Antimagic Labelings of Weighted and Oriented Graphs*

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Abstract

A graph G is *k–weighted–list–antimagic* if for any vertex weighting $\omega\colon V(G) \to \mathbb{R}$ and any list assignment $L\colon E(G) \to 2^{\mathbb{R}}$ with $|L(e)| \geq |E(G)| + k$ there exists an edge labeling f such that $f(e) \in L(e)$ for all $e \in E(G)$, labels of edges are pairwise distinct, and the sum of the labels on edges incident to a vertex plus the weight of that vertex is distinct from the sum at every other vertex. In this paper we prove that every graph on n vertices having no K_1 or K_2 component is $\lfloor \frac{4n}{3} \rfloor$–weighted–list–antimagic.

An oriented graph G is *k–oriented–antimagic* if there exists an injective edge labeling from $E(G)$ into $\{1, \ldots, |E(G)| + k\}$ such that the sum of the labels on edges incident to and oriented toward a vertex minus the sum of the labels on edges incident to and oriented away from that vertex is distinct from the difference of sums at every other vertex. We prove that every graph on n vertices with no K_1 component admits an orientation that is $\lfloor \frac{2n}{3} \rfloor$–oriented–antimagic.
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1 Introduction

In this paper we consider simple, finite graphs. In a labeling of a graph G, we define the *vertex sum* at a vertex v to be the sum of labels of edges incident to v. A graph G is *antimagic* if there exists a bijective edge labeling from $E(G)$ to $\{1, \ldots, |E(G)|\}$ such that the vertex sums are pairwise distinct. This concept was first introduced by Hartsfield and Ringle in [5]. Excluding K_2, they prove that cycles, paths, complete graphs, and wheels are antimagic and they make the following conjecture.

Conjecture 1.1 ([5]). *Every simple connected graph other than K_2 is antimagic.*

The most significant work toward proving this conjecture is by Alon et al. [2] who prove that there is an absolute constant C such that every graph with n vertices and minimum degree at

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least $C \log n$ is antimagic. They also prove that a graph G on n vertices is antimagic if $n \geq 4$ and $\Delta(G) \geq n-2$. Later, Yilma [10] improved this condition from $n-2$ to $n-3$ when $n \geq 9$.

Toward answering Conjecture 1.1, it is helpful to see how close graph classes are to being antimagic. Several notions have been considered as either a measure of closeness to being antimagic or a variation thereof.

A graph G is *k–antimagic* if there exists an injective edge labeling from $E(G)$ into $\{1, \ldots, |E(G)|+k\}$ such that vertex sums are pairwise distinct. If for any vertex weighting from $V(G)$ into $\mathbb{R}$, there exists a bijective edge labeling from $E(G)$ to $\{1, \ldots, |E(G)|\}$ such that the *weighted vertex sum* at a vertex, which is the vertex sum plus the vertex weight, is distinct from the weighted vertex sum at any other vertex, then G is *weighted–antimagic*. When a graph is described using a combination of variations in this paper, it satisfies the conditions of each variation mentioned in its description. For example, a graph G is k–weighted–antimagic if for any vertex weighting from $V(G)$ into $\mathbb{R}$, there exists an injective edge labeling from $E(G)$ into $\{1, \ldots, |E(G)|+k\}$ such that weighted vertex sums are pairwise distinct.

Note that antimagic is equivalent to 0–antimagic. Wong and Zhu [9] provide a family of connected graphs that is not 1–weighted–antimagic in which each graph in the family has an even number of vertices, and they pose the following questions.

Question 1.2 ([9]). *Is it true that every connected graph $G \neq K_2$ is 2–weighted–antimagic?*

Question 1.3 ([9]). *Is there a connected graph G with an odd number of vertices which is not 1–weighted–antimagic?*

They also prove the following.

Theorem 1.4 ([9]). *Every connected graph $G \neq K_2$ on n vertices is $\left(\left\lceil \frac{3n}{2} \right\rceil - 2\right)$–weighted–antimagic.*

A graph G is *k–list–antimagic* if for any list function $L\colon E(G) \to 2^{\mathbb{R}}$, where $|L(e)| \geq |E(G)|+k$ for all $e \in E(G)$, there exists an edge labeling that assigns each edge e a label from $L(e)$ such that edge labels are pairwise distinct and vertex sums are pairwise distinct. We improve upon Theorem 1.4 by proving the following broader theorem.

Theorem 1.5. *Every graph on n vertices with no K_1 or K_2 component is $\left\lfloor \frac{4n}{3} \right\rfloor$–list–weighted–antimagic.*

Note that Theorem 1.5 includes disconnected graphs.

Introduced in [6], an oriented graph G is *oriented–antimagic* if there exists a bijective edge labeling from $E(G)$ to $\{1, \ldots, |E(G)|\}$ such that oriented vertex sums are pairwise distinct, where an *oriented vertex sum* at a vertex v is the sum of labels of edges incident to and oriented toward v minus the sum of labels of edges incident to and oriented away from v. An *orientation of G* is a directed graph with G as the underlying simple graph.

Hefetz, Mütze, and Schwartz [6] prove that there is a constant C such that every orientation of a graph on n vertices with minimum degree at least $C \log n$ is oriented–antimagic. They also show that every orientation of complete graphs, wheels, stars with at least 4 vertices, and regular graphs of odd degree are oriented–antimagic. In addition, they show that every regular graph on n vertices with even degree and a matching of size $\left\lfloor \frac{n}{2} \right\rfloor$ has an orientation that is oriented–antimagic. They make the following conjecture and ask the subsequent question.

Conjecture 1.6 ([6]). *Every connected undirected graph has an orientation that is oriented–antimagic.*

Question 1.7 ([6]). *Is every connected oriented graph with at least 4 vertices oriented–antimagic?*

Toward Conjecture 1.6, we prove the following.

Theorem 1.8. *Every graph on n vertices admits an orientation that is $\lfloor \frac{2n}{3} \rfloor$–oriented–antimagic.*

We direct the interested reader to [4] for a more thorough history of antimagic labelings and its variations.

Before presenting our results in Sections 2 and 3, we present some useful tools. The primary tool used in the results is Alon's Combinatorial Nullstellensatz.

Theorem 1.9 (Combinatorial Nullstellensatz, [1]). *Let f be a polynomial of degree t in m variables over a field $\mathbb{F}$. If there is a monomial $\prod x_i^{t_i}$ in f with $\sum t_i = t$ whose coefficient is nonzero in $\mathbb{F}$, then f is nonzero at some point of $\bigotimes T_i$, where each T_i is a set of $t_i + 1$ distinct values in $\mathbb{F}$.*

We use the following specific instance of Equation (5.16) in [3] when applying the Combinatorial Nullstellensatz.

Lemma 1.10 ([3]). *The coefficient of the monomial $\prod_{1\le i\le N} x_i^{s(N-1)+i-1}$ in the polynomial $\prod_{1\le i<j\le N} (x_i - x_j)^{2s+1}$ has absolute value $\frac{\big((s+1)N\big)!}{N!(s+1)!^N}$.*

Note that the polynomial in the above lemma is the determinant of the $(2s+1)^{st}$ power of the Vandermonde matrix.

A vertex of degree at least j is called a j^+–vertex. For a property P, a configuration C is *P–reducible* if C does not appear in an edge–minimal graph failing P. An *even (odd) component* in a graph is a component that has an even (odd) number of vertices. A vertex v is in edge e, denoted $v \in e$, if e is incident to v. We use notation from [8] unless otherwise specified.

2 Antimagic Results

The main results of this paper rely on an inductive argument. To avoid complications of creating isolated vertices or K_2 components when deleting edges we define the following concept. A graph G is *k–quasi–antimagic* if there exists an injective edge labeling from $E(G)$ into $\{1, \ldots, |E(G)| + k\}$ such that vertex sums are pairwise distinct for pairs of non–isolated vertices that are not adjacent in a K_2 component. Notice that if a graph has no isolated vertex and no K_2 component, k–quasi–antimagic is equivalent to k–antimagic.

Lemma 2.1. *If G is a graph on n vertices and $\Delta(G) \le 2$ then G is $\lfloor \frac{4n}{3} \rfloor$–list–weighted–quasi–antimagic.*

Proof. It suffices to prove the lemma for graphs with $\delta(G) \ge 1$, since adding isolated vertices increases n without adding any additional labeling requirements.

Let G have m edges. Given $1 \le \delta(G) \le \Delta(G) \le 2$, every component of G is a path or cycle and has at least 2 vertices. Let $e_1, \ldots, e_q$ be the q isolated edges of G, $d_1, \ldots, d_r$ be the

r even components of G each having at least 4 vertices, and $c_1, \ldots, c_s$ be the odd components of G. Let $\omega\colon V(G) \to \mathbb{R}$ be a vertex weighting and $L\colon E(G) \to 2^{\mathbb{R}}$ be a list function such that $|L(e)| \geq m + \left\lfloor \frac{4n}{3} \right\rfloor$ for all $e \in E(G)$.

Let E' be a matching in G of maximum size and let $E'' = E(G) - E'$. Notice that $e_1, \ldots, e_q$ are in E'. Thus we may suppose $E' = \{e_1, \ldots, e_k\}$, where $k \geq q$. In particular, $k = \frac{n-s}{2}$. Also define v_i for each $i \in [s]$ to be the unique vertex in c_i such that v_i is not incident to any edge in E'.

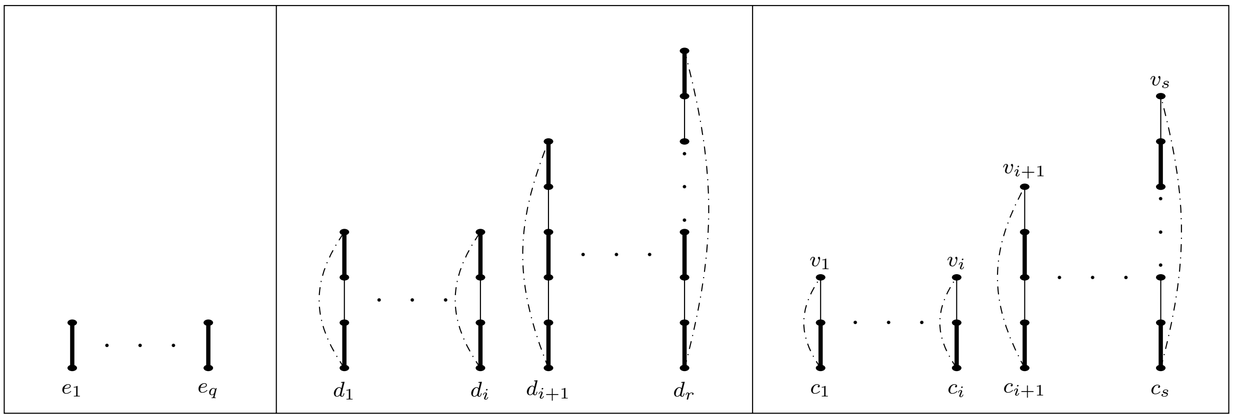

Figure 1: Components (paths or cycles) of G with edges in the maximum matching E' in bold.

In the first stage of this proof, we create an injective edge labeling into $\{1, \ldots, m + \left\lfloor \frac{4n}{3} \right\rfloor\}$ from the edges of E'', that is the edges in the d_i and c_i components that are not in the matching. Note that $|E''| = m - \frac{n-s}{2}$. We create a labeling on E'' iteratively in the following way. For edge $e = yz \in E''$, we label e from $L(e)$ such that (1) the label assigned to e is not already assigned to a different edge, of which there are at most $|E''| - 1$, (2) neither y nor z attains the same weighted vertex sum as that of its neighbor $u \notin \{y, z\}$, if such a neighbor exists, and (3) if $v_i \in \{y, z\}$ for some $i \in [s]$, then the new weighted vertex sum at v_i must be distinct from the weighted vertex sum at v_j for each $j \neq i$. With these three restrictions, there are at most $(|E''| - 1) + 2 + (s - 1)$ values that are not allowed when labeling each edge in E''. Since $|L(e)| \geq m + \left\lfloor \frac{4n}{3} \right\rfloor$ for each edge e and $s \leq \frac{n}{3}$, we have

$$|E''| + s = m - \frac{n-s}{2} + s = m + \frac{3s}{2} - \frac{n}{2} \leq m < |L(e)|.$$

Therefore, such a labeling on E'' is possible.

The second stage of this proof is to label the edges of the maximum matching E' in G. Let $f''\colon E'' \to \mathbb{R}$ be the partial edge labeling described above and $\omega''\colon V(G) \to \mathbb{R}$ be the weighted vertex sums obtained by adding the partial edge labeling to the original vertex weights according to incidence. From the iterative labeling of the first stage, the vertices not incident to any edge in E', $v_1, \ldots, v_s$, have pairwise distinct weighted vertex sums. For each $i \in [k]$, let x_i be the variable for the labeling of edge e_i in E'. Two edge labels or two final vertex sums are the same in G

precisely at zeroes of the polynomial

$$\begin{aligned} g(x_1,\ldots,x_k) &= \prod_{1\le i<j\le k}\left[(x_i-x_j)\prod_{\substack{u\in e_i\\ u'\in e_j}}(x_i+\omega''(u)-x_j-\omega''(u'))\right] \\ &\quad\cdot\prod_{1\le i\le k}\left[\prod_{e\in E''}(x_i-f''(e))\cdot\prod_{1\le j\le s}\prod_{u\in e_i}(x_i+\omega''(u)-\omega''(v_j))\right]. \end{aligned}$$

One can check that a term in the first bracketed product is zero for some particular $i<j$ if and only if either e_i and e_j have been given the same labels or the final vertex sum of an endpoint of e_i matches the final vertex sum of an endpoint of e_j. A term in the second bracketed product is zero for a particular i if and only if the label x_i is already used in E'' or one endpoint of e_i has the same final vertex sum as v_j for any $j\in[s]$. Note that the maximum degree in g is $\binom{k}{2}\cdot 5+k\cdot(2s+m-k)$.

The monomials of maximum degree in g have the same coefficients as they do in polynomial

$$h(x_1,\ldots,x_k)=\prod_{1\le i<j\le k}(x_i-x_j)^5\cdot\prod_{1\le i\le k}x_i^{2s+m-k}.$$

By Lemma 1.10, the monomial

$$x_1^{2(k-1)+(2s+m-k)}x_2^{2(k-1)+1+(2s+m-k)}\cdots x_k^{3(k-1)+(2s+m-k)}$$

has nonzero coefficient in h and thus in g. Note that each edge has a set of at least $m+\left\lfloor\frac{4n}{3}\right\rfloor$ available labels. Recall that $k=\frac{n-s}{2}$ and $s\le\frac{n}{3}$. Hence

$$\begin{aligned} 3(k-1)+(2s+m-k) &= m+n+s-3 \\ &\le m+n+\left\lfloor\frac{n}{3}\right\rfloor-3 \\ &< m+\left\lfloor\frac{4n}{3}\right\rfloor. \end{aligned}$$

By Theorem 1.9, G has a $\left\lfloor\frac{4n}{3}\right\rfloor$–list–weighted–quasi–antimagic labeling. □

Lemma 2.2. *A 3^+–vertex is $\left\lfloor\frac{4n}{3}\right\rfloor$–list–weighted–quasi–antimagic–reducible.*

Proof. Let G be an edge–minimal graph on m edges that is not $\left\lfloor\frac{4n}{3}\right\rfloor$–list–weighted–quasi–antimagic. Suppose that v is a 3^+–vertex with neighbors u_1, u_2, and u_3. Let $G'=G-\{vu_1,vu_2,vu_3\}$. By the choice of G, G' is $\left\lfloor\frac{4n}{3}\right\rfloor$–list–weighted–quasi–antimagic. Let $\omega\colon V(G)\to\mathbb{R}$ and $L\colon E(G)\to 2^{\mathbb{R}}$ such that $|L(e)|\ge m+\left\lfloor\frac{4n}{3}\right\rfloor$ for all $e\in E(G)$. Thus there is a labeling f of $E(G')$ using labels in the lists of its edges that is a $\left\lfloor\frac{4n}{3}\right\rfloor$–list–weighted–quasi–antimagic labeling of G'. We apply the Combinatorial Nullstellensatz to extend f to an edge labeling of G which is $\left\lfloor\frac{4n}{3}\right\rfloor$–list–weighted–quasi–antimagic.

Let x_1, x_2, and x_3 correspond to the labels of edges vu_1, vu_2, and vu_3, respectively. Using $S_{G'}(v)$ to denote the weighted vertex sum of v in G', we define the following polynomial in which respective factors ensure a distinct edge labeling for edges in $\{vu_1,vu_2,vu_2\}$, distinct weighted sums for any pair between $V(G) - \{v, u_1, u_2, u_3\}$ and $\{v, u_1, u_2, u_3\}$, any pair between v and $\{u_1, u_2, u_3\}$, and any pair in $\{u_1, u_2, u_3\}$.

$$\begin{aligned} g(x_1, x_2, x_3) = &\prod_{1 \le i < j \le 3} (x_i - x_j) \cdot \prod_{w \notin \{v, u_1, u_2, u_3\}} (S_{G'}(v) + x_1 + x_2 + x_3 - S_{G'}(w)) \\ &\prod_{i=1}^{3} \prod_{w \notin \{v, u_1, u_2, u_3\}} (x_i + S_{G'}(u_i) - S_{G'}(w)) \cdot \prod_{i=1}^{3} (S_{G'}(v) + x_1 + x_2 + x_3 - x_i - S_{G'}(u_i)) \\ &\prod_{1 \le i < j \le 3} (x_i + S_{G'}(u_i) - x_j - S_{G'}(u_j)). \end{aligned}$$

By construction, $g(x_1, x_2, x_3) = 0$ when $x_i \in L(vu_i) - \{f(e) \colon e \in E(G')\}$ if and only if labels chosen for x_1, x_2, and x_3 do not create a $\lfloor \frac{4n}{3} \rfloor$–list–weighted–quasi–antimagic labeling. Note that

$$\deg(g) \le \binom{3}{2} + (n-4) + 3(n-4) + 3 + \binom{3}{2} = 4n - 7.$$

Therefore the coefficient of any monomial $x_1^a x_2^b x_3^c$, where $a + b + c = 4n - 7$ in g is the same as its coefficient in the polynomial

$$h(x_1, x_2, x_3) = x_1^{n-4} x_2^{n-4} x_3^{n-4} (x_1 + x_2 + x_3)^{n-4} \prod_{1 \le i < j \le 3} (x_i - x_j)^2 (x_i + x_j).$$

Set $a = 4n - 7 - 2 \lfloor \frac{4n-7}{3} \rfloor + 1$, $b = \lfloor \frac{4n-7}{3} \rfloor$, and $c = \lfloor \frac{4n-7}{3} \rfloor - 1$. Using a CAS, it is straightforward to verify that $x_1^a x_2^b x_3^c$ has a nonzero coefficient in h, hence also in g. (Sage [7] code used by the authors is available upon request.)

Define $L'(vu_i) = L(vu_i) - \{f(e) \colon e \in E(G')\}$. Since $|L(vu_i)| \ge m + \lfloor \frac{4n}{3} \rfloor$, we have $|L'(vu_i)| \ge \lfloor \frac{4n}{3} \rfloor + 3$. Thus, by Theorem 1.9, there are labels $f(vu_1)$, $f(vu_2)$, and $f(vu_3)$ in $L'(vu_1)$, $L'(vu_2)$, and $L'(vu_3)$, respectively, for which $g(f(vu_1), f(vu_2), f(vu_3))$ is nonzero. Therefore we obtain a $\lfloor \frac{4n}{3} \rfloor$–list–weighted–quasi–antimagic labeling of G, contradicting the choice of G. □

Instead of proving Theorem 1.5, we prove the following stronger theorem.

Theorem 2.3. *Every graph on n vertices is $\lfloor \frac{4n}{3} \rfloor$–list–weighted–quasi–antimagic.*

Proof. Suppose not and let G be an edge–minimal counterexample. By Lemma 2.2, $\Delta(G) \le 2$. However, our assumption contradicts Lemma 2.1. □

Remark: Taking a similar approach to that in Lemma 2.2 may be advantageous in showing that a d^+ vertex is $\left\lfloor \frac{(d+1)n}{d} \right\rfloor$–list–weighted–quasi–antimagic–reducible.

3 Oriented Antimagic Results

For oriented graphs, we define a slightly different notion of k–quasi–antimagic. An oriented graph G is *k–quasi–oriented–antimagic* if there exists an injective edge labeling from $E(G)$ into $\{1, \ldots, |E(G)| + k\}$ such that the oriented vertex sums are pairwise distinct for pairs of non–isolated vertices, and we call such a labeling a *k–quasi–oriented–antimagic* labeling. The proof of the following lemma is similar to that of Lemma 2.1.

Lemma 3.1. *Let G be a graph with n vertices with $\Delta(G) \leq 2$. The graph G has an orientation that is $\lfloor \frac{2n}{3} \rfloor$–quasi–oriented–antimagic.*

Proof. It suffices to prove the lemma for graphs with $\delta(G) \geq 1$, since adding isolated vertices increases n without adding any additional labeling requirements.

Since $1 \leq \delta(G) \leq \Delta(G) \leq 2$, every component of G is a path or cycle and has at least 2 vertices. Let G have m edges, where $e_1, \ldots, e_q$ are the q isolated edges of G, let $d_1, \ldots, d_r$ be the r even components of G each having at least 4 vertices, and let $c_1, \ldots, c_s$ be the odd components of G. We consider an arbitrary orientation of G, although we may flip the orientation of a few edges in the final stages of this proof. Let E' be a matching in G of maximum size and let $E'' = E(G) - E'$. Notice that $e_1, \ldots, e_q$ are in E'. Thus we may suppose $E' = \{e_1, \ldots, e_k\}$, where $k \geq q$. In particular, $k = \frac{n-s}{2}$. For each $i \in [s]$, define v_i to be the unique vertex in c_i such that v_i is not incident to any edge in E'.

In the first stage of this proof, we create an injective edge labeling on the edges of E'', that is the edges in the d_i and c_i components that are not in the matching. Note that $|E''| = m - \frac{n-s}{2}$. We create the labeling on E'' in such a manner that the oriented vertex sums at vertices not incident to any edge in E' are pairwise distinct. Label the edges of E'' iteratively using labels from $\{1, \ldots, m + \lfloor \frac{2n}{3} \rfloor\}$ such that the label assigned to edge $e = yz \in E''$ is distinct from labels already assigned, of which there are at most $|E''| - 1$, and such that if $v_i \in \{y, z\}$ for some $i \in [s]$, the oriented vertex sum at v_i is distinct from the oriented vertex sum at v_j for each $j \neq i$. With these two restrictions, there are at most $(|E''| - 1) + (s - 1)$ values avoided by labeling e. Since $s \leq \frac{n}{3}$, we have

$$|E''| + s - 2 < m - \frac{n-s}{2} + s = m + \frac{3s}{2} - \frac{n}{2} \leq m < m + \left\lfloor \frac{2n}{3} \right\rfloor.$$

Therefore, such a labeling on E'' is possible.

The second stage of this proof is to label the edges of the maximum matching E' in G. Let $f''\colon E'' \to \mathbb{R}$ be the partial edge labeling and $\omega''\colon V(G) \to \mathbb{R}$ be the oriented vertex sums obtained from the partial edge labeling. From the iterative labeling described in the first stage, the vertices not incident to any edge in E', $v_1, \ldots, v_s$, have pairwise distinct oriented vertex sums. For each $i \in [k]$, let x_i be the variable for the labeling of edge e_i in E'. For each edge e in the orientation of G under consideration, let e^+ denote the endpoint of e toward which e is oriented and let e^- denote the endpoint of e away from which e is oriented. Two edge labels or two final oriented vertex sums are the same in G precisely at zeroes of the polynomial

$$\begin{aligned}
g(x_1, \ldots, x_k) \quad = \quad & \prod_{1 \leq i \leq k} (x_i + \omega''(e_i^+) + x_i - \omega''(e_i^-)) \\
& \cdot \prod_{1 \leq i < j \leq k} (x_i^2 - x_j^2) \\
& \cdot \prod_{1 \leq i < j \leq k} \Big[(x_i + \omega''(e_i^+) - x_j - \omega''(e_j^+))(x_i + \omega''(e_i^+) + x_j - \omega''(e_j^-)) \\
& \qquad\qquad (-x_i + \omega''(e_i^-) + x_j - \omega''(e_j^-))(-x_i + \omega''(e_i^-) - x_j - \omega''(e_j^+)) \Big] \\
& \cdot \prod_{1 \leq i \leq k} \prod_{1 \leq j \leq s} (x_i + \omega''(e_i^+) - \omega''(v_j))(-x_i + \omega''(e_i^-) - \omega''(v_j)) \\
& \cdot \prod_{1 \leq i \leq k} \prod_{e \in E''} (x_i^2 - (f''(e))^2)
\end{aligned}$$

of degree $k + \binom{k}{2}2 + \binom{k}{2}4 + 2ks + 2k(m-k)$. Note that the factors from $\prod_{1\le i<j\le k}(x_i^2 - x_j^2)$ and $\prod_{1\le i\le k}\prod_{e\in E''}(x_i^2 - (f''(e))^2)$ guarantee that labels chosen for edges have distinct absolute values, a fact that will be used to complete the desired edge labeling. The highest degree monomials of g have the same coefficients as they do in the polynomial

$$h(x_1,\dots,x_k) = (-1)^{sk}2^k \cdot \prod_{1\le i<j\le k}(x_i^2 - x_j^2)^3 \cdot \prod_{1\le i\le k} x_i^{1+2s+2(m-k)}.$$

By Lemma 1.10, the monomial

$$x_1^{2[(k-1)]+(1+2s+2(m-k))} x_2^{2[(k-1)+1]+(1+2s+2(m-k))} \cdots x_k^{2[2(k-1)]+(1+2s+2(m-k))}$$

has a nonzero coefficient in h and thus in g. For all $1 \le i \le k$, let $T(e_i) = \{\pm 1, \dots, \pm(m + \lfloor \frac{2n}{3} \rfloor)\}$. Recall that $k = \frac{n-s}{2}$. Since $s \le \lfloor \frac{n}{3} \rfloor$, we have

$$\begin{aligned} 2[2(k-1)] + (1 + 2s + 2(m-k)) &= 2(m+k+s) - 3 \\ &< 2\left(m + \frac{n+s}{2}\right) - 1 \\ &\le 2\left(m + \left\lfloor \frac{2n}{3} \right\rfloor\right). \end{aligned}$$

By Theorem 1.9, there are $f(e_i)$ in $T(e_i)$ such that $h(f(e_1),\dots,f(e_k))$ is nonzero. If $f(e_i) < 0$, switch the initial orientation of that edge and take the absolute value of $f(e_i)$ to obtain a $\lfloor \frac{2n}{3} \rfloor$–quasi–oriented–antimagic labeling of G. □

Lemma 3.2. *A 3^+–vertex in a graph G on n vertices is reducible for the property that there exists an orientation of G that is $\lfloor \frac{2n}{3} \rfloor$–quasi–oriented–antimagic.*

Proof. Let G be an edge–minimal graph that has no orientation that is $\lfloor \frac{2n}{3} \rfloor$–quasi–oriented–antimagic and let $|E(G)| = m$. Suppose that v is a 3^+–vertex with neighbors u_1, u_2, and u_3. Let $G' = G - \{vu_1, vu_2, vu_3\}$. By the choice of G, G' has an orientation D' that is $\lfloor \frac{2n}{3} \rfloor$–quasi–oriented–antimagic. Thus there is a labeling f of $E(D')$ using labels in the set $\{1, \dots, m-3+\lfloor \frac{2n}{3} \rfloor\}$ that is a $\lfloor \frac{2n}{3} \rfloor$–quasi–oriented–antimagic labeling on D'. We apply the Combinatorial Nullstellensatz to find an orientation of G that is $\lfloor \frac{2n}{3} \rfloor$–quasi–oriented–antimagic, in which the orientation and edge labeling on G' are D' and f, respectively.

Let x_1, x_2, and x_3 correspond to the labels of edges vu_1, vu_2, and vu_3, respectively. Consider the following polynomial in which $\omega''(v)$ denotes the oriented vertex sum at v in G':

$$\begin{aligned} g(x_1,x_2,x_3) = &\prod_{1\le i<j\le 3}(x_i^2 - x_j^2) \cdot \prod_{z\notin\{v,u_1,u_2,u_3\}}(\omega''(v) + x_1 + x_2 + x_3 - \omega''(z)) \\ &\prod_{i=1}^{3}\prod_{z\notin\{v,u_1,u_2,u_3\}}(-x_i + \omega''(u_i) - \omega''(z)) \cdot \prod_{i=1}^{3}(\omega''(v) + x_1 + x_2 + x_3 + x_i - \omega''(u_i)) \\ &\prod_{1\le i<j\le 3}(-x_i + \omega''(u_i) + x_j - \omega''(u_j)). \end{aligned}$$

Note that

$$\deg(g) \le 2\binom{3}{2} + (n-4) + 3(n-4) + 3 + \binom{3}{2} = 4n - 4.$$

Therefore the coefficient of any monomial $x_1^a x_2^b x_3^c$, where $a+b+c = 4n-4$ in g is the same as its coefficient in the polynomial

$$h(x_1,x_2,x_3) = (-x_1)^{n-4} \cdot (-x_2)^{n-4} \cdot (-x_3)^{n-4} \cdot (x_1+x_2+x_3)^{n-4} \cdot \prod_{1\le i<j\le 3} (x_i^2 - x_j^2)(x_j - x_i)$$
$$\cdot\, (2x_1+x_2+x_3)\cdot(x_1+2x_2+x_3)\cdot(x_1+x_2+2x_3).$$

Set $a = 4n-4-2\left\lfloor \frac{4n-4}{3} \right\rfloor$, $b = \left\lfloor \frac{4n-4}{3} \right\rfloor$, and $c = \left\lfloor \frac{4n-4}{3} \right\rfloor$. Using a CAS, it is straightforward to verify that $x_1^a x_2^b x_3^c$ has a nonzero coefficient in h, hence also in g. (Sage [7] code used by the authors is available upon request.)

Define $T(vu_i) = \left\{\pm 1, \ldots, \pm\left(m + \left\lfloor \frac{2n}{3} \right\rfloor\right)\right\} - \{\pm f(e) \colon e \in E(G')\}$. Since $|T(vu_i)| \ge 2\left\lfloor \frac{2n}{3} \right\rfloor + 6$, by Theorem 1.9 there are labels $f(vu_1)$, $f(vu_2)$, and $f(vu_3)$ in $T(vu_1)$, $T(vu_2)$, and $T(vu_3)$, respectively, for which $g\big(f(vu_1), f(vu_2), f(vu_3)\big)$ is nonzero.

For $i \in \{1,2,3\}$, if $f(vu_i) > 0$ we orient vu_i from u_i to v, and if $f(vu_i) < 0$ we orient vu_i from v to u_i. The assignment of the label $|f(vu_i)|$ to each vu_i completes an extension of f to a $\left\lfloor \frac{2n}{3} \right\rfloor$–quasi–oriented–antimagic labeling of G, contradicting the choice of G. □

Instead of proving Theorem 1.8, we prove the following stronger theorem.

Theorem 3.3. *Every graph on n vertices admits an orientation that is $\left\lfloor \frac{2n}{3} \right\rfloor$–quasi–oriented–antimagic.*

Proof. Suppose not and let G be an edge–minimal counterexample. By Lemma 3.2, $\Delta(G) \le 2$. However, our assumption contradicts Lemma 3.1. □

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