Trying to Reduce 2D-model into a solvable 1D: ①

$\dot{V} = \Omega_1 - \Omega_2 + \Omega_3(T-V) - T - V|V| + A\sin\Omega t$
$\dot{T} = \Omega_1 - T(1+|V|) + B\sin\Omega t$
$\dot{\Omega}_2 = -\varepsilon$

$\rightarrow T = \frac{\Omega_1}{1+|V|} + \frac{1}{1+|V|} B\sin\Omega t$ ← Psuedo-equilibrium "quasi-steady approx" This holds since $V$ is slowly changing

$\Rightarrow \dot{V} = \Omega_1 - \Omega_2 - (1-\Omega_3)\left[\frac{\Omega_1}{1+|V|} + \frac{B\sin\Omega t}{1+|V|}\right] - \Omega_3 V - V|V| + A\sin\Omega t$
$\dot{\Omega}_2 = -\varepsilon$

Cullently, NS Bif: $(\Omega_2, V) = (\Omega_1\Omega_3, 0) \Rightarrow T = \Omega_1$

Know $V = \varepsilon x$
$\Rightarrow \varepsilon\dot{x} = \Omega_1 - \Omega_2 - (1-\Omega_3)\left[\frac{\Omega_1}{1+\varepsilon|x|} + \frac{B\sin\Omega t}{1+\varepsilon|x|}\right] - \varepsilon\Omega_3 x - \varepsilon^2 x|x| + A\sin\Omega t$
When $x=0 \Rightarrow 0 = \Omega_1 - \Omega_2 - (1-\Omega_3)\Omega_1$ (for $A=B=0$)
$\rightarrow \Omega_2 = \Omega_1\Omega_3 + \varepsilon m$

$\Rightarrow \varepsilon\dot{x} = \Omega_1 - \Omega_1\Omega_3 - \varepsilon m - (1-\Omega_3)\left[\frac{\Omega_1}{1+\varepsilon|x|} + \frac{B\sin\Omega t}{1+\varepsilon|x|}\right] - \varepsilon\Omega_3 x - \varepsilon^2 x|x| + A\sin\Omega t$
$\dot{m} = -1$
Here, introduce Taylor expansion for $\frac{1}{1+\varepsilon|x|} \approx 1 - \varepsilon|x| + \varepsilon^2 x^2 + O(x^3)$

$\Rightarrow \dot{x} = -m + \Omega_1(1-\Omega_3)|x| + (1-\Omega_3)|x|B\sin\Omega t - \Omega_3 x$
$\quad - \varepsilon(1-\Omega_3)x^2 B\sin\Omega t - \varepsilon\Omega_1(1-\Omega_3)x^2 - \varepsilon x|x|$
$\quad + \varepsilon^{-1}(A-(1-\Omega_3)B)\sin\Omega t$
$\dot{m} = -1$
Multiple scales: $\tau = t$, $R = \varepsilon^{-\lambda} t = \Omega t$
$X_R + \varepsilon^{\lambda} X_\tau = \varepsilon^{\lambda}(-m + \Omega_1(1-\Omega_3)|x| + (1-\Omega_3)|x|B\sin R - \Omega_3 X)$
$\quad - \varepsilon^{\lambda+1}((1-\Omega_3)x^2 B\sin R + \Omega_1(1-\Omega_3)x^2 + x|x|)$
$\quad + \varepsilon^{\lambda-1}(A-(1-\Omega_3)B)\sin R$
$m_\tau = -1$ $\Longrightarrow \lambda > 1$

For $\lambda > 1$; $O(1)$; $X_{0R} = 0$

$$O(\varepsilon^{\lambda}):\ X_{1R} + X_{0T} = -m + n_1(1-n_3)|X_0| + (1-n_3)|X_0| B\sin R - n_3 X_0 + \varepsilon^{-1}(A - (1-n_3)B)\sin R$$

Fredholm

$$\Rightarrow X_1 = P_1(T) - \varepsilon^{-1}(A-(1-n_3)B)\cos R - \overbrace{(1-n_3)|X_0(T)| B\cos R}^{\text{This seems like an issue}}$$

$$X_{0T} = -m + n_1(1-n_3)|X_0| - n_3 X_0$$

But, we have $x \sim X_0 + \varepsilon^{\lambda} P_1(T) - \varepsilon^{\lambda-1}(A-(1-n_3)B)\cos R$

$$\Rightarrow V \sim \varepsilon X - \varepsilon^{\lambda}(A-(1-n_3)B)\cos R$$

Let $K = A-(1-n_3)B$, Center to see $\lambda \le 1$; $V = \varepsilon X - \varepsilon^{\lambda} K\cos R$

$$\Rightarrow \dot{X} = -m + n_1(1-n_3)|X - \varepsilon^{\lambda-1}K\cos\Omega t| + (1-n_3)|X - \varepsilon^{\lambda-1}K\cos\Omega t| B\sin\Omega t - n_3(X - \varepsilon^{\lambda-1}K\cos\Omega t)$$
$$- \varepsilon^{-1}\left[(1-n_3)(\varepsilon X - \varepsilon^{\lambda}K\cos\Omega t)^2 B\sin\Omega t + n_1(1-n_3)(\varepsilon X - \varepsilon^{\lambda}K\cos\Omega t)^2 + \varepsilon X|\varepsilon X - \varepsilon^{\lambda}K\cos\Omega t| - \varepsilon^{\lambda}K\cos\Omega t|\varepsilon X - \varepsilon^{\lambda}K\cos\Omega t|\right]$$

Multiple Scales: $T = t$, $R = \varepsilon^{-\lambda} t = \Omega t$

$$X_R + \varepsilon^{\lambda} X_T = \varepsilon^{\lambda}\left[-m + n_1(1-n_3)|X - \varepsilon^{\lambda-1}K\cos R| + (1-n_3)|X - \varepsilon^{\lambda-1}K\cos R| B\sin R - n_3(X - \varepsilon^{\lambda-1}K\cos R)\right]$$
$$- \varepsilon^{\lambda+1}\left[(1-n_3)X^2 B\sin R + n_1(1-n_3)X^2 + X|X - \varepsilon^{\lambda-1}K\cos R|\right]$$
$$+ \varepsilon^{2\lambda}\left[2(1-n_3)XK\cos R\, B\sin R + n_1(1-n_3)XK\cos R - K\cos R|X - \varepsilon^{\lambda-1}K\cos R|\right]$$
$$- \varepsilon^{3\lambda-1}\left[(1-n_3)K^2\cos^2 R\, B\sin R + n_1(1-n_3)K^2\cos^2 R\right]$$

Here we have both $\varepsilon^{3\lambda-1}$ and $\varepsilon^{2\lambda-1}$ terms outside of $|\cdot|$,

Thus $\lambda \ge 1/2$: $O(1)$: $X_{0R} = 0$

$$O(\varepsilon^{\lambda}):\ X_{1R} + X_{0T} = -m + n_1(1-n_3)|X_0 - \varepsilon^{\lambda-1}K\cos R| + (1-n_3)|X_0 - \varepsilon^{\lambda-1}K\cos R| B\sin R - n_3 X_0$$

This is solvable!

Solving the inner eq: 

$\lambda > 1$: $X_{0\tau} = -m + n_1(1-n_3)|X_0| - n_3 X_0$

$\Rightarrow X_{0m} = m - n_1(1-n_3)|X_0| + n_3 X_0$

Recall from 2D slow: $\begin{pmatrix} X_{om} \\ y_{om} \end{pmatrix} = \begin{pmatrix} n_3 & 1-n_3 \\ n_1 \, \mathrm{Sgn}(X_0) & 1 \end{pmatrix} \begin{pmatrix} X_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} m \\ 0 \end{pmatrix}$

Solving $y_0$ in terms of $X_0$ (equivalent to T in terms of V)

$\Rightarrow y_0 = -n_1 |X_0|$

$\rightarrow X_{0m} = m - n_1(1-n_3)|X_0| + n_3 X_0$

Exact same equation! Purely slow reduced case agrees!

In the purely slow case, we expect tipping to occur for $X > 0$

$\Rightarrow X_{0m} = m - (n_1(1-n_3) - n_3)X_0 \rightarrow X_0 \sim C e^{-(n_1(1-n_3)-n_3)m} + \dfrac{(n_1(1-n_3)-n_3)m - 1}{(n_1(1-n_3)-n_3)^2}$

Where tipping will occur for $m = \ln\varepsilon / (n_1(1-n_3) - n_3)$

$\Rightarrow n_2 = n_1 n_3 + \varepsilon \ln\varepsilon / (n_1(1-n_3) - n_3)$

Solving inner eq:

$$\lambda \leq 1: \; X_{0\tau} = -m + \frac{\eta_1(1-\eta_3)}{2\pi}\int_0^{2\pi} |X_0 - \varepsilon^{\lambda-1} K\cos R|\,dR + \frac{(1-\eta_3)B}{2\pi}\int_0^{2\pi} \sin R\,|X_0 - \varepsilon^{\lambda-1}K\cos R|\,dR - \eta_3 X_0$$

The period of $|\sin X|$ is $\pi \Rightarrow |X_0 - \varepsilon^{\lambda-1}K\cos R|$ also is period $\pi$

Thus $\sin R\,|X_0 - \varepsilon^{\lambda-1}K\cos R|$ is odd with period $\pi \Rightarrow$ second int $=0$

$$\Rightarrow X_{0\tau} = -m + \frac{\eta_1(1-\eta_3)}{2\pi}\int_0^{2\pi} |X_0 - \varepsilon^{\lambda-1}K\cos R|\,dR - \eta_3 X_0$$

Using direct integration,

$$X_{0\tau} = -m + \frac{2\eta_1(1-\eta_3)}{\pi}\left[\sin^{-1}\left(\frac{X_0}{\varepsilon^{\lambda-1}K}\right)X_0 + \sqrt{(\varepsilon^{\lambda-1}K)^2 - X_0^2}\,\right] - \eta_3 X_0$$

Taylor approx: $X_{0\tau} \approx \left(-m + \frac{2\eta_1(1-\eta_3)\varepsilon^{\lambda-1}K}{\pi}\right) - \eta_3 X_0 + \frac{\eta_1(1-\eta_3)}{\pi\varepsilon^{\lambda-1}K}X_0^2$

Forming this into Airy: ① $X_{0\tau} \approx (-m + c_0) - \eta_3 X_0 + c_1 X_0^2$

$$= \left(-m + c_0 - \frac{\eta_3^2}{4c_1}\right) + \left(\sqrt{c_1}\,X_0 - \frac{\eta_3}{2\sqrt{c_1}}\right)^2$$

Let $y_0 = \sqrt{c_1}\,X_0 - \frac{\eta_3}{2\sqrt{c_1}} \Rightarrow y_{0\tau} = \sqrt{c_1}\,X_{0\tau}$

$$y_{0\tau} = \sqrt{c_1}\left[\left(-m + c_0 - \frac{\eta_3^2}{4c_1}\right) + y_0^2\right]$$

$\cancel{\text{Let } \tau = \sqrt{c_1}\,\tilde{\tau} \Rightarrow y_{0\tilde{\tau}} = \left(-m + c_0 - \frac{\eta_3^2}{4c_1}\right) + y_0^2, \; m_{\tilde{\tau}} =}$

Shifting focus onto $m$,

② $y_{0m} = \sqrt{c_1}\left(m + \frac{\eta_3^2}{4c_1} - c_0\right) - \sqrt{c_1}\,y_0^2$

Using either eq 1 or 2 we can apply the result from the smooth case,

$$m_{tip} = (c_1)^{-1/3}(-2.33810\ldots) + \left(c_0 - \frac{n_3^2}{4c_1}\right)$$

$$= -\varepsilon^{(\lambda-1)/3}\left(\frac{\pi K}{n_1(1-n_3)}\right)^{1/3}\cdot(2.33810\ldots) + \varepsilon^{\lambda-1}\frac{n_1(1-n_3)K}{\pi}\left(2 - \left(\frac{\pi n_3}{n_1(1-n_3)}\right)^2\right)$$

$$\Rightarrow n_{2tip} = n_1 n_3 - \varepsilon^{(\lambda+2)/3}\left(\frac{\pi|K|}{n_1(1-n_3)}\right)^{1/3}\cdot(2.33810\ldots) + \frac{n_1(1-n_3)|K|}{\pi\Omega}\left(2 - \left(\frac{\pi n_3}{n_1(1-n_3)}\right)^2\right)$$

This reduced approx. only does good in a very small range of $\varepsilon$ (i.e $\varepsilon \approx 10^{-3}$).

Full inner:

$$\begin{pmatrix} x_{0T} \\ y_{0T} \end{pmatrix} = \begin{pmatrix} -n_3 & -(1-n_3) \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \begin{pmatrix} m \\ \frac{n_1}{\pi\varepsilon^{\lambda-1}|A|}x_0^2 + \frac{2n_1\varepsilon^{\lambda-1}|A|}{\pi} \end{pmatrix}$$

Which if we reduce with $y_0 = \frac{-n_1}{\pi\varepsilon^{\lambda-1}|A|}x_0^2 - \frac{2n_1\varepsilon^{\lambda-1}|A|}{\pi}$

$$\Rightarrow x_{0T} = -m + \frac{2n_1(1-n_3)\varepsilon^{\lambda-1}|A|}{\pi} - n_3 x_0 + \frac{n_1(1-n_3)}{\pi\varepsilon^{\lambda-1}|A|}x_0^2$$

which is the same inner equation (with $K=A$).

This is a result from giving $T = \frac{n_1}{1+|V|} + \frac{B\cos\Omega t}{1+|V|}$, but cos shouldn't be included here.

$\Rightarrow$ Reduced equations agree for $\lambda > 1$ and $\lambda \in [1/2, 1]$