# Additive List Coloring of Planar Graphs with Given Girth

Axel Brandt[1] Jennifer Diemunsch[2] Sogol Jahanbekam[3]

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**Abstract**

An *additive coloring* of a graph $G$ is a labeling of the vertices of $G$ from $\{1, 2, \ldots, k\}$ such that two adjacent vertices have distinct sums of labels on their neighbors. The least integer $k$ for which a graph $G$ has an additive coloring is called the *additive coloring number* of $G$, denoted $\chi_\Sigma(G)$. In 2009, Czerwiński, Grytczuk, and Żelazny conjectured that $\chi_\Sigma(G) \leq \chi(G)$, where $\chi(G)$ is the chromatic number of $G$. In this paper, we improve the current bounds for particular classes of graphs with a strengthening of the results through a list version of additive coloring. We apply the discharging method and the Combinatorial Nullstellensatz to show that every planar graph $G$ with girth at least 5 has $\chi_\Sigma(G) \leq 19$, and for girth at least 6, 7, and 26, $\chi_\Sigma(G)$ is at most 9, 8, and 3, respectively. Our result for the class of non-bipartite planar graphs of girth at least 26 is best possible and proves the conjecture for this class of graphs.
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## 1 Introduction

In this paper we only consider simple, finite, undirected graphs. For such a graph $G$, let $V(G)$ denote the vertex set and $E(G)$ the edge set of $G$. When $G$ is a plane graph, let $F(G)$ be the set of faces of $G$ and $l(f)$ be the length of a face $f$. For brevity when discussing a planar graph $G$, we will abuse notation by assuming $F(G)$ refers to the faces of a fixed plane embedding of $G$. Unless otherwise specified, we refer the reader to [17] for notation and definitions.

We consider a derived vertex coloring in which each vertex receives a color based on assigned labels of its neighbors. An *additive coloring* of a graph $G$ is a labeling of the

[1]Department of Mathematics and Computer Science, Davidson College, axbrandt@davidson.edu, Partially funded by NSF GK-12 Transforming Experiences Grant DGE-0742434
[2]Department of Mathematics, Saint Vincent College, jennifer.diemunsch@stvincent.edu
[3]School of Mathematical Sciences, Rochester Institute of Technology, sxjsma@rit.edu

vertices of $G$ with numbers such that two adjacent vertices have distinct sums of labels on their neighbors. The least integer $k$ for which a graph $G$ has an additive coloring using labels in $\{1, \ldots, k\}$ is called the *additive coloring number* of $G$, denoted $\chi_\Sigma(G)$.

We briefly mention that additive coloring was introduced in the literature as *lucky labeling* by Czerwiński, Grytczuk, and Żelazny [10]. Another name for this coloring, *open distinguishing*, has recently been suggested by Axenovich et al. [5]. The authors have chosen to use additive coloring based on its descriptive name and appearance in the survey by Seamone [14] that offers intuitive notation to relate variations of graph colorings.

In 2009, Czerwiński, Grytczuk, and Żelazny proposed the following conjecture for the additive coloring number of $G$:

**Conjecture 1.1** ([10]). *For every graph $G$, $\chi_\Sigma(G) \leq \chi(G)$.*

If true, complete graphs imply that this conjecture is best possible. This conjecture remains open even for bipartite graphs, for which no constant bound is currently known. However, Czerwiński, Grytczuk, and Żelazny [10] showed the following:

**Theorem 1.2** ([10]). *If $G$ is a bipartite graph with an orientation in which each vertex has out-degree at most $k$, then $\chi_\Sigma(G) \leq k + 1$.*

They also showed that $\chi_\Sigma(G) \leq 2$ when $G$ is a tree, $\chi_\Sigma(G) \leq 3$ when $G$ is bipartite and planar, and $\chi_\Sigma(G) \leq 100, 280, 245, 065$ for every planar graph $G$. Note that if Conjecture 1.1 is true, then $\chi_\Sigma(G) \leq 4$ for any planar graph $G$. The bound for planar graphs was later improved by Bartnicki et al. [6].

**Theorem 1.3** ([6]). *If $G$ is a planar graph, then $\chi_\Sigma(G) \leq 468$.*

The *girth* of a graph is the length of its shortest cycle, which is especially useful in giving a measure of sparseness. In the same paper, Bartnicki et al. [6] prove the following:

**Theorem 1.4** ([6]). *If $G$ is a planar graph of girth at least 13, then $\chi_\Sigma(G) \leq 4$.*

Their proof provides a labeling for an I,F-partition of a graph, a partition of the vertex set in which $I$ is a set of vertices that have pairwise distance greater than 2 and the vertices in $F$ induce a forest. After providing an additive labeling for any I,F-partition, they cite a result of Bu et al. [8] that guarantees the existence of an $I, F$-partition for planar graphs with girth at least 13. Referencing a more recent result on the existence of I,F-partitions gives a stronger result; Brandt et al. [7] guarantee the existence of an I,F-partition for all graphs $G$ with $\text{mad}(G) < \frac{5}{2}$, where $\text{mad}(G) := \max\limits_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$ denotes the *maximum average degree* of $G$. This result implies that $\chi_\Sigma(G) \leq 4$ for all $G$ with $\text{mad}(G) < \frac{5}{2}$. A known relationship between girth and maximum average degree presented in Proposition 3.2 gives the following:

**Theorem 1.5.** *If $G$ is a planar graph with girth at least 10, then $\chi_\Sigma(G) \leq 4$.*

Since the bound on maximum average degree is tight in the sense that there are graphs with maximum average degree $\frac{5}{2}$ that do not have an I,F-partition, we focus on determining bounds on $\chi_\Sigma$ given by various girth assumptions:

**Theorem 1.6.** *Let $G$ be a planar graph with girth $g$.*

1. *If $g \geq 5$, then $\chi_\Sigma(G) \leq 19$.*
2. *If $g \geq 6$, then $\chi_\Sigma(G) \leq 9$.*
3. *If $g \geq 7$, then $\chi_\Sigma(G) \leq 8$.*
4. *If $g \geq 26$, then $\chi_\Sigma(G) \leq 3$.*

These results are also true for a list version of additive coloring, which we discuss in the next paragraph. We briefly remark that the proofs for these bounds use the Combinatorial Nullstellensatz within reducibility arguments of the discharging method. The combination of these two popular techniques is a novel approach that can eliminate a considerable amount of case analysis during discharging arguments. This approach also enables choosability results rather than just colorability results.

In 2013, Akbari et al. [2] proposed the list version of additive coloring. A graph is *additively $k$-choosable* if whenever each vertex is given a list of at least $k$ available integers, then an additive coloring can be chosen from the lists. The *additive choice number* of a graph $G$ is the minimum positive integer $k$ such that $G$ is additive $k$-choosable, and is denoted by $\mathrm{ch}_\Sigma(G)$. Ahadi and Dehghan [1] showed that $\chi_\Sigma$ and $\mathrm{ch}_\Sigma$ can be arbitrarily far apart. Our approach contributes toward the implicit question of which graphs $G$ have $\chi_\Sigma(G) = \mathrm{ch}_\Sigma(G)$. Akbari et al. [2] showed the following:

**Theorem 1.7** ([2])**.** *If $G$ is a forest, then* $\mathrm{ch}_\Sigma(G) \leq 3$*.*

**Theorem 1.8** ([2])**.** *For every graph $G$ with $\Delta(G) \geq 2$, $\mathrm{ch}_\Sigma(G) \leq \Delta(G)^2 - \Delta(G) + 1$.*

Axenovich et al. [5] have recently shown that $\mathrm{ch}_\Sigma(G) \leq 5\Delta(G)+1$ for all planar $G$, which improves Theorem 1.3 when $\Delta(G) \leq 93$.

Similar to the additive coloring number of a graph, Chartrand, Okamoto, and Zhang [9] defined $\sigma(G)$ to be the smallest integer $k$ such that $G$ has an additive coloring using $k$ distinct labels. They showed that $\sigma(G) \leq \chi(G)$. Note that $\sigma(G) \leq \chi_\Sigma(G)$, since with $\chi_\Sigma(G)$ we seek the smallest $k$ such that labels are from $\{1, \dots, k\}$, even if some integers in $\{1, \dots, k\}$ are not used as labels, whereas $\sigma(G)$ considers the fewest distinct labels, regardless of the value of the largest label. They showed that $\sigma(C_n) = \chi(C_n)$ for all $n \geq 3$. Notice that Theorems 1.2 and 1.8 then imply that $\chi_\Sigma(C_k) = \chi(C_k)$ for $k \in \mathbb{N}$. Thus, Theorem 1.6 Part 4 is sharp in that the upper bound on $\mathrm{ch}_\Sigma$ can not be improved.

Various 3–colorings of planar graphs have been obtained under certain girth assumptions. For example, Grötzsch [11] proved that planar graphs with girth at least 4 are 3–colorable and Thomassen [16] proved that planar graphs with girth at least 5 are 3–list–colorable. Combined with Grötzsch's result, our result answers Conjecture 1.1 in the affirmative for non-bipartite planar graphs with girth at least 26.

Determining the additive coloring number of a graph is a natural variation of a well–studied problem posed by Karoński, Łuczak and Thomason [13], in which edge labels from $\{1, \dots, k\}$ are summed at incident vertices to induce a vertex coloring. Karoński, Łuczak and Thomason conjectured that edge labels from $\{1, 2, 3\}$ are enough to yield a proper vertex coloring of graphs with no component isomorphic to $K_2$. This conjecture is known as the 1,2,3–Conjecture and is still open. In 2010, Kalkowski, Karoński and Pfender [12] showed that labels from $\{1, 2, 3, 4, 5\}$ suffice.

In Section 2 we introduce the notation and tools that are used throughout the remainder of the paper. We also give an overview of how we use the discharging method and the Combinatorial Nullstellensatz. In Section 3 we obtain the results of Theorem 1.6.

# 2 Notation and Tools

Let $N_G(v)$ be the open neighborhood of a vertex $v$ in a graph $G$. For a labeling $\ell(u) : V(G) \to \mathbb{Z}$ and for $v \in V(G)$, let $S_G(v) = \sum\limits_{u \in N_G(v)} \ell(u)$. When the context is clear we use $S(v)$ in place of $S_G(v)$. For convenience, a *$j$–vertex*, *$j^-$–vertex*, or *$j^+$–vertex* is a vertex with degree $j$, at most $j$, or at least $j$, respectively. Similarly, a *$j$-neighbor* (respectively *$j^-$–neighbor* or *$j^+$–neighbor*) of $v$ is a $j$–vertex (respectively $j^-$–vertex or $j^+$–vertex) adjacent to $v$.

For sets $A$ and $B$ of real numbers $A \oplus B$ is defined to be the set $\{a + b\colon\ a \in A, b \in B\}$. Likewise, $A \ominus B$ is defined to be the set $\{a - b\colon\ a \in A, b \in B\}$. When $B = \emptyset$, we define $A \oplus B = A \ominus B = A$. We use the following known result from additive combinatorics:

**Proposition 2.1.** *Let $A_1, \ldots, A_r$ be finite sets of real numbers. We have*

$$|A_1 \oplus \cdots \oplus A_r| \geq 1 + \sum_{i=1}^{r} (|A_i| - 1).$$

*Proof.* We apply induction on $\sum\limits_{i=1}^{r} |A_i|$. When $\sum\limits_{i=1}^{r} |A_i| = 1$, all but one $A_i$ are empty, so we have $|A_1 \oplus \cdots \oplus A_r| = 1$, as desired.

Now suppose that $\sum\limits_{i=1}^{r} |A_i| = n$. We may suppose that all $A_i$ are nonempty. Let $a_i$ be the minimum element of $A_i$ for $i \in \{1, \ldots, n\}$. Let $A_1' = A_1 - \{a_1\}$. By the induction hypothesis we have $|A_1' \oplus A_2 \oplus \cdots \oplus A_r| \geq 1 + \sum\limits_{i=1}^{r} (|A_i| - 1) - 1$. However $|A_1 \oplus \cdots \oplus A_r| \geq |\{a_1 + \cdots + a_r\}| + |A_1' \oplus A_2 \oplus \cdots \oplus A_r|$. Therefore $|A_1 \oplus \cdots \oplus A_r| \geq 1 + \sum\limits_{i=1}^{r} (|A_i| - 1)$. $\square$

Note that $A \oplus (-B)$ is the same as $A \ominus B$, where $-B = \{-b : b \in B\}$. This yields the following known corollary:

**Corollary 2.2.** *Let $A$ and $B$ be nonempty sets of positive real numbers. We have $|A \ominus B| \geq |A| + |B| - 1$.*

Throughout, we consider when endpoints of edges need different sums to yield an additive coloring. For this reason, if we know $S(u) \neq S(v)$ for an edge $uv$ of $G$, we say that $uv$ is *satisfied*; $uv$ is *unsatisfied* otherwise.

Our proofs rely on applying the discharging method. This proof technique assigns an initial charge to vertices and possibly faces of a graph and then distributes charge according to a list of discharging rules. A configuration is *$k$–reducible* if it cannot occur in a vertex minimal graph $G$ with $\mathrm{ch}_\Sigma(G) > k$. Note that any $k$–reducible configuration is also $(k+1)$–reducible. When applying the discharging method in Theorem 3.11 we require the following known proposition, which is a simple application of Euler's Formula (see [17]):

**Proposition 2.3.** *Given a planar graph $G$,*

$$\sum_{f \in F(G)} (l(f) - 4) + \sum_{v \in V(G)} (d(v) - 4) = -8.$$

The main tool we use to determine when configurations are $k$-reducible is the Combinatorial Nullstellensatz, which is applied to certain graph configurations.

**Theorem 2.4** (Combinatorial Nullstellensatz [4])**.** *Let $f$ be a polynomial of degree $t$ in $m$ variables over a field $\mathbb{F}$. If there is a monomial $\prod x_i^{t_i}$ in $f$ with $\sum t_i = t$ whose coefficient is nonzero in $\mathbb{F}$, then $f$ is nonzero at some point of $\prod T_i$, where each $T_i$ is a set of $t_i + 1$ distinct values in $\mathbb{F}$.*

# 3 Main Result

We begin by presenting some reducible configurations for general $k \in \mathbb{N}$ that will be used in each subsection. Here and in each subsection, the reducible configurations will use the following notation. Let $\mathcal{L} : V(G) \to 2^{\mathbb{R}}$ be a function on $V(G)$ such that $|\mathcal{L}(v)| = k$ for each $v \in V(G)$. Thus $\mathcal{L}(v)$ denotes a list of $k$ available labels for $v$. In each proof we take $G$ to be a vertex minimal graph with $\mathrm{ch}_\Sigma(G) > k$. Then we define a proper subgraph $G'$ of $G$ with $V(G') \subsetneq V(G)$. By the choice of $G$, $G'$ has an additive coloring $\ell$ such that $\ell(v) \in \mathcal{L}(v)$ for all $v \in V(G')$. This labeling of $G'$ is then extended to an additive coloring of $G$ by defining $\ell(v)$ for $v \in V(G) \setminus V(G')$. We discuss the details of this approach in Lemma 3.1. The remaining reducible configurations are similar in approach, so we include fewer details in the proofs.

**Lemma 3.1.** *The following configurations are $k$–reducible in the class of graphs with girth at least 5:*

*(a) A vertex $v$ with $\sum_{u \in N(v)} d(u) < k$.*

*(b) A vertex $v$ with $r$ neighbors of degree 1 and a set $Q$ of 2–neighbors $\{v_1, \ldots, v_q\}$ each having a $(k-1)^-$–neighbor other than $v$, say $v_1', \ldots, v_q'$, respectively, such that $v_1', \ldots, v_q'$ are independent and $1 + r(k-1) + \sum_{v_i \in Q} (k - d(v_i') - 1) > d(v)$.*

*Proof.* Assume $G$ is a vertex minimal graph with $\mathrm{ch}_\Sigma(G) > k$ containing the configuration described in (a). Let $G' = G - \{v\}$. Since $G$ is vertex minimal, $\mathrm{ch}_\Sigma(G') \le k$. Let $\ell$ be an additive coloring from $\mathcal{L}$ on $V(G')$. Our aim is to choose $\ell(v)$ from $\mathcal{L}(v)$ to extend the additive coloring of $G'$ to an additive coloring of $G$. Note that the only unsatisfied edges of $G$ are those incident to neighbors of $v$. Let $e$ be an edge incident to a neighbor $u$ of $v$. If $e = uv$, then $e$ is satisfied when $\ell(v) \neq \sum_{w \in N(v)} \ell(w) - S_{G'}(u)$. If $e = uw$ for some $w \neq v$, then $e$ is satisfied when $\ell(v) \neq S_{G'}(w) - S_{G'}(u)$. Thus picking $\ell(v)$ distinct from at most $\sum_{u \in N(v)} d(u)$ values ensures that all edges of $G$ are satisfied. Since $\sum_{u \in N(v)} d(u) < k$ there exists $\ell(v)$ in $\mathcal{L}(v)$ that can be used to extend the additive coloring of $G'$ to an additive coloring of $G$. Therefore $\mathrm{ch}_\Sigma(G) \leq k$, a contradiction.

Now assume $G$ is a vertex minimal graph with girth at least 5 and $\mathrm{ch}_\Sigma(G) > k$ containing the configuration described in (b). Let $R$ be the set of $r$ 1–neighbors of $v$. Let $G' = G - (R \cup Q)$. Since $\mathrm{girth}(G) \geq 5$, $Q$ is independent. Therefore for each $i \in \{1, \ldots, q\}$ there are at least $|\mathcal{L}(v_i)| - d_G(v_i')$ choices for $\ell(v_i)$ that ensure all edges incident to $v_i'$ are satisfied in $G$. Consider $vw$ in $E(G)$. If $w \in V(G')$, $vw$ is satisfied when $\sum_{x \in R \cup Q} \ell(x) \neq S_{G'}(w) - S_{G'}(v)$. Also, if $w \in R$, then $vw$ is satisfied when $\sum_{x \in R \cup Q} \ell(x) \neq \ell(v) - S_{G'}(v)$. If $w = v_i$ for some $v_i \in Q$, $vw$ is satisfied when $\sum_{x \in R \cup Q} \ell(x) \neq \ell(v) + \ell(w) - S_{G'}(v)$. Therefore, we must avoid at most $d(v)$ values for $\sum_{x \in R \cup Q} \ell(x)$ in order to satisfy all edges incident to $v$. Recall that each vertex in $R$ and $Q$ have $k$ and $k - d(v_i')$ labels, respectively, that avoid restricted sums. Proposition 2.1 guarantees at least $1 + r(k-1) + \sum_{v_i \in Q'} (k - d(v_i') - 1)$ available values for $\sum_{w \in R} \ell(w) + \sum_{w \in Q} \ell(w)$. Since, by assumption, $1 + r(k-1) + \sum_{v_i \in Q} (k - d(v_i') - 1) > d(v)$, there is at least one choice for $\ell(w)$ for each $w$ in $R \cup Q$ that completes an additive coloring of $G$. Thus $\mathrm{ch}_\Sigma(G) \leq k$, a contradiction. $\square$

Recall that the *maximum average degree* of a graph $G$, denoted $\mathrm{mad}(G)$, is the maximum of the average degrees over all subgraphs of $G$. The girth of a planar graph $G$ gives a bound on the maximum average degree of $G$. The following proposition is a simple application of Euler's formula (see [17]) and gives a relationship between these two parameters:

**Proposition 3.2.** *If $G$ is a planar graph with girth $g$, then* $\mathrm{mad}(G) < \frac{2g}{g-2}$.

## 3.1 Planar and Girth 5 implies $\mathrm{ch}_\Sigma \leq 19$

**Lemma 3.3.** *A configuration that is an induced cycle with vertices $v_1v_2v_3v_4v_5$ such that $d(v_1) \leq 17$, $d(v_2) = d(v_5) = 2$, $d(v_3) \leq 7$, and $d(v_4) \leq 7$ is 19–reducible.*

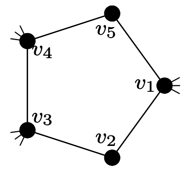


Figure 1: A reducible configuration.

*Proof.* Let $G$ be a vertex minimal graph with $\mathrm{ch}_\Sigma(G) > 19$ and let $\mathcal{L} : V(G) \to 2^{\mathbb{R}}$ be a list assignment on $V(G)$ with $|\mathcal{L}(v)| \geq 19$ for all $v \in V(G)$. Suppose to the contrary that $G$ contains the configuration in Figure 1. Since the most restrictions on labels occurs when $d(v_1) = 17$ and $d(v_3) = d(v_4) = 7$, we assume this is the case. Let $G' = G - \{v_2, v_5\}$. Let $\ell : V(G') \to \mathbb{R}$ be an additive coloring of $G'$ such that $\ell(v) \in \mathcal{L}(v)$ for each $v \in V(G)$. The unsatisfied edges are those incident to $v_1, \ldots, v_5$. The following function has factors corresponding to the unsatisfied edges where $x_2$ and $x_5$ represent labels of $v_2$ and $v_5$, respectively:

$$\begin{aligned} f(x_2, x_5) =& (S_{G'}(v_1) + x_2 + x_5 - \ell(v_1) - \ell(v_3)) \times (\ell(v_1) + \ell(v_3) - x_2 - S_{G'}(v_3)) \\ & \times (x_2 + S_{G'}(v_3) - x_5 - S_{G'}(v_4)) \times (x_5 + S_{G'}(v_4) - \ell(v_1) - \ell(v_4)) \\ & \times (\ell(v_1) + \ell(v_4) - x_2 - x_5 - S_{G'}(v_1)) \times \prod_{w \in N_{G'}(v_4) - \{v_3\}} (S_{G'}(w) - S_{G'}(v_4) - x_5) \\ & \times \prod_{w \in N_{G'}(v_1)} (S_{G'}(w) - S_{G'}(v_1) - x_2 - x_5) \times \prod_{w \in N_{G'}(v_3) - \{v_4\}} (S_{G'}(w) - S_{G'}(v_3) - x_2) \end{aligned}$$

The coefficient of $x_2^{16} x_5^{14}$ in $f(x_2, x_5)$ is the same as the coefficient of $x_2^{10} x_5^{8}$ in $-(x_2 + x_5)^{17}(x_2 - x_5)$, which is $\binom{17}{10} - \binom{17}{9}$. By Theorem 2.4 there is a choice of labels for $\ell(v_2)$ from a list of size 17 and $\ell(v_5)$ from a list of size 15, therefore $\mathrm{ch}_\Sigma(G) \leq 19$, a contradiction. $\square$

We will also require a large independent set, which is given from the following theorem.

**Theorem 3.4** ([15]). *Every planar triangle-free graph on $n$ vertices has an independent set of size at least $\frac{n+1}{3}$.*

**Theorem 3.5.** *If $G$ is a planar graph with* $\mathrm{girth}(G) \geq 5$*, then* $\mathrm{ch}_\Sigma(G) \leq 19$.

*Proof.* Let $G$ be a planar graph with girth at least 5 and suppose that $G$ is vertex minimal with $\mathrm{ch}_\Sigma(G) > 19$. By Proposition 3.2, $\mathrm{mad}(G) < 10/3$. Assign each vertex $v$ an initial charge $d(v)$, and apply the following discharging rules:

(R1) Each 1–vertex receives $7/3$ charge from its neighbor.

(R2) Each 2–vertex

- (a) with two $8^+$–neighbors receives $2/3$ charge from each neighbor.
- (b) with a $4^-$–neighbor and a $15^+$–neighbor receives $4/3$ charge from its $15^+$–neighbor.

(c) with a $10^+$–neighbor and a neighbor of degree 5, 6, or 7 receives 1 charge from its $10^+$–neighbor and 1/3 charge from its other neighbor.

(R3) Each 3–vertex receives 1/3 charge from a $6^+$–neighbor.

A contradiction with $\mathrm{mad}(G) < 10/3$ occurs if the discharging rules reallocate charge so that every vertex has final charge at least 10/3; we show that this is the case.

By Lemma 3.1 (a), each 1–vertex has a $19^+$–neighbor, 2–vertices have neighbors with degree sum at least 19, and 3–vertices have at least one $6^+$–neighbor. Thus, by the discharging rules, $3^-$–vertices have final charge 10/3. Since 4–vertices neither give nor receive charge, they have final charge 4.

Vertices of degree $d$ with $d \in \{5, 6, 7\}$ give charge when incident to $3^-$–vertices. By the discharging rules, they give away at most $d/3$ charge. This results in a final charge of at least $d - \frac{d}{3} = \frac{2d}{3} \geq \frac{10}{3}$, since $d \geq 5$.

Vertices of degree $d$ with $d \in \{8, 9\}$ may loose charge to $3^-$-vertices. By Lemma 3.1 (a) each 9–vertex has at least one $3^+$–neighbor. Also, each 8–vertex has at least two $3^+$–neighbors or at least one $4^+$–neighbor. By the discharging rules, the final charge of any 9–vertex is at least $9-8\cdot\frac{2}{3}-\frac{1}{3} = \frac{10}{3}$ and the final charge of any 8–vertex is at least $\min\{8-6\cdot\frac{2}{3}-2\cdot\frac{1}{3}, 8-7\cdot\frac{2}{3}\} = \frac{10}{3}$.

By Lemma 3.1 (b) vertices of degree $d$ with $d \in \{10, 11\}$ have no 2–neighbors with a $7^-$–neighbor. Thus, these vertices have final charge at least $d - \frac{2d}{3} = \frac{d}{3} \geq \frac{10}{3}$, since $d \geq 10$.

Let $v$ have degree $d$ where $d \in \{12, 13, 14\}$. By Lemma 3.1 (b), $v$ has no 2–neighbor with a $4^-$–neighbor. By Lemma 3.1 (b) and Lemma 3.3, $v$ has at most two 2–neighbors each having a $7^-$–neighbor. By the discharging rules $v$ has final charge at least $d - 2(1) - (d-2)\left(\frac{2}{3}\right) = \frac{d-2}{3} \geq \frac{10}{3}$, since $d \geq 12$.

Similarly, by Lemma 3.1 (b) and Lemma 3.3 vertices of degree 15, 16, or 17 have at most one 2–neighbor with a $4^-$–neighbor and at most two 2–neighbors with a $7^-$–neighbor. Thus these vertices give at most $1(\frac{4}{3}) + (d-1)\cdot\frac{2}{3}$ charge. Hence they have final charge at least $\frac{d-2}{3} \geq \frac{13}{3}$, since $d \geq 15$.

Finally, consider an $18^+$–vertex $v$ of degree $d$. Let $r$ be the number of 1–neighbors of $v$. Let $U = \{u_1, u_2, \ldots, u_q\}$ be the set of 2–neighbors of $v$. For each $u_i$ let $N(u_i) - \{v\} = \{u_i'\}$. Let $T = \{u_i' \in U : \ d(u_i') \leq 7\}$ and let $|T| = t$. Since $G[T]$ is planar with girth at least 5, Theorem 3.4 guarantees at least $\frac{t+1}{3}$ vertices in $T$ that form an independent set. By Lemma 3.1 (b), $d \geq 18r + 11(\frac{t+1}{3}) + 1$. Thus

$$d \geq 18r + \frac{11}{3}t + \frac{14}{3}. \tag{1}$$

The final charge of $v$ is at least $d - \frac{7}{3}r - \frac{4}{3}t - \frac{2}{3}(d-r-t)$. Hence $v$ has final charge at least $\frac{d}{3} - \frac{5}{3}r - \frac{2}{3}t$. From (1), $\frac{d}{3} - \frac{5}{3}r - \frac{2}{3}t \geq \frac{13}{3}r + \frac{5}{9}t + \frac{14}{9}$. When $r \geq 1$ or $t \geq 4$, the final charge is at least $\frac{10}{3}$. When $r = 0$ and $t \leq 3$, the vertex $v$ has final charge at least $d - \frac{4}{3}t - \frac{2}{3}(d-t) \geq \frac{d-6}{3} \geq \frac{12}{3}$, since $d \geq 18$. $\square$

### 3.2 Planar and Girth 6 implies $\mathrm{ch}_\Sigma \le 9$

**Lemma 3.6.** *The following configurations are* 8*-reducible in the class of graphs of girth at least* 6*:*

*(a) A* 6*–vertex* $v$ *having six* 2*-neighbors one of which has a* $3^-$*–neighbor.*

*(b) A* 7*–vertex* $v$ *having seven* 2*-neighbors two of which have* $4^-$*–neighbors.*

*Proof.* Let $G$ be a vertex minimal graph of girth at least 6 with $\mathrm{ch}_\Sigma(G) > 8$ and let $\mathcal{L} : V(G) \to 2^{\mathbb{R}}$ be a list assignment on $V(G)$ with $|\mathcal{L}(v)| \ge 8$ for all $v \in V(G)$. To the contrary suppose $G$ contains the configuration described in (a). Let $u$ be a 2–neighbor of $v$ having a $3^-$–neighbor. Let $G' = G - \{u, v\}$. Let $\ell : V(G') \to \mathbb{R}$ be an additive coloring of $G'$ such that $\ell(v) \in \mathcal{L}(v)$ for each $v \in V(G)$.

To obtain a contradiction, we extend the labeling $\ell$ in $G'$ to an additive labeling in $G$. The only unsatisfied edges of $G$ are those incident to neighbors of $u$ or $v$. To satisfy the unsatisfied edges not incident to $u$ or $v$, we avoid at most two values from $\mathcal{L}(u)$ and at most five values from $\mathcal{L}(v)$. Note that $|\mathcal{L}(u)| \ge 8$ and $|\mathcal{L}(v)| \ge 8$. Thus there are at least six labels available for $u$ and at least three available for $v$. To satisfy the edges incident to $u$ or $v$, $\ell(u) - \ell(v)$ must avoid at most seven values. Corollary 2.2 gives at least eight values for $\ell(u) - \ell(v)$ from available labels. Thus there are labels that complete an additive coloring of $G$. Hence $\mathrm{ch}_\Sigma(G) \le 8$, a contradiction.

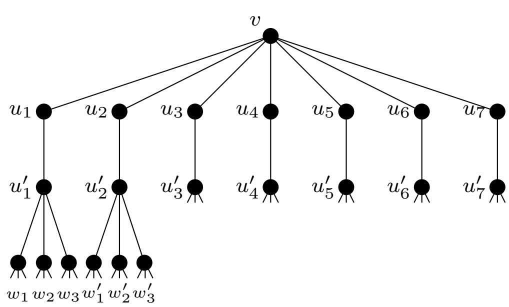


Figure 2: An 8–reducible configuration.

Now, we prove part (b). To the contrary suppose $G$ contains the configuration described in (b). Let $u_1, \ldots, u_7$ be the 2–neighbors of $v$ whose other neighbors are $u'_1, \ldots, u'_7$, respectively, where $u'_1$ and $u'_2$ are $4^-$–vertices. Since the most restrictions on labels occurs when $d(u'_1) = d(u'_2) = 4$, we assume this is the case. Let $N(u'_1) - \{u_1\} = \{w_1, w_2, w_3\}$ and $N(u'_2) - \{u_2\} = \{w'_1, w'_2, w'_3\}$ (see Figure 2). Consider $G' = G - \{v, u_1, u_2\}$. The following function has factors that correspond to unsatisfied edges, where $x$, $y$, and $z$ represent the possible values of $\ell(v)$, $\ell(u_1)$, and $\ell(u_2)$, respectively:

$$\begin{aligned} f(x,y,z) = &\prod_{i=1}^{7}\left(y+z+\sum_{j=3}^{7}\ell(u_j)-x-\ell(u_i')\right)\times\prod_{i=3}^{7}(x+\ell(u_i')-S_{G'}(u_i')) \\ &\times\prod_{i=1}^{3}(y+S_{G'}(u_1')-S_{G'}(w_i))\times\prod_{i=1}^{3}(z+S_{G'}(u_2')-S_{G'}(w_i')) \\ &\times(x+\ell(u_1')-y-S_{G'}(u_1'))\times(x+\ell(u_2')-z-S_{G'}(u_2')) \end{aligned}$$

The coefficient of $x^7y^6z^7$ in $f(x,y,z)$ is equal to its coefficient in $(y+z-x)^7x^5y^3z^3(x-y)(x-z)$, which is 490. By Theorem 2.4, there is a choice of labels for $\ell(v)$, $\ell(u_1)$, and $\ell(u_2)$ from lists of size at least 8 that make $f$ nonzero. Thus these labels induce an additive coloring of $G$. Hence $\mathrm{ch}_\Sigma(G) \le 8$, a contradiction. □

**Theorem 3.7.** *If $G$ is a planar graph with* $\mathrm{girth}(G) \ge 6$*, then* $\mathrm{ch}_\Sigma(G) \le 9$.

*Proof.* Let $G$ be a planar graph with girth at least 6 and suppose $G$ is vertex minimal with $\mathrm{ch}_\Sigma(G) > 9$. By Proposition 3.2, $\mathrm{mad}(G) < 3$. Assign each vertex $v$ an initial charge of $d(v)$ and apply the following discharging rules:

(R1) Each 1–vertex receives 2 charges from its neighbor.

(R2) Each 2–vertex

- (a) with one $8^+$–neighbor and one $5^-$–neighbor receives 1 charge from its $8^+$–neighbor.
- (b) with one $7^+$–neighbor and one $4^-$–neighbor receives 1 charge from its $7^+$–neighbor.
- (c) with one $6^+$–neighbor and one $3^-$–neighbor receives 1 charge from its $6^+$–neighbor.
- (d) receives 1/2 charge from each neighbor, otherwise.

A contradiction with $\mathrm{mad}(G) < 3$ occurs if the discharging rules reallocate charge so that every vertex has final charge at least 3; we show this is the case.

By Lemma 3.1 (a) each 1–vertex has a $9^+$–neighbor and each 2–vertex has neighbors with degree sum at least 9. Under the discharging rules, 1–vertices and 2–vertices gain charge 2 and 1, respectively, and 3–vertices neither gain nor lose charge. Thus, $3^-$–vertices have final charge 3.

By Lemma 3.1 (b) each 4–vertex $v$ has no 1–neighbor and has at most one 2–neighbor whose other neighbor is a $6^-$–vertex. Therefore each 4–vertex has final charge at least $4-\frac{1}{2}$. Similarly, each 5–vertex has no 1–neighbor and has at most four 2–neighbors having another $7^-$–neighbor. Therefore each 5–vertex has final charge at least $5-4(\frac{1}{2})$, as desired.

If $v$ is a 6–vertex, then by Lemma 3.1, $v$ has no 1–neighbor. Moreover, by Lemma 3.6, if $v$ has six 2–neighbors, none of them has a $3^-$–neighbor. Hence $v$ has charge at least $6-\max\{1+4(\frac{1}{2}), 6(\frac{1}{2})\}$, which is 3 as desired.

Similarly by Lemma 3.1, a 7–vertex $v$ has no 1–neighbor. Moreover, by Lemma 3.6, if $v$ has seven 2–neighbors, at most one of them has a $4^-$–neighbor. Thus $v$ has charge at least $7 - \max\{2 + 4(\frac{1}{2}), 1 + 6(\frac{1}{2}), 7(\frac{1}{2})\}$, which is at least 3 as desired.

Finally, if $v$ is a $d$–vertex with $d \geq 8$, then by Lemma 3.1 (b) we have

$$d \geq 8r + 3q + 1, \tag{2}$$

where $r$ is the number of 1–neighbors and $q$ is the number of 2–neighbors having a $5^-$-neighbor. The final charge on $v$ is at least $d - 2r - q - \frac{1}{2}(d - r - q) = \frac{d}{2} - \frac{3}{2}r - \frac{1}{2}q$. Thus by (2) $v$ has final charge at least $\frac{1}{2}(8r + 3q + 1) - \frac{3}{2}r - \frac{1}{2}q = \frac{5}{2}r + q + \frac{1}{2}$. When $r \geq 1$ or $q \geq 3$, this final charge is at least 3. If $r = 0$ and $q \leq 2$ then $v$ has final charge at least $d - 2 - \frac{1}{2}(d - 2) = \frac{d-2}{2} \geq 3$, since $d \geq 8$. □

## 3.3 Planar and Girth 7 implies $\mathrm{ch}_\Sigma \leq 8$

**Theorem 3.8.** *If $G$ is a planar graph with* $\mathrm{girth}(G) \geq 7$*, then* $\mathrm{ch}_\Sigma(G) \leq 8$.

*Proof.* Let $G$ be a planar graph with girth at least 7 and suppose $G$ is a vertex minimal planar graph with $\mathrm{ch}_\Sigma(G) > 9$. By Proposition 3.2, $\mathrm{mad}(G) < 14/5$. Assign each vertex $v$ an initial charge of $d(v)$ and apply the following discharging rules:

(R1) Each 1–vertex receives 9/5 charge from its neighbor.

(R2) Each 2–vertex

- (a) with one $3^-$–neighbor and one $6^+$–neighbor receives 4/5 charge from its $6^+$–neighbor.
- (b) with one 3–neighbor and one 5–neighbor receives 1/5 and 3/5 charge, respectively.
- (c) with two 4–neighbors receives 2/5 charge from each neighbor.
- (d) with one 4–neighbor and one $5^+$–neighbor receives 1/5 and 3/5 charge, respectively.
- (e) with two $5^+$–neighbors receives 2/5 charge from each neighbor.

A contradiction with $\mathrm{mad}(G) < 14/5$ occurs if the discharging rules reallocate charge so that every vertex has final charge at least 14/5; we show this is the case.

By Lemma 3.1 (a) each 1–vertex has an $8^+$–neighbor and each 2–vertex has neighbors with degree sum at least 8. Under the discharging rules, 1–vertices and 2–vertices gain 9/5 and 4/5 charge, respectively. If $v$ is a 3–vertex, then by Lemma 3.1 (b), $v$ has at most one 2–neighbor with a 5–neighbor. Thus $v$ gives at most 1/5 charge. Hence, $3^-$–vertices have final charge at least 14/5.

If $v$ is a 4–vertex, then by Lemma 3.1 (b) $v$ has at most one 2–neighbor with a $4^-$–neighbor other than $v$. Thus $v$ has final charge at least $4 - 1\left(\frac{2}{5}\right) - 3\left(\frac{1}{5}\right) = 3$.

If $v$ is a 5–vertex, then by Lemma 3.1 (b) $v$ has at most one 2–neighbor with a $4^-$–neighbor. Thus $v$ has final charge at least $5 - 1\left(\frac{3}{5}\right) - 4\left(\frac{2}{5}\right) \geq \frac{14}{5}$.

If $v$ is a 6–vertex, then by Lemma 3.1 (b) $v$ has at most one 2–neighbor with a $4^-$–neighbor. Thus $v$ has final charge at least $6 - 1\left(\frac{4}{5}\right) - 5\left(\frac{2}{5}\right) = \frac{16}{5}$.

If $v$ is a 7–vertex, then by Lemma 3.1 (b) $v$ has at most one 2–neighbor with a $3^-$–neighbor, and has at most two 2–neighbors with a $4^-$–neighbor. Thus $v$ has final charge at least $7 - 1\left(\frac{4}{5}\right) - 1\left(\frac{3}{5}\right) - 5\left(\frac{2}{5}\right) = \frac{18}{5}$.

If $v$ is an 8–vertex, then by Lemma 3.1 (b) $v$ has at most one 1–neighbor, at most two 2–neighbors with a $3^-$–neighbor, and at most two 2–neighbors with a $4^-$–neighbor. Moreover, if $v$ has a 1–neighbor, then $v$ does not have a 2–neighbor with a $3^-$–neighbor. Since the discharging rules allocate charge to neighbors with these constraints, $v$ has final charge at least $8 - \max\left\{1\left(\frac{9}{5}\right) + 7\left(\frac{2}{5}\right), 2\left(\frac{4}{5}\right) + 6\left(\frac{2}{5}\right)\right\} = \frac{17}{5}$.

If $v$ is a $d$–vertex with $d \geq 9$, then by Lemma 3.1 (b) $v$ has at most $\frac{d}{8}$ 1–neighbors, at most $\frac{d}{4}$ neighbors that are either a 1–vertex or a 2–vertex with a $3^-$–neighbor, and at most $\frac{d}{3}$ neighbors that are either a 1–vertex or a 2–vertex with a $4^-$–neighbor. Since $v$ gives more charge to neighbors of low degree, we assume $v$ has as many low degree neighbors as possible. Hence $v$ has final charge at least $d - \frac{d}{8}\left(\frac{9}{5}\right) - \left(\frac{d}{4} - \frac{d}{8}\right)\left(\frac{4}{5}\right) - \left(\frac{d}{3} - \frac{d}{4}\right)\left(\frac{3}{5}\right) - \left(d - \frac{d}{3}\right)\left(\frac{2}{5}\right) = \frac{43}{120}d$, which is at least 3 since $d \geq 9$. Therefore, all vertices have final charge at least 14/5 and we obtain a contradiction. □

## 3.4 Planar and Girth 26 implies $\mathrm{ch}_\Sigma \leq 3$

**Lemma 3.9.** *Let $P(t_2, \ldots, t_{n-1})$ be the path $v_1 \cdots v_n$ such that for each $i$ in $\{2, \ldots, n-1\}$ the vertex $v_i$ has $t_i$ 1–neighbors and $d(v_i) = 2 + t_i$. The configurations $P(1, 0, 1)$, $P(1, 1, 1)$, $P(1, 1, 0, 0)$, $P(0, 1, 0, 0)$, $P(1, 0, 0, 0)$, and $P(0, 0, 0, 0, 0)$ are 3–reducible.*

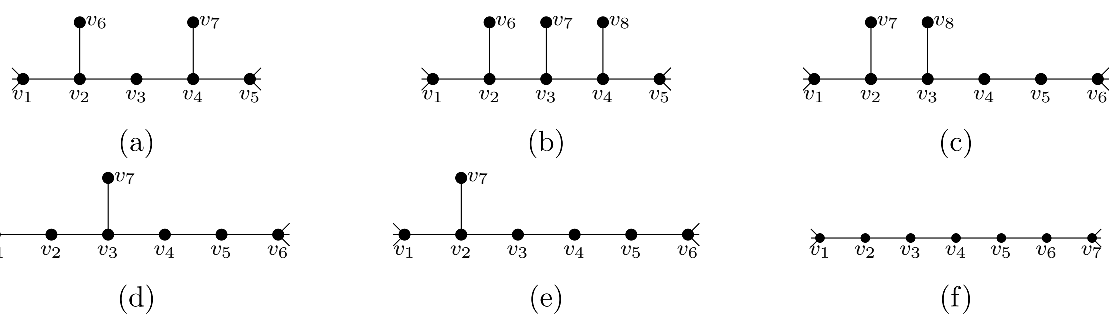


Figure 3: Some 3–reducible configurations.

*Proof.* Let $G$ be a vertex minimal graph with $\mathrm{ch}_\Sigma(G) > 3$ and let $\mathcal{L} : V(G) \to 2^{\mathbb{R}}$ be a list assignment on $V(G)$ with $|\mathcal{L}(v)| \geq 3$ for all $v \in V(G)$. To the contrary suppose $G$ contains $P(1, 0, 1)$, see Figure 3a. Let $v_6$ and $v_7$ be the neighbors of $v_2$ and $v_4$, respectively. Consider $G' = G - \{v_3, v_6, v_7\}$. Let $\ell\colon V(G') \to \mathbb{R}$ be an additive coloring of $G'$ such that $\ell(v) \in \mathcal{L}(v)$ for each $v \in V(G)$. The only unsatisfied edges of $G$ are those incident to $v_2$ and $v_3$. The following function has factors that correspond to unsatisfied edges, where $x_3, x_6, x_7$ represent the possible values of $\ell(v_3), \ell(v_6), \ell(v_7)$, respectively:

$$\begin{aligned} f(x_3,x_6,x_7) = {} & (S_{G'}(v_1) - \ell(v_1) - x_3 - x_6) \times (\ell(v_1) + x_3 + x_6 - \ell(v_2)) \\ & \times (\ell(v_1) + x_3 + x_6 - \ell(v_2) - \ell(v_4)) \times (\ell(v_2) + \ell(v_4) - S_{G'}(v_4) - x_3 - x_7) \\ & \times (\ell(v_5) + x_3 + x_7 - \ell(v_4)) \times (\ell(v_5) + x_3 + x_7 - S_{G'}(v_5)) \end{aligned}$$

The coefficient of $x_3^2x_6^2x_7^2$ in $f(x_3,x_6,x_7)$ is 9. Theorem 2.4 gives $\mathrm{ch}_\Sigma \leq 3$, a contradiction. Thus these labels induce an additive coloring of $G$. Hence $\mathrm{ch}_\Sigma(G) \leq 3$, a contradiction.

For the following, we simply present the proper subgraph $G'$, the function $f$ derived from the configuration, the monomial, and its coefficient. In each function $f$, $x_i$ corresponds to the label of $v_i$.

Suppose $G$ contains $P(1,1,1)$, see Figure 3b. Let $G' = G - \{v_3, v_6, v_7, v_8\}$.

$$\begin{aligned} f(x_3,x_6,x_7,x_8) = {} & (S_{G'}(v_1) - \ell(v_1) - x_3 - x_6) \times (\ell(v_1) + x_3 + x_6 - \ell(v_2)) \\ & \times (\ell(v_1) + x_3 + x_6 - \ell(v_2) - x_7 - \ell(v_4)) \times (\ell(v_2) + \ell(v_4) + x_7 - x_3) \\ & \times (\ell(v_2) + \ell(v_4) + x_7 - \ell(v_5) - x_3 - x_8) \times (\ell(v_5) + x_3 + x_8 - \ell(v_4)) \\ & \times (\ell(v_5) + x_3 + x_8 - S_{G'}(v_5)) \end{aligned}$$

The coefficient of $x_3^2x_6^2x_7x_8^2$ is 15.

Suppose $G$ contains $P(1,1,0,0)$, see Figure 3c. Let $G' = G - \{v_3, v_4, v_7, v_8\}$.

$$\begin{aligned} f(x_3,x_4,x_7,x_8) = {} & (x_3 + x_7 + \ell(v_1) - S_{G'}(v_1)) \times (x_3 + x_7 + \ell(v_1) - \ell(v_2)) \\ & \times (x_4 + x_8 + \ell(v_2) - x_3 - x_7 - \ell(v_1)) \times (x_4 + x_8 + \ell(v_2) - x_3) \\ & \times (x_4 + x_8 + \ell(v_2) - x_3 - \ell(v_5)) \times (x_3 + \ell(v_5) - x_4 - \ell(v_6)) \\ & \times (x_4 + \ell(v_6) - S_{G'}(v_6)) \end{aligned}$$

The coefficient of $x_3x_4^2x_7^2x_8^2$ is 8.

Suppose $G$ contains $P(0,1,0,0)$, see Figure 3d. Let $G' = G - \{v_3, v_4, v_7\}$.

$$\begin{aligned} f(x_3,x_4,x_7) = {} & (S_{G'}(v_1) - \ell(v_1) - x_3) \times (\ell(v_1) + x_3 - \ell(v_2) - x_7 - x_4) \\ & \times (\ell(v_2) + x_7 + x_4 - x_3) \times (\ell(v_2) + x_7 + x_4 - x_3 - \ell(v_5)) \\ & \times (x_3 + \ell(v_5) - x_4 - \ell(v_6)) \times (x_4 + \ell(v_6) - S_{G'}(v_6)) \end{aligned}$$

The coefficient of $x_3^2x_4^2x_7^2$ is 6.

Suppose $G$ contains $P(1,0,0,0)$, see Figure 3e. Let $G' = G - \{v_3, v_4, v_7\}$.

$$\begin{aligned} f(x_3,x_4,x_7) = {} & (S_{G'}(v_1) - \ell(v_1) - x_3 - x_7) \times (\ell(v_2) - \ell(v_1) - x_3 - x_7) \\ & \times (\ell(v_2) + x_4 - \ell(v_1) - x_3 - x_7) \times (\ell(v_2) + x_4 - x_3 - \ell(v_5)) \\ & \times (\ell(v_6) + x_4 - x_3 - \ell(v_5)) \times (\ell(v_6) + x_4 - S_{G'}(v_6)) \end{aligned}$$

The coefficient of $x_3^2x_4^2x_7^2$ is 7.

Suppose $G$ contains $P(0,0,0,0,0)$, see Figure 3f. Let $G' = G - \{v_3, v_4, v_5\}$.

$$f(x_3,x_4,x_5) = (S_{G'}(v_1) - \ell(v_1) - x_3) \times (\ell(v_1) + x_3 - \ell(v_2) - x_4) \times (\ell(v_2) + x_4 - x_3 - x_5) \\ \times (x_3 + x_5 - x_4 - \ell(v_6)) \times (x_4 + \ell(v_6) - x_5 - \ell(v_7)) \times (x_5 + \ell(v_7) - S_{G'}(v_7))$$

The coefficient of $x_3^2 x_4^2 x_5^2$ is 7. Theorem 2.4 implies that these configurations are 3–reducible. □

We call a $d$-vertex *lonely* if it is in exactly one face of $G$. We say that a non–lonely $3^+$–vertex $v$ is *unique* to a face $f$ of $G$ if it is incident to a cut–edge $uv$ such that $d(u) > 1$ and $uv$ is also in $f$.

**Lemma 3.10.** *Let $f$ be a face in a planar graph $G$ with $e_c$ cut–edges such that $f$ has $s$ lonely vertices, and $t$ $3^+$–vertices unique to $f$. We have $s + \frac{t}{2} \leq e_c$.*

*Proof.* We apply induction on $e_c$. If $e_c = 0$, then $s = t = 0$ and the inequality holds. In the following two cases, given some face $f$ containing a cut–edge $uv$, let $G'$ be the graph obtained by contracting the edge $uv$ to a vertex $w$. Let $f'$ be the face in $G'$ corresponding to $f$. Let $s'$ and $t'$ be the number of lonely vertices in $f'$ and the number of $3^+$–vertices unique to $f'$, respectively.

**Case 1**: $u$ or $v$ is lonely.
Without loss of generality assume $u$ is lonely. If $v$ is also lonely, then $w$ is lonely and therefore $s' = s-1$. If $v$ is not lonely, then $w$ is not lonely and still $s' = s-1$. Vertices unique to $f$ are not affected by the contraction, thus $t' = t$. Since $f'$ has $e_c - 1$ cut–edges, by the induction hypothesis $s' + \frac{t'}{2} \leq e_c - 1$. Therefore, $s + \frac{t}{2} \leq e_c$.

**Case 2**: $u$ and $v$ are unique to $f$.
Since $u$ and $v$ are not lonely, $w$ is not lonely and $s' = s$. After contracting $uv$, either $w$ is unique to $f$ and $t' = t-1$ or $w$ is not unique to $f$ and $t' = t-2$, which yields $t'+1 \leq t \leq t'+2$. By the induction hypothesis, $s' + \frac{t'}{2} \leq e_c - 1$. Since $t \leq t'+2$, we have $s + \frac{t}{2} \leq e_c$, as desired. □

**Theorem 3.11.** *If $G$ is a planar graph with* $\mathrm{girth}(G) \geq 26$*, then* $\mathrm{ch}_\Sigma(G) \leq 3$.

*Proof.* Let $G$ be planar with girth at least 26 and suppose $G$ is vertex minimal with $\mathrm{ch}_\Sigma(G) > 3$. Assign each vertex $v$ an initial charge $d(v)$, each face $f$ an initial charge $l(f)$, and apply the following discharging rules:

(R1) Each 1-vertex receives 2 charges from its incident face and 1 charge from its neighbor.

(R2) Each 2-vertex receives 2 charges from its incident face if it is lonely; it receives 1 from each incident face otherwise.

(R3) Each 3-vertex with a 1-neighbor and

  (a) incident to two faces receives 1 charge from each incident face.

  (b) incident to one face receives 2 charges from its face.

(R4) Each 3-vertex without a 1-neighbor and

(a) incident to three faces receives $\frac{1}{3}$ charge from each incident face.

(b) incident to two faces receives $\frac{1}{2}$ charge from each incident face.

(c) incident to one face receives 1 charge from its face.

(R5) Each 4-vertex that has a 1-neighbor and is

(a) incident to three faces receives $\frac{1}{3}$ charge from each incident face.

(b) lonely or unique to some face $f$ receives 1 charge from $f$.

(R6) Each 5-vertex that has two 1-neighbors and is

(a) incident to three faces receives $\frac{1}{3}$ charge from each incident face.

(b) lonely or unique to some face $f$ receives 1 charge from $f$.

A contradiction with Proposition 2.3 occurs if the discharging rules reallocate charge so that every vertex and face has charge at least 4; we show this is the case.

By Lemma 3.1 (a) a 1–vertex has a $3^+$–neighbor. By Lemma 3.1 (b) a $4^-$–vertex has at most one 1–neighbor, a 5–vertex has at most two 1–neighbors, and in general a $d$–vertex has at most $\frac{d-1}{2}$ neighbors of degree 1. Since vertices only give charge to 1–neighbors, $6^+$–vertices have final charge at least 4. Note that according to our discharging rules, if $v$ is a $d$–vertex having at least one 1-neighbor with $d \in \{3, 4, 5\}$, then $v$ receives enough charge from the faces so that its final charge is at least 4. Thus all vertices have final charge at least 4 under the discharging rules.

We turn our attention to the final charge of faces. By Theorem 1.7 and the choice of $G$, $G$ is connected and each face contains at least one cycle. Therefore, each face has length at least 26. Let $R_f$ be the set of vertices incident to a face $f$ that are either a 2–vertex or a 3–vertex that is not lonely and has one 1–neighbor. Let $f$ be a face with $s$ lonely vertices, $t$ unique vertices, and $r$ vertices in $R_f$. By Lemma 3.10 $f$ has at least $s + \frac{t}{2}$ cut edges. Thus,

$$l(f) \geq 26 + 2s + t. \tag{3}$$

A combination of the reducible configurations in Lemma 3.9 implies that there are at most four consecutive vertices from $R_f$ in any cycle of $f$. Thus

$$r \leq \left\lfloor \frac{4}{5}(l(f) - 2s - t) \right\rfloor. \tag{4}$$

By the discharging rules, $f$ has final charge at least

$$l(f) - 2s - t - r - \frac{1}{3}(l(f) - 2s - t - r) = \frac{2}{3}l(f) - \frac{4}{3}s - \frac{2}{3}t - \frac{2}{3}r.$$

By (4),

$$\frac{2}{3}l(f) - \frac{4}{3}s - \frac{2}{3}t - \frac{2}{3}r \geq \frac{2}{3}l(f) - \frac{4}{3}s - \frac{2}{3}t - \frac{2}{3}\left\lfloor \frac{4}{5}(l(f) - 2s - t) \right\rfloor. \tag{5}$$

Therefore the final charge of $f$ is at least

$$\frac{2}{3}l(f) - \frac{4}{3}s - \frac{2}{3}t - \frac{2}{3}\left(\frac{4}{5}(l(f) - 2s - t)\right) = \frac{2}{15}(l(f) - 2s - t),$$

which is at least 4 when $l(f) - 2s - t \geq 30$. When $l(f) - 2s - t \in \{26, \ldots, 29\}$, (5) gives final charge at least 4. $\square$

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