Matrix Facts

An $m \times n$ **Matrix** $A = (a_{ij})$ is an array of numbers with m rows and n columns and entries a_{ij} for $i = 1...m$ and $j = 1...n$

Multiplication If A is $m \times p$ and B is $p \times n$ then the $m \times n$ matrix $C = AB$ has entries $c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj}$

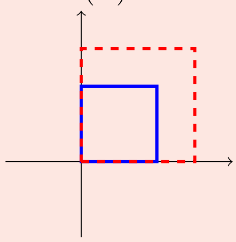
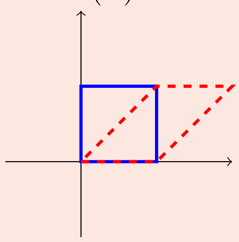
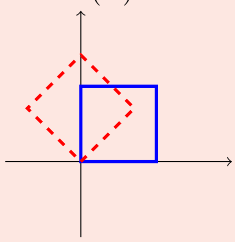
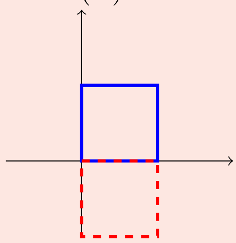
Matrix multiplication is not commutative and so AB is not always BA but it is associative $A(BC) = (AB)C$

The **Transpose** A^T of an $m \times n$ matrix$A$ is an $n \times m$ with the rows and columns swapped so that a_{ij} becomes a_{ji}

Transpose facts: $(A^T)^T = A$ $\quad (A+B)^T = A^T + B^T$ $\quad (AB)^T = B^T A^T$ $\quad A^T A$ is $n \times n$ $\quad AA^T$ is $m \times m$

For 2×2 Matrices A $= \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the **Determinant** $\det(A) = |A| = ad - bc$ and **Inverse** $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

The modulus of the determinant $|\det(T)|$=change in area. If $|\det(T)| = 1$ A is a **Rigid Transformation**

Transformations	Scaling by a factor k $T = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ $\det(T) = k^2$	Shear in x by k $T = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ $\det(T) = 1$	Rotation counter-clockwise by θ $T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\det(T) = 1$	Reflection in the x-axis $T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\det(T) = -1$
				
Inverses	Scaling by a factor $1/k$ $T^{-1} = \begin{bmatrix} 1/k & 0 \\ 0 & 1/k \end{bmatrix}$ $\det(T^{-1}) = 1/k^2$	Shear in x by $-k$ $T^{-1} = \begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$ $\det(T^{-1}) = 1$	Rotation clockwise by θ $T^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ $\det(T^{-1}) = 1$	Reflection in the x-axis $T^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\det(T^{-1}) = -1$

For Real Square $n \times n$ Matrices A

A **Diagonal** matrix is 0 except on the diagonal a_{ii}

An **Identity** matrix I is a diagonal matrix with only 1's on the diagonal so $AI = IA = A$

The **Inverse** A^{-1} if it exists, is such that $AA^{-1} = A^{-1}A = I$

The **Minor** M_{ij} is the determinant of A with row i and column j removed

The **Cofactor** alternates bewteen + or - the minor so $C_{ij} = (-1)^{i+j} M_{ij}$

The **Determinant** can be computed using the **Laplace expansion** whereby each entry of a column is multiplied by its cofactor and summing these $\det(A) = |A| = \sum_{j=1}^{n} C_{ij} a_{ij}$

For example for 3×3 matrices $\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

The **Adjoint** or **Adjugate** Matrix is the transpose of the matrix of cofactors (C_{ji})

The **Inverse** is the adjoint divided by the determinant $A^{-1} = 1/|A|(C_{ji})$

Determinants	Inverses
$\lvert A^T \rvert = \lvert A \rvert$	$(A^T)^{-1} = (A^{-1})^T$
$\lvert AB \rvert = \lvert A \rvert \lvert B \rvert$	$(AB)^{-1} = B^{-1}A^{-1}$
$\lvert kA \rvert = k^n \lvert A \rvert$	$(kA)^{-1} = k^{-1}A^{-1}$
$\lvert D \rvert = \Pi d_{ii}$	$D^{-1} = 1/d_{ii}$
$\lvert A^{-1} \rvert = \lvert A \rvert^{-1}$	

Row (r) and Column (c) operations

Operation	$\lvert A \rvert$
$k \times$r or c	$k\lvert A \rvert$
swap r or c	$-\lvert A \rvert$
add $k\times$ r or c to another r or c	unchanged

Identity

$$\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

Diagonal

$$\begin{pmatrix} \ddots & & \\ & \ddots & \\ & & \ddots \end{pmatrix}$$

Tridiagonal

$$\begin{pmatrix} \ddots & \ddots & \\ \ddots & \ddots & \ddots \\ & \ddots & \ddots \end{pmatrix}$$

Upper Triangular

$$\begin{pmatrix} \ddots & \ddots & \ddots \\ & \ddots & \ddots \\ & & \ddots \end{pmatrix}$$

Lower Triangular

$$\begin{pmatrix} \ddots & & \\ \ddots & \ddots & \\ \ddots & \ddots & \ddots \end{pmatrix}$$

All elements which are not shown are 0

Kernel = **Null Space** is $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$ **Rank** is dim of col/row space

Rank-Nullity Theorem rank(A)+nullity(A)=dim(A)

Non-Singular	**Singular**
$\lvert A \rvert \neq 0$	$\lvert A \rvert = 0$
A^{-1} exists	No A^{-1}
Columns linearly independent	Columns linearly dependent
Rows linearly independent	Rows linearly dependent
A has full rank n	A has rank $< n$
$A\mathbf{x} = \mathbf{b}$ has one solution	$A\mathbf{x} = \mathbf{b}$ has 0 or ∞ solutions
$A\mathbf{x} = \mathbf{0}$ only if $\mathbf{x} = \mathbf{0}$	$A\mathbf{x} = \mathbf{0}$ has other solutions
A has **nullity**=dim(kernel)$= 0$	A has nullity > 0
0 is not an eigenvalue	0 is an eigenvalue

Row Echelon

$$\begin{pmatrix} \cdots & \cdots & \cdots \\ & . & \cdots \\ & & .. \end{pmatrix}$$

Reduced Row Echelon

$$\begin{pmatrix} 1\,.. & \cdots & \cdots \\ & 1\,. & \cdots \\ & & 1\,.. \end{pmatrix}$$

Solving $A\mathbf{x} = \mathbf{b}$

Inverse	$\mathbf{x} = A^{-1}\mathbf{b}$	Slow
Cramer's Rule	$x_i = \lvert A_i\rvert/\lvert A\rvert$ where A_i has col i replaced with $\mathbf{b}$	Slow
Gaussian Elimination	Reduce $(A\lvert\mathbf{b})$ to Row Echelon form with row/col ops	Fast
LU-Decomposition	If $A = LU$ solve $L\mathbf{c} = \mathbf{b}$ then $U\mathbf{x} = \mathbf{c}$	Fast
Iterative Methods	$A = M - N$ Iterate $M\mathbf{x}_{k+1} = N\mathbf{x}_k + \mathbf{b}$	Fast for Sparse

Eigenvalues λ and **Eigenvectors** $\mathbf{x}$ are such that $A\mathbf{x} = \lambda\mathbf{x}$ and the **Characteristic Equation** $det(A - \lambda I) = 0$
The **Trace** $\text{tr}(A)$ is the sum of the elements on the diagonal $\text{tr}(A) = \sum \lambda_i$ and $|A| = \prod \lambda_i$
Complex eigenvalues come in conjugate pairs. Eigenvalues of a diagonal or a triangular matrix are diagonal entries a_{ii}
n distinct eigenvalues is necessary but not sufficient condition for n linearly independent eigenvectors.
A is **Similar** to B if there is a T such that $B = T^{-1}AT$. Similar matrices have the same eigenvalues, rank and det
The **Modal** matrix P has eigenvectors in columns. The **Spectral** matrix D is diagonal with eigenvalues on the diagonal
A is **Diagonalisable** if it is similar to a diagonal matrix so $A = PDP^{-1}$ with P modal matrix and D the spectral
A is **Normal** if it has n orthogonal eigenvectors. Equivalently $A^TA = AA^T$ Also $A = PDP^T$ Its singular values=$|\lambda_i|$
A **Symmetric** matrix is such that $A = A^T$ A **Skew-Symmetric** matrix is such that $A = -A^T$.
Real symmetric matrices are normal. The eigenvalues of a real symmetric matrix are real.
Orthogonal if $A^TA = AA^T = I$ and so the transpose is the inverse. The rows/cols are orthogonal.
For all non-zero $\mathbf{v}$ **Positive Definite** $\mathbf{v}^TA\mathbf{v} > 0$ and so $\lambda_i > 0$ **Negative Definite** $\mathbf{v}^TA\mathbf{v} < 0$ and so $\lambda_i < 0$
For all non-zero $\mathbf{v}$ **Positive Semi-Definite** $\mathbf{v}^TA\mathbf{v} \geq 0$ and so $\lambda_i \geq 0$ **Negative Semi-Definite** $\mathbf{v}^TA\mathbf{v} \leq 0$ and so $\lambda_i \leq 0$
Positive Definite => Semi-Positive Definite => Symmetric => Normal => Diagonisable

For Non-Square $n \times m$ Matrix A

No Eigenvalues or Eigenvectors. No determinant. No inverse.
If A is a real matrix then A^TA and AA^T are real symmetric square semi-positive definite
Singular Values are the square roots of the eigenvlaues of A^TA
A **Semi-orthogonal** matrix is such that either $A^TA = I$ or $AA^T = I$. The longer of the rows or cols are orthogonal.
Norm $||A|| = \max ||A\mathbf{x}||$ for $||\mathbf{x}|| = 1$ and so $||A||_1$ is max column sum $||A||_\infty$ is the max row sum $||A||_2$ is $\sqrt{\lambda_{max}}$ for A^TA

Decomposition	Applies to		
Eigen	Real Square Normal	$A = PDP^T$	P is modal D is spectral
LU	Non-Singular Square	$A = LU$	L lower triangular U Upper triangular
LU with Pivoting	Square	$A = PLU$	P is **Permutation** Matrix which swaps rows
QR	Real Square	$A = QR$	Q is orthogonal R is upper triangular
LDL^T	Symmetric Positive-Definite	$A = LDL^T$	L lower triangular D diagonal $D_{ii} > 0$
Singular Value	Real Non-Square	$A = U\Sigma V^T$	U, V semi-orthogonal Σ diagonal of singular values

Jacobian J	**Hessian** H
For $\mathbf{f}(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^m$	For $f(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$
$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$	$\begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial f^2}{\partial x_n^2} \end{pmatrix}$
$m \times n$	Square Symmetric

At a stationary point:
If H is positive-definite then it is a maximum
If H is negative-definite then it is a minimum
If H has positive and negative eigenvalues then it is a saddle
If H has a zero eigenvalue then inconclusive

For Complex Matrices

Square A is **Hermitian** if it is equal to its **Conjugate Transpose** A^* Its eigenvalues are real
Square A is **Skew-Hermitian Matrix** if $A^* = -A$. Its eigenvalues are purely imaginary
Singular Values are the square roots of the eigenvalues of A^*A
A is **Unitary** if $A^*A = AA^* = I$ A is **Normal** if $AA^* = A^*A$ then A is diagonalizable by a unitary matrix
A Hermitian matrix A is **Positive Definite** if $\mathbf{x}^*A\mathbf{x} > 0$ for all nonzero $\mathbf{x}$
Positive Definite => Semi-Positive Definite => Hermitian => Normal => Diagonisable

Matrix Groups		
General Linear Group	$GL(n)$	$n \times n$ invertible
Special Linear Group	$SL(n)$	$GL(n)$ with $\lvert A\rvert = 1$
Unitary Group	$U(n)$	$GL(n)$ with $A^*A = I$
Special Unitary Group	$SU(n)$	$U(n)$ with $\lvert A\rvert = 1$
Orthogonal Group	$O(n)$	$GL(n)$ with $A^TA = I$
Special Orthogonal Group	$SO(n)$	$O(n)$ with $\det(A) = 1$

Wilson's notorious W

$$\begin{pmatrix} 5 & 7 & 6 & 5 \\ 7 & 10 & 8 & 7 \\ 6 & 8 & 10 & 9 \\ 5 & 7 & 9 & 10 \end{pmatrix}$$

maths_is_not_fun