interpretation to the relaxation times and volumes of the box. (Maybe go into more detail here?) (Also maybe talk about the range of these parameters as well as the significance of $\eta_3 <,=> 1$.)
The parameter η_2 has turns out to be much more interesting as different values were found to cause major qualitative and quantitative changes in the dynamics of the system. It has been discovered to cause sudden changes at two different points in the system, these being both a smooth and non-smooth saddle-node bifurcation.
Another way to see this system is in terms of the variable $V = T - S$, which now looks at the difference between the non-dimensionalized quantities T and S. This leads to the system:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \eta_1 - T(1 + |V|) \tag{3}$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = (\eta_1 - \eta_2) - V|V| - T + \eta_3(T - V) \tag{4}$$



(a) The equilibrium solution for T

(b) The equilibrium solution for V

Figure 2: The equilibria of the non-dimensionalized system with $\eta_1 = 4$ and $\eta_3 = .375$.

where the equation for V now becomes the focus due to it containing the interaction between every constant and variable which can also be seen in the equilibria of the system in Fig. 2. A close look at Fig. 2a shows non-smooth behavior happening at $V = 0$ and the bifurcation parameter η_2 shows the two bifurcations in Fig. 2b. In this plot, both the upper and lower branches of the equilibrium are stable with the middle branch being unstable. The smoothness of each bifurcation is apparent and arises from the non-smooth behavior of the absolute value in the defining dynamics of LABEL EQ STOMMEL.
Much is known about the Stommel model in the case where η_2 is fixed to be a constant value throughout the analysis but realistically this is not the case. This value will change very slowly and here is where the focus of this paper lies. A system with a parameter known to cause a bifurcation, varying the parameter no longer allows the bifurcation to occur in the standard sense. Instead, these conditions give rise to a smooth but rapid change in the system and where this occurs is called a tipping point.

2.2 Slowly Varying Tipping

Tipping points have recently been discovered to occur in a wide variety of systems and have become a big staple in catastrophe theory. They can aid in predicting the future of a system and even be a warning for irreversible change. A tipping point thus share similar characteristics of a bifurcation and may or may not occur close to the standard bifurcation location
(EXPLAIN WHEN TIPPING OCCURS)

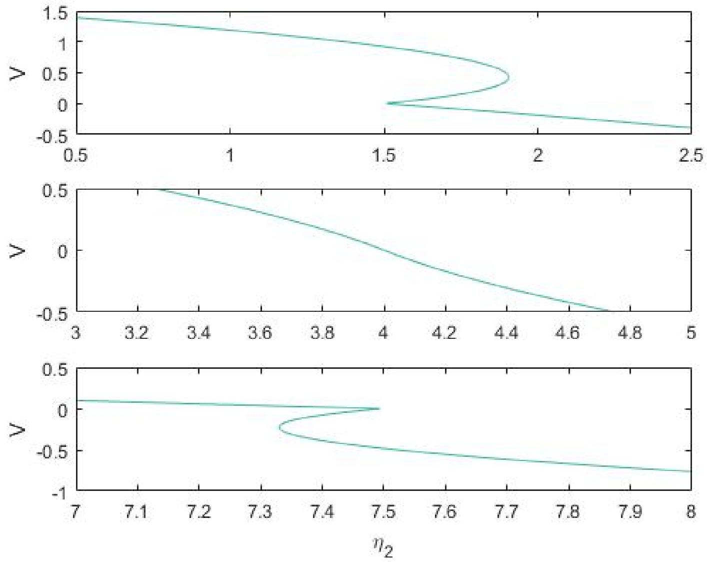


Figure 3: The choice in η_3 turns out to dictate the orientation of the problem, each panel has a fixed $\eta_1 = 4$, but it is the choice in $\eta_3 = .375,\ 1,\ 1.875$ that gives the top, middle and bottom respectfully.

3 Results

3.1 One Dimensional Model

We consider a system that is topologically equivalent to the more complex two dimensional Stommel model,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\mu + 2|x| - x|x| + A\sin(\Omega t), \quad \frac{\mathrm{d}\mu}{\mathrm{d}t} = -\epsilon, \tag{5}$$
$$x(0) = x_i, \quad \mu(0) = \mu_i,$$

where the constants used are the drift rate, $0 < \epsilon \ll 1$, A is the amplitude of oscillation and Ω is the frequency.
The system (5) is generalized from a basic model that contains both a smooth and non-smooth saddle-node bifurcation. This type of behavior gives the topological equivalence to the Stommel model and hence good reason to test generalizations on. In each case, emphasis is put on the non-smooth component of the model give further insight

3.1.1 Static μ and Bifurcations

Consider the system (5) with $A = 0$ and $\epsilon = 0$, which is our basic system with a static μ and no forcing. Setting the system equal to zero, we find there are two stable equilibrium branches. Denote these x_l and x_u for lower and upper respectfully,

$$x_l = 1 - \sqrt{1+\mu}, \quad x_u = 1 + \sqrt{1-\mu}$$

and a single unstable branch,

$$x_{un} = 1 - \sqrt{1-\mu},$$

where x_l is valid for $\mu \geq 0$ and x_u for $\mu \leq 1$. Thus this system always has a stable equilibrium for every choice in the parameter, but has a region of bi-stability for $0 \leq \mu \leq 1$. This indicates that both edges of this region are bifurcations, $\mu_{ns} = 0$ and $\mu_s = 1$, as they cause shifts in stability. Upon further inspection, the points $(x, \mu) = (0, 0)$ and $(x, \mu) = (1, 1)$ are non-smooth and smooth saddle-node bifurcations respectfully.

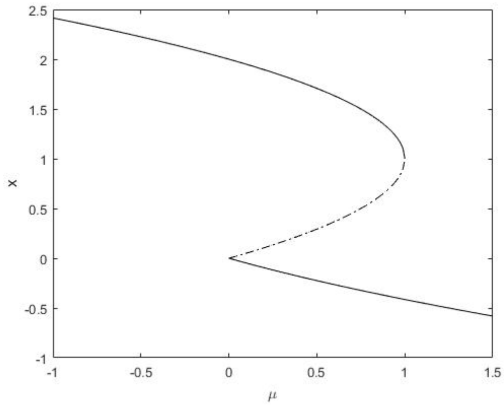


Figure 4: This is the one-dimensional bifurcation diagram and we see the upper and lower equilibrium branches as well as the unstable branch. The non-smooth bifurcation occurs at (0,0) and the smooth bifurcation occurs at (1,1). Both of which are saddle-nodes due to the annihilating equilibria.

3.1.2 Slowly varying μ

Consider (5) with $A = 0$ and $\epsilon > 0$. Here the parameter is allowed to change and thus a bifurcation no longer occurs. Instead, it should be expected that a tipping point occurs nearby the previous bifurcation points. The smooth saddle-node is well understood (see Zhu & Kuske [5]), so let us consider an approach towards discovering the non-smooth tipping point. To do so we begin by allowing the initial conditions $x_i = 1 - \sqrt{1+\mu_i}$ and $\mu_i > 0$. Now our calculations will be centered around the stable lower branch and have emphasis on $x < 0$. Now we consider a rescaling approach where its useful to introduce the notion of slow time, $t = \epsilon\tau$, with $x < 0$, the system (5) becomes

$$\epsilon\frac{\mathrm{d}x}{\mathrm{d}\tau} = -\mu - 2x + x^2, \quad \frac{\mathrm{d}\mu}{\mathrm{d}\tau} = -1. \tag{6}$$

Next, we use an asymptotic expansion in terms of the present small quantity ϵ,

$$x \sim x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots \tag{7}$$

(SEE OTHER PAPERS FOR BETTER PHRASING) This expansion captures the slow variable's dynamics and separate them by magnitude. Thus, substituting (7) into (6), we get the following system of equations.

$$\begin{aligned}
O(1): 0 = -\mu - 2x_0 + x_0^2 \Rightarrow &\quad x_0 = 1 - \sqrt{1+\mu} \\
O(\epsilon): x_{0\tau} = -2x_1 + 2x_1x_0 \Rightarrow &\quad x_1 = \frac{1}{4(1+\mu)} \\
O(\epsilon^2): x_{1\tau} = -2x_2 + 2x_0x_2 + x_1^2 \Rightarrow &\quad x_2 = \frac{-3}{32(1+\mu)^{5/2}}
\end{aligned}$$

Thus, (7) becomes

$$x \sim 1 - \sqrt{1+\mu} + \epsilon\frac{1}{4(1+\mu)} - \epsilon^2\frac{3}{32(1+\mu)^{5/2}} + \dots \tag{8}$$

Since the dynamics of x in (5) change at $x = 0$, this solution is valid only for $x < 0$ and $\mu > 0$. This gives rise to a critical point at $(x_c, \mu_c) = (0, 0)$ which is the non-smooth bifurcation, and a local analysis about this point is necessary.
Time had already been scaled but now scaling our spatial variable is a must to conduct the local analysis. A systematic search for this scaling leads to both the spatial variable and the parameter being scaled by ϵ, thus let $x = \epsilon z$ and $\mu = \epsilon m$. From this our system (5) becomes,

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -m + 2z - \epsilon z^2, \quad \frac{\mathrm{d}m}{\mathrm{d}t} = -1. \tag{9}$$

An important method to capture the interacting dynamics between the system and the parameter is to change the variable to be in terms of the parameter. Introducing this into our approach, we get a system that we can find it's leading order solution to,

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\mathrm{d}z}{\mathrm{d}m}\frac{\mathrm{d}m}{\mathrm{d}t}$$

$$-\frac{\mathrm{d}z}{\mathrm{d}m} = -m + 2z - \epsilon z^2.$$

$$z = Ce^{-2m} + \frac{m}{2} - \frac{1}{4}$$

Finally writing this in terms of the original coordinates gives,

$$\frac{x}{\epsilon} \sim Ce^{-2\mu/\epsilon} + \frac{\mu}{2\epsilon} - \frac{1}{4}. \tag{10}$$

The tipping point can be found when this solution (10) begins growing exponentially, so we focus on when the exponential term becomes large ($O(1/\epsilon)$),

$$e^{-2\mu/\epsilon} \sim O(1/\epsilon) \Rightarrow 2\mu \sim \epsilon \log(\epsilon). \tag{11}$$

With this result, we compare our estimate to numerical results to evaluate it's performance for a varying size of ϵ.

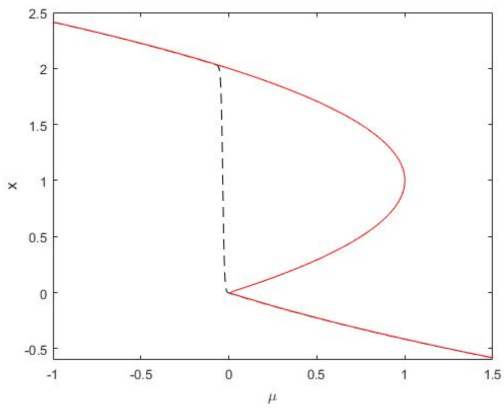


Figure 5: The numerical solution (black dotted line) to (5) with $A = 0$ and $\epsilon = .01$. For reference, the original bifurcation is given as the red solid line. The tipping occurs slightly after where the bifurcation would have occurred, here at $\mu = -.0114$ where our prediction is at $\mu = -.0230$.

(Tipping criteria isn't very good here, makes for a bit of error)

3.1.3 High Frequency Oscillatory Forcing

Consider the system (5) with $A = O(1)$ and $\epsilon = 0$, where we have oscillatory forcing but no drifting parameter; also assume that $\Omega \gg 1$ as to have a high frequency. Where the previous section required rescaling the problem due to the slowly varying dynamics, here we have dynamics occurring on both a regular time scale and a 'fast' scale. This naturally suggests a multiple scales approach where we will call our regular time $\tau = t$ and our 'fast' time $T = \Omega t$, then we search for a solution that is dependent on these scales, $x(t) = x(\tau, T)$. Introducing this method with the appropriate variable change,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} + \frac{\mathrm{d}x}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}\tau} + \Omega\frac{\mathrm{d}x}{\mathrm{d}T}$$

$$\frac{\mathrm{d}x}{\mathrm{d}T} = \Omega^{-1}\left(-\frac{\mathrm{d}x}{\mathrm{d}\tau} - \mu + 2|x| - x|x| + A\sin(T)\right). \tag{12}$$

From this system (12), we see this small quantity, Ω^{-1}, appear which suggests the asymptotic expansion be in powers of this quantity,

$$x \sim x_0 + \Omega^{-1} x_1 + \Omega^{-2} x_2 + \dots \tag{13}$$

Using (13) in our multiple scales system (12), we separate by orders of Ω to get,

$$O(1) : x_{0T} = 0 = R_0(\tau, T) \tag{14}$$

$$O(\Omega^{-1}) : x_{1T} = -x_{0\tau} - \mu + 2|x_0| - x_0|x_0| + A\sin(T) = R_1(\tau, T) \tag{15}$$

$$O(\Omega^{-2}) : x_{2T} = -x_{1\tau} - \mu + 2|x_1| - x_0|x_1| - x_1|x_0| = R_2(\tau, T) \tag{16}$$

Now that we have an equation on each order, we must be able to solve each one but further restrict our solution from having resonant or linearly growing terms. This will assure that the terms in the asymptotic expansion are compatible with one another and we have a robust solution. A common method to guarantee a solution on each order can be found with less than linearly growing terms is the Fredholm alternative which gives a solvability condition on each order,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T R_i(\tau, u)\, du = 0, \quad \forall i.$$

We also must recall that our focus is on the non-smooth behavior and thus we restrict the solution to follow along the lower stable equilibrium branch, $x < 0$. (SEE APPENDIX?) Applying these conditions to each order we can find the terms in (13),

$$x \sim 1 - \sqrt{1+\mu} - \Omega^{-1} A\cos(T) + O(\Omega^{-2}). \tag{17}$$

With this expansion, we probe it to see when it may become invalid (i.e the terms in the series begin reordering). Upon a closer look, we see that when $\mu \sim O(\Omega^{-1})$, the leading order term is now $x_0 \sim O(\Omega^{-1})$ and thus we have the scaling for an inner expansion,

$$\mu = \Omega^{-1} m, \quad x = \Omega^{-1} y.$$

Using this scaling along with the same approach as the outer solution, we have an inner multiple scales system which also suggests an expansion,

$$\frac{\mathrm{d}y}{\mathrm{d}T} = \Omega^{-1}\left(-\frac{\mathrm{d}y}{\mathrm{d}\tau} - m + 2|y|\right) - \Omega^{-2} y|y| + A\sin(T), \tag{18}$$

$$y \sim y_0 + \Omega^{-1} y_1 + \Omega^{-2} y_2 + \dots \tag{19}$$

Like before, we use (19) in (18) and collect by orders of Ω,

$$O(1) : \frac{\mathrm{d}y_0}{\mathrm{d}T} = A\sin(T) \tag{20}$$

$$O(\Omega^{-1}) : \frac{\mathrm{d}y_1}{\mathrm{d}T} = -\frac{\mathrm{d}y_0}{\mathrm{d}\tau} - m + 2|y_0| \tag{21}$$

But we now arrive at a more difficult equation to solve as the result of (FIGURE OUT HOW TO INTRODUCE THE INTEGRAL PROBLEM)

$$\frac{\mathrm{d}v_0}{\mathrm{d}\tau} = -m + 2\lim_{T\to\infty} \int_0^T |A\cos(u) + v_0(\tau)|\, du \tag{22}$$

Here we must consider two cases that determine the difficulty of the integrand, case I: if the unknown function is large enough to keep the interior from ever changing signs and case II: if it is too small and the interior can change sign.

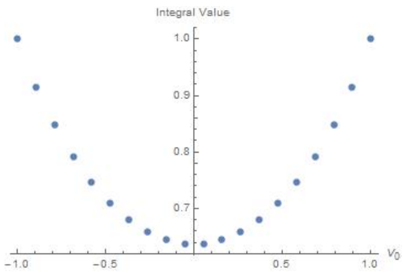
Case I: $|V_0| \geq |A|$

Case II: $|V_0| < |A|$

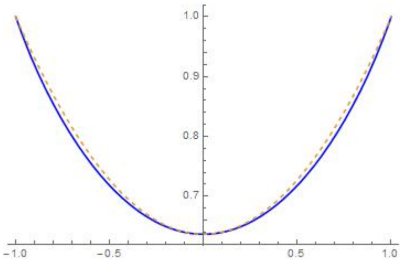
4 Some examples to get started

4.1 How to include Figures

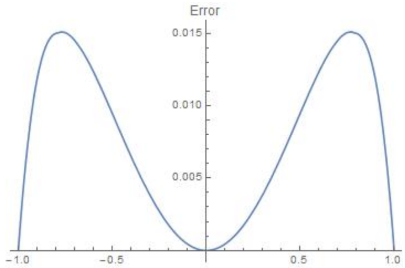
First you have to upload the image file from your computer using the upload link the project menu. Then use the includegraphics command to include it in your document. Use the figure environment and the caption command to add a number and a caption to your figure. See the code for Figure



(a) Interior points for V_0



(b) Quadric (orange dotted) and Chebyshev (blue solid) interpolation.



(c) The resulting error.

Figure 6: Here is a comparison of a quadratic interpolation to chebyshev interpolation given a range of 20 interior points.

Item	Quantity
Widgets	42
Gadgets	13

Table 1: An example table.

4.2 How to add Comments

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Comments can also be added to the margins of the compiled PDF using the todo command, as shown in the example on the right. You can also add inline comments:

Here's a comment in the margin!

This is an inline comment.

4.3 How to add Tables

Use the table and tabular commands for basic tables — see Table 1, for example.

4.4 How to write Mathematics

LaTeX is great at typesetting mathematics. Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed random variables with $\mathrm{E}[X_i] = \mu$ and $\mathrm{Var}[X_i] = \sigma^2 < \infty$, and let

$$S_n = \frac{X_1 + X_2 + \cdots + X_n}{n} = \frac{1}{n} \sum_i^n X_i$$

denote their mean. Then as n approaches infinity, the random variables $\sqrt{n}(S_n - \mu)$ converge in distribution to a normal $\mathcal{N}(0, \sigma^2)$.

4.5 How to create Sections and Subsections

Use section and subsections to organize your document. Simply use the section and subsection buttons in the toolbar to create them, and we'll handle all the formatting and numbering automatically.

4.6 How to add Lists

You can make lists with automatic numbering ...

1. Like this,
2. and like this.

... or bullet points ...

- Like this,
- and like this.

4.7 How to add Citations and a References List

You can upload a `.bib` file containing your BibTeX entries, created with JabRef; or import your Mendeley, CiteULike or Zotero library as a `.bib` file. You can then cite entries from it, like this:. Just remember to specify a bibliography style, as well as the filename of the `.bib`. [4] [2] [3] [5] [1]
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References

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[3] Richard Haberman. Slowly varying jump and transition phenomena associated with algebraic bifurcation problems. *SIAM Journal on Applied Mathematics*, 37(1):69–106, 1979.

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[5] Jielin Zhu, Rachel Kuske, and Thomas Erneux. Tipping points near a delayed saddle node bifurcation with periodic forcing. *SIAM journal on applied dynamical systems*, 14(4):2030–2068, 2015.